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# The Atiyah Class of a dg-Vector Bundle

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# THE ATIYAH CLASS OF A DG-VECTOR BUNDLE

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*En hommage à Charles-Michel Marle à l'occasion de son quatre-vingtième anniversaire*

Abstract. We introduce the notions of Atiyah class and Todd class of a differential graded vector bundle with respect to a differential graded Lie algebroid. We prove that the space of vector fields $\mathfrak{X}(\mathcal{M})$ on a dg-manifold $\mathcal{M}$ with homological vector field $Q$ admits a structure of $L_\infty[1]$-algebra with the Lie derivative $L_Q$ as unary bracket $\lambda_1$, and the Atiyah cocycle $\operatorname{At}_{\mathcal{M}}$ corresponding to a torsion-free affine connection as binary bracket $\lambda_2$.

## 1. dg-manifolds and dg-vector bundles

A $\mathbb{Z}$-graded manifold $\mathcal{M}$ with base manifold $M$ is a sheaf of $\mathbb{Z}$-graded, graded-commutative algebras $\{\mathcal{R}_U | U \subset M \text{ open}\}$ over $M$, locally isomorphic to $C^\infty(U) \otimes \hat{S}(V^\vee)$, where $U \subset M$ is an open submanifold, $V$ is a $\mathbb{Z}$-graded vector space, and $\hat{S}(V^\vee)$ denotes the graded algebra of formal polynomials on $V$. By $C^\infty(\mathcal{M})$, we denote the $\mathbb{Z}$-graded, graded-commutative algebra of global sections. By a dg-manifold, we mean a $\mathbb{Z}$-graded manifold endowed with a homological vector field, i.e. a vector field $Q$ of degree $+1$ satisfying $[Q, Q] = 0$.

*Example* 1.1. *Let $A \to M$ be a Lie algebroid over $\mathbb{C}$. Then $A[1]$ is a dg-manifold with the Chevalley–Eilenberg differential $d_{\mathrm{CE}}$ as homological vector field. In fact, according to Vaĭntrob* [12]*, there is a bijection between the Lie algebroid structures on the vector bundle $A \to M$ and the homological vector fields on the $\mathbb{Z}$-graded manifold $A[1]$.*

*Example* 1.2. *Let $s$ be a smooth section of a vector bundle $E \to M$. Then $E[-1]$ is a dg-manifold with the contraction operator $i_s$ as homological vector field.*

*Example* 1.3. *Let $\mathfrak{g} = \sum_{i \in \mathbb{Z}} \mathfrak{g}_i$ be a $\mathbb{Z}$-graded vector space of finite type, i.e. each $\mathfrak{g}_i$ is a finite-dimensional vector space. Then $\mathfrak{g}[1]$ is a dg-manifold if and only if $\mathfrak{g}$ is an $L_\infty$-algebra.*

A dg-vector bundle is a vector bundle in the category of dg-manifolds. We refer the reader to [10, 4] for details on dg-vector bundles. The following example is essentially due to Kotov–Strobl [4].

*Example* 1.4. *Let $A \to M$ be a gauge Lie algebroid with anchor $\rho$. Then $A[1] \to T[1]M$ is a dg-vector bundle, where the homological vector fields on $A[1]$ and $T[1]M$ are the Chevalley–Eilenberg differentials.*

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The example above is a special case of a general fact [10], that LA-vector bundles [6, 7, 8] (also known as VB-algebroids [2]) give rise to dg-vector bundles.

Given a vector bundle $\mathcal{E} \xrightarrow{\pi} \mathcal{M}$ of graded manifolds, its space of sections, denoted $\Gamma(\mathcal{E})$, is defined to be $\bigoplus_{j\in\mathbb{Z}} \Gamma_j(\mathcal{E})$, where $\Gamma_j(\mathcal{E})$ consists of degree preserving maps $s \in \mathrm{Hom}(\mathcal{M}, \mathcal{E}[-j])$ such that $(\pi[-j]) \circ s = \mathrm{id}_{\mathcal{M}}$, where $\pi[-j] : \mathcal{E}[-j] \to \mathcal{M}$ is the natural map induced from $\pi$; see [10] for more details. When $\mathcal{E} \to \mathcal{M}$ is a dg-vector bundle, the homological vector fields on $\mathcal{E}$ and $\mathcal{M}$ naturally induce a degree 1 operator $\mathcal{Q}$ on $\Gamma(\mathcal{E})$, making $\Gamma(\mathcal{E})$ a dg-module over $C^\infty(\mathcal{M})$. Since the space $\Gamma(\mathcal{E}^\vee)$ of linear functions on $\mathcal{E}$ generates $C^\infty(\mathcal{E})$, the converse is also true.

