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Triangle-free Uniquely 3-Edge Colorable Cubic Graphs

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TRIANGLE-FREE UNIQUELY 3-EDGE COLORABLE CUBIC GRAPHS

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Abstract. This paper presents infinitely many new examples of triangle-free uniquely 3-edge colorable cubic graphs. The only such graph previously known was given by Tutte in 1976.

1. History

Recall that a *cubic* graph is 3-regular, that a *proper 3-edge coloring* assigns colors to edges such that no two incident edges receive the same color, that *edge-Kempe chains* are maximal sequences of edges that alternate between two colors, and that a *Hamilton cycle* includes all vertices of a graph.

It is well known that a cubic graph with a Hamilton cycle is 3-edge colorable, as the Hamilton cycle is even (and thus 2-edge colorable) and its complement is a matching (that can be monochromatically colored). A uniquely 3-edge colorable cubic graph must have exactly three Hamilton cycles, each an edge-Kempe chain in one of the $\binom{3}{2}$ pairs of colors. The converse is not true, as a cubic graph may have some colorings with Hamilton edge-Kempe chains and other colorings with non-Hamilton edge-Kempe chains; examples are given in [12].

The literature classifying uniquely 3-edge colorable cubic graphs is sparse; there is no complete characterization [7]. It is well known that the property of being uniquely 3-edge colorable is invariant under application of $\Delta - Y$ transformations. It was conjectured that every simple planar cubic graph with exactly three Hamilton cycles contains a triangle [13, Cantoni], and also that every simple planar uniquely 3-edge colorable cubic graph contains a triangle [3]. The latter conjecture is proved in [4], where it is also shown that if a simple planar cubic graph has exactly three Hamilton cycles, then it contains a triangle if and only if it is uniquely 3-edge colorable.

Tutte, in a 1976 paper about the average number of Hamilton cycles in a graph [13], offhandedly remarks that one example of a nonplanar triangle-free uniquely 3-edge colorable cubic graph is the generalized Petersen graph $P(9,2)$, pictured in Figure 1. He describes the graph as two 9-cycles $a_0 \dots a_8$, $b_0 \dots b_8$, with additional edges $a_i b_{2i}$ and index arithmetic done modulo 9.

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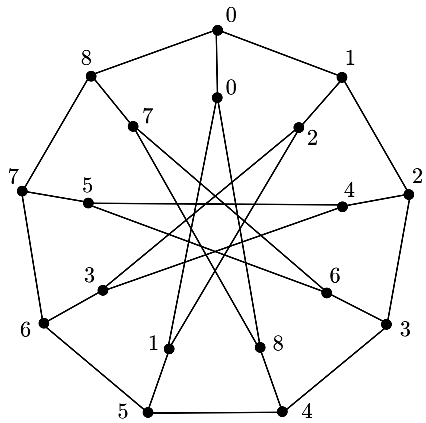

FIGURE 1. The generalized Petersen graph $P(9, 2)$ labeled with Tutte's indices.

(The generalized Petersen graph $P(m, 2)$ is defined analogously, and in fact the known cubic graphs with exactly three Hamilton cycles and multiple distinct 3-edge colorings are $P(6k + 3, 2)$ for $k > 1$ [12].) It appears that the search for examples of triangle-free nonplanar uniquely 3-edge colorable cubic graphs ended with Tutte, or at least that any further efforts have been unsuccessful. Multiple sources ([6], [7], [9]) note that Tutte's example is the only known triangle-free nonplanar example. It has been conjectured [3] that $P(9, 2)$ is the only example. In Section 2 we give infinitely many such graphs.

2. New examples of triangle-free nonplanar uniquely 3-edge colorable cubic graphs

In [2] the authors introduced the following construction. Consider two cubic graphs G_1, G_2, and form $G_1 \curlyvee G_2$ by choosing a vertex v_i in G_i $(i = 1, 2)$, removing v_i from G_i $(i = 1, 2)$, and adding a matching of three edges joining the three neighbors of v_1 with the three neighbors of v_2. Of course there are many ways to choose v_1, v_2, and many ways to identify their incident edges, so the construction is not unique. However, it is reversible; given a cubic graph G with a 3-edge cut, we may decompose $G = G_1 \curlyvee G_2$. In that paper we proved the following.

