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Artículo de Investigación

HEURISTIC INSTRUCTION FOR WAVE EQUATION PROBLEM-SOLVING USING VARIABLE SEPARATION METHOD

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ABSTRACT

This study focuses on heuristic instruction as a method of logical thought processes. The objective is to raise and establish a framework for problem-solving in the application of separation of variables for the wave equation. This leads to a simple and easy pathway to make the topic at a manageable level for the students. This paper proposes a base problem, over which the heuristic instruction is applied. Eventually, the result is the determination of a system that allows a simple approach to avoid difficulties in solving problems in which the separation of variables in the equation of the wave is employed. In order to determine the impact of heuristic in students learning gain, two groups of 20 students each (experimental and control) were subject to a pre-and post-test. To the control group, the scheme proposed was not employed. In contrast to the experimental the proposed approach was employed. The results reveal that the heuristic scheme for wave equation problem-solving has a significant impact on students' learning process.

Key words: Physics, Heuristic Instruction, Differential Equations, Wave Equation.

INSTRUCCIÓN HEURÍSTICA PARA RESOLVER PROBLEMAS DE ECUACIÓN DE ONDA MEDIANTE EL MÉTODO SEPARACIÓN DE VARIABLES

RESUMEN

Este estudio se centra en la instrucción heurística como un método de procesos de pensamiento lógico. El objetivo es plantear y establecer un marco para la resolución de problemas en la aplicación de separación de variables para la ecuación de onda. Esto conduce a un camino simple y fácil para que el tema sea manejable para los estudiantes. Este documento propone un problema de base, sobre el cual se aplica la instrucción heurística. Eventualmente, el resultado es la determinación de un esquema que evite dificultades para resolver problemas en los que se emplea la separación de variables en la ecuación de la onda. Con el fin de determinar el impacto de la heurística en la ganancia de aprendizaje de los estudiantes, dos grupos de 20 estudiantes cada uno (experimental y de control) fueron sujetos a una prueba previa y posterior. Para el grupo de control, el esquema propuesto no fue empleado. Por el contrario, para el grupo experimental, se empleó el enfoque propuesto. Los resultados revelan que el esquema heurístico para la resolución de problemas de ecuaciones de ondas tiene un impacto significativo en el proceso de aprendizaje de los estudiantes.

Palabras clave: Física, instrucción heurística, ecuaciones diferenciales, ecuación de onda.

INSTRUÇÃO HEURISTICA PARA RESOLVER PROBLEMAS DA EQUAÇÃO DE ONDA MEDIANTE O MÉTODO SEPARAÇÃO DE VARIAVEIS

RESUMO

Este estudo se baseia na instrução heurística como um método de processos de pensamento lógico. O objetivo é estabelecer um marco para a resolução de problemas na aplicação de separação de variáveis para a equação de onda. Conduzindo um caminho simples e fácil para que o tema seja de fácil manipulação para os estudantes. Este documento propõe um problema de base sobre o qual se aplica a instrução heurística. Eventualmente, o resultado é a determinação de um esquema que evita as dificuldades para resolver problemas nos quais é empregada a separação de variáveis na equação de onda. Com a finalidade de determinar o impacto da heurística no aprendizado dos estudantes, dois grupos de 20 estudantes cada um (experimental e de controle) foram sujeitos a uma prova previa e posterior. Para o grupo de controle, o esquema proposto. Os resultados revelam que o esquema heurístico para a resolução de problemas de equações de ondas têm um impacto significativo no processo de aprendizado dos estudantes.

Palavra chaves: Física, instrução heurística, equações diferenciais, equação de onda.

1. INTRODUCTION

Teaching Thinking means to be on the student to establish links between certain objects and tasks, and the corresponding response actions (Cabrera *et al*., 2006). Problem-solving is a complex mental process (Gilar, 2003) since is a pathway to apply the knowledge in real situations. When it comes to solving-problems of partial differential equations in order to describe a physical phenomenon, it becomes difficult for the student because this involves the application of many criteria such as the variables separation, Fourier series, Laplace transform, etc.

Nowadays, students solve problems based on a previous example given during the lectures. Nevertheless, they do not realize about what they do, that is if some condition change in the problem they may not be able to solve it. On the other hand, instructors must conscious that they are not programing machines, and presenting problem-solving as a recursive process is not the solution. Therefore, the relevance of this study lies in the development of a scheme or algorithm for problem-solving of wave equation using the variable separation method, which can be created based on basic concepts and logic criterion. The heuristic appears as a comprehensive rule of practical reasoning which can be employed for problem-solving. For any topic, both students and instructors must be protagonists of the heuristic instruction in order to teach and learn in a conscious and planned way.

The heuristic comes from the Greek εὑρίσκειν, meaning to find, discover, invent, etc. This is a systemic strategy for immediately positive innovations, is defined as the art and science of discovery and invention, in addition to problem-solving through lateral thinking or divergent thinking, which constitutes creativity.

