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## Absence of Open Strings in a Lattice-Free Simulation of Cosmic String Formation

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Lattice-based string formation algorithms can, at least in principle, be reduced to the study of the statistics of the corresponding aperiodic random walk. Since in three or more dimensions such walks are transient, this approach necessarily generates a population of open strings. To investigate whether open strings are an artifact of the lattice we develop an alternative lattice-free simulation of string formation. Replacing the lattice with a graph generated by a minimal dynamical model of a first-order phase transition we obtain results consistent with the hypothesis that the energy density in string is due to a scale-invariant Brownian distribution of closed loops alone. [S0031-9007(96)00118-4]

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Cosmic strings are the longest standing candidate for a source of primordial cosmological perturbations arising from particle physics beyond the standard model. First described by Kibble almost two decades ago [1] they have yet to be demonstrated to be incompatible with any of the subsequent cosmological observations, from the anisotropies in the cosmic microwave background to the distribution of luminous matter on the largest scales. The field theoretic requirements of the basic model are simple: the spontaneous breaking of some symmetry of our theory yielding a degenerate vacuum manifold whose first homotopy group is nontrivial. The single free parameter in the theory is then the energy scale of the, typically grand unified theory, symmetry breaking. The complexity of the ensuing cosmology (in part responsible for the model's longevity) lies in the need to track the nonlinear evolution of the string-bearing field through the symmetry breaking phase transition and on to the epoch of any observable consequences.

The starting point for any analysis of string cosmology is in the statistics of the strings' initial distribution. Unless terminating at a pair of monopoles or spatial singularities, all strings must be either closed loops or open (and hence infinite). Here we are interested in the initial distribution of strings both between and within each of these populations. We can trivially define a fraction  $0 \leq f_c \leq 1$  of the string energy density to be in closed loops, leaving  $f_o = 1 - f_c$  in open string; this is the quantity of most interest in this Letter. Further, it is easy to show, under the assumption that the string loops are Brownian and scale invariant, that the number density of loops with lengths in the range  $[l, l + \delta l)$  must go as [2]

$$\delta n \propto l^{-5/2} \delta l. \quad (1)$$

Numerical simulations of string formation assign phases taken from the  $S_1$  manifold of vacuum states to the vertices of some three-dimensional lattice. (Note that by "lattice" we mean a regular array of connected

vertices, while the oxymoron "random lattice" is termed a graph.) Along the lattice edges the field is taken to follow the shortest path on the manifold—the so-called geodesic rule. Each face of the lattice is then inspected to determine whether, in traversing its boundary, the field covers the entire vacuum manifold. If so then a zero of the field is necessarily present somewhere within the face, and a string segment is located passing through it. Such a segment connects the centers of the adjacent lattice volume elements associated with the face, and taken together all the segments join to form a population of complete connected strings. If the lattice is periodic then all the strings are closed loops; otherwise strings which intersect the lattice boundaries simply terminate there.

The original simulation [2] took phases at random from the minimal three-point discretization of  $S_1$  on a cubic lattice, resolving the ambiguity when four string segments met in a single cube by chance. Later refinements to this approach included the use of a tetrahedral lattice [3–5], a continuous vacuum manifold [3,6], and a dynamical allocation of the field phases [7,8]. However, the key common feature of all these simulations is that the field is set at the vertices of a lattice, and hence that the strings form along the edges of its dual. In every case such simulations do indeed generate strings which are Brownian and scale invariant, with the majority of the string energy density being in open strings. The exact fraction depends on the details of the simulation, and in particular on the geometry of the lattice and its dual. For example, the simple cubic lattice with its simple cubic dual gives  $f_o \sim 0.8$  [2], while the tetrahedral lattice with tetrakaidekahedral dual gives  $f_o \sim 0.63$  [4,5].

Since such simulations construct strings as sequences of vertices on a lattice, it is not surprising that they can also be represented as random walks. Since there is an excluded volume around the path of a single string, giving a directional bias away from its origin, we might naively expect the ensemble to have the statistics of a self-avoiding random walk. However, if the strings

are dense the associated ensemble of excluded volumes removes this bias, and the statistics turn out to be those of a Brownian random walk instead. Moreover, it is known that in three or more dimensions an aperiodic random walk on a lattice cannot be recurrent [9], and so has a nonzero probability of not returning to the origin, giving a nonzero fraction of open string. Exploiting this representation, analyses of the statistics of Brownian random walks on the appropriate lattices give similar results, with  $f_o \geq 0.6$  [3,10].

Since open strings appear to be an inevitable consequence of the lattice-based formation algorithms, it is desirable to consider formulations which do not involve a lattice, and use them to test the alternative hypothesis that the string energy density may be accounted for solely in terms of a scale-invariant Brownian population of closed loops. The results of one such algorithm are presented in this Letter.

*Algorithm.*—The use of a lattice in string formation simulations allows us to know in advance where the strings may be found, and hence to bypass the dynamics of the phase transition. Without a lattice we must therefore include the dynamics explicitly, and for computational tractability we adopt the simplest possible dynamical model of a first-order phase transition. Spherically symmetric bubbles of the true vacuum (with phases chosen at random from the minimally discretized manifold) are nucleated at random space-time events in a periodic false vacuum background. They are then taken to expand uniformly at the speed of light for the duration of the simulation, colliding with one another, trapping regions of false vacuum, and generating strings.

