College of the Holy Cross
CrossWorks

Mathematics Department Faculty Scholarship | Mathematics

1976

Taut Immersions of Non-Compact Surfaces into a Euclidean 3-Space

Thomas E. Cecil
College of the Holy Cross, tcecil@holycross.edu

Follow this and additional works at: http://crossworks.holycross.edu/math_fac_scholarship

Part of the Applied Mathematics Commons

Recommended Citation
Cecil, Thomas E. Taut immersions of noncompact surfaces into a Euclidean 3-space. J. Differential Geom. 11 (1976), no. 3, 451-459.

This Article is brought to you for free and open access by the Mathematics at CrossWorks. It has been accepted for inclusion in Mathematics Department Faculty Scholarship by an authorized administrator of CrossWorks.

J. DIFFERENTIAL GEOMETRY
11 (1976) 451-459

TAUT IMMERSIONS OF NONCOMPACT SURFACES INTO A EUCLIDEAN 3-SPACE

THOMAS E. CECIL

Let f be a smooth (C^∞) immersion of a smooth surface M into a Euclidean n-space R^n. For $p \in R^n$, $x \in M$, the function $L_p(x)$ is defined by $L_p(x) = d(f(x), p)^2$, where d is the Euclidean distance in R^n.

The immersion f is said to be proper if the inverse image under f of every compact subset of R^n is compact. Following the terminology of Carter and West [2] the immersion in said to be taut, if f is proper and every Morse function L_p, $p \in R^n$, has the minimum number of critical points required by the Morse inequalities [12, p. 270]. This definition is valid for noncompact as well as compact surfaces M.

For M compact and connected, an immersion $f: M \to R^n$ is taut if and only if $f(M)$ has the spherical 2-piece property, that is, no hyperplane or hypersphere in R^n divides $f(M)$ into more than two pieces (see [1] for more detail).

Primarily through the work of Kuiper [10] and Banchoff [1], all taut immersions of compact surfaces into R^n are known (see [2, p. 711] for a complete listing). In particular, Banchoff proved by considering the spherical 2-piece property that the image of a taut immersion of a compact connected surface in R^3 must be either a Euclidean sphere or a cyclide of Dupin.

The cyclides of Dupin will be discussed in detail in § 1. For now it will suffice to say that a compact cyclide is either a torus of revolution or a surface obtained by inverting such a torus in a sphere whose center is now on the torus. A complete noncompact cyclide is either a circular cylinder or a surface obtained by inverting a torus of revolution in a sphere whose center is on the torus.

The result of Banchoff was generalized by Nomizu and Rodriguez [13] who showed that a taut immersion of an m-sphere in R^n must, in fact, be a Euclidean sphere $S^m \subset R^{m+1} \subset R^n$. Several other generalizations were obtained by Carter and West [2]. The author has also found characterizations of certain submanifolds of hyperbolic space [3] and complex projective space [4] in terms of the distance functions in those spaces.

In this paper, we are concerned with taut immersions of noncompact surfaces. Carter and West [2, p. 710] have proven that if $f: M \to R^n$ is a taut immersion of a noncompact surface, then $f(M) \subset R^4 \subset R^n$. If $f(M)$ is not actually contained in R^3, then $f(M) = P(V)$, where V is a Veronese surface

Received December 9, 1974, and, in revised form, May 30, 1975.

[10, p. 163] in S^4 and P is stereographic projection from S^4 to R^4 with respect to a point on V.

Hence the study of all possible taut immersions $f: M \to R^n$, where M is noncompact, has been reduced to the study of all taut immersions of M into R^3. In this paper, we obtain all such immersions as follows:

Theorem 1. *Let $f: M \to R^3$ be a taut immersion of a connected noncompact surface. Then $f(M)$ is either a hyperplane or a complete cyclide of Dupin.*

We remark that combined with Banchoff's result for compact surfaces, the above theorem implies that if $f: M \to R^3$ is a taut immersion of a connected surface, then $f(M)$ is either totally umbilic or a complete cyclide of Dupin.

The author would like to thank K. Nomizu for suggesting this problem, and T. Banchoff for a very enlightening discussion of this matter.