A problem is formed within a psychological structure, as follows: 1) It starts from data, 2) the analysis of data, 3) Establishing relationships between data, 4) Debugging information, 5) development of a particular strategy suited to the problem. Under this framework, choice and decision making are crucial, as they provide guidance to possible solutions of the problem, which is closely related with heuristics (Rocio & Elizabeth, 2004).

The use of heuristics instruction in the teaching-learning process is critical because it helps to achieve: 1) student's cognitive independence; 2) The integration of new knowledge; 3) Training of mental abilities such as intuition, productivity, originality of the solutions, creativity, and so on. This last point, is the most relevant since at this level the student has developed an intellectual ability to the highest level. This way of problem-solving was not well received in academic circles, apparently due to their limited logical and mathematical rigor. However, thanks to its practical potential to solve real problems were slowly opening the door to heuristic methods, especially from the 60s of XX century (Abrosio, 2011). Currently, in Math continue the development of heuristic methods and are increasing the range of their applications, and their variety of approaches such as Celia & Richard (2017). The relevance of the heuristics lies in the visualization and solve problems that were previously too complex and unthinkable in previous generations.

2. METHODOLOGY

2.1. Research Participants

This study incorporated the participation of 40 students (34 men and 6 women) from an Ecuadorian University. The students were between 20 and 25 years old and they were registered in a course of Calculus III. Two groups were formed, group one (CG), which is the control group and group two (EG), called experimental group. The main objective for this lecture is to learn wave equation problem-solving using variable separation method.

2.2. Research Instrument

A quasi-experimental type design was used with a pre-test-intervention-post-test. The test employed is as given in Annex 1 and the time given to finish the test was 30 minutes. The lecture given for the control group was using traditional teaching method, while for the experimental group was heuristic instruction following the designed scheme. The students are evaluated following the rubric given in Annex 2.

2.3. Hypothesis

H1: In problem-solving for the wave equation, the proposed heuristic scheme produces a better academic performance on students, in comparison with traditional methodology teaching scheme.

H01: In problem-solving for the wave equation, the proposed heuristic scheme produces the same academic performance on students, as the traditional methodology teaching scheme.

2.4. Heuristic Scheme Proposed

In Math, a partial differential equation (sometimes abbreviated as PDE) is a relationship between a mathematical function of several independent variables x,y,z,t, and the partial derivatives of u with respect to these variables. PDEs are employed to describe the behavior of physical processes that are usually distributed in space-time. The variable separation method is relatively simple and powerful enough to solve not homogeneous differential equations. The method is based on the following statement: Let the function $f(x, y, z, t, \dots)$, then there must be a function $g(x), h(y), i(z) \cdot j(t) \dots$ such that: $f(x, y, z, t, \dots) = g(x) \cdot h(y) \cdot i(z) \cdot j(t) \dots$ (Zill, 1997; Chanillo, Franchi, Lu, Perez & Sawyer, 2017). Based on this criterion the heuristic starts to take place.

This study does not develop a general theorem to solve problems using variable separation method for the wave equation, rather it has developed a problem (given in Annex 1) that is used as a base to develop a heuristic scheme. Process analysis in solving problems by using heuristic instruction have revealed the following key points:

A. Read and interpret the problem then determine the objective

Start solving a problem is the most difficult part for many students because they ignore the reading and focus their attention on mathematical formulations. They do not read the problem carefully, and so, they cannot understand the problem and they do not know what to do.

The heuristic instruction indicates to start from data (Montero, 2011), but this is not enough, a proper interpretation of the variables involved in the problem is also needed, and obviously, it is relevant to keep in mind that data need to be processed and this eventually will lead to a result.

For the problem in Annex 1, the data to be considered and interpreted are:

$$\begin{cases} u_{tt} = c^2 u_{xx} & , x \in (0,\pi) \quad t \in \mathbb{R} \\ u(x,0) = e^x & , x \in (0,\pi) \\ u_t(x,0) = 0 & , x \in (0,\pi) \\ u_x(0,t) = 0 & , \quad t \in \mathbb{R} \\ u_x(\pi,t) = 0 & , \quad t \in \mathbb{R} \end{cases} \tag{1}$$

The objective is to determine u_{xt}

B. Understand and apply the variable separation method, to obtain a ratio and find a wave equation solution

Once the data and the objective of the problem are set, the next step is to develop a plan in order to get a solution. PDE allows to reach the solution of the problem, in the heuristic, this is known as search strategies (Francisco, 1999). Applying the variable separation method to the problem given in Annex 1:

$$u(x,t) = X(x)\,T(t) \tag{2}$$

The problem states $u_{tt} = c^2 u_{xx}$.This mathematical formulation in combination with (2):

$$X(x) \cdot T''(t) = c^2 X''(x) \cdot T(t) \tag{3}$$

Consequently, it is possible to do the following relationship:

