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THE NUMBER OF PERMUTATIONS REALIZED BY A SHIFT

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ABSTRACT. A permutation π is realized by the shift on N symbols if there is an infinite word on an N-letter alphabet whose successive left shifts by one position are lexicographically in the same relative order as π. The set of realized permutations is closed under consecutive pattern containment. Permutations that cannot be realized are called forbidden patterns. It was shown in [1] that the shortest forbidden patterns of the shift on N symbols have length $N + 2$. In this paper we give a characterization of the set of permutations that are realized by the shift on N symbols, and we enumerate them according to their length.

1. INTRODUCTION AND DEFINITIONS

The original motivation for this paper comes from an innovative application of pattern-avoiding permutations to dynamical systems (see [1, 3]), which is based on the following idea. Given a piecewise monotone map on a one-dimensional interval, consider the finite sequences (orbits) that are obtained by iterating the map, starting from any point in the interval. It turns out that the relative order of the entries in these sequences cannot be arbitrary. This means that, for any given such map, there will be some order patterns that will never appear in any orbit. The set of such patterns, which we call forbidden patterns, is closed under consecutive pattern containment. These facts can be used to distinguish random from deterministic time series.

A natural question that arises is how to determine, for a given map, what its forbidden patterns are. While this problem is wide open in general, in the present paper we study it for a particular kind of maps, called (one-sided) shift systems. Shift systems are interesting for two reasons. One one hand, they exhibit all important features of low-dimensional chaos, such as sensitivity to initial conditions, strong mixing, and a dense set of periodic points. On the other hand, they are natural maps from a combinatorial perspective, and the study of their forbidden patterns can be done in an elegant combinatorial way.

Forbidden patterns in shift systems were first considered in [1]. The authors prove that the smallest forbidden pattern of the shift on N symbols has length $N + 2$. They also conjecture that, for any N, there are exactly six forbidden patterns of minimal length. In the present paper we give a complete characterization of forbidden patterns of shift systems, and enumerate them with respect to their length.

We will start by defining consecutive pattern containment, forbidden patterns in maps, and shift systems, and introducing some notations and background. In Section 2 we give a formula for the parameter that determines how many symbols are needed in order for a permutation to be realized by a shift. This characterizes allowed and forbidden patterns of shift maps. In Section 3 we give another equivalent characterization involving a transformation on permutations, and we prove that the shift on N symbols has six forbidden patterns of minimal length $N + 2$, as conjectured in [1]. In Section 4 we give a formula for the number of patterns of a given length that are realized by the binary shift. In Section 5 we generalize the results from Section 4 to the shift on N symbols, for arbitrary N. We end the paper mentioning some conjectures and open questions in Section 6.

1.1. **Permutations and consecutive patterns.** We denote by $\mathcal{S}_n$ the set of permutations of $\{1, 2, \ldots, n\}$. If $\pi \in \mathcal{S}_n$, we will write its one-line notation as $\pi = [\pi(1), \pi(2), \ldots, \pi(n)]$ (or $\pi = \pi(1)\pi(2)\ldots\pi(n)$ if it creates no confusion). The use of square brackets is to distinguish it from the cycle notation, where π is written as a product of cycles of the form $(i, \pi(i), \pi^2(i), \ldots, \pi^{k-1}(i))$, with $\pi^k(i) = i$. For example, $\pi = [2, 5, 1, 7, 3, 6, 4] = (1, 2, 5, 3)(4, 7)(6)$.

Given a permutation $\pi = \pi(1)\pi(2)\ldots\pi(n)$, let $D(\pi)$ denote the *descent set* of π, that is, $D(\pi) = \{i \,:\, 1 \le i \le n-1, \ \pi(i) > \pi(i+1)\}$. Let $\mathrm{des}(\pi) = |D(\pi)|$ be the number of descents. The Eulerian polynomials are defined by

$$A_n(x) = \sum_{\pi \in \mathcal{S}_n} x^{\mathrm{des}(\pi)+1}.$$

Its coefficients are called the Eulerian numbers. The descent set and the number of descents can be defined for any sequence of integers $a = a_1 a_2 \ldots a_n$ by letting $D(a) = \{i \,:\, 1 \le i \le n-1, \ a_i > a_{i+1}\}$.

Let X be a totally ordered set, and let $x_1, \ldots, x_n \in X$ with $x_1 < x_2 < \cdots < x_n$. Any permutation of these values can be expressed as $[x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)}]$, where $\pi \in \mathcal{S}_n$. We define its *reduction* to be $\rho([x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)}]) = [\pi(1), \pi(2), \ldots, \pi(n)] = \pi$. Note that the reduction is just a relabeling of the entries with the numbers from 1 to n, keeping the order relationships among them. For example $\rho([4, 7, 1, 6.2, \sqrt{2}]) = [3, 5, 1, 4, 2]$. If the values $y_1, \ldots, y_n$ are not all different, then $\rho([y_1, \ldots, y_n])$ is not defined.

Given two permutations $\sigma \in \mathcal{S}_m$, $\pi \in \mathcal{S}_n$, with $m \ge n$, we say that σ *contains π as a consecutive pattern* is there is some i such that $\rho([\sigma(i), \sigma(i+1), \ldots, \sigma(i+n-1)]) = \pi$. Otherwise, we say that σ *avoids π as a consecutive pattern*. The set of permutations in $\mathcal{S}_n$ that avoid π as a consecutive pattern is denoted by $\mathrm{Av}_n(\pi)$. We let $\mathrm{Av}(\pi) = \bigcup_{n \ge 1} \mathrm{Av}_n(\pi)$. Consecutive pattern containment was first studied in [5], where the sets $\mathrm{Av}_n(\pi)$ are enumerated for certain permutations π.

1.2. **Allowed and forbidden patterns in maps.** Let f be a map $f : X \to X$. Given $x \in X$ and $n \ge 1$, we define

$$\mathrm{Pat}(x, f, n) = \rho([x, f(x), f^2(x), \ldots, f^{n-1}(x)]),$$

provided that there is no pair $1 \le i < j \le n$ such that $f^{i-1}(x) = f^{j-1}(x)$. If there is such a pair, then $\mathrm{Pat}(x, f, n)$ is not defined. When it is defined, we have $\mathrm{Pat}(x, f, n) \in \mathcal{S}_n$. If $\pi \in \mathcal{S}_n$ and there is some $x \in X$ such that $\mathrm{Pat}(x, f, n) = \pi$, we say that π is *realized* by f (at x), or that π is an *allowed pattern* of f. The set of all permutations realized by f is denoted by $\mathrm{Allow}(f) = \bigcup_{n \ge 1} \mathrm{Allow}_n(f)$, where

$$\mathrm{Allow}_n(f) = \{\mathrm{Pat}(x, f, n) : x \in X\} \subseteq \mathcal{S}_n.$$

The remaining permutations are called *forbidden patterns*, and denoted by $\mathrm{Forb}(f) = \bigcup_{n \ge 1} \mathrm{Forb}_n(f)$, where $\mathrm{Forb}_n(f) = \mathcal{S}_n \setminus \mathrm{Allow}_n(f)$.

We are introducing some variations to the notation and terminology used in [1, 3]. The main change is that our permutation $\pi = \mathrm{Pat}(x, f, n)$ is essentially the inverse of the permutation of $\{0, 1, \ldots, n-1\}$ that the authors of [1] refer to as the *order pattern defined by x*. Our convention, aside from simplifying the notation, will be more convenient from a combinatorial point of view. The advantage is that now the set $\mathrm{Allow}(f)$ is closed under consecutive pattern containment, in the standard sense used in the combinatorics literature, and we no longer need to talk about *outgrowth forbidden patterns* like in [1]. Indeed, if $\sigma \in \mathrm{Allow}(f)$ and σ contains τ as a consecutive pattern, then $\tau \in \mathrm{Allow}(f)$. An equivalent statement is that if $\pi \in \mathrm{Forb}(f)$, then $\mathrm{Allow}(f) \subseteq \mathrm{Av}(\pi)$. The minimal elements of $\mathrm{Forb}(f)$, i.e., those permutations in $\mathrm{Forb}(f)$ that avoid all other patterns in $\mathrm{Forb}(f)$, will be called *basic forbidden patterns* of f. The set of these patterns will be denoted $\mathrm{BF}(f)$. Note that basic patterns are the inverses of root patterns as defined in [1].

Let us consider now the case in which X is a closed interval in $\mathbb{R}$, with the usual total order on real numbers. An important incentive to study the set of forbidden patterns of a map comes from the following result, which is a consequence of [3].

Proposition 1.1. *If $I \subset \mathbb{R}$ is a closed interval and $f : I \to I$ is piecewise monotone, then $\mathrm{Forb}(f) \neq \emptyset$.*

Recall that piecewise monotone means that there is a finite partition of I into intervals such that f is continuous and strictly monotone on each of those intervals. It fact, it is shown in [3] that for such a map, the number of allowed patterns of f grows at most exponentially, i.e., there is a constant C such that $|\mathrm{Allow}_n(f)| < C^n$ for n large enough. The value of C is related to the *topological entropy* of f (see [3] for details). Since the growth of the total number of permutations of length n is super-exponential, the above proposition follows.

Proposition 1.1, together with the above observation that $\mathrm{Allow}(f)$ is closed under consecutive pattern containment, provides an interesting connection between dynamical systems on one-dimensional interval maps and pattern avoiding permutations. An important application is that forbidden patterns can be used to distinguish random from deterministic time series. Indeed, in a sequence $(x_1, x_2, x_3, \dots)$ where each x_i has been chosen independently at random from some continuous probability distribution, any pattern $\pi \in \mathcal{S}_n$ appears as $\pi = \rho([x_i, x_{i+1}, \dots, x_{i+n-1}])$ for some i with nonvanishing probability, and this probability approaches one as the length of the sequence increases. On the other hand, if the sequence has been generated by defining $x_{k+1} = f(x_k)$ for $k \geq 1$, where $f : I \to I$ is a piecewise monotone map, then Proposition 1.1 guarantees that some patterns (in fact, most of them) will never appear in the sequence. The practical aspect of these applications has been considered in [2].

A structural property of the set of allowed patterns of a map is that it is closed under consecutive pattern containment. A new and interesting direction of research is to study more properties of the sets $\mathrm{Allow}(f)$. Some natural problems that arise are the following.

1. Understand how $\mathrm{Allow}(f)$ and $\mathrm{BF}(f)$ depend on the map f.
2. Describe and/or enumerate (exactly or asymptotically) $\mathrm{Allow}(f)$ and $\mathrm{BF}(f)$ for a particular f.
3. Among the sets of patterns Σ such that $\mathrm{Av}_n(\Sigma)$ grows at most exponentially in n (this is a necessary condition), characterize those for which there exists a map f such that $\mathrm{BF}(f) = \Sigma$.
4. Given a map f, determine the length of its smallest forbidden pattern.
5. Study how $\mathrm{Allow}(f)$ is related to the set of permutations realized by periodic orbits of f, for which several results generalizing Sharkovskii's theorem are known (see [7, 4]).

Most of this paper is devoted to solving problem 2 for a specific family of maps, that we describe next. Before that, it is worth mentioning that problem 5 is briefly considered in [1], where it is explained that for given $\pi \in \mathcal{S}_n$, the endpoints of the components of the set $\{x \in I : \mathrm{Pat}(x, f, n) = \pi\}$ are points of period n and (iterated) preimages of points of period less than n.

1.3. **One-sided shifts.** We will concentrate on the set of allowed patterns of certain maps called *one-sided shift maps*, or simply *shifts* for short. For a detailed definition of the associated dynamical system, called the *one-sided shift space*, we refer the reader to [1].

The totally ordered set X considered above will now be the set $\mathcal{W}_N = \{0, 1, \dots, N-1\}^{\mathbb{N}}$ of infinite words on N symbols, equipped with the lexicographic order. Define the (one-sided) shift transformation

$$\Sigma_N : \quad \begin{array}{ccc} \mathcal{W}_N & \longrightarrow & \mathcal{W}_N \\ w_1 w_2 w_3 \dots & \mapsto & w_2 w_3 w_4 \dots \end{array}$$

We will use Σ instead of Σ_N when it creates no confusion.