1. The cyclides of Dupin

Let $f: M \to R^3$ be a smooth immersion of a smooth surface, and ξ a unit normal to $f(M)$ at $f(x)$. A point $p \in R^3$ is called a focal point of multiplicity μ if $p = f(x) + (1/\lambda)\xi$ where λ is a principal curvature of M at x of multiplicity μ. The collection of all focal points of (M, x) for all $x \in M$ is called the focal surface of M. In general, the focal surface consists of two surfaces (or sheets) corresponding to the two principal curvatures of M.

We say that the sheet of the focal surface corresponding to the principal curvature λ is degenerate at $x \in M$ if $\lambda(x) = 0$ or if $\lambda(x) \neq 0$ and the sheet of the focal surface corresponding to λ is not a regular surface at the focal point $p = f(x) + (1/\lambda)\xi$.

A straightforward classical computation (see, for example, [5, pp. 310–314]) shows that the following conditions are equivalent on any connected open subset U of M, which contains no umbilic points.

(a) The principal curvatures are constant along their corresponding lines of curvature in U.

(b) All lines of curvature in U lie on circles or straight lines.

(c) Both sheets of the focal surface are degenerate at every point of U.

In fact, the computation shows that if the above conditions hold on U and neither of the principal curvatures is identically 0 on U, then the two sheets of the focal surface of U are a pair of so-called focal conics, or focal curves of a quadric surface. More specifically, the two conics are either an ellipse and a hyperbola situated in perpendicular planes such that the vertex of the hyperbola is a focus of the ellipse and the corresponding vertex of the ellipse is a focus of the hyperbola, or two parabolas in perpendicular planes such that the vertex of each parabola is the focus of the other. Moreover, U must be simultaneously the envelope of two 1-parameter families of spheres whose centers lie on the two focal conics, respectively. The family of spheres may, of course, include some planes.

A cyclide of Dupin was originally defined by Dupin as a surface which is the envelope of a family of spheres tangent to three fixed spheres. A cyclide of Dupin must be either a torus of revolution, a circular cylinder, a circular cone, or a surface obtained by inverting one of the above in a Euclidean sphere. (See [5, pp. 312–314], [7, pp. 217–219], [8, pp. 111–116] for more detail and other characterizations of the cyclides.)

Here by a torus of revolution we mean a torus obtained by rotating a circle around an axis in the plane of the circle which does not meet the circle. We also recall that inversion of R^3 with respect to a sphere centered at a point $p \in R^3$ of radius ρ is a map of $R^3 - \{p\}$ onto itself which sends a point q to a point $I(q)$ on the ray from p through q such that $|q - p||I(q) - p| = \rho^2$.

The compact cyclides are the torus and those surfaces obtained by inverting the torus in a sphere whose center is not on the torus. The complete non-compact cyclides are the circular cylinder and those surfaces obtained by inverting a torus in a sphere whose center is on the torus.

The cone and the inversions of the cylinder and the cone are not complete; they do not contain either one or two of their limit points. In all cases, however, the principal curvature $\lambda_1(x)$ approaches ∞ as x approaches a missing limit point of a cyclide; that is, the lines of curvature corresponding to λ_1 are circles whose radii are decreasing to 0. We also note that none of the cyclides contains an umbilic point.

One proves rather easily that a connected surface is a cyclide of Dupin if and only if the surface is simultaneously the envelope of two 1-parameter families of spheres whose centers lie along a pair of focal conics [5, pp. 312–313]. This includes the degenerate cases of the cylinder and the cone where one of the conics vanishes and the corresponding family of spheres is entirely composed of planes.

Hence the equivalent conditions (a), (b), (c) are satisfied on a connected open subset U of M containing no umbilic points if and only if U is a part of a cyclide of Dupin. This is a local result, however, and the author knows of no previous publication of the following global version of the result which is necessary for these considerations.

Theorem 2. *Let M be a connected complete Riemannian 2-manifold isometrically immersed in R^3, and assume that M is not totally umbilic. If the equivalent conditions* (a), (b) *and* (c) *are satisfied on every connected open subset of M containing no umbilic points, then M is embedded as a complete cyclide of Dupin.*

Proof. Let x be a nonumbilic point of M. We begin by showing that any line of curvature of M through x can be extended to arbitrarily large lengths.

