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The k-Dominating Graph

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Abstract. Given a graph G, the *k-dominating graph of* G, $D_k(G)$, is defined to be the graph whose vertices correspond to the dominating sets of G that have cardinality at most k. Two vertices in $D_k(G)$ are adjacent if and only if the corresponding dominating sets of G differ by either adding or deleting a single vertex. The graph $D_k(G)$ aids in studying the reconfiguration problem for dominating sets. In particular, one dominating set can be reconfigured to another by a sequence of single vertex additions and deletions, such that the intermediate set of vertices at each step is a dominating set if and only if they are in the same connected component of $D_k(G)$. In this paper we give conditions that ensure $D_k(G)$ is connected.

1. Introduction

Let G be a graph and $S \subseteq V(G)$. Then S is a *dominating set* of G if and only if every vertex in $V(G)\backslash S$ is adjacent to a vertex in S. The *domination number* of G, denoted $\gamma(G)$, is the minimum cardinality of a dominating set of G. The *upper domination number* of G, denoted $\Gamma(G)$, is the maximum cardinality of a minimal dominating set of G. We use the term γ-*set* to refer to a dominating set of cardinality $\gamma(G)$, and Γ-*set* to refer to a minimal dominating set of cardinality $\Gamma(G)$. There is a wealth of literature about domination and variations (see, for example [9]). It is easy to construct minimal dominating sets using a greedy approach, but determining $\gamma(G)$ is NP-complete in general. Our interest here is in relationships between dominating sets. In particular, given dominating sets S and T, is there a sequence of dominating sets $S_0 = S_1, S_2, \ldots S_k = T$ such that each S_{i+1} is obtained from S_i by deleting or adding a single vertex.

This work is similar in flavour to recent work in graph colouring. Given a graph H and a positive integer k, the *k-colouring graph of* H, denoted $G_k(H)$, has vertices corresponding to the (proper) k-vertex-colourings of H. Two vertices in $G_k(H)$ are adjacent if and only if the corresponding vertex colourings of G differ on precisely one vertex. The connectedness of k-colouring graphs has been studied, as has the hamiltonicity (see, for example [4, 5, 6, 8]). When $G_k(H)$ is connected, then a Markov process can be defined on it that leads to an approximation for the number of k–colourings of H.

A *reconfiguration problem* asks whether (when) one feasible solution to a problem can be transformed into another by some allowable set of moves, while maintaining feasibility at all steps. The complexity of reconfiguration of various colouring problems in graphs has been studied in a variety of papers including [2, 4, 5, 10]. For many graph problems, such as independent sets and vertex covers, determining whether one feasible solution can be reconfigured to another is hard for general graphs as is shown in [11]. In this paper we show that for bipartite and chordal graphs any minimal dominating set can be reconfigured to any other.

Let G be a graph, and $k \geq \gamma(G)$ an integer. We define the *k-dominating graph of* G, $D_k(G)$, to be the graph whose vertices correspond to the dominating sets of G that have cardinality at most k. Two vertices in $D_k(G)$ are adjacent if and only if the corresponding dominating sets of G differ by either adding or deleting a single vertex, i.e., if A and B are dominating sets of G, then AB is an edge of $D_k(G)$ if and only if there exists a vertex $u \in V(G)$ so that $(A \backslash B) \cup (B \backslash A) = \{u\}$. The graph $D_k(G)$ is a subgraph of the Hasse Diagram of all subsets of $V(G)$ of cardinality k or less. The Hasse Diagram itself is $D_n(K_n)$.

Two different graphs defined on dominating sets have recently been studied in [7, 14]. In both these papers, the *γ-graph* of G, denoted $\gamma[G]$, has vertices corresponding to the dominating sets of cardinality $\gamma(G)$, but the edge sets are defined differently. In [14] there is an edge between two such sets S and T if and only if S is obtained from T by exchanging any one vertex for another, while in [7] there is an edge between two sets S and T only if the swapped vertices are adjacent in the original graph. In this paper, instead of exchanging vertices, we permit individual additions and deletions, allowing dominating sets of varying sizes, and edges only between dominating sets whose cardinalities differ by ± 1. In the last section of this paper we describe the relationship among $D_k(G)$, $G[\gamma]$ as defined in [14], and another related graph.

