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Lie Sphere Geometry and Dupin Hypersurfaces

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Abstract

These notes were originally written for a short course held at the Institute of Mathematics and Statistics, University of São Paulo, S.P. Brazil, January 9–20, 2012. The notes are based on the author's book [17], *Lie Sphere Geometry With Applications to Submanifolds*, Second Edition, published in 2008, and many passages are taken directly from that book. The notes have been updated from their original version to include some recent developments in the field.

A hypersurface M^{n-1} in Euclidean space $\mathbf{R}^n$ is proper Dupin if the number of distinct principal curvatures is constant on M^{n-1}, and each principal curvature function is constant along each leaf of its principal foliation. The main goal of this course is to develop the method for studying proper Dupin hypersurfaces and other submanifolds of $\mathbf{R}^n$ within the context of Lie sphere geometry. This method has been particularly effective in obtaining classification theorems of proper Dupin hypersurfaces.

1 Lie Sphere Geometry

1.1 Introduction

In his doctoral dissertation, published in *Math. Ann.* in 1872, Lie [55] introduced his geometry of oriented hyperspheres in Euclidean space $\mathbf{R}^n$ in

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the context of his work on contact transformations. Lie established a bijective correspondence between the set of all oriented hyperspheres, oriented hyperplanes and point spheres in $\mathbf{R}^n \cup \{\infty\}$, and the set of all points on the quadric hypersurface Q^{n+1} in real projective space $\mathbf{P}^{n+2}$ given by the equation $\langle x, x \rangle = 0$, where $\langle\ ,\ \rangle$ is an indefinite scalar product with signature $(n+1, 2)$ on $\mathbf{R}^{n+3}$. Equivalently, one can study the space of all oriented hyperspheres and point spheres in S^n.

In this short-course, we give Lie's construction in detail, and discuss its applications to the modern study of Dupin hypersurfaces. A hypersurface M in $\mathbf{R}^n$ (or S^n) is said to be *proper Dupin* if the number g of distinct principal curvatures is constant on M and each distinct principal curvature is constant along each leaf of its corresponding principal foliation. Examples of proper Dupin hypersurfaces in $\mathbf{R}^n$ are the images under stereographic projection of isoparametric (constant principal curvatures) hypersurfaces in the sphere S^n, including the cyclides of Dupin [39] in $\mathbf{R}^3$.

Thorbergsson [107] showed that the number g of distinct principal curvatures of a compact proper Dupin hypersurface M in S^n must be $1, 2, 3, 4$ or 6, the same as Münzner's [70]–[71] restriction on the number of distinct principal curvatures of an isoparametric hypersurface in S^n. In the cases $g = 1, 2, 3$, compact Dupin hypersurfaces in S^n have been completely classified, and they are all equivalent to isoparametric hypersurfaces under Lie sphere transformations (see Pinkall [79]–[81], Cecil and Ryan [27]–[28], Miyaoka [59], Cecil and Chern [20], Cecil and Jensen [25]).

The classifications of compact proper Dupin hypersurfaces with $g = 4$ or 6 principal curvatures have not yet been completed, although Stolz [103] ($g = 4$) and Grove and Halperin [43] ($g = 6$) proved that the multiplicities of the principal curvatures must be the same as for an isoparametric hypersurface. Several local and global partial classifications have been found in the cases $g > 3$ (see, for example, Niebergall [68]–[69], Riveros and Tenenblat [90]–[91], Riveros, Rodrigues and Teneblat [89], Dajczer, Florit and Tojeiro [37], Cecil, Chi and Jensen [22]–[23]).

In this short-course, we will discuss various local and global classification results for proper Dupin hypersurfaces in S^n that have been obtained in the context of Lie sphere geometry. The course is based primarily on the author's book [17], *Lie Sphere Geometry With Applications to Submanifolds*, Second Edition, Springer, New York, 2008. Many passages in these notes are direct quotations from this book.

1.2 Preliminaries

We begin with some preliminary remarks on indefinite scalar product spaces and projective geometry. The fundamental result from linear algebra concerns the rank and signature of a bilinear form (see, for example, Nomizu [72, p. 108], Chapter 3 of Artin [3] or O'Neill [76, pp. 46–53]).

Theorem 1.1. *Suppose that* $(\ ,\ )$ *is a bilinear form on a real vector space* V *of dimension* n*. Then there exists a basis* $\{e_1, \ldots, e_n\}$ *of* V *such that:*
1. $(e_i, e_j) = 0$ *for* $i \neq j$*,*
2. $(e_i, e_i) = 1$ *for* $1 \leq i \leq p$*,*
3. $(e_j, e_j) = -1$ *for* $p + 1 \leq j \leq r$*,*
4. $(e_k, e_k) = 0$ *for* $r + 1 \leq k \leq n$*.*

The numbers r and p are determined solely by the bilinear form; r is called the *rank*, $r - p$ is called the *index*, and the ordered pair $(p, r - p)$ is called the *signature* of the bilinear form. The theorem shows that any two spaces of the same dimension with bilinear forms of the same signature are isometrically isomorphic. A *scalar product* is a nondegenerate bilinear form, i.e., a form with rank equal to the dimension of V. For the sake of brevity, we will often refer to a scalar product as a "metric." Usually, we will be dealing with the scalar product space $\mathbf{R}^n_k$ with signature $(n - k, k)$ for $k = 0, 1$ or 2. However, at times we will consider subspaces of $\mathbf{R}^n_k$ on which the bilinear form is degenerate.

Let (x, y) be the indefinite scalar product on the Lorentz space $\mathbf{R}^{n+1}_1$ defined by

$$(x, y) = -x_1 y_1 + x_2 y_2 + \cdots + x_{n+1} y_{n+1}, \tag{1}$$

where $x = (x_1, \ldots, x_{n+1})$ and $y = (y_1, \ldots, y_{n+1})$. We will call this scalar product the *Lorentz metric*. A vector x is said to be *spacelike*, *timelike* or *lightlike*, respectively, depending on whether (x, x) is positive, negative or zero. We will use this terminology even when we are using a metric of different signature. In Lorentz space, the set of all lightlike vectors, given by the equation,

$$x_1^2 = x_2^2 + \cdots + x_{n+1}^2, \tag{2}$$

forms a cone of revolution, called the *light cone*. Lightlike vectors are often called *isotropic* in the literature, and the cone is called the *isotropy cone*. Timelike vectors are "inside the cone" and spacelike vectors are "outside the cone."

If x is a nonzero vector in $\mathbf{R}^{n+1}_1$, let $x^\perp$ denote the orthogonal complement of x with respect to the Lorentz metric. If x is timelike, then the metric restricts to a positive definite form on $x^\perp$, and $x^\perp$ intersects the light cone only at the origin. If x is spacelike, then the metric has signature $(n-1,1)$ on $x^\perp$, and $x^\perp$ intersects the cone in a cone of one less dimension. If x is lightlike, then $x^\perp$ is tangent to the cone along the line through the origin determined by x. The metric has signature $(n-1,0)$ on this n-dimensional plane.

Lie sphere geometry is defined in the context of real projective space $\mathbf{P}^n$, so we now briefly review some important concepts from projective geometry. We define an equivalence relation on $\mathbf{R}^{n+1} - \{0\}$ by setting $x \simeq y$ if $x = ty$ for some nonzero real number t. We denote the equivalence class determined by a vector x by $[x]$. Projective space $\mathbf{P}^n$ is the set of such equivalence classes, and it can naturally be identified with the space of all lines through the origin in $\mathbf{R}^{n+1}$. The rectangular coordinates $(x_1, \ldots, x_{n+1})$ are called *homogeneous coordinates* of the point $[x]$, and they are only determined up to a nonzero scalar multiple. The affine space $\mathbf{R}^n$ can be embedded in $\mathbf{P}^n$ as the complement of the hyperplane $(x_1 = 0)$ at infinity by the map $\phi : \mathbf{R}^n \to \mathbf{P}^n$ given by $\phi(u) = [(1, u)]$. A scalar product on $\mathbf{R}^{n+1}$, such as the Lorentz metric, determines a polar relationship between points and hyperplanes in $\mathbf{P}^n$. We will also use the notation $x^\perp$ to denote the polar hyperplane of $[x]$ in $\mathbf{P}^n$, and we will call $[x]$ the *pole* of $x^\perp$.

If x is a lightlike vector in $\mathbf{R}^{n+1}_1$, then $[x]$ can be represented by a vector of the form $(1, u)$ for $u \in \mathbf{R}^n$. Then the equation $(x, x) = 0$ for the light cone becomes $u \cdot u = 1$ (Euclidean dot product), i.e., the equation for the unit sphere in $\mathbf{R}^n$. Hence, the set of points in $\mathbf{P}^n$ determined by lightlike vectors in $\mathbf{R}^{n+1}_1$ is naturally diffeomorphic to the sphere S^{n-1}.

1.3 Möbius geometry of unoriented spheres

We want to study the space of all (unoriented) hyperspheres in Euclidean n-dimensional space $\mathbf{R}^n$ and in the unit sphere $S^n \subset \mathbf{R}^{n+1}$. These two spaces of spheres are closely related via stereographic projection, as we now describe. We will always assume that $n \geq 2$.

We denote the Euclidean dot product of two vectors u and v in $\mathbf{R}^n$ by $u \cdot v$. We first consider stereographic projection $\sigma : \mathbf{R}^n \to S^n - \{P\}$, where S^n is the unit sphere in $\mathbf{R}^{n+1}$ given by $y \cdot y = 1$, and $P = (-1, 0, \ldots, 0)$ is the south pole of S^n. The well-known formula for $\sigma(u)$ is

$$\sigma(u) = \left(\frac{1 - u \cdot u}{1 + u \cdot u}, \frac{2u}{1 + u \cdot u} \right).$$

We next embed $\mathbf{R}^{n+1}$ into $\mathbf{P}^{n+1}$ by the embedding ϕ mentioned in the previous section. Thus, we have the map $\phi\sigma : \mathbf{R}^n \to \mathbf{P}^{n+1}$ given by

$$\phi\sigma(u) = \left[\left(1, \frac{1 - u \cdot u}{1 + u \cdot u}, \frac{2u}{1 + u \cdot u} \right) \right] = \left[\left(\frac{1 + u \cdot u}{2}, \frac{1 - u \cdot u}{2}, u \right) \right]. \quad (3)$$

Let $(z_1, \ldots, z_{n+2})$ be homogeneous coordinates on $\mathbf{P}^{n+1}$ and $(\ ,\ )$ the Lorentz metric on the space $\mathbf{R}^{n+2}_1$. Then $\phi\sigma(\mathbf{R}^n)$ is just the set of points in $\mathbf{P}^{n+1}$ lying on the n-sphere Σ given by the equation $(z, z) = 0$, with the exception of the *improper point* $[(1, -1, 0, \ldots, 0)]$ corresponding to the south pole P. We will refer to the points in Σ other than $[(1, -1, 0, \ldots, 0)]$ as *proper points*, and will call Σ the *Möbius sphere* or *Möbius space*.

The basic framework for the Möbius geometry of unoriented spheres is as follows. Suppose that ξ is a spacelike vector in $\mathbf{R}^{n+2}_1$. Then the polar hyperplane $\xi^\perp$ to $[\xi]$ in $\mathbf{P}^{n+1}$ intersects the sphere Σ in an $(n-1)$-sphere S^{n-1}. This sphere S^{n-1} is the image under $\phi\sigma$ of an $(n-1)$-sphere in $\mathbf{R}^n$, unless it contains the improper point $[(1, -1, 0, \ldots, 0)]$, in which case it is the image under $\phi\sigma$ of a hyperplane in $\mathbf{R}^n$. Hence, we have a bijective correspondence between the set of all spacelike points in $\mathbf{P}^{n+1}$ and the set of all hyperspheres and hyperplanes in $\mathbf{R}^n$.

We want to find specific formulas for this correspondence. Consider the sphere in $\mathbf{R}^n$ with center p and radius $r > 0$ given by the equation

$$(u - p) \cdot (u - p) = r^2. \quad (4)$$

We wish to translate this into an equation involving the Lorentz metric and the corresponding polarity relationship on $\mathbf{P}^{n+1}$. A direct calculation shows that equation (4) is equivalent to the equation

$$(\xi, \phi\sigma(u)) = 0, \quad (5)$$

where ξ is the spacelike vector,

$$\xi = \left(\frac{1 + p \cdot p - r^2}{2}, \frac{1 - p \cdot p + r^2}{2}, p \right), \quad (6)$$

and $\phi\sigma(u)$ is given by equation (3). Thus, the point u is on the sphere given by equation (4) if and only if $\phi\sigma(u)$ lies on the polar hyperplane of $[\xi]$. Note that the first two coordinates of ξ satisfy $\xi_1 + \xi_2 = 1$, and that $(\xi, \xi) = r^2$. Although ξ is only determined up to a nonzero scalar multiple, we can conclude that $\eta_1 + \eta_2$ is not zero for any $\eta \simeq \xi$.

Conversely, given a spacelike point $[z]$ with $z_1 + z_2$ nonzero, we can determine the corresponding sphere in $\mathbf{R}^n$ as follows. Let $\xi = z/(z_1 + z_2)$ so that $\xi_1 + \xi_2 = 1$. Then from equation (6), the center of the corresponding sphere is the point $p = (\xi_3, \ldots, \xi_{n+2})$, and the radius is the square root of (ξ, ξ).

Next suppose that η is a spacelike vector with $\eta_1 + \eta_2 = 0$. Then

$$(\eta, (1, -1, 0, \ldots, 0)) = 0.$$

Thus, the improper point $\phi(P)$ lies on the polar hyperplane of $[\eta]$, and the point $[\eta]$ corresponds to a hyperplane in $\mathbf{R}^n$. Again we can find an explicit correspondence. Consider the hyperplane in $\mathbf{R}^n$ given by the equation

$$u \cdot N = h, \quad |N| = 1. \tag{7}$$

A direct calculation shows that (7) is equivalent to the equation

$$(\eta, \phi\sigma(u)) = 0, \text{where } \eta = (h, -h, N). \tag{8}$$

Thus, the hyperplane (7) is represented in the polarity relationship by $[\eta]$. Conversely, let z be a spacelike point with $z_1 + z_2 = 0$. Then $(z, z) = v \cdot v$, where $v = (z_3, \ldots, z_{n+2})$. Let $\eta = z/|v|$. Then η has the form (8) and $[z]$ corresponds to the hyperplane (7). Thus we have explicit formulas for the bijective correspondence between the set of spacelike points in $\mathbf{P}^{n+1}$ and the set of hyperspheres and hyperplanes in $\mathbf{R}^n$.

Of course, the fundamental invariant of Möbius geometry is the angle. The study of angles in this setting is quite natural, since orthogonality between spheres and planes in $\mathbf{R}^n$ can be expressed in terms of the Lorentz metric. Let S_1 and S_2 denote the spheres in $\mathbf{R}^n$ with respective centers p_1 and p_2 and respective radii r_1 and r_2. By the Pythagorean Theorem, the two spheres intersect orthogonally if and only if

$$|p_1 - p_2|^2 = r_1^2 + r_2^2. \tag{9}$$

If these spheres correspond by equation (6) to the projective points $[\xi_1]$ and $[\xi_2]$, respectively, then a calculation shows that equation (9) is equivalent to the condition

$$(\xi_1, \xi_2) = 0. \tag{10}$$

A hyperplane π in $\mathbf{R}^n$ is orthogonal to a hypersphere S precisely when π passes through the center of S. If S has center p and radius r, and π is given by the equation $u \cdot N = h$, then the condition for orthogonality is just $p \cdot N = h$. If S corresponds to $[\xi]$ as in (6) and π corresponds to $[\eta]$ as in (8), then this equation for orthogonality is equivalent to $(\xi, \eta) = 0$. Finally, if two planes π_1 and π_2 are represented by $[\eta_1]$ and $[\eta_2]$ as in (8), then the orthogonality condition $N_1 \cdot N_2 = 0$ is equivalent to the equation $(\eta_1, \eta_2) = 0$.

A *Möbius transformation* is a projective transformation of $\mathbf{P}^{n+1}$ which preserves the condition $(\eta, \eta) = 0$. By Theorem 1.4 to follow, a Möbius transformation also preserves the relationship $(\eta, \xi) = 0$, and it maps spacelike points to spacelike points. Thus it preserves orthogonality (and hence angles) between spheres and planes in $\mathbf{R}^n$. Later we will see that the group of Möbius transformations is isomorphic to $O(n+1, 1)/\{\pm I\}$, where $O(n+1, 1)$ is the group of orthogonal transformations of the Lorentz space $\mathbf{R}^{n+2}_1$.

Note that a Möbius transformation takes lightlike vectors to lightlike vectors, and so it induces a conformal diffeomorphism of the sphere Σ onto itself. It is well known that the group of conformal diffeomorphisms of the sphere is precisely the Möbius group.

1.4 Lie geometry of oriented spheres

We now turn to the construction of Lie's geometry of oriented spheres in $\mathbf{R}^n$. Let W^{n+1} be the set of vectors in $\mathbf{R}^{n+2}_1$ satisfying $(\zeta, \zeta) = 1$. This is a hyperboloid of revolution of one sheet in $\mathbf{R}^{n+2}_1$. If α is a spacelike point in $\mathbf{P}^{n+1}$, then there are precisely two vectors $\pm\zeta$ in W^{n+1} with $\alpha = [\zeta]$. These two vectors can be taken to correspond to the two orientations of the oriented sphere or plane represented by α, although we have not yet given a prescription as to how to make the correspondence. To do this, we need to introduce one more coordinate. First, embed $\mathbf{R}^{n+2}_1$ into $\mathbf{P}^{n+2}$ by the embedding $z \mapsto [(z, 1)]$. If $\zeta \in W^{n+1}$, then

$$-\zeta_1^2 + \zeta_2^2 + \cdots + \zeta_{n+2}^2 = 1,$$

so the point $[(\zeta, 1)]$ in $\mathbf{P}^{n+2}$ lies on the quadric Q^{n+1} in $\mathbf{P}^{n+2}$ given in homogeneous coordinates by the equation

$$\langle x, x \rangle = -x_1^2 + x_2^2 + \cdots + x_{n+2}^2 - x_{n+3}^2 = 0. \tag{11}$$

The manifold Q^{n+1} is called the *Lie quadric*, and the scalar product determined by the quadratic form in (11) is called the *Lie metric* or *Lie scalar* *product*. We will let $\{e_1, \ldots, e_{n+3}\}$ denote the standard orthonormal basis for the scalar product space $\mathbf{R}^{n+3}_2$ with metric $\langle\ ,\ \rangle$. Here e_1 and e_{n+3} are timelike and the rest are spacelike.

We shall now see how points on Q^{n+1} correspond to the set of oriented hyperspheres, oriented hyperplanes and point spheres in $\mathbf{R}^n \cup \{\infty\}$. Suppose that x is any point on the quadric with homogeneous coordinate $x_{n+3} \neq 0$. Then x can be represented by a vector of the form $(\zeta, 1)$, where the Lorentz scalar product $(\zeta, \zeta) = 1$. Suppose first that $\zeta_1 + \zeta_2 \neq 0$. Then in Möbius geometry $[\zeta]$ represents a sphere in $\mathbf{R}^n$. If as in equation (6), we represent $[\zeta]$ by a vector of the form

$$\xi = \left(\frac{1 + p \cdot p - r^2}{2}, \frac{1 - p \cdot p + r^2}{2}, p\right),$$

then $(\xi, \xi) = r^2$. Thus ζ must be one of the vectors $\pm\xi/r$. In $\mathbf{P}^{n+2}$, we have

$$[(\zeta, 1)] = [(\pm\xi/r, 1)] = [(\xi, \pm r)].$$

We can interpret the last coordinate as a signed radius of the sphere with center p and unsigned radius $r > 0$. In order to be able to interpret this geometrically, we adopt the convention that a positive signed radius corresponds to the orientation of the sphere represented by the inward field of unit normals, and a negative signed radius corresponds to the orientation given by the outward field of unit normals. Hence, the two orientations of the sphere in $\mathbf{R}^n$ with center p and unsigned radius $r > 0$ are represented by the two projective points,

$$\left[\left(\frac{1 + p \cdot p - r^2}{2}, \frac{1 - p \cdot p + r^2}{2}, p, \pm r\right)\right] \tag{12}$$

in Q^{n+1}. Next if $\zeta_1 + \zeta_2 = 0$, then $[\zeta]$ represents a hyperplane in $\mathbf{R}^n$, as in equation (8). For $\zeta = (h, -h, N)$, with $|N| = 1$, we have $(\zeta, \zeta) = 1$. Then the two projective points on Q^{n+1} induced by ζ and $-\zeta$ are

$$[(h, -h, N, \pm 1)]. \tag{13}$$

These represent the two orientations of the plane with equation $u \cdot N = h$. We make the convention that $[(h, -h, N, 1)]$ corresponds to the orientation given by the field of unit normals N, while the orientation given by $-N$ corresponds to the point $[(h, -h, N, -1)] = [(-h, h, -N, 1)]$.

Thus far we have determined a bijective correspondence between the set of points x in Q^{n+1} with $x_{n+3} \neq 0$ and the set of all oriented spheres and planes in $\mathbf{R}^n$. Suppose now that $x_{n+3} = 0$, i.e., consider a point $[(z, 0)]$, for $z \in \mathbf{R}^{n+2}_1$. Then $\langle x, x \rangle = (z, z) = 0$, and $[z] \in \mathbf{P}^{n+1}$ is simply a point of the Möbius sphere Σ. Thus we have the following bijective correspondence between objects in Euclidean space and points on the Lie quadric:

Euclidean	**Lie**	
points: $u \in \mathbf{R}^n$	$\left[\left(\frac{1+u\cdot u}{2}, \frac{1-u\cdot u}{2}, u, 0\right)\right]$	
∞	$[(1, -1, 0, 0)]$	(14)
spheres: center p, signed radius r	$\left[\left(\frac{1+p\cdot p-r^2}{2}, \frac{1-p\cdot p+r^2}{2}, p, r\right)\right]$	
planes: $u \cdot N = h$, unit normal N	$[(h, -h, N, 1)]$	

In Lie sphere geometry, points are considered to be spheres of radius zero, or point spheres. From now on, we will use the term *Lie sphere* or simply "sphere" to denote an oriented sphere, oriented plane or a point sphere in $\mathbf{R}^n \cup \{\infty\}$. We will refer to the coordinates on the right side of equation (14) as the *Lie coordinates* of the corresponding point, sphere or plane. In the case of $\mathbf{R}^2$ and $\mathbf{R}^3$, respectively, these coordinates were classically called *pentaspherical* and *hexaspherical* coordinates (see [5]).

At times it is useful to have formulas to convert Lie coordinates back into Cartesian equations for the corresponding Euclidean object. Suppose first that $[x]$ is a point on the Lie quadric with $x_1 + x_2 \neq 0$. Then $x = \rho y$, for some $\rho \neq 0$, where y is one of the standard forms on the right side of the table above. From the table, we see that $y_1 + y_2 = 1$, for all proper points and all spheres. Hence if we divide x by $x_1 + x_2$, the new vector will be in standard form, and we can read off the corresponding Euclidean object from the table. In particular, if $x_{n+3} = 0$, then $[x]$ represents the point $u = (u_3, \ldots, u_{n+2})$ where

$$u_i = x_i/(x_1 + x_2), \quad 3 \leq i \leq n+2. \tag{15}$$

If $x_{n+3} \neq 0$, then $[x]$ represents the sphere with center $p = (p_3, \ldots, p_{n+2})$ and signed radius r given by

$$p_i = x_i/(x_1 + x_2), \quad 3 \leq i \leq n+2; \quad r = x_{n+3}/(x_1 + x_2). \tag{16}$$

Finally, suppose that $x_1 + x_2 = 0$. If $x_{n+3} = 0$, then the equation $\langle x, x \rangle = 0$ forces x_i to be zero for $3 \leq i \leq n+2$. Thus $[x] = [(1, -1, 0, \ldots, 0)]$, the improper point. If $x_{n+3} \neq 0$, we divide x by x_{n+3} to make the last coordinate 1. Then if we set $N = (N_3, \ldots, N_{n+2})$ and h according to

$$N_i = x_i/x_{n+3}, \quad 3 \leq i \leq n+2; \quad h = x_1/x_{n+3}, \tag{17}$$

the conditions $\langle x, x \rangle = 0$ and $x_1 + x_2 = 0$ force N to have unit length. Thus $[x]$ corresponds to the hyperplane $u \cdot N = h$, with unit normal N and h as in equation (17).

1.5 Geometry of hyperspheres in S^n and H^n

In some ways it is simpler to use the sphere S^n rather than $\mathbf{R}^n$ as the base space for the study of Möbius or Lie sphere geometry. This avoids the use of stereographic projection and the need to refer to an improper point or to distinguish between spheres and planes. Furthermore, the correspondence in the table in equation (14) can be reduced to a single formula (21) below.

As in §1.3, we consider S^n to be the unit sphere in $\mathbf{R}^{n+1}$, and then embed $\mathbf{R}^{n+1}$ into $\mathbf{P}^{n+1}$ by the canonical embedding ϕ. Then $\phi(S^n)$ is the Möbius sphere Σ, given by the equation $(z, z) = 0$ in homogeneous coordinates. First we find the Möbius equation for the unoriented hypersphere in S^n with center $p \in S^n$ and spherical radius ρ, for $0 < \rho < \pi$. This hypersphere is the intersection of S^n with the hyperplane in $\mathbf{R}^{n+1}$ given by the equation

$$p \cdot y = \cos \rho, \quad 0 < \rho < \pi. \tag{18}$$

Let $[z] = \phi(y) = [(1, y)]$. Then

$$p \cdot y = \frac{-(z, (0, p))}{(z, e_1)}.$$

Thus equation (18) can be rewritten as

$$(z, (\cos \rho, p)) = 0. \tag{19}$$

Therefore, a point $y \in S^n$ is on the hyperplane determined by equation (18) if and only if $\phi(y)$ lies on the polar hyperplane in $\mathbf{P}^{n+1}$ of the point

$$[\xi] = [(\cos\rho, p)]. \tag{20}$$

To obtain the two oriented spheres determined by equation (18) note that

$$(\xi, \xi) = -\cos^2\rho + 1 = \sin^2\rho.$$

Noting that $\sin\rho \neq 0$, we let $\zeta = \pm\xi/\sin\rho$. Then the point $[(\zeta, 1)]$ is on the quadric Q^{n+1}, and

$$[(\zeta, 1)] = [(\xi, \pm\sin\rho)] = [(\cos\rho, p, \pm\sin\rho)].$$

We can incorporate the sign of the last coordinate into the radius and thereby arrange that the oriented sphere S with signed radius $\rho \neq 0, -\pi < \rho < \pi$, and center p corresponds to a point in Q^{n+1} as follows:

$$S \longleftrightarrow [(\cos\rho, p, \sin\rho)]. \tag{21}$$

The formula still makes sense if the radius $\rho = 0$, in which case it yields the point sphere $[(1, p, 0)]$. This one formula (21) plays the role of all the formulas given in equation (14) in the preceding section for the Euclidean case.

As in the Euclidean case, the orientation of a sphere S in S^n is determined by a choice of unit normal field to S in S^n. Geometrically, we take the positive radius in (21) to correspond to the field of unit normals which are tangent vectors to geodesics from $-p$ to p. Each oriented sphere can be considered in two ways, with center p and signed radius $\rho, -\pi < \rho < \pi$, or with center $-p$ and the appropriate signed radius $\rho \pm \pi$.

Given a point $[x]$ in the quadric Q^{n+1}, we now determine the corresponding hypersphere in S^n. Multiplying by -1, if necessary, we may assume that the first homogeneous coordinate x_1 of x satisfies $x_1 \geq 0$. If $x_1 > 0$, then we see from (21) that the center p and signed radius $\rho, -\pi/2 < \rho < \pi/2$, satisfy

$$\tan\rho = x_{n+3}/x_1, \quad p = (x_2, \ldots, x_{n+2})/(x_1^2 + x_{n+3}^2)^{1/2}. \tag{22}$$

If $x_1 = 0$, then $x_{n+3} \neq 0$, so we can divide by x_{n+3} to obtain a point of the form $(0, p, 1)$. This corresponds to the oriented hypersphere with center p and signed radius $\pi/2$, which is a great sphere in S^n.

To treat oriented hyperspheres in hyperbolic space H^n, we let $\mathbf{R}_1^{n+1}$ be the Lorentz subspace of $\mathbf{R}_1^{n+2}$ spanned by the orthonormal basis $\{e_1, e_3, \ldots, e_{n+2}\}$. Then H^n is the hypersurface

$$\{y \in \mathbf{R}_1^{n+1} \mid (y, y) = -1,\ y_1 \geq 1\},$$

on which the restriction of the Lorentz metric $(\ ,\ )$ is a positive definite metric of constant sectional curvature -1 (see [51, Vol. II, p. 268–271] for more detail). The distance between two points p and q in H^n is given by

$$d(p, q) = \cosh^{-1}(-(p, q)).$$

Thus the equation for the unoriented sphere in H^n with center p and radius ρ is

$$(p, y) = -\cosh\rho. \tag{23}$$

As before with S^n, we first embed $\mathbf{R}_1^{n+1}$ into $\mathbf{P}^{n+1}$ by the map

$$\psi(y) = [y + e_2].$$

Let $p \in H^n$ and let $z = y + e_2$ for $y \in H^n$. Then we have

$$(p, y) = (z, p)/(z, e_2).$$

Hence equation (23) is equivalent to the condition that $[z] = [y + e_2]$ lies on the polar hyperplane in $\mathbf{P}^{n+1}$ to

$$[\xi] = [p + \cosh\rho\ e_2].$$

Following exactly the same procedure as in the spherical case, we find that the oriented hypersphere S in H^n with center p and signed radius ρ corresponds to a point in Q^{n+1} as follows:

$$S \longleftrightarrow [p + \cosh\rho\ e_2 + \sinh\rho\ e_{n+3}]. \tag{24}$$

There is also a stereographic projection τ with pole $-e_1$ from H^n onto the unit disk D^n in $\mathbf{R}^n = \text{Span}\ \{e_3, \ldots, e_{n+2}\}$ given by

$$\tau(y_1, y_3, \ldots, y_{n+2}) = (y_3, \ldots, y_{n+2})/(y_1 + 1). \tag{25}$$

The metric g induced on D^n in order to make τ an isometry is the usual Poincaré metric.

From the point of view of Klein's Erlangen Program, all three of these geometries, Euclidean, spherical and hyperbolic, are subgeometries of Lie sphere geometry (see [17, pp. 46–49]).

1.6 Oriented contact of spheres

In Möbius geometry, the principal geometric quantity is the angle. In Lie sphere geometry, the corresponding central concept is that of oriented contact of spheres. Two oriented spheres S_1 and S_2 in $\mathbf{R}^n$ are in *oriented contact* if they are tangent to each other and they have the same orientation at the point of contact.

If p_1 and p_2 are the respective centers of S_1 and S_2, and r_1 and r_2 are their respective signed radii, then the analytic condition for oriented contact is

$$|p_1 - p_2| = |r_1 - r_2|. \tag{26}$$

An oriented sphere S with center p and signed radius r is in oriented contact with an oriented hyperplane π with unit normal N and equation $u \cdot N = h$ if π is tangent to S and their orientations agree at the point of contact. Analytically, this is just the equation

$$p \cdot N = r + h. \tag{27}$$

Two oriented planes π_1 and π_2 are in oriented contact if their unit normals N_1 and N_2 are the same. Two such planes can be thought of as two oriented spheres in oriented contact at the improper point.

A proper point u in $\mathbf{R}^n$ is in oriented contact sphere or plane if it lies on the sphere or plane. Finally, the improper point is in oriented contact with each plane, since it lies on each plane.