Let $\gamma(s)$ be a line of curvature through x parametrized by arc-length with $\gamma(0)=x$. Since M is complete, the line of curvature γ can be extended indefinitely unless there exists a value s_0 such that $\lim_{s\to s_0}\gamma(s) = y$ is an umbilic point of M.

Suppose $\lim_{s\to s_0} \gamma(s) = y$ is umbilic, but $\gamma(s)$ is not umbilic for $0 \leq s < s_0$. Since the set of nonumbilic points of M is open, there exists a connected open set $U \subset M$ of nonumbilic points such that $\gamma(s) \in U$ for $0 \leq s < s_0$. Let λ_1, λ_2 denote the principal curvatures of M in the neighborhood U.

By the local version of this theorem, U must be part of a cyclide N. The point y is a limit point of U, and hence of N. Suppose N is complete; then $y \in N$. Let k_1, k_2 denote the principal curvatures of N. Since N contains no umbilic points, $k_1(y) \neq k_2(y)$.

Since $U \subset M$ and $U \subset N$, we have $\lambda_i(\gamma(s)) = k_i(\gamma(s))$ for $0 \leq s < s_0$. Since λ_i and k_i are continuous functions on M and N, respectively, taking the limit as s approaches s_0 in the above equation yields $\lambda_i(y) = k_i(y)$. This implies that $\lambda_1(y) \neq \lambda_2(y)$, and y is not umbilic.

The only possibility remaining is that N is not a complete cyclide and y is a limit point of N which is not in N. But then $\lim_{s\to s_0} k_1(\gamma(s)) = \infty$, whereas $\lim_{s\to s_0} \lambda_1(\gamma(s)) = \lambda_1(y)$ is a real number. Since the two limits must be equal, this case is impossible. Thus we conclude that γ can be extended indefinitely.

The curve γ contains no umbilic points; thus, for each $s \in R$, there is a second line of curvature $\eta_s(t)$ through the point $\gamma(s)$, where t is the arc-length parameter. As the above proof shows, each $\eta_s(t)$ can be defined for $-\infty < t < \infty$.

Let S be the connected subset of M composed of the union of all the curves η_s, $-\infty < s < \infty$. Since S contains no umbilics, S is contained in an open connected subset U of M, which contains no umbilic points. By the local theorem, U is an open subset of a cyclide N, and the curve γ as well as each curve η_s is a circle or straight line in R^3. Since γ and each η_s are defined for all real parameter values, it is clear that the set S is, in fact, the entire cyclide N. The cyclide N must be one of the complete cyclides since all of its lines of curvature can be extended indefinitely. We know that $N \subset M$, but a complete 2-manifold cannot be embedded as a proper subset of another 2-manifold [9, p. 176], and thus $M = N$.

2. Preliminary results on taut immersions

The following facts are true, with one noted exception, for immersions $f: M^k \to R^n$ where k, n are arbitrary. We will be content, however, to only consider the case when M is a surface.

Thus, let $f: M \to R^n$ be an immersion of a smooth surface. The following facts about the L_p functions are well known (see, for example, [11, pp. 32–38]). The function L_p has a critical point at $x \in M$ if and only if p lies on a normal line to $f(M)$ at $f(x)$. This critical point is degenerate if and only if p is a focal point of (M, x). The index theorem for L_p states that if L_p has a nondegenerate critical point at x, then the index of L_p at x is equal to the number of focal points counting multiplicities on the line segment from $f(x)$ to p.

Finally, the set of points p such that L_p is a Morse function on M is dense in R^n.

An important consequence of the above-mentioned facts for our considerations is the following proven independently by Nomizu and Rodriguez [13, p. 199] and Carter and West [2, p. 708].