A natural first problem is to determine conditions that ensure that $D_k(G)$ is connected. In particular, is there a smallest value, $d_0(G)$, such that $D_k(G)$ is connected for all $k \geq d_0(G)$? Notice that the connectedness of $D_k(G)$ does not guarantee the connectedness of $D_{k+1}(G)$. For example, consider $K_{1,n}$, the star on $n \geq 3$ vertices. Figure 1 shows $D_3(K_{1,3})$, where vertices are represented by copies of $K_{1,3}$, and the dominating sets are indicated by the solid circles. The unique $\Gamma(K_{1,n-1})$ set is an isolated vertex in $D_\Gamma(K_{1,n-1})$, so $D_\Gamma(K_{1,n-1}) = D_{n-1}(K_{1,n-1})$ is

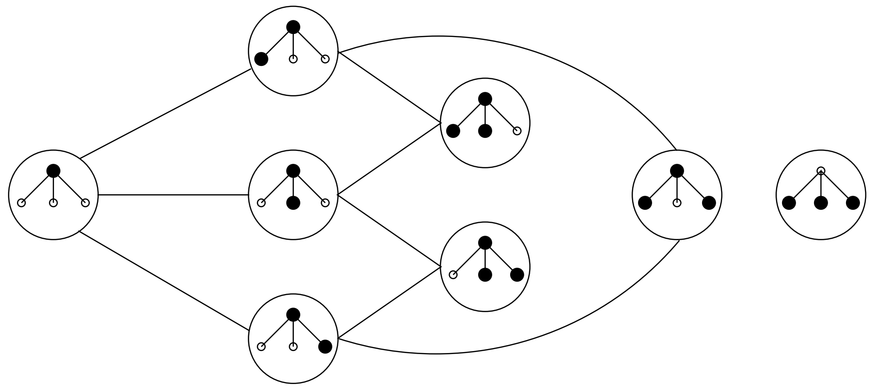

FIGURE 1. $D_3(K_{1,3})$.

not connected. However, $D_j(K_{1,n-1})$ is connected for each j, $1 \leq j \leq n-2$. This example also shows that, in general, there is no function $f(\gamma(G))$ such that $D_k(G)$ is connected for all $k \geq f(\gamma(G))$.

In this paper we show that $D_k(G)$ is connected whenever $k \geq \min\{|V(G)|-1, \Gamma(G)+\gamma(G)\}$. Moreover, for bipartite and chordal graphs, $D_k(G)$ is connected whenever $k \geq \Gamma(G)+1$. Indeed we have yet to find an example of any graph G for which $D_{\Gamma+1}(G)$ is not connected.

We consider only simple graphs, G, with vertex set $V(G)$, edge set $E(G)$, and $|V(G)| = n$. For basic graph theory notation and definitions see [3]. When G is clear from the context we use, for example, V, E and Γ to denote $V(G), E(G)$ and $\Gamma(G)$, respectively.

2. Preliminary Results

We begin with some definitions and basic results.

Definition 1. *Let G be a graph, $k \geq \gamma$, and A, B dominating sets of G of cardinality at least k. We write $A \leftrightarrow B$ if there is a path in $D_k(G)$ joining A and B.*

Proposition 1. *For $A, B \in D_k(G)$,*
(i) $A \leftrightarrow B$ if and only if $B \leftrightarrow A$;
(ii) if $A \subseteq B$, then $A \leftrightarrow B$ and $B \leftrightarrow A$.

To see that $d_0(G)$ exists, notice that if G is a graph with n vertices, then $D_n(G)$ is connected, since for every dominating set A of G, $A \leftrightarrow V(G)$. In fact, we obtain a better upper bound on $d_0(G)$.

