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Single Vertex Flat Foldability

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Abstract

Origami is the art of folding paper into particular shapes and designs.The paper will demonstrate the use of unfolded and folded crease patterns and the circuits these crease patterns create. This paper will explore the criteria required for flat foldability with a single vertex. The results will show the need for an even number of creases and a difference of 2 between total mountain and valley creases. Other results discussed in this paper include the number of layers at any point must be even and the Kawasaki-Justin Theorem using the alternating sum of the angles to obtain flat foldability.

1 Introduction

Origami is the folding of paper into different shapes and figures. Normal associations with the term origami produces ideas of 3D shapes or figures. However the beginning stages of the folding involved flat folds. A flat fold involves making several folds in a piece of paper ultimately resulting in the paper lying flat in one plane. There are many folds that can make a piece of paper fold flat. However, when using only a single vertex there is certain criteria that is required for a flat foldability to be obtained. The next section, section 2, explains the specifics of how to fold the paper, shows base cases, and diagrams of examples of flat single vertex folds. Section 3 shows and proves certain criteria required to have a piece of paper fold flat. These include needing an even number of total creases, needing a certain amount of mountain creases M and valley creases V, and needing M and V to differ by 2. Other results we found are the sum of alternating angles equals zero and the number of layers at any point on the folded paper results in an even number of layers.

2 Background

A single sheet of origami paper is colored on one side and white on the other. Defining the colored side of the paper to be the top then any crease that is convex will be called a M, mountain crease and any crease that is concave will be called a

V, valley crease. Figure 1 shows an example of what mountain and valley creases look like.

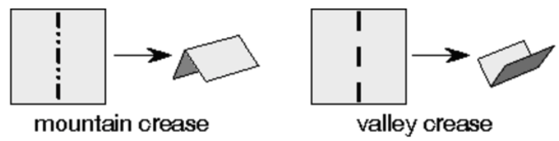

Figure 1: Mountain and Valley Creases

The total number of creases C is defined by the sum of the number of mountain and valley creases (i.e. M+V=C). We define a crease as going from the vertex to an edge. Creases do not need to go from one edge of the paper to the other. If a single fold creates a crease that goes from one edge of the paper to the other we define this to be two creases not one since a crease is said to go from the vertex to the edge and not from edge to edge. The vertex is defined by the intersection of two or more creases. In this study we will be only using one vertex. This means that every crease created must come from a single point on the paper and extend to the edge of the paper. Folds that involve creases extending from another point other than the single vertex identified has introduced another vertex and will not hold for this research.

2.1 Base cases

A flat piece of paper that has not been folded is the zero case in this study and is disregarded as a true case. Figure 2 shows an example of what this looks like.

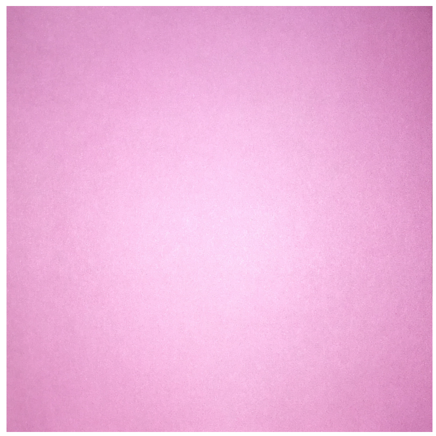

Figure 2: Zero case

Our base case for this study is a piece of paper folded in half once as shown in Figure 3. There is one vertex in this example. However, we cannot define the vertex explicitly because in the base case there is an infinitely many number of vertex points, these points being anywhere along the fold.

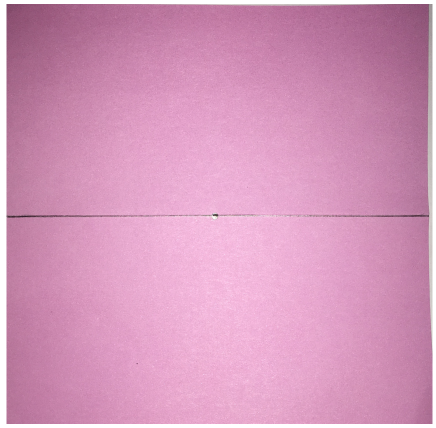

Figure 3: Base Case

So without loss of generality identify a single vertex as the midpoint of our fold. By identifying a vertex we can now see that the base case has 2 mountain creases and 0 valley creases.