Proposition 1. *Let $p \in R^n$, and assume the function L_p has a nondegenerate critical point $x \in M$ of index k. Then there exists a point $q \in R^n$ such that L_q is a Morse function which has a critical point z of index k (q and z may be chosen as close to p and x, respectively, as desired).*

Next we give a precise definition of a taut immersion. Let ϕ be a Morse function on a surface M such that the set

$$M_r(\phi) = \{x \in M \mid \phi(x) \leq r\}$$

is compact for all $r \in R$. For $r \in R$, let

$$m_k(\phi, r) = \text{the number of critical points of } \phi \text{ of index } k \text{ on } M_r(\phi) \ .$$

For an arbitrary field F, let

$$b_k(\phi, r, F) = \dim_F (H_k(M_r(\phi); F) = \text{the } k\text{th } F\text{-Betti number of } M_r(\phi) \ ,$$
$$\beta_k(\phi, r) = \max \{b_k(\phi, r, F) \mid F \text{ is a field}\} \ .$$

The Morse inequalities (see, for example, [12, p. 270]) are

$$m_k(\phi, r) \geq \beta_k(\phi, r) \ , \qquad \text{where } r \in R, \text{ and } k = 0, 1, 2 \ .$$

If an immersion $f\colon M \to R^n$ is proper, than $M_r(L_p)$ is compact for any $p \in R^n$, $r \in R$. Following Carter and West, we make the following definition which is valid for compact or noncompact surfaces M.

Definition. An immersion $f\colon M \to R^n$ is said to be *taut*, if f is proper and

$$m_k(L_p, r) = \beta_k(L_p, r) \ , \qquad k = 0, 1, 2 \ ,$$

for every Morse function L_p and every $r \in R$.

We remark that if dimension $M > 2$, then the above definition must be reworded slightly because M may not satisfy Kuiper's condition $3A$, [10, p. 152], that is, there may not exist a field F such that

$$\beta_k(L_p, r) = b_k(L_p, r, F)$$

for all L_p, r, k. See [2, p. 702] for a proper formulation of the definition when dimension $M > 2$. When dimension $M = 2$, the above equation is always satisfied for the field $F = Z_2$.

Let ϕ be a Morse function on M such that $M_r(\phi)$ is compact for all $r \in R$. Then Kuiper [10, p. 153] proved that

$$m_k(\phi, r) = b_k(\phi, r, Z_2)\ , \qquad k = 0, 1, 2\ ,$$

for all $r \in R$ if and only if the map

$$H_k(M_r(\phi)\,;\, Z_2) \longrightarrow H_k(M\,;\, Z_2)\ , \qquad k = 0, 1, 2\ ,$$

induced by the inclusion $M_r(\phi) \subset M$ is injective for all $r \in R$.

Applying this result to the subject of taut immersions yields the following.

Proposition 2. *Let $f: M \to R^n$ be a proper immersion. Then f is taut if and only if for every Morse function L_p and every $r \in R$, the map*

$$H_k(M_r(L_p)\,;\, Z_2) \longrightarrow H_k(M\,;\, Z_2)\ , \qquad k = 0, 1, 2\ ,$$

induced by the inclusion $M_r(L_p) \subset M$ is injective.

Finally, a basic result about taut immersions which will be useful in this context is the following obtained by Carter and West [2, p. 708].

Proposition 3. *Let $f: M \to R^n$ be a taut immersion, and let $p \in R^n$ such that L_p has a critical point at $x \in M$. Suppose there are no focal points of (M, x) on the open line segment from $f(x)$ to p. Then $L_p(x) \leq L_p(y)$ for all $y \in M$.*

3. Proof of Theorem 1

Let $f: M \to R^3$ be a taut immersion of a connected noncompact surface M. We first remark that Carter and West [2, p. 703] have shown that a taut immersion must be an embedding. Hence we will refer hereafter to f as a taut embedding. We also note that the assumption that f is proper implies that $f(M)$ with the induced Riemannian metric is complete.

If f is totally umbilic, then the proof is finished. Assume, then, that f is not totally umbilic. The proof of Theorem 1 is accomplished by showing that condition (a) of § 1 is satisfied on every open connected subset U of M, which contains no umbilics, i.e., the principal curvatures are constant along the lines of curvature in U. Then Theorem 1 will follow from Theorem 2.

Proposition 4. *Suppose L_p is a Morse function on M. Then L_p can not have a critical point of index 2 on M.*

Proof. Assume L_p has a nondegenerate critical point of index 2 at $x_0 \in M$. Choose $r > L_p(x_0)$. Since the embedding f is taut and L_p has at least one critical point of index 2 on $M_r(L_p)$, the rank of $H_2(M_r(L_p)\,;\, Z_2)$ is at least 1. By Proposition 2, the rank of $H_2(M\,;\, Z_2)$ is at least 1. However, Morse [12, p. 271] has shown that $H_2(M\,;\, F) = 0$ for any noncompact surface M and any field F. Hence we have a contradiction, and the proof is complete.