Suppose that S_1 and S_2 are two Lie spheres which are represented in the standard form given in equation (14) by $[k_1]$ and $[k_2]$. One can check directly that in all cases, the analytic condition for oriented contact is equivalent to the equation

$$\langle k_1, k_2 \rangle = 0. \tag{28}$$

It follows from the linear algebra of indefinite scalar product spaces that the Lie quadric contains projective lines but no linear subspaces of higher dimension (see Corollary 1.1 below). The set of oriented spheres in $\mathbf{R}^n$ corresponding to the points on a line on Q^{n+1} forms a so-called *parabolic pencil* of spheres.

We will show that each parabolic pencil contains exactly one point sphere. Furthermore, if this point sphere is a proper point p in $\mathbf{R}^n$, then the pencil contains exactly one hyperplane π. The pencil consists of all oriented hyperspheres in oriented contact with π at the point p.

The fundamental result needed from linear algebra is the following. Note that a subspace of a scalar product space is called *lightlike* if it consists of only lightlike vectors.

Theorem 1.2. *Let $(\ , \ )$ be a scalar product of signature $(n-k, k)$ on a real vector space V. Then the maximal dimension of a lightlike subspace of V is the minimum of the two numbers k and $n-k$.*

Proof. First, note that the theorem holds for scalar products having signature $(n-k, k)$ if and only if it holds for scalar products of signature $(k, n-k)$, since changing the signs of the quantities (e_i, e_i) for an orthonormal basis does not change the set of lightlike vectors.

Thus, we now assume that $k \leq n-k$ and do the proof by induction on the index k. The theorem is clearly true for scalar products of index 0, since the only lightlike vector is 0 itself. Assume now that the theorem holds for all spaces with a scalar product of index $k-1$, and let V be a scalar product space of index $k \geq 1$. Let W be a lightlike subspace of V of maximal dimension, and let v be a timelike vector in V. Then the scalar product restricts to a scalar product of index $k-1$ on the hyperplane $U = v^{\perp}$, and $W \cap U$ is a lightlike subspace of U. By the induction hypothesis, dim $W \cap U \leq k-1$ and therefore, dim $W \leq k$, as desired. On the other hand, it is easy to exhibit a lightlike subspace of V of dimension k. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis for V with $\{e_1, \ldots, e_k\}$ timelike and the rest spacelike. For $1 \leq i \leq k$, let $v_i = e_i + e_{k+i}$. Then the span of $\{v_1, \ldots, v_k\}$ is a lightlike subspace of dimension k. $\square$

Corollary 1.1. *The Lie quadric contains projective lines but no linear subspaces of higher dimension.*

Proof. This follows immediately from Theorem 1.2, since a linear subspace of $\mathbf{P}^{n+2}$ of dimension $k-1$ that lies on the quadric corresponds to a lightlike vector subspace of dimension k in $\mathbf{R}^{n+3}_2$. $\square$

Theorem 1.2 also implies the following result concerning the orthogonal complement of a line on the quadric. This was pointed out by Pinkall [78, p. 24].

Corollary 1.2. *Let ℓ be a line on $\mathbf{P}^{n+2}$ that lies on the quadric Q^{n+1}.*
(a) *If $[x] \in \ell^{\perp}$ and $[x]$ is lightlike, then $[x] \in \ell$.*
(b) *If $[x] \in \ell^{\perp}$ and $[x]$ is not on ℓ, then $[x]$ is spacelike.*

Proof. (a) Suppose that $[x]$ is a lightlike point in $\ell^\perp$ but not on ℓ. Then the 2-dimensional linear lightlike subspace spanned by $[x]$ and ℓ lies on the quadric, contradicting Corollary 1.1.
(b) Suppose that $[x]$ is in $\ell^\perp$ but not on ℓ. From (a) we know that $[x]$ is either spacelike or timelike. Suppose that $[x]$ is timelike. Then the Lie metric $\langle\ ,\ \rangle$ has signature $(n+1,1)$ on the vector space $x^\perp$, and $x^\perp$ contains the 2-dimensional lightlike vector space that projects to ℓ. This contradicts Theorem 1.2. $\square$

The key result in establishing the relationship between the points on a line in Q^{n+1} and the corresponding parabolic pencil of spheres in $\mathbf{R}^n$ is the following.

Theorem 1.3. (a) *The line in $\mathbf{P}^{n+2}$ determined by two points $[k_1]$ and $[k_2]$ of Q^{n+1} lies on Q^{n+1} if and only if the the spheres corresponding to $[k_1]$ and $[k_2]$ are in oriented contact, i.e., $\langle k_1, k_2\rangle = 0$.*
(b) *If the line $[k_1, k_2]$ lies on Q^{n+1}, then the parabolic pencil of spheres in $\mathbf{R}^n$ corresponding to points on $[k_1, k_2]$ is precisely the set of all spheres in oriented contact with both $[k_1]$ and $[k_2]$.*

Proof. (a) The line $[k_1, k_2]$ consists of the points of the form $[\alpha k_1 + \beta k_2]$, where α and β are any two real numbers, at least one of which is not zero. Since $[k_1]$ and $[k_2]$ are on Q^{n+1}, we have

$$\langle \alpha k_1 + \beta k_2, \alpha k_1 + \beta k_2\rangle = 2\alpha\beta\langle k_1, k_2\rangle.$$

Thus the line is contained in the quadric if and only if $\langle k_1, k_2\rangle = 0$.
(b) Let $[\alpha k_1 + \beta k_2]$ be any point on the line. Since $\langle k_1, k_2\rangle = 0$ by (a), we easily compute that $[\alpha k_1 + \beta k_2]$ is orthogonal to both $[k_1]$ and $[k_2]$. Hence, the corresponding sphere is in oriented contact with the spheres corresponding to $[k_1]$ and $[k_2]$. Conversely, suppose that the sphere corresponding to a point $[k]$ on the quadric is in oriented contact with the spheres corresponding to $[k_1]$ and $[k_2]$. Then $[k]$ is orthogonal to every point on the line $[k_1, k_2]$, and so $[k]$ is on the line $[k_1, k_2]$ by Corollary 1.2 (a). $\square$

Given any timelike point $[z]$ in $\mathbf{P}^{n+2}$, the scalar product $\langle\ ,\ \rangle$ has signature $(n+1,1)$ on $z^\perp$. Hence, $z^\perp$ intersects Q^{n+1} in a Möbius space. We now show that any line on the quadric intersects such a Möbius space at exactly one point.

Corollary 1.3. *Let $[z]$ be a timelike point in $\mathbf{P}^{n+2}$ and ℓ a line that lies on Q^{n+1}. Then ℓ intersects $z^{\perp}$ at exactly one point.*

Proof. Any line in projective space intersects a hyperplane in at least one point. We simply must show that ℓ is not contained in $z^{\perp}$. But this follows from Theorem 1.2, since $\langle\ ,\ \rangle$ has signature $(n+1,1)$ on $z^{\perp}$, and therefore $z^{\perp}$ cannot contain the 2-dimensional lightlike vector space that projects to ℓ. $\square$

As a consequence, we obtain the following corollary.

Corollary 1.4. *Every parabolic pencil contains exactly one point sphere. Furthermore, if the point sphere is a proper point, then the pencil contains exactly one plane.*

Proof. The point spheres are precisely the points of intersection of Q^{n+1} with $e_{n+3}^{\perp}$. Thus each parabolic pencil contains exactly one point sphere by Corollary 1.3. The hyperplanes correspond to the points in the intersection of Q^{n+1} with $(e_1-e_2)^{\perp}$. The line ℓ on the quadric corresponding to the given parabolic pencil intersects this hyperplane at exactly one point unless ℓ is contained in the hyperplane. But ℓ is contained in $(e_1-e_2)^{\perp}$ if and only if the improper point $[e_1-e_2]$ is in $\ell^{\perp}$. By Corollary 1.2 (a), this implies that the point $[e_1-e_2]$ is on ℓ. Hence, if the point sphere of the pencil is not the improper point, then the pencil contains exactly one hyperplane. $\square$

By Corollary 1.4 and Theorem 1.3, we see that if the point sphere in a parabolic pencil is a proper point p in $\mathbf{R}^n$, then the pencil consists precisely of all spheres in oriented contact with a certain oriented plane π at p. Thus, one can identify the parabolic pencil with the point (p,N) in the unit tangent bundle to $\mathbf{R}^n$ where N is the unit normal to the oriented plane π. If the point sphere of the pencil is the improper point, then the pencil must consist entirely of planes. Since these planes are all in oriented contact, they all have the same unit normal N. Thus the pencil can be identified with the point (∞,N) in the unit tangent bundle to $\mathbf{R}^n\cup\{\infty\}=S^n$.

It is also useful to have this correspondence between parabolic pencils and elements of the unit tangent bundle T_1S^n expressed in terms of the spherical metric on S^n. Suppose that ℓ is a line on the quadric. From Corollary 1.3 and equation (21), we see that ℓ intersects both $e_1^{\perp}$ and $e_{n+3}^{\perp}$ at exactly one point.

So the corresponding parabolic pencil contains exactly one point sphere and one great sphere, represented respectively by the points,

$$[k_1] = [(1, p, 0)], \quad [k_2] = [(0, \xi, 1)].$$

The fact that $\langle k_1, k_2 \rangle = 0$ is equivalent to the condition $p \cdot \xi = 0$, i.e., ξ is tangent to S^n at p. Hence the parabolic pencil of spheres corresponding to ℓ can be identified with the point (p, ξ) in $T_1 S^n$. The points on the line ℓ can be parametrized as

$$[K_t] = [\cos t\ k_1 + \sin t\ k_2] = [(\cos t, \cos t\ p + \sin t\ \xi, \sin t)].$$

From equation (21), we see that $[K_t]$ corresponds to the sphere in S^n with center

$$p_t = \cos t\ p + \sin t\ \xi, \tag{29}$$

and signed radius t. These are precisely the spheres through p in oriented contact with the great sphere corresponding to $[k_2]$. Their centers lie along the geodesic in S^n with initial point p and initial velocity vector ξ.

1.7 Lie sphere transformations

A *Lie sphere transformation* is a projective transformation of $\mathbf{P}^{n+2}$ which takes Q^{n+1} to itself. In terms of the geometry of $\mathbf{R}^n$, a Lie sphere transformation maps Lie spheres to Lie spheres. (Here the term "Lie sphere" includes oriented spheres, oriented planes and point spheres.) Furthermore, since it is projective, a Lie sphere transformation maps lines on Q^{n+1} to lines on Q^{n+1}. Thus, it preserves oriented contact of spheres in $\mathbf{R}^n$. We will first show that the group G of Lie sphere transformations is isomorphic to $O(n+1,2)/\{\pm I\}$, where $O(n+1,2)$ is the group of orthogonal transformations of $\mathbf{R}^{n+3}_2$.

Pinkall [82] proved the so-called "Fundamental Theorem of Lie sphere geometry," which states that any line preserving diffeomorphism of Q^{n+1} is the restriction to Q^{n+1} of a projective transformation, that is, a transformation of the space of oriented spheres which preserves oriented contact must be a Lie sphere transformation. We will not give the proof here, but refer the reader to Pinkall's [82, p. 431] paper or the book [17, p. 28].

Recall that a linear transformation $A \in GL(n+1)$ induces a projective transformation $P(A)$ on $\mathbf{P}^n$ defined by $P(A)[x] = [Ax]$. The map P is a homomorphism of $GL(n+1)$ onto the group $PGL(n)$ of projective transformations of $\mathbf{P}^n$. It is well known (see, for example, Samuel [95, p. 6]) that the kernel of P is the group of all nonzero scalar multiples of the identity transformation I.

The fact that the group G is isomorphic to $O(n+1,2)/\{\pm I\}$ follows immediately from the following theorem. Here we let $\langle\ ,\ \rangle$ denote the scalar product on $\mathbf{R}^n_k$ (see [17, p. 26] for a proof).

Theorem 1.4. *Let A be a nonsingular linear transformation on the indefinite scalar product space $\mathbf{R}^n_k, 1 \leq k \leq n-1$, such that A takes lightlike vectors to lightlike vectors.*
(a) *Then there is a nonzero constant λ such that $\langle Av, Aw\rangle = \lambda\langle v, w\rangle$ for all v, w in $\mathbf{R}^n_k$.*
(b) *Furthermore, if $k \neq n-k$, then $\lambda > 0$.*

Remark 1.1. In the case $k = n-k$, conclusion (b) does not necessarily hold. For example, suppose that $\{v_1, \ldots, v_k, w_1, \ldots, w_k\}$ is an orthornormal basis for $\mathbf{R}^{2k}_k$ with $v_1, \ldots, v_k$ timelike and $w_1, \ldots, w_k$ spacelike. Then the linear map T defined by $Tv_i = w_i, Tw_i = v_i$, for $1 \leq i \leq k$, preserves lightlike vectors, but the corresponding $\lambda = -1$.

From Theorem 1.4 we immediately obtain the following corollary.

Corollary 1.5. (a) *The group G of Lie sphere transformations is isomorphic to $O(n+1,2)/\{\pm I\}$.*
(b) *The group H of Möbius transformations is isomorphic to $O(n+1,1)/\{\pm I\}$.*

Proof. (a) Suppose $\alpha = P(A)$ is a Lie sphere transformation. By Theorem 1.4, we have $\langle Av, Aw\rangle = \lambda\langle v, w\rangle$ for all v, w in $\mathbf{R}^{n+3}_2$, where λ is a positive constant. Set B equal to $A/\sqrt{\lambda}$. Then B is in $O(n+1,2)$ and $\alpha = P(B)$. Thus, every Lie sphere transformation can be represented by an orthogonal transformation. Conversely, if $B \in O(n+1,2)$, then $P(B)$ is clearly a Lie sphere transformation. Now let $\Psi : O(n+1,2) \to G$ be the restriction of the homomorphism P to $O(n+1,2)$. Then Ψ is surjective with kernel equal to the intersection of the kernel of P with $O(n+1,2)$, i.e., kernel $\Psi = \{\pm I\}$.
(b) This follows from Theorem 1.4 in the same manner as (a) with the Lorentz metric being used instead of the Lie metric. □

Remark 1.2. *On Möbius transformations in Lie sphere geometry.*
A Möbius transformation is a transformation on the space of unoriented spheres, i.e., the space of projective classes of spacelike vectors in $\mathbf{R}^{n+2}_1$.

Hence, each Möbius transformation naturally induces two Lie sphere transformations on the space Q^{n+1} of oriented spheres. Specifically, if A is in $O(n+1,1)$, then we can extend A to a transformation B in $O(n+1,2)$ by setting $B=A$ on $\mathbf{R}_1^{n+2}$ and $B(e_{n+3})=e_{n+3}$. In terms of matrix representation with respect to the standard orthonormal basis, B has the form

$$B=\left[\begin{array}{cc} A & 0 \\ 0 & 1 \end{array}\right]. \tag{30}$$

Note that while A and $-A$ induce the same Möbius transformation, the Lie transformation $P(B)$ is not the same as the Lie transformation $P(C)$ induced by the matrix

$$C=\left[\begin{array}{cc} -A & 0 \\ 0 & 1 \end{array}\right] \simeq \left[\begin{array}{cc} A & 0 \\ 0 & -1 \end{array}\right],$$

where $\simeq$ denotes equivalence as projective transformations. Hence, the Möbius transformation $P(A)=P(-A)$ induces two Lie transformations, $P(B)$ and $P(C)$. Finally, note that $P(B)=\Gamma P(C)$, where Γ is the Lie transformation represented in matrix form by

$$\Gamma=\left[\begin{array}{cc} I & 0 \\ 0 & -1 \end{array}\right] \simeq \left[\begin{array}{cc} -I & 0 \\ 0 & 1 \end{array}\right].$$

From equation (14), we see that Γ has the effect of changing the orientation of every oriented sphere or plane. We will call Γ the *change of orientation transformation* or in German "Richtungswechsel." Hence, the two Lie sphere transformations induced by the Möbius transformation $P(A)$ differ by this change of orientation factor. Thus, the group of Lie transformations induced from Möbius transformations is isomorphic to $O(n+1,1)$ and is a double covering of the Möbius group H. This group consists of those Lie transformations that map $[e_{n+3}]$ to itself. Since such a transformation must also take $e_{n+3}^{\perp}$ to itself, this is precisely the group of Lie transformations which take point spheres to point spheres. When working in the context of Lie sphere geometry, we will often refer to these transformations as "Möbius transformations."

1.8 Inversions

In this section, we will show that the group G of Lie sphere transformations and the group H of Möbius transformations are generated by inversions. This follows from the fact that the corresponding orthogonal groups are generated by reflections in hyperplanes. In fact, every orthogonal transformation on $\mathbf{R}^n_k$ is a product of at most n reflections, a result due to Cartan and Dieudonné (see Theorem 1.5 below).

For the moment, let $\langle\ ,\ \rangle$ denote the scalar product of signature $(n-k,k)$ on $\mathbf{R}^n_k$. A hyperplane π in $\mathbf{R}^n_k$ is called *nondegenerate* if the scalar product restricts to a nondegenerate form on π. A hyperplane π is nondegenerate if and only if its pole ξ is not lightlike (see, for example, [17, p. 10]). Now let ξ be a unit spacelike or unit timelike vector in $\mathbf{R}^n_k$. The *reflection* Ω_π of $\mathbf{R}^n_k$ in the hyperplane π with pole ξ is defined by the formula

$$\Omega_\pi x = x - \frac{2\langle x,\xi\rangle\xi}{\langle \xi,\xi\rangle}. \tag{31}$$

Note that we do not define reflection in degenerate hyperplanes, i.e., those which have lightlike poles. It is clear that Ω_π fixes every point in π and that $\Omega_\pi\xi = -\xi$. A direct computation shows that Ω_π is in $O(n-k,k)$ and that $\Omega_\pi^2 = I$.

We now state the theorem of Cartan and Dieudonné. (For a proof, see Chapter 3 of E. Artin's book [3], Cartan [12, pp. 10–12], or [17, p. 32].)

Theorem 1.5. *Every orthogonal transformation of $\mathbf{R}^n_k$ is the product of at most n reflections in hyperplanes.*

The Lie sphere transformation induced by a reflection Ω_π in $O(n+1,2)$ is called a *Lie inversion.* Similarly, a Möbius transformation induced by a reflection in $O(n+1,1)$ is called a *Möbius inversion.* An immediate consequence of Corollary 1.5 and Theorem 1.5 is the following.

Theorem 1.6. *The Lie sphere group G and the Möbius group H are both generated by inversions.*

We now give a geometric description of these inversions and other important types of Lie sphere transformations. We begin with a geometric description of Möbius inversions.

An orthogonal transformation in $O(n+1,1)$ induces a projective transformation on $\mathbf{P}^{n+1}$ which maps the Möbius sphere Σ to itself. A Möbius inversion is the projective transformation induced by a reflection Ω_π in $O(n+1,1)$. For the sake of brevity, we will also denote this projective transformation by

Ω_π instead of $P(\Omega_\pi)$. Let ξ be a spacelike point in $\mathbf{P}^{n+1}$ with polar hyperplane π. The hyperplane π intersects the Möbius sphere Σ in a hypersphere S^{n-1}. The Möbius inversion Ω_π, when interpreted as a transformation on $\mathbf{R}^n$, is just ordinary inversion in the hypersphere S^{n-1}. We will now recall the details of this transformation.

Since the Möbius sphere is homogeneous, all inversions in planes with spacelike poles act in essentially the same way. Let us consider the special case where S^{n-1} is the sphere of radius $r > 0$ centered at the origin in $\mathbf{R}^n$. Then by formula (6), the spacelike point ξ in $\mathbf{P}^{n+1}$ corresponding to S^{n-1} has homogeneous coordinates

$$\xi = (1 - r^2, 1 + r^2, 0)/2.$$

Let u be a point in $\mathbf{R}^n$ other than the origin. By equation (3), the point u corresponds to the point in $\mathbf{P}^{n+1}$ with homogeneous coordinates

$$x = (1 + u \cdot u, 1 - u \cdot u, 2u)/2.$$

The formula for Ω_π in homogeneous coordinates is

$$\Omega_\pi x = x - \frac{2(x, \xi)}{(\xi, \xi)}\xi, \tag{32}$$

where (,) is the Lorentz metric. A straightforward calculation shows that $\Omega_\pi x$ is the point in $\mathbf{P}^{n+1}$ with homogeneous coordinates

$$(1 + v \cdot v, 1 - v \cdot v, 2v)/2,$$

where $v = (r^2/|u|^2)u$. Thus, the Euclidean transformation induced by Ω_π maps u to the point v on the ray through u from the origin satisfying the equation $|u||v| = r^2$. From this, it is clear that the fixed points of Ω_π are precisely the points of the sphere S^{n-1}. Viewed in the projective context, this is immediately clear from equation (32), since $\Omega_\pi x = x$ if and only if $(x, \xi) = 0$. In general, an inversion of $\mathbf{R}^n$ in the hypersphere of radius r centered at a point p maps a point $u \neq p$ to the point v on the ray through u from p satisfying

$$|u - p||v - p| = r^2.$$

Another special case is when the unit spacelike vector ξ lies in the Euclidean space $\mathbf{R}^n$ spanned by $\{e_3, \ldots, e_{n+2}\}$. Then the "sphere" corresponding to

$[\xi]$ according to formula (8) is the hyperplane V through the origin in $\mathbf{R}^n$ perpendicular to ξ. In this case the Möbius inversion Ω_π is just ordinary Euclidean reflection in the hyperplane V.

A third noteworthy inversion is the change of orientation transformation Γ (see Remark 1.2) determined by the hyperplane π orthogonal to e_{n+3}.

1.9 Subgroups of the Lie sphere group

We next present an alternative way to view the Lie sphere group G by decomposing it into certain natural subgroups. To do this, we need the concept of a linear complex of spheres. The complex determined by a point ξ in $\mathbf{P}^{n+2}$ is the set of all spheres represented by points x in the Lie quadric Q^{n+1} satisfying the equation $\langle x, \xi \rangle = 0$.

The complex is said to be *elliptic* if ξ is spacelike, *hyperbolic* if ξ is timelike,and *parabolic* if ξ is lightlike. Since the Lie sphere group G acts transitively on each of the three types of points, each linear complex of a given type looks like every other complex of the same type.

A typical example of an elliptic complex is obtained by taking $\xi = e_{n+2}$. A sphere S in $\mathbf{R}^n$ represented by a point x in Q^{n+1} satisfies the equation $\langle x, \xi \rangle = 0$ if and only if its coordinate $x_{n+2} = 0$ in $\mathbf{R}^n$, i.e., the center of S lies in the hyperplane $\mathbf{R}^{n-1}$ with equation $x_{n+2} = 0$ in $\mathbf{R}^n$. The linear complex consists of all spheres and planes orthogonal to this plane, including the points of the plane itself as a special case. A Lie sphere transformation T maps each sphere in the complex to another sphere in the complex if and only if $e_{n+2}^\perp$ is an invariant subspace of T. Since T can be represented by an orthogonal transformation, this is equivalent to $T[e_{n+2}] = [e_{n+2}]$. Thus T is determined by its action on $e_{n+2}^\perp$. Let $\mathbf{R}_2^{n+2}$ denote the vector subspace $e_{n+2}^\perp$ in $\mathbf{R}_2^{n+3}$ endowed with the metric $\langle \ , \ \rangle$ inherited from $\mathbf{R}_2^{n+3}$, and let $O(n, 2)$ denote the group of orthogonal transformations of the space $\mathbf{R}_2^{n+2}$. A transformation A in $O(n, 2)$ can be extended to $\mathbf{R}_2^{n+3}$ by setting $Ae_{n+2} = e_{n+2}$. This gives an isomorphism between $O(n, 2)$ and the group of Lie sphere transformations which fix the elliptic complex. This group is a double covering of the group of Lie sphere transformations of the Euclidean space $\mathbf{R}^{n-1}$ orthogonal to e_{n+2} in $\mathbf{R}^n$.

A typical example of a hyperbolic complex is the case $\xi = e_{n+3}$. This complex consists of all point spheres. A second example is the complex corresponding to $\xi = (-r, r, 0, \ldots, 0, 1)$. This complex consists of all oriented spheres with signed radius r. The group of Lie sphere transformations which map this hyperbolic complex to itself consists of all transformations which map the projective point ξ to itself. This group is isomorphic to the Möbius subgroup of G, as discussed in Remark 1.2.

The parabolic complex determined by a point ξ in Q^{n+1} consists of all spheres in oriented contact with the sphere corresponding to ξ. A noteworthy example is the case $\xi = (1, -1, 0, \ldots, 0)$, the improper point. This complex consists of all oriented hyperplanes in $\mathbf{R}^n$. A Lie sphere transformation which fixes this complex is called a *Laguerre transformation*, and the group of such Laguerre transformations is called the *Laguerre group*.

An important type of Laguerre transformation is the *parallel transformation* P_t, which fixes the center of every sphere in $\mathbf{R}^n$ and adds t to its signed radius, where t is any real number. Using equation (14) to represent the oriented sphere with center $p \in \mathbf{R}^n$ and signed radius r, one can check that the matrix representing P_t with respect to the standard basis of $\mathbf{R}^{n+3}_2$ is (see [17, p. 46]),

$$P_t = \begin{bmatrix} 1-(t^2/2) & -t^2/2 & 0\ldots 0 & -t \\ t^2/2 & 1+(t^2/2) & 0\ldots 0 & t \\ 0 & 0 & I & 0 \\ t & t & 0\ldots 0 & 1 \end{bmatrix}. \tag{33}$$

This is parallel transformation with respect to the metric on the Euclidean space $\mathbf{R}^n$. For both the spherical and hyperbolic metrics, there is also parallel transformation P_t that adds t to the signed radius of each sphere while keeping the center fixed. As we saw in §1.5, the sphere in S^n with center p and signed radius ρ is represented by the point $[(\cos\rho, p, \sin\rho)]$ in Q^{n+1}. One easily checks that *spherical parallel transformation* P_t is accomplished by the following transformation in $O(n+1, 2)$:

$$\begin{aligned} P_t e_1 &= \cos t\ e_1 + \sin t\ e_{n+3}, \\ P_t e_{n+3} &= -\sin t\ e_1 + \cos t\ e_{n+3}, \\ P_t e_i &= e_i, \quad 2 \leq i \leq n+2. \end{aligned} \tag{34}$$

In H^n the sphere with center $p \in H^n$ and signed radius ρ corresponds to the point $[p + \cosh\rho\ e_2 + \sinh\rho\ e_{n+3}]$ in Q^{n+1}. Thus *hyperbolic parallel transformation* is accomplished by the following transformation:

$$\begin{aligned} P_t e_i &= e_i, \quad i = 1, 3, \ldots, n+2. \\ P_t e_2 &= \cosh t\ e_2 + \sinh t\ e_{n+3}, \\ P_t e_{n+3} &= \sinh t\ e_2 + \cosh t\ e_{n+3}. \end{aligned} \tag{35}$$

We close this section with some theorems that describe the Lie sphere group in various ways. Recall that the subgroup of Möbius transformations consists of those Lie sphere transformations that map point spheres to point spheres. These are precisely the Lie sphere transformations that map the point $[e_{n+3}]$ to itself. As we saw in Remark 1.2, this Möbius group is isomorphic to $O(n+1,1)$.

The following theorem demonstrates the important role played by Möbius transformations and parallel transformations in generating the Lie sphere group (see Cecil–Chern [19] or [17, p. 49]).

Theorem 1.7. *Any Lie sphere transformation α can be written as*

$$\alpha = \phi P_t \psi,$$

where ϕ and ψ are Möbius transformations and P_t is some Euclidean, spherical or hyperbolic parallel transformation.

Proof. Represent α by a transformation $A \in O(n+1,2)$. If $Ae_{n+3} = \pm e_{n+3}$, then α is a Möbius transformation. If not, then Ae_{n+3} is some unit timelike vector v linearly independent from e_{n+3}. The plane $[e_{n+3}, v]$ in $\mathbf{R}^{n+3}_2$ can have signature $(-,-), (-,+)$ or $(-,0)$. In the case where the plane has signature $(-,-)$, we can write

$$v = -\sin t \ u_1 + \cos t \ e_{n+3},$$

where u_1 is a unit timelike vector orthogonal to e_{n+3}, and $0 < t < \pi$. Let ϕ be a Möbius transformation such that $\phi^{-1}u_1 = e_1$. Then from equation (34), we see that $P_{-t}\phi^{-1}v = e_{n+3}$. Hence,

$$P_{-t}\phi^{-1}\alpha e_{n+3} = e_{n+3},$$

i.e., $P_{-t}\phi^{-1}\alpha$ is a Möbius transformation ψ. Thus, $\alpha = \phi P_t \psi$, as desired.

The other two cases are similar. If the plane $[e_{n+3}, v]$ has signature $(-,0)$, then we can write

$$v = -tu_1 + tu_2 + e_{n+3},$$

where u_1 and u_2 are unit timelike and spacelike vectors, respectively, orthogonal to e_{n+3} and to each other. If ϕ is a Möbius transformation such that $\phi^{-1}u_1 = e_1$ and $\phi^{-1}u_2 = e_2$, then $P_{-t}\phi\alpha$ is a Möbius transformation ψ, where

P_t is the Euclidean parallel transformation given in equation (33). As before, we get $\alpha = \phi P_t \psi$. Finally, if the plane $[e_{n+3}, v]$ has signature $(-,+)$, then

$$v = \sinh t\ u_2 + \cosh t\ e_{n+3},$$

for a unit spacelike vector u_2 orthogonal to e_{n+3}. Let ϕ be a Möbius transformation such that $\phi^{-1}u_2 = e_2$, and conclude that $\alpha = \phi P_t \psi$ for the hyperbolic parallel transformation P_t in equation (35). □

The subgroup of Laguerre transformations consists of those Lie sphere transformations that map hyperplanes to hyperplanes in $\mathbf{R}^n$. These are the Lie sphere transformations that map the improper point $[e_1 - e_2]$ to itself. Each Laguerre transformation corresponds to an affine Laguerre transformation of the space $\mathbf{R}^{n+1}_1$ spanned by $\{e_3, \ldots, e_{n+3}\}$ (see [17, pp. 37–46] for more detail on affine Laguerre transformations).

As before let $\mathbf{R}^n$ denote the Euclidean space spanned by the vectors $\{e_3, \ldots, e_{n+2}\}$. Recall that a *similarity transformation* of $\mathbf{R}^n$ is a mapping ϕ from $\mathbf{R}^n$ to itself, such that for all p and q in $\mathbf{R}^n$, the Euclidean distance $d(p,q)$ is transformed as follows:

$$d(\phi p, \phi q) = \kappa d(p,q),$$

for some constant $\kappa > 0$. Every similarity transformation can be written as a central dilatation followed by an isometry of $\mathbf{R}^n$. The group of Lie sphere transformations induced by similarity transformations is clearly a subgroup of both the Laguerre group and the Möbius group. The next theorem shows that it is precisely the intersection of these two subgroups (see [17, p. 47] for a proof).

Theorem 1.8. (a) *The intersection of the Laguerre group and the Möbius group is the group of Lie sphere transformations induced by similarity transformations of* $\mathbf{R}^n$.
(b) *The group G of Lie sphere transformations is generated by the union of the groups of Laguerre and Möbius.*

2 Submanifolds in Lie Sphere Geometry

In this section, we develop the framework necessary to study submanifolds within the context of Lie sphere geometry. The manifold Λ^{2n-1} of projective lines on the Lie quadric Q^{n+1} has a contact structure, i.e., a globally defined 1-form ω such that $\omega \wedge (d\omega)^{n-1} \neq 0$ on Λ^{2n-1}. This gives rise to a codimension one distribution D on Λ^{2n-1} that has integral submanifolds of dimension $n-1$, but none of higher dimension. These integral submanifolds are called Legendre submanifolds. Any submanifold of a real space-form $\mathbf{R}^n$, S^n or H^n naturally induces a Legendre submanifold, and thus Lie sphere geometry can be used to analyze submanifolds in these spaces. This has been particularly effective in the classification of proper Dupin hypersurfaces.

