

Fine Computability of Probability Distribution Functions and Computability of Probability Distributions on the Real Line

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(Received September 8, 2014)

Abstract

We continue our work in [9] on an effective relationship between the sequence of probability distributions and the corresponding sequence of probability distribution functions. In order to deal with discontinuous distribution functions, we define the notion of Fine topology on the whole real line, and show that, when a probability distribution is associated with a Fine continuous distribution function, the computability of the former and the sequential computability of the latter can be effectively mutually translatable under a certain condition. The effectivity of the translations is secured by the treatment of the sequences of the objects in concern. The equivalences of effective convergences will also be proved.

Keywords: Computable probability distribution, Effective convergence of probability distributions, Probability distribution functions, Fine computable functions, Effective Fine convergence

Introduction

The domains of our discourse are the real line, the real functions and the probability distributions on the real functions.

We are interested in the effective version of probability theory: the computability notion of the probability distributions as well as that of the probability distribution functions, their effective convergences and some related topics.

Recall that the convergence of a sequence of probability distributions, as well as that of a sequence of random variables is a fundamental concept in the theory of probability and statistics. This together with the convergence of a sequence of probability distribution functions is one of the major subjects in an elementary course of probability theory.

A probability distribution is a Borel probability measure on the real line.

For a probability distribution μ , the corresponding probability distribution function F is defined by $F(x) = \mu((-\infty, x])$. A probability distribution function is characterized by the following properties:

- (Fi) monotone non-decreasing;
- (Fii) right continuous;
- (Fiii) $\lim_{x \rightarrow -\infty} F(x) = 0$ and $\lim_{x \rightarrow \infty} F(x) = 1$.

1998 ACM Subject Classification: Computability theory, Distribution functions

This work has been supported in part by Scientific Foundations of JSPS No. 21540152 and No. 20540143.

Subsequently μ , μ_n , ν and ν_n will denote probability distributions, and F , F_n , G and G_n will denote the corresponding probability distribution functions. This correspondence is one-to-one and onto, and the following three convergences are equivalent (cf. [2]). $(\mu(f))$ represents $\int_{\mathbb{R}} f(x)\mu(dx)$, the integral of a real function f with respect to a probability distribution μ .

- (i) $\mu_n \rightarrow \mu$ weakly, that is, $\mu_n(f)$ converges to $\mu(f)$ for an arbitrary bounded continuous function f .
- (ii) $\mu_n \rightarrow \mu$ vaguely, that is, $\mu_n(f)$ converges to $\mu(f)$ for an arbitrary continuous function f with compact support.
- (iii) $F_n(x) \rightarrow F(x)$ at any x which is a continuity point of F .

The computability of probability distributions on the unit interval based on representation theory has been treated in [3], [12], and [15].

In [9], we have defined the computability and the effective convergence of probability distributions and those of the corresponding probability distribution functions in the scheme of Pour-El and Richards [11]. There, we have presented an effective treatment of probability distributions and the corresponding probability distribution functions under the assumption of the existence of the bounded densities. In such a case, it holds that the computability of $\{\mu_m\}$ is equivalent to the sequential computability of $\{F_m\}$ (Theorem 2:[9]). It has also been shown that a computable sequence of probability distributions $\{\mu_m\}$ converges effectively to a probability distribution μ if and only if $\{F_m\}$ converges effectively pointwise to F (Theorem 3:[9]).

The existence of a bounded density for a probability distribution implies the effective uniform continuity of the corresponding distribution function, which secures the desired equivalences above. Familiar distributions such as discrete probability distributions and Dirac distribution, however, possess no density functions. Furthermore, the probability distribution function corresponding to Dirac distribution is discontinuous. Nevertheless, Dirac distribution is computable according to our definition in [9].

This suggests that we should consider a wider class of probability distribution functions with a certain computability property in order to cope with the case of probability distributions which may not have density functions.

We have no general characterization of probability distribution functions which correspond to computable probability distributions, and hence we must start with some appropriate class of functions. It has turned out that the family of Fine computable functions (defined on the whole real line) is a good candidate for such a domain. We will therefore work on the domain of Fine computable sequences of functions (cf. [7], [10]) as the first step. Note that the distribution function of the Dirac probability distribution is Fine computable.

The value of the family of Fine computable sequences of functions lies in the following facts.

- (a) It is endowed with some computability properties and contains some probability distribution functions which correspond to some computable (sequences of) probability distributions.
- (b) The effective convergence of a function sequence can be defined within the domain.
- (c) It is closed under the effective convergence.

In Section 1, we briefly review the theories of the computability of real numbers, functions and distributions (cf. [11], [9]).

In Section 2, we extend the definition of Fine space to the whole real line, and summarize the basic facts of Fine computability (cf. [5], [8], [10], [16]).

In Section 3, we first prove that the sequential Fine computability (Definition 2.4) of probability distribution functions implies the computability of the corresponding probability distributions (Theorem 3.1). Then, we give an example which shows that the converse does not hold (Example 3.4). Finally, we prove that the computability of a sequence of probability distributions implies the sequential Fine computability of the corresponding sequence of probability distribution functions under the condition that the latter is effectively Fine continuous (Definition 2.5, Theorem 3.6).

