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The Yao Graph Y_6 is a Spanner

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Abstract

We prove that Y_6 is a spanner. Y_6 is the Yao graph on a set of planar points, which has an edge from each point x to a closest point y within each of the six angular cones of 60° surrounding x.

1 Overview

The *Yao graph* [Yao82] Y_k is a geometric graph on a finite set of points S in the plane, defined for an integer $k \geq 1$ as follows. Around each point $x \in S$, k sectors are defined by k equally-spaced rays from x, starting with a horizontal ray along the $+x$ axis. The directed graph $\overrightarrow{Y_k}$ connects x to a closest point y in each sector with a directed edge $\overrightarrow{xy}$. Closeness is measured by the Euclidean metric, $|xy|$. Y_k is the undirected version of $\overrightarrow{Y_k}$.

A graph G is a *spanner* for S if, for any pair of points $x, y \in S$, the length of a shortest path between x and y in G is at most a constant t times the Euclidean distance $|xy|$.

It is now known that Y_1, Y_2, and Y_3 are not spanners, and that Y_4 and all Y_k for $k \geq 7$ are spanners. We refer to [BDD$^+$10] for history and references. In this note we show that Y_6 is a spanner.

Let $p(x, y)$ be the length of a shortest path in Y_6 from x to y. In particular we prove:

Theorem 1 *For any two points $a, b \in S$, $p(a, b) \leq t|ab|$, where t is a constant independent of S. In particular, the claim holds for $t = 20.4$.*

The proof is by induction on the length $|ab|$. We imagine sorting the $\binom{n}{2}$ distances determined by points in S. At any stage in the induction proof for the pair of points (a, b), we have established the theorem for all distances strictly smaller than $|ab|$, and we seek to establish that $p(a, b) \leq t|ab|$.

Let $Q_i(a)$ be the half-open cone of angle 60° with apex at a, including the angle range $[i, i+1)60^\circ$, $i = 0, \ldots, 5$, where angles are measured counterclockwise from the $+x$ axis. $\overrightarrow{Y_6}$ includes exactly one directed edge from a to a closest point

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in $Q_i(a)$. If there are several equally-closest points within $Q_i(a)$, then ties are broken arbitrarily.

1.1 Base Case

Lemma 2 (Base) *If (a, b) is a closest pair of points, then $\overrightarrow{ab} \in \overrightarrow{Y_6}$ and so $p(a, b) = |ab|$.*

Proof: Without loss of generality let $b \in Q_0(a)$. If $\overrightarrow{ab} \in \overrightarrow{Y_6}$, then the lemma has been established. So assume that $\overrightarrow{ab} \notin \overrightarrow{Y_6}$; we will derive a contradiction. Because $\overrightarrow{ab} \notin \overrightarrow{Y_6}$, there must be another point $c \in Q_0(a)$ such that $\overrightarrow{ac} \in \overrightarrow{Y_6}$. Because (a, b) is a closest pair, it must be that $|ac| = |ab|$. Let α_1 and α_2 be the angles that ab and ac make with the horizontal respectively. Because both $\alpha_1, a_2 \in [0, 60°)$, necessarily $|\alpha_1 - \alpha_2| < 60°$. Thus $|bc| < |ab| = |ac|$, contradicting the assumption that (a, b) is a closest pair. So in fact it must be that $\overrightarrow{ab} \in \overrightarrow{Y_6}$, and the lemma is established. □

1.2 Main Idea of Proof

It was already established [BDD+10] that Y_7 is a spanner; the sector angles for Y_7 are 51.4°. The main idea of our proof of Theorem 1 is to partition the 60°-sectors of Y_6 into peripheral cones of angle δ, for some fixed $\delta \in (0, 30°)$, leaving a central sector of angle $60° - 2\delta$. We will use $\delta = 5°$ throughout; see Figure 1. When a Y_6 edge falls inside the central sector, induction will