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{c^2 T(t)} = -\lambda^2 \in \mathbb{R} \tag{4}$$

Hence:

$$X''(x) + \lambda^2 X(x) = 0 \tag{5}$$

$$T''(t) + \lambda^2 c^2 T(t) = 0 \tag{6}$$

C. Use the initial and boundary conditions

By imposing that $u(x,t) = X(x) \cdot T(t)$, a proper interpretation to use the initial and boundary conditions is needed. In heuristics instruction, this is known as data analysis. Applying this criterion to the given example:

1) From first the initial condition $u_t(x,0) = 0 \rightarrow X(x) \cdot T'(0) = 0$, by inspection: $T'(0) = 0$;
2) From the second initial condition $u_x(0,t) = 0 \rightarrow X'(0) \cdot T(t) = 0$, by inspection: $X'(0) = 0$;
3) From the boundary condition $u_x(\pi,t) = 0 \rightarrow X'(\pi) \cdot T(t) = 0$, by inspection: $X'(\pi) = 0$.

D. Create a system of equations and solve them separately

Combining (5), (6) and the initial conditions:

$$\begin{cases} X''(x) + \lambda^2 X(x) = 0 \\ X'(0) = X'(\pi) = 0 \end{cases} \tag{7}$$

$$\begin{cases} T''(t) + \lambda^2 c^2 T(t) = 0 \\ T'(0) = 0 \end{cases} \tag{8}$$

At this point, note that the problem comes down to solving systems of equations separately and then perform the operation $X(x)\,T(t)$. Nevertheless solving the system is not easy as it requires the development of high intellectual skills (Zbigniew, 2004).

E. Determine a model to solve the differential equation

From this point, the hardest part of problem-solving begins, since it requires previously known expressions and association models within this heuristic instruction are termed as intellectual skills based on known models. It is known that a differential equation of the type $\frac{d^2x}{dt^2} + \omega^2 x = 0$, has a solution $\mathrm{x} = \mathrm{A}\sin(\omega \mathrm{t}) + \mathrm{B}\cos(\omega \mathrm{t})$, then:

$$X = X_n = C_2 \cos(nx)\ for\ n = 0, 1, 2, 3, \dots \tag{9}$$

$$T = T_n = C_4 \cos(nct)\ for\ n = 0, 1, 2, 3, \dots \tag{10}$$

F. Determine the wave equation and express it as a summation

Once $X(x)$ and are $T(t)$ known, it is possible to find $u(x,t)$ using (2), then:

$$\mathrm{u_n(x,t) = C} \cdot \cos(nx) \cdot cos(nct) \tag{11}$$

Equation (11) can be expressed it in terms of summation:

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \cdot cos(nx) \cdot \cos(nct) \tag{12}$$

G. Express the wave equation as Fourier series

Recalling the Fourier series:

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{T} a_n \cos(n\omega x) + \sum_{n=1}^{T} b_n \sin(n\omega x) \tag{13}$$

Where:

$$a_0 = \frac{2}{T}\int_0^T f(x)dx; \tag{14}$$

$$a_n = \frac{2}{T}\int_0^T f(x)\cos(n\omega x)dx; \tag{15}$$

$$b_n = \frac{2}{T}\int_0^T f(x)\sin(n\omega x)dx \tag{16}$$

Then, (12) can be written as:

$$u(x,0) = a_0 + \sum_{n=1}^{\infty} a_n \cdot cos(nx) \tag{17}$$

This within the heuristic is known as establishing relationships (Arslan, 2010).

H. Define the limits of the integral and solve it to find some missing coefficients

There are some coefficients that maybe missing in the wave equation and in order to determine them physical parameter given in the problem are needed. The given problem presents a rope with $T = \pi$ and $\omega = 1$. With these data, it is possible to solve the integral and proceed to find the missing coefficients and determine the wave equation.

3. RESULTS Y DISCUSSION

3.1. Heuristic Scheme

As a result the following heuristic scheme for problem-solving the wave equation is obtained. This is shown in **Figure 1**.

The grades obtained (over 10) in the pre- and post-test for the control and experimental group are presented in **Figure 2 and Figure 3**, respectively. Then, a descriptive statistic of the grades in the pre- and post-tests for the control and experimental group are obtained. These are presented in **Table 1** and **Table 2**, respectively.