2.1 Contact structure on Λ^{2n-1}

As before, let $\{e_1, \ldots, e_{n+3}\}$ denote the standard orthonormal basis for $\mathbf{R}^{n+3}_2$ with e_1 and e_{n+3} timelike. We consider S^n to be the unit sphere in the Euclidean space $\mathbf{R}^{n+1}$ spanned by $\{e_2, \ldots, e_{n+2}\}$. A *contact element* on S^n is a pair (x, ξ), where $x \in S^n$ and ξ is a unit tangent vector to S^n at x. Thus, the space of contact elements is the unit tangent bundle T_1S^n. We consider T_1S^n to be the $(2n-1)$-dimensional submanifold of $S^n \times S^n \subset \mathbf{R}^{n+1} \times \mathbf{R}^{n+1}$ given by

$$T_1S^n = \{(x, \xi) \mid \quad |x| = 1, \quad |\xi| = 1, \quad x \cdot \xi = 0\}. \tag{36}$$

In general, a $(2n-1)$-dimensional manifold V^{2n-1} is said to be a *contact manifold* if it carries a global 1-form ω such that

$$\omega \wedge (d\omega)^{n-1} \neq 0 \tag{37}$$

at all points of V^{2n-1}. Such a form ω is called a *contact form*. It is known (see, for example, [4, p. 10]) that the unit tangent bundle T_1M of any n-dimensional Riemannian manifold M is a $(2n-1)$-dimensional contact manifold. A contact form ω defines a codimension one distribution D on V^{2n-1},

$$D_p = \{Y \in T_pV^{2n-1} \mid \omega(Y) = 0\}, \tag{38}$$

for $p \in V^{2n-1}$, called the *contact distribution*. This distribution is as far from being integrable as possible, in that there exist integral submanifolds of D of dimension $n-1$ but none of higher dimension (see Theorem 2.1 below). A contact distribution determines the corresponding contact form up to multiplication by a nonvanishing smooth function.

In our particular case, a tangent vector to T_1S^n at a point (x, ξ) can be written in the form (X, Z) where

$$X \cdot x = 0, \quad Z \cdot \xi = 0. \tag{39}$$

Differentiation of the condition $x \cdot \xi = 0$ implies that (X, Z) must also satisfy

$$X \cdot \xi + Z \cdot x = 0. \tag{40}$$

We will show that the form ω defined by

$$\omega(X, Z) = X \cdot \xi, \tag{41}$$

is a contact form on $T_1 S^n$. Thus, at a point (x, ξ), the distribution D is the $(2n-2)$-dimensional space of vectors (X, Z) satisfying $X \cdot \xi = 0$, as well as the equations (39) and (40). Of course, the equation $X \cdot \xi = 0$ together with equation (40) implies that

$$Z \cdot x = 0, \tag{42}$$

for vectors (X, Z) in D. To see that ω satisfies the condition (37), we will identify $T_1 S^n$ with the manifold Λ^{2n-1} of projective lines on the Lie quadric Q^{n+1} and compute $d\omega$ using the method of moving frames. The results in this calculation will turn out to be useful in our general study of submanifolds.

We establish a bijective correspondence between the points of $T_1 S^n$ and the lines on Q^{n+1} by the map

$$(x, \xi) \mapsto [Y_1(x, \xi), Y_{n+3}(x, \xi)], \tag{43}$$

where

$$Y_1(x, \xi) = (1, x, 0), \quad Y_{n+3}(x, \xi) = (0, \xi, 1). \tag{44}$$

The points on a line on Q^{n+1} correspond to a parabolic pencil of spheres in S^n. By formula (21), the point $[Y_1(x, \xi)]$ corresponds to the unique point sphere in the pencil determined by the line $[Y_1(x, \xi), Y_{n+3}(x, \xi)]$, and $Y_{n+3}(x, \xi)$ corresponds to the unique great sphere in the pencil. Since every line on the quadric contains exactly one point sphere and one great sphere by Corollary 1.3, the correspondence in (43) is bijective. We put a differentiable structure on the manifold Λ^{2n-1} in such a way that the map in (43) becomes a diffeomorphism.

We now introduce the method of moving frames in the context of Lie sphere geometry, as in Cecil–Chern [19]. This method has been very useful in proving many important results about Dupin hypersurfaces.

Since we want to define frames on the manifold Λ^{2n-1}, it is better to use frames for which some of the vectors are lightlike, rather than orthonormal frames. To facilitate the exposition, we will use the following range of indices in this section:

$$1 \leq a, b, c \leq n + 3, \quad 3 \leq i, j, k \leq n + 1. \tag{45}$$

A *Lie frame* is an ordered set of vectors $\{Y_1, \ldots, Y_{n+3}\}$ in $\mathbf{R}^{n+3}_2$ satisfying the relations

$$\langle Y_a, Y_b \rangle = g_{ab}, \tag{46}$$

for

$$[g_{ab}] = \begin{bmatrix} J & 0 & 0 \\ 0 & I_{n-1} & 0 \\ 0 & 0 & J \end{bmatrix}, \tag{47}$$

where I_{n-1} is the $(n-1) \times (n-1)$ identity matrix and

$$J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{48}$$

If $(y_1, \ldots, y_{n+3})$ are homogeneous coordinates on $\mathbf{P}^{n+2}$ with respect to a Lie frame, then the Lie metric has the form

$$\langle y, y \rangle = 2(y_1 y_2 + y_{n+2} y_{n+3}) + y_3^2 + \cdots + y_{n+1}^2. \tag{49}$$

The space of all Lie frames can be identified with the group $O(n+1, 2)$ of which the Lie sphere group G, being isomorphic to $O(n+1, 2)/\{\pm I\}$, is a quotient group. In this space, we introduce the *Maurer–Cartan forms* ω_a^b by the equation

$$dY_a = \sum \omega_a^b Y_b, \tag{50}$$

and we adopt the convention that the sum is always over the repeated index. Differentiating equation (46), we get

$$\omega_{ab} + \omega_{ba} = 0, \tag{51}$$

where

$$\omega_{ab} = \sum g_{bc} \omega_a^c. \tag{52}$$

Equation (51) says that the following matrix is skew-symmetric,

$$[\omega_{ab}] = \begin{bmatrix} \omega_1^2 & \omega_1^1 & \omega_1^i & \omega_1^{n+3} & \omega_1^{n+2} \\ \omega_2^2 & \omega_2^1 & \omega_2^i & \omega_2^{n+3} & \omega_2^{n+2} \\ \omega_j^2 & \omega_j^1 & \omega_j^i & \omega_j^{n+3} & \omega_j^{n+2} \\ \omega_{n+2}^2 & \omega_{n+2}^1 & \omega_{n+2}^i & \omega_{n+2}^{n+3} & \omega_{n+2}^{n+2} \\ \omega_{n+3}^2 & \omega_{n+3}^1 & \omega_{n+3}^i & \omega_{n+3}^{n+3} & \omega_{n+3}^{n+2} \end{bmatrix}. \tag{53}$$

Taking the exterior derivative of equation (50) yields the *Maurer–Cartan equations*,

$$d\omega_a^b = \sum \omega_a^c \wedge \omega_c^b. \tag{54}$$

We now produce a contact form on T_1S^n in the context of moving frames. We want to choose a local frame $\{Y_1, \ldots, Y_{n+3}\}$ on T_1S^n with Y_1 and Y_{n+3} given by equation (44). When we transfer this frame to Λ^{2n-1}, it will have the property that for each point $\lambda \in \Lambda^{2n-1}$, the line $[Y_1, Y_{n+3}]$ of the frame at λ is the line on the quadric Q^{n+1} corresponding to λ.

On a sufficiently small open subset U in T_1S^n, we can find smooth mappings,

$$v_i : U \to \mathbf{R}^{n+1}, \quad 3 \leq i \leq n+1,$$

such that at each point $(x, \xi) \in U$, the vectors $v_3(x, \xi), \ldots, v_{n+1}(x, \xi)$ are unit vectors orthogonal to each other and to x and ξ. By equations (39) and (40), we see that the vectors

$$\{(v_i, 0), (0, v_i), (\xi, -x)\}, \quad 3 \leq i \leq n+1, \tag{55}$$

form a basis to the tangent space to T_1S^n at (x, ξ). We now define a Lie frame on U as follows:

$$\begin{aligned} Y_1(x,\xi) &= (1, x, 0), \\ Y_2(x,\xi) &= (-1/2, x/2, 0), \\ Y_i(x,\xi) &= (0, v_i(x,\xi), 0), \quad 3 \leq i \leq n+1, \\ Y_{n+2}(x,\xi) &= (0, \xi/2, -1/2) \\ Y_{n+3}(x,\xi) &= (0, \xi, 1). \end{aligned} \tag{56}$$

We want to determine certain of the Maurer–Cartan forms ω_a^b by computing dY_a on the basis given in (55). In particular, we compute the derivatives dY_1

and dY_{n+3} and find

$$\begin{aligned} dY_1(v_i, 0) &= (0, v_i, 0) = Y_i, \\ dY_1(0, v_i) &= (0, 0, 0), \\ dY_1(\xi, -x) &= (0, \xi, 0) = Y_{n+2} + (1/2)Y_{n+3}, \end{aligned} \tag{57}$$

and

$$\begin{aligned} dY_{n+3}(v_i, 0) &= (0, 0, 0), \\ dY_{n+3}(0, v_i) &= (0, v_i, 0) = Y_i, \\ dY_{n+3}(\xi, -x) &= (0, -x, 0) = (-1/2)Y_1 - Y_2. \end{aligned} \tag{58}$$

Comparing these equations with the equation

$$dY_a = \sum \omega_a^b Y_b,$$

we see that the 1-forms,

$$\{\omega_1^i, \omega_{n+3}^i, \omega_1^{n+2}\}, \quad 3 \leq i \leq n+1, \tag{59}$$

form the dual basis to the basis given in (55) for the tangent space to T_1S^n at (x, ξ). Since $(\xi, -x)$ has length $\sqrt{2}$, we have

$$\omega_1^{n+2}(X, Z) = ((X, Z) \cdot (\xi, -x))/2 = (X \cdot \xi - Z \cdot x)/2,$$

for a tangent vector (X, Z) to T_1S^n at (x, ξ). Using equation (40),

$$X \cdot \xi + Z \cdot x = 0,$$

we see that

$$\omega_1^{n+2}(X, Z) = X \cdot \xi, \tag{60}$$

so ω_1^{n+2} is precisely the form ω in equation (41). We now want to show that ω_1^{n+2} satisfies condition (37). This is a straightforward calculation using the Maurer–Cartan equation (54) for $d\omega_1^{n+2}$ and the skew-symmetry relations (53). By equation (54), we have

$$d\omega_1^{n+2} = \sum \omega_1^c \wedge \omega_c^{n+2}.$$

The skew-symmetry relations (53) imply that $\omega_1^2 = 0$ and $\omega_{n+3}^{n+2} = 0$. Furthermore, in computing $(d\omega_1^{n+2})^{n-1}$, we can ignore any term involving ω_1^{n+2}, since we will eventually take the wedge product with ω_1^{n+2}. Thus in computing the wedge product $d\omega_1^{n+2} \wedge d\omega_1^{n+2}$, we need only to consider

$$(\sum d\omega_1^i \wedge d\omega_i^{n+2}) \wedge (\sum d\omega_1^j \wedge d\omega_j^{n+2}).$$

If $i \neq j$, we have a term of the form

$$\omega_1^i \wedge \omega_i^{n+2} \wedge \omega_1^j \wedge \omega_j^{n+2} = \omega_1^i \wedge (-\omega_{n+3}^i) \wedge \omega_1^j \wedge (-\omega_{n+3}^j) = \omega_1^i \wedge \omega_{n+3}^i \wedge \omega_1^j \wedge \omega_{n+3}^j \neq 0,$$

where the sign changes are due to the skew-symmetry relations (53). The last term is nonzero since each of the factors is in the basis given in (59). Thus we have

$$d\omega_1^{n+2} \wedge d\omega_1^{n+2} = 2 \sum_{i<j} \omega_1^i \wedge \omega_{n+3}^i \wedge \omega_1^j \wedge \omega_{n+3}^j \pmod{\omega_1^{n+2}}. \tag{61}$$

One continues this process by taking the wedge product of (61) with $d\omega_1^{n+2}$. This time there are three sign changes in each term as a result of the skew-symmetry relations (53), and we get

$$(d\omega_1^{n+2})^3 = (-1)^3 (3!) \sum_{i<j<k} \omega_1^i \wedge \omega_{n+3}^i \wedge \omega_1^j \wedge \omega_{n+3}^j \wedge \omega_1^k \wedge \omega_{n+3}^k \pmod{\omega_1^{n+2}}.$$

Continuing this process, one eventually obtains

$$\begin{aligned} \omega_1^{n+2} \quad &\wedge \quad (d\omega_1^{n+2})^{n-1} = \omega_1^{n+2} \wedge (\sum \omega_1^i \wedge \omega_i^{n+2})^{n-1} \qquad (62) \\ &= \quad (-1)^{n-1}(n-1)! \quad \omega_1^{n+2} \wedge \omega_1^3 \wedge \omega_{n+3}^3 \wedge \cdots \wedge \omega_1^{n+1} \wedge \omega_{n+3}^{n+1} \neq 0. \end{aligned}$$

The last form is nonzero because the set (59) is a basis for the cotangent space to T_1S^n at (x, ξ). Finally, note that the form

$$\omega_1^{n+2} = \langle dY_1, Y_{n+3} \rangle, \tag{63}$$

is globally defined on T_1S^n, since Y_1 and Y_{n+3} are globally defined by equation (56), even though the rest of the Lie frame is only defined on the open set U.

As we noted above, we can use the diffeomorphism given in (43) to transfer this Lie frame and contact form ω_1^{n+2} to the manifold Λ^{2n-1} of lines on the Lie quadric. Now suppose that $\{Z_1, \ldots, Z_{n+3}\}$ is an arbitrary Lie frame on the open set U with the property that the line $[Z_1, Z_{n+3}]$ equals the line $[Y_1, Y_{n+3}]$ at all points of U, i.e.,

$$Z_1 = \alpha Y_1 + \beta Y_{n+3}, \quad Z_{n+3} = \gamma Y_1 + \delta Y_{n+3}, \tag{64}$$

for smooth functions $\alpha, \beta, \gamma, \delta$ with $\alpha\delta - \beta\gamma \neq 0$ on U. Let $\{\theta_a^b\}$ be the Maurer–Cartan forms for this Lie frame. Then using the scalar product relations (46), we get

$$\begin{aligned} \theta_1^{n+2} &= \langle dZ_1, Z_{n+3} \rangle = \langle d(\alpha Y_1 + \beta Y_{n+3}), \gamma Y_1 + \delta Y_{n+3} \rangle \\ &= \alpha\delta \langle dY_1, Y_{n+3} \rangle + \beta\gamma \langle dY_{n+3}, Y_1 \rangle = \alpha\delta\omega_1^{n+2} + \beta\gamma\omega_{n+3}^2 \\ &= (\alpha\delta - \beta\gamma)\omega_1^{n+2}. \end{aligned} \tag{65}$$

Thus, θ_1^{n+2} is also a contact form on T_1S^n.

2.2 Definition of Legendre submanifolds

In the last section, we showed that T_1S^n (and hence Λ^{2n-1}) is a contact manifold. A basic result concerning contact manifolds in general is given in (Theorem 2.1) below. Let V^{2n-1} be a contact manifold with contact form ω. Let D be the corresponding contact distribution defined by

$$D_p = \{Y \in T_pV^{2n-1} \mid \omega(Y) = 0\},$$

for $p \in V^{2n-1}$. An immersion $\phi : W^k \to V^{2n-1}$ of a smooth k-dimensional manifold W^k into V^{2n-1} is called an *integral submanifold* of the distribution D if $\phi^*\omega = 0$ on W^k, i.e., for each tangent vector Y at each point $w \in W$, the vector $d\phi(Y)$ is in the distribution D at the point $\phi(w)$. See [17, p. 57] for a proof of the following theorem.

Theorem 2.1. *Let V^{2n-1} be a contact manifold with contact form ω. Then there exist integral submanifolds of the contact distribution D of dimension $n-1$, but none of higher dimension.*

An immersed $(n-1)$-dimensional integral submanifold of the contact distribution D is called a *Legendre submanifold.* We now return to our specific case of the contact manifold T_1S^n. We first want to formulate necessary and sufficient conditions for a smooth map $\mu : M^{n-1} \to T_1S^n$ to be a Legendre submanifold. We consider T_1S^n as a submanifold of $S^n \times S^n$ as in equation (36). Thus we can write $\mu = (f, \xi)$, where f and ξ are both smooth maps from M^{n-1} to S^n.

Theorem 2.2. *A smooth map $\mu = (f, \xi)$ from an $(n-1)$-dimensional manifold M^{n-1} into T_1S^n is a Legendre submanifold if and only if the following three conditions are satisfied.*
(1) *Scalar product conditions: $f \cdot f = 1, \quad \xi \cdot \xi = 1, \quad f \cdot \xi = 0.$*
(2) *Immersion condition: There is no nonzero tangent vector X at any point $x \in M^{n-1}$ such that $df(X)$ and $d\xi(X)$ are both equal to zero.*
(3) *Contact condition: $df \cdot \xi = 0$.*

Proof. By equation (36), the scalar product conditions are precisely the conditions necessary for the image of the map $\mu = (f, \xi)$ to be contained in T_1S^n. Next, since

$$d\mu(X) = (df(X), d\xi(X)),$$

the second condition is precisely what is needed for μ to be an immersion. Finally, from equation (41) we have

$$\omega(d\mu(X)) = df(X) \cdot \xi(x),$$

for each $X \in T_xM^{n-1}$. Hence the condition $\mu^*\omega = 0$ on M^{n-1} is equivalent to the third condition above. $\square$

We now want to translate these conditions into the projective setting, and find necessary and sufficient conditions for a smooth map $\lambda : M^{n-1} \to \Lambda^{2n-1}$ to be a Legendre submanifold. We again make use of the diffeomorphism defined in equation (43) between T_1S^n and Λ^{2n-1}. For each $x \in M^{n-1}$, we know that $\lambda(x)$ is a line on the quadric Q^{n+1}. This line contains exactly one point $[Y_1(x)]$ corresponding to a point sphere in S^n and one point $[Y_{n+3}(x)]$ corresponding to a great sphere in S^n. The map $[Y_1]$ from M^{n-1} to Q^{n+1} is called the *Möbius projection* or *point sphere map* of λ, and likewise, the map $[Y_{n+3}]$ is called the *great sphere map*.

The homogeneous coordinates of these points with respect to the standard basis are given by

$$Y_1(x) = (1, f(x), 0), \quad Y_{n+3}(x) = (0, \xi(x), 1), \tag{66}$$

where f and ξ are both smooth maps from M^{n-1} to S^n defined by formula (66). The map f is called the *spherical projection* of λ, and ξ is called the *spherical field of unit normals*. The maps f and ξ depend on the choice of orthonormal basis $\{e_1, \ldots, e_{n+2}\}$ for the orthogonal complement of e_{n+3}. In this way, λ determines a map $\mu = (f, \xi)$ from M^{n-1} to T_1S^n, and because of the diffeomorphism (43), λ is a Legendre submanifold if and only if μ satisfies the conditions of Theorem 2.2.

It is useful to have conditions for when λ determines a Legendre submanifold that do not depend on the special parametrization of λ by $[Y_1, Y_{n+3}]$. In fact, in most applications of Lie sphere geometry to submanifolds of S^n or $\mathbf{R}^n$, it is better to use a Lie frame $\{Z_1, \ldots, Z_{n+3}\}$ with $\lambda = [Z_1, Z_{n+3}]$, where Z_1 and Z_{n+3} are not the point sphere and great sphere maps. The following projective formulation of the conditions needed for a Legendre submanifold was given by Pinkall [82], where he referred to a Legendre submanifold as a "Lie geometric hypersurface." (See also [17, pp. 59–60] for a proof.)

Theorem 2.3. *Let $\lambda : M^{n-1} \to \Lambda^{2n-1}$ be a smooth map with $\lambda = [Z_1, Z_{n+3}]$, where Z_1 and Z_{n+3} are smooth maps from M^{n-1} into $\mathbf{R}^{n+3}_2$. Then λ determines a Legendre submanifold if and only if Z_1 and Z_{n+3} satisfy the following conditions.*
(1) *Scalar product conditions: For each $x \in M^{n-1}$, the vectors $Z_1(x)$ and $Z_{n+3}(x)$ are linearly independent and*

$$\langle Z_1, Z_1 \rangle = 0, \quad \langle Z_{n+3}, Z_{n+3} \rangle = 0, \quad \langle Z_1, Z_{n+3} \rangle = 0.$$

(2) *Immersion condition: There is no nonzero tangent vector X at any point $x \in M^{n-1}$ such that $dZ_1(X)$ and $dZ_{n+3}(X)$ are both in* Span $\{Z_1(x), Z_{n+3}(x)\}$.
(3) *Contact condition: $\langle dZ_1, Z_{n+3} \rangle = 0$.*

These conditions are invariant under a reparametrization $\lambda = [W_1, W_{n+3}]$, where $W_1 = \alpha Z_1 + \beta Z_{n+3}$ and $W_{n+3} = \gamma Z_1 + \delta Z_{n+3}$, for smooth functions $\alpha, \beta, \gamma, \delta$ on M^{n-1} with $\alpha\delta - \beta\gamma \neq 0$.

2.3 The Legendre map

All oriented hypersurfaces in the sphere S^n, Euclidean space $\mathbf{R}^n$ or hyperbolic space H^n naturally induce Legendre submanifolds of Λ^{2n-1}, as do all submanifolds of codimension $m > 1$ in these spaces. In this section, we study these examples and see, conversely, how a Legendre submanifold naturally induces a smooth map into S^n which may have singularities.

First, suppose that $f : M^{n-1} \to S^n$ is an immersed oriented hypersurface with field of unit normals $\xi : M^{n-1} \to S^n$. The induced Legendre submanifold is given by the map $\lambda : M^{n-1} \to \Lambda^{2n-1}$ defined by

$$\lambda(x) = [Y_1(x), Y_{n+3}(x)],$$

where

$$Y_1(x) = (1, f(x), 0), \quad Y_{n+3}(x) = (0, \xi(x), 1). \tag{67}$$

The map λ is called the *Legendre map* induced by the immersion f with field of unit normals ξ. We will also refer to λ as the the *Legendre lift* of the pair (f, ξ) or the *Legendre submanifold induced by the pair* (f, ξ). It is easy to check that the pair $\{Y_1, Y_{n+3}\}$ satisfies the conditions of Theorem 2.3. Condition (1) is immediate since both f and ξ are maps into S^n, and $\xi(x)$ is tangent to S^n at $f(x)$ for each x in M^{n-1}. Condition (2) is satisfied since

$$dY_1(X) = (0, df(X), 0),$$

for any vector $X \in T_x M^{n-1}$. Since f is an immersion, $df(X) \neq 0$ for a nonzero vector X, and thus $dY_1(X)$ is not in Span $\{Y_1(x), Y_{n+3}(x)\}$. Finally, condition (3) is satisfied since

$$\langle dY_1(X), Y_{n+3}(x) \rangle = df(X) \cdot \xi(x) = 0,$$

because ξ is a field of unit normals to f.

Next, we handle the case of a submanifold $\phi : V \to S^n$ of codimension $m+1$ greater than one. Let B^{n-1} be the unit normal bundle of the submanifold ϕ. Then B^{n-1} can be considered to be the submanifold of $V \times S^n$ given by

$$B^{n-1} = \{(x, \xi) | \phi(x) \cdot \xi = 0, \ d\phi(X) \cdot \xi = 0, \ \text{for all } X \in T_x V\}.$$

The *Legendre lift* of $\phi(V)$ (or *Legendre submanifold induced by* $\phi(V)$) is the map $\lambda : B^{n-1} \to \Lambda^{2n-1}$ defined by

$$\lambda(x, \xi) = [Y_1(x, \xi), Y_{n+3}(x, \xi)], \tag{68}$$

where

$$Y_1(x, \xi) = (1, \phi(x), 0), \quad Y_{n+3}(x, \xi) = (0, \xi, 1). \tag{69}$$

Geometrically, $\lambda(x, \xi)$ is the line on the quadric Q^{n+1} corresponding to the parabolic pencil of spheres in S^n in oriented contact at the contact element $(\phi(x), \xi) \in T_1 S^n$.

As in the case of a hypersurface, condition (1) is easily checked. However, condition (2) is somewhat different. To compute the differentials of Y_1 and Y_{n+3} at a given point (x, ξ), we first construct a local trivialization of B^{n-1} in a neighborhood of (x, ξ). Let $\{\xi_0, \ldots, \xi_m\}$ be an orthonormal frame at x with $\xi_0 = \xi$. Let W be a normal coordinate neighborhood of x in V, as defined in Kobayashi–Nomizu [51, Vol. 1, p. 148], and extend $\xi_0, \ldots, \xi_m$ to orthonormal normal vector fields on W by parallel translation with respect to the normal connection along geodesics in V through x. For any point $w \in W$ and unit normal η to $\phi(V)$ at w, we can write

$$\eta = (1 - \sum_{i=1}^{m} t_i^2)^{1/2} \xi_0 + t_1 \xi_1 + \cdots + t_m \xi_m,$$

where $0 \leq |t_i| \leq 1$, for all i, and $t_1^2 + \cdots + t_m^2 \leq 1$. The tangent space to B^{n-1} at the given point (x, ξ) can be considered to be

$$T_x V \times \text{Span } \{\partial/\partial t_1, \ldots, \partial/\partial t_m\} = T_x V \times \mathbf{R}^m. \tag{70}$$

Since $\xi_0(x) = \xi$, and ξ_0 is parallel with respect to the normal connection, we have for $X \in T_x V$,

$$d\xi_0(X) = d\phi(-A_\xi X),$$

where A_ξ is the shape operator determined by ξ. Thus, we have

$$\begin{aligned} dY_1(X, 0) &= (0, d\phi(X), 0), \\ dY_{n+3}(X, 0) &= (0, d\xi_0(X), 0) = (0, d\phi(-A_\xi X), 0). \end{aligned} \tag{71}$$

Next we compute from equation (69),

$$dY_1(0, Z) = (0, 0, 0), \quad dY_{n+3}(0, Z) = (0, Z, 0). \tag{72}$$

From equations (71) and (72), we see that there is no nonzero vector (X, Z) such that $dY_1(X, Z)$ and $dY_{n+3}(X, Z)$ are both in Span $\{Y_1, Y_{n+3}\}$, and so condition (2) is satisfied. Finally, condition (3) holds since

$$\langle dY_1(X, Z), Y_{n+3}(x, \xi) \rangle = d\phi(X) \cdot \xi = 0.$$

The situation for submanifolds of $\mathbf{R}^n$ or H^n is similar. First, suppose that $F : M^{n-1} \to \mathbf{R}^n$ is an oriented hypersurface with field of unit normals $\eta : M^{n-1} \to \mathbf{R}^n$. As usual, we identify $\mathbf{R}^n$ with the subspace of $\mathbf{R}^{n+3}_2$ spanned by $\{e_3, \ldots, e_{n+2}\}$. The Legendre submanifold induced by (F, η) is the map $\lambda : M^{n-1} \to \Lambda^{2n-1}$ defined by $\lambda = [Y_1, Y_{n+3}]$, where

$$Y_1 = (1 + F \cdot F, 1 - F \cdot F, 2F, 0)/2, \quad Y_{n+3} = (F \cdot \eta, -(F \cdot \eta), \eta, 1). \tag{73}$$

By equation (14), $[Y_1(x)]$ corresponds to the point sphere and $[Y_{n+3}(x)]$ corresponds to the hyperplane in the parabolic pencil determined by the line $\lambda(x)$ for each $x \in M^{n-1}$. The reader can easily verify conditions (1)–(3) of Theorem 2.3 in a manner similar to the spherical case. In the case of a submanifold $\psi : V \to \mathbf{R}^n$ of codimension greater than one, the induced Legendre submanifold is the map λ from the unit normal bundle B^{n-1} to Λ^{2n-1} defined by

$$\lambda(x, \eta) = [Y_1(x, \eta), Y_{n+3}(x, \eta)],$$

where

$$\begin{aligned} Y_1(x, \eta) &= (1 + \psi(x) \cdot \psi(x), 1 - \psi(x) \cdot \psi(x), 2\psi(x), 0)/2, \\ Y_{n+3}(x, \eta) &= (\psi(x) \cdot \eta, -(\psi(x) \cdot \eta), \eta, 1). \end{aligned} \tag{74}$$

The verification that the pair $\{Y_1, Y_{n+3}\}$ satisfies conditions (1)–(3) is similar to that for submanifolds of S^n of codimension greater than one.

Finally, as in §1.5, we consider H^n to be the submanifold of the Lorentz space $\mathbf{R}_1^{n+1}$ spanned by $\{e_1, e_3, \ldots, e_{n+2}\}$ defined as follows:

$$H^n = \{y \in \mathbf{R}_1^{n+1} | (y, y) = -1, \quad y_1 \geq 1\},$$

where $(\ , \ )$ is the Lorentz metric on $\mathbf{R}_1^{n+1}$ obtained by restricting the Lie metric. Let $h : M^{n-1} \to H^n$ be an oriented hypersurface with field of unit normals $\zeta : M^{n-1} \to \mathbf{R}_1^{n+1}$. The Legendre submanifold induced by (h, ζ) is given by the map

$$\lambda : M^{n-1} \to \Lambda^{2n-1},$$

defined by $\lambda = [Y_1, Y_{n+3}]$, where

$$Y_1(x) = h(x) + e_2, \quad Y_{n+3}(x) = \zeta(x) + e_{n+3}. \tag{75}$$

Note that $(h, h) = -1$, so $\langle Y_1, Y_1 \rangle = 0$, while $(\zeta, \zeta) = 1$, so $\langle Y_{n+3}, Y_{n+3} \rangle = 0$. The reader can easily check that the conditions (1)–(3) are satisfied. Finally, if $\gamma : V \to H^n$ is an immersed submanifold of codimension greater than one, then the Legendre submanifold $\lambda : B^{n-1} \to \Lambda^{2n-1}$ is again defined on the unit normal bundle B^{n-1} of the submanifold $\gamma(V)$ in the usual way.

Now suppose that $\lambda : M^{n-1} \to \Lambda^{2n-1}$ is an arbitrary Legendre submanifold. As we have seen, it is always possible to parametrize λ by the point sphere map $[Y_1]$ and the great sphere map $[Y_{n+3}]$ given by

$$Y_1 = (1, f, 0), \quad Y_{n+3} = (0, \xi, 1). \tag{76}$$

This defines two maps f and ξ from M^{n-1} to S^n, which we called the spherical projection and spherical field of unit normals, respectively, in §2.2. Both f and ξ are smooth maps, but neither need be an immersion or even have constant rank. (See [17, p. 63] for an example.)

The Legendre lift of an oriented hypersurface in S^n is the special case where the spherical projection f is an immersion, i.e., f has constant rank $n-1$ on M^{n-1}. In the case of the Legendre lift of a submanifold $\phi : V^k \to S^n$, the spherical projection $f : B^{n-1} \to S^n$ defined by $f(x, \xi) = \phi(x)$ has constant rank k.

If the range of the point sphere map $[Y_1]$ does not contain the improper point $[(1, -1, 0, \ldots, 0)]$, then λ also determines a *Euclidean projection*,

$$F : M^{n-1} \to \mathbf{R}^n,$$

and a *Euclidean field of unit normals*,

$$\eta : M^{n-1} \to \mathbf{R}^n.$$

These are defined by the equation $\lambda = [Z_1, Z_{n+3}]$, where

$$Z_1 = (1 + F \cdot F, 1 - F \cdot F, 2F, 0)/2, \quad Z_{n+3} = (F \cdot \eta, -(F \cdot \eta), \eta, 1). \quad (77)$$

Here $[Z_1(x)]$ corresponds to the unique point sphere in the parabolic pencil determined by $\lambda(x)$, and $[Z_{n+3}(x)]$ corresponds to the unique plane in this pencil. As in the spherical case, the smooth maps F and η need not have constant rank.