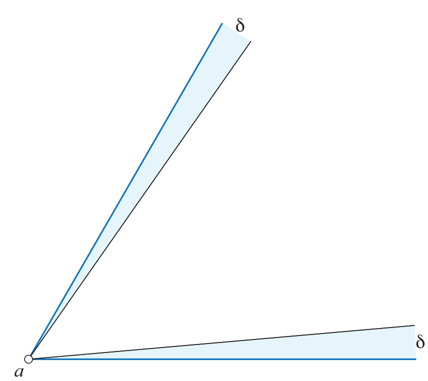

Figure 1: The δ-cones for $Q_0(a)$.

apply, because an edge within the central sector makes definite progress toward the goal in that sector, ensuring that the remaining distance to be covered is strictly smaller than the original. This leaves Y_6 edges falling within the δ-cones, at nearly 60° multiples. Such edges could conceivably not make progress toward the goal. For example, following one edge of an equilateral triangle leaves one exactly as far away from the other corner as at the start. However, we will see that when all relevant edges fall with the δ-cones near $60°$, the restricted geometric structure ensures that progress toward the goal is indeed made, and again induction applies.

2 Triangle Lemma

The primary induction step relies on an elementary triangle lemma. We are seeking to bound the length of a path from a to b, and the $\overrightarrow{Y_6}$ edge from a within the sector that includes b is $\overrightarrow{ac}$. We want to apply the induction hypothesis to the path from c to b. The basic geometry is illustrated in Figure 2. Here is the

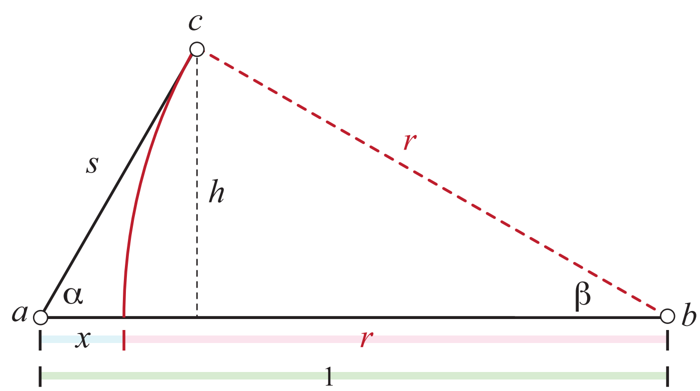

Figure 2: Notation for triangle $\triangle abc$. Here the dimensions have been normalized so that $|ab| = 1$.

key lemma.

Lemma 3 (Triangle) *Let $\triangle abc$ be labeled as in Figure 2, with $\alpha \le 60°$ and $\beta \le 60°$. Let δ be a fixed positive angle strictly smaller than $30°$. Then, if either α or β is bounded away from $60°$ by δ, that is, if $\alpha < 60° - \delta$ or if $\beta < 60° - \delta$, then the ratio s/x is less than some constant t dependent only on δ.*

For $\delta = 5°$, we will see that $t = 20.4$ suffices. Half of Lemma 3 follows from Lemma 10 in [BDD+10], but we will derive it separately to make this note self-contained. In some sense the only novelty in this note is extending that Lemma 10 to apply when either α or β is bounded away from $60°$.

We defer a proof of Lemma 3 to Section 4, and here provide only the intuition. If $\alpha = \beta = 60°$, then $x = 0$ and s/x is unbounded. When either α or β is bounded away from $60°$ by δ, then x is bounded away from zero, and because s itself is bounded ($s = |ac| \le |ab|$), the ratio s/x is bounded by a constant t.

With this lemma available, induction is possible, as follows.

Lemma 4 (Induction Step) *In the situation described in Lemma 3, if $ac \in Y_6$, then we may use induction on $p(c, b)$ to conclude that $p(a, b) \le t|ab|$.*

Proof: Using the notation in Lemma 3, we know that $x > 0$ (because at least one of α or β is strictly smaller than $60°$). Because $x = 1 - r$, $r < 1$ in the normalized triangle. Thus $|cb| < |ab|$, and we may apply induction to bound $p(c, b)$. We apply Lemma 3 to bound $|ac|$ in terms of x: since $|ac|/x < t$, $|ac| < tx$.

$$\begin{aligned} p(a,b) &\leq |ac| + p(c,b) \\ &\leq tx + t|cb| \\ &= t(x + |cb|) \\ &= t|ab| \end{aligned}$$

□

We will henceforth use the symbol $\boxed{\textit{Induct}}$ as shorthand for applying Lemma 4 to a triangle equivalent to that in Figure 2.