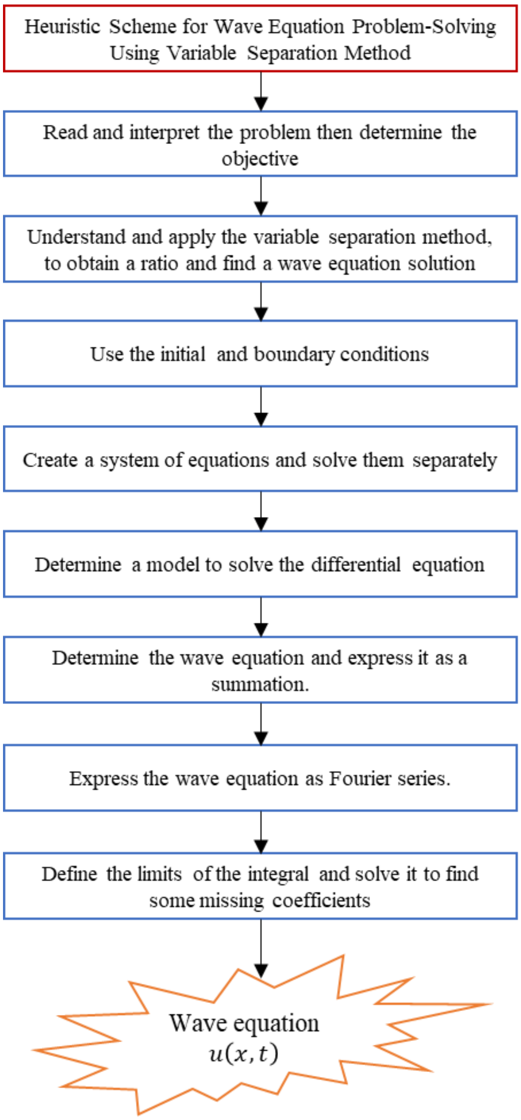

FIGURE 1. Heuristic scheme for wave equation problem-solving using variable separation method.

3.2. Pre- and Post-Tests Statistical Values

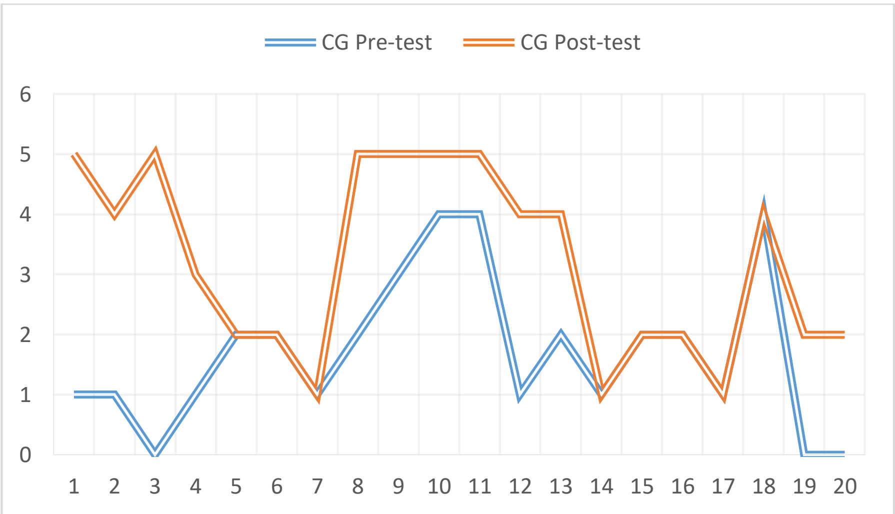

Figure 2. Control group grades obtained.

Table 1. Descriptive Statistics of the Pre- and Post-Tests CG.

Statistical Values	Pre-test	Post-test
Participants	1.70	3.20
Mean	1.50	3.50
Median	1.00	5.00
Mode	1.26	1.54
Standard Deviation	1.59	2.38
Variance	4.00	5.00
Maximum	0.00	1.00
Minimum	4.00	4.00
Range	1.70	3.20

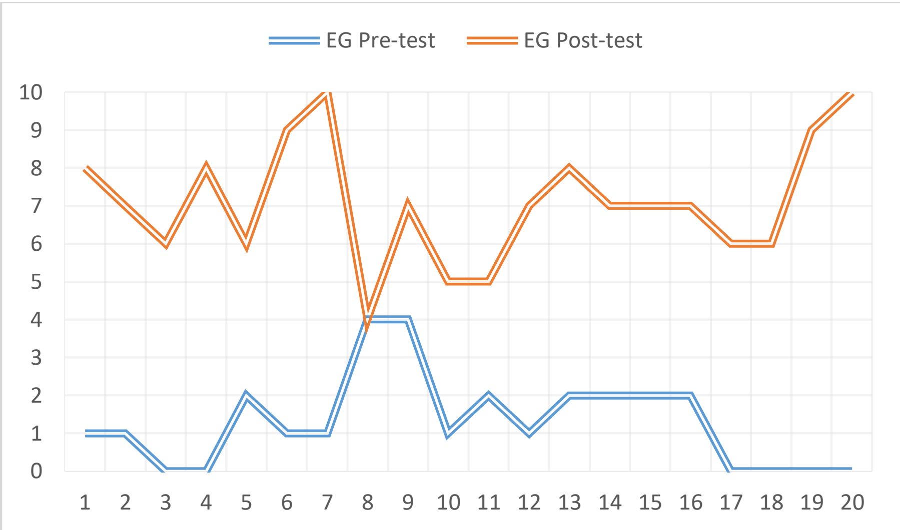

Figure 3. Experimental group grades obtained.