Lemma 2. *If G has at least two independent edges, then $D_{n-1}(G)$ is connected.*

Proof. Note that if $x \in V$ is not an isolated vertex, then $V\backslash\{x\}$ is a dominating set of G. Suppose that S and T are two dominating sets of G. If $|S \cup T| \leq n-1$, then by Proposition 1, $S \leftrightarrow S \cup T \leftrightarrow T$. If $|S \cup T| = n$, then let $S' \supseteq S$, and $T' \supseteq T$ be sets of cardinality $n-1$, say $S' = V\backslash\{s\}$ and $T' = V\backslash\{t\}$. It suffices to show that $S' \leftrightarrow T'$. Since S' and T' are dominating sets, neither s nor t is an isolated vertex. If $V\backslash\{s,t\}$ is a dominating set then clearly $S', V\backslash\{s,t\}, T'$ is a path in $D_{n-1}(G)$. Otherwise it must be the case that, without loss of generality, t is the only neighbor of s. By assumption, there is another edge $uv \in E$ where $u, v \in S' \cap T'$. Then a path in $D_{n-1}(G)$ is $S' = V\backslash\{s\}, V\backslash\{s,u\}, V\backslash\{u\}, V\backslash\{u,t\}, V\backslash\{t\} = T'$. □

The empty graph, $\overline{K_n}$, has only one dominating set, namely, $V(\overline{K_n})$. Hence $D_k(\overline{K_n})$ exists only when $k = n$, in which case it is the trivial graph. For all other graphs there are values of $k \geq \gamma$ for which $D_k(G)$ is disconnected.

Lemma 3. *For any graph G with at least one edge, $D_\Gamma(G)$ is not connected.*

Proof. Since G has at least one edge, $D_\Gamma(G)$ has at least two vertices. Let S be a Γ-set of G. Then no proper subset of S is a dominating set of G, and thus S is an isolated vertex in $D_\Gamma(G)$. □

Note that if all edges of G are incident with a common vertex, then G is the union of a star with a (possibly empty) independent set of vertices, and hence $\Gamma = n-1$; by Lemma 3, $D_{n-1}(G)$ is disconnected. Thus, the assumption in Lemma 2 that G has two independent edges is necessary.

Since any dominating set of cardinality greater than Γ has a subset of cardinality Γ that is a dominating set, we get the following result.

Lemma 4. *If $k > \Gamma(G)$ and $D_k(G)$ is connected, then $D_{k+1}(G)$ is connected.*

It is possible to obtain a better upper bound on $d_0(G)$, as shown in the next theorem.

Theorem 5. *For any graph G with at least at least two disjoint edges, if $k \geq \min\{n-1, \Gamma(G) + \gamma(G)\}$, then $D_k(G)$ is connected.*

Proof. If $\Gamma + \gamma > n-1$, then the result is immediate from Lemma 2. Otherwise, let S be a γ-set of G, $k \geq \Gamma + \gamma$, and let A be an arbitrary dominating set of G with $|A| \leq k$. It suffices to show that there is a walk in $D_k(G)$ from A to S.

Choose $A_1 \subseteq A$ to be a minimal dominating set of G, and consider the four sets $A, A_1, A_1 \cup S$ and S. Then $|A| \leq k$, $|A_1| \leq \Gamma$, $|A_1 \cup S| \leq \Gamma + \gamma$, and $|S| = \gamma$, so each set has cardinality at most k, and hence is a vertex in $D_k(G)$. Furthermore, $A \supseteq A_1 \subseteq (A_1 \cup S) \supseteq S$, so $A \leftrightarrow A_1$, $A_1 \leftrightarrow (A_1 \cup S)$ and $(A_1 \cup S) \leftrightarrow S$. The union of these three paths produces a walk in $D_k(G)$ from A to S. Thus there is a walk (and hence a path) from A to S for any dominating set A with $|A| \leq k$, and hence $D_k(G)$ is connected. □

Corollary 6. *For any graph G with at least two disjoint edges,* $\Gamma(G) + 1 \leq d_0(G) \leq \min\{n-1, \Gamma(G) + \gamma(G)\}$.

In the following sections we show that if G is bipartite or chordal, then $d_0(G) = \Gamma + 1$.

3. Bipartite Graphs

Theorem 7. *For any non-trivial bipartite graph G, $D_{\Gamma+1}(G)$ is connected.*

Proof. Suppose that G has k isolated vertices, and let G' be the graph obtained from G by deleting all isolated vertices. Since the isolated vertices must be elements in every dominating set of G, it follows that $\Gamma(G') = \Gamma(G) - k$, and that $D_{\Gamma(G)+1}(G)$ is connected if and only if $D_{\Gamma(G')+1}(G')$ is connected.