3 Proof of Theorem 1

The proof of Theorem 1 handles the cases where Y_6 edges from a or from b fall in the central portion of the relevant sectors, and so satisfy Lemma 3, and so Lemma 4 applies. After exhausting all these cases, we are left in a very special situation, for which induction may also be applied (for different reasons).

Let $b \in Q_0(a)$ without loss of generality. If $\overrightarrow{ab} \in \overrightarrow{Y_6}$, then $p(a, b) = |ab|$ and we are finished.

Assuming otherwise, there must be a point $c \in Q_0(a)$ such that $\overrightarrow{ac} \in \overrightarrow{Y_6}$ and $|ac| \leq |ab|$. For the remainder of the proof, we are in this situation, with $ac \in Y_6$ and $|ac| \leq |ab|$. The proof now partitions into two three parts: (1) when only $Q_0(a)$ is relevant and leads to $\boxed{\textit{Induct}}$; (2) when $Q_2(b)$ leads to $\boxed{\textit{Induct}}$; (3) when we fall into the final special situation.

(1) The $Q_0(a)$ Sector. Consider $\triangle abc$ as previously illustrated in Figure 2.

If either b or c is not in one of the δ-cones of $Q_0(a)$, then $\alpha = \angle bac < 60° - \delta$. $\boxed{\textit{Induct.}}$

So now assume that both b and c lie in δ-cones of $Q_0(a)$.

If they both lie within the same δ-cone (Figure 3(a)), then again α is small: $\boxed{\textit{Induct.}}$

So without loss of generality let b lie in the lower δ-cone, and c in the upper δ-cone of $Q_0(a)$; see Figure 3(b).

(2) The $Q_2(b)$ Sector. Now we consider $Q_2(b)$, the sector with apex at b aiming back toward a. See Figure 4.

Because b may subtend an angle as large as δ at a with horizontal, the "upper 2δ-cone" of $Q_2(b)$ becomes the relevant region. If c is not in the upper 2δ-cone of $Q_2(b)$ (as in Figure 4), then $\triangle abc$ satisfies Lemma 3 with $\beta < 60° - \delta$:

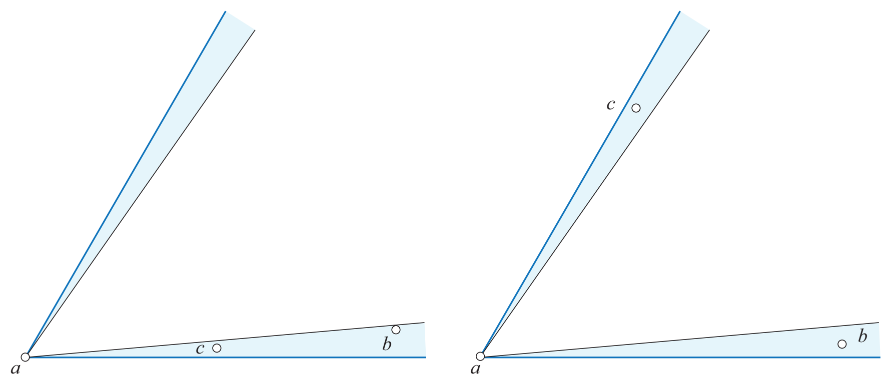

Figure 3: (a) b and c in the same δ-cone. b and c in different δ-cones.

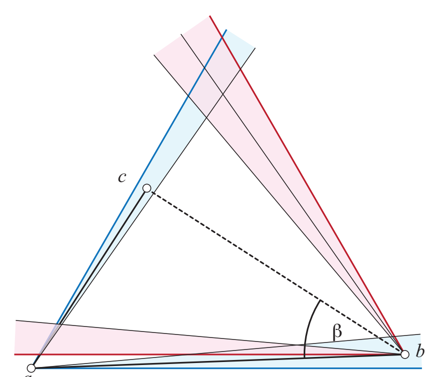

Figure 4: $Q_0(a)$ and $Q_2(b)$. Here β is small.

