2012

π/2-Angle Yao Graphs are Spanners

Prosenjit Bose
Carleton University

Mirela Damian
Villanova University

Karim Douïeb
Carleton University

Joseph O'Rourke
Smith College, jorourke@smith.edu

Ben Seamone
Carleton University

See next page for additional authors

$\pi/2$-Angle Yao Graphs are Spanners

Prosenjit Bose [*] Mirela Damian [†] Karim Douïeb [‡] Joseph O'Rourke [§]
Ben Seamone [¶] Michiel Smid [‖] Stefanie Wuhrer [**]

Abstract

We show that the Yao graph Y_4 in the L_2 metric is a spanner with stretch factor $8(29+23\sqrt{2})$. Enroute to this, we also show that the Yao graph Y_4^∞ in the L_∞ metric is a planar spanner with stretch factor 8.

1 Introduction

Let V be a finite set of points in the plane and let $G = (V, E)$ be the complete Euclidean graph on V. We will refer to the points in V as *nodes*, to distinguish them from other points in the plane. The *Yao graph* [6] with an integer parameter $k > 0$, denoted Y_k, is defined as follows. At each node $u \in V$, any k equally-separated rays originating at u define k cones. In each cone, pick a shortest edge uv, if there is one, and add to Y_k the directed edge $\overrightarrow{uv}$. Ties are broken arbitrarily. Most of the time we ignore the direction of an edge uv; we refer to the directed version $\overrightarrow{uv}$ of uv only when its origin (u) is important and unclear from the context. We will distinguish between Y_k, the Yao graph in the Euclidean L_2 metric, and Y_k^∞, the Yao graph in the L_∞ metric. Unlike Y_k however, in constructing Y_k^∞ ties are broken by always selecting the most counterclockwise edge; the reason for this choice will become clear in Section 2.

For a given subgraph $H \subseteq G$ and a fixed $t \geq 1$, H is called a *t-spanner* for G if, for any two nodes $u, v \in V$, the shortest path in H from u to v is no longer than t times the length of uv. The value t is called the *dilation* or the *stretch factor* of H. If t is constant, then H is called a *length spanner*, or simply a *spanner*.

The class of graphs Y_k has been much studied. Bose et al. [1] showed that, for $k \geq 9$, Y_k is a spanner with stretch factor $\frac{1}{\cos\frac{2\pi}{k} - \sin\frac{2\pi}{k}}$. In the appendix, we improve the stretch factor and show that, in fact, Y_k is a spanner for any $k \geq 7$. Recently, Molla [4] showed that Y_2 and Y_3 are not

[*]School of Computer Science, Carleton University, Ottawa, Canada. jit@scs.carleton.ca. Supported by NSERC.

[†]Department of Computer Science, Villanova University, Villanova, USA. mirela.damian@villanova.edu. Supported by NSF grant CCF-0728909.

[‡]School of Computer Science, Carleton University, Ottawa, Canada. kdouieb@ulb.ac.be. Supported by NSERC.

[§]Department of Computer Science, Smith College, Northampton, USA. orourke@cs.smith.edu. Supported by NSERC.

[¶]School of Mathematics and Statistics, Carleton University, Ottawa, Canada. bseamone@connect.carleton.ca.

[‖]School of Computer Science, Carleton University, Ottawa, Canada. michiel@scs.carleton.ca. Supported by NSERC.

[**]NRC Institute for Information Technology, Ottawa, Canada. Stefanie.Wuhrer@nrc-cnrc.gc.ca.

spanners, and that Y_4 is a spanner with stretch factor $4(2+\sqrt{2})$, for the special case when the nodes in V are in convex position (see also [2]). The authors conjectured that Y_4 is a spanner for arbitrary point sets. In this paper, we settle their conjecture and prove that Y_4 is a spanner with stretch factor $8(29+23\sqrt{2})$.

The paper is organized as follows. In Section 2, we prove that the graph Y_4^∞ is a spanner with stretch factor 8. In Section 3, we prove, in a sequence of Lemmas, several properties for the graph Y_4. Finally, in Section 4, we use the properties of Section 3 to prove that for every edge ab in Y_4^∞, there exists a path between a and b in Y_4, whose length is not much more than the Euclidean distance between a and b. By combining this with the result of Section 2, it follows that Y_4 is a spanner.

2 Y_4^∞: in the L_∞ Metric

In this section we focus on Y_4^∞, which has a nicer structure compared to Y_4. First we prove that Y_4^∞ is planar. Then we use this property to show that Y_4^∞ is an 8-spanner. To be more precise, we prove that for any two nodes a and b, the graph Y_4^∞ contains a path between a and b whose length (in the L_∞-metric) is at most $8|ab|_\infty$.

We need a few definitions. We say that two edges ab and cd *properly cross* (or *cross*, for short) if they share a point other than an endpoint (a, b, c or d); we say that ab and cd *intersect* if they share a point (either an interior point or an endpoint). Let $Q_1(a)$, $Q_2(a)$, $Q_3(a)$ and $Q_4(a)$ be the

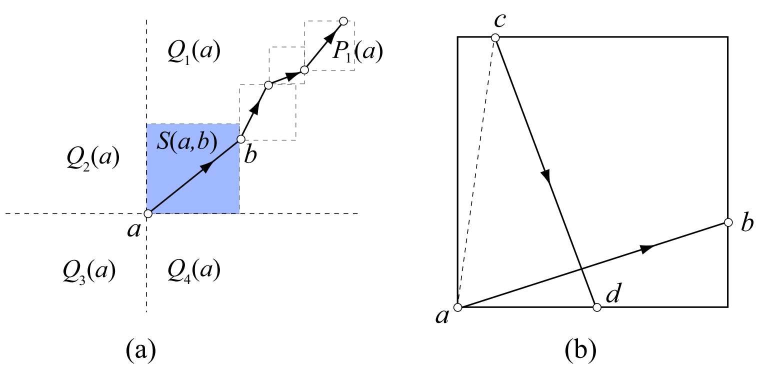

Figure 1: (a) Definitions: $Q_i(a)$, $P_i(a)$ and $S(a,b)$. (b) Lemma 1: ab and cd cannot cross.

four quadrants at a, as in Figure 1a. Let $P_i(a)$ be the path that starts at point a and follows the directed Yao edges in quadrant Q_i. Let $P_i(a,b)$ be the subpath of $P_i(a)$ that starts at a and ends at b. Let $|ab|_\infty$ be the L_∞ distance between a and b. Let $sp(a,b)$ denote a shortest path in Y_4^∞ between a and b. Let $S(a,b)$ denote the open square with corner a whose boundary contains b, and let $\partial S(a,b)$ denote the boundary of $S(a,b)$. These definitions are illustrated in Figure 1a. For a node $a \in V$, let $x(a)$ denote the x-coordinate of a and $y(a)$ denote the y-coordinate of a.

Lemma 1 *Y_4^∞ is planar.*

Proof. The proof is by contradiction. Assume the opposite. Then there are two edges $\overrightarrow{ab}, \overrightarrow{cd} \in Y_4^\infty$ that cross each other. Since $\overrightarrow{ab} \in Y_4^\infty$, $S(a,b)$ must be empty of nodes in V, and similarly for $S(c,d)$. Let j be the intersection point between ab and cd. Then $j \in S(a,b) \cap S(c,d)$, meaning that $S(a,b)$

and $S(c,d)$ must overlap. However, neither square may contain a, b, c or d. It follows that $S(a,b)$ and $S(c,d)$ coincide, meaning that c and d lie on $\partial S(a,b)$ (see Figure 1b). Since cd intersects ab, c and d must lie on opposite sides of ab. Thus either ac or ad lies counterclockwise from ab. Assume without loss of generality that ac lies counterclockwise from ab; the other case is identical. Because $S(a,c)$ coincides with $S(a,b)$, we have that $|ac|_\infty = |ab|_\infty$. In this case however, Y_4^∞ would break the tie between ac and ab by selecting the most counterclockwise edge, which is $\overrightarrow{ac}$. This contradicts the fact that $\overrightarrow{ab} \in Y_4^\infty$. ∎

It can be easily shown that each face of Y_4^∞ is either a triangle or a quadrilateral (except for the outer face). We skip this proof however, since we do not make use of this property in this paper.

Theorem 1 *Y_4^∞ is an 8-spanner.*

Proof. We show that, for any pair of points $a, b \in V$, $|sp(a,b)|_\infty < 8|ab|_\infty$. The proof is by induction on the pairwise distance between the points in V. Assume without loss of generality that $b \in Q_1(a)$, and $|ab|_\infty = |x(b) - x(a)|$. Consider the case in which ab is a closest pair of points in V (the base case for our induction). If $ab \in Y_4^\infty$, then $|sp(a,b)|_\infty = |ab|_\infty$. Otherwise, there must be $ac \in Y_4^\infty$, with $|ac|_\infty = |ab|_\infty$. But then $|bc|_\infty < |ab|_\infty$ (see Figure 2a), a contradiction.