Finally, if the range of the Euclidean projection F lies inside some disk Ω in $\mathbf{R}^n$, then one can define a hyperbolic projection and hyperbolic field of unit normals by placing a hyperbolic metric on Ω.

2.4 Curvature spheres and Dupin hypersurfaces

To motivate the definition of a curvature sphere we consider the case of an oriented hypersurface $f : M^{n-1} \to S^n$ with field of unit normals $\xi : M^{n-1} \to S^n$. The shape operator of f at a point $x \in M^{n-1}$ is the symmetric linear transformation $A : T_x M^{n-1} \to T_x M^{n-1}$ defined by the equation

$$df(AX) = -d\xi(X), \quad X \in T_x M^{n-1}. \quad (78)$$

Often we consider f to be an embedding and suppress the mention of f. Then we identify the tangent vector X with $df(X)$. In that case, we get the familiar formulation of the shape operator,

$$AX = -d\xi(X) = -D_X \xi, \tag{79}$$

where D is the Euclidean covariant derivative.

The eigenvalues of A are called the *principal curvatures*, and the corresponding eigenvectors are called the *principal vectors*. We next recall the notion of a focal point of an immersion. For each real number t, define a map

$$f_t : M^{n-1} \to S^n,$$

by

$$f_t = \cos t\ f + \sin t\ \xi. \tag{80}$$

For each $x \in M^{n-1}$, the point $f_t(x)$ lies an oriented distance t along the normal geodesic to $f(M^{n-1})$ at $f(x)$. A point $p = f_t(x)$ is called a *focal point of multiplicity* $m > 0$ *of* f *at* x if the nullity of df_t is equal to m at x. Geometrically, one thinks of focal points as points where nearby normal geodesics intersect. It is well known that the location of focal points is related to the principal curvatures. Specifically, if $X \in T_x M^{n-1}$, then by equation (78) we have

$$df_t(X) = \cos t\ df(X) + \sin t\ d\xi(X) = df(\cos t\ X - \sin t\ AX). \tag{81}$$

Thus, $df_t(X)$ equals zero for $X \neq 0$ if and only if $\cot t$ is a principal curvature of f at x, and X is a corresponding principal vector. Hence, $p = f_t(x)$ is a focal point of f at x of multiplicity m if and only if $\cot t$ is a principal curvature of multiplicity m at x. Note that each principal curvature

$$\kappa = \cot t, \quad 0 < t < \pi,$$

produces two distinct antipodal focal points on the normal geodesic with parameter values t and $t + \pi$. The oriented hypersphere centered at a focal point p and in oriented contact with $f(M^{n-1})$ at $f(x)$ is called a *curvature sphere* of f at x. The two antipodal focal points determined by κ are the two centers of the corresponding curvature sphere. Thus, the correspondence between principal curvatures and curvature spheres is bijective. The multiplicity of the curvature sphere is by definition equal to the multiplicity of the corresponding principal curvature.

We now consider these ideas as they apply to the Legendre lift of an oriented hypersurface f with field of unit normals ξ. As in equation (67), we have $\lambda = [Y_1, Y_{n+3}]$, where

$$Y_1 = (1, f, 0), \quad Y_{n+3} = (0, \xi, 1). \tag{82}$$

For each $x \in M^{n-1}$, the points on the line $\lambda(x)$ can be parametrized as

$$[K_t(x)] = [\cos t \; Y_1(x) + \sin t \; Y_{n+3}(x)] = [(\cos t, f_t(x), \sin t)], \tag{83}$$

where f_t is given in equation (80). By equation (21), the point $[K_t(x)]$ in Q^{n+1} corresponds to the oriented sphere in S^n with center $f_t(x)$ and signed radius t. This sphere is in oriented contact with the oriented hypersurface $f(M^{n-1})$ at $f(x)$. Given a tangent vector $X \in T_x M^{n-1}$, we have

$$dK_t(X) = (0, df_t(X), 0). \tag{84}$$

Thus, $dK_t(X) = (0, 0, 0)$ if and only if $df_t(X) = 0$, i.e., $p = f_t(x)$ is a focal point of f at x. Hence, we have shown the following.

Lemma 2.1. *The point $[K_t(x)]$ in Q^{n+1} corresponds to a curvature sphere of the hypersurface f at x if and only if $dK_t(X) = (0, 0, 0)$ for some nonzero vector $X \in T_x M^{n-1}$.*

This characterization of curvature spheres depends on the parametrization of λ given by $\{Y_1, Y_{n+3}\}$ as in equation (82), and it has only been defined in the case where the spherical projection f is an immersion. Since it is often desirable to use a different parametrization of λ, we would like a definition of curvature sphere which is independent of the parametrization of λ. We would also like a definition that is valid for an arbitrary Legendre submanifold. This definition is given in the following paragraph.

Let $\lambda : M^{n-1} \to \Lambda^{2n-1}$ be a Legendre submanifold parametrized by the pair $\{Z_1, Z_{n+3}\}$, as in Theorem 2.3. Let $x \in M^{n-1}$ and $r, s \in \mathbf{R}$ with $(r, s) \neq (0, 0)$. The sphere,

$$[K] = [rZ_1(x) + sZ_{n+3}(x)],$$

is called a *curvature sphere* of λ at x if there exists a nonzero vector X in $T_x M^{n-1}$ such that

$$r \; dZ_1(X) + s \; dZ_{n+3}(X) \in \text{Span } \{Z_1(x), Z_{n+3}(x)\}. \tag{85}$$

The vector X is called a *principal vector* corresponding to the curvature sphere $[K]$.

Now consider a reparametrization of the form $\lambda = [W_1, W_{n+3}]$, where $W_1 = \alpha Z_1 + \beta Z_{n+3}$ and $W_{n+3} = \gamma Z_1 + \delta Z_{n+3}$, for smooth functions $\alpha, \beta, \gamma, \delta$ on M^{n-1} with $\alpha\delta - \beta\gamma \neq 0$, as in Theorem 2.3. Then, since

$$\begin{aligned} dW_1 &= \alpha dZ_1 + \beta dZ_{n+3} + (d\alpha)Z_1 + (d\beta)Z_{n+3}, \\ dW_{n+3} &= \gamma dZ_1 + \delta dZ_{n+3} + (d\gamma)Z_1 + (d\delta)Z_{n+3}, \end{aligned} \tag{86}$$

we see that the definition of curvature sphere given above is invariant under such a reparametrization.

Furthermore, if we take the special parametrization $Z_1 = Y_1$, $Z_{n+3} = Y_{n+3}$ given in equation (82), then condition (85) holds if and only if $r\ dY_1(X) + s\ dY_{n+3}(X)$ actually equals $(0, 0, 0)$. Thus, this definition is a generalization of the condition in Lemma 2.1.

From equation (85), it is clear that the set of principal vectors corresponding to a given curvature sphere $[K]$ at x is a subspace of T_xM^{n-1}. This set is called the *principal space* corresponding to the curvature sphere $[K]$. Its dimension is the *multiplicity* of $[K]$.

Remark 2.1. The definition of curvature sphere can be developed in the context of Lie sphere geometry without any reference to submanifolds of S^n (see Cecil–Chern [19] for details). In that case, one begins with a Legendre submanifold $\lambda : M^{n-1} \to \Lambda^{2n-1}$ and considers a curve $\gamma(t)$ lying in M^{n-1}. The set of points in Q^{n+1} lying on the set of lines $\lambda(\gamma(t))$ forms a ruled surface in Q^{n+1}. One then considers conditions for this ruled surface to be developable. This leads to a system of linear equations whose roots determine the curvature spheres at each point along the curve.

We next want to show that the notion of curvature sphere is invariant under Lie sphere transformations. Let $\lambda : M^{n-1} \to \Lambda^{2n-1}$ be a Legendre submanifold parametrized by $\lambda = [Z_1, Z_{n+3}]$. Suppose $\beta = P(B)$ is the Lie sphere transformation induced by an orthogonal transformation B in the group $O(n+1, 2)$. Since B is orthogonal, it is easy to check that the maps, $W_1 = BZ_1$, $W_{n+3} = BZ_{n+3}$, satisfy the conditions (1)–(3) of Theorem 2.3. We will denote the Legendre submanifold defined by $\{W_1, W_{n+3}\}$ by

$$\beta\lambda : M^{n-1} \to \Lambda^{2n-1}.$$

The Legendre submanifolds λ and $\beta\lambda$ are said to be *Lie equivalent*. In terms of Euclidean geometry, suppose that V and W are two immersed submanifolds of $\mathbf{R}^n$ (or of S^n or H^n). We say that V and W are *Lie equivalent* if their induced Legendre submanifolds are Lie equivalent.

Consider λ and β as above, so that $\lambda = [Z_1, Z_{n+3}]$ and $\beta\lambda = [W_1, W_{n+3}]$. Note that for a tangent vector $X \in T_x M^{n-1}$ and for real numbers $(r, s) \neq (0, 0)$, we have

$$r\ dW_1(X) + s\ dW_{n+3}(X) = B(r\ dZ_1(X) + s\ dZ_{n+3}(X)), \qquad (87)$$

since B is linear. Thus, we see that

$$r\ dW_1(X) + s\ dW_{n+3}(X) \in \text{Span}\ \{W_1(x), W_{n+3}(x)\}$$

if and only if

$$r\ dZ_1(X) + s\ dZ_{n+3}(X) \in \text{Span}\ \{Z_1(x), Z_{n+3}(x)\}.$$

This immediately implies the following theorem.

Theorem 2.4. *Let $\lambda : M^{n-1} \to \Lambda^{2n-1}$ be a Legendre submanifold and β a Lie sphere transformation. The point $[K]$ on the line $\lambda(x)$ is a curvature sphere of λ at x if and only if the point $\beta[K]$ is a curvature sphere of the Legendre submanifold $\beta\lambda$ at x. Furthermore, the principal spaces corresponding to $[K]$ and $\beta[K]$ are identical.*

An important special case is when the Lie sphere transformation is a spherical parallel transformation P_t, as given in equation (34),

$$\begin{array}{rcl} P_t e_1 & = & \cos t\ e_1 + \sin t\ e_{n+3}, \\ P_t e_{n+3} & = & -\sin t\ e_1 + \cos t\ e_{n+3}, \\ P_t e_i & = & e_i, \quad 2 \leq i \leq n+2. \end{array} \qquad (88)$$

Recall that P_t has the effect of adding t to the signed radius of each sphere in S^n while keeping the center fixed.

Suppose that $\lambda : M^{n-1} \to \Lambda^{2n-1}$ is a Legendre submanifold parametrized by the point sphere and great sphere maps $\{Y_1, Y_{n+3}\}$, as in equation (82). Then $P_t\lambda = [W_1, W_{n+3}]$, where

$$W_1 = P_t Y_1 = (\cos t, f, \sin t), \quad W_{n+3} = P_t Y_{n+3} = (-\sin t, \xi, \cos t). \qquad (89)$$

Note that W_1 and W_{n+3} are not the point sphere and great sphere maps for $P_t\lambda$. Solving for the point sphere map Z_1 and the great sphere map Z_{n+3} of $P_t\lambda$, we find

$$\begin{aligned} Z_1 &= \cos t \, W_1 - \sin t \, W_{n+3} = (1, \cos t \, f - \sin t \, \xi, 0), \\ Z_{n+3} &= \sin t \, W_1 + \cos t \, W_{n+3} = (0, \sin t \, f + \cos t \, \xi, 1). \end{aligned} \tag{90}$$

From this, we see that $P_t\lambda$ has spherical projection and spherical unit normal field given, respectively, by

$$\begin{aligned} f_{-t} &= \cos t \, f - \sin t \, \xi = \cos(-t) f + \sin(-t)\xi, \\ \xi_{-t} &= \sin t \, f + \cos t \, \xi = -\sin(-t) f + \cos(-t)\xi. \end{aligned} \tag{91}$$

The minus sign occurs because P_t takes a sphere with center $f_{-t}(x)$ and radius $-t$ to the point sphere $f_{-t}(x)$. We call $P_t\lambda$ a *parallel submanifold* of λ. Formula (91) shows the close correspondence between these parallel submanifolds and the *parallel hypersurfaces* f_t to f, in the case where f is an immersed hypersurface. The spherical projection f_t has singularities at the focal points of f, but the parallel submanifold $P_t\lambda$ is still a smooth submanifold of Λ^{2n-1}.

The following theorem, due to Pinkall [82, p. 428] (see also [17, pp. 70–72] for a proof), shows that the number of these singularities is bounded for each $x \in M^{n-1}$. This theorem is clear if the original spherical projection f is an immersion, but it requires proof if f has singularities.

Theorem 2.5. *Let $\lambda : M^{n-1} \to \Lambda^{2n-1}$ be a Legendre submanifold with spherical projection f and spherical unit normal field ξ. Then for each $x \in M^{n-1}$, the parallel map,*

$$f_t = \cos t \, f + \sin t \, \xi,$$

fails to be an immersion at x for at most $n-1$ values of $t \in [0, \pi)$.

Here $[0, \pi)$ is the appropriate interval because of the phenomenon mentioned earlier that each principal curvature of an immersion produces two distinct antipodal focal points in the interval $[0, 2\pi)$. We next state some important consequences of this theorem that are obtained by passing to a parallel submanifold, if necessary, and then applying well-known results concerning immersed hypersurfaces in S^n.

Corollary 2.1. *Let $\lambda : M^{n-1} \to \Lambda^{2n-1}$ be a Legendre submanifold. Then:*
(a) *at each point $x \in M^{n-1}$, there are at most $n-1$ distinct curvature spheres $K_1, \ldots, K_g$,*
(b) *the principal vectors corresponding to a curvature sphere K_i form a subspace T_i of the tangent space $T_x M^{n-1}$,*
(c) *the tangent space $T_x M^{n-1} = T_1 \oplus \cdots \oplus T_g$,*
(d) *if the dimension of a given T_i is constant on an open subset U of M^{n-1}, then the principal distribution T_i is integrable on U,*
(e) *if $\dim T_i = m > 1$ on an open subset U of M^{n-1}, then the curvature sphere map K_i is constant along the leaves of the principal foliation T_i.*

Proof. In the case where the spherical projection f of λ is an immersion, the corollary follows from known results concerning hypersurfaces in S^n and the correspondence between the curvature spheres of λ and the principal curvatures of f. Specifically, (a)–(c) follow from elementary linear algebra applied to the (symmetric) shape operator A of the immersion f. As to (d) and (e), Ryan [93, p. 371] showed that the principal curvature functions on an immersed hypersurface are continuous. Nomizu [73] then showed that any continuous principal curvature function κ_i which has constant multiplicity on an open subset U in M^{n-1} is smooth, as is its corresponding principal distribution (see also, Singley [99]). If the multiplicity m_i of κ_i equals one on U, then T_i is integrable by the theory of ordinary differential equations. If $m_i > 1$, then the integrability of T_i, and the fact that κ_i is constant along the leaves of T_i are consequences of Codazzi's equation (Ryan [93], see also Cecil–Ryan [29, p. 139] and Reckziegel [86]–[88]).

Note that (a)–(c) are pointwise statements, while (d)–(e) hold on an open set U if they can be shown to hold in a neighborhood of each point of U. Now let x be an arbitrary point of M^{n-1}. If the spherical projection f is not an immersion at x, then by Theorem 2.5, we can find a parallel transformation P_{-t} such that the spherical projection f_t of the Legendre submanifold $P_{-t}\lambda$ is an immersion at x, and hence on a neighborhood of x. By Theorem 2.4, the corollary also holds for λ in this neighborhood of x. Since x is an arbitrary point, the corollary is proved. □

Let $\lambda : M^{n-1} \to \Lambda^{2n-1}$ be an arbitrary Legendre submanifold. A connected submanifold S of M^{n-1} is called a *curvature surface* if at each $x \in S$, the tangent space $T_x S$ is equal to some principal space T_i. For example, if $\dim T_i$ is constant on an open subset U of M^{n-1}, then each leaf of the principal foliation T_i is a curvature surface on U. Curvature surfaces are plentiful, since the results of Reckziegel [87] and Singley [99] imply that there is an open dense subset Ω of M^{n-1} on which the multiplicities of the curvature spheres are locally constant. On Ω, each leaf of each principal foliation is a curvature surface.

It is also possible to have a curvature surface S which is not a leaf of a principal foliation, because the multiplicity of the corresponding curvature sphere is not constant on a neighborhood of S, as in the following example.

Example 2.1. *A curvature surface that is not a leaf of a principal foliation.* Let T^2 be a torus of revolution in $\mathbf{R}^3$, and embed $\mathbf{R}^3$ into $\mathbf{R}^4 = \mathbf{R}^3 \times \mathbf{R}$. Let η be a field of unit normals to T^2 in $\mathbf{R}^3$. Let M^3 be a tube of sufficiently small radius $\varepsilon > 0$ around T^2 in $\mathbf{R}^4$, so that M^3 is a compact smooth embedded hypersurface in $\mathbf{R}^4$. The normal space to T^2 in $\mathbf{R}^4$ at a point $x \in T^2$ is spanned by $\eta(x)$ and $e_4 = (0,0,0,1)$. The shape operator A_η of T^2 has two distinct principal curvatures at each point of T^2, while the shape operator A_{e_4} of T^2 is identically zero. Thus the shape operator A_ζ for the normal

$$\zeta = \cos\theta \; \eta(x) + \sin\theta \; e_4,$$

at a point $x \in T^2$, is given by

$$A_\zeta = \cos\theta \; A_{\eta(x)}.$$

From the formulas for the principal curvatures of a tube (see Cecil–Ryan [29, p. 131]), one finds that at all points of M^3 where $x_4 \neq \pm\varepsilon$, there are three distinct principal curvatures of multiplicity one, which are constant along their corresponding lines of curvature (curvature surfaces of dimension one). One of these principal curvatures is $\mu = -1/\varepsilon$ resulting from the tube construction. However, on the two tori, $T^2 \times \{\pm\varepsilon\}$, the principal curvature $\kappa = 0$ has multiplicity two. These two tori are curvature surfaces for this principal curvature κ, since the principal space corresponding to κ is tangent to each torus at every point. These two tori are not leaves of a principal foliation, however, since the leaves of a foliation must all have the same dimension. The Legendre submanifold λ induced by this embedding of M^3 in $\mathbf{R}^4$ has the same properties.

Part (e) of Corollary 2.1 has the following generalization, the proof of which is obtained by invoking the theorem of Ryan [93] mentioned in the proof of Corollary 2.1, with obvious minor modifications.

Corollary 2.2. *Suppose that S is a curvature surface of dimension $m > 1$ in a Legendre submanifold. Then the corresponding curvature sphere is constant along S.*

A hypersurface $f : M^{n-1} \to S^n$ is said to be *Dupin* if:

(a) along each curvature surface, the corresponding principal curvature is constant.

The hypersurface M is called *proper Dupin* if, in addition to Condition (a), the following condition is satisfied:

(b) the number g of distinct principal curvatures is constant on M.

On an open subset U on which Condition (b) holds, Condition (a) is equivalent to requiring that each curvature surface in each principal foliation be an open subset of a metric sphere in S^n of dimension equal to the multiplicity of the corresponding principal curvature. Condition (a) is also equivalent to the condition that along each curvature surface, the corresponding curvature sphere map is constant. Finally, on U, Condition (a) is equivalent to requiring that for each principal curvature κ, the image of the focal map f_κ is a smooth submanifold of S^n of codimension $m+1$, where m is the multiplicity of κ. See Cecil–Ryan [29, pp. 132–151] for proofs of these results.

One consequence of the results given above is that like isoparametric hypersurfaces, all proper Dupin hypersurfaces are algebraic. For simplicity, we take the ambient manifold to be $\mathbf{R}^n$. The theorem states that a connected proper Dupin hypersurface $f : M \to \mathbf{R}^n$ must be contained in a connected component of an irreducible algebraic subset of $\mathbf{R}^n$ of dimension $n-1$. Pinkall [80] sent the author a letter in 1984 that contained a sketch of a proof of this result. However, a proof was not published until recently by Cecil, Chi and Jensen [24], who used methods of real algebraic geometry to give a complete proof based on Pinkall's sketch. The proof makes use of the various principal foliations whose leaves are open subsets of spheres to construct an analytic algebraic parametrization of a neighborhood of $f(x)$ for each point $x \in M$. In contrast to the situation for isoparametric hypersurfaces, however, a connected proper Dupin hypersurface in S^n does not necessarily lie in a compact connected proper Dupin hypersurface, as we will later in these notes.

An important class of proper Dupin hypersurfaces are the isoparametric hypersurfaces in S^n, and those hypersurfaces in $\mathbf{R}^n$ obtained from isoparametric hypersurfaces in S^n via stereographic projection. For example, the well-known ring cyclides of Dupin in $\mathbf{R}^3$ are obtained from a standard product torus $S^1(r) \times S^1(s) \subset S^3$, $r^2 + s^2 = 1$, in this way. A special case is the torus of revolution in $\mathbf{R}^3$ in Example 2.1. On the torus, there are $g = 2$ distinct principal curvatures at each point, and each principal curvature is constant along each leaf of its corresponding principal foliation. These leaves are latitude circles for one principal curvature and longitude circles for the other principal curvature.

Remark 2.2. *A Dupin hypersurface that is not proper Dupin.*
The tube $M^3 \subset \mathbf{R}^4$ over the torus in Example 2.1 is an example of a Dupin hypersurface that is not proper Dupin. At points of M^3 except those on the top and bottom tori $T^2 \times \{\pm\varepsilon\}$, there are three distinct principal curvatures that are each constant along their corresponding principal curves (which are circles). However, on $T^2 \times \{\pm\varepsilon\}$, there are only two distinct principal curvatures, $\kappa = 0$ of multiplicity two, and $\mu = -1/\varepsilon$ of multiplicity one. Thus, M^3 is not proper Dupin, since the number of distinct principal curvatures is not constant on M^3. The hypersurface M^3 is Dupin, however, since along each curvature surface (including $T^2 \times \{\pm\varepsilon\}$), the corresponding principal curvature is constant.

We generalize these definitions to the context of Lie sphere geometry by defining a Legendre submanifold $\lambda : M^{n-1} \to \Lambda^{2n-1}$ to be a *Dupin submanifold* if:

(a) along each curvature surface, the corresponding curvature sphere is constant.

The Legendre submanifold λ is called *proper Dupin* if, in addition to Condition (a), the following condition is satisfied:

(b) the number g of distinct curvature spheres is constant on M.

Of course, the Legendre lift of a Dupin hypersurface in S^n, $\mathbf{R}^n$ or H^n is Dupin in the sense defined here, but this definition is more general, because the spherical projection of a Dupin submanifold need not be an immersion. Corollary 2.2 shows that the only curvature surfaces which must be considered in checking the Dupin property (a) are those of dimension one.

The Legendre lift of the torus of revolution $T^2 \subset \mathbf{R}^3$ in Example 2.1 above is a proper Dupin submanifold. On the other hand, the Legendre lift of the tube M^3 over T^2 is Dupin, but not proper Dupin, since the number of distinct curvature spheres is not constant on M^3.

The following theorem shows that both the Dupin and proper Dupin conditions are invariant under Lie sphere transformations, and many important classification results for Dupin submanifolds have been obtained in the setting of Lie sphere geometry.

Theorem 2.6. *Let $\lambda : M^{n-1} \to \Lambda^{2n-1}$ be a Legendre submanifold and β a Lie sphere transformation.*
(a) If λ is Dupin, then $\beta\lambda$ is Dupin.
(b) If λ is proper Dupin, then $\beta\lambda$ is proper Dupin.

Proof. By Theorem 2.4, a point $[K]$ on the line $\lambda(x)$ is a curvature sphere of λ at $x \in M$ if and only if the point $\beta[K]$ is a curvature sphere of $\beta\lambda$ at x, and the principal spaces corresponding $[K]$ and $\beta[K]$ are identical. Since these principal spaces are the same, if S is a curvature surface of λ corresponding to a curvature sphere map $[K]$, then S is also a curvature surface of $\beta\lambda$ corresponding to a curvature sphere map $\beta[K]$, and clearly $[K]$ is constant along S if and only if $\beta[K]$ is constant along S. This proves part (a) of the theorem. Part (b) also follows immediately from Theorem 2.4, since for each $x \in M$, the number g of distinct curvature spheres of λ at x equals the number of distinct curvatures spheres of $\beta\lambda$ at x. So if this number g is constant on M for λ, then it is constant on M for $\beta\lambda$. □

2.5 Lie curvatures and isoparametric hypersurfaces

In this section,we introduce certain natural Lie invariants of Legendre submanifolds which have been useful in the study of Dupin and isoparametric hypersurfaces.

Let $\lambda : M^{n-1} \to \Lambda^{2n-1}$ be an arbitrary Legendre submanifold. As before, we can write $\lambda = [Y_1, Y_{n+3}]$ with

$$Y_1 = (1, f, 0), \quad Y_{n+3} = (0, \xi, 1), \tag{92}$$

where f and ξ are the spherical projection and spherical field of unit normals, respectively. At each point $x \in M^{n-1}$, the points on the line $\lambda(x)$ can be written in the form,

$$\mu Y_1(x) + Y_{n+3}(x), \tag{93}$$

i.e., take μ as an inhomogeneous coordinate along the projective line $\lambda(x)$. Of course, Y_1 corresponds to $\mu = \infty$. The next two theorems give the relationship between the coordinates of the curvature spheres of λ and the principal curvatures of f, in the case where f has constant rank. In the first theorem, we assume that the spherical projection f is an immersion on M^{n-1}. By Theorem 2.5, we know that this can always be achieved locally by passing to a parallel submanifold.

Theorem 2.7. *Let $\lambda : M^{n-1} \to \Lambda^{2n-1}$ be a Legendre submanifold whose spherical projection $f : M^{n-1} \to S^n$ is an immersion. Let Y_1 and Y_{n+3} be the point sphere and great sphere maps of λ as in equation (92). Then the curvature spheres of λ at a point $x \in M^{n-1}$ are*

$$[K_i] = [\kappa_i Y_1 + Y_{n+3}], \quad 1 \leq i \leq g,$$

where $\kappa_1, \ldots, \kappa_g$ are the distinct principal curvatures at x of the oriented hypersurface f with field of unit normals ξ. The multiplicity of the curvature sphere $[K_i]$ equals the multiplicity of the principal curvature κ_i.

Proof. Let X be a nonzero vector in $T_x M^{n-1}$. Then for any real number μ,

$$d(\mu Y_1 + Y_{n+3})(X) = (0, \mu \, df(X) + d\xi(X), 0).$$

This vector is in Span $\{Y_1(x), Y_{n+3}(x)\}$ if and only if

$$\mu \, df(X) + d\xi(X) = 0,$$

i.e., μ is a principal curvature of f with corresponding principal vector X. $\square$

A second noteworthy case is when the point sphere map Y_1 is a curvature sphere of constant multiplicity m on M^{n-1}. By Corollary 2.1, the corresponding principal distribution is a foliation, and the curvature sphere map $[Y_1]$ is constant along the leaves of this foliation. Thus the map $[Y_1]$ factors through an immersion $[W_1]$ from the space of leaves V of this foliation into Q^{n+1}. We can write

$$W_1 = (1, \phi, 0),$$

where $\phi : V \to S^n$ is an immersed submanifold of codimension $m + 1$. The manifold M^{n-1} is locally diffeomorphic to an open subset of the unit normal bundle B^{n-1} of the submanifold ϕ, and λ is essentially the Legendre lift of the submanifold $\phi(V)$, as defined in §2.3. The following theorem relates the curvature spheres of λ to the principal curvatures of ϕ. Recall that the point sphere and great sphere maps for λ are given as in equation (69) by

$$Y_1(x,\xi) = (1, \phi(x), 0), \quad Y_{n+3}(x,\xi) = (0, \xi, 1). \tag{94}$$

Theorem 2.8. *Let $\lambda : B^{n-1} \to \Lambda^{2n-1}$ be the Legendre lift of the immersed submanifold $\phi(V)$ in S^n of codimension $m+1$. Let Y_1 and Y_{n+3} be the point sphere and great sphere maps of λ as in equation (94). Then the curvature spheres of λ at a point $(x,\xi) \in B^{n-1}$ are*

$$[K_i] = [\kappa_i Y_1 + Y_{n+3}], \quad 1 \leq i \leq g,$$

where $\kappa_1, \ldots, \kappa_{g-1}$ are the distinct principal curvatures of the shape operator A_ξ, and $\kappa_g = \infty$. For $1 \leq i \leq g-1$, the multiplicity of the curvature sphere $[K_i]$ equals the multiplicity of the principal curvature κ_i, while the multiplicity of $[K_g]$ is m.

Proof. To find the curvature spheres of λ, we employ the local trivialization of B^{n-1} used to obtain the decomposition of the tangent space to B^{n-1} at (x,ξ) given in equation (70):

$$T_xV \times \text{Span}\,\{\partial/\partial t_1, \ldots, \partial/\partial t_m\} = T_xV \times \mathbf{R}^m,$$

First, note that $dY_1(0,Z)$ equals 0 for any $Z \in \mathbf{R}^m$, since Y_1 depends only on x. Hence, Y_1 is a curvature sphere, as expected. Furthermore, since

$$dY_1(X,0) = (0, d\phi(X), 0)$$

is never in Span $\{Y_1(x,\xi), Y_{n+3}(x,\xi)\}$ for a nonzero $X \in T_xV$, the multiplicity of the curvature sphere Y_1 is m. If we let $[K_g] = [\kappa_g Y_1 + Y_{n+3}]$ be this curvature sphere, then we must take $\kappa_g = \infty$ to get $[Y_1]$. Using equation (71), we find the other curvature spheres at (x,ξ) by computing

$$d(\mu Y_1 + Y_{n+3})(X,0) = (0, d\phi(\mu X - A_\xi X), 0).$$

From this it is clear that $[\mu Y_1 + Y_{n+3}]$ is a curvature sphere with principal vector $(X,0)$ if and only if μ is a principal curvature of A_ξ with corresponding principal vector X. $\square$

Given these two theorems, we define a *principal curvature* of a Legendre submanifold $\lambda : M^{n-1} \to \Lambda^{2n-1}$ at a point $x \in M^{n-1}$ to be a value κ in the set $\mathbf{R} \cup \{\infty\}$ such that $[\kappa Y_1(x) + Y_{n+3}(x)]$ is a curvature sphere of λ at x, where Y_1 and Y_{n+3} are as in equation (92).

These principal curvatures of a Legendre submanifold are not Lie invariant and depend on the special parametrization for λ given in equation (92). However, R. Miyaoka [60] pointed out that the cross-ratios of the principal curvatures are Lie invariant. In order to formulate Miyaoka's theorem, we need to introduce some notation. Suppose that β is a Lie sphere transformation. The Legendre submanifold $\beta\lambda$ has point sphere and great sphere maps given, respectively, by

$$Z_1 = (1, h, 0), \quad Z_{n+3} = (0, \zeta, 1),$$

where h and ζ are the spherical projection and spherical field of unit normals for $\beta\lambda$. Suppose that

$$[K_i] = [\kappa_i Y_1 + Y_{n+3}], \quad 1 \leq i \leq g,$$

are the distinct curvature spheres of λ at a point $x \in M^{n-1}$. By Theorem 2.4, the points $\beta[K_i], 1 \leq i \leq g$, are the distinct curvature spheres of $\beta\lambda$ at x. We can write

$$\beta[K_i] = [\gamma_i Z_1 + Z_{n+3}], \quad 1 \leq i \leq g.$$

These γ_i are the principal curvatures of $\beta\lambda$ at x.

For four distinct numbers a, b, c, d in $\mathbf{R} \cup \{\infty\}$, we adopt the notation

$$[a, b; c, d] = \frac{(a-b)(d-c)}{(a-c)(d-b)} \tag{95}$$

for the cross-ratio of a, b, c, d. We use the usual conventions involving operations with ∞. For example, if $d = \infty$, then the expression $(d-c)/(d-b)$ evaluates to one, and the cross-ratio $[a, b; c, d]$ equals $(a-b)/(a-c)$.

Miyaoka's theorem can now be stated as follows.