In Section 4, we focus on the convergence problems. Let $\{F_m\}$ be a sequentially Fine computable sequence of probability distribution functions and let F be a sequentially Fine computable probability distribution function, with the corresponding distributions $\{\mu_m\}$ and μ respectively. Then, $\{\mu_m\}$ converges effectively to μ if $\{F_m\}$ converges effectively dyadic-irrationally pointwise to F (Definition 4.4, Theorem 4.5). On the other hand, if we assume further that F is effectively Fine continuous, then the effective convergence of $\{\mu_m\}$ to μ implies effective dyadic-irrationally pointwise Fine convergence of $\{F_m\}$ to F (Theorem 4.6).

1. Preliminaries

Let us first state the overall assumption that we will work with the real numbers and real functions, unless otherwise stated.

We first review briefly the introductory part of the computability theory on the real line developed by Pour-El and Richards [11] as well as some basics of computable probability distributions on the real line. A sequence of rational numbers $\{r_n\}$ is said to be *recursive* if there exist recursive functions α, β and γ such that $r_n = (-1)^{\gamma(n)} \frac{\beta(n)}{\alpha(n)}$. A sequence of real numbers $\{x_{m,n}\}$ is said to *converge effectively* to $\{x_m\}$ if there exists a recursive function $\alpha(m, k)$ such that $n \geq \alpha(m, k)$ implies $|x_{m,n} - x_m| < \frac{1}{2^k}$. A sequence of real numbers $\{x_m\}$ is said to be *computable* if there exists a recursive double sequence of rational numbers $\{r_{m,n}\}$ which converges effectively to $\{x_m\}$.

We adopt the definition of the computability of continuous real functions by Pour-El and Richards in Chapter 0 of [11]. In general, an object a is called computable if the sequence $\{a, a, \dots, a, \dots\}$ is computable.

A sequence of (real) functions $\{f_m\}$ is said to be *computable*, if it is (i) *sequentially computable*, that is, $\{f_m(x_n)\}$ is computable for any computable sequence of real numbers $\{x_n\}$, and (ii) *effectively continuous*, that is, there exists a recursive function $\alpha(m, p, k)$ such that $x, y \in [-p, p]$ and $|x - y| < \frac{1}{2^{\alpha(m, p, k)}}$ imply $|f_m(x) - f_m(y)| < \frac{1}{2^k}$. $\alpha(m, p, k)$ is called a modulus of effective continuity of $\{f_m\}$.

A sequence of (real) functions $\{f_m\}$ is said to be *uniformly computable*, if it is (i) sequentially computable and (ii) *effectively uniformly continuous*, that is, there exists a recursive function $\alpha(m, k)$ such that $|x - y| < \frac{1}{2^{\alpha(m, k)}}$ implies $|f_m(x) - f_m(y)| < \frac{1}{2^k}$. We call this $\alpha(m, k)$ a modulus of uniform continuity.

We say that $\{f_m\}$ is a *computable sequence of functions with compact support* $\{K_m\}$ if $\{f_m\}$ is a computable sequence of functions, $\{K_m\}$ is a recursive sequence of positive integers and $f_m(x) = 0$ for $|x| \geq K_m$. It is easy to prove that a computable sequence of functions with compact support is uniformly computable and the sequence of their maximums is a computable sequence of real numbers.

Now, we cite some definitions and properties in [9], which will be used in the main context

of this article.

Definition 1.1. (*Computability of Probability Distribution*) We say that a sequence of probability distributions $\{\mu_m\}$ is computable if it satisfies the following *vague sequential computability*: $\{\mu_m(f_n)\}$ is computable for any computable sequence of functions $\{f_n\}$ with compact support.

We say that a sequence of functions $\{f_n\}$ is *effectively bounded*, if there is a recursive function $M(n)$ with $|f_n(x)| \leq M(n)$ for all x and n .

Theorem 1.2. $\{\mu_m\}$ is vaguely sequentially computable if and only if it is weakly sequentially computable, that is, $\{\mu_m(f_n)\}$ is a computable double sequence of real numbers for any effectively bounded computable sequence of functions $\{f_n\}$.

Theorem 1.3. If a sequence of probability distribution functions $\{F_m\}$ is sequentially computable, then the corresponding sequence of probability distributions $\{\mu_m\}$ is computable.

Definition 1.4. (*Effective convergence of a sequence of probability distributions*)

A sequence of probability distributions $\{\mu_m\}$ is said to *effectively converge* to a probability distribution μ if $\{\mu_m(f_n)\}$ converges effectively to $\{\mu(f_n)\}$ for any computable sequence of functions $\{f_n\}$ with compact support.

Theorem 1.5. Let $\{\mu_m\}$ be a computable sequence of probability distributions which converges effectively to a probability distribution μ . Then μ is computable.

Theorem 1.6. Let $\{\mu_m\}$ be a computable sequence of probability distributions and μ be a probability distribution. Then, the effective converge of $\{\mu_m\}$ to μ is equivalent to the effective weak convergence, that is, $\{\mu_m(f_n)\}$ converges effectively to $\{\mu(f_n)\}$ for any effectively bounded computable sequence of functions $\{f_n\}$.