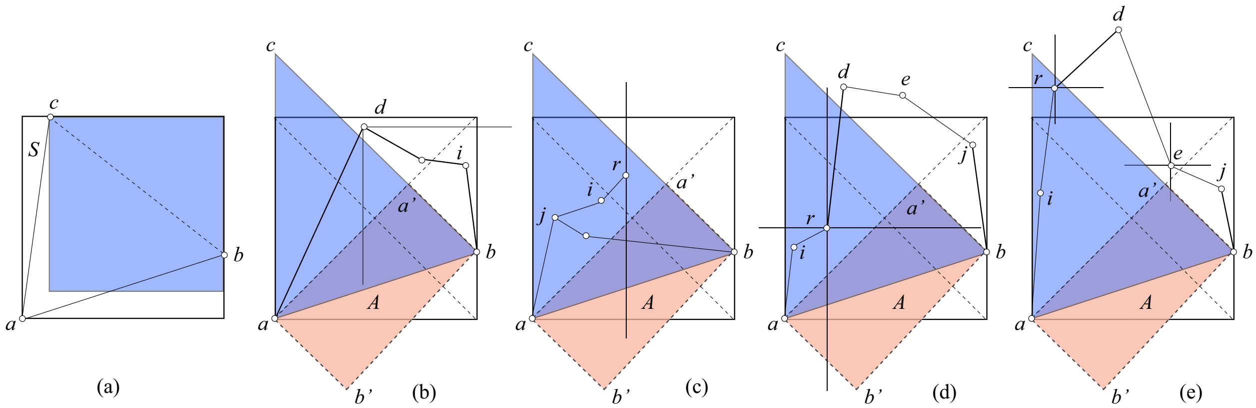

Figure 2: (a) Base case. (b) $\triangle abc$ empty (c) $\triangle abc$ non-empty, $P_{ar} \cap P_2(b) = \{j\}$ (d) $\triangle abc$ non-empty, $P_{ar} \cap P_2(b) = \emptyset$, e above r (e) $\triangle abc$ non-empty, $P_{ar} \cap P_2(b) = \emptyset$, e below r.

Assume now that the inductive hypothesis holds for all pairs of points closer than $|ab|_\infty$. If $ab \in Y_4^\infty$, then $|sp(a,b)|_\infty = |ab|_\infty$ and the proof is finished. If $ab \notin Y_4^\infty$, then the square $S(a,b)$ must be nonempty.

Let A be the rectangle $ab'ba'$ as in Figure 2b, where ba' and bb' are parallel to the diagonals of S. If A is nonempty, then we can use induction to prove that $|sp(a,b)|_\infty <= 8|ab|_\infty$ as follows. Pick $c \in A$ arbitrary. Then $|ac|_\infty + |cb|_\infty = |x(c) - x(a)| + |x(b) - x(c)| = |ab|_\infty$, and by the inductive hypothesis $sp(a,c) \oplus sp(c,b)$ is a path in Y_4^∞ no longer than $8|ac|_\infty + 8|cb|_\infty = 8|ab|_\infty$; here $\oplus$ represents the concatenation operator. Assume now that A is empty. Let c be at the intersection between the line supporting ba' and the vertical line through a (see Figure 2b). We discuss two cases, depending on whether $\triangle abc$ is empty of points or not.

Case 1: $\triangle abc$ is empty of points. Let $ad \in P_1(a)$. We show that $P_4(d)$ cannot contain an edge crossing ab. Assume the opposite, and let $st \in P_4(d)$ cross ab. Since $\triangle abc$ is empty, s must lie above bc and t below ab, therefore $|st|_\infty \geq |y(s) - y(t)| > |y(s) - y(b)| = |sb|_\infty$, contradicting the fact that $st \in Y_4^\infty$. It follows that $P_4(d)$ and $P_2(b)$ must meet in a point $i \in P_4(d) \cap P_2(b)$ (see Figure 2b). Now note that $|P_4(d,i) \oplus P_2(b,i)|_\infty \leq |x(d) - x(b)| + |y(d) - y(b)| < 2|ab|_\infty$. Thus we have that

$$|sp(a,b)|_\infty \leq |ad \oplus P_4(d,i) \oplus P_2(b,i)|_\infty < |ab|_\infty + 2|ab|_\infty = 3|ab|_\infty.$$

Case 2: $\triangle abc$ is nonempty. In this case, we seek a short path from a to b that does not cross to the underside of ab. This is to avoid oscillating paths that cross ab arbitrarily many times. Let r be the rightmost point that lies inside $\triangle abc$. Arguments similar to the ones used in Case 1 show that $P_3(r)$ cannot cross ab and therefore it must meet $P_1(a)$ in a point i. Then $P_{ar} = P_1(a,i) \oplus P_3(r,i)$ is a path in Y_4^∞ of length

$$|P_{ar}|_\infty < |x(a) - x(r)| + |y(a) - y(r)| < |ab|_\infty + 2|ab|_\infty = 3|ab|_\infty. \tag{1}$$

The term $2|ab|_\infty$ in the inequality above represents the fact that $|y(a) - y(r)| \leq |y(a) - y(c)| \leq 2|ab|_\infty$. Consider first the simpler situation in which $P_2(b)$ meets P_{ar} in a point $j \in P_2(b) \cap P_{ar}$ (see Figure 2c). Let $P_{ar}(a,j)$ be the subpath of P_{ar} extending between a and j. Then $P_{ar}(a,j) \oplus P_2(b,j)$ is a path in Y_4^∞ from a to b, therefore

$$|sp(a,b)|_\infty \leq |P_{ar}(a,j) \oplus P_2(b,j)|_\infty < 2|y(j) - y(a)| + |ab|_\infty \leq 5|ab|_\infty.$$

Consider now the case when $P_2(b)$ does not intersect P_{ar}. We argue that, in this case, $Q_1(r)$ may not be empty. Assume the opposite. Then no edge $st \in P_2(b)$ may cross $Q_1(r)$. This is because, for any such edge, $|sr|_\infty < |st|_\infty$, contradicting $st \in Y_4^\infty$. This implies that $P_2(b)$ intersects P_{ar}, again a contradiction to our assumption.

We have established that $Q_1(r)$ is nonempty. Let $rd \in P_1(r)$. The fact that $P_2(b)$ does not intersect P_{ar} implies that d lies to the left of b. The fact that r is the rightmost point in $\triangle abc$ implies that d lies outside $\triangle abc$ (see Figure 2d). It also implies that $P_4(d)$ shares no points with $\triangle abc$. This along with arguments similar to the ones used in case 1 show that $P_4(d)$ and $P_2(b)$ meet in a point $j \in P_4(d) \cap P_2(b)$. Thus we have found a path

$$P_{ab} = P_1(a,i) \oplus P_3(r,i) \oplus rd \oplus P_4(d,j) \oplus P_2(b,j) \tag{2}$$

extending from a to b in Y_4^∞. If $|rd|_\infty = |x(d) - x(r)|$, then $|rd|_\infty < |x(b) - x(a)| = |ab|_\infty$, and the path P_{ab} has length

$$|P_{ab}|_\infty \leq 2|y(d) - y(a)| + |ab|_\infty < 7|ab|_\infty. \tag{3}$$

In the above, we used the fact that $|y(d) - y(a)| = |y(d) - y(r)| + |y(r) - y(a)| < |ab|_\infty + 2|ab|_\infty$. Suppose now that

$$|rd|_\infty = |y(d) - y(r)|. \tag{4}$$

In this case, it is unclear whether the path P_{ab} defined by (2) is short, since rd can be arbitrarily long compared to ab. Let e be the clockwise neighbor of d along the path P_{ab} (e and b may coincide). Then e lies below d, and either $de \in P_4(d)$, or $ed \in P_2(e)$ (or both).

1. If e lies above r, or at the same level as r (i.e., $e \in Q_1(r)$, as in Figure 2d), then

$$|y(e) - y(r)| \quad < \quad |y(d) - y(r)| \tag{5}$$

Since $rd \in P_1(r)$ and e is in the same quadrant of r as d, we have $|rd|_\infty \le |re|_\infty$. This along with inequalities (4) and (5) implies $|re|_\infty > |y(e) - y(r)|$, which in turn implies $|re|_\infty = |x(e) - x(r)| \le |ab|_\infty$, and so $|rd|_\infty \le |ab|_\infty$. Then inequality (3) applies here as well, showing that $|P_{ab}|_\infty < 7|ab|_\infty$.

2. If e lies below r (as in Figure 2e), then

$$|ed|_\infty \ge |y(d) - y(e)| \ge |y(d) - y(r)| = |rd|_\infty. \tag{6}$$

Assume first that $ed \in P_2(e)$, or $|ed|_\infty = |x(e) - x(d)|$. In either case,

$$|ed|_\infty \le |er|_\infty < 2|ab|_\infty.$$

This along with inequality (6) shows that $|rd|_\infty < 2|ab|_\infty$. Substituting this upper bound in (2), we get

$$|P_{ab}|_\infty \le 2|y(d) - y(a)| + 2|ab|_\infty < 8|ab|_\infty.$$

Assume now that $ed \notin P_2(e)$, and $|ed|_\infty = |y(e) - y(d)|$. Then $ee' \in P_2(e)$ cannot go above d (otherwise $|ed|_\infty < |ee'|_\infty$, contradicting $ee' \in P_2(e)$). This along with the fact $de \in P_4(d)$ implies that $P_2(e)$ intersects P_{ar} in a point k. Redefine

$$P_{ab} = P_{ar}(a, k) \oplus P_2(e, k) \oplus P_4(e, j) \oplus P_2(b, j)$$

Then P_{ab} is a path in Y_4^∞ from a to b of length

$$|P_{ab}| \le 2|y(r) - y(a)| + |ab|_\infty \le 5|ab|_\infty.$$

We have established that $|sp(a, b)|_\infty \le |P_{ab}|_\infty < 8|ab|_\infty$. This concludes the proof. ∎

This theorem will be employed in Section 4.

3 Y_4: in the L_2 Metric

In this section we establish basic properties of Y_4. The ultimate goal of this section is to show that, if two edges in Y_4 cross, there is a short path between their endpoints (Lemma 8). We begin with a few definitions.

