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An introduction to the theory of quantum groups

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An Introduction to the Theory of Quantum Groups

by

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Abstract

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This thesis is meant to be an introduction to the theory of quantum groups, a new and exciting field having deep relevance to both pure and applied mathematics. Throughout the thesis, basic theory of requisite background material is developed within an overarching categorical framework. This background material includes vector spaces, algebras and coalgebras, bialgebras, Hopf algebras, and Lie algebras. The understanding gained from these subjects is then used to explore some of the more basic, albeit important, quantum groups. The thesis ends with an indication of how to proceed into the deeper areas of the theory.

Acknowledgements

First, and foremost, I would like to thank my wife, for her patience and encouragement throughout this process. A very deep and special thanks goes to Dr. Ron Gentle, my thesis advisor, for all his help, suggestions and insight. Similarly, I would like to thank Dr. Dale Garraway for his input and for serving, with Dr. Achin Sen, on my thesis committee.

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For my wife Tara

Chapter 1

The Advent of Quantum Groups

1.1 Introduction

At the writing of this thesis the theory of quantum groups is a young and burgeoning area of study. The excitement surrounding the theory is due to its implications for both pure and applied mathematics. My particular interest in the subject was aroused on both accounts. I was actually introduced to the concept while reading a book on the mathematical structure of quantum mechanics. Given my affinity for both pure mathematics and mathematical physics, quantum groups was a very clear choice for me. And what aspiring mathematician wouldn't be thrilled to participate in a new and exciting area of research? This, however, is a double edged sword since although there is a lot of potential for one to contribute, brand new mathematical ideas are generally very involved, complicated, abstract and just plain difficult. They are built on and blossom from deep, as well as broad, mathematical ideas. One cannot hope to simply dive in, but needs a diverse wealth of mathematical background just to get started.

Suffice it to say, this is what I very clearly discovered while writing this thesis and, to a large extent, is why it turned out so long. Even despite the length, I was only able to address the very basics of the theory. Nevertheless, the study was most worthwhile and enlightening, giving me occasion to greatly expand my mathematical knowledge and understanding as well as deepen my understanding of what I was taught in my course work. It is my hope that the reader will gain a similar benefit from exploring this thesis and will appreciate the beauty of this most fascinating subject.

1.1.1 Basic Description

To some extent, quantum groups almost sound like science fiction, especially given the weirdness surrounding the discoveries of quantum physics. So, just what are these exciting new structures called *quantum groups*? It's always good to be honest at the outset of a significant undertaking. With that said, the reader might be disappointed to learn that there is no rigorous, universally accepted definition of the term *quantum group*. However, this has not prevented the development of a rich, powerful and elegant theory with an ever broadening horizon of application. Interestingly, there is also a significant collection of examples for which mathematicians in general can say, "Yeah, that's a quantum group." The situation is reminiscent of the more common difficulty of defining terms like "love". Nevertheless, we can often identify very clear examples. This is not to say that identifying quantum groups is merely a matter of judgement, but only that there are several fruitful and fascinating approaches to the subject which lead to broadly similar structures. For instance, one might take a purely algebraic approach or one might view the matter from a functional-analytic perspective. What is universally agreed upon is that the underlying ideas of quantum groups are (a) algebraic and non-commutative geometry, (b) deformations of "classical" objects and (c) the category of quantum groups should correspond to the opposite category of the category of Hopf algebras. These will become clearer in the next section when we explore the relatively short history of this exciting area of study.

The name *quantum group* is actually something of a misnomer, since they are not really groups at all. In light of (c) above, one common interpretation of quantum groups is that they are a particular kind of *Hopf algebra*, which one can intuitively think of as a structure rich generalization of a group. In general, a Hopf algebra may or may not be commutative or cocommutative. By a "special kind", then, we mean that quantum groups are Hopf algebras of the non-commutative *and* non-cocommutative type.

Now, there are several ways to understand how quantum groups generalize standard groups. First, the reader might recall that groups have a strong affiliation with symmetries. That is to say, groups can be thought of as collections of transformations which act on other objects. Quantum groups also possess this ability to *act* on structures. The difference, however, is that whereas all transformations in a group are invertible, such is not the case with quantum groups. Thus, with groups, it is always possible, by definition, to define an inverse map on the group in question and in case the group is abelian the inverse map becomes an automorphism. For quantum groups one has a similar, albeit weaker, version of an inverse mapping. This mapping is called an antipode and, unlike an inverse mapping, it is not required that the antipode applied to itself be the identity. Instead, only certain linear combinations become invertible. This is referred to by [1] as a "non-local linearized inverse".

Even though the antipode is a relaxed version of the inverse, some remarkable properties are preserved in the generalization from groups to quantum groups. For instance, [1] notes that like groups, quantum groups can act on themselves in an adjoint representation. Also, the antipode, like the inverse, provides a corresponding conjugate representation for every representation of a quantum group.

In this thesis we shall see extensive use of a very important concept in many areas of mathematics called the *tensor product*, which is, in essence, the most general bilinear operation possible. This provides a second way of understanding how quantum groups generalize regular groups. Specifically, representations of groups are known to admit a tensor product. This holds for representations of quantum groups as well. As before, however, there are some underlying modifications that come with this due to the greater generality of quantum groups. Naturally, physicists probably lean toward this understanding of quantum groups given the prevalence of tensors in the field.