Proposition 5. *Suppose p is a focal point of (M, x). Then L_p has an absolute minimum at x.*

Proof. We first note that there cannot be any focal points of (M, x) on the open line segment from $f(x)$ to p. For suppose such a focal point exists; if q is a point beyond p on the normal line to $f(M)$ at $f(x)$, then L_q has a non-degenerate critical point of index 2 at x. By Proposition 1, there is a Morse function L_s having a critical point of index 2. This is impossible by Proposition 4, so that there are no focal points of (M, x) between $f(x)$ and p. Hence Proposition 3 implies that L_p has an absolute minimum at x.

Proposition 6. *Let U be a connected open subset of M which contains no umbilics. Then the principal curvatures are constant along their corresponding lines of curvature in U.*

Proof. Let β be a line of curvature in U, and λ the corresponding principal curvature. If $\lambda \equiv 0$ on β, then λ is constant along β as desired.

Suppose $\lambda(x_0) \neq 0$ for some point x_0 on β. Then the set

$$B = \{x \in \beta \mid \lambda(x) = \lambda(x_0)\}$$

is closed in the relative topology of β since λ is continuous. We will show that B is also relatively open and hence $B = \beta$, which implies that λ is constant along β.

Let y be an arbitrary point in B, $V \subset U$ a neighborhood of y such that $\lambda \neq 0$ on V, and X the unit principal vector field on V corresponding to λ. We will show $X(\lambda) = 0$ on V, and thus λ is constant along its lines of curvature in V. In particular $\lambda(x) = \lambda(y) = \lambda(x_0)$ for all x in $\beta \cap V$, thus proving that B is open in the relative topology of β.

For simplicity, we identify V with its image under the embedding f. Let $x \in V$ and let γ be the normal section of M at x obtained by intersecting V with the plane spanned by $X(x)$ and $\xi(x)$, the unit normal to M at x. Parametrize γ by arc-length parameter s so that $\gamma(0) = x$ and $\gamma'(0) = X(x)$, where the prime denotes differentiation with respect to s.

Let $k(s)$ denote the curvature function of γ, and let $\lambda(s) = \lambda(\gamma(s))$. We know $k(0) = \lambda(0)$, and it is also true that $k'(0) = \lambda'(0)$. To see this, consider the normal curvature at $\gamma(s)$ in the direction $\gamma'(s)$ which is given by the formula

$$k_n(s) = \langle A(\gamma'(s)), \gamma'(s)\rangle ,$$

where A is the shape operator of M, and $\langle\ ,\ \rangle$ is the Euclidean inner product. Meusnier's theorem states that

$$k_n(s) = k(s) \cos(\varphi(s)) ,$$

where φ is the angle between the principal normal to γ at $\gamma(s)$ and $\xi(\gamma(s))$. The fact that $\xi(x)$ is the principal normal to γ at x implies $\varphi(0) = 0$, and hence $k_n'(0) = k'(0)$.

On the other hand, taking Euler's formula for $k_n(s)$ and differentiating yields

$k_n'(0) = \lambda'(0)$. Thus $k'(0) = \lambda'(0)$. Since the value of $X(\lambda)$ at x is precisely $\lambda'(0)$, the proof will be complete once it is shown that $k'(0) = 0$.

The point $p = x + (1/\lambda)\xi(x)$ is a focal point of (M, x). Let C be the osculating circle to γ at x; the point p is the center of C. It is known (see, for example, [6, p. 84]) that γ crosses C at x unless $k'(0) = 0$. Thus, if $k'(0) \neq 0$, then there is a value s_0 such that $L_p(\gamma(s_0)) < L_p(x)$. This contradicts the fact that L_p must have a minimum at x by Proposition 5, and we conclude that $k'(0) = 0$.

Proposition 6 states that condition (a) is satisfied on every open connected subset of M which contains no umbilics. By Theorem 2, we conclude that $f(M)$ is a complete cyclide of Dupin.