Let $Q(a, b)$ denote the infinite quadrant with origin at a that contains b. For a pair of nodes $a, b \in V$, define recursively a directed path $\mathcal{P}(a \to b)$ from a to b in Y_4 as follows. If $a = b$, then $\mathcal{P}(a \to b) = null$. If $a \ne b$, there must exist $\overrightarrow{ac} \in Y_4$ that lies in $Q(a, b)$. In this case, define

$$\mathcal{P}(a \to b) = \overrightarrow{ac} \oplus \mathcal{P}(c \to b).$$

Recall that $\oplus$ represents the concatenation operator. This definition is illustrated in Figure 3a. Fischer et al. [3] show that $\mathcal{P}(a \to b)$ is well defined and lies entirely inside the square centered at b whose boundary contains a.

For any node $a \in V$, let $D(a, r)$ denote the open disk centered at a of radius r, and let $\partial D(a, r)$ denote the boundary of $D(a, r)$. Let $D[a, r] = D(a, r) \cup \partial D(a, r)$. For any path P and any pair of nodes a and b on P, let $P[a, b]$ denote the subpath of P that starts at a and ends at b. Let $R(a, b)$ denote the closed rectangle with diagonal ab.

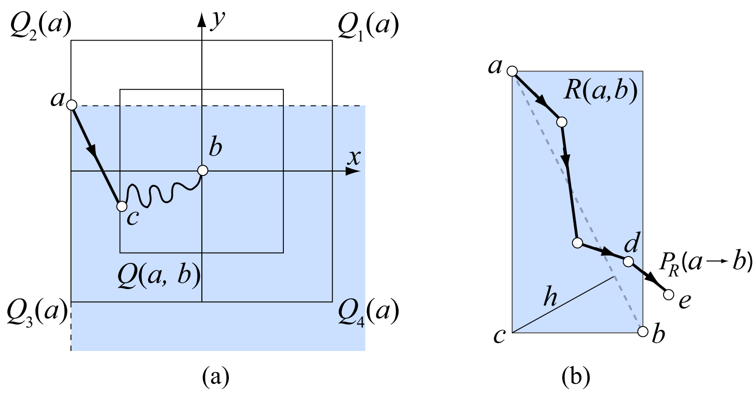

Figure 3: Definitions. (a) $Q(a,b)$ and $\mathcal{P}(a \to b)$. (b) $\mathcal{P}_R(a \to b)$.

For a fixed pair of nodes $a, b \in V$, define a path $\mathcal{P}_R(a \to b)$ as follows. Let $e \in V$ be the first node along $\mathcal{P}(a \to b)$ that is not strictly interior to $R(a,b)$. Then $\mathcal{P}_R(a \to b)$ is the subpath of $\mathcal{P}(a \to b)$ that extends between a and e. In other words, $\mathcal{P}_R(a \to b)$ is the path that follows the Y_4 edges pointing towards b, truncated as soon as it leaves the rectangle with diagonal ab, or as it reaches b. Formally,

$$\mathcal{P}_R(a \to b) = \mathcal{P}(a \to b)[a, e]$$

This definition is illustrated in Figure 3b.

Our proofs will make use of the following two propositions.

Proposition 1 *The sum of the lengths of crossing diagonals of a nondegenerate (necessarily convex) quadrilateral abcd is strictly greater than the sum of the lengths of either pair of opposite sides:*

$$\begin{aligned} |ac| + |bd| &> |ab| + |cd| \\ |ac| + |bd| &> |bc| + |da| \end{aligned}$$

This can be proved by partitioning the diagonals into two pieces each at their intersection point, and then applying the triangle inequality twice.

Proposition 2 *For any triangle $\triangle abc$, the following inequalities hold:*

$$|ac|^2 \begin{cases} < |ab|^2 + |bc|^2, & \text{if } \angle abc < \pi/2 \\ = |ab|^2 + |bc|^2, & \text{if } \angle abc = \pi/2 \\ > |ab|^2 + |bc|^2, & \text{if } \angle abc > \pi/2 \end{cases}$$

This proposition follows immediately from the Law of Cosines applied to triangle $\triangle abc$.

Lemma 2 *For each pair of nodes $a, b \in V$,*

$$|\mathcal{P}_R(a \to b)| \leq |ab|\sqrt{2} \tag{7}$$

Furthermore, each edge of $\mathcal{P}_R(a \to b)$ is no longer than $|ab|$.

Proof. Let c be one of the two corners of $R(a, b)$, other than a and b. Let $\overrightarrow{de} \in \mathcal{P}_R(a \to b)$ be the last edge on $\mathcal{P}_R(a \to b)$, which necessarily intersects $\partial R(a, b)$ (note that it is possible that $e = b$). Refer to Figure 3b. Then $|de| \le |db|$, otherwise $\overrightarrow{de}$ could not be in Y_4. Since db lies in the rectangle with diagonal ab, we have that $|db| \le |ab|$, and similarly for each edge on $\mathcal{P}_R(a \to b)$. This establishes the latter claim of the lemma. For the first claim of the lemma, let

$$p = \mathcal{P}_R(a \to b)[a, d] \oplus db$$

Since $|de| \le |db|$, we have that $|\mathcal{P}_R(a \to b)| \le |p|$. Since p lies entirely inside $R(a, b)$ and consists of edges pointing towards b, we have that p is an xy-monotone path. It follows that $|p| \le |ac| + |cb|$. We now show that $|ac| + |cb| \le |ab|\sqrt{2}$, thus establishing the first claim of the lemma.

Let $x = |ac|$ and $y = |cb|$. Then the inequality $|ac| + |cb| \le |ab|\sqrt{2}$ can be written as $x + y \le \sqrt{2x^2 + 2y^2}$, which is equivalent to $(x - y)^2 \ge 0$. This latter inequality obviously holds, completing the proof of the lemma. ■

Lemma 3 *Let $a, b, c, d \in V$ be four disjoint nodes such that $\overrightarrow{ab}, \overrightarrow{cd} \in Y_4$, $b \in Q_i(a)$ and $d \in Q_i(c)$, for some $i \in \{1, 2, 3, 4\}$. Then ab and cd cannot cross each other.*

Proof. We may assume without loss of generality that $i = 1$ and c is to the left of a. The proof is by contradiction. Assume that ab and cd cross each other. Let j be the intersection point between ab and cd (see Figure 4a). Since $j \in Q_1(a) \cap Q_1(c)$, it follows that $d \in Q_1(a)$ and $b \in Q_1(c)$. Thus $|ab| \le |ad|$, because otherwise, $\overrightarrow{ab}$ cannot be in Y_4. By Proposition 1 applied to the quadrilateral $adbc$,

$$|ad| + |cb| < |ab| + |cd|$$

This along with the fact that $|ab| \le |ad|$ implies that $|cb| < |cd|$, contradicting the fact that $\overrightarrow{cd} \in Y_4$. ■

The next four lemmas (4–8) each concern a pair of crossing Y_4 edges, culminating (in Lemma 8) in the conclusion that there is a short path in Y_4 between a pair of endpoints of those edges.

Lemma 4 *Let a, b, c and d be four disjoint nodes in V such that $\overrightarrow{ab}, \overrightarrow{cd} \in Y_4$, and ab crosses cd. Then the following are true: (i) the ratio between the shortest side and the longer diagonal of the quadrilateral $acbd$ is no greater than $1/\sqrt{2}$, and (ii) the shortest side of the quadrilateral $acbd$ is strictly shorter than either diagonal.*

Proof. The first part of the lemma is a well-known fact that holds for any quadrilateral (see [5], for instance). For the second part of the lemma, let ab be the shorter of the diagonals of $acbd$, and assume without loss of generality that $\overrightarrow{ab} \in Q_1(a)$. Imagine two disks $D_a = D(a, |ab|)$ and $D_b = D(b, |ab|)$, as in Figure 4b. If either c or d belongs to $D_a \cup D_b$, then the lemma follows: a shortest quadrilateral edge is shorter than $|ab|$.

So suppose that neither c nor d lies in $D_a \cup D_b$. In this case, we use the fact that cd crosses ab to show that $\overrightarrow{cd}$ cannot be an edge in Y_4. Define the following regions (see Figure 4b):

$$\begin{aligned}
R_1 &= (Q_1(a) \cap Q_2(b)) \backslash (D_a \cup D_b) \\
R_2 &= (Q_2(a) \cap Q_3(b)) \backslash (D_a \cup D_b) \\
R_3 &= (Q_4(a) \cap Q_3(b)) \backslash (D_a \cup D_b) \\
R_4 &= (Q_1(a) \cap Q_4(b)) \backslash (D_a \cup D_b)
\end{aligned}$$

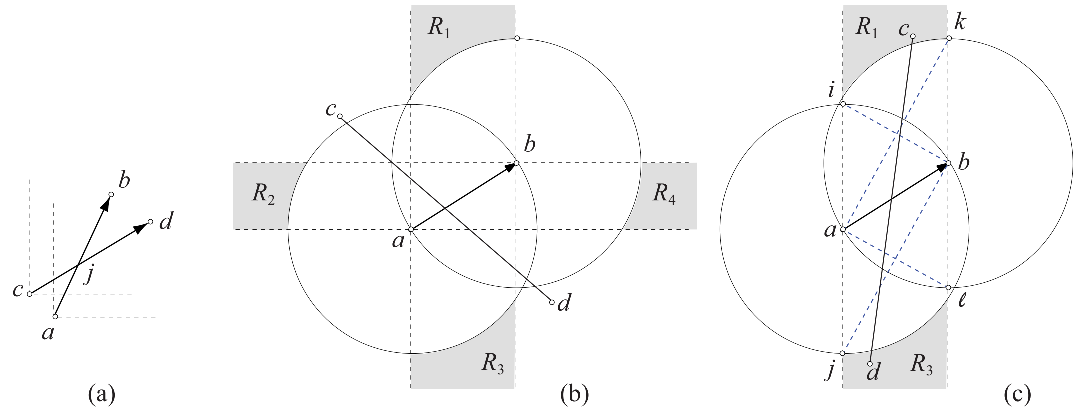

Figure 4: (a) Lemma 3. (b) Lemma 4: $c \notin R_1 \cup R_2 \cup R_3 \cup R_4$ (c) Lemma 4: $c \in R_1$.

