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**Abstract.** – We demonstrate that generalized entanglement [Barnum *et al.*, Phys. Rev. A **68**, 032308 (2003)] provides a natural and reliable indicator of quantum chaotic behavior. Since generalized entanglement depends directly on a choice of preferred observables, exploring how generalized entanglement increases under dynamical evolution is possible without invoking an auxiliary coupled system or decomposing the system into arbitrary subsystems. We find that, in the chaotic regime, the long-time saturation value of generalized entanglement agrees with random matrix theory predictions. For our system, we provide physical intuition into generalized entanglement within a single system by invoking the notion of extent of a state. The latter, in turn, is related to other signatures of quantum chaos.

Central to the study of quantum chaos [1] and broadly significant to fundamental quantum theory [2], is the determination of *distinctive* signatures that unambiguously identify quantum systems whose classical limit exhibits chaotic, versus regular, dynamics. Such signatures are discovered by contrasting quantized versions of classically chaotic and non-chaotic systems. A well-established *static* signature of quantum chaos is the accurate description of a chaotic operators' eigenvalue and eigenvector element statistics by random matrix theory (RMT) [1,3]. A *dynamic* indicator of quantum chaos is the fidelity decay behavior [4–10]. While both approaches have led to deep insights into quantum chaos and its relation to the underlying classical dynamics, they suffer from intrinsic weaknesses. Eigenvector statistics, for example, is basis-dependent. The effectiveness of fidelity decay as an indicator of quantum chaos is strongly influenced by the form of the perturbation. Indeed, regular systems may show chaotic fidelity decay behavior depending on the type of perturbation [8].

A signature of quantum chaos which need not be subject to the above weaknesses and is very natural from a quantum information standpoint is entanglement generation. Chaotic evolution tends to produce states whose statistical properties are similar to those of *random* pure states. Because such states tend to be highly entangled [11], we expect that quantum

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analogs of classically chaotic systems generate greater amounts of entanglement than quantum analogs of non-chaotic ones. This has been confirmed both statically and dynamically. Statically, by directly analyzing the entangling capabilities of the evolution operator, and dynamically by studying the evolution of specific initial states [12–19]. However, all of these studies require a preferred tensor product structure in the ambient Hilbert space, in order for the standard definition of entanglement to be applicable. Thus, some of the above studies arbitrarily decompose the system into subsystems [18, 19], while others couple the system to be studied to another system [12–17]. The latter method introduces the coupling strength as an extra degree of freedom, which can cause strongly chaotic systems to *not* adhere to the proposed chaos indicator. Ultimately, both of these methods effectively impose an external architecture onto the system rather then studying the system on its own terms.

A notion of *generalized entanglement* (GE) able to overcome the limitations of the usual subsystem-based setting has been proposed in [20]. GE extends the observation that standard entanglement can be defined in terms of expectation values of a distinguished set of observables, removing the need for a preferred subsystem decomposition. GE measures constructed from algebras of fermionic operators have provided new diagnostic tools for probing many-body correlations in quantum phase transitions [21], and have contributed to the understanding of standard multipartite entanglement in disordered spin lattices [22].

In this Letter, we establish GE production with respect to appropriate observable sets as an indicator of quantum chaos which removes the above-mentioned weaknesses. In particular, because the GE framework relies only on *convex* structure of the spaces of quantum states and observables, GE is able to be defined within the system *alone*, without resorting to coupling additional systems or imposing arbitrary subsystems. We demonstrate how GE clearly differentiates between fully chaotic, partially chaotic, and regular behavior using the paradigmatic case of a *quantum kicked top* (QKT) [23]. Furthermore, we show that the behavior of the chaotic QKT follows the RMT prediction. Finally, we provide a physical justification by comparing GE to the notion of *extent* of a state, introduced by Peres [24], and recently related to fidelity decay [9].

The starting point to define GE is to realize that standard entangled pure states of a composite quantum system $S$ look mixed to observers whose means to control and measure $S$ are constrained to local operations on individual subsystems: To specify a pure entangled state requires knowledge of the correlations, which are expectations of non-local operators. By thinking of pure states as one-dimensional (extremal) projectors in the set of density operators for $S$, entanglement implies a loss of purity (extremality) upon restricting to local expectations only. A similar characterization can be provided without making reference to a subsystem decomposition for $S$. Let $S$ be defined on a Hilbert space $\mathcal{H}$, and let $\Omega$ denote a *generic* set of observables. Then any pure state $|\psi\rangle \in \mathcal{H}$ induces a *reduced* state that determines only the expectations of operators in $\Omega$. In analogy with the standard case, $|\psi\rangle$ is said to be *generalized unentangled relative to* $\Omega$ if its reduced state is pure, generalized entangled otherwise [20].

