On second order duality for nondifferentiable minimax fractional programming problems involving type-I functions

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Abstract

We introduce second order (C, α, ρ, d) type-I functions and formulate a second order dual model for a nondifferentiable minimax fractional programming problem. The usual duality relations are established under second order $(F, \alpha, \rho, d)/(C, \alpha, \rho, d)$ type-I assumptions. By citing a nontrivial example, it is shown that a second order (C, α, ρ, d) type-I function need not be (F, α, ρ, d) type-I. Several known results are obtained as special cases.

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1 Introduction

An optimization problem in which the objective function is the ratio of two functions is a fractional programming problem. It has a wide number of applications in engineering and economics where a ratio of physical or economic functions must be minimised to measure the efficiency or productivity of the system. In mathematical programming, optimization problems in which both a minimization and maximization process is performed are known as minimax (or minmax) problems. Du and Pardalos [5] provided theory, algorithms and applications of some minimax problems. Schmitendorf [13] formulated the following static minimax problem and established necessary optimality conditions:

$$\text{minimise} \quad f(x) = \sup_{y \in Y} \phi(x, y) \quad \text{subject to} \quad x \in X \subset R^n ,$$

where $\phi : R^n \times R^l \to R$, $g : R^n \to R^m$ are twice continuously differentiable functions, Y is a subset of R^l and $X = \{x \in R^n : g(x) \leqslant 0\}$.

Several different minimax fractional programming problems have been studied and duality relations were obtained under various generalized convexity assumptions [3, 7, 8, 9]. Hachimi and Aghezzaf [6] introduced second order (F, α, ρ, d) type-I functions which generalize convexity. Later, Ahmad et al. [2] formulated a second order dual model for a nondifferentiable minimax programming problem and proved duality relations under (F, α, ρ, d) type-I functions. Recently, Sharma and Gulati [14] discussed duality results for a minimax fractional programming problem using type-I univex functions.

We first introduce second order (C, α, ρ, d) type-I functions. A numerical nontrivial example illustrates the existence of such functions. We then formulate a second order dual model involving a vector $r \in R^n$ for a nondifferentiable multiobjective fractional programming problem and established weak, strong and strict converse duality theorems under second order $(F, \alpha, \rho, d)/(C, \alpha, \rho, d)$ type-I functions.

2 Preliminaries

Throughout this article, gradients and Hessian matrices of the functions f, g, h and ϕ are with respect to the variable x. For instance, $\nabla f(x, y)$ means $\nabla_x f(x, y)$. Here, R^n denotes the n dimensional Euclidean space, R_+ is the set of nonnegative real numbers and $M = \{1, 2, \ldots, m\}$.

Definition 1 (Ahmad et al. [2]). *A functional* $F : X \times X \times R^n \mapsto R$, *where* $X \subseteq R^n$, *is sublinear with respect to the third variable if for all* $(x, z) \in X \times X$

- $F_{x,z}(a_1 + a_2) \leqslant F_{x,z}(a_1) + F_{x,z}(a_2)$ *for all* $a_1, a_2 \in R^n$ *; and*
- $F_{x,z}(\alpha a) = \alpha F_{x,z}(a)$ *for all* $\alpha \in R_+$ *and* $a \in R^n$ *.*

We now rewrite the definition of second order (F, α, ρ, d) type-I functions introduced by Hachimi and Aghezzaf [6]. Let F be a sublinear functional with respect to the third variable, $\alpha^1, \alpha^2 : X \times X \to R_+ \backslash \{0\}$, $d : X \times X \to R_+$ and $\rho^1, \rho_j^2 \in R$ for $j \in M$. Let $\phi : X \to R$ and $g_j : X \to R$ for $j \in M$ be twice differentiable functions.

