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Band Unfoldings and Prismatoids: A Counterexample

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Abstract

This note shows that the hope expressed in [ADL$^+$07]—that the new algorithm for edge-unfolding any polyhedral band without overlap might lead to an algorithm for unfolding any prismatoid without overlap—cannot be realized. A prismatoid is constructed whose sides constitute a nested polyhedral band, with the property that every placement of the prismatoid top face overlaps with the band unfolding.

1 Introduction

An *edge unfolding* of a polyhedron is a cutting of its surface along edges so that it unfolds flat to a single, non-overlapping piece in the plane. Few classes of convex polyhedra are known to have edge unfoldings. In particular, it is unknown if the class of prismatoids have edge unfoldings. A *prismatoid* is the convex hull of two convex polygons A and B lying in parallel planes. See [DO07, Chap. 22] for background on this problem.

Recently an edge unfolding result has been obtained for bands. A *polyhedral band* is the intersection of the surface of a convex polyhedron with the space between two parallel planes, as long as this space does not contain any vertices of the polyhedron. In [ADL$^+$04] [ADL$^+$07], it was established that *nested polyhedral bands*, where the top polygon projects orthogonally inside the bottom or vice versa, have an edge unfolding. This result was extended in [Alo05] to remove the nesting restriction, and to permit vertices along the top and bottom rims.

A prismatoid can be viewed as a polyhedral band closed by top and bottom convex polygons A and B. As there has been no success achieving a "volcano unfolding" of a prismatoid [DO07, pp. 321–3], the suggestion made in [ADL$^+$07] is natural:

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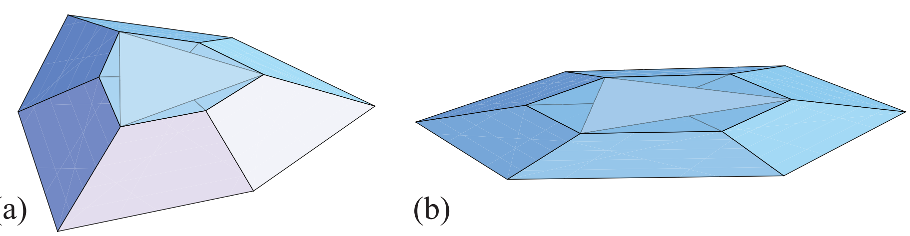
"with it established that arbitrary bands can be unfolded without overlap, it remains interesting to see whether this can lead to a non-overlapping unfolding of prismatoids, including the top and bottom faces. It is natural to hope that these faces could be nestled on opposite sides of the unfolded band, but we do not know how to ensure non-overlap."

The purpose of this note is to dash this hope by constructing a prismatoid, whose side faces constitute a nested polyhedral band, such that, for every edge unfolding of the band, the top A cannot be attached to the band without overlap.

2 Prismatoid Description

There are two key aspects to the design of the prismatoid counterexample. First, the top face A has three acute angles. It is a hexagon formed by replacing each side of an equilateral triangle with two nearly collinear edges. Figure 1(a) illustrates the basic structure, exaggerating the 3D height. Second, the prismatoid is nearly flat, as in (b) of the figure.



(a) (b)

Figure 1: (a) The prismatoid with $z/y = 1/2$; (b) The prismatoid with $z/y = 0.18$ leading to $\delta = \epsilon \approx 1°$.

Figure 2 shows an overhead view of the prismatoid, and establishes notation. The six vertices of A are $(a_0, \ldots, a_5)$, and each is connected to its counterpart b_i on the bottom face B. The side faces $(a_i, b_i, b_{i+1}, a_{i+1})$ are each parallelograms. Three parameters determine the dimensions of the shape:

1. z, the distance between the planes containing A and B.
2. y, the y-distance between a_0 and b_0, as illustrated in the figure.
3. h, the offset of a_0 with respect to the edge of the equilateral triangle inscribed in A (again illustrated in the figure).

Let δ be the curvature, i.e., the angle deficit, at a_0, and let ϵ be the curvature at a_1. Cutting and flattening a vertex opens it by an amount equal to the curvature. By choosing z/y small, we can make these curvatures small. And—although this is not essential—adjustment of the parameters permits achieving $\delta = \epsilon$ (e.g., as in Figure 1(b)).

The counterexample works for a variety of combinations of the three parameters. No attempt has been made to delimit the precise set of parameter combinations that lead to overlap, as choosing δ and ϵ sufficiently small suffices.