We may therefore restrict our attention to graphs with no isolated vertices. Choose a bipartition (X, Y) of G such that X is as small as possible. Then Y and X are minimal dominating sets of G, with $\Gamma \geq |Y| \geq \frac{n}{2}$ and $|X| \leq \frac{n}{2}$.

Let S be an arbitrary vertex in $D_{\Gamma+1}(G)$. We prove that there is a walk in $D_{\Gamma+1}(G)$ between S and X. Choose S_1 to be a dominating set such that $|S_1| = \Gamma$, $S_1 \leftrightarrow S$ and $|S_1 \cap X|$ is as large as possible. We will show that $X \subseteq S_1$ and so in fact $X = S_1$.

Consider the partition $\{X \cap S_1, X \backslash S_1, Y \cap S_1, Y \backslash S_1\}$ of $V(G)$. Since S_1 is a dominating set and G is bipartite, the vertices in $X \backslash S_1$ are dominated by the set $Y \cap S_1$, and the vertices in $Y \backslash S_1$ are dominated by the set $X \cap S_1$. Since G is bipartite and $|S_1| = \Gamma$, $|S_1| \geq \frac{n}{2}$. Thus

$$|X \cap S_1| + |Y \cap S_1| \geq \frac{n}{2}.$$

Also, since $|X| \leq \frac{n}{2}$,

$$|X \cap S_1| + |X \backslash S_1| \leq \frac{n}{2},$$

and it follows that

$$|Y \cap S_1| \geq |X \backslash S_1|.$$

Let $|Y \cap S_1| = m$ and $|X \backslash S_1| = l$ and assume that $|S_1 \cap X| < |X|$. We show, roughly, that we can replace a vertex in $Y \cap S_1$ with one in $X \backslash S_1$. Consider the subgraph H of G induced by $(X \backslash S_1) \cup (Y \cap S_1)$. If $\deg_H(y) = 0$ for some vertex $y \in (Y \cap S_1)$, then y is dominated by $X \cap S_1$ because G has no isolated vertices. Choose $x \in X \backslash S_1$, and set $S_2 = (S_1 \cup \{x\}) \backslash \{y\}$. Then S_2 is a dominating set of G, $|S_2| = |S_1| = \Gamma$, and $S_1, S_1 \cup \{x\}, S_2$ is a path in $D_{\Gamma+1}$ from S_1 to S_2. Otherwise, each vertex in $Y \cap S_1$ has degree at least one in H. Let F be a spanning forest in H. Then $|E(F)| \leq m + l - 1 \leq 2m - 1$, implying that the average degree of the vertices in F in $Y \cap S_1$ is less than two. Therefore, there is a vertex $y \in Y \cap S_1$ with $\deg_F(y) = 1$. Let x denote the neighbour of y in F, and define $S_2 = (S_1 \cup \{x\}) \backslash \{y\}$. Then S_2 is a dominating set of G, $|S_2| = |S_1| = \Gamma$, and $S_1, S_1 \cup \{x\}, S_2$ is a path in $D_{\Gamma+1}(G)$ from S_1 to S_2.

In both cases, $|X \cap S_2| > |X \cap S_1|$, which contradicts the choice of S_1. Thus $(X = S_1) \leftrightarrow S$. $\square$

4. Chordal Graphs

Recall that a graph is *chordal* if and only if every cycle of length more than three has a chord. Equivalently, a graph is chordal if and only if it contains no induced cycle of length at least four. This immediately implies that every induced subgraph of a chordal graph is chordal. There are particular properties of chordal graphs that allow us to prove that for any chordal graph G, $d_0(G) = \Gamma + 1$.

For a graph G, we denote by $\alpha(G)$ the *independence number* of G, i.e., the cardinality of a maximum independent set in G; $\omega(G)$ denotes the *clique number* of G, the number of vertices in a largest complete subgraph of G; $\chi(G)$ denotes the chromatic number of G. Finally, $\overline{\chi}(G)$ denotes the *clique covering* number of G, i.e., the minimum number of complete graphs needed to cover the vertices in G.