A natural way to quantify GE is to relate $|\psi\rangle$ to $\Omega$ via the (square) length of the projection $|\psi\rangle\langle\psi|$ onto $\Omega$ [20]. We shall focus on the case where $\Omega \equiv \mathfrak{h}$ is a real Lie algebra faithfully represented on $\mathcal{H}$, linearly spanned by a set $\{A_\ell\}$, $\ell = 1, \ldots, L$, of Hermitian operators, orthogonal with respect to the trace norm [21]. The *purity of* $|\psi\rangle$ *relative to* $\mathfrak{h}$ ($\mathfrak{h}$-purity) is

$$P_\mathfrak{h}(|\psi\rangle) = \mathtt{K} \sum_{\ell=1}^{L} \langle\psi|A_\ell|\psi\rangle^2 = \mathtt{K} \sum_{\ell=1}^{L} \langle A_\ell\rangle^2 \,, \tag{1}$$

where the constant $\mathtt{K} > 0$ ensures that the maximum value of $P_\mathfrak{h}$ is 1. States of maximum purity are generalized unentangled relative to $\mathfrak{h}$. If the latter is a semisimple Lie algebra acting

irreducibly on $\mathcal{H}$, then any generalized unentangled state has extremal length, and belongs to the family of *generalized coherent states* (GCSs) [25].

The quantum chaotic system we explore is the QKT, used in many previous studies of quantum chaos in general [1] and entanglement generation in particular. In contrast to the present work, however, previous studies of QKT entanglement either explored coupled kicked tops [12, 17], or a realization of the QKT in term of spin 1/2 susystems [18]. The dynamical variables of the QKT are the three components of the angular momentum vector, $\mathbf{J} = (J_x, J_y, J_z)$, with $|\mathbf{J}| = J$ constant. The dynamics of the classical kicked top is a locus of points on the surface of the unit sphere spanned by $\mathbf{J}/J$, with the relative size of the non-chaotic and chaotic regions depending on the kick strength, $k$. The kicked top is fully non-chaotic for $k \lesssim 2.7$, has both chaotic and non-chaotic regions for $2.7 \lesssim k \lesssim 4.2$, and is fully chaotic for $k \gtrsim 4.2$ [6]. QKT evolution is generated by the Floquet operator [23]

$$U_{\texttt{QKT}} = e^{-i\pi J_y/2} e^{-ikJ_z^2/2J}, \quad \hbar = 1, \tag{2}$$

in a Hilbert space $\mathcal{H}_N$ of dimension $N = 2J + 1$. $\mathcal{H}_N$ furnishes a spin-$J$ irreducible representation of SU(2), thus, it is natural to investigate $\mathfrak{h} = \mathfrak{su}(2)$ as a preferred algebra for this system. From Eq. (1), the $\mathfrak{su}(2)$-purity is

$$P_{\mathfrak{su}(2)}(|\psi\rangle) = \frac{1}{J^2} \sum_{\ell=x,y,z} \langle\psi|J_\ell|\psi\rangle^2, \tag{3}$$

where $\texttt{K} = J^{-2}$ is chosen so that $P_{\mathfrak{su}(2)} = 1$ for angular momentum GCSs, defined by the eigenvalue equation $(\mathbf{n} \cdot \mathbf{J})|\psi\rangle = J|\psi\rangle$, $\mathbf{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, $\theta \in [0, \pi], \phi \in [-\pi, \pi]$ [2]. Thus, $\text{GE}_{\mathfrak{su}(2)} = 1 - P_{\mathfrak{su}(2)}$.

In the chaotic regime, RMT predicts the asymptotic state of the QKT to be described by a random pure state uniformly drawn according to the Haar measure on SU($N$). Following the general procedure for estimating the expected GE in typical pure states [26], or exploiting the fact that the above expectation values have been previously studied within RMT [1, 17], the average $\mathfrak{su}(2)$-GE is found to be $\overline{\text{GE}}_{\mathfrak{su}(2)} = 1 - 1/2J$.

We begin by exploring a QKT with a mixed phase space, $k = 3$. Fig. 1 contrasts the $\text{GE}_{\mathfrak{su}(2)}$ growth as a function of time for GCSs centered in the chaotic versus regular region of the classical phase space. States in the chaotic region quickly approach the $\text{GE}_{\mathfrak{su}(2)}$ value predicted by RMT, whereas states in the regular region generate much less GE. GCSs at the "edge of quantum chaos" [27], the border between the chaotic and regular phase space regions, demonstrate intermediate behavior.

