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ISOSPECTRAL DEFORMATIONS OF CLOSED RIEMANNIAN MANIFOLDS WITH DIFFERENT SCALAR CURVATURE

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To the memory of Hubert Pesce (1966–1997), a gifted colleague and deeply missed friend.

Introduction.

To what extent does the eigenvalue spectrum of a compact Riemannian manifold determine the geometry of the manifold? Various global geometric invariants, such as dimension, volume, and total scalar curvature, are known to be spectrally determined. Moreover, various manifolds such as round spheres of dimension less than or equal to six and 2-dimensional

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flat tori are uniquely determined by their spectra. However, we show:

MAIN THEOREM. — *For $n \geq 4$, there exist continuous d-parameter families $\{g_t\}$ of isospectral, non-isometric Riemannian metrics on the manifold $S^n \times T$, where T is the 2-dimensional torus and S^n is the n-dimensional sphere. Here d is of order at least $O(n^2)$. These metrics are non-homogeneous. For some of the deformations, the maximum scalar curvature of g_t depends non-trivially on t.*

The metrics can be chosen arbitrarily close to the standard metric, *i.e.*, to the product of the flat metric on the torus and the round metric on the sphere.

The first example of isospectral Riemannian manifolds was a pair of sixteen dimensional flat tori [M]. The past dozen years have seen an explosion of new examples. See, for example, [BT], [Bu], [GWW], [GW1], [GW2], [Gt1], [Gt2], [I], [Sch], [Su], [V]. However, until recently, all known isospectral manifolds were at least locally isometric; in particular, all isospectral closed manifolds had a common Riemannian cover. This was due primarily to the fact that most examples could be explained by Sunada's method [Su] or its generalizations [DG2]. The Sunada methods rely almost exclusively on representation theory, with the result that isospectral manifolds constructed using these methods must be locally isometric. See the expository articles [Be], [Br], [D], [G1], or [GGt] for more information about isospectral manifolds in general.

Then Szabo [Sz] constructed pairs of isospectral compact manifolds with boundary that are not locally isometric, and Gordon and Wilson [GW3] generalized his construction to obtain continuous families of such manifolds. Finite families of closed isospectral manifolds with different local geometry were given in [G2], [G3] and [GW3].

The examples described in the Main Theorem above have several new features:

- They give the first examples of continuous isospectral deformations of closed manifolds for which the metrics are not locally isometric.

- They are the first examples of isospectral manifolds with different scalar curvature.

The isospectral manifolds in the Main Theorem are the boundaries of the isospectral manifolds constructed in [GW3].

After completing a draft of this paper, the authors learned that

Z. Szabo has independently shown that the boundaries of the manifolds he constructed in [Sz] are isospectral. Like the isospectral deformations considered here, the pairs of isospectral metrics that he constructed are not locally homogeneous nor locally isometric, although they do have the same maximum and minimum scalar curvature. Szabo, moreover, explicitly computed the spectrum of these isospectral metrics. This work is included in a revised version of the article [Sz].

Construction of isospectral metrics.

The isospectral manifolds constructed in [GW3] are domains in certain two-step nilpotent Lie groups. They may be described as follows.

NOTATION 1. — *Let* $\mathfrak{z} = \mathbf{R}^k$ *and* $\mathfrak{v} = \mathbf{R}^m$ *with their standard inner products, and let* $j : \mathfrak{z} \to \mathrm{so}(\mathfrak{v})$ *be a linear map. Let* $\mathfrak{g}$ *be the orthogonal direct sum* $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{z}$. *Define a Lie bracket on* $\mathfrak{g}$ *by declaring* $\mathfrak{z}$ *to be central and defining* $[\,,\,] : \mathfrak{v} \times \mathfrak{v} \to \mathfrak{z}$ *by*

$$\langle [x, y], z \rangle = \langle j(z)x, y \rangle$$

for all x *and* y *in* $\mathfrak{v}$ *and* z *in* $\mathfrak{z}$. *This bracket gives* $\mathfrak{g}$ *the structure of a two-step nilpotent Lie algebra, and all two-step nilpotent Lie algebras may be constructed by this method. We denote* $\mathfrak{g}$ *with this structure by* $\mathfrak{g}(j)$. *An element* $x \in \mathfrak{v}$ *is central if and only if* $j(z)x = 0$ *for all* $z \in \mathfrak{z}$. *Without loss of generality, we assume* $\mathfrak{v}$ *intersects the center trivially; i.e. ,* $\mathfrak{z}$ *coincides with the center of* $\mathfrak{g}(j)$. *(Note that* $\mathfrak{g}(j)$ *may still have an abelian factor, as we are not assuming that* j *is non-singular.) Let* $G(j)$ *denote the simply-connected Lie group with Lie algebra* $\mathfrak{g}(j)$. *The Lie group exponential map* $\exp : \mathfrak{g}(j) \to G(j)$ *is a diffeomorphism. Moreover, the center of* $G(j)$ *is isomorphic to* $\mathbf{R}^k$ *and* $\exp_{|\mathfrak{z}}$ *is a linear isomorphism from* $\mathfrak{z}$ *to the center of* $G(j)$.