2. Fine computabilities

Fine topology and Fine computabilities were originally defined on the interval $[0, 1)$ ([5], [8], [10]). There is no difficulty in extending them to the whole real line \mathbb{R} .

We call an interval of the form $[\frac{i}{2^k}, \frac{j}{2^\ell})$ a dyadic interval, where k and ℓ are positive integers and i and j are integers.

The topology generated by the set of all dyadic intervals is called *Fine topology* on the real line.

Let $I(k, i) = [\frac{i}{2^k}, \frac{i+1}{2^k})$ and let $J(x, k)$ be the unique $I(k, i)$ which contains x . If we denote the integer part of a real number y with $[y]$, then $i = [2^k x]$ and $J(x, k) = [\frac{i}{2^k}, \frac{i+1}{2^k})$. $\{J(x, k)\}$ serves as a fundamental neighborhood system of x in this topology. It is also an effective uniformity, and hence Fine topology is an effective uniform topology ([13], [16]).

Fine computability on the real line \mathbb{R} is defined in terms of Fine topology analogously to the Euclidean computability defined by Pour-El and Richards (Section 1, [1], [6], [7], [8]).

Definition 2.1. (*Effective Fine convergence of real numbers*) A sequence of real numbers $\{x_{n,m}\}$ is said to Fine converge effectively to a sequence $\{x_n\}$ if there exists a recursive function $\alpha(n, k)$ such that $x_{n,m} \in J(x_n, k)$ for all $m \geq \alpha(n, k)$.

Definition 2.2. (*Fine computable sequence of real numbers*) A sequence of real numbers $\{x_n\}$ is called Fine computable if there exists a recursive sequence of rational numbers $\{r_{n,m}\}$ which Fine converges effectively to $\{x_n\}$.

We say a sequence of real numbers $\{x_n\}$ is *dyadic rational* if all x_n are dyadic rational and *dyadic irrational* if all x_n are dyadic irrational. We abbreviate ‘dyadic’ to ‘d’.

Definition 2.3. (*Recursive sequence of d-rationals*) A (double) sequence $\{r_{n,m}\}$ is called a recursive sequence of d-rationals if $r_{n,m} = (-1)^{\gamma(n,m)} \frac{\beta(n,m)}{2^{\alpha(n,m)}}$ for some recursive functions α, β and γ .

We list some properties of Fine computable sequences of real numbers.

Fact 1. A sequence of real numbers $\{x_n\}$ is Fine computable if and only if there exists a recursive sequence of d-rational numbers $\{r_{n,m}\}$ which Fine converges effectively to $\{x_n\}$.

Fact 2. A real number is Fine computable if and only if it is computable.

Fact 3. A Fine computable sequence is computable. The converse is not necessarily true ([1], [6]).

Fact 4. For a sequence of d-irrationals, computability and Fine computability are equivalent.

We say that a computable sequence is an effective separating set if it forms a dense subset.

Fact 5. An effective enumeration of all d-rationals $\{e_i\}$ forms an effective separating set with respect to Fine topology.

Although the Fine computability of real sequences is not necessarily closed under arithmetic operations, we can claim the following.

Fact 6. If $\{x_n\}$ and $\{y_n\}$ are Fine computable sequences of real numbers, then $\{\max\{x_n, y_n\}\}$ and $\{\min\{x_n, y_n\}\}$ are Fine computable.

Fact 7. Suppose that $\{x_n\}$ is a Fine computable sequence.

- (1) If $\{r_n\}$ is a recursive sequence of d-rational numbers, then $\{x_n + r_n\}$ is Fine computable.
- (2) For an integer k , $\{\frac{x_n}{2^k}\}$ is Fine computable.

We define some notions of the Fine computability of a sequence of functions ([4], [5], [8], [10]).

Definition 2.4. (*Sequential Fine computability*) We say that a sequence of functions $\{f_n\}$ is sequentially Fine computable if $\{f_n(x_m)\}$ is computable for any Fine computable sequence $\{x_m\}$.

Let $\{e_i\}$ be an effective enumeration of all d-rationals.

Definition 2.5. (*Effective Fine continuity*) A sequence of functions $\{f_n\}$ is said to be effectively Fine continuous if there exists a recursive function $\alpha(n, k, i)$ which satisfies the following (a) and (b).

- (a) $x \in J(e_i, \alpha(n, k, i))$ implies $|f_n(x) - f_n(e_i)| < 2^{-k}$.
- (b) $\bigcup_{i=1}^{\infty} J(e_i, \alpha(n, k, i)) = \mathbb{R}$ for each n, k .

Definition 2.6. (*Effective locally uniform Fine continuity*) A sequence of functions $\{f_n\}$ is said to be effectively locally uniformly Fine continuous if there exist recursive functions $\alpha(n, k, i)$ and $\beta(n, i)$ which satisfy the following (a) and (b).

- (a) For all i, n and k , $|f_n(x) - f_n(y)| < 2^{-k}$ if $x, y \in J(e_i, \beta(n, i))$ and $y \in J(x, \alpha(n, k, i))$.
- (b) $\bigcup_{i=1}^{\infty} J(e_i, \beta(n, i)) = \mathbb{R}$ for each n .