Definition 2 (Hachimi and Aghezzaf [6]). *Function* (ϕ, g) *is second order* (F, α, ρ, d) *type-I at* $z \in X$ *if for all* $x \in X$ *there exists* $p \in R^n$ *such* *that*

$$\phi(x)-\phi(z)+\tfrac{1}{2}p^T\nabla^2\phi(z)p \geqslant F_{x,z}\big(\alpha^1(x,z)[\nabla\phi(z)+\nabla^2\phi(z)p]\big)+\rho^1 d(x,z)\,,$$
$$-g_j(z)+\tfrac{1}{2}p^T\nabla^2 g_j(z)p \geqslant F_{x,z}\big(\alpha^2(x,z)[\nabla g_j(z)+\nabla^2 g_j(z)p]\big)+\rho_j^2 d(x,z)\,,$$

for each $j \in M$ *.*

Definition 3. *Function* (ϕ, g) *is semistrictly second order* (F,α,ρ,d) *type-I at* $z \in X$ *if for all* $x \in X$ *there exists* $p \in R^n$ *such that*

$$\phi(x)-\phi(z)+\tfrac{1}{2}p^T\nabla^2\phi(z)p > F_{x,z}\big(\alpha^1(x,z)[\nabla\phi(z)+\nabla^2\phi(z)p]\big)+\rho^1 d(x,z)\,,$$
$$-g_j(z)+\tfrac{1}{2}p^T\nabla^2 g_j(z)p \geqslant F_{x,z}\big(\alpha^2(x,z)[\nabla g_j(z)+\nabla^2 g_j(z)p]\big)+\rho_j^2 d(x,z)\,,$$

for each $j \in M$ *.*

Yuan et al. [15] introduced (C,α,ρ,d) convexity and proved necessary and sufficient optimality conditions for a nondifferentiable multiobjective fractional programming problem. In the framework of this definition, Chinchuluun et al. [4] studied nonsmooth multiobjective fractional programming problems. Later, Long [12] established duality relations for a class of nondifferentiable multiobjective fractional programming problems involving (C,α,ρ,d) convex functions.

We now present (C,α,ρ,d) type-I functions, after defining convexity in the function C.

Definition 4 (Yuan et al. [15])**.** *A function* $C : X \times X \times R^n \to R$ *is convex on* R^n *iff for any fixed* $(x,z) \in X \times X$ *and for any* $y_1, y_2 \in R^n$ *,*

$$C_{x,z}[\lambda y_1 + (1-\lambda)y_2] \leqslant \lambda C_{x,z}(y_1) + (1-\lambda)C_{x,z}(y_2)\,,$$

for all $\lambda \in (0,1)$ *.*

Suppose the real valued function $d : X \times X \to R_+$ satisfies $d(x,z) = 0$ iff $x = z$ and let $C : X \times X \times R^n \to R$ be a convex function such that $C_{x,z}(0) = 0$ for any $(x,z) \in X \times X$.

Definition 5. *Function* (ϕ, g) *is second order* (C, α, ρ, d) *type-I at* $z \in X$ *if for all* $x \in X$ *there exists* $p \in R^n$ *such that*

$$\frac{1}{\alpha^1(x,z)}\left[\phi(x) - \phi(z) + \tfrac{1}{2}p^T\nabla^2\phi(z)p\right] \geqslant C_{x,z}[\nabla\phi(z) + \nabla^2\phi(z)p] + \frac{\rho^1 d(x,z)}{\alpha^1(x,z)},$$

$$\frac{1}{\alpha^2(x,z)}\left[-g_j(z) + \tfrac{1}{2}p^T\nabla^2 g_j(z)p\right] \geqslant C_{x,z}[\nabla g_j(z) + \nabla^2 g_j(z)p] + \frac{\rho_j^2 d(x,z)}{\alpha^2(x,z)},$$

for each $j \in M$.

Definition 6. *Function* (ϕ, g) *is semistrictly second order* (C, α, ρ, d) *type-I at* $z \in X$ *if for all* $x \in X$ *there exists* $p \in R^n$ *such that*

$$\frac{1}{\alpha^1(x,z)}\left[\phi(x) - \phi(z) + \tfrac{1}{2}p^T\nabla^2\phi(z)p\right] > C_{x,z}[\nabla\phi(z) + \nabla^2\phi(z)p] + \frac{\rho^1 d(x,z)}{\alpha^1(x,z)},$$

$$\frac{1}{\alpha^2(x,z)}\left[-g_j(z) + \tfrac{1}{2}p^T\nabla^2 g_j(z)p\right] \geqslant C_{x,z}[\nabla g_j(z) + \nabla^2 g_j(z)p] + \frac{\rho_j^2 d(x,z)}{\alpha^2(x,z)},$$

for each $j \in M$.