Yet another way of understanding quantum groups is as self-dual objects. For instance, Hopf algebras have the property that their dual linear spaces are also algebras. This view has natural applications in physical quantum theory. Similarly, the notion of a quantum group as a sort of non-commutative geometry is essential to quantum field theories. To aid in our understanding of what this all means let us start from the beginning.

1.2 Background

One of the "miracles" of reality is that it appears to be "written" in the language of mathematics. It is not a rare occasion that a bit of mathematics is developed with no physical application in mind, yet sometime later finds itself as an indispensable description of some aspect of our universe and its workings. However, this is not a one way street. It can also happen that scientific undertakings inform our mathematics, inspiring new ideas about what is or may be possible. The theory of quantum groups happens to be an occasion of this. Their birthplace is in the work of theoretical and mathematical physics, it being no accident that the adjective "quantum" suggests a strong kinship with quantum mechanics in particular. Let us begin therefore with a synopsis of the transition from classical to quantum mechanics.

The quantum revolution began in the 1920's. Without plowing through the details of the various experiments which called out for a complete overhaul of our understanding of reality, suffice it to say that the intuitive heritage built and handed down to us failed miserably when applied at the atomic level. Our goal here will be to survey the accompanying mathematical shift, which ultimately led to the advances in mathematics considered in this thesis.

1.2.1 The Mathematical Structure of Classical Mechanics

A large part of physics involves studying physical systems and how they evolve over time. Toward this end, one considers the *phase space* of a physical system which is a manifold consisting of points representing all possible states of a particular system. Each state/point P is described by a set of canonical variables $\{p, q\}$ where $p = (p_1, ..., p_n)$ and $q = (q_1, ..., q_n)$. Physical quantities that can be measured (e.g. position and momentum) are called *observables* of the system and refer to polynomials in p, q along with real continuous functions $f(p, q)$ on the phase space. The important mathematical structures studied in this context are the theory of functions and first order differential equations on phase space manifolds.

Arising out of this is an algebra of observables or, more precisely, these observables give rise to an *abelian* algebra A of real (or more generally complex) continuous functions on the phase space. One of the immediate downfalls of this approach is the erroneous assumption that the canonical variable can be measured with infinite precision, hence the identification of a state with a unique point. What we have so far described, then, is merely an idealization. In practice, infinite precision cannot be obtained which means that there is always a measure of error involved. This leads to statistical mechanics which deals with probability distributions rather than strict points. What exactly this means and/or entails, however, is irrelevant to this thesis. The important issue is that the associated algebra and hence geometry in the classical case is commutative.

1.2.2 The Mathematical Structure of Quantum Mechanics

The gist of quantum mechanics is that observables can only be measured with finite precision. For instance, if one wants to measure the position of an electron with greater and greater accuracy, then more energy must be inserted into the system which inevitably changes its momentum. Ultimately, this means that there is a tradeoff in how accurately one can measure both position and momentum simultaneously. This idea is summarized by the now famous *Heisenberg uncertainty relations*. The resulting implication is that it matters in what order one measures observables like position and momentum, which is characterized by the *Heisenberg commutation relations*

$$q_j p_k - p_k q_j = i\hbar\delta_{jk}\mathbf{1}$$

where $\mathbf{1}$ is the multiplicative identity of the algebra, δ_{jk} is the Kronecker delta function and $\hbar$ is Planck's constant. The implications of this simple commuting relation are enormous. It essentially says that the foundations of reality require a non-commutative language and that the so called classical mechanics can be viewed as a limiting case. That is, the algebra of observables becomes commutative as $\hbar \to 0$. In this sense, then, we can think of the quantum case as a particular *deformation* of classical mechanics.

Of course, the above is a monumental simplification of the transition from classical to quantum mechanics. For those interested in a more in depth treatment see [2] and [3].

1.2.3 Quantum Groups Emerge

Fast forward to the early 1980's. One of the problems of interest was understanding exactly solvable models in quantum mechanics, which involves integrable systems. Two key tools for this area of study are the quantum inverse scattering method and the quantum Yang-Baxter equation. From this emerged the first quantum group to be written down, namely the quantum analogue or q-analogue of $SU(2)$ which is the special unitary group of degree 2 consisting of all 2×2 unitary matrices with determinant 1. Unitary matrices are such that for any matrix $U \in SU(2)$

$$UU^{\dagger} = U^{\dagger}U = I$$

where $U^{\dagger}$ indicates the conjugate transpose of U. The key participants were Kulish, Reshetikhin, Sklyanin, Takhtajan, and L.D. Faddeev working with quantum inverse scattering to study integral systems in quantum field theory. In short, the quantum inverse scattering method is a means of finding exact solutions of two-dimensional models in quantum field theory and statistical physics. While inverse scattering was central to the development of quantum groups, the details are a bit physics heavy. The reader is therefore referred to [4] for more details regarding the physical theory behind inverse scattering.