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2.2 Diagrams

There are three main ways to look at and diagram single flat vertex folds.

Figure 4 shows an unfolded diagram of what a single vertex flat fold looks like. For sake of clarity solid lines denote a mountain crease and lines with a dash pattern denotes a valley crease. Each crease is labeled C_1,...,C_n (where order of labeling does not matter but it used for clarity). The example fold in figure 4 shows a fold with a total of 6 creases, 4 mountain, and 2 valley. We will use this example to aid in understanding moving forward in the paper.

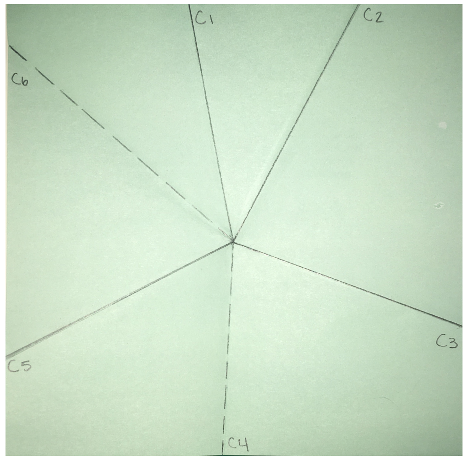

Figure 4: Unfolded

Figure 5 shows the paper from Figure 4 completely folded. This view allows us to see how the unfolded paper looks when it is folded flat and clearly see the single vertex.

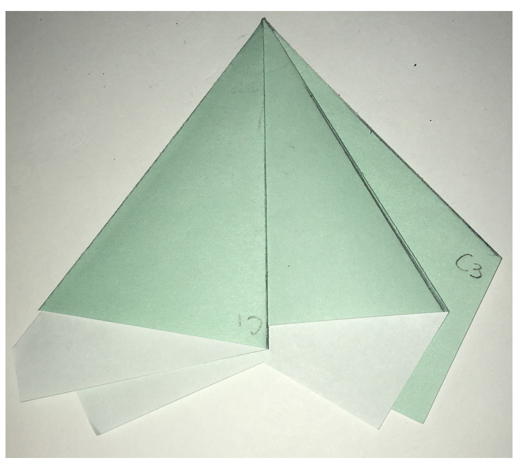

Figure 5: Folded

Figure 6 is a diagram of the circuit created by the folds from the example in Figure 4. The circuit shows the circular pattern created by the creases of the folded paper.

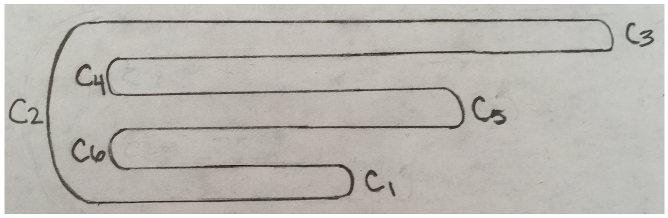

Figure 6: Circuit

To better understand how the circuit is created it helps to take a piece of paper folded flat (as seen in Figure 5) and cutting off the tip or the vertex. This cut can be seen in Figure 7.

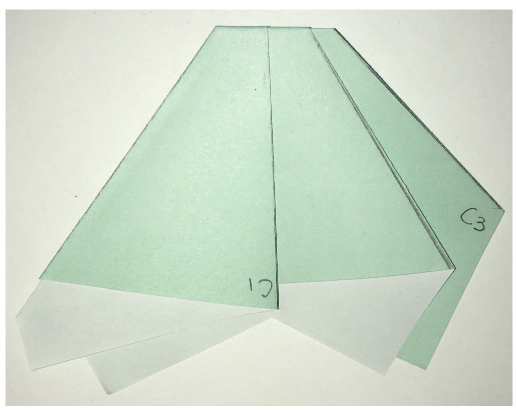

Figure 7: Folded with top cut

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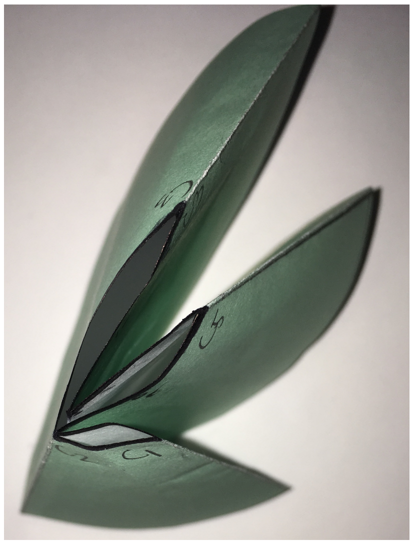