The following are easily verified.

Remark 1. *If S is an independent set in G, $\mathcal{C}$ a clique cover of G, and $|S| = |\mathcal{C}|$, then*

$$\alpha(G) = |S| = |\mathcal{C}| = \overline{\chi}(G).$$

Remark 2. *For a graph G and its complement $\overline{G}$,*

$$\alpha(G) = \omega(\overline{G}) \text{ and } \chi(G) = \overline{\chi}(\overline{G}).$$

Chordal graphs fall into the class of *perfect graphs*. By definition, a graph G is *perfect* if and only if $\chi(H) = \omega(H)$ for every induced subgraph H of G. The *Perfect Graph Theorem*, conjectured by Berge [1] and verified by Lovász [13], states that a graph is perfect if and only if its complement is perfect. Thus (by Remark 2), an equivalent definition of a perfect graph is that G is perfect if and only if $\alpha(H) = \overline{\chi}(H)$ for all induced subgraphs H of G.

Let G be a chordal graph. Then G is perfect, and hence

$$\alpha(H) = \overline{\chi}(H)$$

for every induced subgraph H of G. Before proceeding with our theorem for chordal graphs, we need one additional result.

Theorem 8 (Jacoboson and Peters [12]). *For any chordal graph G, $\alpha(G) = \Gamma(G)$.*

Combining this with the *Perfect Graph Theorem* implies that for any chordal graph G,

$$\alpha(G) = \Gamma(G) = \overline{\chi}(G).$$

Theorem 9. *For any non-trivial chordal graph G, $D_{\Gamma+1}(G)$ is connected.*

Proof. Since G is chordal, $\alpha(G) = \Gamma(G) = \overline{\chi}(G)$. Let S be a maximum independent set in G. Then S is also a minimal dominating set, and we may write $S = \{s_1, s_2, \dots, s_\Gamma\}$. Now let $\mathcal{C} = \{H_1, H_2, \dots, H_\Gamma\}$ be a clique cover of G with a minimum number of cliques. Without loss of generality, we may assume that s_i is a vertex in H_i and that the cliques are vertex disjoint.

To show that $D_{\Gamma+1}(G)$ is connected, it suffices to show that there is a path in $D_{\Gamma+1}(G)$ from an arbitrary dominating set A to the set S. We proceed by induction on Γ.

Suppose G is a chordal graph with $\Gamma = 1$. Then G is a complete graph, so any vertex forms a dominating set. It follows that $D_2(G)$ is connected.

Now suppose that G is a chordal graph with $\Gamma > 1$. Let A be a dominating set of G of cardinality at most $\Gamma + 1$, and let $A_1 \subseteq A$ be a minimal dominating set of G. Since $|A_1| \leq \Gamma$, there exists some i for which $|V(H_i) \cap A_1| \leq 1$.

Case 1. Suppose that for some i, $|V(H_i) \cap A_1| = 0$. Without loss of generality, $|V(H_1) \cap A_1| = 0$. Let $G' = G - V(H_1)$. Then $S' = S \backslash \{s_1\}$ is a maximum independent set in G' and $\mathcal{C}' = \{H_2, H_3, \dots, H_\Gamma\}$ is a clique cover of G'. Since $|S'| = |\mathcal{C}'|$, it follows from Remark 1 that $|S'| = \alpha(G')$. Furthermore, since G' is chordal, $\alpha(G') = \Gamma' := \Gamma(G') = \Gamma - 1$.

Since $|V(H_1) \cap A_1| = 0$, A_1 is a dominating set of G', and $|A_1| \leq \Gamma = \Gamma' + 1$. By the induction hypothesis, $D_{\Gamma'+1}(G')$ is connected. Let

$$A_1, B_1, B_2, \ldots, B_k, S'$$

be a path in $D_{\Gamma'+1}(G')$ from A_1 to S'. Then

$$A_1 \cup \{s_1\}, B_1 \cup \{s_1\}, B_2 \cup \{s_1\}, \ldots, B_k \cup \{s_1\}, S$$

is a path in $D_{\Gamma+1}(G)$ from $A_1 \cup \{s_1\}$ to S. It is clear that there is a walk in $D_{\Gamma+1}(G)$ from A to A_1 to $A_1 \cup \{s_1\}$; combining this with the path from $A_1 \cup \{s_1\}$ to S gives us a walk, and hence a path, from A to S in $D_{\Gamma+1}(G)$.