With respect to the global coordinate system $G(j) \to \mathfrak{v} \oplus \mathfrak{z}$ *defined by* $\exp(x + z) \mapsto (x, z)$ *for* $x \in \mathfrak{v}$ *and* $z \in \mathfrak{z}$, *the Lie group multiplication is given by*

$$(x, z) \cdot (x', z') = \left(x + x', z + z' + \frac{1}{2}[x, x']\right).$$

The inner product on $\mathfrak{g}(j)$ *defines a left-invariant metric on* $G(j)$, *i.e., a metric for which the left translations by group elements are isometries.*

Let $\mathcal{L}$ be a lattice of full rank in $\mathfrak{z}$. Identify $\mathcal{L}$ with $\exp(\mathcal{L})$, *a discrete central subgroup of $G(j)$. The quotient $\overline{G(j)} = G(j)/\mathcal{L}$ is again a two-step nilpotent Lie group, and it inherits a Riemannian metric from its covering $G(j)$. Its center is isomorphic to the k-dimensional torus $\mathfrak{z}/\mathcal{L}$, and $\overline{G(j)}$ is diffeomorphic to $\mathfrak{v} \times (\mathfrak{z}/\mathcal{L})$. Let B be the closed unit ball in $\mathfrak{v}$ and S its $(m-1)$-dimensional boundary sphere. Set*

$$M(j) = \{(x, \bar{z}) \in \overline{G(j)} : x \in B \text{ and } \bar{z} \in \mathfrak{z}/\mathcal{L}\}$$

and

$$N(j) = \{(x, \bar{z}) \in \overline{G(j)} : x \in S \text{ and } \bar{z} \in \mathfrak{z}/\mathcal{L}\}.$$

The manifolds $M(j)$ were studied extensively in [GW3]. The $M(j)$ are manifolds with boundary that are locally homogeneous; their local geometry is that of the Lie group $G(j)$. We study their boundaries $N(j)$ here. Note that $N(j)$ is diffeomorphic to $S^n \times T^k$ where T^k is a torus of dimension $k = \dim(\mathfrak{z})$ and where $n = m - 1$.

PROPOSITION 2. — *The map $\pi_j : N(j) \to S$ given by $(x, \bar{z}) \to x$ is a Riemannian submersion with respect to the canonical round metric on S associated with the inner product on $\mathfrak{v}$. The fibers are totally geodesic flat tori that are orbits of a flat toral group $T(j) \simeq \mathfrak{z}/\mathcal{L}$ of isometries acting freely on $N(j)$.*

Proof. — The analogous statement for the nilmanifold $\overline{G(j)}$ is well-known; *i.e.*, the map $\overline{G(j)} \to \mathfrak{v}$ given by $(x, \bar{z}) \mapsto x$ is a Riemannian submersion with respect to the Euclidean metric on $\mathfrak{v}$, and the fibers are totally geodesic flat tori. The fibers are orbits of the toral group of isometries given by translations by the elements $(0, \bar{z})$, $\bar{z} \in \mathfrak{z}/\mathcal{L}$. The manifold $N(j) \subset \overline{G(j)}$ is the restriction of the torus bundle $\overline{G(j)}$ over B to the subspace $S \subset B$, and the statement follows. □

DEFINITION 3. — *Let $\mathfrak{v}$, $\mathfrak{z}$ and $\mathcal{L}$ be as above.*

(i) *A pair j, j' of linear maps from $\mathfrak{z}$ to* $\mathrm{so}(\mathfrak{v})$ *is called* $\mathcal{L}$-equivalent *if there exist orthogonal linear operators A on $\mathfrak{v}$ and C on $\mathfrak{z}$ such that $C(\mathcal{L}) = \mathcal{L}$ and*

$$Aj(z)A^{-1} = j'(C(z))$$

for all $z \in \mathfrak{z}$. We call the pair (A, C) an $\mathcal{L}$-equivalence between j and j'.

(ii) *We say j is* isospectral *to j', denoted $j \sim j'$, if for each $z \in \mathfrak{z}$, the eigenvalue spectra (with multiplicities) of $j(z)$ and $j'(z)$ coincide, i.e.,*

there exists an orthogonal linear operator A_z for which

$$A_z j(z) A_z^{-1} = j'(z).$$

Remark 4. — We address the question of when $N(j)$ is isometric to $N(j')$ later in the paper. For now, we note the following facts about $\overline{G(j)}$ and $\overline{G(j')}$:

(i) Given orthogonal transformations A of $\mathfrak{v}$ and C of $\mathfrak{z}$ such that $C(\mathcal{L}) = \mathcal{L}$, one easily checks that the following statements are equivalent:

(a) The pair (A, C) is an $\mathcal{L}$-equivalence between j and j'.