Figure 8: Paper view of circuit

Once the top of the paper is cut off we can now look at an areal view of our fold as seen in Figure 8. From Figure 8 we can see how we can trace each layer of paper around until we make an entire circle. This overview is used to help construct the circuit in Figure 6. If we start tracing on the layer of paper closest to the bottom left hand corner and move to the right we can construct the exact circuit in Figure 6. Looking at the vertex a zig-zag circuit pattern is created from the mountain and valley creases. The circuit starts at an arbitrary point p and moves around through all the creases rotating either clockwise or counterclockwise depending on the type of crease. The circuit shows how we can make our way around the creases of the vertex.

3 Results

In order for a piece of paper to fold flat with only a single vertex there must be a certain amount and certain types of creases involved.

Theorem 3.1. *In a flat single vertex fold the number of M, mountain creases and V, valley creases differ by 2. i.e.* $|V - M| = 2$

Proof. Consider a single vertex flat fold with C, creases labeled C_1,...,C_n. Using the circular circuit start at a point p at 0 degrees. Without loss of generality begin moving around the circuit to the right. Every time you encounter a M, mountain crease rotate -180 degrees (counterclockwise). Every time you encounter a V, valley crease rotate 180 degrees (clockwise). To get back to the beginning point p a 360 degree rotation must be completed. Thus 180V - 180M= 360. Thus $|V - M| = 2$. Therefore, the number of M and V creases differs by 2. □

This conclusion is why in Figure 3 of the base we define there to be two mountain creases. To better understand how this proof uses the circuit it will help to look at our example in Figures 4-8. Take Figure 6 and pick an arbitrary point p. This is shown in Figure 9.

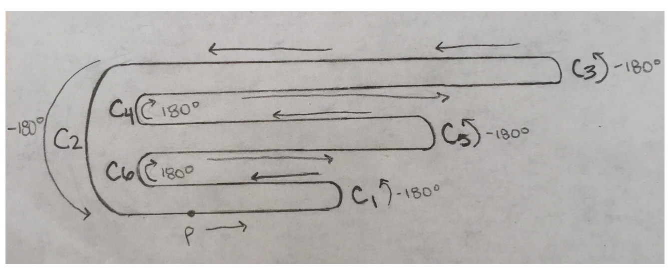

Figure 9: Circuit example

Starting at that arbitrary point p begin to move to the right of our circuit. The first crease we encounter is C_1. From Figure 4 we know C_1 is a mountain crease so we rotate and turn -180 (counterclockwise) degrees and continue until we encounter

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our next crease C_6. Again looking at Figure 4 we know C_6 is a valley crease so we rotate and turn 180 (clockwise) and continue until we encounter another crease. Continuing in this matter we will only be able to complete the circuit and come back to point p if we encounter each crease once and fully complete the circuit. If we look at figure 4 we can clearly see how this theorem holds true in our example. There are 4 mountain creases and 2 valley creases making for a difference of 2.

Theorem 3.1 ia known as the Maekawa-Justin Theorem and an alternate proof can be found in *Project Origami* [1].

Theorem 3.2. *In a flat single vertex fold the number of total creases, C, must be even.*

Proof. Using Theorem 3.1 we know M-V=2. Thus, M=2+V. Then, we know the total number of creases, C=M+V.

$$\Rightarrow C = (2 + V) + V$$

$$\Rightarrow C = 2 + 2V$$

$$\Rightarrow C = 2(1 + V)$$

Therefore, C is an even number. Thus the total number of creases must be even.

□

Theorem 3.2 is known as the Even Degree Theorem and an alternate proof can be found in *Project Orgiami* [1].

Next we consider the layers formed. By putting our pencil anywhere on the folded paper the number of layers under the pencil at that point is even. Looking at our original example from Figure 4 and the circuit that fold created as seen in Figure 6, we will now put in a perpendicular line L. This can be seen in Figure 10.