Given $w \in \mathcal{W}_N$, $n \geq 1$, and $\pi \in \mathcal{S}_n$, we have from the above definition that $\mathrm{Pat}(w, \Sigma, n) = \pi$ if, for all indices $1 \leq i, j \leq n$, $\Sigma^{i-1}(w) < \Sigma^{j-1}(w)$ if and only if $\pi(i) < \pi(j)$. For example,

(1) $$\mathrm{Pat}(2102212210\dots, \Sigma, 7) = [4, 2, 1, 7, 5, 3, 6],$$

because the relative order of the successive shifts is

$$
\begin{array}{ll}
2102212210\ldots & 4 \\
102212210\ldots & 2 \\
02212210\ldots & 1 \\
2212210\ldots & 7 \\
212210\ldots & 5 \\
12210\ldots & 3 \\
2210\ldots & 6,
\end{array}
$$

regardless of the entries in place of the dots. The case $N = 1$ is trivial, since the only allowed pattern of Σ_1 is the permutation of length 1. In the rest of the paper, we will assume that $N \geq 2$.

If $x \in \{0, 1, \ldots, N - 1\}$, we will use the notation $x^\infty = xxx\ldots$. If $w \in \mathcal{W}_N$, then w_n denotes the n-th letter of w, and we write $w = w_1 w_2 w_3 \ldots$. We will also write $w_{[k,\ell]} = w_k w_{k+1} \ldots w_\ell$, and $w_{[k,\infty)} = w_k w_{k+1} \ldots$. Note that $w_{[k,\infty)} = \Sigma^{k-1}(w)$.

Let $\Upsilon \subset \mathcal{W}_N$ be the set of all words of the form $u(N-1)^\infty$ except the word $(N-1)^\infty$. As shown in [1], $\mathcal{W}_N \setminus \Upsilon$ is closed under one-sided shifts, and the map

$$
\begin{array}{rccc}
\varphi: & \mathcal{W}_N \setminus \Upsilon & \longrightarrow & [0,1] \\
& w_1 w_2 w_3 \ldots & \mapsto & \sum_{i \geq 0} w_i N^{-(i+1)}
\end{array}
$$

is an order-preserving isomorphism (here $[0,1]$ denotes the closed unit interval). The map

$$
\varphi \circ \Sigma_N \circ \varphi^{-1} : [0,1] \to [0,1]
$$

is the so-called *sawtooth map* $x \mapsto Nx \mod 1$, which is piecewise linear (see Figure 1). By Proposition 1.1, this map has forbidden patterns. It is not hard to see that Σ_N has the same set of forbidden patterns as this map, and in particular $\mathrm{Forb}(\Sigma_N) \neq \emptyset$. Forbidden patterns of shift systems were first studied in [1], where the main result is that the smallest forbidden patterns of Σ_N have length $N + 2$.
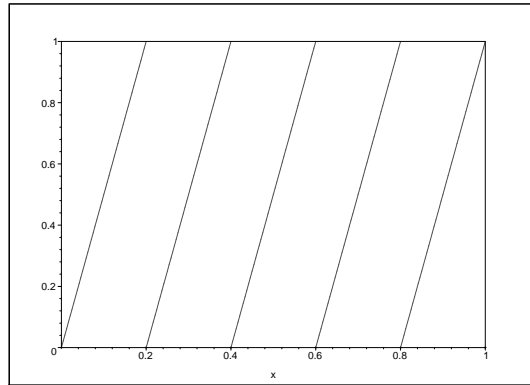

FIGURE 1. The sawtooth map $x \mapsto Nx \mod 1$ for $N = 5$.

Proposition 1.2 ([1]). *Let $N \geq 2$. We have that*

(a) $\mathrm{Forb}_n(\Sigma_N) = \emptyset$ *for every $n \leq N + 1$,*
(b) $\mathrm{Forb}_n(\Sigma_N) \neq \emptyset$ *for every $n \geq N + 2$.*

Example 1. One can check that the smallest forbidden patterns of Σ_4 are

$$
615243, 324156, 342516, 162534, 453621, 435261.
$$

Recall that a word $w \in \{0, 1, \ldots, N - 1\}^k$ is *primitive* if it cannot be written as a power of any proper subword, i.e., it is not of the form $w = u^m$ for any $m > 1$, where the exponent indicates

concatenation of u with itself m times. Let $\psi_N(k)$ denote the number of primitive words of length k over an N-letter alphabet. It is well known that

$$\psi_N(k) = \sum_{d|k} \mu(d) N^{k/d},$$

where μ denotes the Möbius function.

2. THE NUMBER OF SYMBOLS NEEDED TO REALIZE A PATTERN

Given a permutation $\pi \in \mathcal{S}_n$, let $N(\pi)$ be the smallest number N such that $\pi \in \text{Allow}(\Sigma_N)$. The value of $N(\pi)$ indicates what is the minimum number of symbols needed in the alphabet in order for π to be realized by a shift. For example, if $\pi = [4, 2, 1, 7, 5, 3, 6]$, then $N(\pi) \leq 3$ because of (1), and it is not hard to see that $N(\pi) = 3$. The main result in this section is a formula for $N(\pi)$.

Theorem 2.1. *Let $n \geq 2$. For any $\pi \in \mathcal{S}_n$, $N(\pi)$ is given by*

(2) $$N(\pi) = 1 + |A(\pi)| + \Delta(\pi),$$

where

$A(\pi) = \{a : 1 \leq a \leq n-1 \text{ such that if } i = \pi^{-1}(a), j = \pi^{-1}(a+1), \text{ then } i, j < n \text{ and } \pi(i+1) > \pi(j+1)\},$

and $\Delta(\pi) = 0$ except in the following three cases, in which $\Delta(\pi) = 1$:

(I) $\pi(n) \notin \{1, n\}$, and if $i = \pi^{-1}(\pi(n) - 1)$, $j = \pi^{-1}(\pi(n) + 1)$, then $\pi(i + 1) > \pi(j + 1)$;
(II) $\pi(n) = 1$ and $\pi(n - 1) = 2$; or
(III) $\pi(n) = n$ and $\pi(n - 1) = n - 1$.

Note that $A(\pi)$ is the set of entries a in the one-line notation of π such that the entry following $a + 1$ is smaller than the entry following a. For example, if $\pi = [4, 3, 6, 1, 5, 2]$, then $A(\pi) = \{3, 4, 5\}$, so Theorem 2.1 says that $N(\pi) = 1 + 3 + 0 = 4$. The following lemmas will be useful in the proof.

Lemma 2.2. *Suppose that $\text{Pat}(w, \Sigma, n) = \pi$, where $w \in \mathcal{W}_N$ and $\pi \in \mathcal{S}_n$. Define $w' = w'_1 w'_2 \ldots$ by $w'_i = N - 1 - w_i$ for $i \geq 1$, and $\pi' = [\pi'(1), \ldots, \pi'(n)]$ by $\pi'(j) = n + 1 - j$ for $1 \leq j \leq n$. Then, $\text{Pat}(w', \Sigma, n) = \pi'$.*

Proof. It is clear from the definition of w' that for any $i, j \geq 1$, $\Sigma^{i-1}(w) < \Sigma^{j-1}(w)$ if and only if $\Sigma^{i-1}(w') > \Sigma^{j-1}(w')$. Thus, $\text{Pat}(w', \Sigma, n)$ is the only permutation $\pi' \in \mathcal{S}_n$ that satisfies that $\pi(i) < \pi(j)$ if and only if $\pi'(i) > \pi'(j)$, for all $1 \leq i, j \leq n$. $\qquad \square$

Lemma 2.3. *Suppose that $\text{Pat}(w, \Sigma, n) = \pi$. If $1 \leq i, j < n$, $\pi(i) < \pi(j)$, and $\pi(i + 1) > \pi(j + 1)$, then $w_i < w_j$.*

Proof. Clearly $\pi(i + 1) > \pi(j + 1)$ implies that $w_{[i+1,\infty)} > w_{[j+1,\infty)}$ lexicographically. Similarly, $\pi(i) < \pi(j)$ implies that $w_{[i,\infty)} = w_i w_{[i+1,\infty)} < w_j w_{[j+1,\infty)} = w_{[j,\infty)}$. This can only happen if $w_i < w_j$. $\qquad \square$

Lemma 2.4. *Suppose that $\text{Pat}(w, \Sigma, n) = \pi$. If $1 \leq i < k \leq n$ are such that $|\pi(i) - \pi(k)| = 1$, then the word $w_{[i,k-1]}$ is primitive.*

Proof. If $w_{[i,k-1]}$ is not primitive, we can write $w_{[i,k-1]} = q^r$ for some $r > 1$. Let $v = w_{[k,\infty)}$, and let $j = i + |q|$. We have that $w_{[i,\infty)} = q^r v$, $w_{[j,\infty)} = q^{r-1} v$, and $w_{[k,\infty)} = v$.

If $\pi(i) < \pi(j)$, then $q^r v < q^{r-1} v$, which, by canceling the prefix q^{r-l}, implies that $q^l v < q^{l-1} v$ for any $1 \leq l \leq r$. By iteration of this we have that $q^{r-1} v < v$, and thus $\pi(i) < \pi(j) < \pi(k)$, which is a contradiction with $|\pi(i) - \pi(k)| = 1$. If $\pi(i) > \pi(j)$, then the same argument with the inequalities reversed shows that $q^{r-1} v > v$, so $\pi(i) > \pi(j) > \pi(k)$, which is a contradiction as well. $\qquad \square$

We will prove Theorem 2.1 in two parts. In Subsection 2.1 we prove that $1 + |A(\pi)| + \Delta(\pi)$ is a lower bound on $N(\pi)$, and in Subsection 2.2 we prove that it is an upper bound as well.

2.1. **Proof of** $N(\pi) \geq 1 + |A(\pi)| + \Delta(\pi)$**.** To show that the right hand side of (2) is a lower bound on $N(\pi)$, we will prove that if $w \in \mathcal{W}_N$ is such that $\mathrm{Pat}(w, \Sigma, n) = \pi$, then necessarily $N \geq 1 + |A(\pi)| + \Delta(\pi)$. This fact is a consequence of the following lemma.

Lemma 2.5. *Suppose that* $\mathrm{Pat}(w, \Sigma, n) = \pi$, *and let* $b = \pi(n)$. *The entries of* w *satisfy*

$$(3) \qquad\qquad w_{\pi^{-1}(1)} \leq w_{\pi^{-1}(2)} \leq \cdots \leq w_{\pi^{-1}(n)},$$

with strict inequalities $w_{\pi^{-1}(a)} < w_{\pi^{-1}(a+1)}$ *for each* $a \in A(\pi)$. *Additionally, if* $\Delta(\pi) = 1$, *then in each of the three cases from Theorem 2.1 we have, respectively, that*

> (I) *one of the inequalities* $w_{\pi^{-1}(b-1)} \leq w_n \leq w_{\pi^{-1}(b+1)}$ *is strict;*
> (II) $\cdots \leq w_{n+2} \leq w_{n+1} \leq w_n \leq w_{n-1}$ *and one of these inequalities is strict;*
> (III) $w_{n-1} \leq w_n \leq w_{n+1} \leq w_{n+2} \leq \cdots$ *and one of these inequalities is strict.*

In all cases, the entries of w *must satisfy* $|A(\pi)| + \Delta(\pi)$ *strict inequalities.*

Proof. The condition $\mathrm{Pat}(w, \Sigma, n) = \pi$ is equivalent to

$$(4) \qquad\qquad w_{[\pi^{-1}(1),\infty)} < w_{[\pi^{-1}(2),\infty)} < \cdots < w_{[\pi^{-1}(n),\infty)},$$

which clearly implies equation (3). If we remove the term w_n from (3), we get

$$(5) \quad \begin{cases} \text{(a)} & w_{\pi^{-1}(1)} \leq w_{\pi^{-1}(2)} \leq \cdots \leq w_{\pi^{-1}(b-1)} \leq w_{\pi^{-1}(b+1)} \leq \cdots \leq w_{\pi^{-1}(n)} & \text{if } b \notin \{1, n\}, \\ \text{(b)} & w_{\pi^{-1}(2)} \leq w_{\pi^{-1}(3)} \leq \cdots \leq w_{\pi^{-1}(n)} & \text{if } b = 1, \\ \text{(c)} & w_{\pi^{-1}(1)} \leq w_{\pi^{-1}(2)} \leq \cdots \leq w_{\pi^{-1}(n-1)} & \text{if } b = n. \end{cases}$$

For every $a \in A(\pi)$, the inequality $w_{\pi^{-1}(a)} < w_{\pi^{-1}(a+1)}$ in (5) has to be strict, by Lemma 2.3 with $i = \pi^{-1}(a)$ and $j = \pi^{-1}(a+1)$. Let us now see that in the three cases when $\Delta(\pi) = 1$, an additional strict inequality must be satisfied.