$\boxed{\textit{Induct.}}$ Note that this conclusion follows even if c is in the small region outside of $Q_2(b)$: the angle β at b is then very small.

Let $d \in Q_2(b)$ be the point such that $\overrightarrow{bd} \in \overrightarrow{Y_6}$. We now consider possible locations for d.

If $d = c$, then $p(a, b,) \le |ac| + |cb| \le 2|ab|$, and we are finished. So assume henceforth that d is distinct from c.

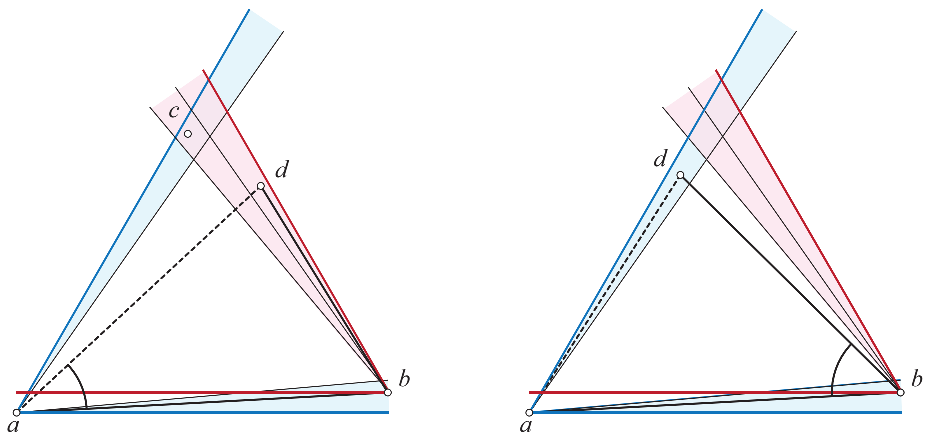

Figure 5: (a) d not in the upper δ-cone of $Q_0(a)$: $\angle bad$ is small. (b) d not in the upper 2δ-cone of $Q_2(b)$: $\angle abd$ is small.

If d is not in the upper δ-cone of $Q_0(a)$ (Figure 5(a)), then $\triangle abd$ satisfies Lemma 3 with the roles of a and b reversed: bd takes a step toward a, with the angle at a satisfying $\angle bad < 60° - \delta$: $\boxed{\textit{Induct.}}$

If d is not in the upper 2δ-cone of $Q_2(b)$ (Figure 5(b)), then $\triangle abd$ satisfies Lemma 3 again with the roles of a and b reversed and this time the angle at b bounded away from $60°$, $\angle abd < 60° - \delta$: $\boxed{\textit{Induct.}}$

(3) Remaining Situation. So now we are left in the following situation, with $\overrightarrow{ac} \in \overrightarrow{Y_6}$ and $\overrightarrow{bd} \in \overrightarrow{Y_6}$:

- b is in the lower δ-cone of $Q_0(a)$.
- $c \ne d$.
- c is in the upper δ-cone of $Q_0(a)$.
- d is in the upper δ-cone of $Q_0(a)$.
- c is in the upper 2δ-cone of $Q_2(b)$.
- d is in the upper 2δ-cone of $Q_2(b)$.