Function (ϕ, g) is (semistrictly) second order $(F, \alpha, \rho, d)/(C, \alpha, \rho, d)$ type-I over X iff it is (semistrictly) second order $(F, \alpha, \rho, d)/(C, \alpha, \rho, d)$ type-I at every point in X.

Remark 7. If C is sublinear with respect to the third variable, then Definitions 5 and 6 are identical to Definitions 2 and 3, respectively.

Remark 8. Since the functional F is sublinear with respect to the third variable, it is convex, as defined in Definition 4. Further, since $\alpha^1, \alpha^2 > 0$, every (F, α, ρ, d) type-I function is (C, α, ρ, d) type-I. But the converse need not be true. This is seen from the following example.

Example 9. Let $X = R$. Let $\phi : X \to R$ and $g : X \to R$ where $\phi(x) = x^2 - 2\sin^2 x$ and $g(x) = \cos^2 x - 2x$. Suppose $\alpha^1, \alpha^2 : X \times X \to R_+ \backslash \{0\}$ and $C : X \times X \times R^n \to R$ are $\alpha^1(x,z) = 1/20$, $\alpha^2(x,z) = 1/3$ and $C_{x,z}(a) = a^2/24$. Let $d : X \times X \to R_+$ be $d(x,z) = (x-z)^2$. For $p = -1$, $\rho^1 = -1/20$, $\rho^2 = -1$

and $z = 0.5\pi$,

$$\frac{1}{\alpha^1(x,z)}\left[\phi(x) - \phi(z) + \tfrac{1}{2}p^T\nabla^2\phi(z)p\right] - C_{x,z}[\nabla\phi(z) + \nabla^2\phi(z)p] - \frac{\rho^1 d(x,z)}{\alpha^1(x,z)}$$
$$= 20x^2 + 40\cos^2 x + 60 - 5\pi^2 - \tfrac{1}{24}(\pi-6)^2 + (x-0.5\pi)^2 \geqslant 0\,,$$

for all $x \in X$, and

$$\frac{1}{\alpha^2(x,z)}\left[-g(z) + \tfrac{1}{2}p^T\nabla^2 g(z)p\right] - C_{x,z}[\nabla g(z) + \nabla^2 g(z)p] - \frac{\rho^2 d(x,z)}{\alpha^2(x,z)}$$
$$= \tfrac{7}{3} + 3\pi + 3(x-0.5\pi)^2 \geqslant 0\,,$$

for all $x \in X$. Hence, (ϕ, g) is second order (C, α, ρ, d) type-I but (ϕ, g) is not second order (F, α, ρ, d) type-I at $z = 0.5\pi$ as C is not sublinear with respect to the third argument.

For $f : R^n \times R^l \to R$, $h : R^n \times R^l \to R$ and $g : R^n \to R^m$ twice continuously differentiable functions, consider the nondifferentiable minimax fractional programming problem (PP):

$$\text{minimise} \quad \psi(x) = \sup_{y \in Y} \frac{f(x,y) + (x^TBx)^{1/2}}{h(x,y) - (x^TDx)^{1/2}} \quad \text{subject to} \quad g(x) \leqslant 0\,,$$

where Y is a compact subset of R^l, B and D are $n \times n$ positive semidefinite matrices, $f(x,y) + (x^TBx)^{1/2} \geqslant 0$ and $h(x,y) - (x^TDx)^{1/2} > 0$ for each $(x,y) \in \mathfrak{J} \times Y$, where $\mathfrak{J} = \{x \in R^n : g(x) \leqslant 0\}$. For each $(x,y) \in \mathfrak{J} \times Y$ we define

$$J(x) = \{j \in M : g_j(x) = 0\}\,,$$
$$Y(x) = \left\{y \in Y : \frac{f(x,y) + (x^TBx)^{1/2}}{h(x,y) - (x^TDx)^{1/2}} = \sup_{z \in Y} \frac{f(x,z) + (x^TBx)^{1/2}}{h(x,z) - (x^TDx)^{1/2}}\right\},$$
$$K(x) = \Bigg\{(s,t,\tilde{y}) \in N \times R^s_+ \times R^{ls} : 1 \leqslant s \leqslant n+1\,, t = (t_1, t_2, \ldots, t_s) \in R^s_+\,,$$
$$\sum_{i=1}^{s} t_i = 1\,, \tilde{y} = (\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_s)\,, \tilde{y}_i \in Y(x)\,, i = 1, 2, \ldots, s\Bigg\}.$$