**Lemma 1.5.** *Let $\mathcal{E} \to \mathcal{M}$ be a vector bundle object in the category of graded manifolds and suppose $\mathcal{M}$ is a dg-manifold. If $\Gamma(\mathcal{E})$ is a dg-module over $C^\infty(\mathcal{M})$, then $\mathcal{E}$ admits a natural dg-manifold structure such that $\mathcal{E} \to \mathcal{M}$ is a dg-vector bundle. In fact, the categories of dg-vector bundles and of locally free dg-modules are equivalent.*

In this case, the degree $+1$ operator $\mathcal{Q}$ on $\Gamma(\mathcal{E})$ gives rise to a cochain complex

$$\cdots \to \Gamma_i(\mathcal{E}) \xrightarrow{\mathcal{Q}} \Gamma_{i+1}(\mathcal{E}) \to \cdots,$$

whose cohomology group will be denoted by $H^\bullet(\Gamma(\mathcal{E}), \mathcal{Q})$.

In particular, the space $\mathfrak{X}(\mathcal{M})$ of vector fields on a dg-manifold $(\mathcal{M}, Q)$ (i.e. graded derivations of $C^\infty(\mathcal{M})$), which can be regarded as the space of sections $\Gamma(T\mathcal{M})$, is naturally a dg-module over $C^\infty(\mathcal{M})$ with the Lie derivative $L_Q : \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$ playing the role of the degree $+1$ operator $\mathcal{Q}$.

Thus we have the following

**Corollary 1.6.** *For every dg-manifold $(\mathcal{M}, Q)$, the Lie derivative $L_Q$ makes $\Gamma(T\mathcal{M})$ into a dg-module over $C^\infty(\mathcal{M})$ and therefore $T\mathcal{M} \to \mathcal{M}$ is naturally a dg-vector bundle.*

Following the classical case, the corresponding homological vector field on $T\mathcal{M}$ is called the *tangent lift* of $Q$.

Differential graded Lie algebroids are another useful notion. Roughly, a dg-Lie algebroid can be thought of as a Lie algebroid object in the category of dg-manifolds. For more details, we refer the reader to [10], where dg-Lie algebroids are called $Q$-algebroids.

Differential graded foliations constitute an important class of examples of dg-Lie algebroids.

**Lemma 1.7.** *Let $\mathcal{D} \subset T\mathcal{M}$ be an integrable distribution on a graded manifold $\mathcal{M}$. Suppose there exists a homological vector field $Q$ on $\mathcal{M}$ such that $\Gamma(\mathcal{D})$ is stable under $L_Q$. Then $\mathcal{D} \to \mathcal{M}$ is a dg-Lie algebroid with the inclusion $\rho : \mathcal{D} \to T\mathcal{M}$ as its anchor map.*

## 2. Atiyah class and Todd class of a dg-vector bundle

Let $\mathcal{E} \to \mathcal{M}$ be a dg-vector bundle and let $\mathcal{A} \to \mathcal{M}$ be a dg-Lie algebroid with anchor $\rho : \mathcal{A} \to T\mathcal{M}$. An $\mathcal{A}$-connection on $\mathcal{E} \to \mathcal{M}$ is a degree 0 map $\nabla : \Gamma(\mathcal{A}) \otimes \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$ such that