Theorem 2.9. *Let $\lambda : M^{n-1} \to \Lambda^{2n-1}$ be a Legendre submanifold and β a Lie sphere transformation. Suppose that $\kappa_1, \ldots, \kappa_g, g \geq 4$, are the distinct principal curvatures of λ at a point $x \in M^{n-1}$, and $\gamma_1, \ldots, \gamma_g$ are the corresponding principal curvatures of $\beta\lambda$ at x. Then for any choice of four numbers h, i, j, k from the set $\{1, \ldots, g\}$, we have*

$$[\kappa_h, \kappa_i; \kappa_j, \kappa_k] = [\gamma_h, \gamma_i; \gamma_j, \gamma_k]. \tag{96}$$

Proof. The left side of equation (96) is the cross-ratio, in the sense of projective geometry, of the four points $[K_h], [K_i], [K_j], [K_k]$ on the projective line $\lambda(x)$. The right side of equation (96) is the cross-ratio of the images of these four points under β. The theorem now follows from the fact that the projective transformation β preserves the cross-ratio of four points on a line. □

The cross-ratios of the principal curvatures of λ are called the *Lie curvatures* of λ. A set of related invariants for the Möbius group is obtained as follows. First, recall that a Möbius transformation is a Lie sphere transformation that takes point spheres to point spheres. Hence the transformation β in Theorem 2.9 is a Möbius transformation if and only if $\beta[Y_1] = [Z_1]$. This leads to the following corollary of Theorem 2.9.

Corollary 2.3. *Let $\lambda : M^{n-1} \to \Lambda^{2n-1}$ be a Legendre submanifold and β a Möbius transformation. Then for any three distinct principal curvatures $\kappa_h, \kappa_i, \kappa_j$ of λ at a point $x \in M^{n-1}$, none of which equals ∞, we have*

$$\Phi(\kappa_h, \kappa_i, \kappa_j) = (\kappa_h - \kappa_i)/(\kappa_h - \kappa_j) = (\gamma_h - \gamma_i)/(\gamma_h - \gamma_j), \tag{97}$$

where γ_h, γ_i and γ_j are the corresponding principal curvatures of $\beta\lambda$ at the point x.

Proof. First, note that we are using equation (97) to define the quantity Φ. Now since β is a Möbius transformation, the point $[Y_1]$, corresponding to $\mu = \infty$, is taken by β to the point Z_1 with coordinate $\gamma = \infty$. Since β preserves cross-ratios, we have

$$[\kappa_h, \kappa_i; \kappa_j, \infty] = [\gamma_h, \gamma_i; \gamma_j, \infty].$$

The corollary now follows since the cross-ratio on the left in the equation above equals the left side of equation (97), and the cross-ratio on the right above equals the right side of equation (97). □

A ratio Φ of the form (97) is called a *Möbius curvature* of λ. Lie and Möbius curvatures have been useful in characterizing Legendre submanifolds that are Lie equivalent to Legendre submanifolds induced by isoparametric hypersurfaces in spheres.

Recall that an immersed hypersurface in a real space-form, $\mathbf{R}^n$, S^n or H^n, is said to be *isoparametric* if it has constant principal curvatures. An isoparametric hypersurface M in $\mathbf{R}^n$ can have at most two distinct principal curvatures, and M must be an open subset of a hyperplane, hypersphere or a spherical cylinder $S^k \times \mathbf{R}^{n-k-1}$. This was first proven for $n = 3$ by Somigliana [100] in 1919 (see also Levi-Civita [52] (1937) for $n = 3$ and B. Segre [96] (1938) for arbitrary n).

Shortly after the publication of the papers of Levi-Civita and Segre, Cartan [7]–[10] undertook the study of isoparametric hypersurfaces in arbitrary real space-forms $\tilde{M}^n(c), c \in \mathbf{R}$, and we now describe his primary contributions.

Let $f : M^{n-1} \to \tilde{M}^n(c)$ be an isoparametric hypersurface with g distinct principal curvatures $\mu_1, \ldots, \mu_g$, having respective multiplicities $m_1, \ldots, m_g$. If $g > 1$, Cartan showed that for each i, $1 \leq i \leq g$,

$$\sum_{j \neq i} m_j \frac{c + \mu_i \mu_j}{\mu_i - \mu_j} = 0. \tag{98}$$

This important equation, known as *Cartan's identity*, is crucial in Cartan's work on isoparametric hypersurfaces. For example, using this identity, Cartan was able to classify isoparametric hypersurfaces in the cases $c \leq 0$ as follows. In the case $c = 0$, if $g = 1$, then f is totally umbilic, and it is well known that $f(M^{n-1})$ must be an open subset of a hyperplane or hypersphere. If $g \geq 2$, then by taking an appropriate choice of unit normal field ξ, one can assume that at least one of the principal curvatures is positive. If μ_i is the smallest positive principal curvature, then each term $\mu_i \mu_j / (\mu_i - \mu_j)$ in the sum in equation (98) is non-positive, and thus must equal zero. Therefore, there are at most two distinct principal curvatures, and if there are two, then one of them must be zero. Hence, $g = 2$ and one can show $f(M^{n-1})$ is an open subset of a spherical cylinder by standard methods in Euclidean hypersurface theory.

In the case $c = -1$, if $g = 1$, then f is totally umbilic, and it is well known that $f(M^{n-1})$ must be an open subset of a totally geodesic hyperplane, an equidistant hypersurface, a horosphere or a hypersphere in H^n (see, for example, [101, p.114]). If $g \geq 2$, then again one can arrange that at least one of the principal curvatures is positive. Then there must exist a positive principal curvature μ_i such that no principal curvature lies between μ_i and $1/\mu_i$. (Here μ_i is either the largest principal curvature between 0 and 1 or the smallest principal curvature greater than or equal to one.) For this μ_i, each term $(-1 + \mu_i \mu_j)/(\mu_i - \mu_j)$ in the sum in equation (98) is negative unless

$\mu_j = 1/\mu_i$. Thus, there are at most two distinct principal curvatures, and if there are two, then they are reciprocals of each other. Hence, $g = 2$ and one can show that $f(M^{n-1})$ is an open subset of a standard product $S^k \times H^{n-k-1}$ in hyperbolic space H^n (see Ryan [94, pp. 252–253]).

In the sphere S^n, however, Cartan showed that there are many more possibilities. He found examples of isoparametric hypersurfaces in S^n with $1, 2, 3$ or 4 distinct principal curvatures, and he classified compact, connected isoparametric hypersurfaces with $g \leq 3$ principal curvatures as follows. If $g = 1$, then the isoparametric hypersurface M is totally umbilic, and it must be a great or small sphere. If $g = 2$, then M must be a standard product of two spheres,

$$S^k(r) \times S^{n-k-1}(s) \subset S^n, \quad r^2 + s^2 = 1.$$

In the case $g = 3$, Cartan [9] showed that all the principal curvatures must have the same multiplicity $m = 1, 2, 4$ or 8, and the isoparametric hypersurface must be a tube of constant radius over a standard embedding of a projective plane $\mathbf{F}P^2$ into S^{3m+1} (see, for example, Cecil–Ryan [29, pp. 296–299]), where $\mathbf{F}$ is the division algebra $\mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$ (quaternions), $\mathbf{O}$ (Cayley numbers), for $m = 1, 2, 4, 8$, respectively. Thus, up to congruence, there is only one such family for each value of m.

Cartan's theory was further developed by Nomizu [74]–[75], Takagi and Takahashi [105], Ozeki and Takeuchi [77], and most extensively by Münzner [70]–[71], who showed that the number g of distinct principal curvatures of an isoparametric hypersurface must be $1, 2, 3, 4$ or 6. (See also Chapter 3 of Cecil–Ryan [29] or Chapter 3 of [30].)

In the case of an isoparametric hypersurface with four principal curvatures, Münzner proved that the principal curvatures can have at most two distinct multiplicities m_1, m_2. Next Ferus, Karcher and Münzner [40] used representations of Clifford algebras to construct for any positive integer m_1 an infinite series of isoparametric hypersurfaces with four principal curvatures having respective multiplicities (m_1, m_2), where m_2 is nondecreasing and unbounded in each series. As later work (described below) by several researchers would show, this class of *FKM-type* isoparametric hypersurfaces contains all isoparametric hypersurfaces with four principal curvatures with the exception of two homogeneous examples, having multiplicities $(2, 2)$ and $(4, 5)$ (see also [17, pp. 95–112] or [30, pp. 162–180] for a description of the FKM examples). This construction of Ferus, Karcher and Münzner was a generalization of an earlier construction due to Ozeki and Takeuchi [77].

Stolz [103] next proved that the multiplicities (m_1, m_2) of the principal curvatures of an isoparametric hypersurface with four principal curvatures must be the same as those of the hypersurfaces of FKM-type or the two homogeneous exceptions. Cecil, Chi and Jensen [21] then showed that if the multiplicities of an isoparametric hypersurface with four principal curvatures satisfy $m_2 \geq 2m_1 - 1$, then the hypersurface is of FKM-type. (A different proof of this result, using isoparametric triple systems, was given later by Immervoll [48].)

Taken together with known results of Takagi [104] for $m_1 = 1$ and Ozeki and Takeuchi [77] for $m_1 = 2$, this result of Cecil, Chi and Jensen handled all possible pairs of multiplicities except for four cases, the homogeneous pair $(4, 5)$, and the FKM pairs $(3, 4), (6, 9)$ and $(7, 8)$. In a series of recent papers, Chi [31]–[36] completed the classification of isoparametric hypersurfaces with four principal curvatures. Specifically, Chi showed that in the cases $(3, 4)$, $(6, 9)$ and $(7, 8)$, the isoparametric hypersurface must be of FKM-type, and in the case $(4, 5)$, it must be homogeneous.

In the case of an isoparametric hypersurface with six principal curvatures, Münzner showed that all of the principal curvatures must have the same multiplicity m, and Abresch [1] showed that m must equal 1 or 2. By the classification of homogeneous isoparametric hypersurfaces due to Takagi and Takahashi [105], there is only one homogeneous family in each case up to congruence. In the case of multiplicity $m = 1$, Dorfmeister and Neher [38] showed that an isoparametric hypersurface must be homogeneous, thereby completely classifying that case. The proof of Dorfmeister and Neher is quite algebraic in nature, and recently Miyaoka [64] and Siffert [97] have given shorter, more geometric proofs of this result.

Miyaoka [62] also gave a geometric description of the case $m = 1$, showing that a homogeneous isoparametric hypersurface M^6 in S^7 can be obtained as the inverse image under the Hopf fibration $h : S^7 \to S^4$ of an isoparametric hypersurface with three principal curvatures of multiplicity one in S^4. Miyaoka also showed that the two focal submanifolds of M^6 are not congruent, even though they are lifts under h^{-1} of congruent Veronese surfaces in S^4. Thus, these focal submanifolds are two non-congruent minimal homogeneous embeddings of $\mathbf{RP}^2 \times S^3$ in S^7.

After the paper of Dorfmeister and Neher [38] in 1985, it was conjectured that the one homogeneous family in the case $g = 6$, $m = 2$, is the only isoparametric family in this case, but this conjecture resisted proof for a long time. Recently, however, Miyaoka [65] (see also the errata [66]) published a proof that in the case $m = 2$, the isoparametric hypersurface must be homogeneous, thereby completing the classification in the case $g = 6$. The errata [66] pertain to an error in the original proof that was pointed out by Abresch and Siffert (see also [97]–[98]).

The major results in the theory of isoparametric hypersurfaces in spheres up to these most recent results of Chi [31]–[36] (for $g = 4$) and Miyaoka [65]–[66] (for $g = 6$) are described in detail in the survey articles by Thorbergsson [108] and Cecil [18], and in the book by Cecil and Ryan [30].

There is a close relationship between the theory of isoparametric hypersurfaces and the theory of compact proper Dupin hypersurfaces embedded in S^n (or $\mathbf{R}^n$), as we will now describe. First Thorbergsson [107] showed that the restriction $g = 1, 2, 3, 4$ or 6 on the number of distinct principal curvatures also holds for a connected, compact proper Dupin hypersurface M embedded in $S^n \subset \mathbf{R}^{n+1}$. He first showed that M must be *taut*, i.e., every nondegenerate distance function $L_p(x) = |p - x|^2, p \in \mathbf{R}^{n+1}$, has the minimum number of critical points required by the Morse inequalities on M. Using tautness, he then showed that M divides S^n into two ball bundles over the first focal submanifolds on either side of M. This topological situation is all that is required for Münzner's proof of the restriction on g.

Münzner's argument also produces certain restrictions on the cohomology and the homotopy groups of isoparametric hypersurfaces. These restrictions necessarily apply to compact proper Dupin hypersurfaces by Thorbergsson's result. Grove and Halperin [43] later found more topological similarities between these two classes of hypersurfaces. Furthermore, the results of Stolz [103] and Grove-Halperin [43] on the possible multiplicities of the principal curvatures actually only require the assumption that M is a compact proper Dupin hypersurface, and not that the hypersurface is isoparametric.

The close relationship between these two classes of hypersurfaces led to the widely held conjecture that every compact proper Dupin hypersurface $M \subset S^n$ is equivalent by a Lie sphere transformation to an isoparametric hypersurface (see [29, p. 184]). The conjecture is obviously true for $g = 1$, in which case M must be a hypersphere in S^n, and so M itself is isoparametric. In 1978, Cecil and Ryan [27] showed that if $g = 2$, then M must be a cyclide of Dupin, and it is therefore Möbius equivalent to an isoparametric hypersurface in S^n. Then in 1984, Miyaoka [59] showed that the conjecture holds for $g = 3$, although it is not true that M must be Möbius equivalent to an isoparametric hypersurface. Thus, as g increases, the group needed to obtain equivalence with an isoparametric hypersurface gets progressively larger.

The case $g = 4$ resisted all attempts at solution for several years until finally in 1988, counterexamples to the conjecture were discovered independently by Pinkall and Thorbergsson [84] and by Miyaoka and Ozawa [67]. The method of Miyaoka and Ozawa also yields counterexamples to the conjecture with $g = 6$ principal curvatures. In both cases, the counterexamples do not have constant Lie curvatures, and so they cannot be Lie equivalent to an isoparametric hypersurface. (See also [17, pp. 112–123] for a description of these counterexamples to the conjecture.) Research on a revised version of the conjecture that includes the assumption of constant Lie curvatures has been an important part of the development of the theory, as will be discussed in Section 3.7.

Remark 2.3. Nomizu [74] began the study of isoparametric hypersurfaces in pseudo-Riemannian space forms by proving a generalization of Cartan's identity for space-like hypersurfaces in a Lorentzian space form $\tilde{M}_1^n(c)$ of constant sectional curvature c. As a consequence of this identity, Nomizu showed that a space-like isoparametric hypersurface in $\tilde{M}_1^n(c)$ can have at most two distinct principal curvatures if $c \geq 0$. Recently, Li and Xie [54] have shown that this conclusion also holds for space-like isoparametric hypersurfaces in $\tilde{M}_1^n(c)$ for $c < 0$. Magid [57] studied isoparametric hypersurfaces in Lorentz space whose shape operator is not diagonalizable, and Hahn [44] contributed an extensive study of isoparametric hypersurfaces in pseudo-Riemannian space forms of arbitrary signatures. Recently Geatti and Gorodski [41] have extended this theory further by showing that a polar orthogonal representation of a connected real reductive algebraic group has the same closed orbits as the isotropy representation of a pseudo-Riemannian symmetric space.

Münzner's work shows that any connected isoparametric hypersurface in S^n can be extended to a compact, connected isoparametric hypersurface in a unique way. The following is a local Lie geometric characterization of those Legendre submanifolds that are Lie equivalent to the Legendre lift of an isoparametric hypersurface in S^n (see [15] or [17, p. 77]). This theorem has proven to be useful in classification theorems of Dupin hypersurfaces.

Recall that a line in $\mathbf{P}^{n+2}$ is called *timelike* if it contains only timelike points. This means that an orthonormal basis for the 2-plane in $\mathbf{R}_2^{n+3}$ determined by the timelike line consists of two timelike vectors. An example is the line $[e_1, e_{n+3}]$.

Theorem 2.10. *Let $\lambda : M^{n-1} \to \Lambda^{2n-1}$ be a Legendre submanifold with g distinct curvature spheres $[K_1], \ldots, [K_g]$ at each point. Then λ is Lie equivalent to the Legendre lift of an isoparametric hypersurface in S^n if and only if there exist g points $[P_1], \ldots, [P_g]$ on a timelike line in $\mathbf{P}^{n+2}$ such that*

$$\langle K_i, P_i \rangle = 0, \quad 1 \leq i \leq g.$$

Proof. If λ is the Legendre lift of an isoparametric hypersurface in S^n, then all the spheres in a family $[K_i]$ have the same radius ρ_i, where $0 < \rho_i < \pi$. By formula (21), this is equivalent to the condition $\langle K_i, P_i \rangle = 0$, where

$$P_i = \sin \rho_i \; e_1 - \cos \rho_i \; e_{n+3}, \quad 1 \leq i \leq g, \tag{99}$$

are g points on the timelike line $[e_1, e_{n+3}]$. Since a Lie sphere transformation preserves curvature spheres, timelike lines and the polarity relationship, the same is true for any image of λ under a Lie sphere transformation.

Conversely, suppose that there exist g points $[P_1], \ldots, [P_g]$ on a timelike line ℓ such that

$$\langle K_i, P_i \rangle = 0, \quad 1 \leq i \leq g.$$

Let β be a Lie sphere transformation that maps ℓ to the line $[e_1, e_{n+3}]$. Then the curvature spheres $\beta[K_i]$ of $\beta\lambda$ are respectively orthogonal to the points $[Q_i] = \beta[P_i]$ on the line $[e_1, e_{n+3}]$. This means that the spheres corresponding to $\beta[K_i]$ have constant radius on M^{n-1}. By applying a parallel transformation, if necessary, we can arrange that none of these curvature spheres has radius zero. Then $\beta\lambda$ is the Legendre lift of an isoparametric hypersurface in S^n. $\square$

Remark 2.4. In the case where λ is Lie equivalent to the Legendre lift of an isoparametric hypersurface in S^n, one can say more about the position of the points $[P_1], \ldots, [P_g]$ on the timelike line ℓ. Münzner showed that the radii ρ_i of the curvature spheres of an isoparametric hypersurface must be of the form

$$\rho_i = \rho_1 + (i-1)\frac{\pi}{g}, \quad 1 \leq i \leq g, \tag{100}$$

for some $\rho_1 \in (0, \pi/g)$. Hence, after Lie sphere transformation, the $[P_i]$ must have the form (99) for ρ_i as in equation (100).

Since the principal curvatures are constant on an isoparametric hypersurface, the Lie curvatures are also constant. By Münzner's work, the distinct principal curvatures $\kappa_i, 1 \leq i \leq g$, of an isoparametric hypersurface must have the form

$$\kappa_i = \cot \rho_i, \tag{101}$$

for ρ_i as in equation (100). Thus the Lie curvatures of an isoparametric hypersurface can be determined. We can order the principal curvatures so that

$$\kappa_1 < \cdots < \kappa_g. \tag{102}$$

In the case $g = 4$, this leads to a unique Lie curvature Ψ defined by

$$\Psi = [\kappa_1, \kappa_2; \kappa_3, \kappa_4] = (\kappa_1 - \kappa_2)(\kappa_4 - \kappa_3)/(\kappa_1 - \kappa_3)(\kappa_4 - \kappa_2). \tag{103}$$

The ordering of the principal curvatures implies that Ψ must satisfy $0 < \Psi < 1$. Using equations (101) and (103), one can compute that $\Psi = 1/2$ on any isoparametric hypersurface, i.e., the four curvature spheres form a *harmonic set* in the sense of projective geometry (see, for example, [95, p. 59]).

There is, however, a simpler way to compute Ψ. One applies Theorem 2.8 to the Legendre lift of one of the focal submanifolds of the isoparametric hypersurface. By the work of Münzner, each isoparametric hypersurface M^{n-1} embedded in S^n has two distinct focal submanifolds, each of codimension greater than one. The hypersurface M^{n-1} is a tube of constant radius over each of these focal submanifolds. Therefore, the Legendre lift of M^{n-1} is obtained from the Legendre lift of either focal submanifold by parallel transformation. Thus, the Legendre lift of M^{n-1} has the same Lie curvature as the Legendre lift of either focal submanifold. Let $\phi : V \to S^n$ be one of the focal submanifolds. By the same calculation that yields equation (100), Münzner showed that if ξ is any unit normal to $\phi(V)$ at any point, then the shape operator A_ξ has three distinct principal curvatures,

$$\kappa_1 = -1, \quad \kappa_2 = 0, \quad \kappa_3 = 1.$$

By Theorem 2.8, the Legendre lift of ϕ has a fourth principal curvature $\kappa_4 = \infty$. Thus, the Lie curvature of this Legendre submanifold is

$$\Psi = (-1 - 0)(\infty - 1)/(-1 - 1)(\infty - 0) = 1/2.$$

In the case $g = 4$, one can ask what is the strength of the assumption $\Psi = 1/2$ on M^{n-1}. Since Ψ is only one function of the principal curvatures, one would not expect this assumption to classify Legendre submanifolds up to Lie equivalence. However, if one makes additional assumptions, e.g., the Dupin condition, then results can be obtained.

Miyaoka [60] proved that the assumption that Ψ is constant on a compact connected proper Dupin hypersurface M^{n-1} in S^n with four principal curvatures, together with an additional assumption regarding intersections of leaves of the various principal foliations, implies that M^{n-1} is Lie equivalent to an isoparametric hypersurface.

As mentioned above, Thorbergsson [107] showed that for a compact proper Dupin hypersurface in S^n with four principal curvatures, the multiplicities of the principal curvatures must satisfy $m_1 = m_3, m_2 = m_4$, when the principal curvatures are appropriately ordered (see also Stolz [103]), the same as for an isoparametric hypersurface.

Cecil, Chi and Jensen [22] used a different approach than Miyaoka to prove that if M^{n-1} is a compact connected proper Dupin hypersurface in S^n with four principal curvatures and constant Lie curvature, whose multiplicities satisfy $m_1 = m_3 \geq 1$, $m_2 = m_4 = 1$, then M^{n-1} is Lie equivalent to an isoparametric hypersurface. Thus, Miyaoka's additional assumption regarding intersections of leaves of the various principal foliations is not needed in that case. It remains an open question whether Miyaoka's additional assumption can be removed in the case where $m_2 = m_4$ is also allowed to be greater than one, although this has been conjectured to be true by Cecil and Jensen [26, pp. 3–4].

In the same paper [22], Cecil, Chi and Jensen also obtained a local result by showing that if a connected proper Dupin submanifold,

$$\lambda : M^{n-1} \to \Lambda^{2n-1},$$

has four distinct principal curvatures with multiplicities,

$$m_1 = m_3 \geq 1, \quad m_2 = m_4 = 1, \tag{104}$$

and constant Lie curvature $\Psi = 1/2$, and λ is irreducible (in the sense of Pinkall [82], see §3.1), then λ is Lie equivalent to the Legendre lift of an isoparametric hypersurface in S^n. Again the conjecture of Cecil and Jensen [26, pp. 3–4] states that this result also holds if $m_2 = m_4$ is allowed to be greater than one.

The following example of Cecil [15] shows that some additional hypotheses (either compactness or irreducibility) besides $\Psi = 1/2$ are needed to be able to conclude that a proper Dupin hypersurface with four principal curvatures is Lie equivalent to an isoparametric hypersurface. This example is a noncompact proper Dupin submanifold with $g = 4$ distinct principal curvatures and constant Lie curvature $\Psi = 1/2$, which is not Lie equivalent to an open subset of an isoparametric hypersurface with four principal curvatures in S^n. This example is reducible in the sense of Pinkall (see Section 3.1), and it cannot be made compact without destroying the property that the number g of distinct curvatures spheres equals four at each point.

Example 2.2. Let $\phi : V \to S^{n-m}$ be an embedded Dupin hypersurface in S^{n-m} with field of unit normals ξ, such that ϕ has three distinct principal curvatures,

$$\mu_1 < \mu_2 < \mu_3,$$

at each point of V. Embed S^{n-m} as a totally geodesic submanifold of S^n, and let B^{n-1} be the unit normal bundle of the submanifold $\phi(V)$ in S^n. Let $\lambda : B^{n-1} \to \Lambda^{2n-1}$ be the Legendre submanifold induced by the submanifold $\phi(V)$ in S^n. Any unit normal η to $\phi(V)$ at a point $x \in V$ can be written in the form

$$\eta = \cos\theta \ \xi(x) + \sin\theta \ \zeta,$$

where ζ is a unit normal to S^{n-m} in S^n. Since the shape operator $A_\zeta = 0$, we have

$$A_\eta = \cos\theta \ A_\xi.$$

Thus the principal curvatures of A_η are

$$\kappa_i = \cos\theta \ \mu_i, \quad 1 \leq i \leq 3. \tag{105}$$

If $\eta \cdot \xi = \cos\theta \neq 0$, then A_η has three distinct principal curvatures. However, if $\eta \cdot \xi = 0$, then $A_\eta = 0$. Let U be the open subset of B^{n-1} on which $\cos\theta > 0$, and let α denote the restriction of λ to U. By Theorem 2.8, α has four distinct curvature spheres at each point of U. Since $\phi(V)$ is Dupin in S^{n-m}, it is easy to show that α is Dupin (see the tube construction in §3.2 for the details). Furthermore, since $\kappa_4 = \infty$, the Lie curvature Ψ of α at a point (x, η) of U equals the Möbius curvature $\Phi(\kappa_1, \kappa_2, \kappa_3)$. Using equation (105), we compute

$$\Psi = \Phi(\kappa_1, \kappa_2, \kappa_3) = \frac{\kappa_1 - \kappa_2}{\kappa_1 - \kappa_3} = \frac{\mu_1 - \mu_2}{\mu_1 - \mu_3} = \Phi(\mu_1, \mu_2, \mu_3). \tag{106}$$

Now suppose that $\phi(V)$ is a minimal isoparametric hypersurface in S^{n-m} with three distinct principal curvatures. By Münzner's formula (100), these principal curvatures must have the values,

$$\mu_1 = -\sqrt{3}, \quad \mu_2 = 0, \quad \mu_3 = \sqrt{3}.$$

On the open subset U of B^{n-1} described above, the Lie curvature of α has the constant value $1/2$ by equation (106). To construct an immersed proper Dupin hypersurface with four principal curvatures and constant Lie curvature $\Psi = 1/2$ in S^n, we simply take the open subset $\phi_t(U)$ of the tube of radius t around $\phi(V)$ in S^n.

It is not hard to see that this example is not Lie equivalent to an open subset of an isoparametric hypersurface in S^n with four distinct principal curvatures. Note that the point sphere map $[Y_1]$ of α is a curvature sphere of multiplicity m which lies in the linear subspace of codimension $m+1$ in $\mathbf{P}^{n+2}$ orthogonal to the space spanned by e_{n+3} and by those vectors ζ normal to S^{n-m} in S^n. This geometric fact implies that for such a vector ζ, there are only two distinct curvature spheres on each of the lines $\lambda(x,\zeta)$, since $A_\zeta = 0$ (see Theorem 2.8). On the other hand, if $\gamma : M^{n-1} \to \Lambda^{2n-1}$ is the Legendre lift of an isoparametric hypersurface in S^n with four distinct principal curvatures, then there are four distinct curvature spheres on each line $\gamma(x)$, for $x \in M^{n-1}$. Thus, no curvature sphere of γ lies in a linear subspace of codimension greater than one in $\mathbf{P}^{n+2}$, and so γ is not Lie equivalent to α. This change in the number of distinct curvature spheres at points of the form (x,ζ) is precisely why α cannot be extended to a compact proper Dupin submanifold with $g = 4$.

With regard to Theorem 2.10, α comes as close as possible to satisfying the requirements for being Lie equivalent to an isoparametric hypersurface without actually fulfilling them. The principal curvatures $\kappa_2 = 0$ and $\kappa_4 = \infty$ are constant on U. If a third principal curvature were also constant, then the constancy of Ψ would imply that all four principal curvatures were constant, and α would be the Legendre submanifold induced by an isoparametric hypersurface.

Using this same method, it is easy to construct noncompact proper Dupin hypersurfaces in S^n with $g = 4$ and $\Psi = c$, for any constant $0 < c < 1$. If $\phi(V)$ is an isoparametric hypersurface in S^{n-m} with three distinct principal curvatures, then Münzner's formula (100) implies that these principal curvatures must have the values,

$$\mu_1 = \cot(\theta + \frac{2\pi}{3}), \quad \mu_2 = \cot(\theta + \frac{\pi}{3}) \quad \mu_3 = \cot\theta, \quad 0 < \theta < \frac{\pi}{3}. \tag{107}$$

Furthermore, any value of θ in $(0, \pi/3)$ can be realized by some hypersurface in a parallel family of isoparametric hypersurfaces. A direct calculation using equations (106) and (107) shows that the Lie curvature Ψ of α satisfies

$$\Psi = \Phi(\kappa_1, \kappa_2, \kappa_3) = \frac{\kappa_1 - \kappa_2}{\kappa_1 - \kappa_3} = \frac{\mu_1 - \mu_2}{\mu_1 - \mu_3} = \frac{1}{2} + \frac{\sqrt{3}}{2}\tan(\theta - \frac{\pi}{6}),$$

on U. This can assume any value c in the interval $(0, 1)$ by an appropriate choice of θ in $(0, \pi/3)$. An open subset of a tube over $\phi(V)$ in S^n is a proper Dupin hypersurface with $g = 4$ and $\Psi = \Phi = c$. Note that Φ has different values on different hypersurfaces in the parallel family of isoparametric hypersurfaces. Thus these hypersurfaces are not Möbius equivalent to each other by Corollary 2.3. This is consistent with the fact that a parallel transformation is not a Möbius transformation.

3 Dupin Hypersurfaces

In this section, we concentrate on local results that have been obtained using Lie sphere geometry. We present the classification of proper Dupin submanifolds with two principal curvatures (cyclides of Dupin) and describe the classification of proper Dupin hypersurfaces with three or four principal curvatures. These classifications have been obtained by using the method of moving Lie frames.

3.1 Local constructions

Pinkall [82] introduced four constructions for obtaining a Dupin hypersurface W in $\mathbf{R}^{n+m}$ from a Dupin hypersurface M in $\mathbf{R}^n$. We first describe these constructions in the case $m = 1$ as follows.

Begin with a Dupin hypersurface M^{n-1} in $\mathbf{R}^n$ and then consider $\mathbf{R}^n$ as the linear subspace $\mathbf{R}^n \times \{0\}$ in $\mathbf{R}^{n+1}$. The following constructions yield a Dupin hypersurface W^n in $\mathbf{R}^{n+1}$.

(1) Let W^n be the cylinder $M^{n-1} \times \mathbf{R}$ in $\mathbf{R}^{n+1}$.

(2) Let W^n be the hypersurface in $\mathbf{R}^{n+1}$ obtained by rotating M^{n-1} around an axis $\mathbf{R}^{n-1} \subset \mathbf{R}^n$.

(3) Let W^n be a tube in $\mathbf{R}^{n+1}$ around M^{n-1}.

(4) Project M^{n-1} stereographically onto a hypersurface $V^{n-1} \subset S^n \subset \mathbf{R}^{n+1}$. Let W^n be the cone over V^{n-1} in $\mathbf{R}^{n+1}$.

In general, these constructions introduce a new principal curvature of multiplicity one which is constant along its lines of curvature. The other principal curvatures are determined by the principal curvatures of M^{n-1}, and the Dupin property is preserved for these principal curvatures. These constructions can be generalized to produce a new principal curvature of multiplicity m by considering $\mathbf{R}^n$ as a subset of $\mathbf{R}^n \times \mathbf{R}^m$ rather than $\mathbf{R}^n \times \mathbf{R}$.

Although Pinkall gave these four constructions, his Theorem 4 [82, p. 438] showed that the cone construction is redundant, since it is Lie equivalent to a tube. This will be explained further in Remark 3.7. For this reason, we will only study three standard constructions: tubes, cylinders and surfaces of revolution in detail.