Definition 2.7. (*Effective uniform Fine continuity*) A sequence of functions $\{f_n\}$ is said to be effectively uniformly Fine continuous if there exists a recursive function $\alpha(n, k)$ such that, for all n, k and all $x, y, y \in J(x, \alpha(n, k))$ implies $|f_n(x) - f_n(y)| < 2^{-k}$.

Although the definitions of the effective Fine continuities of a sequence of functions apparently depend on the choice of $\{e_i\}$, it can be proved that they are in fact independent of the choice of $\{e_i\}$ (cf. [7], [10]).

Definition 2.8. (*Fine computabilities of sequences of functions*)

- (1) A sequence of functions $\{f_n\}$ is said to be Fine computable if it is sequentially Fine computable and effectively Fine continuous.
- (2) A sequence of functions $\{f_n\}$ is said to be locally uniformly Fine computable if it is sequentially Fine computable and effectively locally uniformly Fine continuous.
- (3) A sequence of functions $\{f_n\}$ is said to be uniformly Fine computable if it is sequentially Fine computable and effectively uniformly Fine continuous.

Definition 2.9. (*Effective Fine convergence of functions*, [8], [10]) We say that a sequence of functions $\{f_n\}$ *Fine converges effectively* to a function f if there exist recursive functions $\beta(k, i)$ and $\gamma(k, i)$ which satisfy the following (a) and (b).

- (a) $x \in J(e_i, \beta(k, i))$ and $n \geq \gamma(k, i)$ imply $|f_n(x) - f(x)| < 2^{-k}$.
- (b) $\bigcup_{i=1}^{\infty} J(e_i, \beta(k, i)) = \mathbb{R}$ for each k .

We can likewise define effective uniform Fine convergence and effective locally uniform Fine convergence.

3. Fine computability of probability distribution functions and the corresponding probability distributions

We have treated in [9] the computability problem of probability distributions with bounded densities. In that case, the corresponding distribution functions are uniformly computable. Here, we do not assume the existence of bounded densities for probability distributions, and try to characterize the corresponding distribution functions in a more general situation.

Theorem 3.1. *If a sequence of probability distribution functions $\{F_m\}$ is sequentially Fine computable, then the corresponding sequence of probability distributions $\{\mu_m\}$ is computable.*

Proof. Let $\{f_n\}$ be a computable sequence of functions with compact support $\{K_n\}$. We denote its modulus of effective uniform continuity with $\alpha(n, k)$. Let us define

$$x_{n,p,i} = -K_n + \frac{i}{2^p}, 0 \leq i \leq 2K_n 2^p \quad \text{and}$$

$$f_{n,p}(x) = \sum_{i=1}^{2K_n 2^p} f_n(x_{n,p,i}) \chi_{(x_{n,p,i-1}, x_{n,p,i}]}(x).$$

We note that $\{x_{n,p,i}\}$ is a recursive sequence of d-rationals and $|f_n(x) - f_{n,p}(x)| < \frac{1}{2^k}$ for all x if $p \geq \alpha(n, k)$.

On the other hand,

$$\mu_m(f_{n,p}) = \sum_{i=1}^{2K_n 2^p} f(x_{n,p,i})(F_m(x_{n,p,i}) - F_m(x_{n,p,i-1}))$$

and hence $\{\mu_m(f_{n,p})\}$ is a computable sequence of real numbers.

If we take $p \geq \alpha(n, k + 1)$, then

$$|\mu_m(f_n) - \mu_m(f_{n,p})| \leq \mu_m(|f_n - f_{n,p}|) \leq \sup_x |f_n(x) - f_{n,p}(x)| \leq \frac{1}{2^{k+1}} < \frac{1}{2^k}.$$

This proves the effective convergence of $\{\mu_m(f_{n,p})\}$ to $\{\mu_m(f_n)\}$. So $\{\mu_m(f_n)\}$ is computable. □

The proof above is similar to that of Theorem 1.3 except that here we need to choose a Fine computable real sequence. Theorem 3.1 is obviously stronger than Theorem 1.3.

The following two examples concern the case where no density is assumed, but the corresponding probability distribution functions are Fine computable.

Example 3.2. (Dirac distribution) Let δ_a be the translated Dirac distribution, that is,

$$\delta_a(A) = \begin{cases} 1 & \text{if } a \in A \\ 0 & \text{otherwise} \end{cases}.$$

Its probability distribution function satisfies

$$D_a(x) = \begin{cases} 0 & (x < a) \\ 1 & (x \geq a) \end{cases}.$$

Note that δ_a has no density function. $\delta_a(f) = f(a)$, and δ_a is computable if a is computable. D_a is neither sequentially computable nor effectively continuous even if a is computable. However, D_a is Fine computable if a is d-rational. □

Example 3.3. Let α be a one-to-one recursive function whose range is not recursive from the set of all positive integers to itself. We notice that $d = \sum_{i=1}^{\infty} \frac{1}{2^{\alpha(i)}} < 1$ and d is not computable. Let us define

$$\mu = \sum_{i=1}^{\infty} \frac{1}{2^{\alpha(i)}} \delta_{(1-\frac{1}{2^{i-1}})} + (1-d) \delta_1.$$

The probability distribution function of μ is

$$F = \sum_{i=1}^{\infty} (\sum_{j=1}^i \frac{1}{2^{\alpha(j)}}) \chi_{[1-\frac{1}{2^{i-1}}, 1-\frac{1}{2^i})} + \chi_{[1, \infty)}.$$

While F is not continuous and hence not computable, F is locally uniformly Fine computable, and so it is necessarily Fine computable. By Theorem 3.1, μ is computable. $F(1-0) = \lim_{x \uparrow 1} F(x) = d$ is not computable. So, F is not uniformly Fine computable. □

The next example shows that the converse of Theorem 3.1 does not hold.