$$\nabla_{fX} s = f \nabla_X s$$

and

$$\nabla_X(fs) = \rho(X)(f)s + (-1)^{|X||f|} f \nabla_X s$$

for all $f \in C^\infty(\mathcal{M})$, $X \in \Gamma(\mathcal{A})$, and $s \in \Gamma(\mathcal{E})$. Here we use the 'absolute value' notation to denote the degree of the argument. When we say that $\nabla$ is of degree 0, we actually mean that $|\nabla_X s| = |X| + |s|$ for every pair of homogeneous elements $X$ and $s$. Such connections always exist since the standard partition of unity argument holds in the context of graded manifolds. Given a dg-vector bundle $\mathcal{E} \to \mathcal{M}$ and an $\mathcal{A}$-connection $\nabla$ on it, we can consider the bundle map $\mathrm{At}_\mathcal{E} : \mathcal{A} \otimes \mathcal{E} \to \mathcal{E}$ defined by

$$\mathrm{At}_\mathcal{E}(X,s) := \mathcal{Q}(\nabla_X s) - \nabla_{\mathcal{Q}(X)} s - (-1)^{|X|} \nabla_X \big(\mathcal{Q}(s)\big), \quad \forall X \in \Gamma(\mathcal{A}), s \in \Gamma(\mathcal{E}). \tag{1}$$

**Proposition 2.1.**
(1) *$\mathrm{At}_\mathcal{E} : \mathcal{A} \otimes \mathcal{E} \to \mathcal{E}$ is a degree $+1$ bundle map and therefore can also be regarded as a degree $+1$ section of $\mathcal{A}^\vee \otimes \mathrm{End}\, \mathcal{E}$.*
(2) *$\mathrm{At}_\mathcal{E}$ is a cocycle: $\mathcal{Q}(\mathrm{At}_\mathcal{E}) = 0$.*
(3) *The cohomology class of $\mathrm{At}_\mathcal{E}$ is independent of the choice of the connection $\nabla$.*

Thus there is a natural cohomology class $\alpha_\mathcal{E} := [\mathrm{At}_\mathcal{E}]$ in $H^1\big(\Gamma(\mathcal{A}^\vee \otimes \mathrm{End}\, \mathcal{E}), Q\big)$. The class $\alpha_\mathcal{E}$ is called the *Atiyah class* of the dg-vector bundle $\mathcal{E} \to \mathcal{M}$ relative to the dg-Lie algebroid $\mathcal{A} \to \mathcal{M}$.

The Atiyah class of a dg-manifold, which is the obstruction to the existence of connections compatible with the differential, was first investigated by Shoikhet [11] in relation with Kontsevich's formality theorem and Duflo formula. More recently, the Atiyah class of a dg-vector bundle appeared in Costello's work [1].

We define the *Todd class* $\mathrm{Td}_\mathcal{E}$ of a dg-vector bundle $\mathcal{E} \to \mathcal{M}$ relative to a dg-Lie algebroid $\mathcal{A} \to \mathcal{M}$ by

$$\mathrm{Td}_\mathcal{E} := \mathrm{Ber}\left(\frac{1 - e^{-\alpha_\mathcal{E}}}{\alpha_\mathcal{E}}\right) \in \prod_{k \geq 0} H^k\big(\Gamma(\wedge^k \mathcal{A}^\vee), Q\big), \tag{2}$$

where Ber denotes the Berezinian [9] and $\wedge^k \mathcal{A}^\vee$ denotes the dg vector bundle $S^k(\mathcal{A}^\vee[-1])[k] \to \mathcal{M}$. One checks that $\mathrm{Td}_\mathcal{E}$ can be expressed in terms of scalar Atiyah classes $c_k = \frac{1}{k!}(\frac{i}{2\pi})^k \operatorname{str} \alpha_\mathcal{E}^k \in H^k\big(\Gamma(\wedge^k \mathcal{A}^\vee), Q\big)$. Here $\operatorname{str} : \mathrm{End}\, \mathcal{E} \to C^\infty(\mathcal{M})$ denotes the supertrace. Note that $\operatorname{str} \alpha_\mathcal{E}^k \in \Gamma(\wedge^k \mathcal{A}^\vee)$ since $\alpha_\mathcal{E}^k \in \Gamma(\wedge^k \mathcal{A}^\vee) \otimes_{C^\infty(\mathcal{M})} \mathrm{End}\, \mathcal{E}$. If $\mathcal{A} = T\mathcal{M}$, we write $\Omega^k(\mathcal{M})$ instead of $\Gamma(\wedge^k T^\vee \mathcal{M})$.