Consider first case (I). Let $i = \pi^{-1}(b-1)$ and $j = \pi^{-1}(b+1)$. Since $\pi(i+1) > \pi(j+1)$, Lemma 2.3 implies that $w_i < w_j$, so the inequality $w_{\pi^{-1}(b-1)} < w_{\pi^{-1}(b+1)}$ (equivalently, $w_i < w_j$) in (5a) has to be strict. In case (II), the leftmost inequality in (4) is $w_{[n,\infty)} < w_{[n-1,\infty)}$. For this to hold, we need $\cdots \leq w_{n+2} \leq w_{n+1} \leq w_n \leq w_{n-1}$ and at least one of these inequalities must be strict. Similarly, in case (III), the rightmost inequality in (4) is $w_{[n-1,\infty)} < w_{[n,\infty)}$. This forces $w_{n-1} \leq w_n \leq w_{n+1} \leq w_{n+2} \leq \cdots$ with at least one strict inequality. $\qquad\square$

We will refer to the $|A(\pi)| + \Delta(\pi)$ strict inequalities in Lemma 2.5 as the *required strict inequalities*. Combined with the weak inequalities from the lemma, they force the number of symbols used in w to be at least $1 + |A(\pi)| + \Delta(\pi)$. Examples 2 and 3 illustrate how this lemma is used.

2.2. **Proof of** $N(\pi) \leq 1 + |A(\pi)| + \Delta(\pi)$**.** We will show how for any given $\pi \in \mathcal{S}_n$ one can construct a word $w \in \mathcal{W}_N$ with $\mathrm{Pat}(w, \Sigma, n) = \pi$, where $N = 1 + |A(\pi)| + \Delta(\pi)$. We need w to satisfy condition (4). Again, let $b = \pi(n)$.

The first important observation is that, if we can only use N different symbols, then the $|A(\pi)| + \Delta(\pi) = N - 1$ required strict inequalities from Lemma 2.5 determine the values of the entries $w_1 w_2 \ldots w_{n-1}$. This fact is restated as Corollary 2.13. Consequently, we are forced to assign values to these entries as follows:

> (a) If $b \notin \{1, n\}$, assign values to the variables in (5a) from left to right, starting with $w_{\pi^{-1}(1)} = 0$ and increasing the value by 1 at each required strict inequality.
> (b) If $b = 1$, assign values to the variables in (5b) from left to right, starting with $w_{\pi^{-1}(2)} = 0$ if $\pi(n-1) \neq 2$, or with $w_{\pi^{-1}(2)} = 1$ if $\pi(n-1) = 2$ (this is needed in order for condition (II) in Lemma 2.5 to hold), and increasing the value by 1 at each required strict inequality.
> (c) If $b = n$, assign values to the variables in (5c) from left to right, starting with $w_{\pi^{-1}(1)} = 0$ and increasing the value by 1 at each required strict inequality. (Note that when $\Delta(\pi) = 1$, the last assigned value is $w_{\pi^{-1}(n-1)} = w_{n-1} = |A(\pi)| = N - 2$.)

It remains to assign the values to w_m for $m \geq n$. Before we do this, let us prove some facts about the entries $w_1 \ldots w_{n-1}$. In the following three lemmas, π is any permutation in $\mathcal{S}_n$ with $N(\pi) = N$ and $w_1 \ldots w_{n-1}$ are the values in $\{0, 1, \ldots, N-1\}$ assigned above in order to satisfy the required strict inequalities.

Lemma 2.6. *Let $i < n$. If $\pi(i) > \pi(i+1)$, then $w_i \geq 1$.*

Proof. In the case that $\pi(n) = 1$ and $\pi(n-1) = 2$, our construction makes $w_j \geq 1$ for $1 \leq j \leq n-1$, so this case is trivial.

In any other case, we will show that in equation (5) there is some required strict inequality to the left of w_i. Let $c = \pi(i)$. Consider the sequence $1, 2, \ldots, c$ and remove the entry $\pi(n)$ if it is in the sequence. Let $a_1, \ldots, a_s$ be the resulting sequence (note that $s = c - 1$ or $s = c$ depending on whether $\pi(n) \leq c$ or not). For $1 \leq \ell \leq s$, let b_ℓ be the entry following a_ℓ in the one-line notation of π. We have that $a_1 < b_1$ and $a_s = c > b_s = \pi(i+1)$. Since $a_{\ell+1} - a_\ell \leq 2$ for every ℓ and all the b_ℓ's are different, there has to be some r such that $b_r > b_{r+1}$. Now the inequality $w_{\pi^{-1}(a_r)} < w_{\pi^{-1}(a_{r+1})}$ must be one of the required strict inequalities, because either it is of the form $w_{\pi^{-1}(a)} < w_{\pi^{-1}(a+1)}$ with $a \in A(\pi)$, or it is $w_h < w_j$ with $h = \pi^{-1}(b-1)$, $j = \pi^{-1}(b+1)$, and $\pi(h+1) > \pi(j+1)$. $\square$

Lemma 2.7. *Let $i < n$. If $\pi(i) < \pi(i+1)$, then $w_i \leq N - 2$.*

Proof. In the case that $\pi(n) = n$ and $\pi(n-1) = n-1$, it follows from our construction that $w_j \leq |A(\pi)| = N - 2$ for $1 \leq j \leq n-1$, so this case is trivial.

In any other case, we will show that in equation (5) there is some required strict inequality to the right of w_i. Let $c = \pi(i)$. Consider the sequence $c, c+1, \ldots, n$ and remove the entry $\pi(n)$ if it is in the sequence. Let $a_1, \ldots, a_s$ be the resulting sequence. For $1 \leq \ell \leq s$, let b_ℓ be the entry following a_ℓ in the one-line notation of π. We have that $a_1 = c < b_1 = \pi(i+1)$ and $a_s > b_s$. Since $a_{\ell+1} - a_\ell \leq 2$ for every ℓ and all the b_ℓ's are different, there has to be some r such that $b_r > b_{r+1}$. Now the inequality $w_{\pi^{-1}(a_r)} < w_{\pi^{-1}(a_{r+1})}$ must be one of the required strict inequalities, as in the proof of Lemma 2.6. $\square$

Lemma 2.8. *If $1 \leq i, j < n$ are such that $\pi(i) < \pi(j)$ and $\pi(i+1) > \pi(j+1)$, then $w_i < w_j$.*

Proof. We will show that in equation (5) there is some required strict inequality between w_i and w_j. Let $a_1, \ldots, a_s$ be the ordered sequence of integers between $\pi(i)$ and $\pi(j)$, both included, after $\pi(n)$ has been removed from it (if it appeared). For $1 \leq \ell \leq s$, let b_ℓ be the entry following a_ℓ in the one-line notation of π. The fact that $\pi(i+1) = b_1 > b_\ell = \pi(j+1)$ implies that there has to be some $1 \leq r < s$ such that $b_r > b_{r+1}$. Now the inequality $w_{\pi^{-1}(a_r)} < w_{\pi^{-1}(a_{r+1})}$ has to be strict, as in the proof of Lemma 2.6. $\square$

Once the values $w_1 \ldots w_{n-1}$ have been determined, there are several ways to assign values to w_m for $m \geq n$. Here we give a few different possibilities. Each one has some restriction on the value of $b = \pi(n)$.

A. Assume that $b \neq n$. Let $k = \pi^{-1}(b+1)$. Let $u = w_1 w_2 \ldots w_{k-1}$ and $p = w_k w_{k+1} \ldots w_{n-1}$. Let m be any integer satisfying $m \geq 1 + \frac{n-2}{n-k}$ (for definiteness, we can pick $m = n-1$). Let $w_A(\pi) = u p^m 0^\infty$.

B. Assume that $b \neq 1$. Let $k = \pi^{-1}(b-1)$. Let $u = w_1 w_2 \ldots w_{k-1}$ and $p = w_k w_{k+1} \ldots w_{n-1}$. Again, let m be such that $m \geq 1 + \frac{n-2}{n-k}$ (for definiteness, we can pick $m = n-1$). Let $w_B(\pi) = u p^m (N-1)^\infty$.

C. Assume that $b = 1$. Let $w_C(\pi) = w_1 w_2 \ldots w_{n-1} 0^\infty$.

D. Assume that $b = n$. Let $w_D(\pi) = w_1 w_2 \ldots w_{n-1} (N-1)^\infty$.

E. Assume that $1 < b < n$, and that $w_{\pi^{-1}(b-1)} < w_{\pi^{-1}(b+1)}$ (this happens when $\Delta(\pi) = 1$). Let $c = w_{\pi^{-1}(b-1)}$. Define $w_E(\pi) = w_1 w_2 \ldots w_{n-1} c(N-1)^\infty$ and $w_F(\pi) = w_1 w_2 \ldots w_{n-1} (c+1) 0^\infty$.

Clearly, $w = w_*(\pi)$ uses N different symbols in each one of the above cases. It remains to prove that $\mathrm{Pat}(w, \Sigma, n) = \pi$, which is equivalent to showing that w satisfies condition (4). Let us now prove that this is the case for $w = w_A(\pi)$, when $b \neq n$.

In the following three lemmas and in Proposition 2.12, π is any permutation in $\mathcal{S}_n$ with $\pi(n) \neq n$, and $w = w_A(\pi)$. Also, k, u, p and m are as defined in case A above.

Lemma 2.9. *The word $p = w_k w_{k+1} \ldots w_{n-1}$ has some nonzero entry.*

Proof. Since $\pi(k) = \pi(n) + 1 > \pi(n)$, there has to be a descent in the sequence $\pi(k), \pi(k + 1), \ldots, \pi(n-1), \pi(n)$. Let $k \leq i \leq n-1$ be such that $\pi(i) > \pi(i+1)$. By Lemma 2.6, $w_i \geq 1$, so we are done. $\square$

Lemma 2.10. *The word $p = w_k w_{k+1} \ldots w_{n-1}$ is primitive.*

Proof. Assume for contradiction that $p = q^r$ for some $r > 1$. Let $s = |q|$. Then, $k \leq i < n - s$, we have that $w_i = w_{i+s}$. There are now two possibilities depending on the relationship between $\pi(k)$ and $\pi(k + s)$.

If $\pi(k) < \pi(k + s)$, then by Lemma 2.8, $\pi(k + 1) < \pi(k + s + 1)$. By iterating this argument, we see that $\pi(i) < \pi(i + s)$ for $k \leq i \leq n - s$. In particular, $\pi(k) < \pi(k + s) < \pi(k + 2s) < \cdots < \pi(k + rs) = \pi(n)$, which contradicts the choice of k as the index such that $\pi(k) = \pi(n) + 1$.

If $\pi(k) > \pi(k+s)$, then similarly by Lemma 2.8 we have that $\pi(i) > \pi(i+s)$ for $k \leq i \leq n-s$. In particular, $\pi(k) > \pi(k+s) > \cdots > \pi(k+rs) = \pi(n)$. Since $r > 1$, this implies that $\pi(k) - \pi(n) \geq 2$, which contradicts again the choice of k. $\square$

Lemma 2.11. *We have that $w_{[n,\infty)} < w_{[k,\infty)}$. Moreover, there is no $1 \leq s \leq n$ such that $w_{[n,\infty)} < w_{[s,\infty)} < w_{[k,\infty)}$.*

Proof. For the first part, note that $w_{[n,\infty)} = p^{m-1} 0^\infty < w_{[k,\infty)} = p^m 0^\infty$ because p has some nonzero entry, by Lemma 2.9. For the second part, assume there is an s such that $p^{m-1} 0^\infty < w_{[s,\infty)} < p^m 0^\infty$. Then $w_{[s,\infty)} = p^{m-1} v$, with $0^\infty < v < p0^\infty$. Since p is primitive (by Lemma 2.10), the initial p in $w_{[s,\infty)}$ cannot overlap simultaneously with the two occurrences of p at the beginning of $w_{[k,\infty)}$. Given that $s \neq k, n$, this only leaves the possibility that $s < k$. If some of the occurrences of p in $w_{[s,\infty)} = p^{m-1} v$ and in $w_{[k,\infty)} = p^m 0^\infty$ coincide, then the condition $v < p0^\infty$ does not hold. The only remaining possibility is that the $m - 1$ initial occurrences of p in $w_{[s,\infty)}$ are entirely contained in $u = w_1 w_2 \ldots w_{k-1}$. However, the choice of m satisfying $m \geq 1 + \frac{n-2}{n-k}$ was made so that $|p^{m-1}| = (n-k)(m-1) \geq n-2$. This only leaves the case that $s = 1$ and $u = p^{m-1}$, but then $v = w_{[k,\infty)} = p^m 0^\infty$, so the condition $v < p0^\infty$ does not hold. $\square$

The next proposition proves that $\mathrm{Pat}(w_A(\pi), \Sigma, n) = \pi$.