Example 3.4. We take the same α as in Example 3.3, and define

$$\nu = (1-d)\delta_0 + \sum_{i=1}^{\infty} \frac{1}{2^{\alpha(i)}} \delta_{\frac{1}{2^i}}.$$

The corresponding probability distribution function G is:

$$G = (1-d)\chi_{[0,1)} + \sum_{i=1}^{\infty} \left(\sum_{j=i}^{\infty} \frac{1}{2^{\alpha(j)}} \right) \chi_{[\frac{1}{2^i}, \frac{1}{2^{i-1}})} + \chi_{[1, \infty)}.$$

$G(0)$ is not computable and hence G is not sequentially Fine computable. Moreover, G is not effectively Fine continuous, and yet we can prove that ν is computable.

For the proof, let f be a bounded computable function and put

$$s_m = \sum_{i=1}^m \frac{1}{2^{\alpha(i)}} f\left(\frac{1}{2^i}\right) + (1 - \sum_{i=1}^m \frac{1}{2^{\alpha(i)}}) f(0).$$

Then $\{s_m\}$ is a computable sequence of real numbers and, by the definition of the integral, it holds that

$$\nu(f) = (1-d)f(0) + \sum_{i=1}^{\infty} \frac{1}{2^{\alpha(i)}} f\left(\frac{1}{2^i}\right),$$

$$\begin{aligned} |\nu(f) - s_m| &= \left| \sum_{i=1}^m \frac{1}{2^{\alpha(i)}} f(0) - df(0) + \sum_{i=m+1}^{\infty} \frac{1}{2^{\alpha(i)}} f\left(\frac{1}{2^i}\right) \right| \\ &= \left| \sum_{i=m+1}^{\infty} \frac{1}{2^{\alpha(i)}} (f\left(\frac{1}{2^i}\right) - f(0)) \right| \leq \sup_{0 < x \leq \frac{1}{2^m}} |f(0) - f(x)| \end{aligned}$$

for each integer m . The last term converges to zero effectively as m tends to infinity by the effective continuity of f . This proves that $\nu(f)$ is computable.

The proof above is also valid for a bounded computable sequence of functions $\{f_n\}$. So ν is computable. \square

Remark 3.5. Example 3.4 shows that the sequential computability of a probability distribution function is stronger than the computability of the corresponding distribution. Indeed, the proof of Theorem 3.1 suggests that the existence of an effective separating set $\{x_n\}$ for which computability of $\{F_m(x_n)\}$ holds is sufficient for the computability of a sequence of probability distributions $\{\mu_m\}$. \square

For later use, let us define functions $w_{c,h}^+$ and $w_{c,h}^-$ for a real number c and a positive real number h as follows:

$$w_{c,h}^+(x) = \begin{cases} 1 & \text{if } x \leq c \\ -\frac{1}{h}(x-c) + 1 & \text{if } c \leq x \leq c+h \\ 0 & \text{if } x \geq c+h \end{cases},$$

$$w_{c,h}^-(x) = \begin{cases} 1 & \text{if } x \leq c-h \\ -\frac{1}{h}(x-c) & \text{if } c-h \leq x \leq c \\ 0 & \text{if } x \geq c \end{cases}.$$

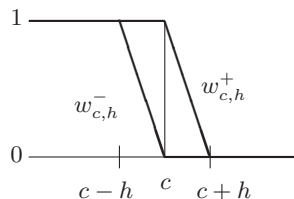


Figure 1:
 $w_{c,h}^+(x)$ and $w_{c,h}^-(x)$

Notice that, for computable c and h , $w_{c,h}^+$ and $w_{c,h}^-$ are computable functions. These functions satisfy

$$\chi_{(-\infty, c-h]}(x) \leq w_{c,h}^-(x) \leq \chi_{(-\infty, c]}(x) \leq w_{c,h}^+(x) \leq \chi_{(-\infty, c+h]}(x).$$

Furthermore, for any probability distribution μ and its probability distribution function F , it holds that

$$F(c - h) \leq \mu(w_{c,h}^-) \leq F(c) \leq \mu(w_{c,h}^+) \leq F(c + h).$$

Suppose μ is computable, c is a computable real number, and $\{h_n\}$ is a computable sequence of positive real numbers which converges effectively and decreasingly to zero. Then $\{\mu(w_{c,h_n}^+)\}$ is a computable sequence of real numbers and converges monotonically downwards to $F(c)$. F is thus right computable in the sense of [17].

The next theorem claims that, although the converse of Theorem 3.1 does not hold in general circumstances, it can hold on a restricted family of distribution functions.