Though the q-analogue of $SU(2)$ was the first discovered quantum group, it was not known as such. The actual name "quantum group" was coined by V.G. Drinfel'd in 1985 who, along with M. Jimbo, also did extensive work in the area of integrable systems. At first, quantum groups were understood to be associative algebras whose defining relations are expressed in terms of a matrix of constants known as a quantum R-matrix. Universal R-matrices are also attributed to Drinfel'd. In the same year, Drinfel'd and Jimbo independently observed that these algebras are really Hopf algebras. Hopf algebras themselves were not novel at this time, but were introduced in the 50's and since the 60's have been examined in depth. While the language of Hopf algebras has more than proved useful, the important feature of these particular Hopf algebras is that they are deformations of universal enveloping algebras of Lie algebras as well as classical matrix groups. This gives an idea behind the motivation of the term *quantum* group, since it closely resembles the notion of quantum mechanics as a deformation of classical mechanics. Drinfel'd introduced this new object along with ground breaking examples at the International Congress of Mathematics in 1986. Not long after, non-commutative deformations of the algebra of functions on $SL_2(\mathbb{C})$ and $SU(2)$ were independently constructed by Yu. I. Manin and S.L. Woronowicz.

These deformations were originally intended to aid in the construction of solutions to the now famous Yang-Baxter equation. This equation is of significant importance to modern theoretical physics. In fact, it can very well be considered the basis of quantum group theory [5], since solutions to the Yang-Baxter equation provide a starting point for the quantum inverse scattering method [6], which, as mentioned above, is what led to the discovery of quantum groups in the first place. Today, it is believed that quantum groups provide the necessary framework for solving the holy grail of physics, namely the unification of quantum mechanics with gravity. This alone makes quantum groups an appealing and intriguing area of study.

Since their inception, however, quantum groups have graduated from their physics nursery to have far reaching effects in pure mathematics. For instance, quantum groups have asserted their influence in such areas as category theory, representation theory, topology, analysis, combinatorics, non-commutative geometry, symplectic geometry and knot theory to name a few. The rapid growth of this theory unfortunately means that this thesis will have to refrain from commenting on most of these fascinating applications and focus on a very narrow slice of the theory. The goal is to present a sufficient algebraic basis for entering this exciting world which is pregnant with possibility and has a richness of theory promising to lead to ever greater discoveries.

1.3 Overview of Approach

Now that we have some idea of where quantum groups came from and their usefulness, it will be good to lay out a general schematic for this thesis. As mentioned above, the course followed in this thesis is algebraic in nature. We shall therefore embrace the view adopted by [7] and [1]. Chapter 2 is meant to be something of a review of essential structures such as vector spaces and modules, but we will also develop some specifically important concepts such as the tensor product and duality.

Beginning in Chapter 3 the material will very likely become less familiar. We introduce the notion of an algebra along with the dual notion of a coalgebra. Some basics of their theory will be considered, but the primary focus will be on the connection between them. This will lead us to Chapter 4 where we first consider bialgebras, which are a precursor to Hopf algebras. Since quantum groups are considered to be a special class of Hopf algebra the greater portion of Chapter 4 endeavors to introduce the theory of Hopf algebras in general, elucidating their unique features and providing a survey of notable results. Admittedly, Chapter 4 is heavily focused on theory, but three central examples of Hopf algebras will be considered to facilitate understanding, namely the group algebra, $GL(2)$ and $SL(2)$, the latter two being related.

A large part of quantum group theory involves Lie groups and Lie algebras. Chapter 5 therefore takes something of a detour to explicate this important facet. While Lie groups are especially relevant to the origin of quantum groups, emphasis is placed more heavily on Lie algebras, since Lie groups lead into analytic methods, while Lie algebras lead into algebraic methods, the latter coinciding with our approach.

Chapter 6 finally introduces the reader to the quantum setting with an invitation to the quantum plane, which is a nice and simple example of a deformation of a classical object, in this case the affine plane. Certain features, such as a quantum calculus, are briefly discussed. In this chapter, the reader will also meet two well known examples of quantum groups which act and coact on the quantum plane. These are $GL_q(2)$ and $SL_q(2)$, which the reader may have gathered, are deformations of $GL(2)$ and $SL(2)$ respectively.

As a grand finale, Chapter 7 gives the reader a taste of the more involved quantum groups. In particular, $U_q(\mathfrak{sl}(2))$ is examined, which draws heavily upon material from Chapter 5, but also calls upon Chapter 4. Again, the central idea is deformation. Ties will also be made to material in Chapter 6 regarding the action of this quantum group on the quantum plane and its duality with $SL_q(2)$.

Below is a category diagram, which provides something of a "road map" for our ensuing investigations.