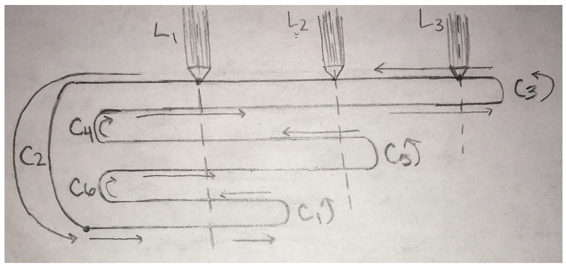

Figure 10: Circuit with line L

We will follow along the circuit once again rotating appropriately as outlined in the proof of Theorem 3.1. From the figure we can see how each encounter of a crease results in a change in direction, either right to left or left to right thus causing at least 2 layers of paper. As we can see in Figure 10 any interesting line L that we pick will always result in an even number of layers. If we look at the base case in Figure three we can see that no matter where we place our pencil we end up with 2 layers which is an even number of layers.

We must note though that this does not imply that the number of layers at the thickest point equals the number of creases. Figure 11 shows an example of a fold that has 6 creases but at the thickest point of the fold the total number of layers is 4, which is an even number but does not equal the total number of creases.

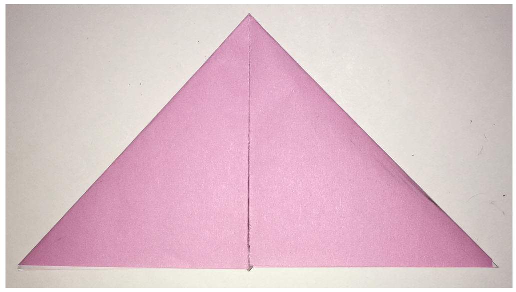

Figure 11: Layers Counterexample

Figure 12 shows the circuit for Figure 11. Here, we can see that if we pick a point at the thickest part of the paper the total number of layers is 4 which does not equal the total number of creases which is 6. However, any point we do pick will result in an even number of layers.

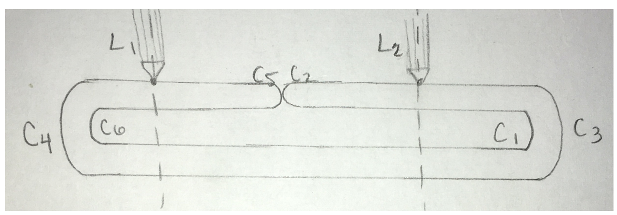

Figure 12: Circuit counterexample

Thus we can only conclude that the number of layers at any point on the folded paper is even.

Theorem 3.3. *The number of layers at any point on the folded paper is even.*

Proof. Using the circular path of the circuit draw an arbitrary perpendicular interesting line L. Then every time line L is crossed in the circuit indicates a new layer of paper. Suppose we start at the intersecting line L on the circuit and move to the right. This movement to the right indicates one layer of paper. Once a crease is encountered rotate the appropriate direction as indicated in the proof for Theorem 3.1. The rotation will lead to a direction shift back to the left causing line L to be crossed again indicating another layer of paper. Thus every shift to the right will cause a shift ot the left and vice versa. For every shift to the right there must be a shift back to the left in order for the circuit to be completed. Continue around the circuit in this matter until you return to the starting point and thus will result in an even number of layers at the intersecting line L. □

We will now look at the angle's between each of the creases. Figure 13 shows the example from Figure 4 with the measurement of each angle calculated simply by using a protractor to measure each angle. Any paper that is flat foldable with a single vertex if the alternating sums of the angles is equal to zero.

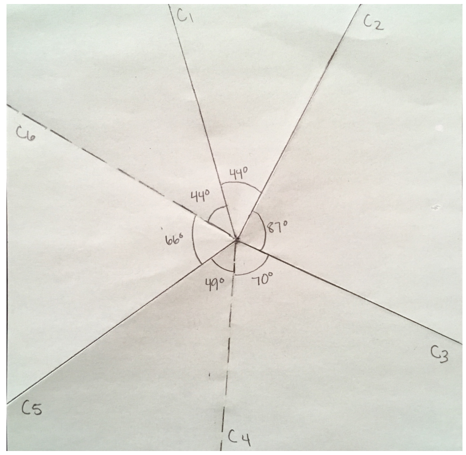

Figure 13: Unfolded with degree measurements

To better see how the alternating sums of the angles is equal to zero let's look at a simpler example. Figure 14 shows an unfolded view of a different flat foldable crease pattern.