If the node c is not inside any of the regions R_i, for $i = \{1, 2, 3, 4\}$, then the nodes a and b are in the same quadrant of c as d. In this case, note that either $\angle cad > \pi/2$ or $\angle cbd > \pi/2$, which implies that either $|ca|$ or $|cb|$ is strictly smaller than $|cd|$. These together show that $\overrightarrow{cd} \notin Y_4$.

So assume that c is in R_i for some $i \in \{1, 2, 3, 4\}$. In this situation, the node d must lie in the region R_j, with $j = (i + 2) \bmod 4$ (with the understanding that $R_0 = R_4$), because otherwise, (i) a and d are in the same quadrant of c and $|ca| < |cd|$ or (ii) b and d are in the same quadrant of c and $|cb| < |cd|$. Either case contradicts the fact $\overrightarrow{cd} \in Y_4$. Consider now the case $c \in R_1$ and $d \in R_3$; the other cases are treated similarly. Let i and j be the intersection points between D_a and the vertical line through a. Similarly, let k and ℓ be the intersection points between D_b and the vertical line through b (see Figure 4c). Since ij is a diameter of D_a, we have that $\angle ibj = \pi/2$ and similarly $\angle kal = \pi/2$. Also note that $\angle cbd \geq \angle ibj = \pi/2$, meaning that $|cd| > |cb|$. Similarly, $\angle cad \geq \angle kal = \pi/2$, meaning that $|cd| > |ca|$. These along with the fact that at least one of a and b is in the same quadrant for c as d, imply that $\overrightarrow{cd} \notin Y_4$. This completes the proof. ∎

Lemma 5 *Let a, b, c, d be four distinct nodes in V, with $c \in Q_1(a)$, such that*

(a) $\overrightarrow{ab} \in Q_1(a)$ and $\overrightarrow{cd} \in Q_2(c)$ are two edges in Y_4 that cross each other.

(b) ad is a shortest side of the quadrilateral $acbd$.

Then $\mathcal{P}_R(a \to d)$ and $\mathcal{P}_R(d \to a)$ have a nonempty intersection.

Proof. The proof consists of two parts showing that the following claims hold: (i) $d \in Q_2(a)$ and (ii) $\mathcal{P}_R(d \to a)$ does not cross ab.

Before we prove these two claims, let us argue that they are sufficient to prove the lemma. Lemma 3 and (i) imply that $\mathcal{P}_R(a \to d)$ cannot cross cd. As a result, $\mathcal{P}_R(a \to d)$ intersects the left side of the rectangle $R(d, a)$. Consider the last edge $\overrightarrow{xy}$ of the path $\mathcal{P}_R(d \to a)$. If this edge crosses the right side of $R(a, d)$, then (ii) implies that y is in the wedge bounded by ab and the upwards vertical ray starting at a; this implies that $|ay| < |ab|$, contradicting the fact that $\overrightarrow{ab}$ is an edge in Y_4. Therefore, $\overrightarrow{xy}$ intersects the bottom side of $R(d, a)$, and the lemma follows (see Figure 5b).

To prove the first claim (i), we observe that the assumptions in the lemma imply that $d \in Q_1(a) \cup Q_2(a)$. Therefore, it suffices to prove that d is not in $Q_1(a)$. Assume to the contrary that

$d \in Q_1(a)$. Since $c \in Q_1(a)$, it must be that $b \in Q_2(c)$; otherwise, $\angle acb \geq \pi/2$, which implies $|ab| > |ac|$, contradicting the fact that $\overrightarrow{ab} \in Y_4$. Let i and j be the intersection points between cd and $\partial D(a, |ab|)$, where i is to the left of j. Since $\angle dbc \geq \angle ibj > \pi/2$, we have $|cb| < |cd|$. This, together with the fact that b and d are in the same quadrant $Q_2(c)$, contradicts the assumption that $\overrightarrow{cd}$ is an edge in Y_4. This completes the proof of claim (i).

Next we prove claim (ii) by contradiction. Thus, we assume that there is an edge $\overrightarrow{xy}$ on the path $\mathcal{P}_R(d \to a)$ that crosses ab. Then necessarily $x \in R(a, d)$ and $y \in Q_1(a) \cup Q_4(a)$. If $y \in Q_4(a)$, then $\angle xay > \pi/2$, meaning that $|xy| > |xa|$, a contradiction to the fact that $\overrightarrow{xy} \in Y_4$. Thus, it must be that $y \in Q_1(a)$, as in Figure 5a. This implies that $|ab| \leq |ay|$, because $\overrightarrow{ab} \in Y_4$.

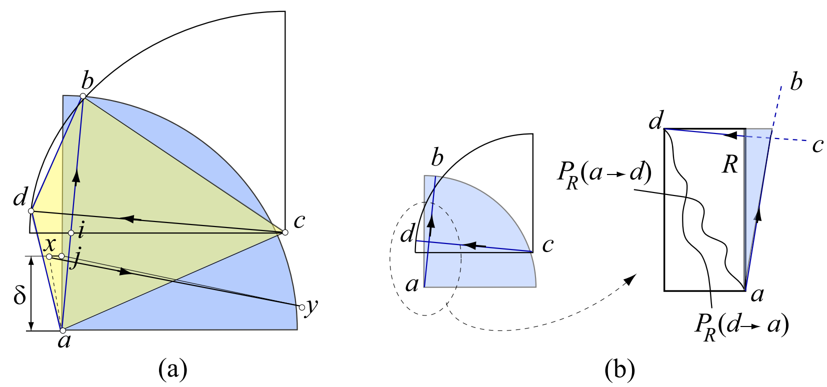

Figure 5: (a) Lemma 5: $xy \in \mathcal{P}_R(d \to a)$ cannot cross ab.

The contradiction to our assumption that $\overrightarrow{xy}$ crosses ab will be obtained by proving that $|xy| > |xa|$. Indeed, this inequality contradicts the fact that $\overrightarrow{xy} \in Y_4$.

Let δ be the distance from x to the horizontal line through a. Our intermediate goal is to show that

$$\delta \leq |ab|/\sqrt{2}. \tag{8}$$

We claim that $\angle acb < \pi/2$. Indeed, if this is not the case, then $|ac| < |ab|$, contradicting the fact that $\overrightarrow{ab}$ is an edge in Y_4. By a similar argument, and using the fact that $\overrightarrow{cd}$ is an edge in Y_4, we obtain the inequality $\angle cbd < \pi/2$. We now consider two cases, depending on the relative lengths of ac and cb.

1. Assume first that $|ac| > |cb|$. If $\angle cad \geq \pi/2$, then $|cd| \geq |ac| > |cb|$, contradicting the fact that $\overrightarrow{cd}$ is an edge in Y_4 (recall that b and d are in the same quadrant of c). Therefore, we have $\angle cad < \pi/2$. Thus far we have established that three angles of the convex quadrilateral $acbd$ are acute. It follows that the fourth one ($\angle adb$) is obtuse. Proposition 2 applied to $\triangle adb$ tells us that

$$|ab|^2 > |ad|^2 + |db|^2 \geq 2|ad|^2,$$

where the latter inequality follows from the assumption that ad is a shortest side of $acbd$ (and, therefore, $|db| \geq |ad|$). Thus, we have that $|ad| \leq |ab|/\sqrt{2}$. This along with the fact that $x \in R(a, d)$ implies inequality (8).

2. Assume now that $|ac| \leq |cb|$. Let i be the intersection point between ab and the horizontal line through c (refer to Figure 5a). Note that $\angle aic \geq \pi/2$ and $\angle bic \leq \pi/2$ (these two angles

sum to π). This along with Proposition 2 applied to triangle $\triangle aic$ shows that

$$|ac|^2 \geq |ai|^2 + |ic|^2.$$

Similarly, Proposition 2 applied to triangle $\triangle bic$ shows that

$$|bc|^2 \leq |bi|^2 + |ic|^2.$$

The two inequalities above along with our assumption that $|ac| \leq |cb|$ imply that $|ai| \leq |bi|$, which in turn implies that $|ai| \leq |ab|/2$, because $|ai|+|ib| = |ab|$. Since x is below i (otherwise, $|cx| < |cd|$, contradicting the fact that $\overrightarrow{cd}$ is an edge in Y_4), we have $\delta \leq |ai|$. It follows that $\delta \leq |ab|/2$.

Finally we derive a contradiction using the now established inequality (8). Let j be the orthogonal projection of x onto the vertical line through a (thus $|aj| = \delta$). Note that $\angle ajy < \pi/2$, because $y \in Q_4(x)$. By Proposition 2 applied to $\triangle ajy$, we have

$$|ay|^2 < |aj|^2 + |jy|^2 = \delta^2 + |jy|^2.$$

Since y and b are in the same quadrant of a, and since $\overrightarrow{ab} \in Y_4$, we have that $|ab| \leq |ay|$. This along with the inequality above and (8) implies that $|jy| \geq |ab|/\sqrt{2} \geq \delta$. By Proposition 2 applied to $\triangle xjy$, we have $|xy|^2 > |xj|^2 + |jy|^2 \geq |xj|^2 + \delta^2 = |xj|^2 + |ja|^2 = |xa|^2$. It follows that $|xy| > |xa|$, contradicting our assumption that $\overrightarrow{xy} \in Y_4$. ∎

Lemma 6 *Let a, b, c, d be four distinct nodes in V, with $c \in Q_1(a)$, such that*

(a) $\overrightarrow{ab} \in Q_1(a)$ and $\overrightarrow{cd} \in Q_3(c)$ are two edges in Y_4 that cross each other.