## 3. Atiyah class and Todd class of a dg-manifold

Consider a dg-manifold $(\mathcal{M}, Q)$. According to Lemma 1.7, its tangent bundle $T\mathcal{M}$ is indeed a dg-Lie algebroid. By the *Atiyah class of a dg-manifold* $(\mathcal{M}, Q)$, denoted $\alpha_\mathcal{M}$, we mean the Atiyah class of the tangent dg-vector bundle $T\mathcal{M} \to \mathcal{M}$ with respect to the dg-Lie algebroid $T\mathcal{M}$. Similarly, the Atiyah 1-cocycle of a dg manifold $\mathcal{M}$ corresponding to an affine connection on $\mathcal{M}$ (i.e. a $T\mathcal{M}$-connection on $T\mathcal{M} \to \mathcal{M}$) is the 1-cocycle defined as in Eq. (1).

**Lemma 3.1.** *Suppose $V$ is a vector space. The only connection on the graded manifold $V[1]$ is the trivial connection.*

*Proof.* Since the graded algebra of functions on $V[1]$ is $\wedge(V^\vee)$, every vector $v \in V$ determines a degree $-1$ vector field $\iota_v$ on $V[1]$, which acts as a contraction operator on $\wedge(V^\vee)$. The $C^\infty(V[1])$-module of all vector fields on $V[1]$ is generated by its subset $\{\iota_v\}_{v \in V}$. It follows that a connection $\nabla$ on $V[1]$ is completely determined by the knowledge of $\nabla_{\iota_v}\iota_w$ for all $v, w \in V$. Since the degree of every vector field $\nabla_{\iota_v}\iota_w$ must be $-2$ and there are no nonzero vector fields of degree $-2$, it follows that $\nabla_{\iota_v}\iota_w = 0$. □

Given a finite-dimensional Lie algebra $\mathfrak{g}$, consider the dg-manifold $(\mathcal{M}, Q)$, where $\mathcal{M} = \mathfrak{g}[1]$ and $Q$ is the Chevalley-Eilenberg differential $d_{\mathrm{CE}}$. The following result can be easily verified using the canonical trivalization $T\mathcal{M} \cong \mathfrak{g}[1] \times \mathfrak{g}[1]$.

**Lemma 3.2.** *Let $(\mathcal{M}, Q) = (\mathfrak{g}[1], d_{\mathrm{CE}})$ be the canonical dg-manifold corresponding to a finite-dimensional Lie algebra $\mathfrak{g}$. Then,*

$$H^k(\Gamma(T^\vee\mathcal{M} \otimes \operatorname{End} T\mathcal{M}), Q) \cong H^{k-1}_{\mathrm{CE}}(\mathfrak{g}, \mathfrak{g}^\vee \otimes \mathfrak{g}^\vee \otimes \mathfrak{g}),$$

*and*

$$H^k(\Omega^k(\mathcal{M}), Q) \cong (S^k\mathfrak{g}^\vee)^{\mathfrak{g}}.$$

**Proposition 3.3.** *Let $(\mathcal{M}, Q) = (\mathfrak{g}[1], d_{\mathrm{CE}})$ be the canonical dg-manifold corresponding to a finite-dimensional Lie algebra $\mathfrak{g}$. Then the Atiyah class $\alpha_{\mathfrak{g}[1]}$ is precisely the Lie bracket of $\mathfrak{g}$ regarded as an element of $(\mathfrak{g}^\vee \otimes \mathfrak{g}^\vee \otimes \mathfrak{g})^{\mathfrak{g}} \cong H^1\big(\Gamma(T^\vee\mathcal{M} \otimes \operatorname{End} T\mathcal{M}), Q\big)$. Consequently, the isomorphism*

$$\prod_k H^k\big(\Omega^k(\mathcal{M}), Q\big) \xrightarrow{\cong} \big(\widehat{S}(\mathfrak{g}^\vee)\big)^{\mathfrak{g}}$$