Theorem 3.6. *Let $\{\mu_m\}$ be a computable sequence of probability distributions. If the corresponding sequence of probability distribution functions $\{F_m\}$ is effectively Fine continuous, then $\{F_m\}$ is sequentially Fine computable.*

Proof. Assume that $\{F_m\}$ is effectively Fine continuous with respect to $\alpha(m, k, i)$, that is, $x \in J(e_i, \alpha(m, k, i))$ implies $|F_m(x) - F_m(e_i)| < \frac{1}{2^k}$ and $\bigcup_{i=1}^\infty J(e_i, \alpha(m, k, i)) = \mathbb{R}$.

Assume further that $\{\mu_m\}$ is computable and $\{x_n\}$ is a Fine computable sequence of real numbers.

For each m, n, k , we can find effectively $i = i(m, n, k)$ such that $x_n \in J(e_i, \alpha(m, k + 1, i))$. With such i , $J(e_i, \alpha(m, k + 1, i)) = [\frac{\ell}{2^{\alpha(m, k + 1, i)}}, \frac{\ell + 1}{2^{\alpha(m, k + 1, i)}})$ for some integer $\ell = \ell(m, k, i)$, which is obtained effectively. Therefore, $\{\frac{\ell + 1}{2^{\alpha(m, k + 1, i)}}\}$ is a recursive d-rational sequence.

Put $y_{m,n,k} = \frac{1}{2}(x_n + \frac{\ell + 1}{2^{\alpha(m, k + 1, i)}})$ and $z_{m,n,k} = \mu(w_{x_n,h}^+)$, where $h = \frac{1}{2}(\frac{\ell + 1}{2^{\alpha(m, k + 1, i)}} - x_n)$. Then $\{y_{m,n,k}\}$ is Fine computable by virtue of Fact 7 and $\{z_{m,n,k}\}$ is a computable sequence of real numbers. By definition,

$$x_n, y_{m,n,k} \in J(e_i, \alpha(m, k + 1, i)),$$

$$\chi_{(-\infty, x_n]} \leq w_{x_n,h}^+ \leq \chi_{(-\infty, y_{m,n,k}]}$$

Therefore,

$$|F_m(x_n) - z_{m,n,k}| \leq |F_m(x_n) - F_m(y_{m,n,k})| \leq |F_m(x_n) - F_m(e_i)| + |F_m(e_i) - F_m(y_{m,n,k})| < \frac{1}{2^k}.$$

Thus, $\{z_{m,n,k}\}$ converges effectively to $\{F_m(x_n)\}$ and hence $\{F_m(x_n)\}$ is computable. \square

4. Effective convergence of probability distributions and probability distribution functions

We start this section with the following examples of convergent sequences of probability distributions.

Example 4.1. Let δ_a be the translated Dirac distribution in Example 3.2. If we define $\mu_m = \delta_{\frac{1}{2^m}}$, then $\mu_m(f) = f(\frac{1}{2^m})$ converges to $f(0) = \delta_0(f)$ for a continuous f . This convergence is effective if f is computable and hence $\{\mu_m\}$ converges effectively to δ_0 .

Since $F_m(x) = 0$ if $x < \frac{1}{2^m}$ and $F_m(x) = 1$ if $x \geq \frac{1}{2^m}$, $\{F_m(0)\}$ does not converge to $D_0(0)$, although $\{F_m(x)\}$ converges to $D_0(x)$ uniformly on $(-\infty, 0) \cup [\frac{1}{n}, \infty)$ for any positive integer n . \square

The above example shows that the convergence of probability distributions does not imply the convergence of probability distribution functions.

Example 4.2. The uniform probability distribution on $[a, b]$ is the distribution with density $u_{a,b}(x) = \frac{1}{b-a} \chi_{[a,b]}(x)$. Its distribution function $U_{a,b}$ is given by

$$U_{a,b}(x) = \begin{cases} 0 & \text{if } x < a \\ \frac{x-a}{b-a} & \text{if } a \leq x < b \\ 1 & \text{if } x \geq b \end{cases} .$$

$u_{a,b}(x)$ is bounded, but not computable even if a and b are computable. On the other hand, $U_{a,b}$ is uniformly computable if a and b are computable.

Take a computable a and put $b_m = a + \frac{1}{m}$. If we denote the corresponding distribution μ_m , then $\{\mu_m\}$ converges effectively to δ_a , while $U_{a,a+\frac{1}{m}}(a) = 0$ does not converge to $D_a(a) = 1$. \square

Example 4.3. Let μ be the probability distribution in Example 3.3. If we define μ_m by

$$\mu_m = \sum_{i=1}^m \frac{1}{2^{\alpha(i)}} \delta_{(1-\frac{1}{2^{i-1}})} + (1 - \sum_{i=1}^m \frac{1}{2^{\alpha(i)}}) \delta_1,$$

then $\{\mu_m\}$ converges effectively to μ .