(b) ad is a shortest side of the quadrilateral $acbd$.

Then $\mathcal{P}_R(d \to a)$ does not cross ab.

Proof. We first show that $d \notin Q_3(a)$. Assume the opposite. Since $c \in Q_1(a)$ and $d \in Q_3(a)$, we have that $\angle cad > \pi/2$. This implies that $|ca| < |cd|$, which along with the fact that $a, d \in Q_3(c)$ contradict the fact that $\overrightarrow{cd} \in Y_4$. Also note that $d \notin Q_1(a)$, since in that case ab and cd could not intersect. In the following we discuss the case $d \in Q_2(a)$; the case $d \in Q_4(a)$ is symmetric.

A first observation is that c must lie below b; otherwise $|cb| < |cd|$ (since $\angle cbd > \pi/2$), which would contradict the fact that $\overrightarrow{cd} \in Y_4$. We now prove by contradiction that there is no edge in $\mathcal{P}_R(d \to a)$ crossing ab. Assume the contrary, and let $\overrightarrow{xy} \in \mathcal{P}_R(d \to a)$ be such an edge. Then necessarily $x \in R(a, d)$ and $\overrightarrow{xy} \in Q_4(x)$. Note that y cannot lie below a; otherwise $|xa| < |xy|$ (since $\angle xay > \pi/2$), which would contradict the fact that $\overrightarrow{xy} \in Y_4$. Also y must lie outside $D(c, |cd|) \cap Q(c, d)$, otherwise $\overrightarrow{cd}$ could not be in Y_4. These together show that y sits to the right of c. See Figure 6(a). Then the following inequalities regarding the quadrilateral $xayb$ must hold:

(i) $|by| > |bc|$, due to the fact that $\angle bcy > \pi/2$.

(ii) $|bx| \geq |bd|$ ($|bx| = |bd|$ if x and d coincide). If x and d are distinct, the inequality $|bx| > |bd|$ follows from the fact that $|cx| \geq |cd|$ (since x is outside $D(c, |cd|)$), and Proposition 1 applied to the quadrilateral $xcbd$:

$$|bd| + |cx| < |bx| + |cd|$$

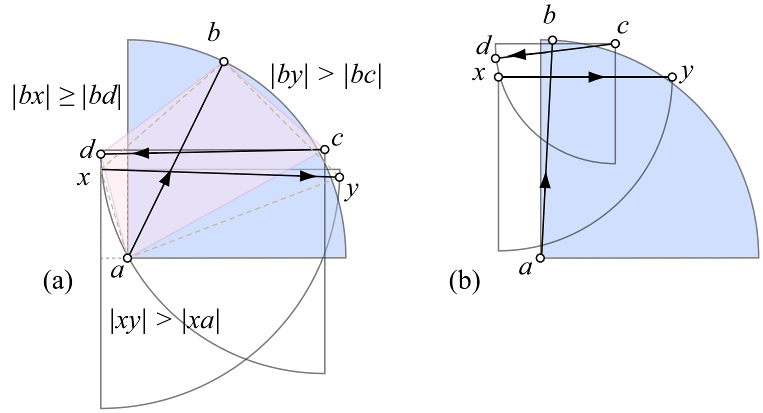

Figure 6: Lemma 6: (a) $\mathcal{P}_R(d \to a)$ does not cross ab. (b) If ad is not the shortest side of $acbd$, the lemma conclusion might not hold.

Inequalities (i) and (ii) show that by and bx are longer than sides of the quadrilateral $acbd$, and so they must be longer than the shortest side of $acbd$, which by assumption (b) of the lemma is ad: $\min\{|bx|, |by|\} \geq |ad| \geq |ax|$ (this latter inequality follows from the fact that $x \in R(d, a)$). Also note that $|ab| \leq |ay|$, since $\overrightarrow{ab} \in Y_4$ and y lies in the same quadrant of a as b. The fact that both diagonals of $xayb$ are in Y_4 enables us to apply Lemma 4(ii) to conclude that ay is not a shortest side of the quadrilateral $xayb$. Thus xa is a shortest side of the quadrilateral $xayb$, and we can use Lemma 4(ii) to claim that

$$|xa| < \min\{|xy|, |ab|\} \leq |xy|.$$

This contradicts our assumption that $\overrightarrow{xy} \in Y_4$. ∎

Figure 6(b) shows that the claim of the lemma might be false without assumption (b). The next lemma relies on all of Lemmas 2–6.

Lemma 7 *Let $a, b, c, d \in V$ be four distinct nodes such that $\overrightarrow{ab} \in Y_4$ crosses $\overrightarrow{cd} \in Y_4$, and let xy be a shortest side of the quadrilateral $abcd$. Then there exist two paths $\mathcal{P}_x$ and $\mathcal{P}_y$ in Y_4, where $\mathcal{P}_x$ has x as an endpoint and $\mathcal{P}_y$ has y as an endpoint, with the following properties:*

- *(a) $\mathcal{P}_x$ and $\mathcal{P}_y$ have a nonempty intersection.*
- *(b) $|\mathcal{P}_x| + |\mathcal{P}_y| \leq 3\sqrt{2}|xy|$.*
- *(c) Each edge on $\mathcal{P}_x \cup \mathcal{P}_y$ is no longer than $|xy|$.*

Proof. Assume without loss of generality that $b \in Q_1(a)$. We discuss the following exhaustive cases:

1. $c \in Q_1(a)$, and $d \in Q_1(c)$. In this case, ab and cd cannot cross each other (by Lemma 3), so this case is finished.
2. $c \in Q_1(a)$, and $d \in Q_2(c)$, as in Figure 7a. Since ab crosses cd, $b \in Q_2(c)$. Since $\overrightarrow{ab} \in Y_4$, $|ab| \leq |ac|$. Since $\overrightarrow{cd} \in Y_4$, $|cd| \leq |cb|$. These along with Lemma 4 imply that ad and db are the only candidates for a shortest edge of $acbd$.

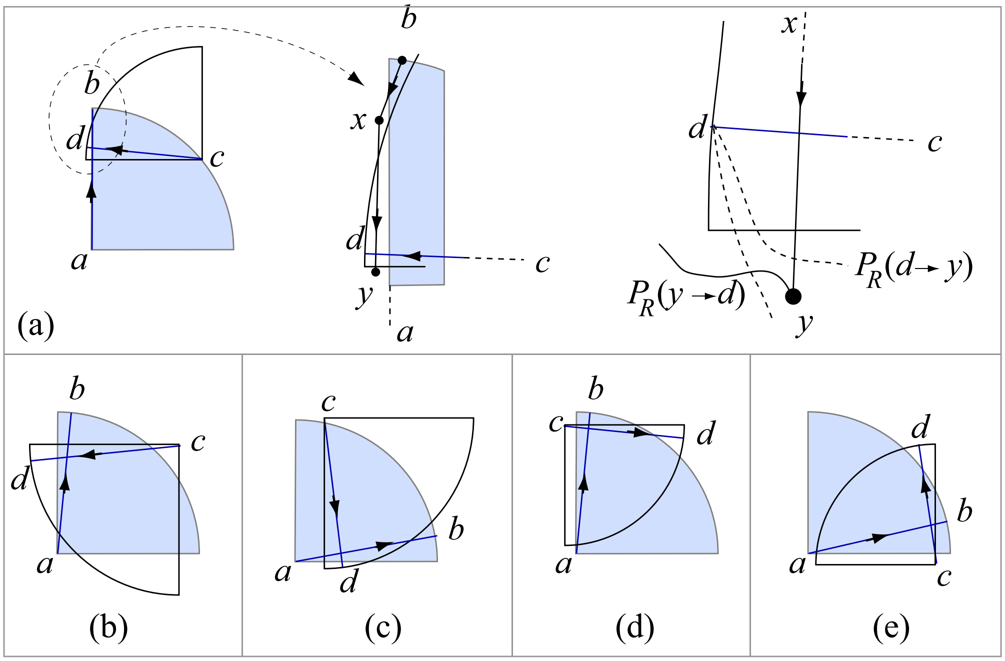

Figure 7: Lemma 7: (a) $c \in Q_1(a)$, and $d \in Q_2(c)$ (b)$c \in Q_1(a)$, and $d \in Q_3(c)$ (c) $c \in Q_2(a)$ (d) $c \in Q_4(a)$.

Assume first that ad is a shortest edge of $acbd$. By Lemma 3, $\mathcal{P}_a = \mathcal{P}_R(a \to d)$ does not cross cd. It follows from Lemma 5 that $\mathcal{P}_a$ and $\mathcal{P}_d = \mathcal{P}_R(d \to a)$ have a nonempty intersection. Furthermore, by Lemma 2, $|\mathcal{P}_a| \le |ad|\sqrt{2}$ and $|\mathcal{P}_d| \le |ad|\sqrt{2}$, and no edge on these paths is longer than $|ad|$, proving the lemma true for this case.