Theorem 2.1. *[3.8 of [2]] Let G_1, G_2 be cubic graphs and a_i the number of 3-edge colorings of G_i. Then $G_1 \curlyvee G_2$ has $a_1 a_2$ edge colorings.*

Define $G^{\curlyvee}$ to be the infinite family of graphs consisting of all graphs of the form $G \curlyvee G \curlyvee \cdots \curlyvee G$. This leads to the following corollaries of Theorem 2.1.

Theorem 2.2. *If G is a uniquely 3-edge colorable graph, then all graphs in $G^{\curlyvee}$ are uniquely 3-edge colorable.*

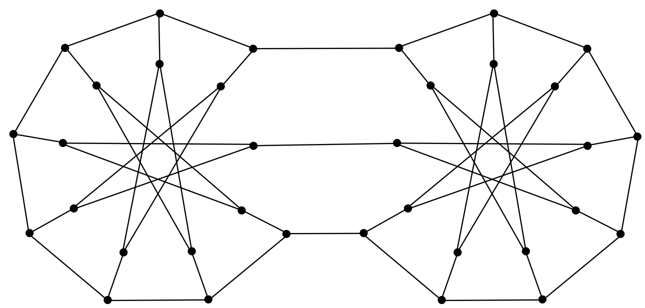

FIGURE 2. A nonplanar, triangle-free, uniquely 3-edge colorable graph with 34 vertices.

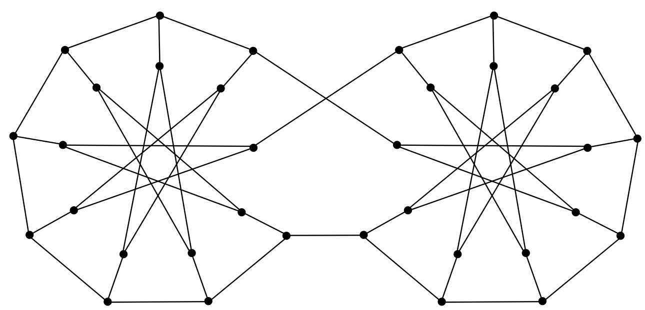

FIGURE 3. A nonplanar, triangle-free, uniquely 3-edge colorable graph with 34 vertices that is nonisomorphic to that shown in Figure 2.

Proof. The proof proceeds by induction on the number of copies of G. □

Corollary 2.3. *All members of the infinite family $P(9,2)^{\curlyvee}$ are uniquely 3-edge colorable.*

Note.
In [5], Goldwasser and Zhang proved that if a uniquely 3-edge colorable graph has an edge cut of size 3 or 4 such that each remaining component contains a cycle, then the graph can be decomposed into two smaller uniquely 3-edge colorable graphs. It seems they did not observe the reverse construction.

2.1. **Examples and Properties.** The smallest member of $P(9,2)^{\curlyvee}$ is of course $P(9,2)$, which has 18 vertices. For every integer $k > 1$ there are multiple graphs in $P(9,2)^{\curlyvee}$ with $16k+2$ vertices. Nonisomorphic examples with $k = 2$ are shown in Figures 2 and 3.

The graphs in $P(9,2)^{\curlyvee}$ are clearly all nonplanar. We show next that there are graphs in $P(9,2)^{\curlyvee}$ of every other orientable and nonorientable genus.