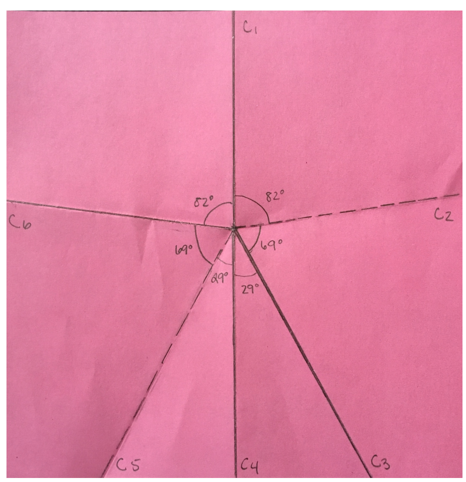

Figure 14: Unfolded with degree measurements

Each angle appears in a pair. Thus if we alternate the sum of the angles we get 82° - 82° + 69° - 69° + 29° - 29° = 0° . This example makes it very easy to see how when alternating the sums of the angles we will get zero since each angle gets canceled out by its pair

We may be led to believe that all folds will result in pairs of angles that will clearly cancel each other out resulting in zero but let's look back at our original example in Figure 13. There is only one pair of similar angles in Figure 13. The other 4 angles do not have pairs.

We will now modify the circuit in Figure 6 to show the angular distance between each crease. This is shown in Figure 15.

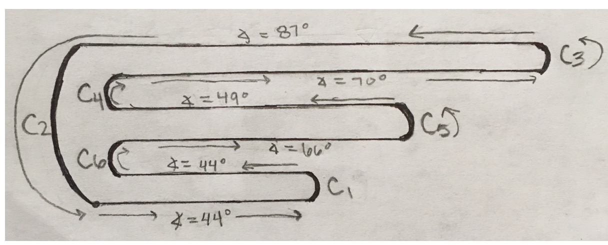

Figure 15: Circuit with angles

From this circuit we can see how our movement this time around the circuit does not just focus on what direction we are moving but how far we are moving. The distance between each curve is the measurement of the angle. From this example we get: 44° - 44° + 66° - 49° + 70° - 87° = 0°

Our base case has two angles both 180°. Thus 180° - 180° = 0

Theorem 3.4 (Kawasaki-Justin). *A set of an even number of creases meeting at a vertex folds flat if, and only if, the alternating sum of the angles is zero:* $\theta_1 - \theta_2 + \theta_3 - \theta_4 + ... + \theta_{n\text{-}1} - \theta_n = 0$ *[2]*

Proof. Consider a single vertex flat fold with consecutive angles $\alpha_1 ... \alpha_\mathrm{n}$ Using the circuit start at a crease and without loss of generality begin moving to the right paying attention to the angular distance traveled. Each crease that is encountered results in a reversal in direction leading to alternating sums of the angular distance traveled. If you continue around the circuit in this manner you will end up back where you started. Thus each angle will either be added or subtracted.

Working the other way we now assume that the alternating sum of the determined angles is zero. In order to show that the paper can fold flat we must create a mountain-valley crease pattern. Working from one cut end work around the paper and draw alternating mountain and valley creases until the other cut end is reached. Then beginning at the first cut end begin to fold the paper according to the creases assigned. This would be like folding the paper accordian style until you reach the other cut end. Once the paper is folded the two cut ends will line up to be glued back together assigning a mountain or valley crease to the two cut ends that now make one crease and results in a paper that is folded flat. □

An alternate proof for the Kawasaki-Justin Theorem can be found in *Project Origami* [1]. Figures 16 and 17 show how the second part of the Kawasaki-Justin

Theorem works. Figure 16 shows the cut that should be made from the edge of the paper to the vertex. We can then see the alternating mountain and valley crease assignment