*maps the Todd class $\mathrm{Td}_{\mathfrak{g}[1]}$ onto the Duflo element of $\mathfrak{g}$.*

*Example* 3.4. *Consider a dg-manifold of the form $\mathcal{M} = (\mathbb{R}^{m|n}, Q)$. Let $(x_1, \cdots, x_m; x_{m+1} \cdots x_{m+n})$ be coordinate functions on $\mathbb{R}^{m|n}$, and write $Q = \sum_k Q_k(x)\frac{\partial}{\partial x_k}$. Then the Atiyah 1-cocycle associated to the trivial connection $\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j} = 0$ is given by*

$$\mathrm{At}_{\mathcal{M}}\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = (-1)^{|x_i|+|x_j|}\sum_k \frac{\partial^2 Q_k}{\partial x_i \partial x_j}\frac{\partial}{\partial x_k} \tag{3}$$

*As we can see from (3), the Atiyah 1-cocycle $\mathrm{At}_{\mathcal{M}}$ includes the information about the homological vector field of second-order and higher.*

## 4. Atiyah class and homotopy Lie algebras

Let $\mathcal{M}$ be a graded manifold. A $(1,2)$-tensor of degree $k$ on $\mathcal{M}$ is a $\mathbb{C}$-linear map $\alpha : \mathfrak{X}(\mathcal{M}) \otimes_{\mathbb{C}} \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$ such that $|\alpha(X, Y)| = |X| + |Y| + k$ and

$$\alpha(fX, Y) = (-1)^{k|f|} f\alpha(X, Y) = (-1)^{|f||X|}\alpha(X, fY).$$

We denote the space of $(1,2)$-tensors of degree $k$ by $\mathcal{T}^{1,2}_k(\mathcal{M})$, and the space of all $(1,2)$-tensors by $\mathcal{T}^{1,2}(\mathcal{M}) = \bigoplus_k \mathcal{T}^{1,2}_k(\mathcal{M})$.

The torsion of an affine connection $\nabla$ is given by

$$T(X, Y) = \nabla_X Y - (-1)^{|X||Y|}\nabla_Y X - [X, Y]. \tag{4}$$

The torsion is an element in $\mathcal{T}^{1,2}_0(\mathcal{M})$. Given any affine connection, one can define its opposite affine connection $\nabla^{\mathrm{op}}$, given by

$$\nabla^{\mathrm{op}}_X Y = \nabla_X Y - T(X, Y) = [X, Y] + (-1)^{|X||Y|}\nabla_Y X. \tag{5}$$

The average $\frac{1}{2}(\nabla + \nabla^{\mathrm{op}})$ is a torsion-free affine connection. This shows that torsion-free affine connections always exist on graded manifolds.

In this section, we show that, as in the classical situation considered by Kapranov in [3, 5], the Atiyah 1-cocycle of a dg-manifold gives rise to an interesting homotopy Lie algebra. As in the last section, let $(\mathcal{M}, Q)$ be a dg-manifold and let $\nabla$ be an affine connection on $\mathcal{M}$. The following can be easily verified by direct computation.

(1) The anti-symmetrization of the Atiyah 1-cocycle $\operatorname{At}_{\mathcal{M}}$ is equal to $L_Q T$, so $\operatorname{At}_{\mathcal{M}}$ is graded antisymmetric up to an exact term. In particular, if $\nabla$ is torsion-free, we have

$$\operatorname{At}_{\mathcal{M}}(X,Y) = (-1)^{|X||Y|} \operatorname{At}_{\mathcal{M}}(Y,X).$$