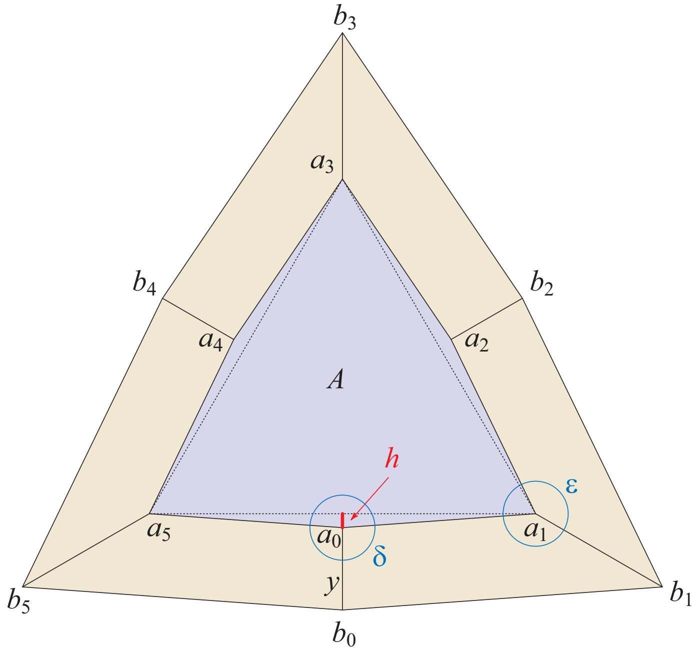


Figure 2: Overhead view of prismatoid. If an origin is set at the midpoint of the bottom side of the equilateral triangle on A, then $a_0 = (0, -h, 0)$ and $b_0 = (0, -(h+y), -z)$.

3 Unfolding

Because of the symmetry of A, there are only two distinct band edge-cuts that need be considered: cutting at the apex of the equilateral triangle, say, edge a_3b_3, or cutting at a side vertex, say, edge a_0b_0. Because we choose δ and ϵ to be small, the band opens only a little. Figure 3(a-c) illustrates the opening at a_3, and (d-f) the opening at a_0. For the opening at a_3, there are only three distinct edges at which to attach A: along edge a_0a_1, or a_1a_2, or a_2a_3. All three attachments lead to overlaps. Similarly, cutting at a_0 leaves three distinct attachment edges, a_3a_4, a_4a_5, and a_5a_0, each of which leads to overlap.

The key here is the three acute angles of A, for it is acute angles which lead to overlap. One might be cut (e.g., a_3), but that still leaves two to thwart all placements. The example can be viewed as an extension of the (non-nested) prismatoid described in Figs. 5, 6, and 7 of [BO07], which has two acute angles, and so thwarted one particular unfolding but not all unfoldings.

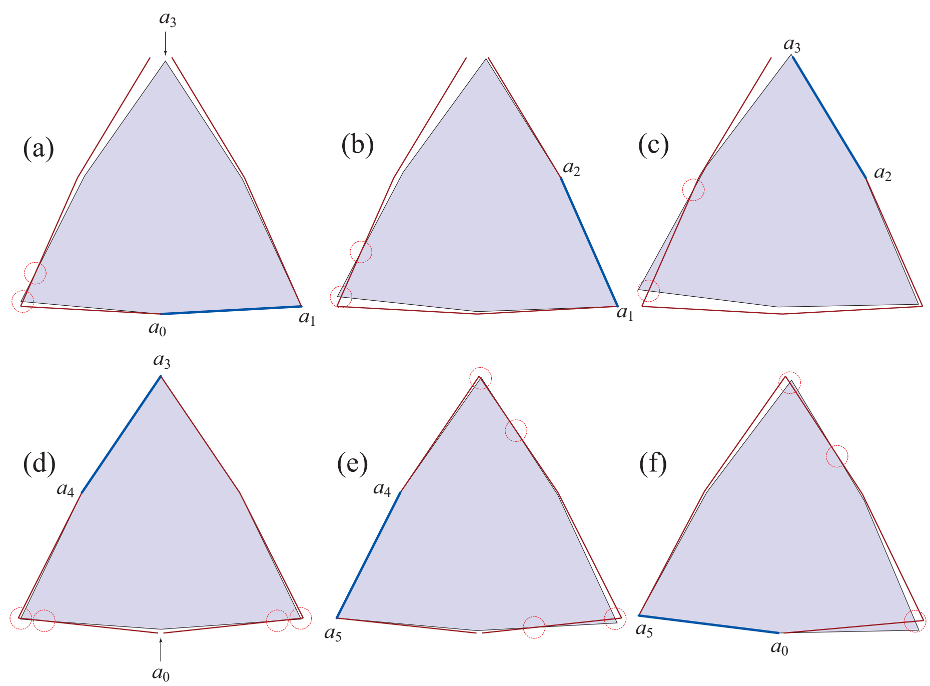


Figure 3: Placements of A when a_3 is cut (top row) and when a_0 is cut (bottom row). The band rim is shown red; the band itself lies outside the rim (cf. Fig. 1(b)). The attachment edge of the band to A is blue. Circles indicate overlap. In this figure, $\delta = \epsilon = 1^\circ$.

References

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[Alo05] Greg Aloupis. *Reconfigurations of Polygonal Structures*. PhD thesis, McGill Univ., School Comput. Sci., 2005.

[BO07] Alex Benton and Joseph O'Rourke. Unfolding polyhedra via cut-tree truncation. In *Proc. 19th Canad. Conf. Comput. Geom.*, pages 77–80, 2007.

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