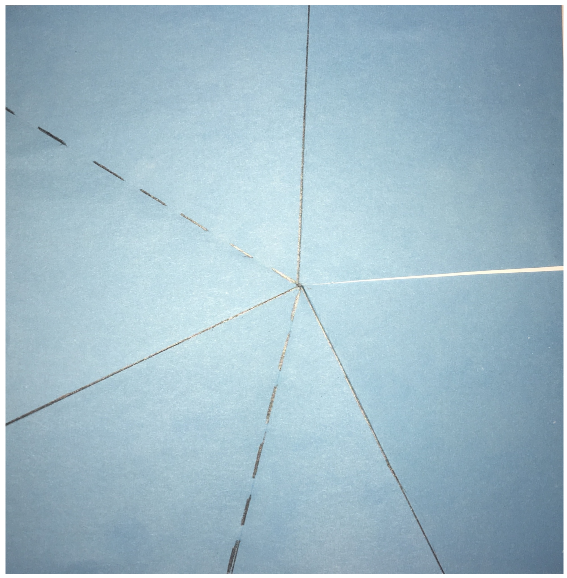

Figure 16: Unfolded crease pattern with one cut crease

If we then fold the paper according to the alternating crease assignment we will get the folded paper in Figure 17. Here, we can see how the two cut edges seamlessly match up on the right side of the folded paper creating a last mountain crease and folding completely flat.

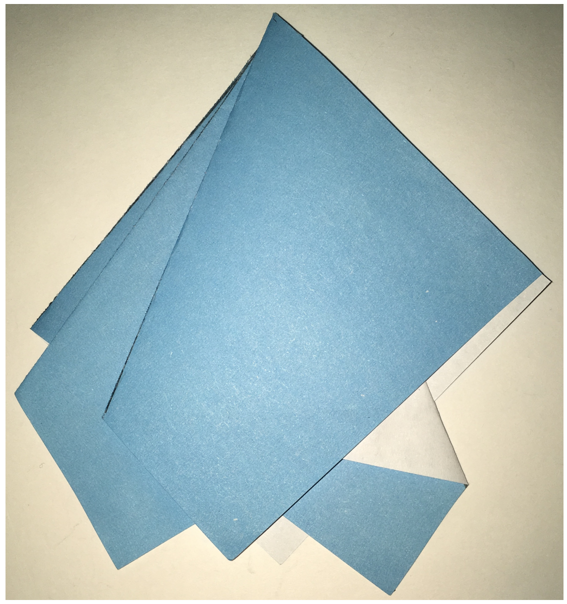

Figure 17: Folded crease pattern with one cut crease

A fold that could potentially present a counterexample to the Kawasaki-Justin Theorem is shown in Figure 17.

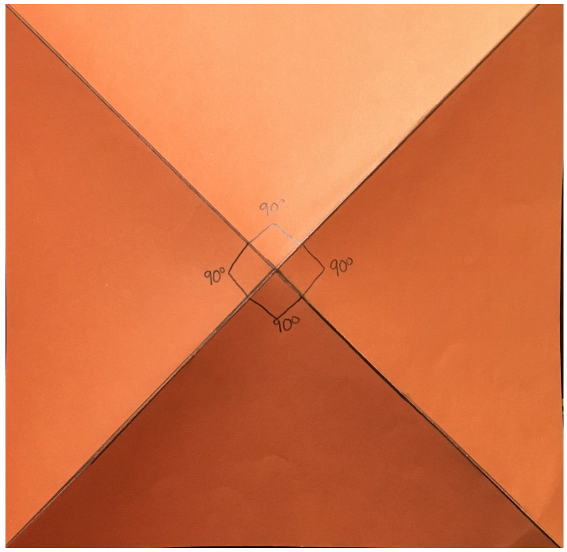

Figure 18: Four mountain crease fold

This Figure shows a crease pattern that contains four mountain creases and four 90° angles. Thus 90° - 90° + 90° - 90° = 0. However, this crease pattern does not allow the paper to fold flat. We would want to conclude in this case that this is a counterexample proving the Kawasaki-Justin Theorem inaccurate but this fold in fact cannot be used as a counterexample. The Kawasaki-Justin Theorem states that if the alternating sum of the angles is equal to zero than there is a set of an even number of creases meeting at a vertex folds flat. Since the theorem does not claim that if the alternating sum of the angles is equal to zero than all crease patterns will fold flat we cannot use this example as a counterexample.

References

[1] Thomas Hull, 2006: *Project Origami*. A K Peters, Ltd., 235.

[2] Joseph O'Rourke, 2011: *How to Fold it*. Everbest, 177.