Proposition 2.12. *If $1 \leq i, j \leq n$ are such that $\pi(i) < \pi(j)$, then $w_{[i,\infty)} < w_{[j,\infty)}$.*

Proof. Let $S(i, j)$ be the statement "$\pi(i) < \pi(j)$ implies $w_{[i,\infty)} < w_{[j,\infty)}$". We want to prove $S(i, j)$ for all $1 \leq i, j \leq n$ with $i \neq j$. We consider three cases.

- *Case $i = n$.* Assume that $\pi(n) < \pi(j)$. By Lemma 2.11 we know that $w_{[n,\infty)} < w_{[k,\infty)}$. If $j = k$ we are done. If $j \neq k$, then $\pi(n) < \pi(j)$ implies that $\pi(k) < \pi(j)$, since $\pi(k) = \pi(n)+1$. So, if $S(k, j)$ holds, then $S(n, j)$ must hold as well. We have reduced $S(n, j)$ to $S(k, j)$. Equivalently, $\neg S(n, j) \Rightarrow \neg S(k, j)$, where $\neg$ denotes negation.
- *Case $j = n$.* Assume that $\pi(i) < \pi(n)$, so in particular $i \neq k$. By the second part of Lemma 2.11, in order to prove that $w_{[i,\infty)} < w_{[n,\infty)}$ it is enough to show that $w_{[i,\infty)} < w_{[k,\infty)}$. Also, $\pi(i) < \pi(n)$ implies that $\pi(i) < \pi(k)$. We have reduced $S(i, n)$ to $S(i, k)$.
- *Case $i, j < n$.* Assume that $\pi(i) < \pi(j)$. If $w_i < w_j$, then $w_{[i,\infty)} < w_{[j,\infty)}$ and we are done. If $w_i = w_j$, then we know by Lemma 2.8 that $\pi(i + 1) < \pi(j + 1)$. If we can show that $w_{[i+1,\infty)} < w_{[j+1,\infty)}$, then $w_{[i,\infty)} = w_i w_{[i+1,\infty)} < w_j w_{[j+1,\infty)} = w_{[j,\infty)}$. So, we have reduced $S(i, j)$ to $S(i + 1, j + 1)$.

The above three cases prove that for all $1 \leq i, j \leq n - 1$, $\neg S(i, j) \Rightarrow \neg S(g(i), g(j))$, where g is defined for $1 \leq i \leq n - 1$ as

$$g(i) = \begin{cases} i + 1 & \text{if } i < n - 1, \\ k & \text{if } i = n - 1. \end{cases}$$

Suppose now that $S(i, j)$ does not hold for some i, j. Using the first two cases above, we can assume that $1 \leq i, j \leq n - 1$. Then, $S(g^{\ell}(i), g^{\ell}(j))$ fails for every $\ell \geq 1$. Let r be the index for which $\pi(r)$ is maximum among those r with $k \leq r \leq n - 1$. Let ℓ be such that $g^{\ell}(i) = r$ and $k \leq g^{\ell}(j) \leq n - 1$ (such an ℓ always exists). Then $S(g^{\ell}(i), g^{\ell}(j))$ must hold because $\pi(g^{\ell}(i)) \geq \pi(g^{\ell}(j))$. This is a contradiction, so the proposition is proved. $\square$

If $b \neq 1$, proving that $w = w_B(\pi)$ satisfies $\mathrm{Pat}(w, \Sigma, n) = \pi$ is analogous to our argument for $w = w_A(\pi)$. Instead of Lemma 2.6 we would use Lemma 2.7, and the analogue of Lemma 2.9 is the fact that $p = w_k w_{k+1} \ldots w_{n-1}$ has some entry strictly smaller than $N - 1$. Lemma 2.11 would be replaced with the fact that $w_{[k,\infty)} < w_{[n,\infty)}$ and there is no $1 \leq s \leq n$ such that $w_{[k,\infty)} < w_{[s,\infty)} < w_{[n,\infty)}$. With these variations, Proposition 2.12 can be proved similarly for $w = w_B(\pi)$.

We can complete the proof of the upper bound on $N(\pi)$ as follows. Let $\pi \in \mathcal{S}_n$ be given, and let $N = 1 + |A(\pi)| + \Delta(\pi)$. If $\pi(n - 1) > \pi(n)$, let $w = w_A(\pi)$. If $\pi(n - 1) < \pi(n)$, let $w = w_B(\pi)$. Since $\mathrm{Pat}(w, \Sigma, n) = \pi$ and $w \in \mathcal{W}_N$, the theorem is proved.

Example 2. Let $\pi = [4, 3, 6, 1, 5, 2]$. By Theorem 2.1, $N(\pi) = 4$. If $\mathrm{Pat}(w, \Sigma, n) = \pi$, then Lemma 2.5 implies that $w_4 \leq w_6 \leq w_2 < w_1 < w_5 < w_3$, and there are no more required strict inequalities. We assign $w_4 = w_2 = 0$, $w_1 = 1$, $w_5 = 2$, $w_3 = 3$. Since $\pi(5) > \pi(6)$ and $b = \pi(6) = 2$, we can take $w = w_A(\pi)$ (with $m = 2$), so $k = \pi^{-1}(3) = 2$, $u = w_1 = 1$, and $p = w_2 w_3 w_4 w_5 = 0302$. We get $w = up^2 0^{\infty} = 1030203020^{\infty}$.

The following consequence of the proof of Theorem 2.1 will be used in Section 4.

Corollary 2.13. *Let $\pi \in \mathcal{S}_n$, $N = N(\pi)$, and let $w \in \mathcal{W}_N$ be such that $\mathrm{Pat}(w, \Sigma, n) = \pi$. Then the entries $w_1 w_2 \ldots w_{n-1}$ are uniquely determined by π.*

Note that, however, with the conditions of Corollary 2.13, w_n is not always determined. In the case that $\pi(n) \notin \{1, n\}$ and $\Delta(\pi) = 1$, we have two choices for w_n. In general, there is a lot of flexibility in the choice of w_m for $m \geq n$. The choices $w = w_A(\pi)$ and $w = w_B(\pi)$ in the proof of Theorem 2.1 were made to simplify the proof of Proposition 2.12 for all cases at once. It is not hard to modify the proof to see that $w_C(\pi)$, $w_D(\pi)$, $w_E(\pi)$ and $w_F(\pi)$ work as well, in the cases where they are defined. Let us see some examples of different ways to find a word $w \in \mathcal{W}_N$ such that $\mathrm{Pat}(w, \Sigma, n) = \pi$, for given $\pi \in \mathcal{S}_n$ with $N(\pi) = N$.

Example 3. (i) Let $\pi = [8, 9, 3, 1, 4, 6, 2, 7, 5]$. If $\mathrm{Pat}(w, \Sigma, n) = \pi$, then by Lemma 2.5 we must have $w_4 \leq w_7 < w_3 \leq w_5 \leq w_9 \leq w_6 \leq w_8 \leq w_1 < w_2$, with a strict inequality $w_5 < w_6$ caused by the fact that $\Delta(\pi) = 1$. We have $N(\pi) = 4 = 1 + |\{2, 8\}| + 1$. If $w \in \mathcal{W}_4$, the entries $w_4 = w_7 = 0$, $w_3 = w_5 = 1$, $w_6 = w_8 = w_1 = 2$, $w_2 = 3$ are forced. For the remaining entries we could take for example $w_{[9,\infty)} = 13^{\infty}$, obtaining $w = w_E(\pi) = 2310120213^{\infty}$.

(ii) If $\pi = [3, 5, 2, 4, 1]$, then Lemma 2.5 tells us that $w_5 \leq w_3 \leq w_1 < w_4 \leq w_2$ and there are no more required strict inequalities. We can take $w = w_C(\pi) = 010100^{\infty}$.

(iii) If $\pi = [2, 3, 1, 4, 5, 6]$, then by Lemma 2.5 we have that $w_3 < w_1 < w_2 \leq w_4 \leq w_5 \leq w_6$, and there must be some strict inequality in $w_5 \leq w_6 \leq w_7 \leq \cdots$. We can take $w = w_D(\pi) = 1202233^{\infty}$.

Note that in terms of π^{-1}, the set $A(\pi)$ can alternatively be described as

$$A(\pi) = \{i : \pi^{-1}(i) + 1 \text{ appears to the right of } \pi^{-1}(i + 1) + 1 \text{ in the one-line notation for } \pi^{-1}\}.$$

For instance, if π is the permutation in part (i) of the above example, we have $\pi^{-1} = [4, 7, 3, 5, 9, 6, 8, 1, 2]$, so $A(\pi) = \{2, 8\}$.

3. An equivalent characterization

We start this section by giving an expression for $N(\pi)$ that is sometimes more convenient to work with than the one in Theorem 2.1. We denote by $\mathcal{C}_n$ the set of permutations in $\mathcal{S}_n$ whose cycle decomposition consists of a unique cycle of length n. Let $\mathcal{T}_n$ be the set of permutations $\pi \in \mathcal{C}_n$ with one distinguished entry $\pi(i)$, for some $1 \leq i \leq n$. We call the elements of $\mathcal{T}_n$ *marked cycles*. We will use the symbol $\star$ to denote the distinguished entry, both in one-line and in cycle notation. Note that it is not necessary to keep track of its value, since it is determined once we know all the remaining entries. For example, $\mathcal{T}_3 = \{[\star, 3, 1], [2, \star, 1], [2, 3, \star], [\star, 1, 2], [3, \star, 2], [3, 1, \star]\}$. Clearly, $|\mathcal{T}_n| = (n-1)! \cdot n = n!$, since there are n ways to choose the distinguished entry.

Define a map

$$\theta : \begin{array}{ccc} \mathcal{S}_n & \to & \mathcal{T}_n \\ \pi & \mapsto & \hat{\pi} \end{array}$$

as follows. For each $1 \leq i \leq n$ with $i \neq \pi(n)$, let $\hat{\pi}(i)$ be the entry immediately to the right of i in the one-line notation of π. For $i = \pi(n)$, let $\hat{\pi}(i) = \star$ be the distinguished entry.

We can also give the following equivalent definition of $\hat{\pi}$. If $\pi = [\pi(1), \pi(2), \ldots, \pi(n)]$, then $\hat{\pi}$ is the permutation with cycle decomposition $(\pi(1), \pi(2), \ldots, \pi(n))$ with the entry $\pi(1)$ distinguished. We write $\hat{\pi} = (\star, \pi(2), \ldots, \pi(n))$. For example, if $\pi = [8, 9, 2, 3, 6, 4, 1, 5, 7]$, then $\hat{\pi} = (\star, 9, 2, 3, 6, 4, 1, 5, 7)$, or in one-line notation, $\hat{\pi} = [5, 3, 6, 1, 7, 4, \star, 9, 2]$.

The map θ is a bijection between $\mathcal{S}_n$ and $\mathcal{T}_n$, since it is clearly invertible. Indeed, to recover π from $\hat{\pi} \in \mathcal{T}_n$, write $\hat{\pi}$ in cycle notation, replace the $\star$ with the entry in $\{1, \ldots, n\}$ that is missing, and turn the parentheses into brackets, thus recovering the one-line notation of π.

For $\hat{\pi} \in \mathcal{T}_n$, let $\operatorname{des}(\hat{\pi})$ denote the number of descents of the sequence that we get by deleting the $\star$ from the one-line notation of $\hat{\pi}$. That is, if $\hat{\pi} = [a_1, \ldots, a_j, \star, a_{j+1}, \ldots, a_{n-1}]$, then $\operatorname{des}(\hat{\pi}) = |\{i : 1 \leq i \leq n-2, a_i > a_{i+1}\}|$. We can now state a simpler formula for $N(\pi)$.

