Curvature Variation Minimizing Cardinal Spline Curves

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Abstract: Since the free parameter and the two auxiliary points have effect on the shape of the Cardinal spline curves, a natural idea arises to find the optimal parameter and auxiliary points to obtain the smoothest curves. We use curvature variation minimization to achieve this goal in this paper. By minimizing an appropriate approximation of the curvature variation energy, the unique solution can be easily obtained. Some examples show that the Cardinal spline curves with minimum curvature variation are better than the Catmull-Rom spline curves in dealing with interpolation problems.

Keywords: Cardinal spline; Catmull-Rom spline; smooth curve; minimization; curvature variation.

1. Introduction

It is known that the smoothness is an important geometric feature of a curve. The construction of smooth curves is a fundamental issue in computer aided design (CAD) and related application fields, see e.g. [1, 2]. Although the smoothness of a curve is difficult to be expressed in a quantitative way, the general ways to construct the smoothest curves are achieved by minimizing some energy functions [3]. The strain energy (also called bending energy) and curvature variation energy are two widely adopted metrics to describe the smoothness of a curve, see e.g. [4-11].

As we know, the Cardinal spline [12] contains a free parameter (also called tension parameter). We can obtain different shapes of curves by altering the value of the tension parameter in case the data points are kept unchanged. In addition, we often need to add two auxiliary points at the first and the end of the data points in order to ensure the curves can pass all the data points. Thus the two auxiliary points have effect on the shapes of the first and the end segments of the curves. That is to say, we can adjust the shapes of the Cardinal spline curves by the free parameter and the two auxiliary points. However, it might just have to construct the smoothest curves in some cases. Therefore, a natural idea is that we can find the optimal parameter and auxiliary points so that the Cardinal spline curves are as smooth as possible. In this paper, we determine the optimal parameter and auxiliary points by minimizing the curvature variation energy so that we can obtain the smoothest Cardinal spline curves. We observe that the unique solution can be easily obtained by minimizing an appropriate approximation of the curvature variation energy.

The rest of this paper is organized as follows. In Section 2, the influence factors on the shape of the Cardinal spline curves are briefly analyzed. In Section 3, the curvature variation minimization approach is described. In Section 4, some examples are presented to illustrate the effectiveness of the proposed method. A short conclusion is given in Section 5.

2. The Cardinal spline curves

According to [12], the Cardinal spline curves can be defined by

$$\mathbf{r}_i(t) = b_0(t)\mathbf{p}_i + b_1(t)\mathbf{p}_{i+1} + b_2(t)\mathbf{p}_{i+2} + b_3(t)\mathbf{p}_{i+3}, \ (i = 0,1,\cdots,n-3), \tag{1}$$

where $0 \le t \le 1$, $\mathbf{p}_k \in \mathbb{R}^2 (k = 0,1,\cdots,n; n \ge 3)$ are an array of given data points, and

$$\begin{cases} b_0(t) = -\alpha t^3 + 2\alpha t^2 - \alpha t, \\ b_1(t) = (2-\alpha)t^3 + (\alpha-3)t^2 + 1, \\ b_2(t) = (\alpha-2)t^3 + (3-2\alpha)t^2 + \alpha t, \\ b_3(t) = \alpha t^3 - \alpha t^2, \end{cases} \tag{2}$$

with $\alpha = (1-T)/2$ being a free real parameter, T is called the tension parameter.

It is clear that the Cardinal spline is joined by a sequence of individual curves to form a whole curve that is specified by an array of data points and a free parameter (or a tension parameter). When the free parameter is taken as $\alpha = 0.5$ (*i.e.* the tension parameter equal to zero), the corresponding Cardinal spline curves are also called the Catmull-Rom spline curves.

By (1) and (2), we can obtain that

$$\mathbf{r}_i(0) = \mathbf{p}_{i+1}, \ \mathbf{r}_i(1) = \mathbf{p}_{i+2}. \quad (3)$$

$$\mathbf{r}_i'(0) = \alpha(\mathbf{p}_{i+2} - \mathbf{p}_i), \ \mathbf{r}_i'(1) = \alpha(\mathbf{p}_{i+3} - \mathbf{p}_{i+1}). \quad (4)$$

From (3) we know that each individual curve interpolates the second and the third data points, which means the whole curve passes all the given data points except the first data point $\mathbf{p}_0$ and the end data point $\mathbf{p}_n$. Because the curves are generally required to pass all the given points, it is necessary to add an auxiliary point $\mathbf{p}_{-1}$ to the front of $\mathbf{p}_0$ and an auxiliary point $\mathbf{p}_{n+1}$ to the back of $\mathbf{p}_n$ so that the whole curve can pass the first and the end data points.