Table 2. Descriptive Statistics of the Pre- and Post-Tests EG.

Statistical Values	Pre-test	Post-test
Participants	1.30	7.10
Mean	1.00	7.00
Median	1.00	7.00
Mode	1.22	1.62
Standard Deviation	1.48	2.62
Variance	4.00	10.00
Maximum	0.00	4.00
Minimum	4.00	6.00
Range	1.30	7.10

3.3. “t” Test

A “t” test is employed to verify the hypothesis. The post-test data for the experimental and control are contrasted using EXCEL 2013. The results are shown in **Table 3** and it reveals a P value of 4.39699E-6 which is less than 0.05, therefore the null hypothesis is rejected in favor of the alternative hypothesis.

Table 3. "t" test results

	Teaching Scheme	
	Traditional	Heuristic
Mean	3.2	7.1
Variance	2.4	2.6
Observations	20	20
Pearson Correlation	-0.514287463	
Hypothesized Mean Difference	0	
df	19	
t Stat	-6.339820507	
t Critical one-tail	1.729132812	
P(T<=t) two-tail	4.39699E-06	
t Critical two-tail	2.093024054	

3.4. Leaning Gain

In order to quantify the students' learning gain, the Hake factor (g) was employed. The grades obtained in the pre- and post-tests are related as follows (Dellwo, 2010):

$$g = \frac{\theta_{post} - \theta_{pre}}{\theta_{post}} \tag{17}$$

where θ_{post} is the grade obtained in the post-test and θ_{pre} is the grade obtained in the pre-test.

The learning gain for both groups is presented in **Figure 4**. The results reveal that EG presents more learning gain than CG.

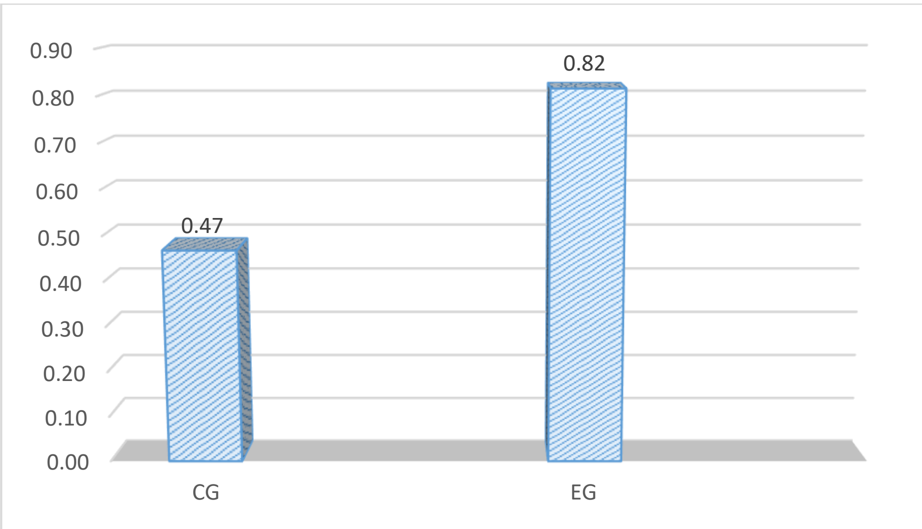

Figure 4. Hake Factor.

4. CONCLUSION

When it comes to solving problems, which aims to determine the wave equation, we need certain skills related to differentiation and integration of real variable functions, and students must recognize that there is no recursive procedure to solve the problem. Nevertheless, it is possible to devise a strategy to facilitate to obtain the solution, and then develop a methodology and summarize it in a comprehensive scheme. This scheme is nothing more than a succession of heuristic indications with a view to have an orientation and to develop the problem. It is relevant to emphasize that the scheme does not guarantee success in solving the problem as largely, as indicates the heuristic is necessary for the individual to develop intellectual operations such as: analyze, synthesize, compare and rank; also requires forms of critical thinking and mathematical science: variation of conditions, search for relations and dependencies, and considerations analogy. The scheme is very useful and avoids certain difficulties in raising strategies in order to find its solution.

Concerning the learning gain obtained (Hake factor) for the CG and EG is 0.47 and 0.82, respectively. Therefore, the proposed schematic heuristic instruction for wave equation problem-solving using variable separation method brings a higher gain learning than traditional methodology. Furthermore, in the “t” test, the p statistic value (4.39699E-06) is less than 0.05, and following the theory related with this test, the null hypothesis is rejected. Therefore, it is possible to state that in problem-solving for the wave equation, the proposed heuristic scheme produces a better academic performance on students, in comparison with traditional methodology teaching scheme.