For the ν in Example 3.4, it holds that

$$\nu_m = (1 - \sum_{i=1}^m \frac{1}{2^{\alpha(i)}}) \delta_0 + \sum_{i=1}^m \frac{1}{2^{\alpha(i)}} \delta_{\frac{1}{2^i}}$$

converges effectively to ν . \square

Definition 4.4. (Effective dyadically irrationally (d -irrationally) pointwise convergence) Let $\{F_m\}$ be a sequence of probability distribution functions and let F be a probability distribution function. We say that $\{F_m\}$ converges effectively d -irrationally pointwise to F if $\{F_m(x_n)\}$ converges effectively to $\{F(x_n)\}$ for any computable d -irrational sequence $\{x_n\}$.

Theorem 4.5. Let $\{F_m\}$ be a sequence of probability distribution functions and F be a probability distribution function. Let $\{\mu_m\}$ and μ be the corresponding probability distributions. If $\{F_m\}$ converges effectively d -irrationally pointwise to F , then $\{\mu_m\}$ converges effectively to μ .

Proof. We consult and effectivize the proof in [14].

We prove that $\{\mu_m(f)\}$ converges effectively to $\mu(f)$ for a computable function f with compact support. Let $\gamma(k)$ be a modulus of uniform continuity of f , K be an integer such that $f(x) = 0$ if $|x| \geq K$ and M be an integer such that $|f(x)| \leq M$ for any x .

Put

$$x_{p,i} = -K - \frac{1}{3} + \frac{i}{2^p}, 0 \leq i \leq (2K+1)2^p.$$

Then $\{x_{p,i}\}$ is a computable d -irrational sequence, and $\{F_m(x_{p,i})\}$ converges effectively to $\{F(x_{p,i})\}$ by virtue of the assumption, that is, there exists a recursive function $\beta(p, i, q)$ such that $m \geq \beta(p, i, q)$ implies

$$|F_m(x_{p,i}) - F(x_{p,i})| < \frac{1}{2^q}. \quad (4.1)$$

Let us define

$$f_p(x) = \sum_{i=1}^{(2K+1)2^p} f(x_{p,i}) \chi_{(x_{p,i-1}, x_{p,i}]}(x). \quad (4.2)$$

Then, for any $p \geq \gamma(k)$, $|f(x) - f_p(x)| < \frac{1}{2^k}$, hence

$$|\mu(f) - \mu(f_p)| \leq \frac{1}{2^k}, \text{ and } |\mu_m(f) - \mu_m(f_p)| \leq \frac{1}{2^k} \text{ for all } m. \tag{4.3}$$

$$|\mu_m(f) - \mu(f)| \leq |\mu_m(f) - \mu_m(f_p)| + |\mu_m(f_p) - \mu(f_p)| + |\mu(f_p) - \mu(f)|. \tag{4.4}$$

By Equation (4.2)

$$\mu(f_p) = \sum_{i=1}^{(2K+1)2^p} f(x_{p,i})(F(x_{p,i}) - F(x_{p,i-1})), \tag{4.5}$$

$$\mu_m(f_p) = \sum_{i=1}^{(2K+1)2^p} f(x_{p,i})(F_m(x_{p,i}) - F_m(x_{p,i-1})). \tag{4.6}$$

For a k with $p = \gamma(k + 1)$ in (4.3), we have

$$|\mu_m(f) - \mu_m(f_{\gamma(k+2)})| \leq \frac{1}{2^{k+2}} \text{ and } |\mu(f_{\gamma(k+2)}) - \mu(f)| \leq \frac{1}{2^{k+2}}. \tag{4.7}$$

For any q and $m \geq \max_{0 \leq i \leq (2K+1)2^{\gamma(k+2)}} \beta(\gamma(k+2), i, q)$, by Equations (4.1), (4.5) and (4.6)

$$\begin{aligned} |\mu_m(f_{\gamma(k+2)}) - \mu(f_{\gamma(k+2)})| &\leq \sum_{i=1}^{(2K+1)2^{\gamma(k+2)}} (|f(x_{\gamma(k+2),i})| |F_m(x_{\gamma(k+2),i}) - F(x_{\gamma(k+2),i})| \\ &\quad + |f(x_{\gamma(k+2),i})| |F_m(x_{\gamma(k+2),i-1}) - F(x_{\gamma(k+2),i-1})|) \\ &\leq \frac{2M(2K+1)2^{\gamma(k+2)}}{2^q}. \end{aligned} \tag{4.8}$$

If we put $q = 6 + M + K + 1 + \gamma(k + 2) + k$ in Equation (4.8), then it holds

$$|\mu_m(f_{\gamma(k+2)}) - \mu(f_{\gamma(k+2)})| < \frac{1}{2^{k+2}}.$$

Combining this with Equations (4.4) and (4.7), we obtain

$$|\mu_m(f) - \mu(f)| < \frac{1}{2^k}.$$

This proves the effective convergence of $\{\mu_m(f)\}$ to $\mu(f)$.

We can easily extend the argument to a sequence of functions $\{f_n\}$. □

Theorem 4.5 corresponds to Proposition 17 in [9]. In fact, Theorem 4.5 is a stronger version of Proposition 17 in [9], since here we assume only d-irrational Fine convergence.

Theorem 4.6. *Let $\{F_m\}$ be a sequentially Fine computable sequence of probability distribution functions and let F be a sequentially Fine computable probability distribution. Let $\{\mu_m\}$ and μ be the corresponding probability distributions. Assume further that F is effectively Fine continuous. Then the effective convergence of $\{\mu_m\}$ to μ implies the effective d-irrationally pointwise convergence of $\{F_m\}$ to F .*

Proof. As in the previous proof, we consult the proof in [14].