(b) The map $\mathfrak{g}(j) \to \mathfrak{g}(j')$ given by $x + z \mapsto A(x) + C(z)$ for $x \in \mathfrak{v}$ and $z \in \mathfrak{z}$ is a Lie algebra isomorphism.

(c) Viewing C as an automorphism of the torus $\mathfrak{z}/\mathcal{L}$, the map $\tau : \overline{G(j)} \to \overline{G(j')}$ given by $\tau(x, \bar{z}) = (A(x), C(\bar{z}))$ is both a Lie group isomorphism and an isometry.

(ii) By [W], any isometry between $\overline{G(j)}$ and $\overline{G(j')}$ is of the form $L_{(x,\bar{z})} \circ \tau$ where $\tau : \overline{G(j)} \to \overline{G(j')}$ is both a Lie group isomorphism and an isometry (and thus arises from an $\mathcal{L}$-equivalence between j and j') and $L_{(x,\bar{z})} : \overline{G(j')} \to \overline{G(j')}$ is left translation by the element $(x, \bar{z})$ of $\overline{G(j')}$. In particular, $\overline{G(j)}$ is isometric to $\overline{G(j')}$ if and only if j and j' are $\mathcal{L}$-equivalent.

(iii) If j is $\mathcal{L}$-equivalent to j', then the isometry $\tau : \overline{G(j)} \to \overline{G(j')}$ defined in (i)(c) restricts to an isometry from $N(j)$ to $N(j')$.

THEOREM 5. — *Fix inner product spaces $\mathfrak{v}$ and $\mathfrak{z}$ and a lattice $\mathcal{L}$ of full rank in $\mathfrak{z}$, and let $j, j' : \mathfrak{z} \to \mathrm{so}(\mathfrak{v})$ be linear maps. If $j \sim j'$, then $N(j)$ is isospectral to $N(j')$.*

Proof. — The analogous statement for $M(j)$ and $M(j')$ is proved in [GW3]. The proof that $N(j)$ and $N(j')$ are isospectral is similar, so we just give the main ideas here. First note that if $\mathfrak{z}$ is one-dimensional and if $j \sim j'$, then j is $\mathcal{L}$-equivalent to j', with C in Definition 3 being the identity map. By Remark 4, the manifolds are isometric, hence trivially isospectral.

Now consider the general case. By Proposition 2, the central torus $T(j) \simeq \mathfrak{z}/\mathcal{L}$ acts on $N(j)$ by isometries, and the manifold $N(j)$ has the structure of a torus bundle over the round sphere S. Moreover, the projection $N(j) \to S$ is a Riemannian submersion with totally geodesic fibers. If $\mathfrak{w}$ is a codimension-one subspace of $\mathfrak{z}$ spanned by lattice vectors

in $\mathcal{L}$, then $\mathfrak{w}$ covers a codimension-one subtorus H of $T(j)$. Let $N_H(j)$ denote the quotient of $N(j)$ by the isometry action of this torus. With the induced metric on $N_H(j)$, the projection $N(j) \to N_H(j)$ is again a Riemannian submersion with totally geodesic fibers.

Now $N_H(j)$ is defined in the same way as $N(j)$, but with $\mathfrak{z}$ replaced by $\mathfrak{w}^\perp = \mathfrak{z} \ominus \mathfrak{w}$, the orthogonal complement of $\mathfrak{w}$ in $\mathfrak{z}$, and j replaced by $j_{|\mathfrak{w}^\perp}$. Since $j_{|\mathfrak{w}^\perp} \sim j'_{|\mathfrak{w}^\perp}$ and since $\mathfrak{w}^\perp$ is one-dimensional, the first part of the proof shows that $N_H(j)$ is isometric to $N_H(j')$.

Functions on $N_H(j)$ pull back to functions on $N(j)$. Since the projection is a Riemannian submersion with totally geodesic fibers, a classical theorem says that eigenfunctions on $N_H(j)$ pull back to eigenfunctions on $N(j)$ with the same eigenvalue. Thus the spectrum of $N_H(j)$ is part of the spectrum of $N(j)$. By a Fourier decomposition argument involving the action of the torus $T(j)$, one shows that the spectra of the various $N_H(j)$ exhaust the spectrum of $N(j)$. We can then use the isospectrality of the $N_H(j)$ with the $N_H(j')$ to deduce the isospectrality of $N(j)$ with $N(j')$.

□

Scalar curvature.