3 Duality model

Consider the dual problem (DP) to the PP:

$$\max_{(s,t,\tilde{y})\in K(z)} \sup_{(z,\mu,\lambda,w,\nu,r,p)\in H_1(s,t,\tilde{y})} \lambda\,,$$

where $H_1(s,t,\tilde{y})$ denotes the set of all $(z,\mu,\lambda,w,\nu,r,p)\in R^n\times R^m_+\times R_+\times R^n\times R^n\times R^n\times R^n$ satisfying

$$\sum_{i=1}^{s} t_i I(z,\tilde{y}_i) + \nabla^2\sum_{i=1}^{s} t_i[f(z,\tilde{y}_i)-\lambda h(z,\tilde{y}_i)]p + \sum_{j=1}^{m}\mu_j\nabla g_j(z) + \nabla^2\sum_{j=1}^{m}\mu_j g_j(z)p = 0\,, \tag{1}$$

$$\sum_{i=1}^{s} t_i G(z,\tilde{y}_i) + \left[\sum_{i=1}^{s} t_i I(z,\tilde{y}_i)\right]^T r - \tfrac{1}{2}p^T\nabla^2\sum_{i=1}^{s} t_i[f(z,\tilde{y}_i)-\lambda h(z,\tilde{y}_i)]p \geqslant 0\,, \tag{2}$$

$$\sum_{j=1}^{m}\mu_j g_j(z) + \left[\sum_{j=1}^{m}\mu_j\nabla g_j(z)\right]^T r - \tfrac{1}{2}p^T\nabla^2\sum_{j=1}^{m}\mu_j g_j(z)p \geqslant 0\,, \tag{3}$$

$$\left[\sum_{i=1}^{s} t_i I(z,\tilde{y}_i)\right]^T r + \left(\sum_{j=1}^{m}\mu_j\nabla g_j(z)\right)^T r \leqslant 0\,, \tag{4}$$

$$w^T B w \leqslant 1 \quad\text{and}\quad \nu^T D\nu \leqslant 1\,, \tag{5}$$

where

$$I(z,\tilde{y}_i) = \nabla f(z,\tilde{y}_i) + Bw - \lambda[\nabla h(z,\tilde{y}_i) - D\nu]\,,$$
$$G(z,\tilde{y}_i) = f(z,\tilde{y}_i) + z^T Bw - \lambda[h(z,\tilde{y}_i) - z^T D\nu]\,.$$

If, for a triplet $(s,t,\tilde{y})\in K(z)$, the set $H_1(s,t,\tilde{y})=\phi$, then we define the supremum over H_1 to be $-\infty$. Now, we establish the duality relations between PP and DP.

Theorem 10 (Weak duality)**.** *Let* x *and* $(z, \mu, \lambda, w, v, s, t, \tilde{y}, r, p)$ *be feasible solutions of* PP *and* DP*, respectively. Assume that any one of the following four conditions hold:*

1. $\{G(\cdot, \tilde{y}_i)\,, g_j(\cdot)\,, i = 1, 2, \ldots, s\,, j = 1, 2, \ldots, m\}$ *is second order* (F, α, ρ, d) *type-I at* z *and* $\sum_{i=1}^{s} t_i\rho_i^1 + \sum_{j=1}^{m} \mu_j\rho_j^2 \geqslant 0\,;$
2. $\{\sum_{i=1}^{s} t_i G(\cdot, \tilde{y}_i)\,, g_j(\cdot)\,, j = 1, 2, \ldots, m\}$ *is second order* (F, α, ρ, d) *type-I at* z *and* $\rho^1 + \sum_{j=1}^{m} \mu_j\rho_j^2 \geqslant 0\,;$
3. $\{G(\cdot, \tilde{y}_i)\,, g_j(\cdot)\,, i = 1, 2, \ldots, s\,, j = 1, 2, \ldots, m\}$ *is second order* (C, α, ρ, d) *type-I at* z *and* $\sum_{i=1}^{s} t_i\rho_i^1 + \sum_{j=1}^{m} \mu_j\rho_j^2 \geqslant 0\,;$
4. $\{\sum_{i=1}^{s} t_i G(\cdot, \tilde{y}_i)\,, g_j(\cdot)\,, j = 1, 2, \ldots, m\}$ *is second order* (C, α, ρ, d) *type-I at* z *and* $\rho^1 + \sum_{j=1}^{m} \mu_j\rho_j^2 \geqslant 0\,.$