We assume that $\{\mu_m\}$ effectively converges to μ and that F is effectively Fine continuous with respect to $\alpha(k, i)$, and prove that $\{F_m(c)\}$ converges effectively to $F(c)$ for any (Fine) computable d-irrational c . This argument can be easily generalized to a Fine computable sequence of d-irrational numbers.

Let us take the functions $w_{c,h}^+(x)$ and $w_{c,h}^-(x)$ in Figure 1. It holds that, for any $h > 0$,

$$\begin{aligned} \mu_m(w_{c,h}^-) &\leq F_m(c) \leq \mu_m(w_{c,h}^+), \\ F(c-h) &\leq \mu(w_{c,h}^-) \leq F(c) \leq \mu(w_{c,h}^+) \leq F(c+h). \end{aligned} \tag{4.9}$$

If c and h are computable, then $\mu_m(w_{c,h}^+)$ and $\mu_m(w_{c,h}^-)$ respectively converge effectively to $\mu(w_{c,h}^+)$ and $\mu(w_{c,h}^-)$, respectively, due to Theorem 1.6.

On the other hand, we can find effectively $i = i(k)$ and $\ell = \ell(k)$ such that $c \in J(e_i, \alpha(k, i)) = [\frac{\ell}{2^{\alpha(k, i)}}, \frac{\ell+1}{2^{\alpha(k, i)}})$. It holds then that

$$\frac{\ell}{2^{\alpha(k, i)}} < c < \frac{\ell+1}{2^{\alpha(k, i)}} \quad \text{and} \quad |F(x) - F(e_i)| < \frac{1}{2^k}$$

for any $x \in J(e_i, \alpha(k, i))$ by virtue of the effective Fine continuity of F . Let us take $h_k = \frac{1}{2} \min\{c - \frac{\ell}{2^{\alpha(k, i)}}, \frac{\ell+1}{2^{\alpha(k, i)}} - c\}$. Then $\{c + h_k\}$ and $\{c - h_k\}$ are Fine computable d-irrational sequences and they are contained in $J(e_i, \alpha(k, i))$. Hence, by the assumption on $\{\mu_m\}$ and μ , $\{\mu_m(w_{c, h_k}^+)\}$ and $\{\mu_m(w_{c, h_k}^-)\}$ respectively converge effectively to $\mu(w_{c, h_k}^+)$ and $\mu(w_{c, h_k}^-)$. We denote the corresponding moduli of convergence with $\beta^+(k, j)$ and $\beta^-(k, j)$ respectively. This means that $m \geq \beta^+(k, j)$ implies $|\mu_m(w_{c, h_k}^+) - \mu(w_{c, h_k}^+)| < \frac{1}{2^j}$ and $m \geq \beta^-(k, j)$ implies $|\mu_m(w_{c, h_k}^-) - \mu(w_{c, h_k}^-)| < \frac{1}{2^j}$. Inequalities in (4.9) imply

$$|\mu(w_{c, h_k}^-) - \mu(w_{c, h_k}^+)| \leq |F(c + h_k) - F(c - h_k)| \leq |F(c + h_k) - F(e_i)| + |F(e_i) - F(c - h_k)| < \frac{2}{2^k}. \quad (4.10)$$

If $m \geq \max\{\beta^+(k+2, k+2), \beta^-(k+2, k+2)\}$, then by virtue of (4.9) again

$$\mu(w_{c, h_{k+2}}^-) - \frac{1}{2^{k+2}} < \mu_m(w_{c, h_{k+2}}^-) \leq F_m(c) \leq \mu_m(w_{c, h_{k+2}}^+) < \mu(w_{c, h_{k+2}}^+) + \frac{1}{2^{k+2}}.$$

This with $\mu(w_{c, h_{k+2}}^-) \leq F(c) \leq \mu(w_{c, h_{k+2}}^+)$ implies

$$|F_m(c) - F(c)| < |\mu(w_{c, h_{k+2}}^-) - \mu(w_{c, h_{k+2}}^+)| + \frac{1}{2^{k+2}}.$$

Combining this with (4.10), we obtain $|F_m(c) - F(c)| < \frac{3}{2^{k+2}} < \frac{1}{2^k}$. □

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実数上の確率分布関数のファイン計算可能性と実軸上の 確率分布の計算可能性

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要 旨

本論文では、文献 [9] における実数上の確率分布列と対応する確率分布関数列の間の実効的関連の研究を継続する。不連続関数の計算可能性概念の導入のために、実数全体にファイン位相を与える。主な結果は、確率分布関数がファイン連続な確率分布に対しては、確率分布関数の計算可能性と確率分布の列計算可能性が、ある条件のもとで、互いに実効的に移行可能であること、および、移行の実効性は列に対しても成り立つことである。さらに、実効的収束の同値性も示される。ファイン計算可能関数を扱う理由はその族が実効的収束について閉じているからである。

キーワード：計算可能確率分布，確率分布の実効的収束，確率分布関数，ファイン計算可能関数，実効的ファイン収束