Case 2. We may now assume that for every i, $1 \leq i \leq \Gamma$, $|V(H_i) \cap A_1| \geq 1$. However, since $|A_1| \leq \Gamma$, this implies that $|A_1| = \Gamma$ and that $|V(H_i) \cap A_1| = 1$ for each i.

We define a sequence of dominating sets $A_2, \ldots, A_\Gamma$ such that A_{i+1} is either equal to A_i, or adjacent to A_i in $D_{\Gamma+1}$. For $i = 1, 2, \ldots, \Gamma$, if $V(H_i) \cap A_i = \{s_i\}$, set $A_{i+1} = A_i$. On the other hand, if $V(H_i) \cap A_i \neq \{s_i\}$, then set

$$A_{i+1} = A_i \cup \{s_i\} \backslash (V(H_i) \cap A_i).$$

Then $A_i, A_i \cup \{s_i\}, A_i \cup \{s_i\} \backslash (V(H_i) \cap A_i) = A_{i+1}$ is a path in $D_{\Gamma+1}(G)$ between A_i and A_{i+1}. As in Case 1, it is clear that there is a path in $D_{\Gamma+1}(G)$ from A to A_1; the union of this path with the paths from A_i to A_{i+1}, $1 \leq i \leq \Gamma$, gives us a walk, and hence a path, from A to $A_{\Gamma+1} = S$ in $D_{\Gamma+1}(G)$. □

5. Other graphs from dominating sets

Given a graph G, a *γ–graph* of G, denoted $G[\gamma]$, is defined in [14]. The graph $G[\gamma]$ has vertices corresponding to the γ-sets of G; two such sets S and T are are adjacent in $G[\gamma]$ if there exist $s \in S$ and $t \in T$ such that $T = (S \backslash \{s\}) \cup \{t\}$. As mentioned in Section 1, a different definition for $G[\gamma]$ is given in [7]. We generalize the graph given in [14] as follows. Define $X_k(G)$ to be the graph whose vertices correspond to all the dominating sets of G of cardinality k, with an edge between two dominating sets, S and T, if there exist $s \in S$ and $t \in T$ such that $T = (S \backslash \{s\}) \cup \{t\}$. Clearly, $X_\gamma(G) = G[\gamma]$. In this section we consider the relationship between $X_k(G)$ and $D_j(G)$ for $j \geq k$.

Lemma 10. *Let A and B be dominating sets of a graph G with $|A| = |B| = l$. If $A \leftrightarrow B$ in $D_{l+1}(G)$ then there exists a walk between A and B in $D_{l+1}(G)$ that contains only dominating sets of cardinality l or $l+1$.*

Proof. Denote by $\mathcal{W}$ the set of ordered pairs (A, B) such that
(i) A and B are dominating sets of G of cardinality l, and
(ii) no path in $D_{l+1}(G)$ from A to B contains any other dominating set of cardinality l.
We first show that the lemma is true for pairs in $\mathcal{W}$. Choose $(A, B) \in \mathcal{W}$. Write $A_0 = A$ and $A_r = B$, and suppose that $A_0, A_1, A_2, \dots, A_{r-1}, A_r$ is a path in $D_{l+1}(G)$.

Case 1. Suppose $A_1 = A_0 \cup \{x\}$. Then $|A_1| = l+1$ so $|A_2| = l$. Hence $A_2 = B$ and the path uses only dominating sets of cardinality l and $l+1$.

Case 2. Suppose $A_1 = A_0 \backslash \{y\}$ and $A_2 = A_1 \cup \{x\}$. Then $A_2 = B$ and the path $A_0, A_0 \cup \{x\}, (A_0 \cup \{x\}) \backslash \{y\} = B$ uses only dominating sets of cardinality l and $l+1$.