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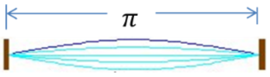
Annex 1

Proposed problem with the corresponding solution

There is a string of length $L = \pi$ and is released, without impulse, from the position $f(x) = e^x$

We denote c, as the speed at which the waves travel up the rope. The equations that model this problem are:

$$\begin{cases} u_{tt} = c^2 u_{xx} & , x \in (0,\pi) \quad t \in \mathbb{R} \\ u(x,0) = e^x & , x \in (0,\pi) \\ u_t(x,0) = 0 & , x \in (0,\pi) \\ u_x(0,t) = 0 & , \quad t \in \mathbb{R} \\ u_x(\pi,t) = 0 & , \quad t \in \mathbb{R} \end{cases}$$



Determine the solution describing the wave equation.

Solution:

Read and interpret the problem then determine the objective.

The data to be considered and interpreted is:

$$\begin{cases} \mathrm{u_{tt}} = \mathrm{c^2 u_{xx}} & , \mathrm{x} \in (0,\pi) \quad \mathrm{t} \in \mathbb{R} \\ \mathrm{u(x,0)} = \mathrm{e^x} & , \mathrm{x} \in (0,\pi) \\ \mathrm{u_t(x,0)} = 0 & , \mathrm{x} \in (0,\pi) \\ \mathrm{u_x(0,t)} = 0 & , \quad \mathrm{t} \in \mathbb{R} \\ \mathrm{u_x(\pi,t)} = 0 & , \quad \mathrm{t} \in \mathbb{R} \end{cases}$$

The objective is to determine the wave equation.

Understand and apply the variable separation method, to obtain a ratio and find a wave equation solution.

The basic idea of the method is to seek solutions in the form of separate variables of the homogeneous part of the problem to solve.

$$u(x,t) = X(x) \cdot T(t)$$

The mathematical formulation $u_{tt} = c^2 u_{xx}$, in combination with the variable separation method criterion:

$$X(x) \cdot T''(t) = c^2 X''(x) \cdot T(t),$$

Consequently, the following relationship can be done:

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{c^2 T(t)} = -\lambda^2 \in \mathbb{R}$$

Thus:

$$X''(x) + \lambda^2 X(x) = 0$$

$$T''(t) + \lambda^2 c^2 T(t) = 0$$

Use the initial conditions and boundary conditions.

By imposing that the function $u(x,t) = X(x)T(t)$ and using this with the initial conditions:

- The wave equation $u_{tt} = c^2 u_{xx}$, we get that $X(x) \cdot T''(t) = c^2 X''(x) \cdot T(t)$, then
$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{c^2 T(t)} = -\lambda^2 \in \mathbb{R}$$
- The initial condition $u_t(x,0) = 0$, by inspection $T'(0) = 0$.
- The boundary condition $u_x(0,t) = 0$, by inspection $X'(0) = 0$.
- The boundary condition $u_x(\pi,t) = 0$, by inspection $X'(\pi) = 0$.

Create a system of equations and solve them separately.

Therefore, it produces two separate problems:

$$(1)\begin{cases} X''(x) + \lambda^2 X(x) = 0 \\ X'(0) = X'(\pi) = 0 \end{cases} \quad , \quad (2)\begin{cases} T''(t) + \lambda^2 c^2 T(t) = 0 \\ T'(0) = 0 \end{cases}$$

Determine a model to solve the differential equation.

The solution, for the equation $X''(x) + \lambda^2 X(x) = 0$ is $X = C_1\, sen(\lambda x) + C_2 \cos(\lambda x)$ and using the boundaries conditions:

$$X = C_1\, sen(\lambda x) + C_2 \cos(\lambda x)$$

$$\Longrightarrow X' = \lambda C_1 \cos(\lambda x) - \lambda C_2 \sin(\lambda x)$$

$$Boundary\ condition: X'(0) = 0$$

$$\Rightarrow 0 = \lambda C_1 \cos\big(\lambda(0)\big) - \lambda C_2 \sin\big(\lambda(0)\big)$$

$$\Rightarrow 0 = \lambda C_1 - \lambda C_2(0)$$

$$\Rightarrow \lambda C_1 = 0 \ \Rightarrow \lambda = 0 \ \vee \ C_1 = 0,$$

$$however\ \frac{X''(x)}{X(x)} = -\lambda^2 \in \mathbb{R},$$

$$so\ \lambda\ can\ not\ be\ zero$$

$$\Rightarrow C_1 = 0$$

$$Boundary\ condition: X'(\pi) = 0$$

$$\Rightarrow 0 = \lambda C_1 \cos\big(\lambda(\pi)\big) - \lambda C_2 \sin\big(\lambda(\pi)\big)$$