We next express the scalar curvature function scal of the manifold $N(j)$ in terms of the scalar and Ricci curvatures $\widetilde{\mathrm{scal}}$ and $\widetilde{\mathrm{Ric}}$ of the ambient nilmanifold $\overline{G(j)}$. Since $\overline{G(j)}$ is locally homogeneous, $\widetilde{\mathrm{scal}}$ is a constant function; we denote its constant value also by $\widetilde{\mathrm{scal}}$. The Ricci curvature $\widetilde{\mathrm{Ric}}$ does not depend on the point and may be viewed as a bilinear form on the Lie algebra $\mathfrak{g}(j)$.

PROPOSITION 6. — *Using Notation 1, let* $m = \dim(\mathfrak{v})$. *The scalar curvature* scal *of* $N(j)$ *is given at the point* $p = (x, \bar{z})$ *of* $N(j)$ *by*

$$\mathrm{scal}(p) = \widetilde{\mathrm{scal}} + (m-1)(m-2) - \widetilde{\mathrm{Ric}}(x, x).$$

Proof. — Let ν denote a unit normal vector field to $N(j)$ in $\overline{G(j)}$. Denote by $\widetilde{\nabla}$, ∇ the Levi–Civita connections of $\overline{G(j)}$ and $N(j)$ respectively.

A routine calculation using the second fundamental form shows that for p in $N(j)$,

$$\mathrm{scal}(p) = \widetilde{\mathrm{scal}} - 2\widetilde{\mathrm{Ric}}(\nu_p, \nu_p) + (\mathrm{trace}(\widetilde{\nabla}\nu_{|\,\mathrm{T}_p N(j)}))^2 - \|\widetilde{\nabla}\nu_{|\,\mathrm{T}_p N(j)}\|^2. \tag{6.1}$$

(The norm in the last term is that on tensors of type (1,1), *i.e.* , the L^2 norm on matrices relative to an orthonormal basis.)

We remark that for any two-step nilpotent Lie group G with Lie algebra $\mathfrak{g}$ and any v in $\mathfrak{g}$, one has the identity

$$\exp_{*v} = L_{\exp(v)*}\Big(\mathrm{Id} - \frac{1}{2}\,\mathrm{ad}_v\Big). \tag{6.2}$$

Let $p = (x, \bar{z})$ and recall that $\mathrm{T}_p\overline{G(j)} = L_{p*}(\mathfrak{g}(j))$, where L_p is left translation by p. Using (6.2), one easily checks that as a subspace of $L_{p*}(\mathfrak{g}(j))$, $\mathrm{T}_pN(j) = L_{p*}(x^\perp)$, where $x^\perp$ is the orthogonal complement of $\mathrm{span}\{x\}$ in $\mathfrak{g}(j)$. The outward unit normal vector field is then given by $\nu_p = L_{p*}(x)$.

Let $y \in x^\perp \cap \mathfrak{v}$ and $w \in \mathfrak{z}$. We view x, y, $\widetilde{\nabla}_y x$, w, and $\widetilde{\nabla}_w x$ as left-invariant vector fields on $\overline{G(j)}$, so we write, e.g., y_p for $L_{p*}y$. The curve $c(t) = \exp((\cos t)x + (\sin t)\big(y + \frac{1}{2}[x,y]\big) + \bar{z}\big)$ has initial velocity y_p. Viewing ν as a vector field along c and computing the covariant derivative yields

$$\widetilde{\nabla}_{y_p}\nu = y_p + (\widetilde{\nabla}_y x)_p.$$

Similarly,

$$\widetilde{\nabla}_{w_p}\nu = (\widetilde{\nabla}_w x)_p.$$

In summary, for $p = (x, \bar{z}) \in N(j)$ and $u \in x^\perp = L_{p*}^{-1}(\mathrm{T}_pN(j))$,

$$\widetilde{\nabla}_u\nu = L_{p*}(\mathrm{proj}_{\mathfrak{v}}\, u + \widetilde{\nabla}_u x),$$

where x is viewed as a left-invariant vector field.

Recall (see [E], Section 2) that for $x, y \in \mathfrak{v}$ and $w \in \mathfrak{z}$,

$$\widetilde{\nabla}_y x = \frac{1}{2}[y,x] \quad \text{and} \quad \widetilde{\nabla}_w x = -\frac{1}{2}j(w)x. \tag{6.3}$$

Thus $\widetilde{\nabla}x$ sends $\mathfrak{v} \to \mathfrak{z}$ and $\mathfrak{z} \to \mathfrak{v}$, and it follows immediately that $\mathrm{trace}(\widetilde{\nabla}x) = 0$, and that $\mathrm{trace}(\widetilde{\nabla}\nu_{|\,\mathrm{T}_pN(j)}) = m - 1$. Likewise, $\|(\widetilde{\nabla}\nu)_{|\,\mathrm{T}_pN(j)}\|^2 = (m-1) + \|\widetilde{\nabla}x\|^2$.