(2) The degree $1+|X|$ operator $\operatorname{At}_{\mathcal{M}}(X,-)$ need not be a derivation of the degree $+1$ 'product' $\mathfrak{X}(\mathcal{M}) \otimes_{\mathbb{C}} \mathfrak{X}(\mathcal{M}) \xrightarrow{\operatorname{At}_{\mathcal{M}}} \mathfrak{X}(\mathcal{M})$. However, the Jacobiator

$$(X,Y,Z) \mapsto \operatorname{At}_{\mathcal{M}}\big(X, \operatorname{At}_{\mathcal{M}}(Y,Z)\big) - \big\{(-1)^{|X|+1} \operatorname{At}_{\mathcal{M}}\big(\operatorname{At}_{\mathcal{M}}(X,Y), Z\big) + (-1)^{(|X|+1)(|Y|+1)} \operatorname{At}_{\mathcal{M}}\big(Y, \operatorname{At}_{\mathcal{M}}(X,Z)\big)\big\},$$

of $\operatorname{At}_{\mathcal{M}}$, which vanishes precisely when $\operatorname{At}_{\mathcal{M}}(X,-)$ is a derivation of $\operatorname{At}_{\mathcal{M}}$, is equal to $\pm L_Q(\nabla \operatorname{At}_{\mathcal{M}})$. Hence $\operatorname{At}_{\mathcal{M}}$ satisfies the graded Jacobi identity up to the exact term $L_Q(\nabla \operatorname{At}_{\mathcal{M}})$.

Armed with this motivation, we can now state the main result of this note.

**Theorem 4.1.** *Let $(\mathcal{M}, Q)$ be a dg-manifold and let $\nabla$ be a torsion-free affine connection on $\mathcal{M}$. There exists a sequence $(\lambda_k)_{k\geq 2}$ of maps $\lambda_k \in \operatorname{Hom}(S^k(T\mathcal{M}), T\mathcal{M}[-1])$ starting with $\lambda_2 := \operatorname{At}_{\mathcal{M}} \in \operatorname{Hom}(S^2(T\mathcal{M}), T\mathcal{M}[-1])$ which, together with $\lambda_1 := L_Q : \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$, satisfy the $L_\infty[1]$-algebra axioms. As a consequence, the space of vector fields $\mathfrak{X}(\mathcal{M})$ on a dg-manifold $(\mathcal{M}, Q)$ admits an $L_\infty[1]$-algebra structure with the Lie derivative $L_Q$ as unary bracket $\lambda_1$ and the Atiyah cocycle $\operatorname{At}_{\mathcal{M}}$ as binary bracket $\lambda_2$.*

To prove Theorem 4.1, we introduce a Poincaré–Birkhoff–Witt map for graded manifolds.

It was shown in [5] that every torsion-free affine connection $\nabla$ on a smooth manifold $M$ determines an isomorphism of coalgebras (over $C^\infty(M)$)

$$\operatorname{pbw}^\nabla : \Gamma(S(TM)) \xrightarrow{\cong} D(M), \tag{6}$$

called the Poincaré–Birkhoff–Witt (PBW) map. Here $D(M)$ denotes the space of differential operators on $M$.

Geometrically, an affine connection $\nabla$ induces an exponential map $TM \to M \times M$, which is a well-defined diffeomorphism from a neighborhood of the zero section of $TM$ to a neighborhood of the diagonal $\Delta(M)$ of $M \times M$. Sections of $S(TM)$ can be viewed as fiberwise distributions on $TM$ supported on the zero section, while $D(M)$ can be viewed as the space of source-fiberwise distributions on $M \times M$ supported on the diagonal $\Delta(M)$. The map $\operatorname{pbw}^\nabla : \Gamma(S(TM)) \to D(M)$ is simply the push-forward of fiberwise distributions through the exponential map $\exp^\nabla : TM \to M \times M$ and is clearly an isomorphism of coalgebras over $C^\infty(M)$.

Even though, for a *graded* manifold $\mathcal{M}$ endowed with a torsion-free affine connection $\nabla$, we lack an exponential map $\exp^\nabla : T\mathcal{M} \to \mathcal{M} \times \mathcal{M}$, a PBW map can still be defined purely algebraically thanks to the iteration formula introduced in [5].