Consider now the case when db is a shortest edge of $acbd$ (see Figure 7a). Note that d is below b (otherwise, $d \in Q_2(c)$ and $|cd| > |cb|$) and, therefore, $b \in Q_1(d)$). By Lemma 3, $\mathcal{P}_d = \mathcal{P}_R(d \to b)$ does not cross ab. If $\mathcal{P}_b = \mathcal{P}_R(b \to d)$ does not cross cd, then $\mathcal{P}_b$ and $\mathcal{P}_d$ have a nonempty intersection, proving the lemma true for this case. Otherwise, there exists $\overrightarrow{xy} \in \mathcal{P}_R(b \to d)$ that crosses cd (see Figure 7a). Define

$$\begin{aligned}\mathcal{P}_b &= \mathcal{P}_R(b \to d) \oplus \mathcal{P}_R(y \to d)\\ \mathcal{P}_d &= \mathcal{P}_R(d \to y)\end{aligned}$$

By Lemma 3, $\mathcal{P}_R(y \to d)$ does not cross cd. Then $\mathcal{P}_b$ and $\mathcal{P}_d$ must have a nonempty intersection. We now show that $\mathcal{P}_b$ and $\mathcal{P}_d$ satisfy conditions (b) and (c) of the lemma. Proposition 1 applied on the quadrilateral $xdyc$ tells us that

$$|xc| + |yd| < |xy| + |cd|$$

We also have that $|cx| \ge |cd|$, since $\overrightarrow{cd} \in Y_4$ and x is in the same quadrant of c as d. This along with the inequality above implies $|yd| < |xy|$. Because $xy \in \mathcal{P}_R(b \to d)$, by Lemma 2 we have that $|xy| \le |bd|$, which along with the previous inequality shows that $|yd| < |bd|$. This along with Lemma 2 shows that condition (c) of the lemma is satisfied. Furthermore, $|\mathcal{P}_R(y \to d)| \le |yd|\sqrt{2}$ and $|\mathcal{P}_R(d \to y)| \le |yd|\sqrt{2}$. It follows that $|\mathcal{P}_b| + |\mathcal{P}_d| \le 3\sqrt{2}|bd|$.

3. $c \in Q_1(a)$, and $d \in Q_3(c)$, as in Figure 7b. Then $|ac| \geq \max\{ab, cd\}$, and by Lemma 4 ac is not a shortest edge of $acbd$. The case when bd is a shortest edge of $acbd$ is settled by Lemmas 3 and 2: Lemma 3 tells us that $\mathcal{P}_d = \mathcal{P}_R(d \to b)$ does not cross ab, and $\mathcal{P}_b = \mathcal{P}_R(b \to d)$ does not cross cd. It follows that $\mathcal{P}_d$ and $\mathcal{P}_b$ have a nonempty intersection. Furthermore, Lemma 2 guarantees that $\mathcal{P}_d$ and $\mathcal{P}_b$ satisfy conditions (b) and (c) of the lemma.

   Consider now the case when ad is a shortest edge of $acbd$; the case when bc is shortest is symmetric. By Lemma 6, $\mathcal{P}_R(d \to a)$ does not cross ab. If $\mathcal{P}_R(a \to d)$ does not cross cd, then this case is settled: $\mathcal{P}_d = \mathcal{P}_R(d \to a)$ and $\mathcal{P}_a = \mathcal{P}_R(a \to d)$ satisfy the three conditions of the lemma. Otherwise, let $\overrightarrow{xy} \in \mathcal{P}_R(a \to d)$ be the edge crossing cd. Arguments similar to the ones used in case 1 above show that

$$
\begin{aligned}
\mathcal{P}_a &= \mathcal{P}_R(a \to d) \oplus \mathcal{P}_R(y \to d) \\
\mathcal{P}_d &= \mathcal{P}_R(d \to y)
\end{aligned}
$$

   are two paths that satisfy the conditions of the lemma.

4. $c \in Q_1(a)$, and $d \in Q_4(c)$, as in Figure 7c. Note that a horizontal reflection of Figure 7c, followed by a rotation of $\pi/2$, depicts a case identical to case 1, which has already been settled.

5. $c \in Q_2(a)$, as in Figure 7d. Note that Figure 7d rotated by $\pi/2$ depicts a case identical to case 1, which has already been settled.

6. $c \in Q_3(a)$. Then it must be that $d \in Q_1(c)$, otherwise cd cannot cross ab. By Lemma 3 however, ab and cd may not cross, unless one of them is not in Y_4.

7. $c \in Q_4(a)$, as in Figure 7e. Note that a vertical reflection of Figure 7e depicts a case identical to case 1, so this case is settled as well.

Having exhausted all cases, we conclude that the lemma holds. ∎

We are now ready to establish the main lemma of this section, showing that there is a short path between the endpoints of two intersecting edges in Y_4.

Lemma 8 *Let $a, b, c, d \in V$ be four distinct nodes such that $\overrightarrow{ab} \in Y_4$ crosses $\overrightarrow{cd} \in Y_4$, and let xy be a shortest side of the quadrilateral $abcd$. Then Y_4 contains a path $p(x, y)$ connecting x and y, of length*

$$|p(x,y)| \leq \frac{6}{\sqrt{2}-1} \cdot |xy|.$$

Furthermore, no edge on $p(x, y)$ is longer than $|xy|$.

Proof. Let $\mathcal{P}_x$ and $\mathcal{P}_y$ be the two paths whose existence in Y_4 is guaranteed by Lemma 7. By condition (c) of Lemma 7, no edge on $\mathcal{P}_x$ and $\mathcal{P}_y$ is longer than $|xy|$. By condition (a) of Lemma 7, $\mathcal{P}_x$ and $\mathcal{P}_y$ have a nonempty intersection. If $\mathcal{P}_x$ and $\mathcal{P}_y$ share a node $u \in V$, then the path

$$p(x,y) = \mathcal{P}_x[x,u] \oplus \mathcal{P}_y[y,u]$$

is a path from x to y in Y_4 no longer than $3\sqrt{2}|xy|$; the length restriction follows from guarantee (b) of Lemma 7. Otherwise, let $\overrightarrow{a'b'} \in \mathcal{P}_x$ and $\overrightarrow{c'd'} \in \mathcal{P}_y$ be two edges crossing each other. Let $x'y'$

be a shortest side of the quadrilateral $a'c'b'd'$, with $x' \in \mathcal{P}_x$ and $y' \in \mathcal{P}_y$. Lemma 7 tells us that $|a'b'| \le |xy|$ and $|c'd'| \le |xy|$. These along with Lemma 4 imply that

$$|x'y'| \le |xy|/\sqrt{2}. \tag{9}$$

This enables us to derive a recursive formula for computing a path $p(x, y) \in Y_4$ as follows:

$$p(x,y) = \begin{cases} x, & \text{if } x = y \\ \mathcal{P}_x[x, x'] \oplus \mathcal{P}_y[y, y'] \oplus p(x', y'), & \text{if } x \neq y \end{cases} \tag{10}$$

Next we use induction on the length of xy to prove the claim of the lemma. The base case corresponds to $x = y$, case in which $p(x, y)$ degenerates to a point and $|p(x, y)| = 0$. To prove the inductive step, pick a shortest side xy of a quadrilateral $acbd$, with $\overrightarrow{ab}, \overrightarrow{cd} \in Y_4$ crossing each other, and assume that the lemma holds for all such sides shorter than xy. Let $p(x, y)$ be the path determined recursively as in (10). By the inductive hypothesis, we have that $p(x', y')$ contains no edges longer than $|x'y'| \le |xy|$, and

$$|p(x', y')| \le \frac{6}{\sqrt{2}-1}|x'y'| \le \frac{6}{2-\sqrt{2}}|xy|.$$

This latter inequality follows from (9). This along with Lemma 7 and formula (10) implies

$$|p(x, y)| \le (3\sqrt{2} + \frac{6}{2-\sqrt{2}}) \cdot |xy| = \frac{6}{\sqrt{2}-1} \cdot |xy|.$$

This completes the proof. ∎

4 Y_4^∞ and Y_4

We prove that every individual edge of Y_4^∞ is spanned by a short path in Y_4. This, along with the result of Theorem 1, establishes that Y_4 is a spanner.

Fix an edge $\overrightarrow{ab} \in Y_4^\infty$. Define an edge or a path as *short* if its length is within a constant factor of $|ab|$. In our proof that ab is spanned by a short path in Y_4, we will make use of the following three statements (which will be proved in the appendix).

S1 If ab is short, then $\mathcal{P}_R(a \to b)$, and therefore its reverse, $\mathcal{P}_R^{-1}(a \to b)$, are short by Lemma 2.

S2 If $ab \in Y_4$ and $cd \in Y_4$ are short, and if ab intersects cd, Lemma 8 shows that then there is a short path between any two of the endpoints of these edges.

S3 If $p(a, b)$ and $p(c, d)$ are short paths that intersect, then there is a short path P between any two of the endpoints of these paths, by **S2**.

Lemma 9 *For any edge $ab \in Y_4^\infty$, there is a short path $p(a, b) \in Y_4$ of length*

$$|p(a, b)| \le (29 + 23\sqrt{2})|ab|.$$

Proof. For the sake of clarity, we only prove here that there is a short path $p(a,b)$, and defer the calculations of the actual stretch factor of $p(a,b)$ to the appendix. Assume without loss of generality that $\overrightarrow{ab} \in Y_4^\infty$, and $\overrightarrow{ab} \in Q_1(a)$. If $\overrightarrow{ab} \in Y_4$, then $p(a,b) = ab$ and the proof is finished. So assume the opposite, and let $\overrightarrow{ac} \in Q_1(a)$ be the edge in Y_4; since $Q_1(a)$ is nonempty, $\overrightarrow{ac}$ exists. Because $\overrightarrow{ac} \in Y_4$ and b is in the same quadrant of a as c, we have that

$$\begin{array}{rcll} |ac| & \le & |ab| & \text{(i)} \\ |bc| & \le & |ac|\sqrt{2} & \text{(ii)} \end{array} \tag{11}$$

Thus both ac and bc are short. And this in turn implies that $\mathcal{P}_R(b \to c)$ is short by **S1**. We next focus on $\mathcal{P}_R(b \to c)$. Let $b' \notin R(b,c)$ be the other endpoint of $\mathcal{P}_R(b \to c)$. We distinguish three cases.