Finally, let $z_1, \ldots, z_k$ be an orthonormal basis of $\mathfrak{z}$. It is known that (see [E], Section 2)

$$\widetilde{\mathrm{Ric}}(x,x) = \frac{1}{2}\sum_{i=1}^{k}\langle j(z_i)^2 x, x\rangle\,. \tag{6.4}$$

Equations (6.3) and (6.4) and Notation 1 imply

$$\|\widetilde{\nabla}x\|^2 = -\widetilde{\mathrm{Ric}}(x,x)$$

and Proposition 6 follows. □

COROLLARY 7. — *Fix inner product spaces $\mathfrak{v}$ and $\mathfrak{z}$ and a lattice $\mathcal{L}$ of full rank in $\mathfrak{z}$, and let $j, j' : \mathfrak{z} \to \mathrm{so}(\mathfrak{v})$ be linear maps. The maximum, respectively minimum, values of the scalar curvature of $N(j)$ and $N(j')$ coincide if and only if the minimum, respectively maximum, eigenvalues of the Ricci tensors of $\overline{G(j)}$ and $\overline{G(j')}$ restricted to $\mathfrak{v} \otimes \mathfrak{v}$ coincide.*

Examples 8. — We briefly review Example 2.3 of [GW3] and show that some of the isospectral deformations constructed there have changing maximal scalar curvature.

Take $\mathfrak{z} = \mathbf{R}^2$ and $\mathfrak{v} = \mathbf{R}^6$ with their standard ordered bases and standard inner product. For $a, b \in \mathrm{so}(6)$ and $s, t \in \mathbf{R}$, define $j_{a,b}(s,t) = sa + tb$. Each linear map $j : \mathbf{R}^2 \to \mathrm{so}(6)$ is of the form $j = j_{a,b}$ for some $a, b \in \mathrm{so}(6)$. Fix for the remainder of the discussion an element $a \in \mathrm{so}(6)$ that is in block diagonal form with 2×2 diagonal blocks $\begin{bmatrix} 0 & -a_i \\ a_i & 0 \end{bmatrix}, 1 \leq i \leq 3$, where $0 < a_1 < a_2 < a_3$. Consider all matrices $b \in \mathrm{so}(6)$ of the form

$$b = \begin{bmatrix} 0 & 0 & b_{12} & 0 & b_{13} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -b_{12} & 0 & 0 & 0 & b_{23} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -b_{13} & 0 & -b_{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

with $(b_{12}, b_{13}, b_{23}) \in \mathbf{R}^3$. From [GW3], we know that if b and b' are of this form, then $j_{a,b} \sim j_{a,b'}$ if and only if there exists a real number u in $I = \left[\max\left(\frac{-b_{12}^2}{a_2^2 - a_1^2}, \frac{-b_{23}^2}{a_3^2 - a_2^2} \right), \frac{b_{13}^2}{a_3^2 - a_1^2} \right]$ satisfying

$$(*) \qquad \begin{aligned} (b'_{12})^2 &= b_{12}^2 + u(a_2^2 - a_1^2), \\ (b'_{13})^2 &= b_{13}^2 + u(a_1^2 - a_3^2), \\ (b'_{23})^2 &= b_{23}^2 + u(a_3^2 - a_2^2). \end{aligned}$$

If we fix any b for which I has non-empty interior and, for each $u \in I$, define a one-parameter family $b(u)$ as the unique solution b' of the above equations for which $b_{ij}(u)$ has the same sign as b_{ij} for all i, j, it follows that $u \to j_{a,b(u)}$ is a 1-parameter isospectral deformation of $j_{a,b}$.

Using (6.4) above, we see that $\widetilde{\mathrm{Ric}}(u)(x,x) = \frac{1}{2}\langle (a^2 + b(u)^2)x, x\rangle$ for $x \in \mathfrak{v}$. For example, letting $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, $b_{12} = 0$, $b_{13} = 1$, and $b_{23} = 0$, we obtain, with respect to the standard basis on $\mathfrak{v}$,

$$\widetilde{\mathrm{Ric}}(u) = -\frac{1}{2}\begin{bmatrix} 2-5u & 0 & \sqrt{5u-40u^2} & 0 & -\sqrt{15}u & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \sqrt{5u-40u^2} & 0 & 4+8u & 0 & \sqrt{3u-24u^2} & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ -\sqrt{15}u & 0 & \sqrt{3u-24u^2} & 0 & 10-3u & 0 \\ 0 & 0 & 0 & 0 & 0 & 9 \end{bmatrix}$$

for $u \in [0, \frac{1}{8}]$. One easily checks that the eigenvalues of $\widetilde{\mathrm{Ric}}(u)$ change non-trivially with u; in particular, -5 is an eigenvalue of $\widetilde{\mathrm{Ric}}(0)$, but the eigenvalues of $\widetilde{\mathrm{Ric}}(u)$ have absolute values strictly less than 5 for $0 < u < \frac{1}{8}$. By Corollary 7, the maximum value of the scalar curvature of $N(j_{a,b(u)})$ changes non-trivially with u, showing that $N(j_{a,b(u)})$ is a non-trivial isospectral deformation.