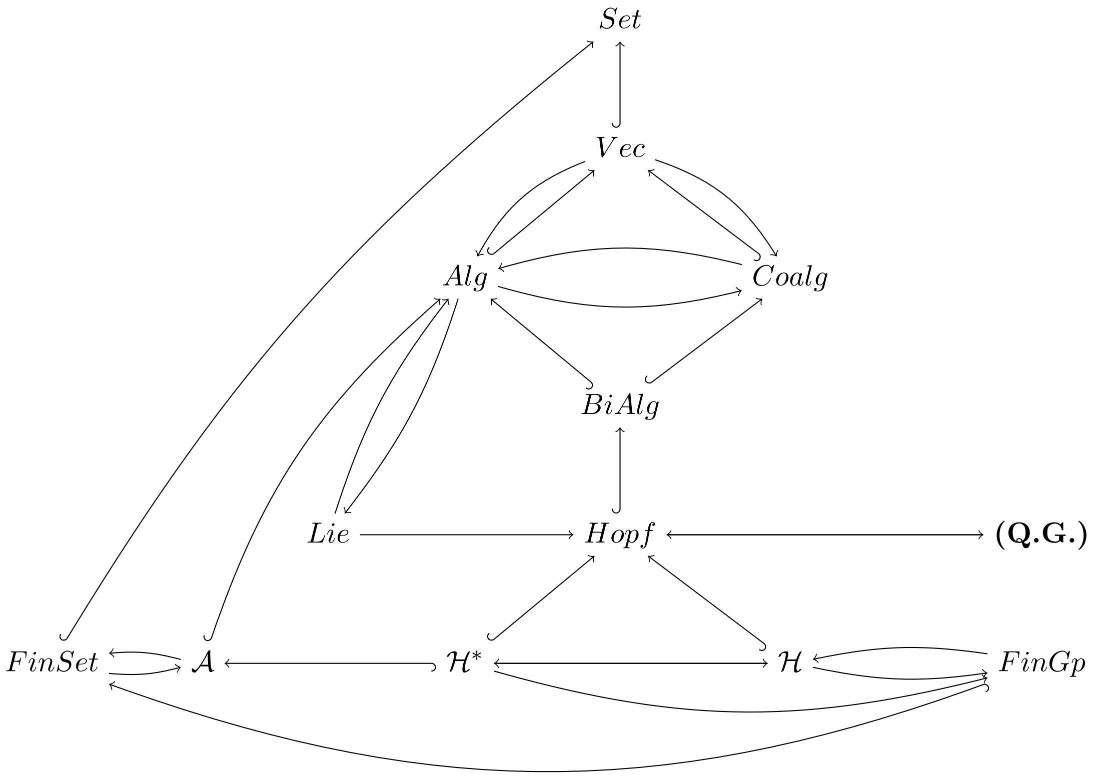

Figure 1.1

At this point the connecting arrows have been left blank to reduce clutter and confusion. As we progress, the goal will be to illuminate and examine these connections in order to gain a sufficient understanding of the objects residing in **(Q.G.)**, namely quantum groups.

It is hoped that, at worst, the reader finds this thesis an interesting read, but more optimistically an inspiration to immerse his/herself ever deeper into this challenging and exciting area of research and study. Because of its nascent status there is still much to be discovered. But every journey of discovery needs a place to start so let us begin this journey together and uncover the foundations of quantum group theory.

Chapter 2

The Basics: Vector Spaces and Modules

Though there are several approaches to quantum groups, it is best to first take stock of and understand the essential ingredients that go toward the theory. In this chapter we introduce and develop some of these basic or foundational concepts required for grasping the theory of quantum groups. Although the reader is hopefully acquainted with much of this material, the aim will be to provide a sufficient refresher as well as to extend the reader's understanding so that the later, more difficult concepts are not so opaque. The main focus of this chapter will be key aspects of the theory of vector spaces, especially the development of the *tensor product* which will be heavily used throughout. A cursory treatment of modules is also given since, while not of primary importance, they do show up in some important areas of consideration.

2.1 Vector Spaces

Central to our study is the familiar notion of a vector space. It will be assumed that the reader possesses a working understanding of these objects, but some time will be taken to establish some of the more abstract areas of the theory. Let's begin by agreeing on some notation.

As a matter of common knowledge, vector spaces come with arrows, morphisms or maps which allow one vector space to be transformed into another. Apropos, we call these *linear transformations*.

Definition 2.1 (Linear Transformation)**.** Let V and W be vector spaces over a field κ. A function $\tau : V \to W$ is a *linear transformation* (or *linear morphism*) if

$$\tau(\lambda u + \gamma v) = \lambda\tau(u) + \gamma\tau(v)$$

for all scalars $\lambda, \gamma \in \kappa$ and all vectors $u, v \in V$. The set of all linear transformations from V to W is denoted by $\mathcal{L}(V, W)$ or $\hom_\kappa(V, W)$. In this last case, the κ is often suppressed if there is no danger of confusion. A linear transformation $\tau \in \mathcal{L}(V, V)$ (or $\hom_\kappa(V, V)$) is called a *linear operator* on V and the set of all linear operators on V is usually abbreviated to $\mathcal{L}(V)$. Alternatively, we can think of linear operators on a space V as endomorphisms and so it is also common to write $\mathrm{End}(V)$.

A result which is often useful is that $\hom(\kappa, V) \cong V$ for any vector space V. Note that $f \in \hom(\kappa, V)$ is determined by what it does to $1 \in \kappa$. So, if $\{v_i\}_{i\in I}$ is a basis for V, then let $\hat{v}_i \in \hom(\kappa, V)$ be the map $\hat{v}_i(1) = v_i$. The set $\{\hat{v}_i\}_{i\in I}$ thus forms a basis for $\hom(\kappa, V)$ and from this the isomorphism follows.