Furthermore, suppose $\alpha^1(x, z) = \alpha^2(x, z)\,,$ *then*

$$\sup_{\tilde{y}\in Y} \frac{f(x, \tilde{y}) + (x^T Bx)^{1/2}}{h(x, \tilde{y}) - (x^T Dx)^{1/2}} \geqslant \lambda\,.$$

Proof: Suppose, contrary to the theorem,

$$\sup_{\tilde{y}\in Y} \frac{f(x, \tilde{y}) + (x^T Bx)^{1/2}}{h(x, \tilde{y}) - (x^T Dx)^{1/2}} < \lambda\,,$$

then,

$$f(x, \tilde{y}_i) + (x^T Bx)^{1/2} - \lambda[h(x, \tilde{y}_i) - (x^T Dx)^{1/2}] < 0\,,$$

for all $\tilde{y}_i \in Y(x)$ with $i = 1, 2, \ldots, s$. It follows from $t_i \geqslant 0$, $i = 1, 2, \ldots, s$, that

$$t_i\{f(x, \tilde{y}_i) + (x^T Bx)^{1/2} - \lambda[h(x, \tilde{y}_i) - (x^T Dx)^{1/2}]\} \leqslant 0\,,$$

with at least one strict inequality, since $t = (t_1, t_2, \ldots, t_s) \neq 0$. Taking the summation over i and using (5),

$$\sum_{i=1}^{s} t_i\{f(x, \tilde{y}_i) + x^T Bw - \lambda[h(x, \tilde{y}_i) - x^T Dv]\} = \sum_{i=1}^{s} t_i G(x, \tilde{y}_i) < 0\,. \qquad (6)$$

Condition 1: By the second order (F, α, ρ, d) type-I assumption on $\{G(\cdot, \tilde{y}_i)$, $g_j(\cdot)$, $i = 1, 2, \ldots, s$, $j = 1, 2, \ldots, m\}$ at z, for $i = 1, 2, \ldots, s$,

$$\begin{aligned} & G(x, \tilde{y}_i) - G(z, \tilde{y}_i) + \tfrac{1}{2} p^T \nabla^2 [f(z, \tilde{y}_i) - \lambda h(z, \tilde{y}_i)] p \\ & \geqslant F_{x,z}\big(\alpha^1(x, z)\{I(z, \tilde{y}_i) + \nabla^2 [f(z, \tilde{y}_i) - \lambda h(z, \tilde{y}_i)] p\}\big) + \rho_i^1 d(x, z) , \end{aligned} \tag{7}$$

and, for $j = 1, 2, \ldots, m$,

$$-g_j(z) + \tfrac{1}{2} p^T \nabla^2 g_j(z) p \geqslant F_{x,z}\big(\alpha^2(x, z)[\nabla g_j(z) + \nabla^2 g_j(z) p]\big) + \rho_j^2 d(x, z) . \tag{8}$$

Multiplying (7) by $t_i \geqslant 0$, $i = 1, 2, \ldots, s$, multiplying (8) by $\mu_j \geqslant 0$, $j = 1, 2, \ldots, m$, taking summations over i and j and using the sublinearity of F, we obtain

$$\begin{aligned} & \sum_{i=1}^{s} t_i G(x, \tilde{y}_i) - \sum_{i=1}^{s} t_i G(z, \tilde{y}_i) + \tfrac{1}{2} p^T \nabla^2 \sum_{i=1}^{s} t_i [f(z, \tilde{y}_i) - \lambda h(z, \tilde{y}_i)] p \\ & \geqslant F_{x,z}\left[\alpha^1(x, z)\left(\sum_{i=1}^{s} t_i I(z, \tilde{y}_i) + \nabla^2 \sum_{i=1}^{s} t_i [f(z, \tilde{y}_i) - \lambda h(z, \tilde{y}_i)] p\right)\right] \\ & \quad + \sum_{i=1}^{s} t_i \rho_i^1 d(x, z) , \end{aligned} \tag{9}$$

$$\begin{aligned} & -\sum_{j=1}^{m} \mu_j g_j(z) + \tfrac{1}{2} p^T \nabla^2 \sum_{j=1}^{m} \mu_j g_j(z) p \\ & \geqslant F_{x,z}\left[\alpha^2(x, z)\left(\sum_{j=1}^{m} \mu_j \nabla g_j(z) + \nabla^2 \sum_{j=1}^{m} \mu_j g_j(z) p\right)\right] + \sum_{j=1}^{m} \mu_j \rho_j^2 d(x, z) . \end{aligned} \tag{10}$$