Large-dimensional families.

Finally we turn to the Main Theorem, stated in the introductory remarks. In the proposition below, the notion of *equivalence* of linear maps $j, j' : \mathfrak{z} \to \mathfrak{v}$ is identical to the notion of $\mathcal{L}$-equivalence in Definition 3 except that the map C is not required to preserve a given lattice $\mathcal{L}$.

PROPOSITION 9 [GW3]. — *Let* $\dim \mathfrak{z} = 2$, *and let* $m = \dim \mathfrak{v}$ *be any positive integer other than* $1, 2, 3, 4$, *or* 6. *Let* W *be the real vector space consisting of all linear maps from* $\mathfrak{z}$ *to* $\mathrm{so}(\mathfrak{v})$. *Then there is a Zariski open subset* $\mathcal{O}$ *of* W *(i.e.,* $\mathcal{O}$ *is the complement of the zero locus of some non-zero polynomial function on* W*) such that each* $j_0 \in \mathcal{O}$ *belongs to a* d*-parameter family of isospectral, inequivalent elements of* W. *Here* $d \geq m(m-1)/2 - [m/2]([m/2]+2) > 1$. *In particular,* d *is of order at least* $O(m^2)$.

Although the expression for d gives 0 when $m = 6$, Example 8 gives continuous families of isospectral, inequivalent j maps when $m = 6$.

Letting $n = m-1$, choosing any lattice $\mathcal{L}$ in $\mathfrak{z}$, and applying Theorem 5, the families of isospectral j maps in Proposition 9 give rise to continuous

families of isospectral manifolds $N(j)$, or equivalently to continuous families of isospectral metrics on $S^n \times T$ where $T \simeq \mathfrak{z}/\mathcal{L}$ is a 2-torus and $n = m-1$. To complete the proof of the Main Theorem, we need to show that the manifolds in these deformations are not isometric. Recall that by Remark 4, $\overline{G(j)}$ and $\overline{G(j')}$ are isometric if and only if j and j' are $\mathcal{L}$-equivalent. We now address the question of whether $\mathcal{L}$-equivalence of j and j' is necessary in order for the submanifolds $N(j)$ and $N(j')$ to be isometric.

PROPOSITION 10. — *Fix inner product spaces $\mathfrak{v}$ and $\mathfrak{z}$ and a lattice $\mathcal{L}$ of full rank in $\mathfrak{z}$, and let $j, j' : \mathfrak{z} \to \mathrm{so}(\mathfrak{v})$ be linear maps. Suppose that there are only finitely many automorphisms of $\mathfrak{g}(j)$ that restrict to the identity on $\mathfrak{z}$. If $N(j)$ is isometric to $N(j')$, then j is $\mathcal{L}$-equivalent to j'.*

Remark 11. — For any choice of j, the linear map that restricts to $-\mathrm{Id}$ on $\mathfrak{v}$ and to Id on $\mathfrak{z}$ is an automorphism of $\mathfrak{g}(j)$. For generic choices of j, this is the only non-trivial automorphism that restricts to the identity on $\mathfrak{z}$.

Before proving Proposition 10, we note that the Main Theorem now follows immediately:

Proof of Main Theorem. — While the proof of Proposition 9 in [GW3] is not constructive, the only maps $j \in W$ considered there are those that satisfy the genericity condition in Remark 11, hence the hypothesis of Proposition 10. Thus each d-parameter family $\{j_t\}$ of isospectral, inequivalent j maps gives rise to a d-parameter family of isospectral, non-isometric manifolds $N(j_t)$. □

In order to prove Proposition 10, we need the following lemma.

LEMMA 12. — *Fix inner product spaces $\mathfrak{v}$ and $\mathfrak{z}$ and a lattice $\mathcal{L}$ of full rank in $\mathfrak{z}$, and let $j, j' : \mathfrak{z} \to \mathrm{so}(\mathfrak{v})$ be linear maps. If $\tau : N(j) \to N(j')$ is both an isometry and a bundle map with respect to the bundle structures defined in Proposition 2, then τ extends to a map $\tilde{\tau} : \overline{G(j)} \to \overline{G(j')}$ that is both a Lie group isomorphism and an isometry. Moreover, j is $\mathcal{L}$-equivalent to j'.*

Proof of Lemma 12. — Since τ preserves the fiber structures, it induces an isometry on the unit sphere S in $\mathfrak{v}$. Such an isometry is given by an orthogonal transformation A of $\mathfrak{v}$.