While linear transformations are generally important, we will be particularly interested in a special kind of linear transformation known as a *linear functional* or *linear form.*

Definition 2.2 (Linear Functional)**.** Let V be a vector space over κ. A linear transformation $f \in \mathcal{L}(V, \kappa)$, whose values lie in the base field κ is called a *linear functional* (or *functional*) on V. The space of all linear functionals on V is denoted by V^*.

One reason linear functionals are important is because for any vector space V there is a corresponding *dual* vector space $V^* := \hom(V, \kappa)$. Addition and scalar multiplication are given as follows

$$(f + g)(x) = f(x) + g(x)$$
$$(\lambda f)(x) = \lambda f(x)$$

for all $f, g \in V^*, x \in V$ and $\lambda \in \kappa$. Besides "linear functionals", the vectors of V^* are sometimes referred to as *covectors* or *one-forms.*

Right away we hit upon a crucial idea and theme in the study of quantum groups, namely *duality.* The idea of duality in mathematics is pervasive, but nuanced. Crudely speaking, a duality indicates a kind of complementary relationship between two "objects" where results concerning one object translate into "complementary" results for the dual object. It is also often the case that dual objects possess similar or complementary structures. In this context, we can gain some insight as follows. If V is a vector space over a field κ with basis $\{v_i\}_{i\in I}$, then for each basis element v_i we can determine a corresponding covector $v_i^* \in V^*$ which is defined by

$$v_i^*(v_j) := \delta_{ij} \quad \text{[Kronecker map]}$$

The set $\{v_i^*\}_{i\in I}$ has the property of being linearly independent. This can be seen by applying

$$0 = a_{i_1} v_{i_1}^* + \ldots + a_{i_n} v_{i_n} *$$

to a basis vector v_{i_k}, which yields

$$\begin{aligned} 0 &= \sum_{j=1}^{n} a_{i_j} v_{i_j}^*(v_{i_k}) \\ &= \sum_{j=1}^{n} a_{i_j} \delta_{i_j, i_k} \\ &= a_{i_k} \end{aligned}$$

for all i_k. When V is finite dimensional, then $\{v_i^*\}$ is a basis for V^* called the *dual basis* of $\{v_i\}$ and $V \cong V^*$. This is not a natural isomorphism, but depends on the choice of basis. There is, however, a natural isomorphism between V and the double dual V^{**} when the dimension of V is finite. For details see [8]. We'll revisit this after introducing the tensor product. Later, we'll consider duality from a more categorical perspective.

Now that we have an understanding of the dual of a vector space, we introduce an important map known as the transpose of a linear map, which relates vector spaces with their duals.

Definition 2.3 (Transpose of a Linear Map)**.** Let V and W be vector spaces over a field κ with V^* and W^* their respective dual vector spaces. If $f : V \to W$ is a linear map, then the transpose of f, (usually) denoted by f^*, is the linear map $f^* : W^* \to V^*$ defined by

$$f^*(\phi) := \phi \circ f$$
$$V \overset{f}{\mapsto} W \overset{\phi}{\mapsto} \kappa$$

This type of map will make regular appearances throughout this thesis. For now, however, let us give due consideration to some other useful concepts.

2.1.1 Direct Sums

Generally, whenever we have a particular mathematical object of interest it is useful to determine in what ways new objects of this type may be constructed out of old ones.

One common means of doing this is via a direct sum. There are two equivalent ways of looking at a direct sum. One is called the *external* direct sum, while the other is referred to as the *internal* direct sum. They are defined as follows:

Definition 2.4 (External Direct Sum)**.** Let Λ be an indexing set and V_α, with $\alpha \in \Lambda$, be a collection of vector spaces over a field κ. The external direct sum of this collection, denoted by

$$V := [+]_{\alpha\in\Lambda} V_\alpha$$

is the vector space V whose elements are sequences indexed by Λ.

$$V = \{(v_\alpha)_{\alpha\in\Lambda} : v_\alpha \in V_\alpha, \forall\alpha \in \Lambda \text{ and almost all } v_\alpha = 0\}$$

The condition "almost all $v_\alpha = 0$" means that $v_\alpha = 0$ for all but a finite number of α. Operations are component-wise.

Definition 2.5 (Internal Direct Sum)**.** Let V be a vector space. We say that V is the internal direct sum of a family of subspaces $\mathcal{F} := \{S_\alpha : \alpha \in \Lambda\}$ of V if every vector $v \in V$ can be written uniquely (up to order) as a finite sum of vectors from the subspaces in $\mathcal{F}$, that is, if for all $v \in V$,

$$v = u_1 + \ldots + u_n$$

where $u_i \in S_{\alpha_i}$ for a set of distinct $\alpha_i \in \Lambda$ and furthermore, if

$$v = w_1 + \ldots + w_m$$

where $w_i \in S_{\beta_i}$ for a distinct set of $\beta_i \in \Lambda$, then $m = n$ and appropriate reindexing yields that $\alpha_i = \beta_i$ and $w_i = u_i$ for all i. If V is the internal direct sum of $\mathcal{F}$, we write

$$V = \bigoplus_{\alpha\in\Lambda} S_\alpha$$

and refer to each S_α as a direct summand of V.