The exterior (i.e., outside of any other bubble, and so in the false vacuum) intersection of the surfaces of any three bubbles of differing phases generates a string segment with open ends. As the bubbles continue to expand this segment traces the locus of intersection of their surfaces, lengthening, and, except in the case where the three bubbles are the same size, bending (cf. lattice-based simulations where the bubbles are always the same size, and the string segments are necessarily straight). As the bubbles continue to expand they eventually meet a fourth bubble. One and only one of the collisions between this bubble and each pair of the original triplet also generates a string segment. The final four-bubble intersection event then sees the joining of one of the ends of each segment to form a new extended segment. The exterior intersection of the surfaces of any four bubbles can therefore also be viewed as the meeting of four three-bubble intersection loci, and hence of the joining of either zero or two string segments.

The key to this algorithm is the location of the points of intersection of all bubble triplets and quadruplets. However, since the bubbles are spheres of known centers and predictably time-dependent radii these points of intersection can be determined analytically. Having calculated

the location of all the three- and four-bubble intersections in the simulation space-time we can then construct the corresponding string distribution and determine its statistics. Taking the four-bubble intersection points as nodes, and their four constituent three-bubble intersection loci as connections, it is clear that we are generating a graph with fourfold connectivity. Note also that the precise location of the strings on this graph is determined by the particular phase realization, enabling us to use the same graph to generate many different string sets.

This algorithm is the natural extension of that employed in two dimensions to assess the effect of bubble wall speed on vortex formation [11]. Its three-dimensional realization, including the formation of both strings and monopoles, is described in full detail elsewhere [12].

*Implementation.*—This algorithm is clearly much more complex (and computationally intensive) than its lattice-based counterparts; indeed the limiting factor in its application will time and again prove to be its CPU time.

In a finite simulation we can never hope to represent the longest strings—for a string bearing field with correlation length  $\xi$  in a box of side  $L$  any string longer than  $O(L^2/\xi)$  is likely to cross the boundary. If we leave the boundaries open, we can estimate the open string energy density by seeing whether the number density of boundary crossings reaches a nonzero asymptote as the box size increases [2]. If instead we impose periodic boundaries, forcing all strings to be loops, we can use the theoretical scarcity of long closed loops to identify any excess as representing the open strings. We adopt the second approach for a number of reasons. Since the hypothesis under consideration is that the string energy density is due to loops alone, we are looking for exactly the excess that might occur in a periodic simulation. Moreover, the open boundary approach requires runs on a wide range of box sizes to test for an asymptotic limit, and hence falls foul of our CPU-time constraints. Finally, the requirement that all strings be closed provides a useful check that the phase transition has completed and that the bubbles have indeed filled the entire simulation volume by the end of the run, as well as an overall test of the robustness of the code.

Since we wish to calculate many-bubble intersection points, periodicity is implemented by incorporating 26 copies of the simulation volume to surround it. For a simulation of box size  $L$  and duration  $T$  we then consider only those bubbles within  $T$  of the central volume, and only those intersections that occur inside it. We therefore have to balance the wish to maximize the ratio  $L/T$  so as to minimize the periodic copies, and the increased number of bubbles necessary to fill a larger volume in a shorter time. Again the final constraint is CPU time. Note that the simulation size and time are dimensionless parameters and that the physical scale of the system is set by the bubble nucleation rate, and hence by varying the number of bubbles nucleated.

We determine the graphs associated with five simulations each starting with 5000 bubbles in a periodic box of size  $5^3$  with a simulation run time of 1 in units of the box length, each graph requiring approximately 250 h of CPU time on a twin-processor Sun SPARC 10. We then generate the associated string populations from 1000 different random phase realizations in each case, giving some  $N_s \sim 300\,000$  strings. Figure 1 shows the normalized number density  $N(l)$  of all these strings binned by length against the length  $l$ .

We can immediately see that there are three phases present: (i) A low-end tail population of loops with  $l < l_o$ , with  $l_o \sim e^{0.5}$  here. Clearly the loop distribution cannot follow Eq. (1) down to arbitrary short lengths, since this would generate an infinite energy density. This population has never been explicitly identified before, since lattice-based simulations automatically impose a low-end cutoff of the order of a few lattice spacings; it is worth noting that in this simulation the range of loop lengths covers more than 6 orders of magnitude. (ii) A population of scale-invariant Brownian loops with number density falling as  $l^{-5/2}$ . (iii) A high-end tail population of loops with  $l_1 < l < l_{\max}$ , with  $l_1 \sim e^{3.5}$  here (note that  $e^{3.5} \sim 5^2$ , as expected). The truncation of the longest strings (both long finite closed loops and infinite open strings) by the periodic boundaries generates an excess of strings longer than  $l_1$ , so that in this region the loop number density falls more slowly, as  $l^{-\alpha}$  with  $\alpha < -5/2$ .