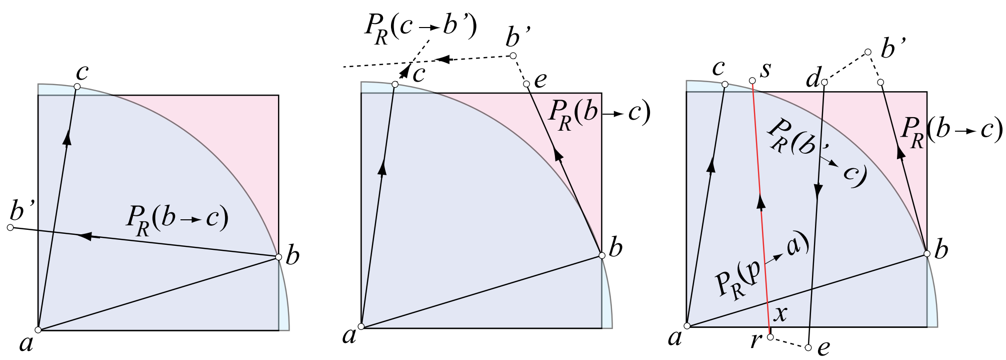

Figure 8: Lemma 9: (a) Case 1: $\mathcal{P}_R(b \to c)$ and ac have a nonempty intersection. (b) Case 2: $\mathcal{P}_R(b' \to a)$ and ab have an empty intersection. (c) Case 3: $\mathcal{P}_R(b' \to a)$ and ab have a non-empty intersection.

Case 1: $\mathcal{P}_R(b \to c)$ and ac intersect. Then by **S3** there is a short path $p(a,b)$ between a and b.

Case 2: $\mathcal{P}_R(b \to c)$ and ac do not intersect, and $\mathcal{P}_R(b' \to a)$ and ab do not intersect (see Figure 8b). Note that because b' is the endpoint of the short path $\mathcal{P}_R(b \to c)$, the triangle inequality on $\triangle abb'$ implies that ab' is short, and therefore $\mathcal{P}_R(b' \to a)$ is short. We consider two cases:

(i) $\mathcal{P}_R(b' \to a)$ intersects ac. Then by **S3** there is a short path $p(a,b')$. So

$$p(a,b) = p(a,b') \oplus \mathcal{P}_R^{-1}(b \to c)$$

is short.

(ii) $\mathcal{P}_R(b' \to a)$ does not intersect ac. Then $\mathcal{P}_R(c \to b')$ must intersect $\mathcal{P}_R(b \to c) \oplus \mathcal{P}_R(b' \to a)$. Next we establish that $b'c$ is short. Let $\overrightarrow{eb'}$ be the last edge of $\mathcal{P}_R(b \to c)$, and so incident to b' (note that e and b may coincide). Because $\mathcal{P}_R(b \to c)$ does not intersect ac, b' and c are in the same quadrant for e. It follows that $|eb'| \le |ec|$ and $\angle b'ec < \pi/2$. These along with Proposition 2 for $\triangle b'ec$ imply that $|b'c|^2 < |b'e|^2 + |ec|^2 \le 2|ec|^2 < 2|bc|^2$ (this latter inequality uses the fact that $\angle bec > \pi/2$, which implies that $|ec| < |bc|$). It follows that

$$|b'c| \;\le\; |bc|\sqrt{2} \;\le\; 2|ac| \qquad \text{(by (11)ii)} \tag{12}$$

Thus $b'c$ is short, and by **S1** we have that $\mathcal{P}_R(c \to b')$ is short. Since $\mathcal{P}_R(c \to b')$ intersects the short path $\mathcal{P}_R(b \to c) \oplus \mathcal{P}_R(b' \to a)$, there is by **S3** a short path $p(c, b)$, and so

$$p(a, b) = ac \oplus p(c, b)$$

is short.

Case 3: $\mathcal{P}_R(b \to c)$ and ac do not intersect, and $\mathcal{P}_R(b' \to a)$ intersects ab (see Figure 8c). If $\mathcal{P}_R(b' \to a)$ intersects ab at a, then $p(a, b) = \mathcal{P}_R(b \to c) \oplus \mathcal{P}_R(b' \to a)$ is short. So assume otherwise, in which case there is an edge $\overrightarrow{de} \in \mathcal{P}_R(b' \to a)$ that crosses ab. Then $d \in Q_1(a)$, $e \in Q_3(a) \cup Q_4(a)$, and e and a are in the same quadrant for d. Note however that e cannot lie in $Q_3(a)$, since in that case $\angle dae > \pi/2$, which would imply $|de| > |da|$, which in turn would imply $\overrightarrow{de} \notin Y_4$. So it must be that $e \in Q_4(a)$.

Next we show that $\mathcal{P}_R(e \to a)$ does not cross ab. Assume the opposite, and let $\overrightarrow{rs} \in \mathcal{P}_R(e \to a)$ cross ab. Then $r \in Q_4(a)$, $s \in Q_1(a) \cup Q_2(a)$, and s and a are in the same quadrant for r. Arguments similar to the ones above show that $s \notin Q_2(a)$, so s must lie in $Q_1(a)$. Let d be the L_∞ distance from a to b. Let x be the projection of r on the horizontal line through a. Then

$$|rs| \geq |rx| + d \geq |rx| + |xa| > |ra| \quad \text{(by the triangle inequality)}$$

Because a and s are in the same quadrant for r, the inequality above contradicts $\overrightarrow{rs} \in Y_4$.

We have established that $\mathcal{P}_R(e \to a)$ does not cross ab. Then $\mathcal{P}_R(a \to e)$ must intersect $\mathcal{P}_R(e \to a) \oplus de$. Note that de is short because it is in the short path $\mathcal{P}_R(b' \to a)$. Thus ae is short, and so $\mathcal{P}_R(a \to e)$ and $\mathcal{P}_R(e \to a)$ are short. Thus we have two intersecting short paths, and so by **S3** there is a short path $p(a, e)$. Then

$$p(a, b) = p(a, e) \oplus \mathcal{P}_R^{-1}(b' \to a) \oplus \mathcal{P}_R^{-1}(b \to c)$$

is short. Calculations deferred to the appendix show that, in each of these cases, the stretch factor for $p(a, b)$ does not exceed $29 + 23\sqrt{2}$. ∎

Our main result follows immediately from Theorem 1 and Lemma 9:

Theorem 2 *Y_4 is a t-spanner, for $t \geq 8(29 + 23\sqrt{2})$.*

5 Conclusion

Our results settle a long-standing open problem, asking whether Y_4 is a spanner or not. We answer this question positively, and establish a loose stretch factor of $8(29 + 23\sqrt{2})$. Experimental results, however, indicate a stretch factor of the order $1 + \sqrt{2}$, a factor of 200 smaller. Finding tighter stretch factors for both Y_4^∞ and Y_4 remain interesting open problems. Establishing whether Y_5 and Y_6 are spanners or not is also open.

References

[1] P. Bose, A. Maheshwari, G. Narasimhan, M. Smid, and N. Zeh. Approximating geometric bottleneck shortest paths. *Computational Geometry: Theory and Applications*, 29:233–249, 2004.

[2] M. Damian, N. Molla, and V. Pinciu. Spanner properties of $\pi/2$-angle Yao graphs. In *Proc. of the 25th European Workshop on Computational Geometry*, pages 21–24, March 2009.

[3] M. Fischer, T. Lukovszki, and M. Ziegler. Geometric searching in walkthrough animations with weak spanners in real time. In *ESA '98: Proc. of the 6th Annual European Symposium on Algorithms*, pages 163–174, 1998.

[4] N. Molla. Yao spanners for wireless ad hoc networks. M.S. Thesis, Department of Computer Science, Villanova University, December 2009.

[5] J.W. Green. A note on the chords of a convex curve. *Portugaliae Mathematica*, 10(3):121–123, 1951.

[6] A.C.-C. Yao. On constructing minimum spanning trees in k-dimensional spaces and related problems. *SIAM Journal on Computing*, 11(4):721–736, 1982.

6 Appendix

6.1 Calculations for the stretch factor of $p(a, b)$ in Lemma 9

We start by computing the stretch factor of the short paths claimed by statements **S2** and **S3**.

S2 If $ab \in Y_4$ and $cd \in Y_4$ are short, and if ab intersects cd, then there is a short path P between any two of the endpoints of these edges, of length

$$|P| \le |ab| + |cd| + 3(2 + \sqrt{2}) \max\{|ab|, |cd|\} \tag{13}$$

This upper bound can be derived as follows. Let xy be a shortest side of the quadrilateral $acbd$. By Lemma 8, Y_4 contains a path $p(x, y)$ no longer than $6(\sqrt{2} + 1)|xy|$. By Lemma 4, $|xy| \le \max\{|ab|, |cd|\}/\sqrt{2}$. These together with the fact that $|P| \le |ab| + |cd| + |p(x, y)|$ yield inequality (13).

S3 If $p(a, b)$ and $p(c, d)$ are short paths that intersect, then there is a short path P between any two of the endpoints of these paths, of length

$$|P| \le |p(a, b)| + |p(c, d)| + 3(2 + \sqrt{2}) \max\{|ab|, |cd|\} \tag{14}$$

This follows immediately from **S2** and the fact that no edge on $p(a, b) \cup p(c, d)$ is longer than $\max\{|ab|, |cd|\}$ (by Lemma 8).