Now, using (2), (4) and (6) in (9) and (3) in (10),

$$F_{x,z}\left[\alpha^1(x,z)\left(\sum_{i=1}^{s} t_i I(z,\tilde{y}_i) + \nabla^2 \sum_{i=1}^{s} t_i[f(z,\tilde{y}_i) - \lambda h(z,\tilde{y}_i)]p\right)\right]$$

$$+ \sum_{i=1}^{s} t_i \rho_i^1 d(x,z) < -\left[\sum_{j=1}^{m} \mu_j \nabla g_j(z)\right]^T r\,, \tag{11}$$

and

$$F_{x,z}\left[\alpha^2(x,z)\left(\sum_{j=1}^{m} \mu_j \nabla g_j(z) + \nabla^2 \sum_{j=1}^{m} \mu_j g_j(z)p\right)\right] + \sum_{j=1}^{m} \mu_j \rho_j^2 d(x,z)$$

$$\leqslant \left[\sum_{j=1}^{m} \mu_j \nabla g_j(z)\right]^T r. \tag{12}$$

Finally, using $\alpha^1(x,z) = \alpha^2(x,z) > 0$, in the addition of (11) and (12) and from the sublinearity of F, $\sum_{i=1}^{s} t_i \rho_i^1 + \sum_{j=1}^{m} \mu_j \rho_j^2 \geqslant 0$ and (1), we have

$$0 = F_{x,z}(0) = F_{x,z}\left(\sum_{i=1}^{s} t_i I(z,\tilde{y}_i) + \nabla^2 \sum_{i=1}^{s} t_i[f(z,\tilde{y}_i) - \lambda h(z,\tilde{y}_i)]p\right.$$

$$\left. + \sum_{j=1}^{m} \mu_j \nabla g_j(z) + \nabla^2 \sum_{j=1}^{m} \mu_j g_j(z)p\right)$$

$$< -\left(\sum_{i=1}^{s} t_i \rho_i^1 + \sum_{j=1}^{m} \mu_j \rho_j^2\right) \frac{d(x,z)}{\alpha^1(x,z)} \leqslant 0\,,$$

which is a contradiction. Hence the theorem is proved. Similarly, the proof of the theorem can be obtained using Condition 2.

Condition 3: Since $\{G(\cdot,\tilde{y}_i)\,, i = 1,2,\dots,s\,, j = 1,2,\dots,m\}$ is second order (C,α,ρ,d) type-I at z, for $i = 1,2,\dots,s$,

$$\frac{1}{\alpha^1(x,z)}\left\{G(x,\tilde{y}_i) - G(z,\tilde{y}_i) + \tfrac{1}{2}p^T\nabla^2[f(z,\tilde{y}_i) - \lambda h(z,\tilde{y}_i)]p\right\} \geqslant C_{x,z}\big(I(z,\tilde{y}_i) + \nabla^2[f(z,\tilde{y}_i) - \lambda h(z,\tilde{y}_i)]p\big) + \frac{\rho_i^1 d(x,z)}{\alpha^1(x,z)}\,, \tag{13}$$

and, for $j = 1,2,\dots,m$,

$$\frac{1}{\alpha^2(x,z)}\left[-g_j(z) + \tfrac{1}{2}p^T\nabla^2 g_j(z)p\right] \geqslant C_{x,z}[\nabla g_j(z) + \nabla^2 g_j(z)p] + \frac{\rho_j^2 d(x,z)}{\alpha^2(x,z)}\,. \tag{14}$$