As $k$ increases, the chaotic sea covers the whole of phase space. Correspondingly, the $\text{GE}_{\mathfrak{su}(2)}$ of *all* states quickly approach the RMT estimation. The inset of Fig. 1 illustrates this for a QKT of $k = 12$. The $\text{GE}_{\mathfrak{su}(2)}$ initially increases as a Gaussian and then plateaus at 0.999, as predicted. In contrast, a QKT with a regular phase space, $k = 1.1$, displays an initial linear $\text{GE}_{\mathfrak{su}(2)}$ growth that typically plateaus well below one, Fig. 2(a).

The above results strongly support the use of GE as a signature of quantum chaos but do not offer a clear insight into the physical meaning of this property in our system. To clarify this concept we establish a relationship between GE and the extent of a state relative to a Hermitian observable $A$ [24, 28]. The latter is defined as

$$\Delta A(|\psi\rangle) = \sqrt{\langle A^2 \rangle - \langle A \rangle^2}. \tag{4}$$

Thus, the extent is the square-root of the variance of $A$ for the state $|\psi\rangle$. The connection to GE is shown by noting that, for $\mathfrak{h}$ irreducible, the $P_{\mathfrak{h}}$-purity is directly related to the *invariant*
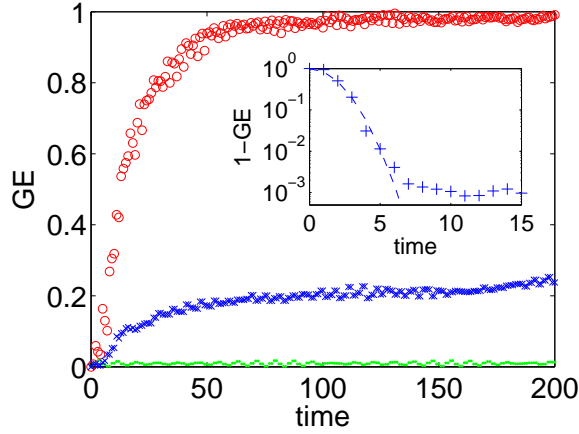
Fig. 1 – $GE_{\mathfrak{su}(2)}$ versus time for representative initial angular momentum GCSs under the evolution of a mixed phase space QKT, $k = 3$, $J = 500$. The $GE_{\mathfrak{su}(2)}$ of the GCS centered in a chaotic region $\theta = 3\pi/5, \phi = -\pi/10$ ($\bigcirc$), quickly approaches one. The state centered in the regular region, $\theta = \pi/2, \phi = 0$, generates very little $GE_{\mathfrak{su}(2)}$ ($\cdot$), and the state centered at the edge of chaos $\theta = \pi/2, \phi = -\pi/10$, exhibits intermediate behavior ($\times$). Inset: Average $\mathfrak{su}(2)$-purity of 90 GCSs under chaotic QKT evolution, $k = 12$, $J = 500$. The $GE_{\mathfrak{su}(2)}$ increases as a Gaussian, $e^{-0.18t^2}$ (dashed line), saturating at $\approx 0.999$, the RMT estimation.
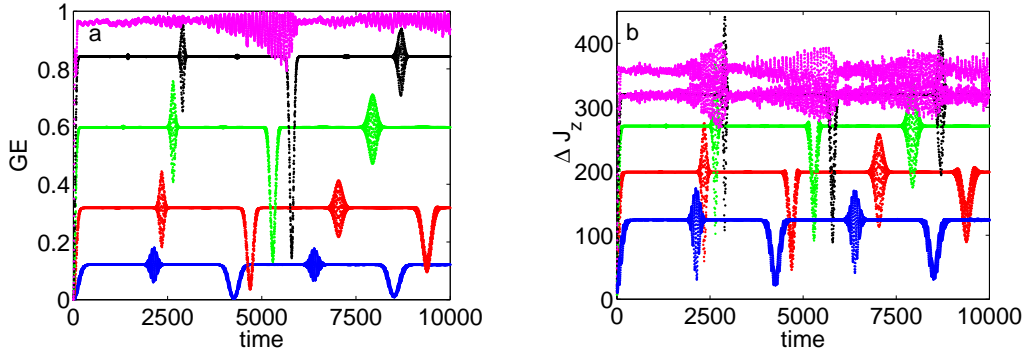