It is clear that the free parameter α has influence on the shape of the whole curve when all the points are remained unchanged. For example, suppose the coordinates (px, py) of the given data points $\mathbf{p}_i$ $(i = 0,1,\cdots,8)$ are $px = [35\ 16\ 15\ 25\ 40\ 65\ 50\ 60\ 80]$, $py = [47\ 40\ 15\ 36\ 15\ 25\ 40\ 42\ 37]$. Figure 1 shows the influence of the free parameter α on the shape of the whole curve, where the two auxiliary points are taken as $\mathbf{p}_{-1} = \mathbf{p}_0$ and $\mathbf{p}_9 = \mathbf{p}_8$.

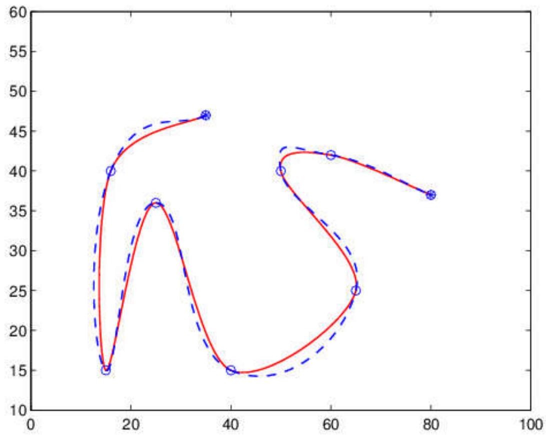


Figure 1. The effect of the free parameter on the shape of the curve, where the parameter of the solid lines is taken as $\alpha = 0.5$, the parameter of the dashed lines is taken as $\alpha = 0.8$.

In addition, we can see that the two auxiliary points $\mathbf{p}_{-1}$ and $\mathbf{p}_{n+1}$ have obvious influence on the shapes of the first and the end segments of the whole curve in case the free parameter is fixed. For the same data points given in Figure 1, the influence of the two auxiliary points $\mathbf{p}_{-1}$ and $\mathbf{p}_{n+1}$ on the shapes of the first and the end segments of the whole curve is shown in Figure 2, where the free parameter is taken as $\alpha = 0.5$.

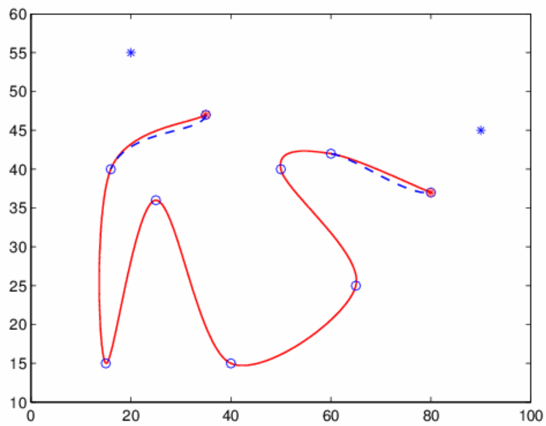


Figure 2. The effect of the two auxiliary points on the shape of the curve, where 'o' represent the given points, '*' represent the two auxiliary points.

Summarizing, the free parameter and the two auxiliary points can be taken as any values according to different application requirements. Alternatively, we can find the optimal parameter and auxiliary points by minimizing some energy functions to obtain the smoothest Cardinal spline curves.

3. Curvature variation minimization

The curvature variation energy of the curve $\mathbf{r}(t)$ is defined by (see e.g. [4])

$$E = \int_0^1 \left(\kappa'(t)\right)^2 \mathrm{d}t , \tag{5}$$

where $\kappa(t) = \dfrac{\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$.

Because the curvature variation energy (5) is highly nonlinear, one may use some approximate forms to simplify the calculation. In [11], the curvature variation energy is approximated by

$$\hat{E} = \int_0^1 \|\mathbf{r}'''(t)\|^2 \mathrm{d}t . \tag{6}$$

In this paper, we use (6) to minimize the curvature variation energy to obtain the smoothest Cardinal spline curves.