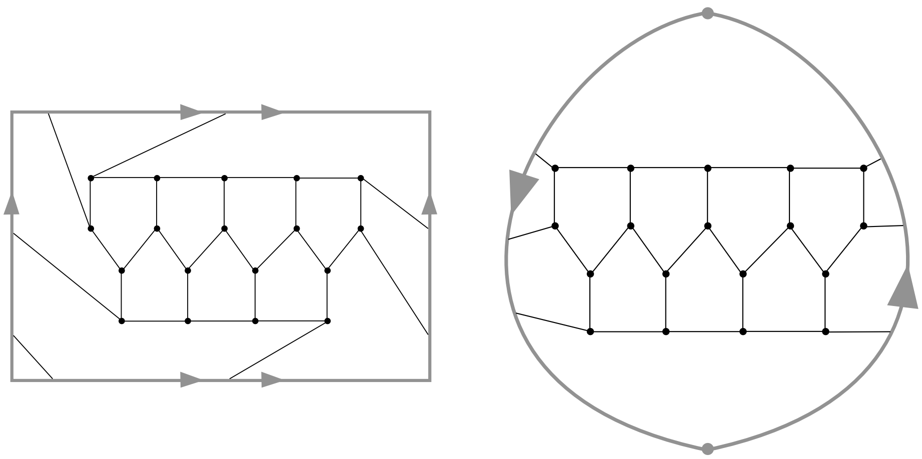

FIGURE 4. Embeddings of $P(9,2)$ on the torus (left) and projective plane (right)

Theorem 2.4. *Every graph in* $P(9,2)^{\curlyvee}$ *with* $16k+2$ *vertices has orientable and nonorientable genus at most* k*. Further, there is a large subfamily of graphs in* $P(9,2)^{\curlyvee}$*, each of which has* $16k+2$ *vertices and orientable and nonorientable genus exactly* k*.*

Proof. We will show by induction that any graph Q_k created using the $\curlyvee$-construction with k copies of $P(9,2)$ has orientable and nonorientable genus at most k. The base case holds because $P(9,2)$ embeds on both the torus (see Figure 4 (left)) and on the projective plane (see Figure 4 (right)).

Now consider Q_k, a graph created using the $\curlyvee$-construction with k copies of $P(9,2)$. Q_k was obtained by removing and associating $v \in P(9,2)$ and some $w \in Q_{k-1}$ via the $\curlyvee$-construction, where Q_{k-1} is some graph created using $k-1$ copies of $P(9,2)$ that has genus $k-1$ or less by inductive hypothesis. Let $\widehat{Q_k}$ be the graph produced by simply identifying the vertices v and w. The graph $\widehat{Q_k}$ has two blocks that meet at this vertex and by Theorem 1 of [1], the genus of $\widehat{Q_k}$ is the sum of the genera of the blocks (so it is k). Replacing the cut vertex by a 3-edge cut to implement the $\curlyvee$ construction does not increase the genus, which completes the proof of the upper bound on genus.

A copy of a subdivision of $K_{3,3}$ is highlighted in the embedding of $P(9,2)$ shown in Figure 5. There are four vertices $\{t_1, t_2, t_3, t_4\}$ whose edges are not involved in the subdivided $K_{3,3}$. Any (or all) of $\{t_1, t_2, t_3, t_4\}$ can be removed and the resulting graph will still have a $K_{3,3}$ minor. If Q_k is formed such that in each copy of $P(9,2)$ only (some) of vertices $\{t_1, t_2, t_3, t_4\}$ are used in the $\curlyvee$ construction then there will still be k disjoint copies of subdivisions of $K_{3,3}$ in Q_k. The genus of a graph is the sum of the genera of its components [1, Cor. 2], so using this construction Q_k has a minor with orientable (resp.

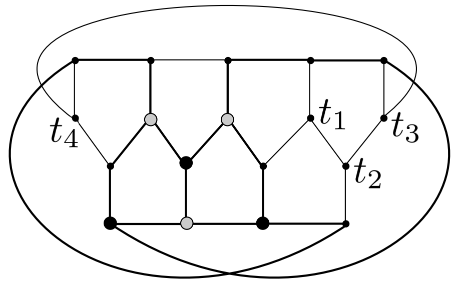

FIGURE 5. $P(9, 2)$ with a copy of a subdivision of $K_{3,3}$ highlighted.

nonorientable) genus exactly k. It is straightforward to draw an embedding of sample Q_k on a surface of orientable or nonorientable genus k.

□

3. CONCLUSION

While we have provided infinitely many examples of triangle-free nonplanar uniquely 3-edge colorable cubic graphs, it is still unknown whether other examples exist. All our examples support Zhang's conjecture [14] that every triangle-free uniquely 3-edge colorable cubic graph contains a Petersen graph minor. That conjecture remains open.

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