Fig. 2 – $GE_{\mathfrak{su}(2)}$ (a), and $z$-extent, $\Delta J_z$ (b) for representative initial angular momentum GCSs under regular QKT evolution, $k = 1.1$, $J = 500$. The states (bottom to top) are centered at $\theta = 3\pi/5, \phi = -2\pi/5, -3\pi/10, -\pi/5, -\pi/10, 0$, which lie on phase space orbits of increasing size. The first four states exhibit linear GE and extent increase, until saturation at a level which depends on the size of the GCS orbit. When the orbit is large, the GCS has a larger spread with respect to $\mathfrak{su}(2)$-observables, hence a larger GE and $\Delta J_z$. After saturation, periodic recurrences in both the GE and extent are seen. The *initial* extent is a good indicator of regular regime fidelity decay behavior [9]. The above four GCSs are mainly composed of a few low-extent eigenvalues and show a Gaussian fidelity decay. The GCS $\theta = 3\pi/5, \phi = 0$, does not display any recurrences. Rather, both the GE and extent exhibit wild oscillations and achieve higher values than the other states. This state is composed of several high-extent QKT eigenvalues and exhibits a power-law fidelity decay under small perturbations [9].

*uncertainty* functional, $(\Delta\mathcal{I})^2$. With respect to the above operator basis [21],

$$(\Delta\mathcal{I})^2(|\psi\rangle) = \sum_\ell [\langle A_\ell^2 \rangle - \langle A_\ell \rangle^2] = \langle C_2 \rangle - P_{\mathfrak{h}}(|\psi\rangle) , \qquad (5)$$

where $C_2$ is the quadratic Casimir invariant of $\mathfrak{h}$, here $C_2 = \mathbf{J} \cdot \mathbf{J}$. Thus, Eqs. (4)-(5) yield

$$\mathrm{GE}_{\mathfrak{su}(2)} = \sum_{\ell=x,y,z} \Delta\Big(\frac{J_\ell}{J}\Big)^2 - \frac{1}{J} . \qquad (6)$$

Eq. (6) clarifies how GE relative to the angular momentum observables is directly related to (identical to, as $J \to \infty$) the squared extent for rescaled observables $J_\ell/J$, *averaged* over $x, y, z$. As suggested in [2, 24], the extent in each direction contains essentially equivalent information for differentiating between regular and chaotic dynamics. For the QKT, this may be seen explicitly: due to the $\pi/2$ rotation in $U_{\mathrm{QKT}}$, $x$ and $z$ are interchanged at every time step, leading to equivalent $z$- and $x$-extent behavior. The $y$-extent is bounded by $(\Delta J_y)^2 \leq J(J + 1) - 2(\Delta J_z)^2$ causing large $x, z$-extents to be correlated with small $y$-extent values. Thus, the behavior of $\mathrm{GE}_{\mathfrak{su}(2)}$ as a quantum chaos indicator should be *qualitatively similar* to the extent behavior for any observable $J_\ell/J$.

The relation between the $z$-extent and $\mathrm{GE}_{\mathfrak{su}(2)}$ is exhibited graphically in Fig. 2 using initial GCSs under regular QKT evolution, $k = 1.1$. The similarity is striking: Both increase linearly as a function of time until some saturation level. Upon saturation, both the extent and $\mathrm{GE}_{\mathfrak{su}(2)}$ plateau, except for periodic recurrences which occur at the same time. An analogous behavior has been numerically verified for extents in the $x, y$ directions.

The relationship between $\mathrm{GE}_{\mathfrak{su}(2)}$ and the extent provides the following intuitive, physical picture for GE and "self-entanglement": $\mathrm{GE}_{\mathfrak{su}(2)}$ is analogous to a measure of the spread of the system's state vector in the phase space associated with the dynamical observables. As an initial GCS evolves to cover more and more of its orbit, the $\mathrm{GE}_{\mathfrak{su}(2)}$ grows. The larger the phase space orbit of the GCS, the larger the entanglement saturation level (Fig. 2). For fully chaotic systems, any typical orbit covers all of phase space, hence the $\mathrm{GE}_{\mathfrak{su}(2)}$ converges to the RMT estimation. In this sense, $\mathrm{GE}_{\mathfrak{su}(2)}$ evolution, *at least starting from states which have a good classical limit*, directly reflects the underlying classical phase space structure. Similar connections between the entanglement growth and the spread of an initial GCS have been made for standard bi-partite entanglement, the rate of entanglement increase being determined by the Lyapunov exponents of the corresponding classical Liouville distribution [14,16,17,29]. Additional insight into single-particle entanglement has been provided in [30].