The final feature to note is that since we have a finite total number of loops  $N_s$  in our simulation there is a minimum measurable nonzero normalized bin number density of  $1/N_s$ , clearly identifiable as the extreme high-end plateau in Fig. 1. As the string number density approaches this the relative error in the measurements can

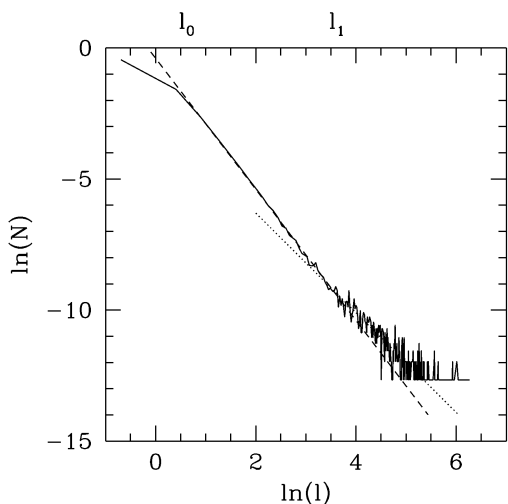


FIG. 1. A log-log plot of the normalized number density  $N(l)$  of strings binned by length against their length  $l$ . The solid line is the simulation data, the dashed line is the best fit to the Brownian regime  $N(l) = 0.67l^{-2.5}$ , and the dotted line is the best fit to the periodicity-distorted regime  $N(l) = 0.082l^{-1.9}$ .

be seen to increase dramatically. These extreme high-end points then correspond to those bins which happen to contain one or more strings, despite having a theoretical  $\delta n(l) < 1/N_s$ .

This is in many ways a familiar picture, quantitatively giving the predicted behavior of the midrange loops and at least qualitatively the deviations at either extreme. The first indication of significant differences comes with the value of  $\alpha$ . Analytic estimates for random walks on a periodic cubic lattice give  $\alpha = 1$  [10], whereas we find  $\alpha \sim 1.9$ . The fact that our long loop distribution falls off much faster than a random walk model would predict indicates that the amount of long string is much less than in such models.

The periodicity of our simulation forces any open string back into the box, where it would appear in the high-end tail. We can now estimate its energy density as the difference between the observed energy density in all strings longer than  $l_1$ ,  $E^{\text{obs}}(l \geq l_1)$ , and the theoretical energy density in loops longer than  $l_1$  alone,

$$E_l^{\text{th}}(l \geq l_1) = k \int_{l_1}^{\infty} l^{-3/2} dl = 2kl_1^{-1/2}. \quad (2)$$

To determine the normalization  $k$  we can use the convergence of the observed and theoretical loop distributions in the region  $l_o \leq l \leq l_1$ , giving

$$k = \frac{E^{\text{obs}}(l_o \leq l \leq l_1)}{2(l_o^{-1/2} - l_1^{-1/2})}. \quad (3)$$

The remarkable result when we do so for each of the five runs is that

$$f_o = \frac{E^{\text{obs}}(l \geq l_1) - E_l^{\text{th}}(l \geq l_1)}{E^{\text{obs}}(l \geq 0)} = 0.0062 \quad (4)$$

(with a variance of 0.0050) compared to the 0.66–0.8 we might expect. We can account for *all* the energy density in the strings as being due to a single population of scale-invariant Brownian loops.

In conclusion, we have used a minimal dynamical model of a first-order phase transition to generate an initial string distribution whose statistics are significantly different from the standard lore. We find that the loops short enough to be unaffected by the box size are scale invariant and Brownian (except at the very shortest lengths) as before, but that now the energy density in the longer loops can also be accounted for simply by this distribution continued out to infinity. The absence of an energy excess in the long loops means that we find no need to include a second population of open strings. Moreover, we can readily see why all lattice-based work necessarily generates open strings, whether or not they should be present, while graph-based work need not. Intriguingly, however, recent lattice simulations show that

it is also possible to achieve a very significant reduction in the fraction of open strings either by increasing the variance of the sizes of the initial phase domains [13] or by decreasing the index of the power spectrum of the string-bearing field [14,15].

Our simulations are certainly constrained by their CPU-time demands. The range of the central, scale-invariant Brownian loop, distribution is narrower than we might like. However, this is an extremely difficult problem to address; the number of bubbles required to fill the simulation increases with the volume  $L^3$ , and the number of four nodes to calculate then increases as  $N^4$ . Since the runs presented here are already taking around 250 h, even an order of magnitude increase in  $L^2$  would be prohibitively expensive.

We can expect the consequences of our results for string cosmologies to be profound. Simulations of string evolution with all the open string removed suggest that such distributions may reach a different scaling solution from the usual one [16]. Furthermore, current models of string-induced density perturbations focus on the wakes of long strings, and if we remove the open string such models will require a much higher string number density to have sufficient large loops to generate the required perturbations. However, this number density is independently bounded from above by the millisecond pulsar limits on gravitational wave production in the decay of small loops. Whether cosmic string scenarios can be made to fit these new constraints remains an open question.

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