Multiplying (13) by $t_i/\tau \geqslant 0$ for $i = 1,2,\dots,s$, and (14) by $\mu_j/\tau \geqslant 0$ for $j = 1,2,\dots,m$, where $\tau = 1 + \sum_{j=1}^m \mu_j$, we obtain, for $i = 1,2,\dots,s$,

$$\frac{1}{\tau\alpha^1(x,z)}\big(t_i\{G(x,\tilde{y}_i) - G(z,\tilde{y}_i) + \tfrac{1}{2}p^T\nabla^2[f(z,\tilde{y}_i) - \lambda h(z,\tilde{y}_i)]p\}\big) \geqslant \frac{t_i}{\tau}C_{x,z}\big(I(z,\tilde{y}_i) + \nabla^2[f(z,\tilde{y}_i) - \lambda h(z,\tilde{y}_i)]p\big) + \frac{t_i\rho_i^1 d(x,z)}{\tau\alpha^1(x,z)}\,, \tag{15}$$

$$\frac{\mu_j}{\tau\alpha^2(x,z)}[-g_j(z) + \tfrac{1}{2}p^T\nabla^2 g_j(z)p] \geqslant \frac{\mu_j}{\tau}C_{x,z}[\nabla g_j(z) + \nabla^2 g_j(z)p] + \frac{\mu_j\rho_j^2 d(x,z)}{\tau\alpha^2(x,z)}\,. \tag{16}$$

Summing (15) over i and (16) over j, using $\alpha^1(x,z) = \alpha^2(x,z)$ and the convexity of C,

$$\frac{1}{\tau\alpha^1(x,z)}\Bigg[\sum_{i=1}^{s} t_i\{G(x,\tilde{y}_i) - G(z,\tilde{y}_i) + \tfrac{1}{2}p^T\nabla^2[f(z,\tilde{y}_i) - \lambda h(z,\tilde{y}_i)]p\}$$
$$- \sum_{j=1}^{m}\mu_j g_j(z) + \tfrac{1}{2}p^T\nabla^2\sum_{j=1}^{m}\mu_j g_j(z)p\Bigg]$$
$$> C_{x,z}\Bigg[\frac{1}{\tau}\Bigg(\sum_{i=1}^{s} t_i I(z,\tilde{y}_i) + \nabla^2\sum_{i=1}^{s} t_i[f(z,\tilde{y}_i) - \lambda h(z,\tilde{y}_i)]p + \sum_{j=1}^{m}\mu_j\nabla g_j(z)$$
$$+\nabla^2\sum_{j=1}^{m}\mu_j g_j(z)p\Bigg)\Bigg] + \Bigg(\sum_{i=1}^{s} t_i\rho_i^1 + \sum_{j=1}^{m}\mu_j\rho_j^2\Bigg)\frac{d(x,z)}{\alpha^1(x,z)\tau}\,. \tag{17}$$

Now, inequalities (2)–(4) yield

$$-\sum_{j=1}^{m}\mu_j g_j(z) + \tfrac{1}{2}p^T\nabla^2\sum_{j=1}^{m}\mu_j g_j(z)p - \sum_{i=1}^{s} t_i G(z,\tilde{y}_i)$$
$$+ \tfrac{1}{2}p^T\nabla^2\sum_{i=1}^{s} t_i[f(z,\tilde{y}_i) - \lambda h(z,\tilde{y}_i)]p \leqslant 0\,. \tag{18}$$

Finally, using (1), (6), (18) and $\sum_{i=1}^{s} t_i\rho_i^1 + \sum_{j=1}^{m}\mu_j\rho_j^2 \geqslant 0$ in (17),

$$0 = C_{x,z}(0) = C_{x,z}\Bigg[\frac{1}{\tau}\Bigg(\sum_{i=1}^{s} t_i I(z,\tilde{y}_i) + \nabla^2\sum_{i=1}^{s} t_i[f(z,\tilde{y}_i) - \lambda h(z,\tilde{y}_i)]p$$
$$+ \sum_{j=1}^{m}\mu_j\nabla g_j(z) + \nabla^2\sum_{j=1}^{m}\mu_j g_j(z)p\Bigg)\Bigg] < 0\,,$$

which is a contradiction. Hence the theorem is proved. Similarly, the proof of the theorem can be obtained using Condition 4. ♠

Theorem 11 (Strong duality). *Assume that* x^* *is an optimal solution of* PP *and* $\nabla g_j(x^*)$ *for* $j \in J(x^*)$ *are linearly independent. Then there exist* $(s^*, t^*, \tilde{y}^*) \in K(x^*)$ *and* $(x^*, \mu^*, \lambda^*, w^*, \nu^*, r^* = 0, p^* = 0) \in H_1(s^*, t^*, \tilde{y}^*)$ *such that* $(x^*, \mu^*, \lambda^*, w^*, \nu^*, s^*, t^*, \tilde{y}^*, r^* = 0, p^* = 0)$ *is a feasible solution of* DP *and the two objectives have the same values. If, in addition, the assumptions of Theorem 10 hold for all feasible solutions* $(x, \mu, \lambda, w, \nu, s, t, \tilde{y}, r, p)$ *of* DP, *then* $(x^*, \mu^*, \lambda^*, w^*, \nu^*, s^*, t^*, \tilde{y}^*, r^* = 0, p^* = 0)$ *is an optimal solution of* DP.