As mentioned, previous studies of QKT entanglement relied on decomposing the system into $N = 2J$ spin $1/2$ systems and restricting to states symmetric under spin exchange [18]. It can then be shown that $\mathrm{GE}_{\mathfrak{su}(2)}$ is *equivalent* to standard global multipartite entanglement as quantified by the Meyer-Wallach measure [31] that is, $\mathrm{GE}_{\mathfrak{su}(2)}$ is proportional to the average linear entropy of entanglement between any spin $1/2$ subsystem and the rest. Yet, the classical picture of spread in phase space is still useful in showing the close relationship between chaos and entanglement generation. Formally, the connection between $\mathrm{GE}_{\mathfrak{su}(2)}$ and other entanglement measures demonstrates how GE unifies different entanglement approaches.

Through the extent, $\mathrm{GE}_{\mathfrak{su}(2)}$ is connected to fidelity decay – a quantum chaos signature which provides a fingerprint of the classical Lyapunov exponent for quantized versions of classically chaotic systems [5,32]. Fidelity is a measure of distance between the states reached from a given initial state $|\psi_i\rangle$ under slightly different evolutions [4], $F(t) = |\langle\psi_i|U^{-t}U_p^t|\psi_i\rangle|^2$, where $U^t = e^{-iH_0 t}$, $U_p^t = e^{-i(H_0+\delta V)t}$ are the unperturbed and perturbed evolutions, and

$\delta$, $V$ are the perturbation strength and Hamiltonian, respectively. At short time, $F(t) = 1 - (\langle V^2 \rangle - \langle V \rangle^2)\delta^2 t^2 + ...$, immediately identifying the square of the *initial V*-extent as the coefficient of the second-order term [24]. While a complete characterization of the operators $V$ able to induce effective dynamical cross-over is lacking [8], perturbations commuting with one of the $J_\ell$ make fidelity a reliable indicator for QKT dynamics [2,6]. This relation between the $\ell$-extent and fidelity decay further supports the validity of $\mathrm{GE}_{\mathfrak{su}(2)}$ as a quantum chaos indicator (see also Fig. 2). We note that fidelity decay of a GCS is also related to the size of the classical phase space orbit [9]. In general, fidelity decay is connected to other signatures of quantum chaos, such as the shape of the local density of states [6] and eigenvector statistics [8]. Montangero *et al* [33] demonstrate the similarity of behavior between fidelity decay and the decay of (bi-partite) entanglement of an initial Bell pair. Here, it is the generalized purity which decays and, as shown, behaves qualitatively similarly to fidelity decay – complementing the results obtained for local purity and fidelity in [22].

The fact that a single spatial direction suffices for identifying a valid fidelity perturbation and extent variable suggests that we examine an observable set consisting of a *single observable* as another candidate for defining GE. This is done by restricting to a (Cartan) subalgebra $\mathfrak{h} = \mathfrak{so}(2) \subset \mathfrak{su}(2)$ generated by a single operator $J_\ell$, say $J_z$. Using Eq. (1), the $\mathfrak{so}(2)$-purity is

$$P_{\mathfrak{so}(2)}(|\psi\rangle) = \frac{1}{J^2}\langle\psi|J_z|\psi\rangle^2 , \quad \mathrm{GE}_{\mathfrak{so}(2)} = 1 - P_{\mathfrak{so}(2)} . \tag{7}$$

Numerical simulations for initially $\mathfrak{so}(2)$-unentangled GCSs ($\theta = 0$) show that $\mathrm{GE}_{\mathfrak{so}(2)}$ is also a valid signature of quantum chaos for this system (data not shown). This indicates the ability of the $\mathfrak{h}$-purity to differentiate between regular and chaotic dynamics without a direct link to variances of observables or invariant uncertainty. Suggestively, the $\mathfrak{so}(2)$-purity has been shown to characterize quantum criticality in the Lipkin-Meshov-Glick model [21], which may also be mapped into a single (pseudo)spin system with $\mathfrak{su}(2)$ dynamical algebra.

The above discussion shows the utility of expectation values and statistical moments of observables in understanding quantum chaos. Uncertainty-based entanglement measures have been suggested outside the GE framework [34] providing links to important quantities such as the Wigner-Yanasi skew information [30,35] and a quantum analog of the Fisher information [36]. A dedicated study connectng GE to such measures will be presented elsewhere. A *general* characterization of a preferred set of observables which can sharply differentiate between chaotic and non-chaotic regimes likewise remains an area for future in-depth analysis. In particular, this will require extending the present study to other quantum chaos models – for instance, kicked rotors and the quantum baker's maps. While further generalizations of the mathematical formalism are likely to be needed (in order to properly define, for example, GCSs for *discrete* groups), we believe that the GE notion has both the flexibility and the potential for identifying quantum chaos signatures in arbitrary physical settings.

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