A Dupin submanifold obtained from a lower-dimensional Dupin submanifold via one of these standard constructions is said to be *reducible*. More generally, a Dupin submanifold which is locally Lie equivalent to such a Dupin submanifold is called *reducible*.

Using these constructions, Pinkall was able to produce a proper Dupin hypersurface in Euclidean space with an arbitrary number of distinct principal curvatures, each with any given multiplicity (see Theorem 3.1 below). In general, these proper Dupin hypersurfaces cannot be extended to compact Dupin hypersurfaces without losing the property that the number of distinct principal curvatures is constant, as we will see when we discuss the constructions in detail. For now, we give a proof of Pinkall's theorem without attempting to compactify the hypersurfaces constructed.

Theorem 3.1. *Given positive integers $m_1, \ldots, m_g$ with*

$$m_1 + \cdots + m_g = n - 1,$$

there exists a proper Dupin hypersurface in $\mathbf{R}^n$ with g distinct principal curvatures having respective multiplicities $m_1, \ldots, m_g$.

Proof. The proof is by an inductive construction, which will be clear once the first few examples are done. To begin, note that a usual torus of revolution T^2 in $\mathbf{R}^3$ is a proper Dupin hypersurface with two principal curvatures. To construct a proper Dupin hypersurface W^3 in $\mathbf{R}^4$ with three principal curvatures, each of multiplicity one, begin with an open subset U of a torus of revolution in $\mathbf{R}^3$ on which neither principal curvature vanishes. Take W^3 to be the cylinder $U \times \mathbf{R}$ in $\mathbf{R}^3 \times \mathbf{R} = \mathbf{R}^4$. Then W^3 has three distinct principal curvatures at each point, one of which is zero. These are clearly constant along their corresponding 1-dimensional curvature surfaces.

To get a proper Dupin hypersurface in $\mathbf{R}^5$ with three principal curvatures having respective multiplicities $m_1 = m_2 = 1$, $m_3 = 2$, one simply takes

$$U \times \mathbf{R}^2 \subset \mathbf{R}^3 \times \mathbf{R}^2 = \mathbf{R}^5.$$

To obtain a proper Dupin hypersurface Z^4 in $\mathbf{R}^5$ with four principal curvatures of multiplicity one, first invert the hypersurface W^3 above in a 3-sphere in $\mathbf{R}^4$, chosen so that the image of W^3 contains an open subset V^3 on which no principal curvature vanishes. The hypersurface V^3 is proper Dupin, since the proper Dupin property is preserved by Möbius transformations. Now take Z^4 to be the cylinder $V^3 \times \mathbf{R}$ in $\mathbf{R}^4 \times \mathbf{R} = \mathbf{R}^5$. $\square$

The proof of this theorem gives an indication of the type of problems that occur when attempting to extend these constructions to produce a compact, proper Dupin hypersurface. In particular, for the cylinder construction, the new principal curvature on the constructed hypersurface W^n is identically zero. Thus, in order for W^n to be proper Dupin, either zero is not a principal curvature at any point of the original hypersurface M^{n-1}, or else zero is a principal curvature of constant multiplicity on M^{n-1}. Otherwise, the principal curvature zero will not have constant multiplicity on W^n, which implies that W^n is not proper Dupin.

3.2 Reducible Dupin submanifolds

In this section, we discuss the standard constructions of Pinkall [82] in more detail. This actually involves some rather technical calculations, and some unexpected subleties arise that are important in the theory. We will state the main theorems and discuss the important aspects of the theory. The reader is referred to [17, pp. 127–148] for complete proofs of these results.

For each construction, we imitate the case where the Euclidean projection of the initial Legendre submanifold λ is an immersion, but we do not assume this. We then determine the curvature spheres of the Legendre submanifold μ obtained from the construction and their respective multiplicities. Although this approach is more complicated than simply working in $\mathbf{R}^n$, it enables us to answer important questions concerning the possibility of constructing compact proper Dupin submanifolds by these methods.

We first give some notation common to all three constructions. This will help us explain the results precisely, even though we do not give all the details of the constructions. Let $\{e_1, \ldots, e_{n+m+3}\}$ be the standard orthonormal basis for $\mathbf{R}^{n+m+3}_2$, with e_1 and e_{n+m+3} timelike. Let $\mathbf{P}^{n+m+2}$ be the projective space determined by $\mathbf{R}^{n+m+3}_2$, with corresponding Lie quadric Q^{n+m+1}. Let $\mathbf{R}^{n+3}_2 \subset \mathbf{R}^{n+m+3}_2$ be the subspace

$$\mathbf{R}^{n+3}_2 = \text{Span}\ \{e_1, \ldots, e_{n+2}, e_{n+m+3}\},$$

and let $\mathbf{P}^{n+2}$ and Q^{n+1} be the corresponding projective space and Lie quadric, respectively. Let Λ^{2n-1} and $\Lambda^{2(n+m)-1}$ be the spaces of projective lines on Q^{n+1} and Q^{n+m+1}, respectively. Finally, let

$$\begin{aligned} \mathbf{R}^n &= \text{Span}\ \{e_3, \ldots, e_{n+2}\}, \\ \mathbf{R}^{n+m} &= \text{Span}\ \{e_3, \ldots, e_{n+m+2}\}. \end{aligned} \tag{108}$$

A. Tubes

We will construct a Legendre submanifold which corresponds to building a tube of radius ε in $\mathbf{R}^{n+m}$ around an $(n-1)$-dimensional submanifold M^{n-1} in $\mathbf{R}^n$. This can be done for any Legendre submanifold λ, although we will assume that λ is a proper Dupin submanifold. We will work with Euclidean projections of the Legendre submanifolds here, but one could just as well use spherical projections and construct a tube of radius ε using the spherical metric (see Remark 3.7). The reader is referred to [17, pp. 127–131] for a proof of the following result. That proof is a good example of submanifold theory in the context of Lie sphere geometry.

Proposition 3.1. *Suppose that $\lambda : M^{n-1} \to \Lambda^{2n-1}$ is a proper Dupin submanifold with g distinct curvature spheres such that the Euclidean projection f is an immersion of M^{n-1} into $\mathbf{R}^n \subset \mathbf{R}^{n+m}$. Then the tube construction*

yields a Dupin submanifold μ defined on the unit normal bundle B^{n+m-1} of $f(M^{n-1})$ in $\mathbf{R}^{n+m}$. The number $\gamma(x,\eta)$ of distinct curvature spheres of μ at a point $(x,\eta) \in B^{n+m-1}$ is as follows:
(a) $\gamma(x,\eta) = 2$, *if η is orthogonal to $\mathbf{R}^n$ in $\mathbf{R}^{n+m}$.*
(b) $\gamma(x,\eta) = g+1$, *otherwise.*

Remark 3.1. The fact that $g = 2$ if η is orthogonal to $\mathbf{R}^n$ in $\mathbf{R}^{n+m}$ is exactly the same type of calculation that occurred in Example 2.1 of a tube over a torus $T^2 \subset \mathbf{R}^3 \subset \mathbf{R}^4$.

Remark 3.2. The new family of curvature spheres resulting from the construction consists of spheres of signed radius $-\varepsilon$ with centers at the points of M^{n-1} (the minus sign is due to the choice of field of outward normals to the tube). In the Lie quadric, this family of curvature spheres lies in a linear subspace E of codimension $m+1$ in the projective space $\mathbf{P}^{n+m+2}$ such that the signature of $\langle\ ,\ \rangle$ on the orthogonal complement $E^\perp$ is $(m,1)$. The reason for the signature is that all the spheres have the same signed radius $-\varepsilon$, so their corresponding points in the Lie quadric are orthogonal to the unit timelike vector $\varepsilon e_1 - \varepsilon e_2 + e_{n+m+3}$. The spheres also have there centers in $\mathbf{R}^n$, so their corresponding points in the Lie quadric are orthogonal to the the m-dimensional (spacelike) orthogonal complement of $\mathbf{R}^n$ in $\mathbf{R}^{n+m}$. Thus, the space $E^\perp$ is given by

$$E^\perp = \text{Span}\ \{e_{n+3}, \ldots, e_{n+m+2}, \varepsilon e_1 - \varepsilon e_2 + e_{n+m+3}\}, \tag{109}$$

on which the scalar product $\langle\ ,\ \rangle$ has signature $(m,1)$.

Remark 3.3. In the case $\varepsilon = 0$, μ is the Legendre lift of the immersion $f(M^{n-1})$ as a submanifold of codimension $m+1$ in $\mathbf{R}^{n+m}$. Theorem 2.8 describes the curvature spheres of μ. The point sphere map $[K_1]$ is a curvature sphere of multiplicity m, which lies in the $(n+1)$-dimensional linear subspace E of $\mathbf{P}^{n+m+2}$ with orthogonal complement $E^\perp$ of signature $(m,1)$ as above with $\varepsilon = 0$. The tubes of radius $\varepsilon \neq 0$ over $f(M^{n-1})$ are parallel submanifolds of μ.

If the Euclidean projection f of λ has constant rank less that $n-1$, then λ is the Legendre lift of an immersed submanifold $\phi : V \to \mathbf{R}^n$ of codimension $\nu + 1$, and the domain of λ is the unit normal bundle B^{n-1} of $\phi(V)$ in $\mathbf{R}^n$. In that case, we get a different result concerning the number of distinct principal curvatures of the tube over $\phi(V)$ in $\mathbf{R}^{n+m}$ due to the fact that the point sphere map of λ is already a curvature sphere map (see [17, pp. 131–132] for more detail).

Proposition 3.2. *Suppose that $\lambda : B^{n-1} \to \Lambda^{2n-1}$ is a proper Dupin submanifold with g distinct curvature spheres induced by an immersed submanifold $\phi(V)$ of codimension $\nu+1$ in $\mathbf{R}^n \subset \mathbf{R}^{n+m}$. Then the tube construction yields a Dupin submanifold μ defined on the unit normal bundle B^{n+m-1} to $\phi(V)$ in $\mathbf{R}^{n+m}$. The number $\gamma(x,\eta)$ of distinct curvature spheres of μ at a point $(x,\eta) \in B^{n+m-1}$ is as follows:*
(a) $\gamma(x,\eta) = 2$, *if η is orthogonal to $\mathbf{R}^n$ in $\mathbf{R}^{n+m}$.*
(b) $\gamma(x,\eta) = g$, *otherwise.*

Remark 3.4. The original purpose of Pinkall's constructions was to increase the number of distinct curvature spheres by one, as in Proposition 3.1. However, as Proposition 3.2 shows, this does not happen when λ is the Legendre lift of a submanifold $\phi(V)$ of codimension greater than in one in $\mathbf{R}^n$. Still we consider the Dupin submanifold μ in Proposition 3.2 to be reducible, since it is obtained from λ by one of the standard constructions. The following is a concrete example of this phenomenon.

Example 3.1 (Tube over a Veronese surface in $S^4 \subset S^5$)**.** We consider the case where V^2 is a Veronese surface embedded in $S^4 \subset S^5$, where S^4 is a great sphere in S^5. We first recall the details of the Veronese surface. Let S^2 be the unit sphere in $\mathbf{R}^3$ given by the equation

$$u^2 + v^2 + w^2 = 1.$$

Consider the map from S^2 into the unit sphere $S^4 \subset \mathbf{R}^5$ given by

$$(u,v,w) \mapsto \left(\sqrt{3}vw, \sqrt{3}wu, \sqrt{3}uv, \frac{\sqrt{3}}{2}(u^2 - v^2), w^2 - \frac{u^2+v^2}{2} \right).$$

This map takes the same value on antipodal points of S^2, so it induces a map $\phi : \mathbf{P}^2 \to S^4$, and one can show that ϕ is an embedding. The surface $V^2 = \phi(\mathbf{P}^2)$ is called a *Veronese surface*. One can show (see, for example, [29, Example 7.3, pp. 296–299]) that a tube over V^2 of radius ε, for $0 < \varepsilon < \pi/3$, in the spherical metric of S^4 is an isoparametric hypersurface M^3 with $g = 3$ distinct principal curvatures (Cartan's isoparametric hypersurface). This hypersurface M^3 is not reducible, because the Veronese surface is substantial (does not lie in a hyperplane) in $\mathbf{R}^5$, so M^3 is not obtained as a result of the tube construction as described above. (See Takeuchi [106] for further discussion of proper Dupin hypersurfaces obtained as tubes over symmetric submanifolds of codimension greater than one in space-forms.)

Now embed $\mathbf{R}^5$ as a hyperplane through the origin in $\mathbf{R}^6$ and let e_6 be a unit normal vector to $\mathbf{R}^5$ in $\mathbf{R}^6$. The surface V^2 is a subset of the unit sphere $S^5 \subset \mathbf{R}^6$. As in the calculations made prior to Proposition 3.2, one can show that a tube over V^2 of radius ε in S^5 is not an isoparametric hypersurface, nor is it even a proper Dupin hypersurface, because the number of distinct principal curvatures is not constant on the unit normal bundle B^4 to V^2 in S^5. Specifically, if μ is the Legendre lift of the submanifold $V^2 \subset S^5$, then μ has two distinct curvature spheres at points in B^4 of the form $(x, \pm e_6)$, and three distinct curvature spheres at all other points of B^4. A tube W^4 over V^2 in S^5 is a reducible Dupin hypersurface, but it is not proper Dupin. At points of W^4 corresponding to the points $(x, \pm e_6)$ in B^4, there are two principal curvatures, both of multiplicity two. At the other points of W^4, there are three distinct principal curvatures, one of multiplicity two, and the others of multiplicity one. Thus, W^4 has an open dense subset U which is a reducible proper Dupin hypersurface with three principal curvatures at each point, but W^4 itself is not proper Dupin.

B. Cylinders

As before, we begin with a proper Dupin submanifold $\lambda : M^{n-1} \to \Lambda^{2n-1}$ with g distinct curvature spheres at each point, and assume that the locus of point spheres does not contain the improper point $[e_1 - e_2]$. We can write the point sphere map $[k_1]$ and the hyperplane map $[k_2]$ in the form

$$k_1 = (1 + f \cdot f, 1 - f \cdot f, 2f, 0)/2, \quad k_2 = (f \cdot \xi, -f \cdot \xi, \xi, 1). \tag{110}$$

and thereby define the Euclidean projection f and the Euclidean field of unit normals ξ as maps from M^{n-1} to $\mathbf{R}^n$. Usually, one thinks of the cylinder built over f in $\mathbf{R}^{n+m} = \mathbf{R}^n \times \mathbf{R}^m$ to be the map from $M^{n-1} \times \mathbf{R}^m$ to $\mathbf{R}^{n+m}$ given by

$$(x, z) \mapsto f(x) + z.$$

In [17, p. 133], we extended this map to a map defined on $M^{n-1} \times S^m$ by working in the context of Lie sphere geometry. This is accomplished by mapping all points in the set $M^{n-1} \times \{\infty\}$ to the improper point in Lie sphere geometry. The Legendre immersion condition (2) of Theorem 2.3 can still be satisfied at points of the form (x, ∞) because the normal vector varies as x varies. However, the Legendre immersion condition (2) is only satisfied at points of the form (x, ∞) for which the map ξ has rank $n-1$ at x.

The calculations to find the curvature spheres resulting from the cylinder construction yield the following results (see [17, pp. 135–136].

Proposition 3.3. *Suppose that $\lambda : M^{n-1} \to \Lambda^{2n-1}$ is a proper Dupin submanifold with g distinct curvature spheres such that the Euclidean field of unit normals ξ is an immersion. Then the cylinder construction yields a Dupin submanifold μ defined on $M^{n-1} \times S^m$. The number $\gamma(x, y)$ of distinct curvature spheres of μ at a point $(x, y) \in M^{n-1} \times S^m$ is as follows:*
(a) $\gamma(x, y) = 2$, *if y is the pole P of the stereographic projection τ from S^m to $\mathbf{R}^m$.*
(b) $\gamma(x, y) = g + 1$, *otherwise.*

Remark 3.5. At each point of the cylinder, the new curvature sphere corresponds to a hyperplane in $\mathbf{R}^{n+m}$ in oriented contact with the cylinder along one of its rulings. In the Lie quadric, this family of curvature spheres lies in a linear subspace E of codimension $m+1$ in the projective space $\mathbf{P}^{n+m+2}$ such that the signature of $\langle\ ,\ \rangle$ on the orthogonal complement $E^{\perp}$ is $(m, 0)$. The reason for the signature is that all the curvature spheres are oriented hyperplanes, so their corresponding points in the Lie quadric are orthogonal to the lightlike vector $e_1 - e_2$ (representing the improper point). Furthermore, the normal vector to each such hyperplane lies in $\mathbf{R}^n$, and so the point $[K]$ in the Lie quadric representing such a hyperplane according to equation (14) is orthogonal to the m-dimensional spacelike space $(\mathbf{R}^n)^{\perp}$ in $\mathbf{R}^{n+m}$. Thus, the space $E^{\perp}$ is given by

$$E^{\perp} = \text{Span}\ \{e_{n+3}, \ldots, e_{n+m+2}, e_1 - e_2\}, \tag{111}$$

on which the scalar product $\langle\ ,\ \rangle$ has signature $(m, 0)$.

The cylinder construction also yields a Dupin submanifold defined on the manifold $M^{n-1} \times \mathbf{R}^m$ if ξ has constant rank $n-1-\nu$, for $\nu \geq 1$. However, the construction does not extend to $M^{n-1} \times \{P\}$ because the Legendre immersion condition (2) is not satisfied at those points. Specifically, if $X \in T_x M^{n-1}$ is a nonzero vector such that $d\xi(X) = 0$, then $dK_1(X, 0)$ and $dK_2(X, 0)$ are both zero at the point (x, P). Furthermore, the number of distinct curvature spheres of the cylinder μ on $M^{n-1} \times \mathbf{R}^m$ is g, not $g+1$, since the hyperplane map is already a curvature sphere of λ. The curvature surfaces of $[K_2]$ are of the form $S \times S^m$, where S is a curvature surface of λ. Thus, we have:

Proposition 3.4. *Suppose that $\lambda : M^{n-1} \to \Lambda^{2n-1}$ is a proper Dupin submanifold with g distinct curvature spheres such that the Euclidean field of unit normals ξ has constant rank $n-1-\nu$, where $\nu \geq 1$. Then the cylinder construction yields a Dupin submanifold μ defined on $M^{n-1} \times \mathbf{R}^m$ with g distinct curvature spheres at each point.*

C. Surfaces of Revolution

As before, we begin with a proper Dupin submanifold $\lambda : M^{n-1} \to \Lambda^{2n-1}$ with g distinct curvature spheres at each point, and assume that the point sphere map $[k_1]$ does not contain the improper point. We write the point sphere map $[k_1]$ and the hyperplane map $[k_2]$ in the form of equation (110), and thereby define the Euclidean projection f and the Euclidean field of unit normals ξ as maps from M^{n-1} to $\mathbf{R}^n$. We now want to construct the Legendre submanifold μ obtained by revolving the *profile submanifold* f around an *axis of revolution*

$$\mathbf{R}^{n-1} \subset \mathbf{R}^n \subset \mathbf{R}^{n+m}.$$

We do not assume that f is an immersion, nor that the image of f is disjoint from the axis $\mathbf{R}^{n-1}$. The calculations in [17, pp. 136–139] yield the following proposition concerning the curvatures spheres of the hypersurface of revolution.

Proposition 3.5. *Suppose that $\lambda : M^{n-1} \to \Lambda^{2n-1}$ is a proper Dupin submanifold with g distinct curvature spheres. The surface of revolution construction yields a Dupin submanifold μ defined on all of $M^{n-1} \times S^m$, except those points where the spheres in the parabolic pencil determined by the line $\lambda(x)$ are all orthogonal to the axis $\mathbf{R}^{n-1}$. For (x, y) in the domain of μ, the number $\gamma(x, y)$ of distinct curvature spheres of μ at (x, y) is as follows:*
(a) $\gamma(x, y) = g + 1$, *if none of the curvature spheres of λ at x are orthogonal to the axis $\mathbf{R}^{n-1}$.*
(b) $\gamma(x, y) = g$, *otherwise.*

Remark 3.6. The new curvature spheres all meet the axis $\mathbf{R}^{n-1}$ orthogonally. In the Lie quadric, this family of curvature spheres lies in a linear subspace E of codimension $m+1$ in the projective space $\mathbf{P}^{n+m+2}$ such that the signature of $\langle \ , \ \rangle$ on the orthogonal complement $E^\perp$ is $(m+1, 0)$. The reason for the signature the points in the Lie quadric representing these curvature spheres all lie in $(m+1)$-dimensional spacelike space orthogonal to $\mathbf{R}^{n-1}$ in $\mathbf{R}^{n+m}$. Thus, the space $E^\perp$ is given by

$$E^\perp = \text{Span } \{e_{n+2}, \ldots, e_{n+m+2}\}, \tag{112}$$

on which $\langle \ , \ \rangle$ has signature $(m+1, 0)$.

Thus, as with the other constructions, there are two cases in which this construction yields a proper Dupin submanifold; either no curvature sphere of λ is ever orthogonal to the axis $\mathbf{R}^{n-1}$ or one of the curvature spheres of λ is always orthogonal to the axis.

Next we state a proposition concerning the surface of revolution construction that will be used in the proof of Theorem 3.4 which states that a compact proper Dupin hypersurface embedded in Euclidean space with $g > 2$ distinct principal curvatures must be irreducible. That result and the following proposition were proved by Cecil, Chi and Jensen [22] (see also [17, pp. 139–141]). The proof of this proposition is rather technical, and we omit it here.

Proposition 3.6. *Suppose that $\mu : M^{n-1} \times S^m \to \Lambda^{2(n+m)-1}$ is a Legendre submanifold that is obtained from a proper Dupin submanifold $\lambda : M^{n-1} \to \Lambda^{2n-1}$ by the the surface of revolution construction. If there exists a Lie sphere transformation β such that the point sphere map of $\beta\mu$ is an immersion, then there exists a Lie sphere transformation α such that the point sphere map of $\alpha\lambda$ is an immersion.*

3.3 Lie sphere geometric criterion for reducibility

We now find a Lie sphere geometric criterion for when a Dupin submanifold is reducible to a lower-dimensional Dupin submanifold. First, note that the totally umbilic case of a proper Dupin submanifold with one distinct curvature sphere is well known. These are all Lie equivalent to the Legendre submanifold induced by an open subset of a standard metric sphere S^{n-1} embedded in $\mathbf{R}^n$. A standard sphere can be obtained from a point by any of the standard constructions. From now on, we will only consider the case in which the number of distinct curvature spheres is greater than one.

We say that a Dupin submanifold η that is obtained from a Dupin submanifold λ by one of the standard constructions is *reducible to* λ. More generally, a Dupin submanifold μ that is Lie equivalent to such a Dupin submanifold η is also said to be *reducible to* λ.

In general, as we see from Propositions 3.1, 3.3 and 3.5, the application of one of the standard constructions to a proper Dupin submanifold with g distinct curvature spheres produces a proper Dupin submanifold with $g+1$ distinct curvature spheres defined on an open subset of $M^{n-1} \times S^m$. (Example 3.1 shows that this is not always the case, however.) Pinkall [82, p. 438] found the following simple criterion for reducibility in this general situation.

Theorem 3.2. *A proper Dupin submanifold $\mu : W^{d-1} \to \Lambda^{2d-1}$ with $g+1 \geq 2$ distinct curvature spheres is reducible to a proper Dupin submanifold λ with g distinct curvature spheres if and only if μ has a curvature sphere $[K]$ of multiplicity $m \geq 1$ that lies in a $(d+1-m)$-dimensional linear subspace of $\mathbf{P}^{d+2}$.*

Proof. Let $n = d-m$ in order to agree with the notation used in the previous section. Since μ has at least two distinct curvature spheres, we have

$$d - 1 - m \geq 1,$$

i.e., $n \geq 2$. For each of the three constructions, it was shown that if the constructed Dupin submanifold η has one more curvature sphere than the original Dupin submanifold λ, then the new curvature sphere $[K]$ has multiplicity m and lies in a $(d+1-m)$-dimensional linear subspace E of $\mathbf{P}^{d+2}$. The same holds for a Dupin submanifold μ that is Lie equivalent to such a Dupin submanifold η.

Conversely, if there exists a curvature sphere $[K]$ of multiplicity m that lies in an $(n+1)$-dimensional linear subspace V of $\mathbf{P}^{d+2}$, then the signature of $\langle\ ,\ \rangle$ on the $(m+1)$-dimensional vector space $V^\perp$ must be $(m+1, 0)$, $(m, 1)$ or $(m, 0)$. Otherwise, $V \cap Q^{d+1}$ is either empty or consists of a single point or a line (see Corollary 1.2). However, the curvature sphere map $[K]$ is an immersion of the $(n-1)$-dimensional space of leaves M^{n-1} of the principal foliation corresponding to $[K]$, and its image cannot be contained in a single line.

If the signature of $\langle\ ,\ \rangle$ on $V^\perp$ is $(m+1, 0)$, then there is a Lie sphere transformation α, induced by an orthogonal transformation $A \in O(d+1, 2)$, which takes $V^\perp$ to the space $E^\perp$ in equation (112). For the Dupin submanifold $\eta = \alpha\mu$, the centers of the curvature spheres in the family $[AK]$ all lie in the space

$$\mathbf{R}^{n-1} = \text{Span } \{e_3, \ldots, e_{n+1}\} \subset \mathbf{R}^n = \text{Span } \{e_3, \ldots, e_{n+2}\} \tag{113}$$

in $\mathbf{R}_2^{n+m+3}$. The proper Dupin submanifold η is an envelope of this family of curvature spheres $[AK]$, with each curvature sphere tangent to the envelope along a leaf of the principal foliation corresponding to $[AK]$. Since the family of curvature spheres $[AK]$ is invariant under $SO(m+1)$, the subgroup of $SO(d)$ consisting of isometries that keep the axis $\mathbf{R}^{n-1}$ pointwise fixed, the envelope of these curvature spheres is also invariant under $SO(m+1)$. Thus η is an open subset of a surface of revolution. The profile submanifold λ in $\mathbf{R}^n$ of this surface of revolution is locally obtained by taking those contact elements in $\mathbf{R}^n$ which are in the image of η. Each curvature surface of $[AK]$ is the orbit of a contact element in the image of λ under the action of the group $SO(m+1)$. Since the multiplicity m of $[AK]$ is accounted for by the action of this group, the profile submanifold has one less curvature sphere than η (and hence μ) at each point.

Similarly, if the signature of $\langle\ ,\ \rangle$ on $V^\perp$ is $(m,1)$, then $V^\perp$ can be mapped by a Lie sphere transformation α induced by $A \in O(d+1,2)$ to the space $E^\perp$ in equation (109). Then each curvature sphere in the family $[AK]$ has radius $-\varepsilon$ and has center in the space $\mathbf{R}^n$ in equation (113). Since the map $[AK]$ factors through an immersion of the space of leaves M^{n-1} of the principal foliation, the locus of centers of these spheres factors through an immersion f of M^{n-1} into $\mathbf{R}^n$. The proper Dupin submanifold $\eta = \alpha\mu$ is an envelope of this family of curvature spheres, and it is obtained from the Legendre submanifold λ induced from the hypersurface f in $\mathbf{R}^n$ via the tube construction. Since the multiplicity of $[AK]$ is accounted for by the tube construction, λ has one less curvature sphere than μ.

Finally, if the signature of $\langle\ ,\ \rangle$ on $V^\perp$ is $(m,0)$, then $V^\perp$ can be mapped by a Lie sphere transformation α induced by $A \in O(d+1,2)$ to the space $E^\perp$ in equation (111). The family $[AK]$ of curvature spheres consists of hyperplanes orthogonal to the space $\mathbf{R}^n$ in equation (113). The proper Dupin submanifold $\eta = \alpha\mu$ is an envelope of this family of hyperplanes, with each hyperplane tangent to the envelope along a leaf of the principal foliation. This family of hyperplanes is invariant under the action of the group H of translations of $\mathbf{R}^d$ in directions orthogonal to $\mathbf{R}^n$, and so is the envelope. Each leaf of the principal foliation is the orbit of a single contact element in $\mathbf{R}^n$ under the action of H. These contact elements in $\mathbf{R}^n$ determine the original proper Dupin submanifold λ from which η is obtained by the cylinder construction. Again, it is clear that λ has one less curvature sphere than μ at each point, since the multiplicity m of the curvature sphere $[AK]$ equals the codimension of $\mathbf{R}^n$ in $\mathbf{R}^d$. $\square$

Pinkall [82, p. 438] also formulated his local criterion for reducibility to handle the case where the number of distinct curvature spheres of μ is the same as the number of distinct curvature spheres of λ, as in Example 3.1. For this theorem, we do not take into account the multiplicity of the curvature sphere $[K]$. The result also holds for a proper Dupin submanifold with one curvature sphere at each point, so we also include that case.

The following version of the proof of Pinkall's criterion for reducibility was published in [22]. This proof makes use of the fact that a proper Dupin submanifold must be algebraic and hence analytic, as mentioned earlier.

Theorem 3.3. *A connected proper Dupin submanifold $\mu : W^{d-1} \to \Lambda^{2d-1}$ is reducible if and only if there exists a curvature sphere $[K]$ of μ that lies in a linear subspace of $\mathbf{P}^{d+2}$ of codimension at least two.*

Proof. First, assume that μ is reducible. By definition this means that for every $x \in W^{d-1}$, there exists a neighborhood of x such that the restriction of μ to this neighborhood is Lie equivalent to the end product of one of the standard constructions. For each of these constructions it was shown that one of the curvature spheres $[K]$ lies in a space of codimension at least two in $\mathbf{P}^{d+2}$. For each $x \in W^{d-1}$, let $m_x \geq 1$ be the largest positive integer such that for some neighborhood U_x of x, the restriction of the curvature sphere map $[K]$ to U_x lies in a linear subspace of codimension $m_x + 1$ in $\mathbf{P}^{d+2}$. Choose x_0 to be a point where m_x attains its maximum value m on W^{d-1}. Then there exist linearly independent vectors $v_1, \ldots, v_{m+1}$ in $\mathbf{R}^{d+3}_2$ such that on a neighborhood U_{x_0} of x_0, we have

$$\langle K, v_i \rangle = 0, \quad 1 \leq i \leq m+1. \tag{114}$$

Since μ is analytic, the curvature sphere map $[K]$ is analytic. Then since the analytic functions $\langle K, v_i \rangle$ equal zero on the open set U_{x_0} in the connected analytic manifold W^{d-1}, they must equal zero on all of W^{d-1}. Thus, equation (114) holds on all of W^{d-1}, and the function $m_x = m$ for all $x \in W^{d-1}$. The curvature sphere $[K]$ lies in the space E of codimension $m+1$ determined by equation (114), and so $[K]$ lies in a linear space of codimension at least two in $\mathbf{P}^{d+2}$.