Since the principal torus bundles $N(j)$ and $N(j')$ over S are topologically trivial, we can choose global sections of $N(j)$ and $N(j')$. The torus action then determines an isomorphism of each fiber of each bundle with the torus $T = \mathfrak{z}/\mathcal{L}$. For each $p \in S$, the isometry τ must restrict to an isometry τ_p of the fiber T_p above p in $N(j)$ to the fiber $T_{A(p)}$ above $A(p)$ in $N(j')$. Now τ_p must be of the form $C_p \circ L_{\bar{z}_p}$, where C_p is an orthogonal transformation of $\mathfrak{z}$ that preserves $\mathcal{L}$, and $L_{\bar{z}_p}$ denotes translation by some element $\bar{z}_p$ of the torus T. Both $\bar{z}_p$ and C_p must vary continuously with p. However, C_p stays within a discrete set and hence must be independent of p; we thus drop the subscript.

We now show that $\overline{G(j)}$ and $\overline{G(j')}$ must be isomorphic. Let x and y be any orthonormal pair of vectors in $\mathfrak{v}$. The 2-plane spanned by these vectors intersects S in a great circle. Choose a lift $\tilde{p}$ of x in $N(j)$. Lift the great circle to a horizontal geodesic σ in $N(j)$ starting at $\tilde{p}$. Since τ carries the vertical space at $\tilde{p}$ to the vertical space at $\tau(\tilde{p})$, it must induce an isomorphism between the horizontal spaces as well; this isomorphism is determined by the orthogonal transformation A of $\mathfrak{v}$. In particular, τ must carry σ to a horizontal geodesic through $\tau(\tilde{p})$; this horizontal geodesic is the lift of the great circle defined by the 2-plane of the vectors $A(x)$ and $A(y)$.

If $\tilde{p} = (x, \bar{z})$, the geodesic σ is given by $\sigma(t) = ((\cos t)x + (\sin t)y, \bar{z}(t))$ with $z(t) = z + t[x, y]/2$. Thus σ in general is not closed unless x and y commute. The displacement in the torus fiber between the initial and final points is given by

$$\sigma(2\pi) - \sigma(0) = \pi[x, y].$$

Similarly,

$$(\tau \circ \sigma)(2\pi) - (\tau \circ \sigma)(0) = \pi[A(x), A(y)].$$

On the other hand,

$$(\tau \circ \sigma)(2\pi) - (\tau \circ \sigma)(0) = C(\sigma(2\pi) - \sigma(0)).$$

These equations are to be understood in the sense that all terms are well-defined modulo $\mathcal{L}$.

We conclude that

$$C([x, y]) \equiv [A(x), A(y)] \pmod{\mathcal{L}} \tag{12.1}$$

for each pair of orthonormal vectors $x, y \in \mathfrak{v}$. We claim that the equivalence is actually an equality, in fact, that

$$C([x, y]) = [A(x), A(y)] \tag{12.2}$$

for all $x, y \in \mathfrak{v}$. If $\dim(\mathfrak{v}) \geq 3$, the set $\mathcal{P}$ of pairs of orthonormal vectors forms a connected subset of $\mathfrak{v} \times \mathfrak{v}$. By continuity of A and C, there must exist a constant vector $a \in \mathcal{L}$ such that $C([x,y]) - [A(x), A(y)] = a$ for all $(x,y) \in \mathcal{P}$. Replacing x by $-x$ in this equation, we see that a must be zero, *i.e.*, that

$$C([x,y]) = [A(x), A(y)]$$

for all $(x,y) \in \mathcal{P}$ and thus for arbitrary vectors $x, y \in \mathfrak{v}$, as claimed. While the case $\dim(\mathfrak{v}) < 3$ is not of interest in our application of Lemma 12, we note that equation (12.2) can also be proven in all dimensions by lifting the isometry τ to an isometry between the universal coverings of $N(j)$ and $N(j')$ and carrying out the entire proof leading up to the equivalence (12.1) at this level.

From equation (12.2), it follows that the linear map from $\mathfrak{g}(j)$ to $\mathfrak{g}(j')$ given by $x + z \mapsto A(x) + C(z)$ for $x \in \mathfrak{v}$ and $z \in \mathfrak{z}$ is a Lie algebra isomorphism as well as an orthogonal map. By Remark 4, the pair (A, C) is an $\mathcal{L}$-equivalence between j and j', and thus $\overline{G(j)}$ is isometric to $\overline{G(j')}$.