Proposition 3.1. *Let $\pi \in \mathcal{S}_n$, $\hat{\pi} = \theta(\pi)$. Then $N(\pi)$ is given by*

$$N(\pi) = 1 + \operatorname{des}(\hat{\pi}) + \epsilon(\hat{\pi}),$$

where

$$\epsilon(\hat{\pi}) = \begin{cases} 1 & \text{if } \hat{\pi} = [\star, 1, \ldots] \text{ or } \hat{\pi} = [\ldots, n, \star], \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We compare this formula with the one in Theorem 2.1. If $\pi(n) \notin \{1, n\}$, then $\epsilon(\hat{\pi}) = 0$, and $\operatorname{des}(\hat{\pi}) = |A(\pi)| + 1$ or $|A(\pi)|$ depending on whether $\pi(i+1) > \pi(j+1)$ or not, where $i = \pi^{-1}(\pi(n) - 1)$ and $j = \pi^{-1}(\pi(n) + 1)$. If $\pi(n) \in \{1, n\}$, then $\operatorname{des}(\hat{\pi}) = |A(\pi)|$ and $\epsilon(\hat{\pi}) = \Delta(\pi)$. In all cases, the two formulas for $N(\pi)$ are equivalent. $\square$

For example, if $\pi = [8, 9, 2, 3, 6, 4, 1, 5, 7]$, then $\hat{\pi} = [5, 3, 6, 1, 7, 4, \star, 9, 2]$ has 4 descents, so $N(\pi) = 1 + 4 + 0 = 5$. If $\pi = [8, 9, 3, 1, 4, 6, 2, 7, 5]$, then $\hat{\pi} = [4, 7, 1, 6, \star, 2, 5, 9, 3]$ has 3 descents, so $N(\pi) = 1 + 3 + 0 = 4$. If $\pi = [3, 4, 2, 1]$, then $\hat{\pi} = [\star, 1, 4, 2]$ has 1 descent, so $N(\pi) = 1 + 1 + 1 = 3$.

If $\pi \in \mathcal{S}_n$, we have by definition that $N(\pi) = \min\{N : \pi \notin \operatorname{Forb}_n(\Sigma_N)\} = \min\{N : \pi \in \operatorname{Allow}_n(\Sigma_N)\}$. As a consequence of Proposition 3.1 we recover Proposition 1.2, which in terms of the statistic $N(\pi)$ can be reformulated as follows.

Corollary 3.2. *Let $n \geq 3$. We have that*

(a) *for every $\pi \in \mathcal{S}_n$, $N(\pi) \leq n - 1$;*
(b) *there is some $\pi \in \mathcal{S}_n$ such that $N(\pi) = n - 1$.*

Proof.   (a) Since $\hat{\pi}$ has $n-1$ entries other than the $\star$, it can have at most $n-2$ descents. Besides, in the cases when $\epsilon(\hat{\pi}) = 1$, we have necessarily $\operatorname{des}(\hat{\pi}) \leq n - 3$. Thus, by Proposition 3.1, $N(\pi) = 1 + \operatorname{des}(\hat{\pi}) + \epsilon(\hat{\pi}) \leq n - 1$ in any case.

(b) If $n = 2m$ is even, we can take for example $\pi = [m, m+1, m-1, m+2, m-2, m+3, \ldots, 1, 2m]$, so $\hat{\pi} = [2m, 2m-1, \ldots, m+1, m-1, m-2, \ldots, 1, \star]$. If $n = 2m+1$ is odd, take $\pi = [m+1, m, m+2, m-1, m+3, m-2, \ldots, 1, 2m+1]$, so $\hat{\pi} = [2m+1, 2m, \ldots, m+2, m, m-1, \ldots, 1, \star]$. In both cases, $\mathrm{des}(\hat{\pi}) = n - 2$, so $N(\pi) = n - 1$.

$\square$

We define $\mathcal{S}_{n,N} = \{\pi \in \mathcal{S}_n : N(\pi) = N\}$. We are interested in the numbers $a_{n,N} = |\mathcal{S}_{n,N}|$. To avoid the trivial cases, we will assume that $n, N \geq 2$. From the definitions,

$$\mathrm{Allow}_n(\Sigma_M) = \bigcup_{N=2}^{M} \mathcal{S}_{n,N}, \qquad \mathrm{Forb}_n(\Sigma_M) = \bigcup_{N=M+1}^{n-1} \mathcal{S}_{n,N}.$$

Since the sets $\mathcal{S}_{n,N}$ are disjoint, we have that

$$|\mathrm{Allow}_n(\Sigma_M)| = \sum_{N=2}^{M} a_{n,N}, \qquad |\mathrm{Forb}_n(\Sigma_M)| = \sum_{N=M+1}^{n-1} a_{n,N}.$$

The first few values of $a_{n,N}$ are given in Table 1. It follows from Lemma 2.2 that all the $a_{n,N}$ are even.

$n \backslash N$	2	3	4	5	6	7
2	2					
3	6					
4	18	6				
5	48	66	6			
6	126	402	186	6		
7	306	2028	2232	468	6	
8	738	8790	19426	10212	1098	6

TABLE 1. The numbers $a_{n,N} = |\{\pi \in \mathcal{S}_n : N(\pi) = N\}|$ for $n \leq 8$.

The next result shows that, independently of n, there are exactly six permutations of length n that require the maximum number of symbols (i.e., $n-1$) in order to be realized. This settles a conjecture from [1]. Given a permutation $\pi \in \mathcal{S}_n$, we will use π^{rc} to denote the permutation such that $\pi^{rc}(i) = n + 1 - \pi(n + 1 - i)$ for $1 \leq i \leq n$. If σ is a marked cycle, then σ^{rc} is defined similarly, where if $\sigma(i)$ is the marked entry of π, then $\sigma^{rc}(n + 1 - i)$ is the marked entry of σ^{rc}. It will be convenient to visualize $\pi \in \mathcal{S}_n$ as an $n \times n$ array with dots in positions $(i, \pi(i))$, for $1 \leq i \leq n$. The first coordinate refers to the row number, which increases from left to right, and the second coordinate is the column number, which increases from bottom to top. Then, the array of π^{rc} is obtained from the array of π by a 180-degree rotation. Of course, the array of π^{-1} is obtained from the one of π by reflecting it along the diagonal $y = x$. Notice also that the cycle structure of π is preserved in π^{-1} and in π^{rc}. A marked cycle can be visualized in the same way, replacing the dot corresponding to the distinguished element with a $\star$.

Proposition 3.3. *For every $n \geq 3$, $a_{n,n-1} = 6$.*

Proof. First we show (this part was already proved in [1]) that $a_{n,n-1} \geq 6$, by giving six permutations in this set. Let $m = \lceil n/2 \rceil$, and let

$$\sigma = [n, n-1, \ldots, m+1, \star, m, m-1, \ldots, 2], \quad \tau = [\star, 1, n, n-1 \ldots, m+2, m, m-1, \ldots, 2] \in \mathcal{T}_n$$

(see Figure 2). Using Proposition 3.1, it is easy to check that if $\hat{\pi} \in \{\sigma, \sigma^{rc}, \sigma^{-1}, (\sigma^{-1})^{rc}, \tau, \tau^{rc}\}$, then $N(\pi) = n - 1$, and that the six permutations in the set are different.

Let us now show that there are no other permutations with $N(\pi) = n - 1$. We know by Proposition 3.1 that $N(\pi) = n - 1$ can only happen if $\mathrm{des}(\hat{\pi}) = n - 2$, or if $\mathrm{des}(\hat{\pi}) = n - 3$ and $\epsilon(\hat{\pi}) = 1$.
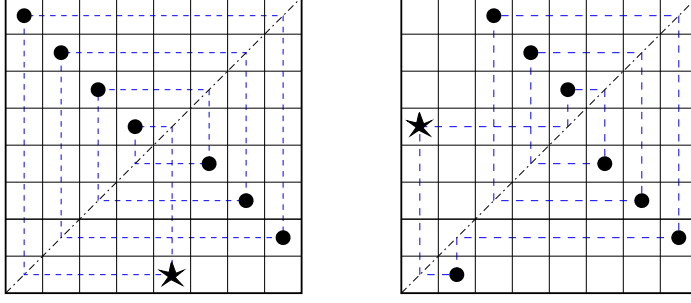
FIGURE 2. The arrays of σ and τ for $n = 8$, with dotted lines indicating the cycle structure.

Case 1: $\mathrm{des}(\hat{\pi}) = n - 2$. In this case, all the entries in $\hat{\pi}$ other that the $\star$ must be in decreasing order. If the distinguished entry is neither $\hat{\pi}(1)$ nor $\hat{\pi}(n)$, then the $\star$ must be replacing either 1 or n; otherwise we would have that $\hat{\pi}(1) = n$ and $\hat{\pi}(n) = 1$, so $\hat{\pi}$ would not be an n-cycle. It follows that in the array of $\hat{\pi}$, the entry corresponding to the $\star$ is either in the top or bottom row, or in the leftmost or rightmost column.

If the $\star$ is replacing 1 (i.e, it is is the bottom row of the array), we claim that the only possible n-cycle in which the other entries are in decreasing order is $\hat{\pi} = \sigma$. Indeed, if we consider the cycle structure of $\hat{\pi} = (1, \hat{\pi}(1), \hat{\pi}^2(1), \ldots, \hat{\pi}^{n-1}(1))$, we see that $\hat{\pi}(1) = n$ and $\hat{\pi}^2(1) = \hat{\pi}(n) = 2$. Now, $\hat{\pi}^i(1) \neq 1$ for $3 \leq i \leq n - 1$, so the decreasing condition on the remaining entries forces $\hat{\pi}^3(1) = \hat{\pi}(2) = n - 1$, $\hat{\pi}^4(1) = \hat{\pi}(n - 1) = 3$, and so on. A similar argument, considering that rotating the array 180 degrees preserves the cycle structure, shows that if the $\star$ is replacing n (i.e, it is in the top row of the array), then necessarily $\hat{\pi} = \sigma^{rc}$.

If the distinguished entry is $\hat{\pi}(1)$ (i.e, it is in the leftmost column of the array), then a symmetric argument, reflecting the array along $y = x$), shows that $\hat{\pi} = \sigma^{-1}$. Similarly, if the distinguished entry is $\hat{\pi}(n)$ (i.e, it is is the rightmost column of the array), then necessarily $\hat{\pi} = (\sigma^{-1})^{rc}$.

Case 2: $\mathrm{des}(\hat{\pi}) = n - 3$ *and* $\epsilon(\hat{\pi}) = 1$. The second condition forces $\hat{\pi} = [\star, 1, \ldots]$ or $\hat{\pi} = [\ldots, n, \star]$. Let us restrict to the first case (the second one can be argued in a similar way if we rotate the array 180 degrees). We must have $\hat{\pi}(3) > \hat{\pi}(4) > \cdots > \hat{\pi}(n)$. We claim that the only such $\hat{\pi}$ that is also an n-cycle is $\hat{\pi} = \tau$. Indeed, looking at the cycle structure $\hat{\pi} = (\hat{\pi}^{-(n-1)}(1), \ldots, \hat{\pi}^{-1}(1), 1)$, we see that $\hat{\pi}^{-1}(1) = 2$. Now, $\hat{\pi}^{-i}(1) \neq 1$ for $2 \leq i \leq n - 1$, so the decreasing condition on the remaining entries forces $\hat{\pi}^{-2}(1) = \hat{\pi}^{-1}(2) = n$, $\hat{\pi}^{-3}(1) = \hat{\pi}^{-1}(n) = 3$, $\hat{\pi}^{-4}(1) = \hat{\pi}^{-1}(3) = n - 1$, and so on. $\quad\square$

4. The number of allowed patterns of the binary shift

In the rest of the paper, we will assume for simplicity that $w_A(\pi)$ and $w_B(\pi)$ are defined taking $m = n - 1$, so they are of the form $up^{n-1}x^\infty$, with $x = 0$ or $x = N - 1$ respectively. The following variation of Lemma 2.11 will be useful later.