From (2), we have

$$\begin{cases} b_0'''(t) = -6\alpha, \\ b_1'''(t) = 12-6\alpha, \\ b_2'''(t) = 6\alpha-12, \\ b_3'''(t) = 6\alpha. \end{cases} \tag{7}$$

By (1) and (7), we can obtain that

$$\begin{aligned} \|\mathbf{r}_i'''(t)\|^2 &= \left\|-6\alpha\mathbf{p}_i + (12-6\alpha)\mathbf{p}_{i+1} + (6\alpha-12)\mathbf{p}_{i+2} + 6\alpha\mathbf{p}_{i+3}\right\|^2 \\ &= \left\|6(-\mathbf{p}_i - \mathbf{p}_{i+1} + \mathbf{p}_{i+2} + \mathbf{p}_{i+3})\alpha + 12(\mathbf{p}_{i+1} - \mathbf{p}_{i+2})\right\|^2 \\ &= 36\alpha^2\|\mathbf{d}_i\|^2 + 144\alpha\mathbf{d}_i\cdot\mathbf{m}_i + 144\|\mathbf{m}_i\|^2 , \end{aligned} \tag{8}$$

where $\mathbf{d}_i := -\mathbf{p}_i - \mathbf{p}_{i+1} + \mathbf{p}_{i+2} + \mathbf{p}_{i+3}$, $\mathbf{m}_i := \mathbf{p}_{i+1} - \mathbf{p}_{i+2}$.

According to (6) and (8), we can express the curvature variation energy of each individual segment of the Cardinal spline curve as

$$\hat{E}_i = \int_0^1 \|\mathbf{r}_i'''(t)\|^2 \mathrm{d}t = 36\alpha^2\|\mathbf{d}_i\|^2 + 144\alpha\mathbf{d}_i\cdot\mathbf{m}_i + 144\|\mathbf{m}_i\|^2 . \tag{9}$$

After adding the two auxiliary points $\mathbf{p}_{-1}$ and $\mathbf{p}_{n+1}$, the curvature variation energy of the whole curve $\mathbf{r}_i(t)$ $(i=-1,0,\cdots,n-2)$ can be expressed as

$$\hat{E} = \sum_{i=-1}^{n-2}\hat{E}_i = 36\alpha^2\sum_{i=-1}^{n-2}\|\mathbf{d}_i\|^2 + 144\alpha\sum_{i=-1}^{n-2}(\mathbf{d}_i\cdot\mathbf{m}_i) + 144\sum_{i=-1}^{n-2}\|\mathbf{m}_i\|^2 . \tag{10}$$

Since the free parameter α and the two auxiliary points $\mathbf{p}_{-1}$ and $\mathbf{p}_{n+1}$ can completely determine the shape of the Cardinal spline curves when the data points $\mathbf{p}_k$ $(k=0,1,\cdots,n)$ are fixed, the following problem could be naturally obtained if we want to minimize the curvature variation energy of the curves,

$$\min \quad f(\alpha,\mathbf{p}_{-1},\mathbf{p}_{n+1}) := 36\alpha^2\sum_{i=-1}^{n-2}\|\mathbf{d}_i\|^2 + 144\alpha\sum_{i=-1}^{n-2}(\mathbf{d}_i\cdot\mathbf{m}_i) + 144\sum_{i=-1}^{n-2}\|\mathbf{m}_i\|^2 . \tag{11}$$

We can use the classical descent method to solve (11). The gradients of f can be calculated by

$$\frac{\partial f}{\partial\alpha} = 72\alpha\left(\|\mathbf{d}_{-1}\|^2 + \sum_{i=0}^{n-3}\|\mathbf{d}_i\|^2 + \|\mathbf{d}_{n-2}\|^2\right) + 144\left(\mathbf{d}_{-1}\cdot\mathbf{m}_{-1} + \sum_{i=0}^{n-3}(\mathbf{d}_i\cdot\mathbf{m}_i) + \mathbf{d}_{n-2}\cdot\mathbf{m}_{n-2}\right), \tag{12}$$

$$\frac{\partial f}{\partial\mathbf{p}_{-1}} = -72\alpha^2\mathbf{d}_{-1} - 144\alpha\mathbf{m}_{-1} , \tag{13}$$

$$\frac{\partial f}{\partial\mathbf{p}_{n+1}} = 72\alpha^2\mathbf{d}_{n-2} + 144\alpha\mathbf{m}_{n-2} . \tag{14}$$

Note that $\frac{\partial f}{\partial \mathbf{p}_{-1}}$ and $\frac{\partial f}{\partial \mathbf{p}_{n+1}}$ contain the partial derivatives of x-coordinates and y-coordinates.