Note, in particular, that if $\beta = \{v_1, ..., v_n\}$ is a basis for V, then $V = \bigoplus_i Span(v_i)$.

Although superficially different, an internal direct sum is isomorphic to its corresponding external direct sum, and therefore, we merely refer to the *direct sum* without qualification. We will adopt the symbol "$\oplus$" in general, since the internal form is often most useful. Even so, the question might naturally arise as to why we need a new summation symbol at all. Why not just use sigma notation? The answer to this is that the symbol "$\oplus$" conceptually captures more than just summation. We illustrate this in the following theorem.

Theorem 2.6. *A vector space V is the direct sum of a family $\mathcal{F} = \{S_\alpha : \alpha \in \Lambda\}$ of subspaces if and only if*

1. *V is the sum of the S_α*

$$V = \sum_{\alpha \in \Lambda} S_\alpha \quad \textit{(i.e. the } S_\alpha \textit{ span } V\textit{)}$$

2. *For each $\alpha \in \Lambda$,*

$$S_\alpha \cap \Big(\sum_{\beta \neq \alpha} S_\beta \Big) = \{0\} \quad \textit{(i.e. the } S_\alpha\textit{'s are independent)}$$

Proof. Suppose that V is the direct sum of a family $\mathcal{F} = \{S_\alpha : \alpha \in \Lambda\}$ of subspaces. Then by definition (1) must hold. To show that (2) holds, let

$$v \in S_\alpha \cap \Big(\sum_{\beta \neq \alpha} S_\beta \Big)$$

Then it must be that $v = u_\alpha$ for some $u_\alpha \in S_\alpha$ *and*

$$v = u_{\beta_1} + \ldots + u_{\beta_n}$$

where $\beta_k \neq \alpha, k \in \{1, ..., n\}$ and $u_{\beta_k} \in S_{\beta_k}$ for all k. But this says that v is expressible in two ways and therefore the uniqueness of v forces both to be zero. Hence (2) holds as desired.

Conversely, suppose that (1) and (2) hold. Then we need only establish uniqueness of expression. Suppose

$$v = u_{\alpha_1} + \ldots + u_{\alpha_n}$$

and

$$v = t_{\beta_1} + \ldots + t_{\beta_m}$$

where $u_{\alpha_i} \in S_{\alpha_i}$ and $t_{\beta_i} \in S_{\beta_i}$. By adding in the appropriate number of zero terms we can equalize the indexing sets so that we just have the indexing set $\{\gamma_1, ..., \gamma_p\}$. We therefore have

$$v = u_{\gamma_1} + \ldots + u_{\gamma_p}$$

and

$$v = t_{\gamma_1} + \ldots + t_{\gamma_p}$$

Subtracting yields

$$(u_{\gamma_1} - t_{\gamma_1}) + \ldots + (u_{\gamma_p} - t_{\gamma_p}) = 0$$

If we solve for any $(u_{\gamma_r} - t_{\gamma_r}) \in S_{\gamma_r}$, we will then have that this element is a sum of vectors from subspaces other than S_{γ_r}, which by (2) means that $u_{\gamma_r} - t_{\gamma_r} = 0$ and hence $u_{\gamma_r} = t_{\gamma_r}$, for all γ_r. We therefore have that v is unique and V is the direct sum of $\mathcal{F}$. □

2.1.2 Quotient Spaces

It is well known that new vector spaces can also be constructed as quotient spaces. We need not say more than this, but two results concerning quotient spaces are worth mentioning. The first concerns something called a universal property of quotient spaces and the second is the familiar First Isomorphism Theorem.

Theorem 2.7. *Let S be a subspace of V and let $\tau \in \mathcal{L}(V, W)$ satisfy $S \subseteq Ker(\tau)$. Then there is a unique linear transformation $\tau' : V/S \to W$ with the property that*

$$\tau' \circ \pi_S = \tau$$

where π_S is the canonical projection of V to V/S. Moreover, $Ker(\tau') = Ker(\tau)/S$ and $Im(\tau') = Im(\tau)$.

This universal property can be pictured in the following diagram:

$$\begin{tikzcd} V \arrow[r, "\tau"] \arrow[d, "\pi_S"'] & W \\ V/S \arrow[ur, dashed, "\tau'"'] & \end{tikzcd}$$

Essentially, it says that $\tau \in \mathcal{L}(V, W)$ can be *factored through* the canonical projection π_S.

Theorem 2.8 (The First Isomorphism Theorem)**.** *Let $\tau : V \to W$ be a linear transformation. Then the linear transformation $\tau' : V/Ker(\tau) \to W$ defined by*

$$\tau'(v + Ker(\tau)) = \tau(v)$$

is injective and

$$V/Ker(\tau) \cong Im(\tau)$$

Since these results are well known in the theory of vector spaces, and more generally for modules (see below), we shall omit the proofs here and simply take them for granted. The interested reader, however, can find proofs for both of these theorems in [8].

2.1.3 Tensor Products

The tensor product can safely be deemed one of the most important foundational concepts to quantum groups. Like direct sums, tensor products provide another very useful way of creating new vector spaces out of old ones. More than this, we shall see that the tensor product often provides a means of creating a new object out of old ones.