Lemma 4.1. *Let* $w = up^{n-1}0^\infty \in \mathcal{W}_N$, *where* $|u| = k - 1$ *and* $|p| = n - k$ *for some* $1 \leq k \leq n - 1$, *and* p *is primitive. If* $\pi = \mathrm{Pat}(w, \Sigma, n)$ *is defined, then* $\pi(n) = \pi(k) - 1$.

Proof. We have that $w_{[k,\infty)} = p^{n-1}0^\infty$ and $w_{[n,\infty)} = p^{n-2}0^\infty$. Since p must have some nonzero entry (otherwise $\mathrm{Pat}(w, \Sigma, n)$ would not be defined), it is clear that $w_{[n,\infty)} < w_{[k,\infty)}$. The same argument as in the proof of Lemma 2.11 can be used to show that there is no $1 \leq s \leq n$ such that $p^{n-2}0^\infty < w_{[s,\infty)} < p^{n-1}0^\infty$. It follows that $\pi(n) = \pi(k) - 1$. $\quad\square$

For $n \geq 2$, the set of patterns of length n that are realized by the shift on two symbols is $\mathrm{Allow}_n(\Sigma_2) = \mathcal{S}_{n,2}$. In this section we enumerate these permutations. Recall that $a_{n,2} = |\mathcal{S}_{n,2}|$ and that $\psi_2(t)$ is the number of primitive binary words of length t.

Theorem 4.2. *For $n \geq 2$,*

$$a_{n,2} = \sum_{t=1}^{n-1} \psi_2(t) 2^{n-t-1}.$$

Proof. Fix $n \geq 2$. We will construct a set $W \subset \mathcal{W}_2$ with the following four properties:

   (i) for all $w \in W$, $\mathrm{Pat}(w, \Sigma_2, n)$ is defined,
   (ii) for all $w, w' \in W$ with $w \neq w'$, we have that $\mathrm{Pat}(w, \Sigma_2, n) \neq \mathrm{Pat}(w', \Sigma_2, n)$,
   (iii) for all $\pi \in \mathrm{Allow}_n(\Sigma_2)$, there is a word $w \in W$ such that $\mathrm{Pat}(w, \Sigma_2, n) = \pi$,
   (iv) $|W| = \sum_{t=1}^{n-1} \psi_2(t) 2^{n-t-1}$.

Properties (i)-(iii) imply that the map

$$\begin{aligned} W &\longrightarrow \mathcal{S}_{n,2} \\ w &\mapsto \mathrm{Pat}(w, \Sigma_2, n) \end{aligned}$$

(6)

is a bijection. Thus, $a_{n,2} = |W|$ and the result will follow from with property (iv).
   Let

$$W = \bigcup_{t=1}^{n-1} \{up^{n-1}x^\infty : u \in \{0,1\}^{n-t-1}, p \in \{0,1\}^t \text{ is a primitive word, and } x = \bar{p}_t\},$$

where we use the notation $\bar{0} = 1$, $\bar{1} = 0$. Given binary words u, p of lengths $n-t-1$ and t respectively, where p is primitive, and $x = \bar{p}_t$, we will denote $v(u,p) = up^{n-1}x^\infty$.
   To see that W satisfies (i), we have to show that for any $w \in W$ and any $1 \leq i < j \leq n$, we have $w_{[i,\infty)} \neq w_{[j,\infty)}$. This is clear because if $x = 0$ (resp. $x = 1$) both $w_{[i,\infty)}$ and $w_{[j,\infty)}$ end with 10^∞ (resp. 01^∞), with the last 1 (resp. 0) being in different positions in $w_{[i,\infty)}$ and $w_{[j,\infty)}$.
   Now we prove that W satisfies (ii). Let u, u' be binary words of lengths $n - t - 1$, $n - t' - 1$, respectively, and let p, p' be primitive binary words of lengths t, t', respectively. Let $w = v(u,p)$ and $w' = v(u',p')$, and let $\pi = \mathrm{Pat}(w, \Sigma_2, n)$, $\pi' = Pat(w', \Sigma_2, n)$. We assume that $w \neq w'$, and want to show that $\pi \neq \pi'$. From $w \neq w'$ it follows that $u \neq u'$ or $p \neq p'$.
   Corollary 2.13 for $N = 2$ implies that if $w_1 w_2 \ldots w_{n-1} \neq w'_1 w'_2 \ldots w'_{n-1}$, then $\mathrm{Pat}(w, \Sigma_2, n) \neq \mathrm{Pat}(w', \Sigma_2, n)$. In particular, if $t = t'$, then $up \neq u'p'$, so $\pi \neq \pi'$.
   We are left with the case that $t \neq t'$ and $up = u'p' = w_1 w_2 \ldots w_{n-1}$. Let us first assume that $w_{n-1} = 1$ (and so $p_t = p'_{t'} = 1$). By Lemma 4.1 with $k = n - t$, we have that $\pi(n) = \pi(n-t) - 1$, and similarly $\pi'(n) = \pi'(n - t') - 1$. If we had that $\pi = \pi'$, then $\pi(n) = \pi'(n)$ and so $\pi(n-t) = \pi'(n - t') = \pi(n - t')$. But $t \neq t'$, so this is a contradiction. In the case $w_{n-1} = 0$, an analogous argument to the proof of Lemma 4.1 implies that $w_{[n-t,\infty)} = p^{n-1}1^\infty < p^{n-2}1^\infty = w_{[n,\infty)}$ and there is no s such that $w_{[s,\infty)}$ is strictly in between the two. Thus, $\pi(n) = \pi(n - t) + 1$, and similarly $\pi'(n) = \pi'(n - t') + 1$, so again $\pi \neq \pi'$.
   To see that W satisfies (iii) we use the construction from the proof of the upper bound in Theorem 2.1. Let $\pi \in \mathrm{Allow}_n(\Sigma_2)$. If $\pi(n - 1) > \pi(n)$, let $w = w_A(\pi) = up^{n-1}0^\infty$. By Lemma 2.6, $w_{n-1} = 1$, so $w \in W$. Similarly, if $\pi(n - 1) < \pi(n)$, let $w = w_B(\pi) = up^{n-1}1^\infty$. By Lemma 2.7, $w_{n-1} = 0$, so $w \in W$. In both cases, $\mathrm{Pat}(w, \Sigma_2, n) = \pi$, so this construction is the inverse of the map (6).
   To prove (iv), observe that the union in the definition of W is a disjoint union. This is because the value of t determines the position of the last entry in w that is not equal to x. For fixed t, there are 2^{n-t-1} choices for u and $\psi_2(t)$ choices for t, so the formula follows. $\qquad \square$

Example 4. For $n = 3$, we have $W = \{\underline{0}\,0\,0\,1^\infty, \underline{0}\,1\,1\,0^\infty, \underline{1}\,0\,0\,1^\infty, \underline{1}\,1\,1\,0^\infty, \ 1\,0\,1\,0\,1^\infty, 0\,1\,0\,1\,0^\infty\}$, where each word is written as $w = \underline{u}\,p\,p\,x^\infty$. The permutations corresponding to these words are $\mathrm{Allow}_3(\Sigma_2) = \{123, 132, 312, 321, \ 231, 213\}$, in the same order.

For $n = 4$, we have

$$W = \{\underline{00}\,0\,0\,0\,1^\infty, \underline{00}\,1\,1\,1\,0^\infty, \underline{01}\,0\,0\,0\,1^\infty, \underline{01}\,1\,1\,1\,0^\infty, \underline{10}\,0\,0\,0\,1^\infty, \underline{10}\,1\,1\,1\,0^\infty, \underline{11}\,0\,0\,0\,1^\infty, \underline{11}\,1\,1\,1\,0^\infty,$$
$$\underline{0}\,01\,01\,01\,0^\infty, \underline{0}\,10\,10\,10\,1^\infty, \underline{1}\,01\,01\,01\,0^\infty, \underline{1}\,10\,10\,10\,1^\infty,$$
$$001\,001\,001\,0^\infty, 010\,010\,010\,1^\infty, 011\,011\,011\,0^\infty, 100\,100\,100\,1^\infty, 101\,101\,101\,0^\infty, 110\,110\,110\,1^\infty\},$$

where each word is written as $w = \underline{u}\,p\,p\,p\,x^\infty$. The permutations corresponding to these words are

$$\text{Allow}_4(\Sigma_2) = \{1234, 1243, 3412, 1432, 4123, 2143, 4312, 4321,$$
$$1342, 1324, 4231, 4213,$$
$$2341, 2413, 2431, 3124, 3142, 3214\}.$$

5. The number of allowed patterns of the shift on N symbols

In this section we give a formula for the numbers $a_{n,N}$, which count permutations that can be realized by the shift on N symbols but not by the shift on $N-1$ symbols. This generalizes Theorem 4.2. For any $N \geq 2$, define

(7)
$$\Omega_{n,N} = \bigcup_{t=1}^{n-1} \{up^{n-1}0^\infty : u \in \{0, 1, \ldots, N-1\}^{n-t-1}, \text{ and } p \in \{0, 1, \ldots, N-1\}^t \text{ is a primitive word}\}.$$

Lemma 5.1. *Let $w = up^{n-1}0^\infty \in \Omega_{n,N}$. Then $\text{Pat}(w, \Sigma, n)$ is defined if and only if $p \neq 0$.*

Proof. If $p = 0$, then $w_{[n-1,\infty)} = w_{[n,\infty)} = 0^\infty$, so $\text{Pat}(w, \Sigma, n)$ is not defined. Otherwise, since p is primitive, it must have some nonzero entry. Let p_ℓ be the rightmost nonzero entry of p. Then, for any $1 \leq i < j \leq n$, we have that $w_{[i,\infty)} \neq w_{[j,\infty)}$, because both $w_{[i,\infty)}$ and $w_{[j,\infty)}$ end with $p_\ell 0^\infty$, with the last nonzero entry being in different positions in $w_{[i,\infty)}$ and $w_{[j,\infty)}$. $\square$

Let $\mathcal{H}'_{n,N} = \{\pi \in \mathcal{S}_{n,N} : \pi(n) = n\}$, and let $h_{n,N} = |\mathcal{H}'_{n,N}|$. Recall the definition of $w_A(\pi)$ from Section 2.2. We have a map

$$\begin{array}{ccc} \mathcal{S}_{n,N} \setminus \mathcal{H}'_{n,N} & \longrightarrow & \Omega_{n,N} \\ \pi & \mapsto & w_A(\pi). \end{array}$$

This is a one-to-one map, since π can be recovered from $w = w_A(\pi)$ using that $\text{Pat}(w, \Sigma, n) = \pi$. Denote its range by

$$\Gamma_{n,N} = \{w_A(\pi) : \pi \in \mathcal{S}_{n,N} \setminus \mathcal{H}'_{n,N}\}.$$

Defining $g_{n,N} = |\Gamma_{n,N}|$, it is clear that

$$g_{n,N} = |\mathcal{S}_{n,N} \setminus \mathcal{H}'_{n,N}| = a_{n,N} - h_{n,N}.$$

Our first goal is to find a formula for $g_{n,N}$.

Let $w = up^{n-1}0^\infty \in \Omega_{n,N}$. By Lemma 5.1, $\text{Pat}(w, \Sigma, n)$ is defined unless $p = 0$. Assuming that $p \neq 0$, let $\pi = \text{Pat}(w, \Sigma, n)$. Clearly $\pi \in \text{Allow}_n(\Sigma_N)$, so $2 \leq N(\pi) \leq N$. Let $M = N(\pi)$, and let $j = N - M$. We can write $\Omega_{n,N}$ as a disjoint union:

(8)
$$\Omega_{n,N} = \bigcup_{j=0}^{N-2} \{w \in \Omega_{n,N} : \pi = \text{Pat}(w, \Sigma, n) \text{ is defined and } N(\pi) = N - j\}$$
$$\cup \{u0^\infty : u \in \{0, 1, \ldots, N-1\}^{n-2}\},$$

where the last term is the set of words w for which $\text{Pat}(w, \Sigma, n)$ is not defined. Let us restrict to the case where $\pi = \text{Pat}(w, \Sigma, n)$ is defined.

Lemma 5.2. *Let $w \in \Omega_{n,N}$. If $\pi = \text{Pat}(w, \Sigma, n)$ is defined, then $\pi(n) \neq n$.*

Proof. We have that $w_{[n,\infty)} = p^{n-2}0^\infty < p^{n-1}0^\infty$, so $\pi(n)$ is not the largest value in π. $\square$

For $j = 0$, we have the following result.