These constraints together imply that c and d are close to one another, as $cabd$ almost forms an equilateral triangle; see Figure 6. In order to apply induction, we need to show that $|cd| < |ab|$. Rather than establish this for an

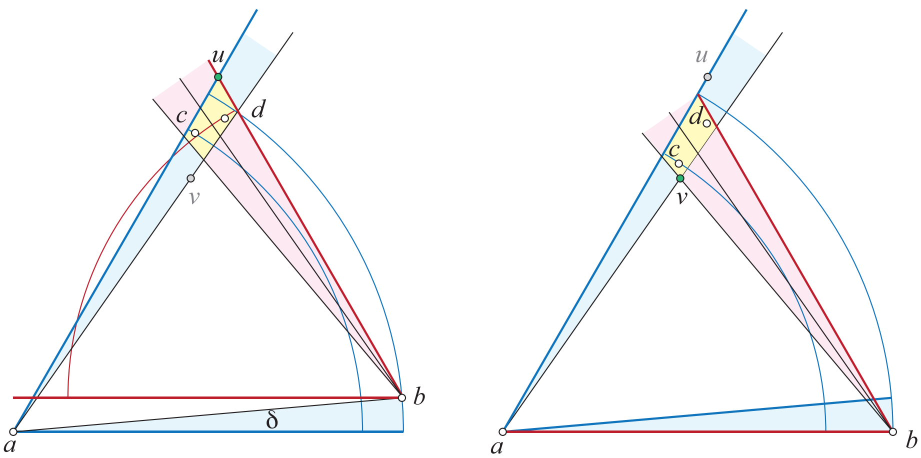

Figure 6: (a) b is at the top edge of the δ-cone of $Q_0(a)$. (b) b is on the bottom edge of the same δ-cone. Points u and v are used in the proof of Lemma 5.

arbitrary δ, when it seems the computations become complex, we opt to fix $\delta = 5°$, and then claim:

Lemma 5 *For $\delta = 5°$, in the situation described above and illustrated in Figure 6,*

$$|cd| < (0.27)|ab| \ .$$

The proof is deferred to Section 5.

We know that $|ac| \le |ab|$ because both c and b are in $Q_0(a)$ and $\overrightarrow{ac} \in \overrightarrow{Y_6}$. We know that $|bd| < |bc|$ because both c and d are in $Q_2(b)$ and $\overrightarrow{bd} \in \overrightarrow{Y_6}$.

We can see that $|bc| \le |ab|$ from Figure 7; so $|bd| \le |ab|$.

So in this special situation (illustrated in Figure 6), we have

$$\begin{aligned}
p(a,b) &\le |ac| + p(c,d) + |bd| \\
&\le 2|ab| + p(c,d) \\
&\le 2|ab| + t|cd| \\
&\le 2|ab| + t(0.27)|ab| \\
&\le 2|ab| + (7.97)|ab| \\
&\le (2 + 7.97)|ab| \\
&< t|ab|
\end{aligned}$$

where we have applied Lemma 5 to bound $|cd|$ and used the explicit value $t = 20.4$ to conclude that $9.97 < t$.

This completes the proof of Theorem 1.

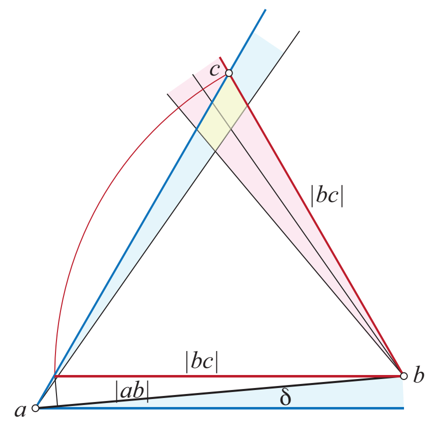

Figure 7: The furthest c could be from b occurs when c lies at the upper extreme boundaries of both the $Q_0(a)$ sector and the $Q_2(b)$ sector. In this situation, $|bc| \leq |ab|$.

4 Proof of Triangle Lemma 3

First we show how half of Lemma 3 follows from Lemma 10 in [BDD$^+$10]. That lemma says that, for α bounded less than 60°,

$$|cb| \leq |ab| - |ac|/t\ .$$

Using the notation in Figure 2, with $|ab| = 1$, this is equivalent to

$$\begin{aligned} r &\leq 1 - s/t \\ tr &\leq t - s \\ s &\leq t(1-r) \\ s/x &\leq t \end{aligned}$$

The last line is the claim of Lemma 3. The value of t established in [BDD$^+$10] is strictly greater than the value we derive here, due to an approximation made in the proof of that Lemma 10 not employed below.