Central to the notion of tensor products is *bilinearity*. To really understand what the tensor product *is* we build it from the ground up. The motivation proceeds as follows.

Definition 2.9. Let U, V and W be vector spaces over κ. Let $U \times V$ be the *cartesian product* of U and V as sets. A set function $f : U \times V \to W$ is bilinear if it is linear in both arguments separately, that is, if

$$f(ru + su', v) = rf(u, v) + sf(u', v)$$

and

$$f(u, rv + sv') = rf(u, v) + sf(u, v')$$

The set of all bilinear functions from $U \times V$ to W is denoted by $\hom_\kappa(U, V; W)$ (or $\hom^{(2)}(U, V; W)$). A bilinear function $f : U \times V \to \kappa$, with values in the base field, is called a *bilinear form* on $U \times V$.

The motivation for defining tensor products is to have a universal property for bilinear functions, as "measured" by linearity. The key is to define a vector space T and a bilinear map $t : U \times V \to T$ so that any bilinear map f with domain $U \times V$ can be factored uniquely through t in accordance with the commuting diagram:

$$\begin{tikzcd} U \times V \arrow[r, "t_{\text{bilinear}}"] \arrow[dr, swap, "f_{\text{bilinear}}"] & T \arrow[d, "\tau_{\text{linear}}"] \\ & W \end{tikzcd}$$

which just says that any bilinear map $f : U \times V \to W$ can be factored in the form

$$f = \tau \circ t$$

where t is fixed and τ is a linear map depending on the chosen f. Athough this composition involves a linear map and a bilinear map, the composition is bilinear since

$$\begin{aligned}
(\tau \circ t)(ru + su', v) &= \tau(t(ru + su', v)) \\
&= \tau(rt(u, v) + st(u', v)) \\
&= r\tau(t(u, v)) + s\tau(t(u', v)) \\
&= r(\tau \circ t)(u, v) + s(\tau \circ t)(u', v)
\end{aligned}$$

The second argument is similarly shown to be linear.

Let us now state this as a formal definition.

Definition 2.10 (Universal Pair for Bilinearity)**.** Let $U \times V$ be the cartesian product of two vector spaces over κ. A pair $(T, t : U \times V \to T)$ where T is a vector space and t is bilinear, is universal for bilinearity if for every bilinear map $g : U \times V \to W$, where W is an arbitrary vector space over κ, there is a unique linear transformation $\tau : T \to W$ for which

$$g = \tau \circ t$$

Having a definition is one thing, but it remains to be seen that there exists such a universal pair for bilinearity within the "universe" of vector spaces. There is more than one way to demonstrate this existence. We shall explore two such methods. The first is much more constructive in nature, while the second is of a more abstract essence, using quotient spaces.

First Proof of Existence. Let $\{u_i\}_{i\in I}$ be a basis for the vector space U and $\{v_j\}_{j\in J}$ a basis for the vector space V. Define a map t on $U \times V$ to the set of *formal* images of t by assigning a *formal* image to each pair of basis elements. That is,

$$(u_i, v_j) \mapsto t(u_i, v_j)$$

In order that t look more like a function we devise the formal notation $u_i \otimes v_j$ to represent this image. Thus

$$t(u_i, v_j) := u_i \otimes v_j$$

This is called the tensor *product* of u_i and v_j. Note that, in some sense, this is only a pseudo-product, since $u_i \otimes v_j$ is not in either U or V. In fact, even if we took $u_i, u_j \in U$

we still find that $u_i \otimes u_j \notin U$. Instead, we define a new vector space T having formal basis $\{u_i \otimes v_j\}_{(i,j)\in I\times J}$. A generic element will thus have the form:

$$\sum_{i=1}^{n} \lambda_i (u_{k_i} \otimes v_{\ell_i})$$

We then extend our map t by bilinearity, and this makes t unique, since bilinear maps are uniquely determined by what they do to the "basis" pairs (u_i, v_j).

For this reason too, if $g : U \times V \to W$ is a bilinear function, then the condition that $g = \tau \circ t$ is equivalent to

$$\tau(u_i \otimes v_j) = g(u_i, v_j)$$

where τ is the linear map $T \to W$ we are constructing. And because $\{u_i \otimes v_j\}_{(i,j)\in I\times J}$ is a basis of T this also uniquely defines a linear map $\tau : T \to W$ so that (T, t) is indeed universal for bilinearity.

Also, if $u = \sum_{i=1}^{n} \lambda_i u_i \in U$ and $v = \sum_{j=1}^{m} \gamma_j v_j \in V$ then we have

$$\begin{aligned}
u \otimes v &:= t(u, v) && (2.1)\\
&= t\Big(\sum_{i=1}^{n} \lambda_i u_i, \sum_{j=1}^{m} \gamma_j v_j\Big) && (2.2)\\
&= \sum_{i,j} \lambda_i \gamma_j (u_i \otimes v_j) \quad \text{[using bilinearity of } t\text{]} && (2.3)
\end{aligned}$$

As a matter of notation, we denote the vector space T by $U \otimes V$ and call it the tensor product of U and V. Here, the element $u \otimes v$ of $U \otimes V$ is known as a *pure tensor*. A generic element of $U \otimes V$ will actually be a finite sum of pure tensors. □

Although this way of defining $U \otimes V$ is straightforward, it has the disadvantage of requiring a choice of a basis for U and V. The next method of defining $U \otimes V$ circumvents this drawback quite elegantly.