Lemma 5.3. *Let $w \in \Omega_{n,N}$. If $\pi = \mathrm{Pat}(w, \Sigma, n)$ is defined and $N(\pi) = N$, then $w = w_A(\pi)$.*

Proof. Write $w = up^{n-1}0^\infty$, and let $w' = w_A(\pi) = u'p'^{n-1}0^\infty \in \mathcal{W}_N$. Since $\mathrm{Pat}(w', \Sigma, n) = \mathrm{Pat}(w, \Sigma, n) = \pi$, Corollary 2.13 implies that $u'p' = w'_1 w'_2 \ldots w'_{n-1} = w_1 w_2 \ldots w_{n-1} = up$, because these entries are uniquely determined by π. Let $t = |p|$ and $t' = |p'|$. From the definition of w_A, we have that $\pi(n) = \pi(n - t') - 1$, and by Lemma 4.1 with $k = n - t$, we have that $\pi(n) = \pi(n - t) - 1$. It follows that $\pi(n - t') = \pi(n - t)$, so $t = t'$. Therefore, $w = w'$. $\qquad\square$

From Lemmas 5.2 and 5.3 we have that

$$\{w \in \Omega_{n,N} : \pi = \mathrm{Pat}(w, \Sigma, n) \text{ is defined and } N(\pi) = N\} = \Gamma_{n,N}.$$

For general j, the situation is slightly more involved.

Lemma 5.4. *Let $w \in \Omega_{n,N}$. If $\pi = \mathrm{Pat}(w, \Sigma, n)$ is defined and $N(\pi) = M = N - j$, then the entries $w_1 w_2 \ldots w_{n-1}$ satisfy*

$$(w_{i_1}, w_{i_2}, \ldots, w_{i_{n-1}}) = (v_{i_1} + c_1, v_{i_2} + c_2, \ldots, v_{i_{n-1}} + c_{n-1})$$

for some $0 \leq c_1 \leq c_2 \leq \cdots \leq c_{n-1} \leq j$, where $i_1, \ldots, i_{n-1}$ are the subindices in equation (5) from left to right, and $v = w_A(\pi) \in \Gamma_{n,M}$.

Proof. First note that by Lemma 5.2, $\pi(n) \neq n$, so $v \in \Gamma_{n,M}$. We know by Theorem 2.1 that $M = 1 + |A(\pi)| + \Delta(\pi)$. By definition of $i_1, i_2, \ldots, i_{n-1}$, equation (5) can be written as

$$(9) \qquad\qquad\qquad w_{i_1} \leq w_{i_2} \leq \cdots \leq w_{i_{n-1}}.$$

By Lemma 2.5, the entries of w have to fulfill (9), plus $M - 1$ required strict inequalities. Similarly, since $\mathrm{Pat}(v, \Sigma_M, n) = \pi$, the entries of v satisfy $v_{i_1} \leq v_{i_2} \leq \cdots \leq v_{i_{n-1}}$ and the same $M - 1$ required strict inequalities. In fact, by Corollary 2.13, the entries $v_1 v_2 \ldots v_{n-1}$ are uniquely determined by these inequalities and the fact that $v \in \mathcal{W}_M$.

It is now easy to see that $w \in \Omega_{n,N}$ satisfies (9) plus the $M - 1$ required strict inequalities if and only if the values $w_1 w_2 \ldots w_{n-1}$ satisfy

$$(w_{i_1}, w_{i_2}, \ldots, w_{i_{n-1}}) = (v_{i_1} + c_1, v_{i_2} + c_2, \ldots, v_{i_{n-1}} + c_{n-1})$$

for some $0 \leq c_1 \leq c_2 \leq \cdots \leq c_{n-1} \leq j$. $\qquad\square$

Note that for $j = 0$, Lemma 5.4 says that the values $w_1 w_2 \ldots w_{n-1}$ are determined by π, which is just Corollary 2.13. For general j, there are $\binom{n+j-1}{j}$ choices for the vector $(c_1, c_2, \ldots, c_{n-1})$. Let $C_{n,j} = \{(c_1, c_2, \ldots, c_{n-1}) : 0 \leq c_1 \leq c_2 \leq \cdots \leq c_{n-1} \leq j\}$.

Lemma 5.5. *For $0 \leq j \leq N - 2$, the map*

$$\begin{array}{ccc} \{w \in \Omega_{n,N} : \pi = \mathrm{Pat}(w, \Sigma, n) \text{ is defined and } N(\pi) = N - j\} & \longrightarrow & \Gamma_{n,N-j} \times C_{n,j} \\ w & \mapsto & (v, (c_1, c_2, \ldots, c_{n-1})) \end{array}$$

defined by Lemma 5.4 is a bijection.

Proof. The map is clearly well defined, since $v = w_A(\pi) \in \Gamma_{n,N-j}$ and $(c_1, \ldots, c_{n-1}) \in C_{n,j}$ are uniquely determined by w. To see that it is invertible, take $(v, (c_1, c_2, \ldots, c_{n-1})) \in \Gamma_{n,N-j} \times C_{n,j}$. Then, $\pi = \mathrm{Pat}(v, \Sigma, n)$ determines the sequence $i_1, i_2, \ldots, i_{n-1}$ of subindices in equation (5), so the first $n - 1$ entries in w must be $(w_{i_1}, w_{i_2}, \ldots, w_{i_{n-1}}) = (v_{i_1} + c_1, v_{i_2} + c_2, \ldots, v_{i_{n-1}} + c_{n-1})$. By Lemma 4.1, the unique word in $w \in \Omega_{n,N}$ with these entries such that $\mathrm{Pat}(w, \Sigma, n) = \pi$ is $w = w_1 w_2 \ldots w_{k-1}(w_k w_{k+1} \ldots w_{n-1})^{n-1}0^\infty$, where $k = \pi^{-1}(\pi(n) + 1)$. $\qquad\square$

Proposition 5.6. *The numbers $g_{n,N}$ satisfy*

$$\sum_{t=1}^{n-1} \psi_N(t) N^{n-t-1} = \sum_{j=0}^{N-2} \binom{n+j-1}{j} g_{n,N-j} + N^{n-2}.$$

Proof. The formula is obtained by taking set cardinalities in equation (8). For the left hand side we use that $|\Omega_{n,N}| = \sum_{t=1}^{n-1} \psi_N(t) N^{n-t-1}$, which follows from the definition (7) of $\Omega_{n,N}$, noticing that it is a disjoint union. By Lemma 5.5,

$$|\{w \in \Omega_{n,N} : \pi = \mathrm{Pat}(w, \Sigma, n) \text{ is defined and } N(\pi) = N - j\}| = |\Gamma_{n,N-j}||C_{n,j}| = g_{n,N-j}\binom{n+j-1}{j}$$

for $0 \leq j \leq N - 2$. Finally, it is clear that $|\{u0^\infty : u \in \{0, 1, \ldots, N - 1\}^{n-2}\}| = N^{n-2}$. $\qquad\square$

From Proposition 5.6 we obtain the following recurrence for $g_{n,N}$, for $N \geq 2$.

$$(10) \qquad g_{n,N} = \sum_{t=1}^{n-1} \psi_N(t) N^{n-t-1} - N^{n-2} - \sum_{j=1}^{N-2} \binom{n+j-1}{j} g_{n,N-j}.$$

To obtain an expression for the general term $g_{n,N}$, we use the following lemma.

Lemma 5.7. *Let $n \geq 1$ and $\{b_N\}_{N \geq 2}$ be fixed. If the sequence $\{r_N\}_{N \geq 2}$ satisfies the recurrence*

$$r_N = b_N - \sum_{j=1}^{N-2} \binom{n+j-1}{j} r_{N-j},$$

then

$$r_N = \sum_{i=0}^{N-2} (-1)^i \binom{n}{i} b_{N-i}$$

for all $N \geq 2$.

Proof. We proceed by induction on N. Assuming that $r_M = \sum_{i=0}^{M-2} (-1)^i \binom{n}{i} b_{M-i}$ for $2 \leq M \leq N - 1$, we have that

$$
\begin{aligned}
r_N &= b_N - \sum_{j=1}^{N-2} \binom{n+j-1}{j} \left(\sum_{i=0}^{N-j-2} (-1)^i \binom{n}{i} b_{N-j-i} \right) \\
&= b_N - \sum_{k=1}^{N-2} \sum_{j=1}^{k} \binom{n+j-1}{j} (-1)^{k-j} \binom{n}{k-j} b_{N-k} \\
&= b_N - \sum_{k=1}^{N-2} (-1)^{k-1} \binom{n}{k} b_{N-k} = \sum_{k=0}^{N-2} (-1)^k \binom{n}{k} b_{N-k},
\end{aligned}
$$

where in the second step we have made the substitution $k = i + j$, and in the third step we have used the identity $\sum_{j=1}^{k} (-1)^{j-1} \binom{n+j-1}{j} \binom{n}{k-j} = \binom{n}{k}$. $\qquad\square$

Substituting $b_N = \sum_{t=1}^{n-1} \psi_N(t) N^{n-t-1} - N^{n-2}$ in Lemma 5.7, we can solve recurrence (10) for $g_{n,N}$.

Corollary 5.8. *For any $n, N \geq 2$,*

$$g_{n,N} = \sum_{i=0}^{N-2} (-1)^i \binom{n}{i} \left(\sum_{t=1}^{n-1} \psi_{N-i}(t)(N-i)^{n-t-1} - (N-i)^{n-2} \right).$$

Recall that $g_{n,N} = a_{n,N} - h_{n,N}$, and that our main goal is to compute $a_{n,N}$, the number of permutations of length n that require N symbols in the alphabet in order to be realized by a shift. The next step will be to find a formula for $h_{n,N} = |\{\pi \in \mathcal{S}_{n,N} : \pi(n) = n\}|$, using similar ideas to the ones we used for $g_{n,N}$. Let $\mathcal{H}_{n,N} = \{\pi \in \mathcal{S}_{n,N} : \pi(n) = 1\}$. By Lemma 2.2, $h_{n,N} = |\mathcal{H}_{n,N}|$. We now define analogues of $\Omega_{n,N}$ and $\Gamma_{n,N}$ corresponding to permutations ending with a 1. Let

$$\Theta_{n,N} = \{w \in \Omega_{n,N} : \pi = \mathrm{Pat}(w, \Sigma, n) \text{ is defined and has } \pi(n) = 1\},$$

$$\Lambda_{n,N} = \{w_A(\pi) : \pi \in \mathcal{H}_{n,N}\}.$$

Since w_A is one-to-one, it is clear that $|\Lambda_{n,N}| = h_{n,N}$.

Proposition 5.9. *For any $n, N \geq 2$, $|\Theta_{n,N}| = (N-1)N^{n-2}$.*

Proof. We will give a bijection between the set

$$R_{n,N} = \{w_1 w_2 \ldots w_{n-1} : w_i \in \{0, 1, \ldots, N-1\} \text{ for } 1 \leq i \leq N-2, \text{ and } w_{n-1} \in \{1, 2, \ldots, N-1\}\}$$

and $\Theta_{n,N}$.

Given $w_1 \ldots w_{n-1} \in R_{n,N}$, we want to construct a word $w = u p^{n-1} 0^\infty \in \Theta_{n,N}$ where $u = w_1 w_2 \ldots w_{k-1}$ and $p = w_k w_{k+1} \ldots w_{n-1}$ for some $1 \leq k \leq n-1$. Let k be the index such that $w_k w_{k+1} \ldots w_{n-1} 0^\infty$ is smallest (in lexicographic order), with $1 \leq k \leq n-1$. Note that these $n-1$ words are all different because the last nonzero entry of $w_k w_{k+1} \ldots w_{n-1} 0^\infty$ is w_{n-1}, which is in position $n-k$. We claim that with this choice of k, the word $p = w_k w_{k+1} \ldots w_{n-1}$ is primitive. Otherwise, if $p = q^m$ for some $m > 1$, then $q^m 0^\infty > q 0^\infty$, contradicting the choice of k. Since $p \neq 0$, Lemma 5.1 guarantees that $\pi = \text{Pat}(w, \Sigma, n)$ is defined. Moreover, $\pi(n) = 1$ because $w_{[n,\infty)} = p^{n-2} 0^\infty < w_{[k,\infty)} = p^{n-1} 0^\infty$, and by the choice of k, $w_{[k,\infty)} < w_{[i,\infty)}$ for all $1 \leq i \leq n-1$ with $i \neq k$. Thus $w \in \Theta_{n,N}$.