The proof of the converse is essentially the same as that of Theorem 3.2. Suppose that μ has a curvature sphere $[K]$ that lies in a linear subspace V of codimension $m+1$, where $m \geq 1$. As in the proof of Theorem 3.2, there exists a Lie sphere transformation α induced by $A \in O(d+1, 2)$ such that α

maps $V^\perp$ to the appropriate space $E^\perp$ in Remarks 3.2, 3.5, 3.6, as determined by the signature of $\langle\ ,\ \rangle$ on $V^\perp$. The proof of Theorem 3.2 deals specifically with the case where $[K]$ has multiplicity m, and so the number of curvature spheres of $\eta = \alpha\mu$ is one greater than the number of curvature spheres of λ. If $[K]$ has multiplicity greater than m, then the curvature sphere $[AK]$ of η is also equal to one of the curvature spheres of η induced from a curvature sphere $[k]$ of λ, and the multiplicity of $[K]$ is $m+l$, where l is the multiplicity of $[k]$. In that case, μ and λ have the same number of curvature spheres, as in Example 3.1. The rest of the proof is quite similar to the proof of Theorem 3.2. $\square$

Remark 3.7. When Pinkall introduced his constructions, he also listed the following construction. Begin with a proper Dupin submanifold λ induced by an embedded proper Dupin hypersurface $M^{n-1} \subset S^n \subset \mathbf{R}^{n+1}$. The new Dupin submanifold μ is the Legendre submanifold induced from the cone C^n over M^{n-1} in $\mathbf{R}^{n+1}$ with vertex at the origin. Theorem 3.3 shows that this construction is locally Lie equivalent to the tube construction as follows. The tube construction is characterized by the fact that one curvature sphere map $[K]$ lies in a d-dimensional linear subspace E of $\mathbf{P}^{d+2}$, whose orthogonal complement has signature $(1,1)$. Geometrically, this means that after a suitable Lie sphere transformation, all the spheres in the family $[K]$ have the same radius and their centers lie in a subspace $\mathbf{R}^{d-1} \subset \mathbf{R}^d$. For the cone construction, the new family $[K]$ of curvature spheres consists of hyperplanes through the origin (corresponding to the point $[e_1 + e_2]$ in Lie sphere geometry) that are tangent to the cone along the rulings. Since the hyperplanes also all pass through the improper point $[e_1 - e_2]$, they correspond to points in the linear subspace E, whose orthogonal complement is as follows:

$$E^\perp = \text{Span } \{e_1 + e_2, e_1 - e_2\}.$$

Since $E^\perp$ is spanned by e_1 and e_2, it has signature $(1,1)$. Thus, the cone construction is Lie equivalent to the tube construction. Finally, there is one more geometric interpretation of the tube construction. Note that a family $[K]$ of curvature spheres that lies in a linear subspace whose orthogonal complement has signature $(1,1)$ can also be considered to consist of spheres in S^d of constant radius in the spherical metric whose centers lie in a hyperplane. The corresponding proper Dupin submanifold can thus be considered to be a tube in the spherical metric over a lower-dimensional submanifold in S^d.

Remark 3.8. In a recent paper [37], Dajczer, Florit and Tojeiro studied reducibility in the context of Riemannian geometry. They formulated a concept of weak reducibility for proper Dupin submanifolds that have a flat normal bundle including proper Dupin hypersurfaces. For hypersurfaces, their definition can be formulated as follows. A proper Dupin hypersurface $f : M^{n-1} \to \mathbf{R}^n$ (or S^n) is said to be *weakly reducible* if, for some principal curvature κ_i with corresponding principal space T_i, the orthogonal complement $T_i^\perp$ is integrable. Dajczer, Florit and Tojeiro show that if a proper Dupin hypersurface $f : M^{n-1} \to \mathbf{R}^n$ is Lie equivalent to a proper Dupin hypersurface with $g+1$ distinct principal curvatures that is obtained via one of the standard constructions from a proper Dupin hypersurface with g distinct principal curvatures, then f is weakly reducible. Thus, reducible implies weakly reducible for such hypersurfaces.

However, one can show that the open set U of the tube W^4 over V^2 in S^5 in Example 3.1 on which there are three principal curvatures at each point is reducible but not weakly reducible, because none of the orthogonal complements of the principal spaces is integrable. Of course, U is not constructed from a proper Dupin submanifold with two curvature spheres, but rather one with three curvature spheres.

In two papers by Cecil and Jensen [25]–[26], the notion of local irreducibility was used. Specifically, a proper Dupin submanifold $\lambda : M^{n-1} \to \Lambda^{2n-1}$ is said to be *locally irreducible* if there does not exist any open subset $U \subset M^{n-1}$ such that the restriction of λ to U is reducible. Theoretically, this is a more restrictive condition than the requirement of irreducibility of λ itself. However, using the analyticity of proper Dupin submanifolds, Cecil, Chi and Jensen [22] proved the following proposition which shows that the concepts of local irreducibility and irreducibility are equivalent.

Proposition 3.7. *Let $\lambda : M^{n-1} \to \Lambda^{2n-1}$ be a connected, proper Dupin submanifold. If the restriction of λ to an open subset $U \subset M^{n-1}$ is reducible, then λ is reducible. Thus, a connected proper Dupin submanifold is locally irreducible if and only if it is irreducible.*

Proof. Suppose there exists an open subset $U \subset M^{n-1}$ on which the restriction of λ is reducible. By Theorem 3.3 there exists a curvature sphere $[K]$ of λ and two linearly independent vectors v_1 and v_2, such that

$$\langle K(x), v_i \rangle = 0, \quad i = 1, 2,$$

for all $x \in U$. Since the curvature sphere map $[K]$ is analytic on M^{n-1}, the functions $\langle K, v_i \rangle$ are analytic on M^{n-1} for $i = 1, 2$. Since these functions are identically equal to zero on the open set U, they must equal zero on all of the connected analytic manifold M^{n-1}. Therefore, by Theorem 3.3, the proper Dupin submanifold $\lambda : M^{n-1} \to \Lambda^{2n-1}$ is reducible. Thus, if λ is irreducible, then it cannot have a reducible open subset, so it must be locally irreducible. $\square$

The considerations above are all of a local nature. We now want to consider the global question of when a compact proper Dupin hypersurface embedded in $\mathbf{R}^d$ or S^d is irreducible. Thorbergsson [107] showed that a compact, connected proper Dupin hypersurface immersed in $\mathbf{R}^d$ or S^d is taut, and therefore it is embedded (see Carter–West [13] or Cecil–Ryan [29, p. 121]). The following theorem was proved by Cecil, Chi and Jensen [22].

Theorem 3.4. *Let W^{d-1} be a compact, connected proper Dupin hypersurface immersed in $\mathbf{R}^d$ with $g > 2$ distinct principal curvatures. Then W^{d-1} is irreducible. That is, the Legendre submanifold induced by the hypersurface W^{d-1} is irreducible.*

Proof. As noted above, tautness implies that an immersed compact, connected proper Dupin hypersurface is embedded in $\mathbf{R}^d$. We will assume that $W^{d-1} \subset \mathbf{R}^d$ is reducible and obtain a contradiction. Let $\mu : W^{d-1} \to \Lambda^{2d-1}$ be the Legendre submanifold induced by the embedded hypersurface $W^{d-1} \subset \mathbf{R}^d$. By the proof of Theorem 3.3, the fact that μ is reducible implies that μ is equivalent by a Lie sphere transformation α to a proper Dupin submanifold $\eta = \alpha\mu : W^{d-1} \to \Lambda^{2d-1}$ that is obtained from a lower-dimensional proper Dupin submanifold $\lambda : M^{n-1} \to \Lambda^{2n-1}$ by one of the three standard constructions. Thus, W^{d-1} is diffeomorphic to $M^{n-1} \times S^m$, where $m = d - n$, and M^{n-1} must be compact, since W^{d-1} is compact. By hypothesis, μ has $g > 2$ distinct curvature spheres at each point, and thus so does η. For η obtained from λ by the tube or cylinder constructions, Propositions 3.1, 3.2 and 3.3 show that there always exist points on $M^{n-1} \times S^m$ at which the number of distinct curvature spheres is two, and therefore η cannot be obtained via the tube or cylinder constructions.

Therefore the only remaining possibility is that η is obtained from λ by the surface of revolution construction. Proposition 3.5 shows that for a surface of revolution, the number j of distinct curvature spheres on M^{n-1}

must be $g-1$ or g. Note that the sum β of the $\mathbf{Z}_2$-Betti numbers of W^{d-1} and M^{n-1} are related by the equation,

$$\beta(W^{d-1}) = \beta(M^{n-1} \times S^m) = 2\beta(M^{n-1}). \tag{115}$$

On the other hand, Thorbergsson showed that for a connected, compact proper Dupin hypersurface embedded in S^d, β is equal to twice the number of distinct curvature spheres. Thus, we have $\beta(W^{d-1}) = 2g$.

We know that η is Lie equivalent to μ and the point sphere map of μ is an immersion. Furthermore, η is obtained from λ by the surface of revolution construction. Thus, by Proposition 3.6, we conclude that there exists a Lie sphere transformation γ such that the point sphere map of $\gamma\lambda$ is an immersion. This point sphere map of $\gamma\lambda$ gives rise to a Euclidean projection,

$$f : M^{n-1} \to \mathbf{R}^n,$$

that is an immersed (and thus embedded) proper Dupin hypersurface. Thus, by Thorbergsson's theorem, we have $\beta(M^{n-1}) = 2j$, where j equals $g-1$ or g. This fact, together with equation (115), implies that it is impossible for W^{d-1} and M^{n-1} to have the same number of distinct curvature spheres, and so M^{n-1} has $g-1$ distinct curvature spheres. Hence, we have

$$\beta(W^{d-1}) = 2g, \quad \beta(M^{n-1}) = 2(g-1) = 2g-2. \tag{116}$$

Combining equations (115) and (116), we get

$$2g = 2(2g-2) = 4g-4,$$

and thus $g = 2$, contradicting the assumption that $g > 2$. Therefore, μ cannot be reducible. $\square$

A hypersurface in S^d is conformally equivalent to its image in $\mathbf{R}^d$ under stereographic projection. Furthermore, the proper Dupin condition is preserved under stereographic projection. Thus, as a corollary of Theorem 3.4, we conclude that a compact, connected isoparametric hypersurface in S^d is irreducible as a Dupin hypersurface if the number g of distinct principal curvatures is greater than two. This was proved earlier by Pinkall in his dissertation [78]. Of course, compactness is not really a restriction for an isoparametric hypersurface, since Münzner [70] has shown that any connected isoparametric hypersurface is contained in a unique compact, connected isoparametric hypersurface. The same is not true for proper Dupin hypersurfaces, since the completion of a proper Dupin hypersurface may not be proper Dupin. Consider, for example, the tube M^3 over a torus $T^2 \subset \mathbf{R}^3 \subset \mathbf{R}^4$ in Example 2.1. The tube M^3 is the completion of the open subset U of M^3 on which there are three distinct principal curvatures of multiplicity one. The set U is a proper Dupin hypersurface (with two connected components), but M^3 is only Dupin, but not proper Dupin. This phenomenon is also made clear by Propositions 3.1, 3.2, 3.3 and 3.5.

There is one other geometric consequence about isoparametric hypersurfaces that is implied by the theorem. Münzner showed that an isoparametric hypersurface $M^{n-1} \subset S^n \subset \mathbf{R}^{n+1}$ is a tube of constant radius in S^n over each of its two focal submanifolds. If $g = 2$, then the isoparametric hypersurface M^{n-1} must be a standard product of two spheres,

$$S^k(r) \times S^{n-k-1}(s) \subset S^n, \quad r^2 + s^2 = 1,$$

and the two focal submanifolds are both totally geodesic spheres, $S^k(1) \times \{0\}$ and $\{0\} \times S^{n-k-1}(1)$. The isoparametric hypersurface M^{n-1} is reducible in two ways, since it can be obtained as a tube of constant radius over each of these focal submanifolds, which are not substantial in $\mathbf{R}^{n+1}$. On the other hand, if an isoparametric hypersurface M^{n-1} has $g \geq 3$ distinct principal curvatures, then each of its focal submanifolds must be substantial in $\mathbf{R}^{n+1}$. Otherwise, M^{n-1} would be reducible to such a non-substantial focal submanifold by the tube construction, contradicting Theorem 3.4.

3.4 Cyclides of Dupin

A proper Dupin submanifold with two distinct curvature spheres of respective multiplicities p and q is called a *cyclide of Dupin of characteristic* (p, q). These are the simplest Dupin submanifolds after the spheres, and they were first studied in $\mathbf{R}^3$ by Dupin [39] in 1822. An example of a cyclide of Dupin of characteristic $(1, 1)$ in $\mathbf{R}^3$ is a torus of revolution. The cyclides were studied by many prominent mathematicians in the nineteenth century, including Liouville [56], Cayley [14], and Maxwell [58], whose paper contains stereoscopic figures of the various types of cyclides.

For cyclides of Dupin in $\mathbf{R}^3$, it was known in the nineteenth century that every connected Dupin cyclide is Möbius equivalent to an open subset of a surface of revolution obtained by revolving a profile circle $S^1 \subset \mathbf{R}^2$ about an axis $\mathbf{R}^1 \subset \mathbf{R}^2 \subset \mathbf{R}^3$. The profile circle is allowed to intersect the axis, thereby introducing Euclidean singularities. However, the corresponding Legendre map into the space of contact elements in $\mathbf{R}^3$ is an immersion, as discussed in §2.3.

Higher-dimensional cyclides of Dupin appeared in the study of isoparametric hypersurfaces in spheres. Cartan proved that an isoparametric hypersurface in a sphere with two curvature spheres must be a standard product of spheres,

$$S^p(r) \times S^q(s) \subset S^n(1) \subset \mathbf{R}^{p+1} \times \mathbf{R}^{q+1} = \mathbf{R}^{p+q+2}, \quad r^2 + s^2 = 1.$$

Cecil and Ryan [27] showed that a compact proper Dupin hypersurface M^{n-1} embedded in S^n with two distinct curvature spheres must be Möbius equivalent to a standard product of spheres. The proof, however, uses the compactness of M^{n-1} in an essential way. Later, Pinkall [82] used Lie sphere geometric techniques to obtain a local classification of the higher-dimensional cyclides of Dupin that is analogous to the classical result. In this section, we will prove Pinkall's theorem and then derive a local Möbius geometric classification from it. Pinkall's result is the following.

Theorem 3.5. (a) *Every connected cyclide of Dupin is contained in a unique compact, connected cyclide of Dupin.*
(b) *Any two cyclides of Dupin of the same characteristic are locally Lie equivalent.*

Before proving the theorem, we consider some models for compact cyclides of Dupin. The results of §3.2 show that one can obtain a cyclide of Dupin of characteristic (p, q) by applying any of the standard constructions (tube, cylinder or surface of revolution) to a p-sphere $S^p \subset \mathbf{R}^{p+1} \subset \mathbf{R}^n$, where $n = p + q + 1$. Another simple model of a cyclide of Dupin is obtained by considering the Legendre lift of a totally geodesic $S^q \subset S^n$, as a submanifold of codimension $p + 1$. Such a sphere is one of the two focal submanifolds of the family of isoparametric hypersurfaces obtained by taking tubes over S^q in S^n. The other focal submanifold is a totally geodesic p-sphere S^p in S^n. We now explicitly parametrize this Legendre submanifold by k_1 and k_2 satisfying the conditions (1)–(3) of Theorem 2.3. Let

$$\{e_1, \ldots, e_{n+3}\}$$

be the standard orthonormal basis for $\mathbf{R}^{n+3}_2$, and let

$$\Omega = \text{Span } \{e_1, \ldots, e_{q+2}\}, \quad \Omega^\perp = \text{Span } \{e_{q+3}, \ldots, e_{n+3}\}. \tag{117}$$

These spaces have signatures $(q+1,1)$ and $(p+1,1)$, respectively. The intersection $\Omega \cap Q^{n+1}$ is given in homogeneous coordinates by

$$x_1^2 = x_2^2 + \cdots + x_{q+2}^2, \quad x_{q+3} = \cdots = x_{n+3} = 0.$$

This set is diffeomorphic to the unit sphere S^q in

$$\mathbf{R}^{q+1} = \text{Span } \{e_2, \ldots, e_{q+2}\},$$

by the diffeomorphism $\phi : S^q \to \Omega \cap Q^{n+1}$, $\phi(v) = [e_1 + v]$. Similarly, $\Omega^\perp \cap Q^{n+1}$ is diffeomorphic to the unit sphere S^p in

$$\mathbf{R}^{p+1} = \text{Span } \{e_{q+3}, \ldots, e_{n+2}\}$$

by the diffeomorphism $\psi : S^p \to \Omega^\perp \cap Q^{n+1}$, $\psi(u) = [u + e_{n+3}]$. The Legendre submanifold $\lambda : S^p \times S^q \to \Lambda^{2n-1}$ is defined by

$$\lambda(u,v) = [k_1, k_2], \text{with } [k_1(u,v)] = [\phi(v)], \quad [k_2(u,v)] = [\psi(u)]. \tag{118}$$

It is easy to check that the Legendre conditions (1)–(3) are satisfied by the pair $\{k_1, k_2\}$. To find the curvature spheres of λ, we decompose the tangent space to $S^p \times S^q$ at a point (u,v) as

$$T_{(u,v)} S^p \times S^q = T_u S^p \times T_v S^q.$$

Then $dk_1(X, 0) = 0$ for all $X \in T_u S^p$, and $dk_2(Y) = 0$ for all Y in $T_v S^q$. Thus, $[k_1]$ and $[k_2]$ are curvature spheres of λ with respective multiplicities p and q. Furthermore, the image of $[k_1]$ lies in the set $\Omega \cap Q^{n+1}$, and the image of $[k_2]$ is contained in $\Omega^\perp \cap Q^{n+1}$. The essence of Pinkall's proof is to show that this type of relationship always holds between the two curvature spheres of a cyclide of Dupin.

Proof of Theorem 3.5. Suppose that $\lambda : M^{n-1} \to \Lambda^{2n-1}$ is a connected cyclide of Dupin of characteristic (p,q) with $p + q = n - 1$. We may take $\lambda = [k_1, k_2]$, where $[k_1]$ and $[k_2]$ are the curvature spheres with respective multiplicities p and q. Each curvature sphere map factors through an immersion of the space of leaves of its principal foliation. Thus, locally on M^{n-1}, we can take a principal coordinate system (u,v) defined on an open set

$$W = U \times V \subset \mathbf{R}^p \times \mathbf{R}^q,$$

such that

(i) $[k_1]$ depends only on v, and $[k_2]$ depends only on u, for all $(u, v) \in W$.

(ii) $[k_1(W)]$ and $[k_2(W)]$ are submanifolds of Q^{n+1} of dimensions q and p, respectively.

Note that, in general, such a principal coordinate system cannot be found in the case of a proper Dupin submanifold with $g > 2$ curvature spheres (see Cecil–Ryan [29, p. 182]).

Now let (u, v) and $(\bar{u}, \bar{v})$ be any two points in W. From (i), we have

$$\langle k_1(u, v), k_2(\bar{u}, \bar{v})\rangle = \langle k_1(v), k_2(\bar{u})\rangle = \langle k_1(\bar{u}, v), k_2(\bar{u}, v)\rangle = 0. \tag{119}$$

Let E be the smallest linear subspace of $\mathbf{P}^{n+2}$ containing the q-dimensional submanifold $[k_1(W)]$. By equation (119), we have

$$[k_1(W)] \subset E \cap Q^{n+1}, \quad [k_2(W)] \subset E^\perp \cap Q^{n+1}. \tag{120}$$

The dimensions of E and $E^\perp$ as subspaces of $\mathbf{P}^{n+2}$ satisfy

$$\dim E + \dim E^\perp = n + 1 = p + q + 2. \tag{121}$$

We claim that $\dim E = q + 1$ and $\dim E^\perp = p + 1$. To see this, suppose first that $\dim E > q + 1$. Then $\dim E^\perp \leq p$, and $E^\perp \cap Q^{n+1}$ cannot contain the p-dimensional submanifold $k_2(W)$. Similarly, assuming that $\dim E^\perp > p + 1$ leads to a contradiction. Thus we have

$$\dim E \leq q + 1, \quad \dim E^\perp \leq p + 1.$$

This and equation (121) imply that $\dim E = q + 1$ and $\dim E^\perp = p + 1$. Furthermore, from the fact that $E \cap Q^{n+1}$ and $E^\perp \cap Q^{n+1}$ contain submanifolds of dimensions q and p, respectively, it is easy to deduce that the Lie inner product $\langle\ ,\ \rangle$ has signature $(q + 1, 1)$ on E and $(p + 1, 1)$ on $E^\perp$. Then since $E \cap Q^{n+1}$ and $E^\perp \cap Q^{n+1}$ are diffeomorphic to S^q and S^p, respectively, the inclusions in equation (120) are open subsets. If A is a Lie sphere transformation that takes E to the space Ω in equation (117), and thus takes $E^\perp$ to $\Omega^\perp$, then $A\lambda(W)$ is an open subset of the standard model in equation (118). Both assertions in Theorem 3.5 are now clear. $\square$

We now turn to the Möbius geometric classification of the cyclides of Dupin. For the classical cyclides in $\mathbf{R}^3$, this was known in the nineteenth century. K. Voss [109] announced the classification in Theorem 3.6 below for the higher-dimensional cyclides, but he did not publish a proof. The theorem follows quite directly from Theorem 3.5 and the results of the previous section on surfaces of revolution. The theorem is phrased in terms embedded hypersurfaces in $\mathbf{R}^n$. Thus we are excluding the standard model given in equation (118), where the Euclidean projection is not an immersion. Of course, the Euclidean projection of a parallel submanifold to the standard model is an embedding. The following proof was also given in [16].

Theorem 3.6. (a) *Every connected cyclide of Dupin M^{n-1} of characteristic (p,q) embedded in $\mathbf{R}^n$ is Möbius equivalent to an open subset of a hypersurface of revolution obtained by revolving a q-sphere $S^q \subset \mathbf{R}^{q+1} \subset \mathbf{R}^n$ about an axis of revolution $\mathbf{R}^q \subset \mathbf{R}^{q+1}$ or a p-sphere $S^p \subset \mathbf{R}^{p+1} \subset \mathbf{R}^n$ about an axis $\mathbf{R}^p \subset \mathbf{R}^{p+1}$.*
(b) *Two such hypersurfaces are Möbius equivalent if and only if they have the same value of $\rho = |r|/a$, where r is the signed radius of the profile sphere S^q and $a > 0$ is the distance from the center of S^q to the axis of revolution.*

Proof. We always work with the Legendre submanifold induced by the embedding of M^{n-1} into $\mathbf{R}^n$. By Theorem 3.5, every connected cyclide is contained in a unique compact, connected cyclide. Thus, it suffices to classify compact, connected cyclides up to Möbius equivalence. Consider a compact, connected cyclide

$$\lambda : S^p \times S^q \to \Lambda^{2n-1}, \quad p+q = n-1,$$

of characteristic (p,q). By Theorem 3.5, there is a linear space E of $\mathbf{P}^{n+2}$ with signature $(q+1,1)$ such that the two curvature sphere maps,

$$[k_1] : S^q \to E \cap Q^{n+1}, \quad [k_2] : S^p \to E^{\perp} \cap Q^{n+1},$$

are diffeomorphisms.

Möbius transformations are precisely those Lie sphere transformations A satisfying $A[e_{n+3}] = [e_{n+3}]$. Thus we decompose e_{n+3} as

$$e_{n+3} = \alpha + \beta, \quad \alpha \in E, \quad \beta \in E^{\perp}. \tag{122}$$

Note that since $\langle \alpha, \beta \rangle = 0$, we have

$$-1 = \langle e_{n+3}, e_{n+3} \rangle = \langle \alpha, \alpha \rangle + \langle \beta, \beta \rangle.$$

Hence, at least one of the two vectors α, β is timelike. First, suppose that β is timelike. Let Z be the orthogonal complement of β in $E^\perp$. Then Z is a $(p+1)$-dimensional vector space on which the restriction of $\langle\ ,\ \rangle$ has signature $(p+1, 0)$. Since $Z \subset e_{n+3}^\perp$, there is a Möbius transformation A such that

$$A(Z) = S = \text{Span } \{e_{q+3}, \ldots, e_{n+2}\}.$$

The curvature sphere map $[Ak_1]$ of the Dupin submanifold $A\lambda$ is a q-dimensional submanifold in the space $S^\perp \cap Q^{n+1}$. By equation (14), this means that these spheres all have their centers in the space

$$\mathbf{R}^q = \text{Span } \{e_3, \ldots, e_{q+2}\}.$$

Note that

$$\mathbf{R}^q \subset \mathbf{R}^{q+1} = \text{Span } \{e_3, \ldots, e_{q+3}\} \subset \mathbf{R}^n = \text{Span } \{e_3, \ldots, e_{n+2}\}.$$

As we see from the proof of Theorem 3.2, this means that the Dupin submanifold $A\lambda$ is a hypersurface of revolution in $\mathbf{R}^n$ obtained by revolving a q-dimensional profile submanifold in $\mathbf{R}^{q+1}$ about the axis $\mathbf{R}^q$. Moreover, since $A\lambda$ has two distinct curvature spheres, the profile submanifold has only one curvature sphere. Thus, it is an umbilical submanifold of $\mathbf{R}^{q+1}$.

Four cases are naturally distinguished by the nature of the vector α in equation (122). Geometrically, these correspond to different singularity sets of the Euclidean projection of $A\lambda$. Such singularities correspond exactly with the singularities of the Euclidean projection of λ, since the Möbius transformation A preserves the rank of the Euclidean projection. Since we have assumed that β is timelike, we know that for all $u \in S^p$,

$$\langle k_2(u), e_{n+3}\rangle = \langle k_2(u), \alpha + \beta\rangle = \langle k_2(u), \beta\rangle \neq 0,$$

because the orthogonal complement of β in $E^\perp$ is spacelike. Thus, the curvature sphere $[Ak_2]$ is never a point sphere. However, it is possible for $[Ak_1]$ to be a point sphere.

Case 1: $\alpha = 0$. In this case, the curvature sphere $[Ak_1]$ is a point sphere for every point in $S^p \times S^q$. The image of the Euclidean projection of $A\lambda$ is precisely the axis $\mathbf{R}^q$. The cyclide $A\lambda$ is the Legendre submanifold induced from $\mathbf{R}^q$ as a submanifold of codimension $p+1$ in $\mathbf{R}^n$. This is, in fact, the standard model of equation (118). However, since the Euclidean projection is not an immersion, this case does not lead to any of the embedded hypersurfaces classified in part (a) of the theorem.

In the remaining cases, we can always arrange that the umbilic profile submanifold is a q-sphere and not a q-plane. This can be accomplished by first inverting $\mathbf{R}^{q+1}$ in a sphere centered at a point on the axis $\mathbf{R}^q$ which is not on the profile submanifold, if necessary. Such an inversion preserves the axis of revolution $\mathbf{R}^q$. After a Euclidean translation, we may assume that the center of the profile sphere is a point $(0, a)$ on the x_{q+3}-axis ℓ in $\mathbf{R}^{q+1}$. The center of the profile sphere cannot lie on the axis of revolution $\mathbf{R}^q$, for then the hypersurface of revolution would be an $(n-1)$-sphere and not a cyclide of Dupin. Thus, we may take $a > 0$.

The map $[Ak_1]$ is the curvature sphere map that results from the surface of revolution construction. The other curvature sphere of $A\lambda$ corresponds exactly to the curvature sphere of the profile sphere, i.e., to the profile sphere itself. This means that the signed radius r of the profile sphere is equal to the signed radius of the curvature sphere $[Ak_2]$. Since $[Ak_2]$ is never a point sphere, we conclude that $r \neq 0$. From now on, we will identify the profile sphere with the second factor S^q in the domain of λ.

Case 2: *α is timelike.* In this case, for all $v \in S^q$, we have

$$\langle k_1(v), e_{n+3} \rangle = \langle k_1(v), \alpha \rangle \neq 0,$$

since the orthogonal complement of α in E is spacelike. Thus the Euclidean projection of $A\lambda$ is an immersion at all points. This corresponds to the case $|r| < a$, when the profile sphere is disjoint from the axis of revolution. Classically these were known as the *ring cyclides* (see [29, pp. 151–166] or [17, pp. 151–159] for more detail). Note that by interchanging the roles of α and β, we can find a Möbius transformation that takes λ to the Legendre submanifold obtained by revolving a p-sphere around an axis $\mathbf{R}^p \subset \mathbf{R}^{p+1} \subset \mathbf{R}^n$.

Case 3: *α is lightlike, but not zero.* Then there is exactly one $v \in S^q$ such that

$$\langle k_1(v), e_{n+3} \rangle = \langle k_1(v), \alpha \rangle = 0. \tag{123}$$

This corresponds to the case $|r| = a$, where the profile sphere intersects the axis in one point. Thus, $S^p \times \{v\}$ is the set of points in $S^p \times S^q$ where the Euclidean projection is singular. Classically these were known as the *limit* *spindle cyclides.*

Case 4: α *is spacelike.* Then the condition (123) holds for points v in a $(q-1)$-sphere $S^{q-1} \subset S^q$. For points in $S^p \times S^{q-1}$, the point sphere map is a curvature sphere, and thus the Euclidean projection is singular. Geometrically, this is the case $|r| > a$, where the profile sphere intersects the axis $\mathbf{R}^q$ in a $(q-1)$-sphere. Classically these were known as the *spindle cyclides.*

Of course, there are also four cases to handle under the assumption that α, instead of β, is timelike. Then the axis will be a subspace $\mathbf{R}^p \subset \mathbf{R}^{p+1}$, and the profile submanifold will be a p-sphere. The roles of p and q in determining the dimension of the singularity set of the Euclidean projection will be reversed. So if $p \neq q$, then only a ring cyclide can be represented as a hypersurface of revolution of both a q-sphere and a p-sphere. This completes the proof of part (a).

To prove part (b), we may assume that the profile sphere S^q of the hypersurface of revolution has center $(0, a)$ with $a > 0$ on the x_{q+3}-axis ℓ. Möbius classification clearly does not depend on the sign of the radius of S^q, since the two hypersurfaces of revolution obtained by revolving spheres with the same center and opposite radii differ only by the change of orientation transformation Γ (see Remark 1.2). We now show that the ratio $\rho = |r|/a$ is invariant under the subgroup of Möbius transformations of the profile space $\mathbf{R}^{q+1}$ which take one such hypersurface of revolution to another. First, note that symmetry implies that a transformation T in this subgroup must take the axis of revolution $\mathbf{R}^q$ to itself and the axis of symmetry ℓ to itself. Since $\mathbf{R}^q$ and ℓ intersect only at 0 and the improper point ∞, the transformation T maps the set $\{0, \infty\}$ to itself. If T maps 0 to ∞, then the composition ΦT, where Φ is an inversion in a sphere centered at 0, is a member of the subgroup of transformations that map ∞ to ∞ and map 0 to 0. By Theorem 1.8, such a Möbius transformation must be a similarity transformation, and so it is the composition of a central dilatation D and a linear isometry Ψ. Therefore, $T = \Phi D \Psi$, and each of the transformations on the right of this equation preserves the ratio ρ. The invariant ρ is the only one needed for Möbius classification, since any two profile spheres with the same value of ρ can be mapped to one another by a central dilatation. $\square$

Remark 3.9. We can obtain a family consisting of one representative from each Möbius equivalence class by fixing $a = 1$ and letting r vary, $0 < r < \infty$.

This is just a family of parallel hypersurfaces of revolution. Taking a negative signed radius s for the profile sphere yields a parallel hypersurface that differs only in orientation from the hypersurface corresponding to $r = -s$. Finally, taking $r = 0$ also gives a parallel submanifold in the family, but the Euclidean projection degenerates to a sphere S^p. This is the case $\beta = 0, \alpha = e_{n+3}$, where the point sphere map equals the curvature sphere $[k_2]$ at every point.

3.5 Local classification in the case g = 3

In this section, we discuss classifications of proper Dupin hypersurfaces with three principal curvatures. In his dissertation, Pinkall gave [78], [81] local classification of proper Dupin hypersurfaces in $\mathbf{R}^4$ up to Lie equivalence (see also Cecil–Chern [20]). This is a fundamental case, and it makes use of the method of moving frames in a way that was not necessary in the case $g = 2$. It is the first case where Lie invariants are necessary in the classification, and it is worthy of careful study. Later classification results of Niebergall [68], [69], Cecil and Jensen [25], [26], and Cecil, Chi and Jensen [22] use a similar approach to that employed here.

Pinkall found a complete local classification up to Lie equivalence for Dupin hypersurfaces with three principal curvatures in $\mathbf{R}^4$. He found one Lie invariant (ρ in our treatment [17, pp. 168–188] of Pinkall's result) that completely determines whether or not the Legendre lift λ of the Dupin hypersurface is reducible. If $\rho \neq 0$, then λ is irreducible. Pinkall proved that the Legendre lifts of any two irreducible proper Dupin hypersurfaces with $g = 3$ in $\mathbf{R}^4$ are locally Lie equivalent, each being Lie equivalent to an open subset of Cartan's isoparametric hypersurface in S^4. If $\rho = 0$, then λ is reducible, and Pinkall showed that there is a 1-parameter family of Lie equivalence classes of reducible proper Dupin hypersurfaces with $g = 3$ in $\mathbf{R}^4$.