$$\Rightarrow 0 = \lambda(0) \cos\big(\lambda(\pi)\big) - \lambda C_2 \sin\big(\lambda(\pi)\big)$$

$$\Longrightarrow 0 = 0 - \lambda C_2 \sin\big(\lambda(\pi)\big)$$

$\Rightarrow \lambda C_2 = 0 \ \vee \ \sin\big(\lambda(\pi)\big) = 0,$

$but\ \lambda C_2\ can\ not\ be\ zero\ because\ it\ exist$

$\Rightarrow \sin\big(\lambda(\pi)\big) = 0$

$\Rightarrow \lambda = n\ siendo\ n = 0, 1, 2, 3, \ldots$

Replacing values λ y C_1, on $X = C_1\ sen(\lambda x) + C_2 \cos(\lambda x)$:

$$\therefore X = X_n = C_2 \cos(nx)\ for\ n = 0, 1, 2, 3, \ldots$$

The solution to the equation $T''(t) + \lambda^2 c^2 T(t) = 0$ is $T = C_3\ sin(\lambda ct) + C_4 \cos(\lambda ct)$ and using the initial condition:

$T = C_3\ sin(\lambda ct) + C_4 \cos(\lambda ct)$

$\Longrightarrow T' = \lambda c C_3 \cos(\lambda cx) - \lambda c C_4 \sin(\lambda cx)$

$Initial\ condition$: $T'(0) = 0$

$\Rightarrow 0 = \lambda c C_3 \cos\big(\lambda c(0)\big) - \lambda c C_4 \sin\big(\lambda c(0)\big)$

$\Rightarrow 0 = \lambda c C_3 - \lambda c C_4(0)$

$\Rightarrow \lambda c C_3 = 0 \ \Rightarrow \lambda = 0 \ \vee \ C_3 = 0,$

$however\ \lambda = n\ being\ n = 0, 1, 2, 3, \ldots$

$\Rightarrow C_3 = 0$

Replacing the values of λ and C_3, on the equation

$T = C_3\ sin(\lambda ct) + C_4 \cos(\lambda ct)$, we get:

$\therefore T = T_n = C_4 \cos(nct)\ para\ n = 0, 1, 2, 3, \ldots$

Determine the wave equation $\mathbf{u(x, t)}$ and express it as a summation.

Now:

$$u(x,t) = X(x) \cdot T(t)$$

$$u_n(x,t) = C_2\ cos(nx) \cdot C_4 \cos(nct)$$

$$u_n(x,t) = C \cdot cos(nx) \cdot \cos(nct) \qquad (3)$$

Express the wave equation as Fourier series.

This function is described in Eq. (3) is called the normal modes are the natural modes of vibration of the string. The natural term means that due to the linearity of the homogeneous problem, the vibration of the string will always be a superposition (sum) of these infinite normal modes, making it possible to write the function as a sum:

$$u(x,t) = \sum_{n=0}^{\infty} a_n \cdot cos(nx) \cdot \cos(nct) = a_0 + \sum_{n=1}^{\infty} a_n \cdot cos(nx) \cdot \cos(nct) \quad (4)$$

$$u(x,0) = \sum_{n=0}^{\infty} a_n \cdot cos(nx) \cdot \cos(nc(0))$$

$$u(x,0) = \sum_{n=0}^{\infty} a_n \cdot cos(nx) = a_0 + \sum_{n=1}^{\infty} a_n \cdot cos(nx)$$

Recalling the Fourier series:

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{T} a_n \cos(n\omega x) + \sum_{n=1}^{T} b_n \sin(n\omega x)$$

Where:

$$a_0 = \frac{2}{T}\int_0^T f(x)dx;\ a_n = \frac{2}{T}\int_0^T f(x)\cos(n\omega x)dx;\ b_n = \frac{2}{T}\int_0^T f(x)\sin(n\omega x)dx;$$

It is indicated as a condition of the problem: $u(x,0) = f(x) = e^x$

So:

$$u(x,0) = \sum_{n=0}^{\infty} a_n \cdot cos(nx) \cdot \cos(0) = a_0 + \sum_{n=1}^{\infty} a_n \cdot cos(nx)$$

Define the limits of the integral and solve it to find some missing coefficients.