**Lemma 4.2.** *Let $\mathcal{M}$ be a $\mathbb{Z}$-graded manifold and let $\nabla$ be a torsion-free affine connection on $\mathcal{M}$. The Poincaré-Birkhoff-Witt map inductively defined by the relations*[1]

$$\mathrm{pbw}^{\nabla}(f) = f, \quad \forall f \in C^{\infty}(\mathcal{M});$$
$$\mathrm{pbw}^{\nabla}(X) = X, \quad \forall X \in \mathfrak{X}(\mathcal{M});$$

*and*

$$\mathrm{pbw}^{\nabla}(X_0 \odot \cdots \odot X_n) = \frac{1}{n+1} \sum_{k=0}^{n} (-1)^{|X_k|(|X_0|+\cdots+|X_{k-1}|)} \big\{ X_k \cdot \mathrm{pbw}^{\nabla}(X_0 \odot \cdots \odot \widehat{X}_k \odot \cdots \odot X_n) - \mathrm{pbw}^{\nabla}\big(\nabla_{X_k}(X_0 \odot \cdots \odot \widehat{X}_k \odot \cdots \odot X_n)\big) \big\},$$

*for all $n \in \mathbb{N}$ and $X_0, \ldots, X_n \in \mathfrak{X}(\mathcal{M})$, establishes an isomorphism*

$$\mathrm{pbw}^{\nabla} : \Gamma(S(T\mathcal{M})) \xrightarrow{\cong} D(\mathcal{M}). \tag{7}$$

*of coalgebras over $C^{\infty}(\mathcal{M})$.*

Now assume that $(\mathcal{M}, Q)$ is a dg-manifold. The homological vector field $Q$ induces a degree $+1$ coderivation of $D(\mathcal{M})$ defined by the Lie derivative:

$$L_Q(X_1 \cdots X_n) = \sum_{k=1}^{n} (-1)^{|X_1|+\cdots+|X_{k-1}|} X_1 \cdots X_{k-1} [Q, X_k] X_{k+1} \cdots X_n. \tag{8}$$

Now using the isomorphism of coalgebras $\mathrm{pbw}^{\nabla}$ as in Eq. (7) to transfer $L_Q$ from $D(\mathcal{M})$ to $\Gamma(S(T\mathcal{M}))$, we obtain $\delta := (\mathrm{pbw}^{\nabla})^{-1} \circ L_Q \circ \mathrm{pbw}^{\nabla}$, a degree 1 coderivation of $\Gamma(S(T\mathcal{M}))$. Finally, dualizing $\delta$, we obtain an operator

$$D : \Gamma(\hat{S}(T^{\vee}\mathcal{M})) \to \Gamma(\hat{S}(T^{\vee}\mathcal{M}))$$

as

$$\Gamma(\hat{S}(T^{\vee}\mathcal{M})) \cong \mathrm{Hom}_{C^{\infty}(\mathcal{M})}(\Gamma(S(T\mathcal{M})), C^{\infty}(\mathcal{M})).$$

**Theorem 4.3.** *Let $(\mathcal{M}, Q)$ be a dg-manifold and let $\nabla$ be a torsion-free affine connection on $\mathcal{M}$.*

(1) *The operator $D$, dual to $(\mathrm{pbw}^{\nabla})^{-1} \circ L_Q \circ \mathrm{pbw}^{\nabla}$, is a degree $+1$ derivation of the graded algebra $\Gamma(\widehat{S}(T^{\vee}\mathcal{M}))$ satisfying $D^2 = 0$.*
(2) *There exists a sequence $\{R_k\}_{k \geq 2}$ of homomorphisms $R_k \in \mathrm{Hom}(S^k T\mathcal{M}, T\mathcal{M}[-1])$, whose first term $R_2$ is precisely the Atiyah 1-cocycle $\mathrm{At}_{\mathcal{M}}$, such that $D = L_Q + \sum_{k=2}^{\infty} \widetilde{R_k}$, where $\widetilde{R_k}$ denotes the $C^{\infty}(\mathcal{M})$-linear operator on $\Gamma(\widehat{S}(T^{\vee}\mathcal{M}))$ corresponding to $R_k$.*

Finally we note that Theorem 4.1 is a consequence of Theorem 4.3.

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