However, we still need to show that τ itself extends to an isometry $\tilde{\tau} : \overline{G(j)} \to \overline{G(j')}$. Let $\alpha : \overline{G(j)} \to \overline{G(j')}$ be the isometry associated with the $\mathcal{L}$-equivalence as in Remark 4(i). Then $\alpha_{|N(j)} : N(j) \to N(j')$ is an isometry. Moreover, $\zeta := \alpha^{-1}_{|N(j)} \circ \tau$ is an isometry of $N(j)$ that preserves every fiber and acts on the fiber over $p \in S$ as translation by $\bar{z}_p$.

Fix a point $q \in S$. Compose ζ with an appropriate element of $T(j)$ to obtain a fiber-preserving isometry β of $N(j)$ that acts trivially on the fiber over q. Fix a point u in this fiber. Then the differential β_{*u} restricts to the identity on the tangent space to the fiber. Moreover, since β induces the identity on S, β_{*u} also restricts to the identity on the horizontal space at u. Hence we have both $\beta(u) = u$ and $\beta_{*u} = \mathrm{Id}$. Since any isometry is uniquely determined by its value and its differential at a single point, it follows that $\beta = \mathrm{Id}$ and so $\zeta \in T(j)$. Thus ζ extends to a left translation of $\overline{G(j)}$ and $\tau = \alpha \circ \zeta$ extends to an isometry of $\overline{G(j)}$. □

Proof of Proposition 10. — Let $\mathrm{Iso}(j)$ be the identity component in the full isometry group of $N(j)$. Since $N(j)$ is compact, $\mathrm{Iso}(j)$ is a compact Lie group containing the toral group $T(j)$. We first show that $T(j)$ is a maximal torus in $\mathrm{Iso}(j)$. Any isometry α that commutes with $T(j)$ must preserve the fiber structure. By Lemma 12, α extends to an isometry $\tilde{\alpha} : \overline{G(j)} \to \overline{G(j)}$. By Remark 4(ii), $\tilde{\alpha} = L_{(0,\bar{z})} \circ \beta$ for some $\bar{z} \in \mathfrak{z}/\mathcal{L}$ and some automorphism β of $\overline{G(j)}$. The left translation $L_{(0,\bar{z})}$ is

the extension to $\overline{G(j)}$ of an element of $T(j)$. Moreover, since α commutes with $T(j)$, the automorphism β commutes with translation by all central elements of $\overline{G(j)}$, and thus β restricts to the identity on the center of $\overline{G(j)}$. By the hypothesis of the proposition, β lies in a finite set of automorphisms. Hence $T(j)$ must have finite index in its centralizer, so $T(j)$ is a maximal torus in $\mathrm{Iso}(j)$.

Now suppose $\tau : N(j) \to N(j')$ is an isometry. Then the map $\hat{\tau} : \mathrm{Iso}(j) \to \mathrm{Iso}(j')$ given by $\hat{\tau}(\beta) = \tau\beta\tau^{-1}$ is an isomorphism from $\mathrm{Iso}(j)$ to $\mathrm{Iso}(j')$ and thus carries $T(j)$ to a maximal torus in $\mathrm{Iso}(j')$. Since $\dim(T(j)) = \dim(\mathfrak{z}) = \dim(T(j'))$, it follows that $T(j')$ is a maximal torus in $\mathrm{Iso}(j')$. Since all maximal tori in a compact Lie group are conjugate, we may assume, after composing τ with an isometry of $N(j')$, that $\hat{\tau}(T(j)) = T(j')$. It follows that τ is a bundle map, and by Lemma 12, j is $\mathcal{L}$-equivalent to j'. □

Remark 13. — Under the hypothesis of the proposition, we actually have that $\mathrm{Iso}(j) = T(j)$. To see this, fix a point $u \in N(j)$, and define $\rho : \mathrm{Iso}(j) \to N(j)$ by $\rho(\alpha) = \alpha(u)$. Then the restriction $\rho_{|T(j)}$ maps $T(j)$ to a fiber in $N(j)$ and induces an isomorphism on fundamental groups $\rho_* : \pi_1(T(j)) \tilde{\to} \pi_1(N(j))$. On the other hand, the inclusion $\iota : T(j) \to \mathrm{Iso}(j)$ of the maximal torus into the compact Lie group $\mathrm{Iso}(j)$ induces an injection on fundamental groups only if $\mathrm{Iso}(j) = T(j)$. (Indeed, a compact Lie group K is a product of a compact semisimple subgroup H and a torus. Any maximal torus T must include a maximal torus in H. Since H has finite fundamental group, the inclusion of a maximal torus in K cannot induce an injection of fundamental groups unless H is trivial. On the other hand, if H is trivial, then K is itself a torus and thus must equal T.) Since $\rho \circ \iota = \rho_{|T(j)}$, we conclude that $T(j) = \mathrm{Iso}(j)$.

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