Proof: Since x^* is an optimal solution of PP and $\nabla g_j(x^*)$ for $j \in J(x^*)$ are linearly independent, then by Theorem 10 and Lai et al. [10] there exist $(s^*, t^*, \tilde{y}^*) \in K(x^*)$ and $(x^*, \mu^*, \lambda^*, w^*, \nu^*, r^* = 0, p^* = 0) \in H_1(s^*, t^*, \tilde{y}^*)$ such that $(x^*, \mu^*, \lambda^*, w^*, \nu^*, s^*, t^*, \tilde{y}^*, r^* = 0, p^* = 0)$ is a feasible solution of DP and the two objectives have same values. Optimality of $(x^*, \mu^*, \lambda^*, w^*, \nu^*, s^*, t^*, \tilde{y}^*, r^* = 0, p^* = 0)$ for DP thus follows from Theorem 10. ♠

Theorem 12 (Strict Converse Duality). *Let* x^* *be an optimal solution of* PP *and* $(z^*, \mu^*, \lambda^*, w^*, \nu^*, s^*, t^*, \tilde{y}^*, r^*, p^*)$ *be an optimal solution of* DP. *Assume that any one of the following four conditions holds.*

1. $\{G(\cdot, \tilde{y}_i^*), g_j(\cdot), i = 1, 2, \ldots, s^*, j = 1, 2, \ldots, m\}$ *is semistrictly second order* (F, α, ρ, d) *type-I at* z^* *and* $\sum_{i=1}^{s^*} t_i^* \rho_i^1 + \sum_{j=1}^{m} \mu_j^* \rho_j^2 \geqslant 0$.
2. $\left\{\sum_{i=1}^{s^*} t_i^* G(\cdot, \tilde{y}_i^*), g_j(\cdot), j = 1, 2, \ldots, m\right\}$ *is semistrictly second order* (F, α, ρ, d) *type-I at* z^* *and* $\rho^1 + \sum_{j=1}^{m} \mu_j^* \rho_j^2 \geqslant 0$.
3. $\{G(\cdot, \tilde{y}_i^*), g_j(\cdot), i = 1, 2, \ldots, s^*, j = 1, 2, \ldots, m\}$ *is semistrictly second order* (C, α, ρ, d) *type-I at* z^* *and* $\sum_{i=1}^{s^*} t_i^* \rho_i^1 + \sum_{j=1}^{m} \mu_j^* \rho_j^2 \geqslant 0$.
4. $\left\{\sum_{i=1}^{s^*} t_i^* G(\cdot, \tilde{y}_i^*), g_j(\cdot), j = 1, 2, \ldots, m\right\}$ *is semistrictly second order* (C, α, ρ, d) *type-I at* z^* *and* $\rho^1 + \sum_{j=1}^{m} \mu_j^* \rho_j^2 \geqslant 0$.

Furthermore, suppose the set of vectors $\{\nabla g_j(x^*), j \in J(x^*)\}$ *is linearly independent and* $\alpha^1(x^*, z^*) = \alpha^2(x^*, z^*)$ *. Then* $z^* = x^*$ *, that is,* z^* *is an optimal solution of* PP*.*

Proof: The proof follows similarly to the proof of Theorem 10 and Theorem 3.3 of Ahmad et al. [2]. ♠

Remark 13. Let B and D be zero matrices of order n, then the model DP becomes the dual models discussed by Hu et al. [8]. Further, if $r = 0$, then our dual models reduce to the problems of Husain et al. [7] and Sharma and Gulati [14]. In addition, if $p = 0$, then DP becomes the dual model considered by Liu and Wu [11]. If $r = 0$ and $p = 0$, then the model DP reduces to the model of Ahmad and Husain [1].

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