Remark. It is clear that a plane or a circular cylinder is tautly immersed in R^3. It is not as obvious, however, that the other noncompact complete cyclides are also tautly immersed. We provide a proof of that fact here.

Let T be a torus of revolution. Banchoff proved that T has spherical 2-piece property [1, p. 203]. Let $I_{x,\rho}$ denote the mapping from $R^3 - \{x\}$ onto itself given by inversion in a sphere of radius ρ and center x. Let M be the image of T under $I_{x,\rho}$ where $x \in T$ and $\rho > 0$.

One easily proves that M is properly embedded in R^3. To complete the proof that M is tautly embedded, we will show that any Morse function L_p has exactly one minimum and no maxima on M. Then, for $r \in R$, let m_k denote the number of critical points of L_p of index k on $M_r(L_p)$, and let b_k denote the rank of $H_k(M_r(L_p); Z_2)$. Thus the Morse inequality

$$\sum_{k=0}^{2} (-1)^k m_k = \sum_{k=0}^{2} (-1)^k b_k$$

and the fact that L_p has one minimum and no maxima on M imply that $m_k = b_k$ for $k = 0, 1, 2$. This is true for any Morse function L_p and any $r \in R$, and hence M is tautly embedded in R^3.

Suppose there is a Morse function L_p having two distinct minima x_0 and x_1 on M. Assume $L_p(x_0) \leq L_p(x_1)$, and let $r = L_p(x_1)$. By the lemma of Morse, there exist an open neighborhood U of x_1 in M, with $x_0 \notin U$, and a sufficiently small $\varepsilon > 0$ such that the set

$$V = \{y \in U \mid L_p(y) \leq r + \varepsilon\}$$

is a closed neighborhood of x_1 in M.

Let S and B be the sphere and closed ball centered at p of radius $(r + \varepsilon)^{1/2}$, respectively. Then $V = U \cap B$, and V is a connected component of $M \cap B$. Since $x_0 \notin V$, but $x_0 \in M \cap B$, there are at least two connected components of $M \cap B$, both of which are compact.

Thus $I_{x,\rho}^{-1}(M \cap B)$ has at least two components, and the sphere (or plane) $I_{x,\rho}^{-1}(S)$ divides T into at least three pieces. This is impossible since T has the spherical 2-piece property, and we conclude that L_p can not have two minima. A similar argument proves that L_p can not have a local maximum, and the proof is complete.

Bibliography

[1] T. F. Banchoff, *The spherical two-piece property and tight surfaces in spheres,* J. Differential Geometry **4** (1970) 193–205.
[2] S. Carter & A. West, *Tight and taut immersions,* Proc. London Math. Soc. **25** (1972) 701–720.
[3] T. Cecil, *A characterization of metric spheres in hyperbolic space by Morse theory,* Tôhoku Math. J. **26** (1974) 341–351.
[4] ——, *Geometric applications of critical point theory to submanifolds of complex projective space,* Nagoya Math. J. **55** (1974) 5–31.
[5] L. Eisenhart, *A treatise on the differential geometry of curves and surfaces,* Constable, London, 1909.
[6] A. Goetz, *Introduction to differential geometry,* Addison-Wesley, Reading, Massachusetts, 1970.
[7] D. Hilbert & S. Cohn-Vossen, *Geometry and the imagination,* Chelsea, New York, 1952.
[8] F. Klein, *Einleitung in die höhere Geometrie.* I, Göttingen, 1893.
[9] S. Kobayashi & K. Nomizu, *Foundations of differential geometry,* Vol. 1, Interscience, New York, 1963.
[10] N. H. Kuiper, *On convex maps,* Nieuw Arch. Wisk. **10** (1962) 147–164.
[11] J. Milnor, *Morse theory,* Ann. of Math. Studies, No. 51, Princeton University Press, Princeton, 1963.
[12] M. Morse & S. Cairns, *Critical point theory in global analysis and differential topology,* Academic Press, New York, 1969.
[13] K. Nomizu & L. Rodriguez, *Umbilical submanifolds and Morse functions,* Nagoya Math. J. **48** (1972) 197–201.

VASSAR COLLEGE