It can be associated with the Fourier series $u(x,0)$ and by inspection of the equation and the graph given in the problem: $T = \pi,\ \omega = 1$ y $b_n = 0$

Then:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} e^x dx = \frac{2}{\pi}(e^{\pi} - e^0)$$

$$a_0 = \frac{2}{\pi}(e^{\pi} - 1) \quad (5)$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} e^x \cos(nx)\, dx \quad (6)$$

To solve the integral of Eq. (6) requires the integration by parts $\int u\, dv = uv - \int v\, du$

$$\int e^x \cos(nx)\, dx$$

$$u = e^x \to du = e^x\, dx$$

$$dv = cos(nx)\, dx \;\to v = \frac{1}{n}\, sin(nx)$$

$$\frac{1}{n}\, e^x sin(nx) - \int \frac{1}{n}\, sin(nx)\, e^x dx$$

$$u = e^x \to du = e^x\, dx$$

$$dv = sin\,(nx)\, dx \;\to v = -\frac{1}{n}\, cos(nx)$$

$$\frac{1}{n}\, e^x sin(nx) - \frac{1}{n}\left(-\frac{1}{n} e^x\, cos(nx) + \int \frac{1}{n}\, cos(nx)\, e^x dx\right)$$

$$\frac{1}{n}\, e^x sin(nx) + \frac{1}{n^2} e^x\, cos(nx) - \frac{1}{n^2}\int\; cos(nx)\, e^x dx$$

Then:

$$\int e^x \cos(nx)\, dx \;= \frac{1}{n}\, e^x sin(nx) + \frac{1}{n^2} e^x\, cos(nx) - \frac{1}{n^2}\int\; cos(nx)\, e^x dx \;+ C$$

$$\int e^x \cos(nx)\, dx + \frac{1}{n^2}\int\; e^x cos(nx)\, dx \;= \frac{1}{n}\, e^x sin(nx) + \frac{1}{n^2} e^x\, cos(nx) + C$$

$$\left(1 + \frac{1}{n^2}\right)\int\; cos(nx)\, e^x dx \;= \frac{1}{n}\, e^x sin(nx) + \frac{1}{n^2} e^x\, cos(nx) + C$$

$$\int\; cos(nx)\, e^x dx \;= \frac{\frac{1}{n}\, e^x sin(nx) + \frac{1}{n^2} e^x\, cos(nx) + C}{1 + \frac{1}{n^2}} = \frac{\frac{1}{n}\, e^x sin(nx) + \frac{1}{n^2} e^x\, cos(nx) + C}{\frac{n^2+1}{n^2}}$$

$$\int\; cos(nx)\, e^x dx \;= \frac{n\, e^x sin(nx) + e^x\, cos(nx) + C}{n^2+1}$$

By replacing the limits of the integral Eq. (6) and solving:

$$a_n = \frac{2}{\pi}\int_0^{\pi} e^x \cos(nx)\, dx = \frac{2}{\pi}\cdot \frac{n\, e^x sin(nx) + e^x\, cos(nx)}{n^2+1}\Bigg|_{0}^{\pi}$$

$$a_n = \left(\frac{2}{\pi(n^2+1)}\right)\; [e^{\pi} sin(n\pi) + e^{\pi}\, cos(n\pi)] - [e^0 sin(0) + e^0\, cos(0)]$$

$$a_n = \left(\frac{2}{\pi(n^2+1)}\right)\; [0 + e^{\pi}\, (-1)^n] - [0+1]$$

$$a_n = \frac{-4e^{\pi}}{\pi(n^2+1)}\; para\; n = 1, 3, 5, \ldots \qquad (7)$$

Replacing the Eq. (5) and Eq. (7) into Eq. (4):

$$u(x,t) = \frac{(e^{\pi}-1)}{\pi} + \sum_{n=1}^{\infty} \frac{-4e^{\pi}}{\pi(n^2+1)} \cdot cos(nx) \cdot \cos(nct) \qquad (8)$$

Eq. (8) is the solution that describes the wave equation.

Annex 2

Rubric employed for the problem given in Annex 1

Task	Points
If the student get: $\frac{X''(x)}{X(x)} = \frac{T''(t)}{c^2T(t)} = -\lambda^2 \in \mathbb{R}$	**0-1**
If the student use the initial and boundaries conditions as follows: • The initial condition $u_t(x,0) = 0$, by inspection $T'(0) = 0$. • The boundary condition $u_x(0,t) = 0$, by inspection $X'(0) = 0$. • The boundary condition $u_x(\pi,t) = 0$, by inspection $X'(\pi) = 0$.	**1-2.5** **(+0.5 each)**
If the student get: $C_1 = 0$ or $X = X_n = C_2 \cos(nx)\ for\ n = 0,1,2,3,\ldots$ or $C_3 = 0$ or $T = T_n = C_4 \cos(nct)\ para\ n = 0,1,2,3,\ldots$	**2.5-6.5** **(+1 each)**
If the student express the wave equation as a Fourier series: $u(x,0) = \sum_{n=0}^{\infty} a_n \cdot cos(nx) \cdot \cos(0) = a_0 + \sum_{n=1}^{\infty} a_n \cdot cos(nx)$	**6.5-8.5** **(depending on the process)**
If the student get the wave equation solution: $u(x,t) = \frac{(e^{\pi}-1)}{\pi} + \sum_{n=1}^{\infty} \frac{-4e^{\pi}}{\pi(n^2+1)} \cdot cos(nx) \cdot \cos(nct)$	**8.0-10.0** **(depending on the process)**