Niebergall [68] next proved that every connected proper Dupin hypersurface in $\mathbf{R}^5$ with three principal curvatures is reducible. Then Cecil and Jensen [25] proved that if M^{n-1} is an irreducible, connected proper Dupin hypersurface in S^n with three distinct principal curvatures of multiplicities m_1, m_2, m_3, then $m_1 = m_2 = m_3 = m$, and M^{n-1} is Lie equivalent to an isoparametric hypersurface in S^n. It then follows from Cartan's classification of isoparametric hypersurfaces with $g = 3$ mentioned in Section 2.5 that $m = 1, 2, 4$ or 8. Note that in the original paper [25], this result was proven under the assumption that M^{n-1} is locally irreducible. However, as noted in Proposition 3.7 above, local irreducibility has now been shown to be equivalent to irreducibility.

The proof of this result of Cecil and Jensen is accomplished by using the method of moving frames in the context of Lie sphere geometry. A key tool in this context is Theorem 2.10, which states that a Legendre submanifold $\lambda : M^{n-1} \to \Lambda^{2n-1}$ with g distinct curvature spheres $K_1, \ldots, K_g$ at each point is Lie equivalent to the Legendre lift of an isoparametric hypersurface in S^n if and only if there exist g points $P_1, \ldots, P_g$ on a timelike line in P^{n+2} such that $< K_i, P_i >= 0$, for $1 \leq i \leq g$. Another important tool is Pinkall's Lie geometric criterion for reducibility given in Theorem 3.3.

An open problem is the classification of reducible Dupin hypersurfaces of arbitrary dimension with three principal curvatures up to Lie equivalence. As noted above, Pinkall [81] found such a classification in the case of $M^3 \subset \mathbf{R}^4$. It may be possible to generalize Pinkall's result to higher dimensions using the approach of [25].

3.6 Local classification in the case g = 4

In this section, we discuss local classification results proper Dupin submanifolds with four curvature spheres, which were obtained in the papers of Cecil and Jensen [26] and Cecil, Chi and Jensen [22].

Let $\lambda : M^{n-1} \to \Lambda^{2n-1}$ be a a connected proper Dupin submanifold with four curvature spheres at each point. Then we can find a Lie frame $\{Y_1, \ldots, Y_{n+3}\}$ in $\mathbf{R}^{n+3}_2$ (see [17, p. 53]) in which the four curvature spheres are Y_1, Y_{n+3}, $Y_1 + Y_{n+3}$ and $Y_1 + \Psi Y_{n+3}$, where Ψ is the Lie curvature of λ. Denote the respective multiplicities of these curvature spheres by m_1, m_2, m_3 and m_4. Corresponding to the one function ρ in the case $g = 3$ above, there are four sets of functions that are crucial in the proof for $g = 4$,

$$F^{\alpha}_{pa}, F^{\mu}_{pa}, F^{\mu}_{\alpha a}, F^{\mu}_{\alpha p}, \tag{124}$$

where $1 \leq a \leq m_1$, $m_1 + 1 \leq p \leq m_1 + m_2$, $m_1 + m_2 + 1 \leq \alpha \leq m_1 + m_2 + m_3$, and $m_1 + m_2 + m_3 + 1 \leq \mu \leq n - 1 = m_1 + m_2 + m_3 + m_4$.

As noted earlier, Thorbergsson [107] showed that for a compact proper Dupin hypersurface in S^n with four principal curvatures, the multiplicities of the principal curvatures must satisfy $m_1 = m_3, m_2 = m_4$, when the principal curvatures are appropriately ordered (see also Stolz [103]). Thus, in the papers [26] and [22], we make that assumption on the multiplicities. We also assume in [22] that the Lie curvature $\Psi = 1/2$, since that is true for an isoparametric hypersurface with four principal curvatures (when the principal curvatures are ordered in the appropriate way).

In [26, pp. 3–4], Cecil and Jensen conjectured that an irreducible connected proper Dupin hypersurface in S^n with four principal curvatures having multiplicities satisfying $m_1 = m_3, m_2 = m_4$ and constant Lie curvature Ψ must be Lie equivalent to an open subset of an isoparametric hypersurface in S^n.

In that paper [26], we proved that the conjecture is true if all the multiplicities are equal to one (see also Niebergall [69] who obtained the same conclusion under additional assumptions). In the second paper [22] mentioned above, we proved that the conjecture is true if $m_1 = m_3 \geq 1$, and $m_2 = m_4 = 1$, and the Lie curvature is assumed to satisfy $\Psi = 1/2$ by proving Theorem 3.7 below. We believe that the conjecture is true in its full generality, but we have not been able to prove that yet.

Theorem 3.7. *Let M be an irreducible connected proper Dupin hypersurface in S^n with four principal curvatures having multiplicities $m_1 = m_3, m_2 = m_4$, and constant Lie curvature $\Psi = 1/2$. Then M is Lie equivalent to an open subset of an isoparametric hypersurface.*

An important step in proving this theorem is that under the assumptions on the multiplicities and the Lie curvature given in the theorem, the corresponding Dupin submanifold λ is reducible if there exists some fixed index, say a, such that

$$F^{\alpha}_{pa} = F^{\mu}_{pa} = F^{\mu}_{\alpha a} = 0, \quad \text{for all } p, \alpha, \mu. \tag{125}$$

Thus, if λ is irreducible, no such index a can exist, and we show after a lengthy argument that λ is Lie equivalent to the Legendre lift of an open subset of an isoparametric hypersurface in S^n by invoking Theorem 2.10.

As noted in Example 2.2, the assumption of irreducibility is necessary in the theorem, since we constructed a reducible proper Dupin hypersurface with four principal curvatures having multiplicities $m_1 = m_3, m_2 = m_4$, and constant Lie curvature $\Psi = 1/2$ in that example.

Using a different approach based on the theory of higher-dimensional Laplace invariants [50], Riveros and Tenenblat [90]–[91] gave a local classification of proper Dupin hypersurfaces M^4 in $\mathbf{R}^5$ with four distinct principal curvatures which are parametrized by lines of curvature.

In our survey paper [18], we discussed some results in the Möbius geometry of submanifolds in S^n that are related to the study of Dupin hypersurfaces as follows. First, C.-P. Wang [110]–[112] studied this subject in a series of papers. Using the method of moving frames, Wang found a complete set of Möbius invariants for surfaces in $\mathbf{R}^3$ without umbilic points [110] and for hypersurfaces in $\mathbf{R}^4$ with three principal curvatures at each point [111]. Then in [112], Wang defined a Möbius invariant metric g and second fundamental form B for submanifolds in S^n. Wang then proved that for hypersurfaces in S^n with $n \geq 4$, the pair (g, B) forms a complete Möbius invariant system which determines the hypersurface up to Möbius transformations.

In a related result, Riveros, Rodrigues and Tenenblat [89] proved that a proper Dupin hypersurface $M^n \subset \mathbf{R}^{n+1}$, $n \geq 4$, with n distinct principal curvatures and constant Möbius curvatures cannot be parametrized by lines of curvature. They also showed that up to Möbius transformations, there is a unique proper Dupin hypersurface $M^3 \subset \mathbf{R}^4$ with three principal curvatures and constant Möbius curvature that is parametrized by lines of curvature. This M^3 is a cone in $\mathbf{R}^4$ over a standard flat torus in the unit sphere $S^3 \subset \mathbf{R}^4$.

In [53], H. Li, Lui, Wang and Zhao introduced the concept of a Möbius isoparametric hypersurface in a sphere S^n. They showed that a (Euclidean) isoparametric hypersurface is automatically Möbius isoparametric, whereas a Möbius isoparametric hypersurface must be proper Dupin. Later Rodrigues and Tenenblat [92] showed that if $M \subset S^n$ is a hypersurface with a constant number g of distinct principal curvatures at each point, where $g \geq 3$, then M is Möbius isoparametric if and only if M is Dupin with constant Möbius curvatures.

Recently significant progress has been made in the classification of Möbius isoparametric hypersurfaces. First, H. Li, Lui, Wang and Zhao [53] showed that a connected Möbius isoparametric hypersurface in S^n with two distinct principal curvatures is Möbius equivalent to an open subset of one of the following three types of hypersurfaces in S^n:

(i) a standard product of spheres $S^k(r) \times S^{n-k-1}(s) \subset S^n$, $r^2 + s^2 = 1$,
(ii) the image under inverse stereographic projection from $\mathbf{R}^n \rightarrow S^n - \{P\}$ of a standard cylinder $S^k(1) \times \mathbf{R}^{n-k-1} \subset \mathbf{R}^n$,
(iii) the image under hyperbolic stereographic projection from $H^n \rightarrow S^n$ of a standard product $S^k(r) \times H^{n-k-1}(\sqrt{1+r^2}) \subset H^n$.

Later Hu and H. Li [46] classified Möbius isoparametric hypersurfaces in S^4, and Hu, H. Li and Wang [47] classified Möbius isoparametric hypersurfaces in S^5. Then Hu and D. Li [45] studied Möbius isoparametric hypersurfaces with three distinct principal curvatures in S^n and found a complete classification of such hypersurfaces in S^6.

3.7 Compact proper Dupin hypersurfaces

In contrast to Pinkall's local existence Theorem 3.1 above, Thorbergsson [107] proved that for a compact Dupin hypersurface embedded in $\mathbf{R}^n$ or S^n, the number g of distinct principal curvatures must be $1, 2, 3, 4$ or 6, the same as for an isoparametric hypersurface in a sphere [70]. Furthermore, the restrictions on the multiplicities of the principal curvatures of isoparametric hypersurfaces are still valid for compact proper Dupin hypersurfaces (see Stolz [103] for $g = 4$ and Grove-Halperin [43] for $g = 6$).

We now describe the classification results that have been obtained for compact proper Dupin hypersurfaces embedded in S^n. In the case of $g = 1$ principal curvature, the hypersurface must be a great or small hypersphere. In the case $g = 2$, Cecil and Ryan [27] showed that a compact proper Dupin hypersurface must be Möbius equivalent to an isoparametric hypersurface, and thus be a cyclide of Dupin. Next Miyaoka [59] showed that a compact proper Dupin hypersurface with $g = 3$ principal curvatures must be equivalent by a Lie sphere transformation to an isoparametric hypersurface. (Later Cecil and Jensen [25] gave a different proof of Miyaoka's result as a consequence of their classification of irreducible proper Dupin hypersurfaces with three principal curvatures.)

Given these results, Cecil and Ryan [29, p.184] conjectured in 1985 that every compact proper Dupin hypersurface is equivalent by a Lie sphere transformation to an isoparametric hypersurface in a sphere. Counterexamples to this conjecture in the case $g = 4$ were obtained by Pinkall and Thorbergsson [84], and independently by Miyaoka and Ozawa [67]. The construction of Miyaoka of Ozawa also yields a counterexample in the case $g = 6$. (See also [17, pp. 112–123] for a description of these counterexamples.)

In the counterexamples to the conjecture, it was shown that the hypersurfaces constructed do not have constant Lie curvatures, and therefore they cannot be Lie equivalent to an isoparametric hypersurface, which obviously has constant Lie curvatures.

This left open the possibility that a compact proper Dupin hypersurface with 4 or 6 six principal curvatures, and constant Lie curvatures must be Lie equivalent to an isoparametric hypersurface. In the case $g = 4$, Miyaoka [60] showed that this is true if the hypersurface also satisfies some additional assumptions on the intersections of the leaves of the various principal foliations. In the same paper, Miyaoka also proved that if the Lie curvature of compact proper Dupin hypersurface with $g = 4$ is constant, then it must equal the value $1/2$.

Later Cecil, Chi and Jensen [23] formulated the following conjecture which remains as an open problem, although partial results have been obtained.

Conjecture 3.1. *Every compact, connected proper Dupin hypersurface in S^n with $g = 4$ or $g = 6$ principal curvatures and constant Lie curvatures is Lie equivalent to an isoparametric hypersurface.*

In [25], Cecil and Jensen proved that conjecture is true for a compact proper Dupin hypersurface with four principal curvatures of multiplicity one. Then Cecil, Chi and Jensen [22], verified the conjecture in the case where the multiplicities satisfy $m_1 = m_3 \geq 1$, $m_2 = m_4 = 1$ to obtain the following theorem.

Theorem 3.8. *Let M be a compact, connected proper Dupin hypersurface in S^n with four principal curvatures having multiplicities $m_1 = m_3 \geq 1$, $m_2 = m_4 = 1$, and constant Lie curvature. Then M is Lie equivalent to an isoparametric hypersurface.*

Note that since the multiplicities of a compact, connected proper Dupin hypersurface with four principal curvatures, must satisfy the conditions $m_1 = m_3$ and $m_2 = m_4$ when the principal curvatures are appropriately ordered. This means that the full conjecture for $g = 4$ would be proven if the assumption that the value of $m_2 = m_4$ is equal to one could be eliminated from the theorem above.

Cecil, Chi and Jensen proved Theorem 3.8 as a consequence of the local classification (Theorem 3.7) of irreducible proper Dupin hypersurfaces with four principal curvatures having the given multiplicities and constant Lie curvature. The fact that the constant Lie curvature must equal $1/2$ in the compact case is due to Miyaoka [60], as mentioned above. The proof of Theorem 3.7 involves some complicated calculations, which become even more complicated if the assumption that $m_2 = m_4 = 1$ is dropped. Even so, this approach to proving Conjecture 3.1 could be successful with some additional insight regarding the structure of the calculations involved.

In the case $g = 6$, we do not know of any results beyond those of Miyaoka [61], who showed that Conjecture 3.1 is true if the hypersurface satisfies some additional assumptions on the intersections of the leaves of the various principal foliations. An approach similar to that used in [22] for the $g = 4$ case is plausible, but the calculations involved would be complicated, unless some new algebraic insight is found to simplify the situation.

References

[1] U. Abresch, *Isoparametric hypersurfaces with four or six distinct principal curvatures*, Math. Ann. 264 (1983), 283–302.

[2] V. I. Arnold, *Mathematical methods of classical mechanics*, Springer, Berlin, 1978.

[3] E. Artin, *Geometric Algebra*, Wiley–Interscience, New York, 1957.

[4] D. Blair, *Contact Manifolds in Riemannian Geometry*, Lecture Notes in Math. 509, Springer, Berlin, 1976.

[5] W. Blaschke, *Vorlesungen über Differentialgeometrie und geometrische Grundlagen von Einsteins Relativitätstheorie*, Vol. 3, Springer, Berlin, 1929.

[6] G. Bol, *Projektive Differentialgeometrie*, Vol. 3, Vandenhoeck and Ruprecht, Göttingen, 1967.

[7] E. Cartan, *Théorie des groupes finis et continus et la géométrie différentielle traitées par la méthode du repère mobile*, Gauthiers–Villars, Paris, 1937.

[8] E. Cartan, *Familles de surfaces isoparamétriques dans les espaces à courbure constante*, Annali di Mat. 17 (1938), 177–191.

[9] E. Cartan, *Sur des familles remarquables d'hypersurfaces isoparamétriques dans les espaces sphériques*, Math. Z. 45 (1939), 335–367.

[10] E. Cartan, *Sur quelque familles remarquables d'hypersurfaces*, C.R. Congrès Math. Liège, 1939, 30–41.

[11] E. Cartan, *Sur des familles d'hypersurfaces isoparamétriques des espaces sphériques à 5 et à 9 dimensions*, Revista Univ. Tucuman, Serie A, 1 (1940), 5–22.

[12] E. Cartan, *The Theory of Spinors*, Hermann, Paris, 1966, reprinted by Dover, New York, 1981.

[13] S. Carter and A. West, *Tight and taut immersions*, Proc. London Math. Soc. 25 (1972), 701–720.

[14] A. Cayley, *On the cyclide*, Quart. J. of Pure and Appl. Math. 12 (1873), 148–165. See also Collected Papers, Vol. 9, 64–78.

[15] T. Cecil, *On the Lie curvatures of Dupin hypersurfaces*, Kodai Math. J. 13 (1990), 143–153.

[16] T. Cecil, *Lie sphere geometry and Dupin submanifolds*, Geometry and Topology of Submanifolds III, 90–107, World Scientific, River Edge, NJ, 1991.

[17] T. Cecil, *Lie sphere geometry, with applications to submanifolds*, Second Edition, Universitext, Springer, New York, 2008.

[18] T. Cecil, Isoparametric and Dupin hypersurfaces, SIGMA 4 (2008), 062, 28 pages, 2008.

[19] T. Cecil and S.-S. Chern, *Tautness and Lie sphere geometry*, Math. Ann. 278 (1987), 381–399.

[20] T. Cecil and S.-S. Chern, *Dupin submanifolds in Lie sphere geometry*, Differential geometry and topology, Proceedings Tianjin 1986–87, Editors B. Jiang et al., Lecture Notes in Math. 1369, 1–48, Springer, Berlin-New York, 1989.

[21] T. Cecil, Q.-S. Chi and G. Jensen, *Isoparametric hypersurfaces with four principal curvatures*, Ann. Math. 166 (2007), 1–76.

[22] T. Cecil, Q.-S. Chi and G. Jensen, *Dupin hypersurfaces with four principal curvatures* II, Geom. Dedicata 128 (2007), 55–95.

[23] T. Cecil, Q.-S. Chi and G. Jensen, *Classifications of Dupin hypersurfaces*, Pure and Applied Differential Geometry, PADGE 2007, Editors F. Dillen and I. Van de Woestyne, 48–56, Shaker Verlag, Aachen, 2007.

[24] T. Cecil, Q.-S. Chi and G. Jensen, *On Kuiper's question whether taut submanifolds are algebraic*, Pacific J. Math. 234 (2008), 229–247.

[25] T. Cecil and G. Jensen, *Dupin hypersurfaces with three principal curvatures*, Invent. Math. 132 (1998), 121–178.

[26] T. Cecil and G. Jensen, *Dupin hypersurfaces with four principal curvatures*, Geom. Dedicata 79 (2000), 1–49.

[27] T. Cecil and P. Ryan, *Focal sets, taut embeddings and the cyclides of Dupin*, Math. Ann. 236 (1978), 177–190.

[28] T. Cecil and P. Ryan, *Conformal geometry and the cyclides of Dupin*, Canadian J. Math. 32 (1980), 767–782.

[29] T. Cecil and P. Ryan, *Tight and Taut Immersions of Manifolds*, Research Notes in Math. 107, Pitman, London, 1985.

[30] T. Cecil and P. Ryan, *Geometry of Hypersurfaces*, Springer, New York, 2015.

[31] Q.-S. Chi, *Isoparametric hypersurfaces with four principal curvatures revisited*, Nagoya Math. J. **193** (2009), 129–154.

[32] Q.-S. Chi, *A note on the paper "Isoparametric hypersurfaces with four principal curvatures"*, Hongyou Wu Memorial Volume, Pacific Journal of Applied Mathematics **3** (2011), 127–134.

[33] Q.-S. Chi, *Isoparametric hypersurfaces with four principal curvatures* II, Nagoya Math. J. **204** (2011), 1–18.

[34] Q.-S. Chi, *A new look at Condition A*, Osaka J. Math. **49** (2012), 133–166.

[35] Q.-S. Chi, *Isoparametric hypersurfaces with four principal curvatures* III, J. Differential Geom. **94** (2013), 469–504.

[36] Q.-S. Chi, *Isoparametric hypersurfaces with four principal curvatures* IV, arXiv:1605.00976v2 [math.DG] 5 Jun 2017.

[37] M. Dajczer, L. Florit and R. Tojeiro, *Reducibility of Dupin submanifolds*, Illinois J. Math. 49 (2005), 759–791.

[38] J. Dorfmeister and E. Neher, *Isoparametric hypersurfaces, case* $g = 6, m = 1$, Communications in Algebra 13 (1985), 2299–2368.

[39] C. Dupin, *Applications de géométrie et de méchanique*, Bachelier, Paris, 1822.

[40] D. Ferus, H. Karcher and H.-F. Münzner, *Cliffordalgebren und neue isoparametrische Hyperflächen*, Math. Z. 177 (1981), 479–502 (see also an English translation by T.E. Cecil, arXiv:1112.2780v1 [mathDG] 13 Dec 2011).

[41] L. Geatti and C. Gorodski, *Polar orthogonal representations of real reductive algebraic groups*, J. Algebra 320 (2008), 3036–3061.

[42] P. Griffiths, *On Cartan's method of Lie groups and moving frames as applied to uniqueness and existence questions in differential geometry*, Duke Math. J. 41 (1974), 775–814.

[43] K. Grove and S. Halperin, *Dupin hypersurfaces, group actions, and the double mapping cylinder*, J. Diff. Geom. 26 (1987), 429–459.

[44] J. Hahn, *Isoparametric hypersurfaces in the pseudo-Riemannian space forms*, Math. Z. 187 (1984), 195–208.

[45] Z. Hu and D. Li, *Möbius isoparametric hypersurfaces with three distinct principal curvatures*, Pacific J. Math. 232 (2007), 289–311.

[46] Z. Hu and H.-Z. Li, *Classification of Möbius isoparametric hypersurfaces in* S^4, Nagoya Math. J. 179 (2005), 147–162.

[47] Z. Hu, H.-Z. Li and C.-P. Wang, *Classification of Möbius isoparametric hypersurfaces in* S^5, Monatsh. Math. 151 (2007), 201–222.

[48] S. Immervoll, *On the classification of isoparametric hypersurfaces with four distinct principal curvatures in spheres*, Ann. Math. 168 (2008), 1011–1024.

[49] G. Jensen, *Higher Order Contact of Submanifolds of Homogeneous Spaces*, Lecture Notes in Math. 610, Springer, Berlin, 1977.

[50] N. Kamran and K. Tenenblat, *Laplace transformation in higher dimensions*, Duke Math. J. 84 (1996), 237–266.

[51] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Vols. I, II, Wiley–Interscience, New York, 1963, 1969.

[52] T. Levi–Civita, *Famiglie di superficie isoparametrische nell'ordinario spacio euclideo*, Atti. Accad. naz. Lincei. Rend. Cl. Sci. Fis. Mat. Natur 26 (1937), 355–362.

[53] H.-Z. Li, H.-L. Liu, C.-P. Wang and G.-S. Zhao, *Möbius isoparametric hypersurfaces in S^{n+1} with two distinct principal curvatures*, Acta Math. Sin. (Engl. Ser.) **18** (2002), 437–446.

[54] Z.-Q. Li and X.-H. Xie, *Space-like isoparametric hypersurfaces in Lorentzian space forms*, Translated from the Chinese original [J. Nanchang Univ. Natur. Sci. Ed. 28 (2004), 113–117], Front. Math. China 1 (2006), 130–137.

[55] S. Lie, *Über Komplexe, inbesondere Linien- und Kugelkomplexe, mit Anwendung auf der Theorie der partieller Differentialgleichungen*, Math. Ann. 5 (1872), 145–208, 209–256 (Ges. Abh. 2, 1–121).

[56] J. Liouville, *Note au sujet de l'article précedént*, J. de Math. Pure et Appl. (1) 12 (1847), 265–290.

[57] M. Magid, *Lorentzian isoparametric hypersurfaces*, Pacific J. Math. 118 (1985), 165–197.

[58] J.C. Maxwell, *On the cyclide*, Quart. J. of Pure and Appl. Math. 34 (1867). See also Collected Works, Vol. 2, 144–159.

[59] R. Miyaoka, *Compact Dupin hypersurfaces with three principal curvatures*, Math. Z. 187 (1984), 433–452.

[60] R. Miyaoka, *Dupin hypersurfaces and a Lie invariant*, Kodai Math. J. 12 (1989), 228–256.

[61] R. Miyaoka, *Dupin hypersurfaces with six principal curvatures*, Kodai Math. J. 12 (1989), 308–315.

[62] R. Miyaoka, *The linear isotropy group of $G_2/SO(4)$, the Hopf fibering and isoparametric hypersurfaces*, Osaka J. Math. 30 (1993), 179–202.

[63] R. Miyaoka, *A new proof of the homogeneity of isoparametric hypersurfaces with $(g, m) = (6, 1)$*, Geometry and Topology of Submanifolds X, 178–199, World Scientific, River Edge, NJ, 2000.

[64] R. Miyaoka, *The Dorfmeister–Neher theorem on isoparametric hypersurfaces*, Osaka J. Math. 46 (2009), 675–715.

[65] R. Miyaoka, *Isoparametric hypersurfaces with $(g, m) = (6, 2)$*, Ann. Math. **177** (2013), 53–110.

[66] R. Miyaoka, *Errata on isoparametric hypersurfaces with $(g, m) = (6, 2)$*, Ann. Math. **183** (2016), 1057–1071.

[67] R. Miyaoka and T. Ozawa, *Construction of taut embeddings and Cecil–Ryan conjecture*, Geometry of Manifolds, ed. K. Shiohama, Perspect. Math. 8, 181–189, Academic Press, New York, 1989.

[68] R. Niebergall, *Dupin hypersurfaces in $\mathbf{R}^5$ I*, Geom. Dedicata 40 (1991), 1–22.

[69] R. Niebergall, *Dupin hypersurfaces in $\mathbf{R}^5$ II*, Geom. Dedicata 41 (1992), 5–38.

[70] H.-F. Münzner, *Isoparametrische Hyperflächen in Sphären*, Math. Ann. 251 (1980), 57–71.

[71] H.-F. Münzner, *Isoparametrische Hyperflächen in Sphären II: Über die Zerlegung der Sphäre in Ballbündel*, Math. Ann. 256 (1981), 215–232.

[72] K. Nomizu, *Fundamentals of Linear Algebra*, McGraw–Hill, New York, 1966.

[73] K. Nomizu, *Characteristic roots and vectors of a differentiable family of symmetric matrices*, Lin. and Multilin. Alg. 2 (1973), 159–162.

[74] K. Nomizu, *Some results in E. Cartan's theory of isoparametric families of hypersurfaces*, Bull. Amer. Math. Soc. 79 (1973), 1184–1188.

[75] K. Nomizu, *Élie Cartan's work on isoparametric families of hypersurfaces*, Proc. Sympos. Pure Math. 27, Part 1, 191–200, Amer. Math. Soc., Providence, RI, 1975.

[76] B. O'Neill, *Semi-Riemannian Geometry*, Academic Press, New York, 1983.

[77] H. Ozeki and M. Takeuchi, *On some types of isoparametric hypersurfaces in spheres* I, II, Tôhoku Math. J. 27 (1975), 515–559 and 28 (1976), 7–55.

[78] U. Pinkall, *Dupin'sche Hyperflächen*, Dissertation, Univ. Freiburg, 1981.

[79] U. Pinkall, *W-Kurven in der ebenen Lie-Geometrie*, Elemente der Mathematik 39 (1984), 28–33 and 67–78.

[80] U. Pinkall, Letter to T. Cecil, Dec. 5, 1984.

[81] U. Pinkall, *Dupin'sche Hyperflächen in* E^4, Manuscr. Math. 51 (1985), 89–119.

[82] U. Pinkall, *Dupin hypersurfaces*, Math. Ann. 270 (1985), 427–440.

[83] U. Pinkall, *Curvature properties of taut submanifolds*, Geom. Dedicata 20 (1986), 79–83.

[84] U. Pinkall and G. Thorbergsson, *Deformations of Dupin hypersurfaces*, Proc. Amer. Math. Soc. 107 (1989), 1037–1043.

[85] U. Pinkall, *Taut 3-manifolds*, Topology 28 (1989), 389–401.

[86] H. Reckziegel, *Krümmungsflächen von isometrischen Immersionen in Räume konstanter Krümmung*, Math. Ann. 223 (1976), 169–181.

[87] H. Reckziegel, *On the eigenvalues of the shape operator of an isometric immersion into a space of constant curvature*, Math. Ann. 243 (1979), 71–82.

[88] H. Reckziegel, *Completeness of curvature surfaces of an isometric immersion*, J. Diff. Geom. 14 (1979), 7–20.

[89] C. M. Riveros, L. A. Rodrigues and K. Tenenblat, *On Dupin hypersurfaces with constant Möbius curvature*, Pacific J. Math. 236 (2008), 89–103.

[90] C. M. Riveros and K. Tenenblat, *On four dimensional Dupin hypersurfaces in Euclidean space*, An. Acad. Brasil Ciênc. 75 (2003), 1–7.

[91] C. M. Riveros and K. Tenenblat, *Dupin hypersurfaces in* $\mathbf{R}^5$, Canad. J. Math. 57 (2005), 1291–1313.

[92] L. A. Rodrigues and K. Tenenblat, *A characterization of Moebius isoparametric hypersurfaces of the sphere*, Monatsh. Math. 158 (2009), 321–327.

[93] P. Ryan, *Homogeneity and some curvature conditions for hypersurfaces*, Tôhoku Math. J. 21 (1969), 363–388.

[94] P. Ryan, *Hypersurfaces with parallel Ricci tensor*, Osaka J. Math. 8 (1971), 251–259.

[95] P. Samuel, *Projective Geometry*, Springer, Berlin, 1988.

[96] B. Segre, *Famiglie di ipersuperficie isoparametrische negli spazi euclidei ad un qualunque numero di demesioni*, Atti. Accad. naz Lincie Rend. Cl. Sci. Fis. Mat. Natur 27 (1938), 203–207.

[97] A. Siffert, *Classification of isoparametric hypersurfaces in spheres with* $(g, m) = (6, 1)$, Proc. Amer. Math. Soc. 144 (2016), 2217–2230.

[98] A. Siffert, *A new structural approach to isoparametric hypersurfaces in spheres*, Ann. Global Anal. Geom. 52 (2017), 425–456.

[99] D. Singley, *Smoothness theorems for the principal curvatures and principal vectors of a hypersurface*, Rocky Mountain J. Math. 5 (1975), 135–144.

[100] C. Somigliana, Sulle relazione fra il principio di Huygens e l'ottica geometrica, *Atti Acc. Sc. Torino* 54 (1918–1919), 974–979. Also in: *Memorie Scelte*, 434–439.

[101] M. Spivak, *A Comprehensive Introduction to Differential Geometry*, Vols. 1–5, Publish or Perish, Boston, 1970–1975.

[102] S. Sternberg, *Lectures on Differential Geometry*, Prentice–Hall, Englewood Cliffs, 1964, 2nd ed., Chelsea, New York, 1983.

[103] S. Stolz, *Multiplicities of Dupin hypersurfaces*, Invent. Math. 138 (1999), 253–279.

[104] R. Takagi, *A class of hypersurfaces with constant principal curvatures in a sphere*, J. Diff. Geom. 11 (1976), 225–233.

[105] R. Takagi and T. Takahashi, *On the principal curvatures of homogeneous hypersurfaces in a sphere*, Differential Geometry in honor of K. Yano, 469–481, Kinokuniya, Tokyo, 1972.

[106] M. Takeuchi, *Proper Dupin hypersurfaces generated by symmetric submanifolds*, Osaka Math. J. 28 (1991), 153–161.

[107] G. Thorbergsson, *Dupin hypersurfaces*, Bull. London Math. Soc. 15 (1983), 493–498.

[108] G. Thorbergsson, *A survey on isoparametric hypersurfaces and their generalizations*, Handbook of Differential Geometry, Vol. I, 963–995, North-Holland, Amsterdam, 2000.

[109] K. Voss, *Eine Verallgemeinerung der Dupinsche Zykliden*, Tagungsbericht 41/1981, Geometrie, Mathematisches Forschungsinstitut, Oberwolfach, 1981.

[110] C.-P. Wang, *Surfaces in Möbius geometry*, Nagoya Math. J. 125 (1992), 53–72.

[111] C.-P. Wang, *Möbius geometry for hypersurfaces in* S^4, Nagoya Math. J. 139 (1995), 1–20.

[112] C.-P. Wang, *Möbius geometry of submanifolds in* S^n, Manuscripta Math. 96 (1998), 517–534.

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