Case 1: $\mathcal{P}_R(b \to c)$ and ac intersect. Then by **S3** we have

$$\begin{array}{rcll} |p(a,b)| & \le & |\mathcal{P}_R(b,c)| + |ac| + 3(2+\sqrt{2}) \max\{|bc|, |ac|\} & \\ & \le & \sqrt{2}|bc| + |ac| + 3(2+\sqrt{2})\sqrt{2}|ac| & \text{(by (7), (11)ii)} \\ & = & 3(3+2\sqrt{2})|ac| \le 3(3+2\sqrt{2})|ab| & \text{(by (11)i)} \end{array}$$

Case 2(i): $\mathcal{P}_R(b \to c)$ and ac do not intersect; $\mathcal{P}_R(b' \to a)$ and ab do not intersect; and $\mathcal{P}_R(b' \to a)$ intersects ac. By **S3**, there is a short path $p(a, b')$ of length

$$\begin{aligned} |p(a,b')| &\leq |\mathcal{P}_R(b',a)| + |ac| + 3(2+\sqrt{2})\max\{|b'a|,|ac|\} \\ &\leq |b'a|\sqrt{2} + |ac| + 3(2+\sqrt{2})\max\{|b'a|,|ac|\} \qquad \text{(by (7))} \end{aligned} \tag{15}$$

Next we establish an upper bound on $|b'a|$. By the triangle inequality,

$$|ab'| < |ac| + |cb'| \leq 3|ac| \qquad \text{(by (12))} \tag{16}$$

Substituting this inequality in (15) yields

$$|p(a,b')| \leq (19 + 12\sqrt{2})|ac| \tag{17}$$

Thus $p(a, b) = p(a, b') \oplus \mathcal{P}_R^{-1}(b \to c)$ is a path in Y_4 of length

$$\begin{aligned} |p(a,b)| &\leq |p(a,b')| + |bc|\sqrt{2} && \text{(by (7))} \\ &\leq |p(a,b')| + 2|ac| && \text{(by (11)ii)} \\ &\leq (21 + 12\sqrt{2})|ac| && \text{(by (17))} \\ &\leq (21 + 12\sqrt{2})|ab| && \text{(by (11)i)} \end{aligned}$$

Case 2(ii): $\mathcal{P}_R(b \to c)$ and ac do not intersect; $\mathcal{P}_R(b' \to a)$ and ab do not intersect; and $\mathcal{P}_R(b' \to a)$ does not intersect ac. Then $\mathcal{P}_R(c \to b')$ must intersect $\mathcal{P}_R(b \to c) \oplus \mathcal{P}_R(b' \to a)$. By **S3** there is a short path $p(c, b)$ of length

$$\begin{aligned} |p(c,b)| &\leq |\mathcal{P}_R(c \to b')| + |\mathcal{P}_R(b \to c)| + |\mathcal{P}_R(b' \to a)| + 3(2+\sqrt{2})\max\{|cb'|,|bc|,|b'a|\} \\ &\leq (|cb'| + |bc| + |b'a|)\sqrt{2} + 3(2+\sqrt{2})\max\{|cb'|,|bc|,|b'a|\} \qquad \text{(by (7))} \end{aligned}$$

Inequalities (11)ii, (12) and (16) imply that $\max\{|cb'|, |bc|, |b'a|\} \leq 3ac$. Substituting in the above, we get

$$\begin{aligned} |p(c,b)| &\leq (2 + \sqrt{2} + 3)\sqrt{2}|ac| + 9(2+\sqrt{2})|ac| \\ &\leq (20 + 14\sqrt{2})|ac| \qquad \text{(by (11)i)} \end{aligned}$$

Thus $p(a, b) = ac \oplus p(c, b)$ is a path in Y_4 from a to b of length

$$|p(a,b)| \leq (21 + 14\sqrt{2})|ac| \leq (21 + 14\sqrt{2})|ab| \qquad \text{(by (11)i)}$$

Case 3: $\mathcal{P}_R(b \to c)$ and ac do not intersect, and $\mathcal{P}_R(b' \to a)$ intersects ab. If $\mathcal{P}_R(b' \to a)$ intersects ab at a, then $p(a, b) = \mathcal{P}_R(b \to c) \oplus \mathcal{P}_R(b' \to a)$ is clearly short and does not exceed the spanning ratio of the lemma. Otherwise, there is an edge $\overrightarrow{de} \in \mathcal{P}_R(b' \to a)$ that crosses ab, and $\mathcal{P}_R(a \to e)$ intersects $\mathcal{P}_R(e \to a) \oplus de$ (as established in the proof of Lemma 9). By **S3** there is a short path $p(a, e)$ of length

$$\begin{aligned} |p(a,e)| &\leq |\mathcal{P}_R(a \to e)| + |\mathcal{P}_R(e \to a)| + |de| + 3(2+\sqrt{2})\max\{|ae|,|de|\} \\ &\leq 2|ae|\sqrt{2} + |de| + 3(2+\sqrt{2})\max\{|ae|,|de|\} \qquad \text{(by (7))} \end{aligned} \tag{18}$$

A loose upper bound on $|ae|$ can be obtained by employing Proposition 1 to the quadrilateral $aebd$: $|ae| + |bd| < |ab| + |de| < |ab| + |ab'|$. Substituting the upper bound for ab' from (16) yields

$$|ae| < |ab| + 3|ac| \leq 4|ab| \tag{19}$$

By Lemma 2, $|de| \leq |ab'|$ (since $de \in \mathcal{P}_R(b' \to a)$), which along with (16) implies

$$|de| \leq 3|ab| \tag{20}$$

Substituting (19) and (20) in (18) yields

$$|p(a,e)| \leq (27 + 20\sqrt{2})|ab|$$

Then

$$p(a,b) = p(a,e) \oplus \mathcal{P}_R^{-1}(b' \to a) \oplus \mathcal{P}_R^{-1}(b \to c)$$

is a path from a to b of length

$$\begin{aligned}
|p(a,b)| &\leq |p(a,e)| + |b'a|\sqrt{2} + |bc|\sqrt{2} && \text{(by (7))} \\
&\leq (27 + 20\sqrt{2})|ab| + 3\sqrt{2}|ab| + 2|ab| && \text{(by (16), (11))} \\
&= (29 + 23\sqrt{2})|ab|
\end{aligned}$$

6.2 Y_k is a Spanner, for $k \geq 7$

Lemma 10 *Let θ be a real number with $0 < \theta < \pi/3$, and let*

$$t = \frac{1 + \sqrt{2 - 2\cos\theta}}{2\cos\theta - 1}.$$

Let a, b, and c be three distinct points in the plane such that $|ac| \leq |ab|$, let $\alpha = \angle bac$, and assume that $0 \leq \alpha \leq \theta$. Then

$$|bc| \leq |ab| - |ac|/t. \tag{21}$$

Proof. Refer to Figure 9. By the Law of Cosines, we have

$$|bc|^2 = |ac|^2 + |ab|^2 - 2|ac| \cdot |ab| \cos\alpha.$$

Since $t > 1$ and $|ac| \leq |ab|$, the right-hand side in (21) is positive, so (21) is equivalent to

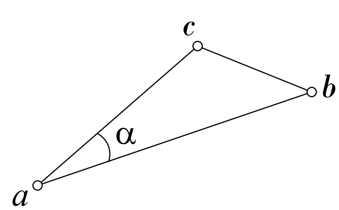

Figure 9: Lemma 1: If $\alpha < 60$ and $|ac| \leq |ab|$, then $|bc| \leq |ab| - |ac|/t$.

$$|bc|^2 \leq (|ab| - |ac|/t)^2 .$$

Thus, we have to show that

$$|ac|^2 + |ab|^2 - 2|ac| \cdot |ab| \cos\alpha \leq (|ab| - |ac|/t)^2 ,$$

which simplifies to

$$\left(1 - 1/t^2\right) |ac| \leq 2(\cos\alpha - 1/t)|ab|. \tag{22}$$

Since $|ac| \leq |ab|$ and $\cos\theta \leq \cos\alpha$, (22) holds if

$$1 - 1/t^2 \leq 2(\cos\theta - 1/t),$$

which can be rewritten as

$$(2\cos\theta - 1)t^2 - 2t + 1 \geq 0. \tag{23}$$

By our choice of t, equality holds in (23). ∎

An immediate consequence of Lemma 10 is the following result.

Theorem 3 *For any θ with $0 < \theta < \pi/3$, the Yao-graph with cones of angle θ, is a t-spanner for*

$$t = \frac{1 + \sqrt{2 - 2\cos\theta}}{2\cos\theta - 1}.$$

Proof. The proof of this claim is by induction on the distances defined by the $\binom{n}{2}$ pairs of nodes. Since $\theta < \pi/3$, any closest pair is connected by an edge in the Yao-graph; this proves the basis of the induction. The induction step follows from Lemma 10. ∎

What happens to the value of t from Lemma 10, if θ gets close to $\pi/3$: Let $\varepsilon = \cos\theta - 1/2$, so that ε is close to zero. Then

$$\begin{aligned} t &= \frac{1}{2\varepsilon} + \sqrt{\frac{1 - 2\varepsilon}{4\varepsilon^2}} \\ &= \frac{1}{2\varepsilon} + \frac{\sqrt{1 - 2\varepsilon}}{2\varepsilon} \\ &\sim \frac{1}{2\varepsilon} + \frac{1 - \varepsilon}{2\varepsilon} \\ &= -\frac{1}{2} + \frac{1}{\varepsilon} \\ &= -\frac{1}{2} + \frac{1}{\cos\theta - 1/2}. \end{aligned}$$