Rather than rely on the above, we offer an independent proof of Lemma 3, which establishes both directions (when α is bounded less than 60°, and when β is bounded less than 60°), using the following calculation:

Lemma 6 *Let $\triangle abc$ be labeled as in Figure 2, with $\alpha \leq 60°$ and $\beta \leq 60°$. Then, with $x = |ab| - |bc|$ and $s = |ac|$,*

$$\frac{s}{x} = \frac{\cos(\beta/2)}{\cos(\alpha + \beta/2)}\ .$$

Proof: Normalize the triangle so that $|ab| = 1$; this does not alter the quantity we seek to compute, s/x. Let $|ac| = s$ and $|bc| = r$ to simplify notation. Then $x = 1 - r$, and $x \geq 0$ because $\beta \leq 60°$. Computing the altitude h of $\triangle abc$ in two ways yields

$$s \sin(\alpha) = r \sin(\beta) \ .$$

Also projections onto ab yield,

$$s \cos(\alpha) + r \cos(\beta) = 1 \ .$$

Solving these two equations simultaneously yields expressions for r and s as functions of α and β:

$$r = \frac{\sin \alpha}{\sin \alpha \cos \beta + \cos \alpha \sin \beta}$$

$$s = \frac{\sin \beta}{\sin \alpha \cos \beta + \cos \alpha \sin \beta}$$

Now we can compute $s/x = s/(1-r)$ as a function of α and β. This simplifies to

$$\frac{s}{x} = \frac{\cos(\beta/2)}{\cos(\alpha + \beta/2)} \ .$$

as claimed. □

Now we would like to use Lemma 6 to compute one upper bound for t that covers all α and β possibilities. Rather than prove the bounds analytically, which would be tedious and unrevealing, we justify four claims by graphs of the function derived in Lemma 6.

Claim 1. For any fixed $\alpha < 60°$, the maximum ratio $t = s/x$ in Lemma 3 occurs when $\beta = 60°$. See Figure 8.

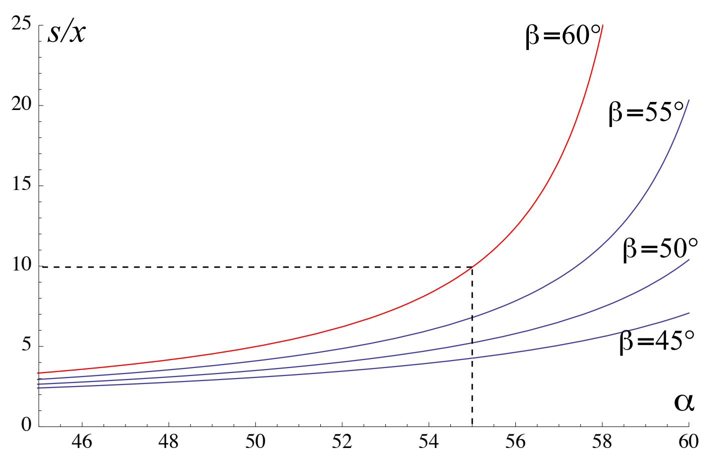

Figure 8: For any value of α, the largest ratio s/x occurs when $\beta = 60°$.

Claim 2. For any fixed $\beta < 60°$, the maximum ratio $t = s/x$ in Lemma 3 occurs when $\alpha = 60°$. See Figure 9.

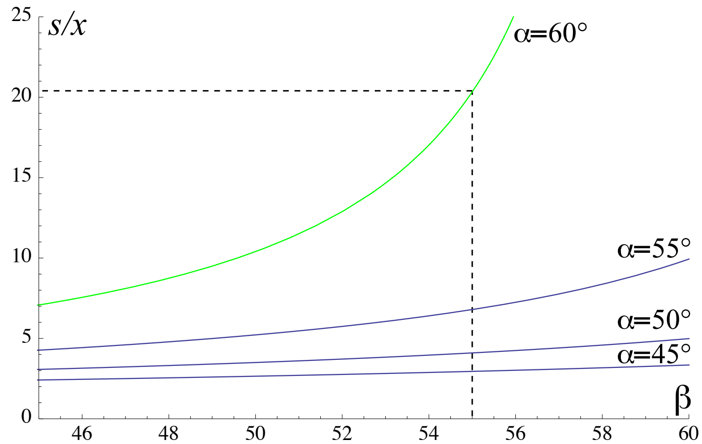

Figure 9: For any value of β, the largest ratio s/x occurs when $\alpha = 60°$.