Because the free parameter α is not zero in general, the following results can be solved from $\nabla f = 0$ with the gradients expressed in (12)-(14),

$$\alpha = -\frac{2\sum_{i=0}^{n-3}(\mathbf{d}_i \cdot \mathbf{m}_i)}{\sum_{i=0}^{n-3}\|\mathbf{d}_i\|^2}, \tag{15}$$

$$\mathbf{d}_{-1} = -\frac{2}{\alpha}\mathbf{m}_{-1}, \tag{16}$$

$$\mathbf{d}_{n-2} = -\frac{2}{\alpha}\mathbf{m}_{n-2}. \tag{17}$$

Recall that $\mathbf{d}_i = -\mathbf{p}_i - \mathbf{p}_{i+1} + \mathbf{p}_{i+2} + \mathbf{p}_{i+3}$, $\mathbf{m}_i = \mathbf{p}_{i+1} - \mathbf{p}_{i+2}$, then the following unique solution of (11) can be deduced from (15)-(17),

$$\alpha^* = -\frac{2\sum_{i=0}^{n-3}\left((-\mathbf{p}_i - \mathbf{p}_{i+1} + \mathbf{p}_{i+2} + \mathbf{p}_{i+3})\cdot(\mathbf{p}_{i+1} - \mathbf{p}_{i+2})\right)}{\sum_{i=0}^{n-3}\|-\mathbf{p}_i - \mathbf{p}_{i+1} + \mathbf{p}_{i+2} + \mathbf{p}_{i+3}\|^2}, \tag{18}$$

$$\mathbf{p}_{-1}^* = \left(1 - \frac{2}{\alpha}\right)(\mathbf{p}_1 - \mathbf{p}_0) + \mathbf{p}_2, \tag{19}$$

$$\mathbf{p}_{n+1}^* = \mathbf{p}_{n-2} - \left(1 - \frac{2}{\alpha}\right)(\mathbf{p}_n - \mathbf{p}_{n-1}). \tag{20}$$

4. Examples

The first example let us consider the same data points given in Fig. 1, the optimal parameter and auxiliary points calculated by (18)-(20) are $\alpha^* = 0.7509$, $\mathbf{p}_{-1}^* = (46.6047, 26.6438)$, $\mathbf{p}_9^* = (83.2681, 31.6830)$. Figure 3 shows the Cardinal spline curves with minimum curvature variation and the Catmull-Rom spline curves (*i.e.* $\alpha = 0.5$).

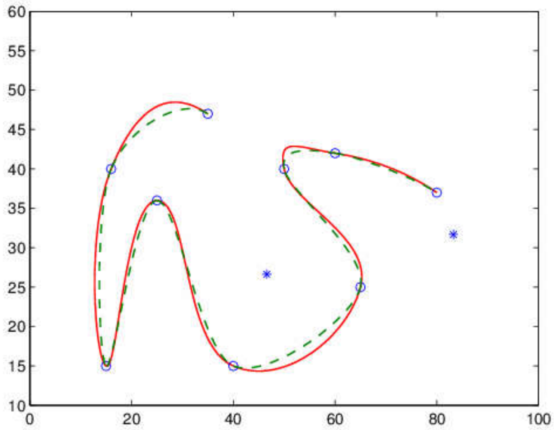


Figure 3. The Cardinal spline curves with minimum curvature variation (solid lines) and the Catmull-Rom spline curves (dashed lines).

The curvature variation energies of the above two curves calculated by (10) are shown in Table 1.