This map is clearly one-to-one, because $w_1 \ldots w_{n-1}$ can be recovered by taking the first $n-1$ entries in w. To show that it is onto, we have to show that if $w \in \Theta_{n,N}$, then $w_{n-1} \neq 0$. Assume for contradiction that $w_{n-1} = 0$. Then, $w_{[n-1,\infty)} = 0 w_{[n,\infty)}$, so $w_{[n-1,\infty)} \leq w_{[n,\infty)}$, which contradicts the fact that $\pi = \text{Pat}(w, \Sigma, n)$ has $\pi(n) = 1$.

Clearly $|R_{n,N}| = (N-1)N^{n-2}$, so the proposition is proved. $\square$

Similarly to the decomposition of $\Omega_{n,N}$ in equation (8), we can decompose $\Theta_{n,N}$ as

$$(11) \qquad \Theta_{n,N} = \bigcup_{j=0}^{N-2} \{w \in \Theta_{n,N} : N(\text{Pat}(w, \Sigma, n)) = N - j\}.$$

For $j = 0$, Lemma 5.3 implies that

$$\{w \in \Theta_{n,N} : N(\text{Pat}(w, \Sigma, n)) = N\} = \Lambda_{n,N}.$$

In general, given $w \in \Theta_{n,N}$, let $\pi = \text{Pat}(w, \Sigma, n)$, and assume that $N(\pi) = N - j$. Then, $\pi \in \mathcal{H}_{n,N-j}$, and the word $v = w_A(\pi)$ given by Lemma 5.4 is in $\Lambda_{n,N-j}$. Restricting to permutations with $\pi(n) = 1$, we obtain the following analogue of Lemma 5.5.

Lemma 5.10. *For $0 \leq j \leq N-2$, the map*

$$\begin{array}{ccc} \{w \in \Theta_{n,N} : N(\text{Pat}(w, \Sigma, n)) = N - j\} & \longrightarrow & \Lambda_{n,N-j} \times C_{n,j} \\ w & \mapsto & (v, (c_1, c_2, \ldots, c_{n-1})) \end{array}$$

defined by Lemma 5.4 is a bijection.

Proposition 5.11. *The numbers $h_{n,N}$ satisfy*

$$(N-1)N^{n-2} = \sum_{j=0}^{N-2} \binom{n+j-1}{j} h_{n,N-j}.$$

Proof. The result follows by taking set cardinalities in equation (11). The left hand side is given by Proposition 5.9. For the right hand side, we use Lemma 5.10 to deduce that

$$|\{w \in \Theta_{n,N} : N(\text{Pat}(w, \Sigma, n)) = N - j\}| = |\Lambda_{n,N-j}||C_{n,j}| = h_{n,N-j} \binom{n+j-1}{j}$$

for $0 \leq j \leq N-2$. $\square$

From Proposition 5.11 we obtain the recurrence

$$h_{n,N} = (N-1)N^{n-2} - \sum_{j=1}^{N-2} \binom{n+j-1}{j} h_{n,N-j},$$

which can be solved using Lemma 5.7 to get the following expression for the general term $h_{n,N}$.

Corollary 5.12. *For any $n, N \geq 2$,*

$$h_{n,N} = \sum_{i=0}^{N-2} (-1)^i \binom{n}{i} (N-i-1)(N-i)^{n-2}.$$

Combining Corollaries 5.8 and 5.12, and using the fact that $a_{n,N} = g_{n,N} + h_{n,N}$, we obtain an expression for the numbers $a_{n,N} = |\{\pi \in \mathcal{S}_n : N(\pi) = N\}|$.

Theorem 5.13. *For any $n, N \geq 2$,*

$$(12) \qquad a_{n,N} = \sum_{i=0}^{N-2} (-1)^i \binom{n}{i} \left((N-i-2)(N-i)^{n-2} + \sum_{t=1}^{n-1} \psi_{N-i}(t)(N-i)^{n-t-1} \right).$$

For $N = 2$, Theorem 5.13 gives $a_{n,2} = \sum_{t=1}^{n-1} \psi_2(t) 2^{n-t-1}$, which agrees with Theorem 4.2.

6. Conjectures and further work

Some general open questions that deserve further study have already been discussed at the end of Section 1.2. Here we mention two curious conjectures that came up while studying forbidden patterns of shift systems. They are derived from experimental evidence, and it would be interesting to find combinatorial proofs.

6.1. **An appearance of the Eulerian numbers.** Using Proposition 3.1, the problem of enumerating $\mathcal{S}_{n,N}$ can be formulated in terms of counting marked cycles $\hat{\pi} \in \mathcal{T}_n$ with respect to $\mathrm{des}(\hat{\pi})$ and $\epsilon(\hat{\pi})$. More precisely,

$$(13) \quad a_{n,N} = |\{\hat{\pi} \in \mathcal{T}_n : \mathrm{des}(\hat{\pi}) = N-1 \text{ and } \epsilon(\hat{\pi}) = 0\}| + |\{\hat{\pi} \in \mathcal{T}_n : \mathrm{des}(\hat{\pi}) = N-2 \text{ and } \epsilon(\hat{\pi}) = 1\}|.$$

Let us take a closer look at rightmost summand. Marked cycles with $\epsilon(\hat{\pi}) = 1$ can be separated into two disjoint sets $\mathcal{E}_n$ and $\mathcal{E}'_n$, depending on whether they are of the form $\hat{\pi} = [\star, 1, \dots]$ or $\hat{\pi} = [\dots, n, \star]$, respectively. The map $\hat{\pi} \mapsto \hat{\pi}^{rc}$ (see the definition above Proposition 3.3) is a bijection between $\mathcal{E}_n$ and $\mathcal{E}'_n$, which preserves the number of descents. Thus, for any $0 \leq k \leq n-3$,

$$|\{\hat{\pi} \in \mathcal{T}_n : \mathrm{des}(\hat{\pi}) = k \text{ and } \epsilon(\hat{\pi}) = 1\}| = 2\,|\{\hat{\pi} \in \mathcal{E}_n : \mathrm{des}(\hat{\pi}) = k\}|.$$

It will be convenient to define the set $\mathcal{T}_n^0$ of n-cycles where one entry has been replaced with 0. The set $\mathcal{T}_n^0$ is essentially the same as $\mathcal{T}_n$, with the only difference that the $\star$ symbol in each element is replaced with a 0. The reason for this change is that now we define a descent of $\sigma = [\sigma(1), \dots, \sigma(n)] \in \mathcal{T}_n^0$ to be a position i such that $\sigma(i) > \sigma(i+1)$, so the 0 entry is no longer skipped in the computation of the descent set $D(\sigma)$ and the number of descents $\mathrm{des}(\sigma)$.

Lemma 6.1. *For $n \geq 3$, the map*

$$\Phi : \quad \begin{matrix} \mathcal{E}_n & \longrightarrow & \mathcal{T}_{n-2}^0 \\ [\star, 1, \hat{\pi}(3), \hat{\pi}(4), \dots, \hat{\pi}(n)] & \mapsto & [\hat{\pi}(3) - 2, \hat{\pi}(4) - 2, \dots, \hat{\pi}(n) - 2] \end{matrix}$$

is well-defined and it is a bijection.

Proof. First we show that if $\hat{\pi} \in \mathcal{E}_n$, then $\Phi(\hat{\pi}) \in \mathcal{T}_{n-2}^0$. Let $j \in \{1, \dots, n\}$ be the missing entry in $\hat{\pi}$, that is, the entry that $\star$ is replacing. Clearly $j \neq 2$, otherwise $\hat{\pi}$ would not be an n-cycle. In particular, there is some $3 \leq i \leq n$ such that $\hat{\pi}(i) = 2$, which produces the 0 entry in $\Phi(\hat{\pi})$. Besides, if we replace this 0 entry in $\Phi(\hat{\pi})$ with $j - 2$, we obtain an $(n-2)$-cycle. This is indeed the same $(n-2)$-cycle that we get starting from the cycle notation of $\hat{\pi}$, replacing the $\star$ with j, deleting

the entries 1 and 2, and shifting the values of the remaining entries down by 2. This proves that $\Phi(\hat{\pi}) \in \mathcal{T}_{n-2}^0$.

The map is clearly invertible, and the fact that $[\hat{\sigma}(1), \hat{\sigma}(2), \ldots, \hat{\sigma}(n-2)] \in \mathcal{T}_{n-2}^0$ guarantees that $[\star, 1, \hat{\sigma}(1) + 2, \hat{\sigma}(2) + 2, \ldots, \hat{\sigma}(n-2) + 2]$ is a marked cycle. $\qquad\square$

For example, if $\hat{\pi} = [\star, 1, 5, 7, 6, 2, 3]$, then $\Phi(\hat{\pi}) = [3, 5, 4, 0, 1]$. Note that from the cycle notation of $\hat{\pi} = (2, 1, \star, 7, 3, 5, 6)$, after replacing the $\star$ with a 4, deleting 2 and 1, and shifting the remaining entries down by 2, we obtain $(2, 5, 1, 3, 4)$, which is the cycle notation of $\Phi(\hat{\pi})$ with the 0 replaced with a 2.

It is clear from the definition of Φ that it preserves the number of descents. As a consequence of this property,
$$|\{\hat{\pi} \in \mathcal{E}_n \,:\, \mathrm{des}(\hat{\pi}) = k\}| = |\{\hat{\pi} \in \mathcal{T}_{n-2}^0 \,:\, \mathrm{des}(\hat{\pi}) = k\}|,$$
so we have reduced the computation of the rightmost summand of (13) to studying the distribution of the number of descents in $\mathcal{T}_{n-2}^0$. Experimental evidence suggests that this is a very well-known distribution:

Conjecture 6.2. *For any n and any subset $D \subseteq \{1, 2, \ldots, n-1\}$,*
$$|\{\sigma \in \mathcal{T}_n^0 \,:\, D(\sigma) = D\}| = |\{\pi \in \mathcal{S}_n \,:\, D(\pi) = D\}|.$$
In particular, the statistic des *has the same distribution in* $\mathcal{T}_n^0$ *as in* $\mathcal{S}_n$*, i.e,*
$$\sum_{\sigma \in \mathcal{T}_n^0} x^{\mathrm{des}(\sigma)+1} = A_n(x),$$
the n-th Eulerian polynomial.

We have checked this conjecture by computer for n up to 9.

6.2. **Divisibility properties.** Even though Theorem 5.13 gives an explicit formula for the numbers $a_{n,N}$, some properties of these numbers are not apparent from the formula. For example, it is not trivial to derive Proposition 3.3 from equation (12). A striking property of the entries of Table 1 (and of other values of $a_{n,N}$ computed for larger n using Theorem 5.13) is that they are all divisible by 6. We conjecture that this is always the case.

Conjecture 6.3. *For every $n, N \geq 3$, $a_{n,N}$ is divisible by 6.*

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References

[1] J.M. Amigó, S. Elizalde and M. Kennel, Forbidden patterns and shift systems, *J. Combin. Theory Ser. A* 115 (2008), pp. 485–504.

[2] J.M. Amigó, S. Zambrano and M.A.F. Sanjuán, True and false forbidden patterns in deterministic and random dynamics, *Europhys. Lett.* 79 (2007), 50001-p1, -p5.

[3] C. Bandt, G. Keller and B. Pompe, Entropy of interval maps via permutations, *Nonlinearity* 15 (2002), pp. 1595-1602.

[4] J. Bobok and M. Kuchta, X-minimal patterns and a generalization of Sharkovskii's theorem, *Fund. Math.* 156 (1998), pp. 33–66.

[5] S. Elizalde and M. Noy, Consecutive patterns in permutations, *Adv. Appl. Math.* 30 (2003), pp. 110-125.

[6] I. Gessel and C. Reutenauer, Counting permutations with given cycle structure and descent set, *J. Combin. Theory Ser. A* 64 (1993), pp. 189–215.

[7] A.N. Sarkovskii, Coexistence of cycles of a continuous map of a line into itself, *Ukrainian Math. J.* 16 (1964), pp. 61–71.

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