Claim 3. The maximum ratio $t = s/x$ in Lemma 3, for any δ, occurs when $\alpha = 60°$ and $\beta = 60° - \delta$. See Figure 10.

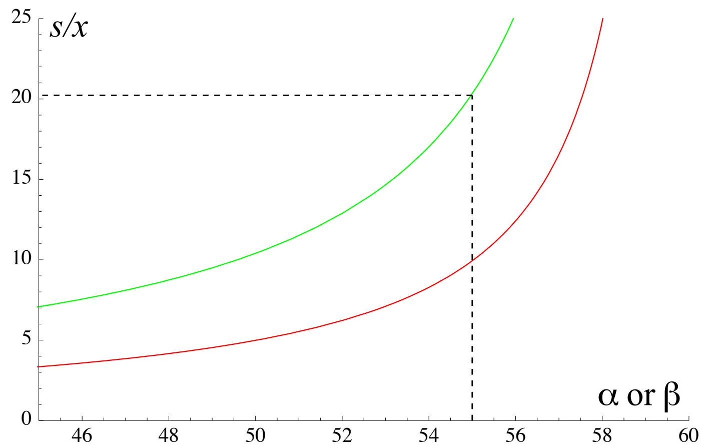

Figure 10: The worst-case curves from Figures 8 and 9.

Claim 4. Fixing $\delta = 5°$, Figure 10 shows that $t = 20.335$. This worst-case value of t occurs in the triangle shown in Figure 11, when $\alpha = 60°$ and $\beta = 55°$.

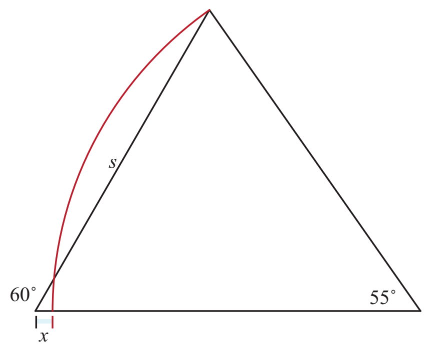

Figure 11: The worst case ratio $s/x = 20.335$, when $\alpha = 60°$ and $\beta = 55°$.

5 Proof of Lemma 5

Because both c and d must lie in the upper δ-cone of $Q_0(a)$ and in the upper 2δ-cone of $Q_2(b)$, they both must lie in the intersection of these cones shaded in Figure 6. Let u be the highest possible point of this region, over all positions of b, and let v be the lowest possible point of this region. These are illustrated in (a) and (b) of the figure respectively. Then we know that $|cd| \leq |uv|$. Explicit computation shows that, when all distances are normalized so that $|ab| = 1$,

$$u = (0.5233, 0.9063) \tag{1}$$
$$v = (0.4549, 0.6496) \tag{2}$$
$$|uv| = 0.2656 \tag{3}$$

This distance is in fact realized in Figure 6.

6 Conclusion

We made no attempt to optimize the value of t. Selecting a larger value of δ would lower t, but we did not determine the largest value of δ that would suffice to make the argument go through at its last step.

Even were this calculation carried out, this proof would still be far from establishing "the truth." We conjecture that Y_6 is a spanner for $t = 2$.

The case Y_5 is the last Yao graph whose "spanner-hood" has not yet been settled. Given that both Y_4 and Y_6 are both spanners, it is natural to conjecture that Y_5 is also a spanner.

Acknowledgements. I thank the authors of [BDD$^+$10] for introducing me to the Y_6 problem.

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