Case 3. Suppose $A_1 = A_0 \backslash \{y\}$ and $A_2 = A_1 \backslash \{z\}$. Let j be the least i for which $A_i \subseteq A_{i+1}$, that is, $A_{j+1} = A_j \cup \{x\}$. For all i, $A_i \cup \{x\}$ is a dominating set since A_i is a dominating set, and for $0 \le i \le j$, $|A_i| \le l$, so $|A_i \cup \{x\}| \le l+1$. Hence the sequence

$$A_0, A_0 \cup \{x\}, A_1 \cup \{x\}, \dots, A_{j-1} \cup \{x\}, A_{j+1}, A_{j+2}, \dots, A_{r-1}, A_r$$

is a path in $D_{l+1}(G)$. But now, $|A_1 \cup \{x\}| = l$, implying $(A, B) \notin \mathcal{W}$.

Now suppose that A and B are dominating sets of G with $|A| = |B| = l$, but with $(A, B) \notin \mathcal{W}$. Write $A_0 = A$ and $A_r = B$, and let $A_0, A_1, \dots, A_r$ be a path between A and B. Let $S_0, S_1, \dots, S_t$ be the subsequence of vertices on this path that are the dominating sets of cardinality l, so $S_0 = A_0$, $S_t = A_r$.

Note that $(S_i, S_{i+1}) \in \mathcal{W}$ for $0 \le i \le t-1$. It follows from Cases 1, 2 and 3 that there is a path between S_i and S_{i+1} using only dominating sets of cardinality l or $l+1$. The union of these paths for $i = 0, 1, 2, \dots, t-1$ results in a walk between A and B in D_{l+1} containing only dominating sets of cardinality l and $l+1$. □

Lemma 11. *Let A and B be dominating sets of G with $|A| = |B| = k$. Then $A \leftrightarrow B$ in $D_{k+1}(G)$ if and only if $A \leftrightarrow B$ in $X_k(G)$.*

Proof. Let S and T be adjacent dominating sets in $X_k(G)$ with $T = (S \backslash \{s\}) \cup \{t\}$. Then $S, S \cup \{t\}, T$ is a path in $D_{k+1}(G)$. Hence $A \leftrightarrow B$ in $X_k(G)$ implies $A \leftrightarrow B$ in $D_{k+1}(G)$.

Conversely, suppose $A \leftrightarrow B$ in $D_{k+1}(G)$. Then by Lemma 10, there is a path $A, A_1, A_2, \dots A_{2r+1}, B$ such that $|A_i| = k+1$ if i is odd, and $|A_i| = k$ if i is even. Hence $A, A_2, A_4, \dots A_{2r}, B$ is a path in $X_k(G)$. □

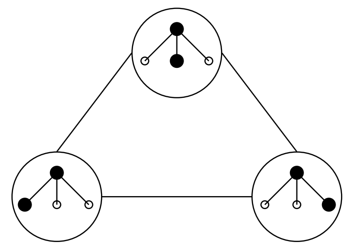

FIGURE 2. $X_2(K_{1,3})$.

Theorem 12. *If $D_{k+1}(G)$ is connected then $X_k(G)$ is connected.*

Proof. The proof follows immediately from Lemma 11. □

The converse of this theorem is false, as illustrated with the graphs $X_2(K_{1,3})$ and $D_3(K_{1,3})$. We see in Figure 2 that $X_2(K_{1,3})$ is connected, while Figure 1 shows that $D_3(K_{1,3})$ is not connected.

6. Directions for further work

In this paper we have just begun the study of dominating graphs. There are a range of questions that should be addressed in future work.

The major open question suggested by this paper is whether $d_0(G) = \Gamma + 1$, for all graphs G. If this is not true, then is there a characterization of when $d_0(G) = \Gamma + 1$? What is the complexity of determining whether two dominating sets of G are in the same connected component of $D_{\Gamma+1}(G)$? When $D_k(G)$ is connected, what is the diameter of $D_k(G)$, i.e, how long is the longest shortest path between dominating sets? Under what conditions is $D_k(G)$ Hamiltonian? Which graphs are $D_k(G)$ for some G? Note that for the star graph, $D_2(K_{1,n}) \cong K_{1,n}$, raising the question: are there other graphs G for which $D_k(G) \cong G$?

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