Table 1. Comparison of the two curves in Figure 3

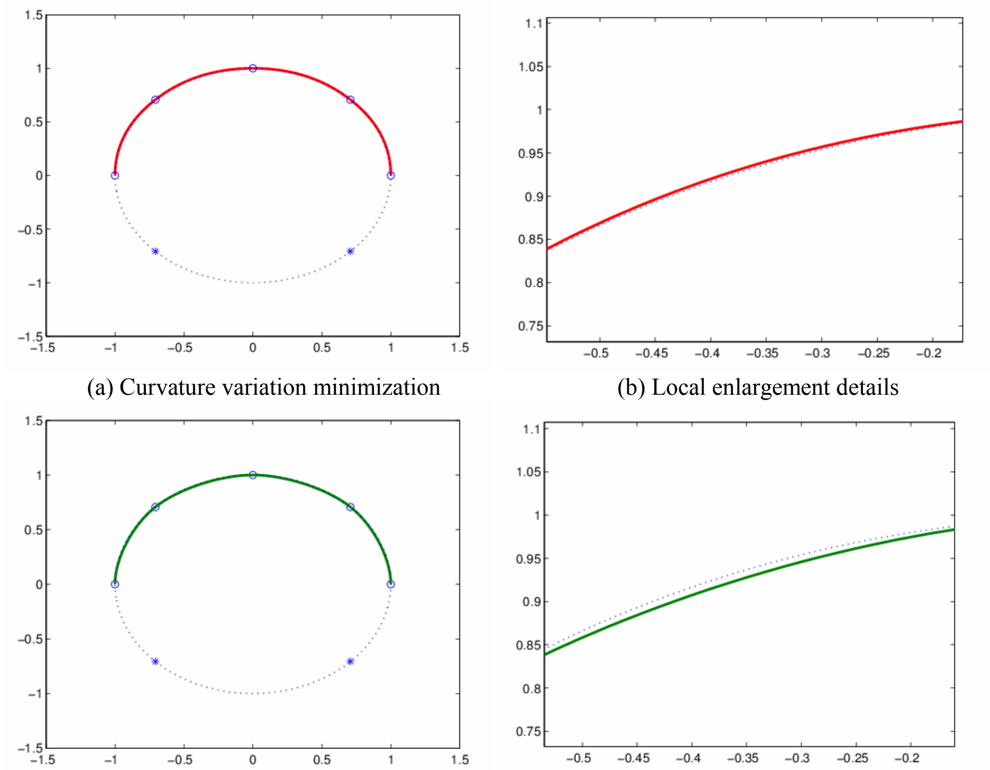
Curves	Value of the parameter	Curvature variation energy
Cardinal spline curves with minimum curvature variation	0.7509	1.9964×10^5
Catmull-Rom spline curves	0.5	2.4081×10^5

From Table 1 we can see that the curvature variation energy of the Cardinal spline curves obtained by curvature variation minimization is less than the Catmull-Rom spline curves, which means the Cardinal spline curves with minimum curvature variation are smoother than the Catmull-Rom spline curves.

The second example let us consider the data points taken from a half unit circle $\mathbf{r}(t) = (\cos t, \sin t)^{\mathrm{T}}$, $t = \frac{\pi}{4} i$, $i = 0,1,2,3,4$. The optimal parameter and auxiliary points calculated by (18)-(20) are

$$\alpha^* = 0.5858,\ \mathbf{p}_{-1}^* = (0.7071, -0.7071),\ \mathbf{p}_5^* = (-0.7071, -0.7071).$$

The Cardinal spline curves obtained by curvature variation minimization, the Catmull-Rom spline curves and the unit circle plot are shown in Figure 4, together with their local enlargement details.



(a) Curvature variation minimization (b) Local enlargement details

(c) Catmull-Rom spline curves (d) Local enlargement details

Figure 4. The Cardinal spline curves obtained by curvature variation minimization (solid lines above), the Catmull-Rom spline curves (solid lines below) and the unit circle plot (dotted lines), together with their local enlargement details (right).

From Figure 4 we can see that the interpolation effect of the Cardinal spline curves with minimum curvature variation minimization is better than the Catmull-Rom spline curves. The curvature variation energies of the Cardinal spline curves obtained by curvature variation minimization and the Catmull-Rom spline curves calculated by (10) are shown in Table 2.

Table 2. Comparison of the two curves in Figure 4

Curves	Value of the parameter	Curvature variation energy
Cardinal spline curves with minimum curvature variation	0.5858	1.8087×10^{-7}
Catmull-Rom spline curves	0.5	7.2364

Table 2 shows that the curvature variation energy of the Cardinal spline curves with minimum curvature variation is much less than the Catmull-Rom spline curves, which means the Cardinal spline curves with minimum curvature variation are smoother than the Catmull-Rom spline curves.

5. Conclusions

This paper presents a method for obtaining the smoothest Cardinal spline curves by minimizing the curvature variation. Our method focuses on how to find the optimal parameter and auxiliary points of the Cardinal spline curves, which is achieved by minimizing an appropriate approximation of the curvature variation energy. Some examples show that the Cardinal spline curves obtained by curvature variation minimization is better than the Catmull-Rom spline curves in dealing with interpolation problems.

Acknowledgements

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