Second Proof of Existence. Let $F_{U\times V}$ be the free vector space over F with basis $U \times V$. Let S be the subspace of $F_{U\times V}$ generated by all vectors of the form

$$r(u, w) + s(v, w) - (ru + sv, w)$$

and

$$r(u, v) + s(u, w) - (u, rv + sw)$$

where $r, s \in F$ and u, v and w are in the appropriate spaces. Now consider the quotient space

$$\frac{F_{U\times V}}{S}$$

Quotienting out by S thus gives the necessary bilinear relations and it is this space that we define to be the tensor product of U and V - i.e. $U \otimes V$. A typical element will have the form:

$$\left(\sum_{(u,v)\in U\times V} \lambda_{(u,v)}(u,v)\right) + S = \sum_{(u,v)\in U\times V} \lambda_{(u,v)}[(u,v)+S]$$

where all but a finite number of $\lambda_{(u,v)} = 0$. But because

$$\lambda(u,v) - (\lambda u, v) \in S \quad \text{and} \quad \lambda(u,v) - (u, \lambda v) \in S$$

we have that

$$\begin{aligned} \lambda_{(u,v)}(u,v) + S &= (\lambda_{(u,v)}u, v) + S \\ &= (u, \lambda_{(u,v)}v) + S \end{aligned}$$

which allows the scalar to be absorbed and we may simply write the elements of $U \otimes V$ as

$$\sum [(u,v)+S]$$

If we denote $(u,v)+S$ by $u \otimes v$, then the elements of $U \otimes V$ are simply

$$\sum u \otimes v$$

In this case the function $t : U \times V \to U \otimes V$ is just the canonical map and will be bilinear due to the fact that we are quotienting $F_{U\times V}$ out by S.

With this way of defining tensor products we now want to show that the pair

$$(U \otimes V, t : U \times V \to U \otimes V)$$

is universal for bilinearity. Consider the diagram given below.

$$\begin{tikzcd}
U \times V \arrow[r, "j"] \arrow[rr, bend left, "t"] \arrow[dr, "f"'] & F_{U\times V} \arrow[r, "\pi"] \arrow[d, dashed, "\sigma"] & U \otimes V \arrow[dl, dashed, "\tau"] \\
& W &
\end{tikzcd}$$

We want to show that any bilinear function f factors through t. Note that $t = \pi \circ j$, where j is the inclusion map and π is the canonical projection map. Since $U \times V$ is a basis of $F_{U\times V}$, there is a unique linear transformation

$$\sigma : F_{U\times V} \to W$$

which extends f, i.e. $\sigma \circ j = f$. This follows from the universal property of vector spaces (see [8]). Furthermore, since f is bilinear and σ is a linear transformation which extends f, then σ will send the vectors generating S to zero. For instance,

$$\begin{aligned}
\sigma\big(r(u,w) + s(v,w) - (ru+sv,w)\big) &= \sigma\big(rj(u,w) + sj(v,w) - j(ru+sv,w)\big) \\
&= r\sigma(j(u,w)) + s\sigma(j(v,w)) - \sigma(j(ru+sv,w)) \\
&= rf(u,w) + sf(v,w) - f(ru+sv,w) = 0
\end{aligned}$$

The linearity of the second argument is similarly shown. Thus, $S \subseteq Ker(\sigma)$ and hence, by Theorem 2.7, there is a unique linear transformation $\tau : U \otimes V \to W$ for which $\tau \circ \pi = \sigma$ and therefore

$$\tau \circ t = \tau \circ \pi \circ j = \sigma \circ j = f$$

Now suppose that there is τ' such that $\tau' \circ t = f$. Then $\sigma' = \tau' \circ \pi$ satisfies

$$\sigma' \circ j = \tau' \circ \pi \circ j = \tau' \circ t = f$$

But the uniqueness of σ implies that $\sigma' = \sigma$, which in turn implies that

$$\tau' \circ \pi = \sigma' = \sigma = \tau \circ \pi$$

and the uniqueness of τ implies that $\tau' = \tau$. □

The key result here is that bilinearity on $U \times V$ is just linearity on $U \otimes V$. That is, there is an isomorphism

$$\hom^{(2)}(U,V;W) \cong \mathcal{L}(U \otimes V, W) \quad \text{as abelian groups}$$

These two seemingly different definitions of a tensor product are equivalent. In fact, *any* two models or constructions of the tensor product are isomorphic. To see this, let $U \otimes_1 V$ and $U \otimes_2 V$ be tensor products resulting from Definition 2.10. Consider the following diagram:

$$\begin{tikzcd}
& U \times V \arrow[dl, "t_1"'] \arrow[d, "t_2"] \arrow[dr, "t_1"] & \\
U \otimes_1 V \arrow[r, dashed, "\tau_1"] \arrow[rr, dashed, bend right, "\tau_3"] & U \otimes_2 V \arrow[r, dashed, "\tau_2"] & U \otimes_1 V
\end{tikzcd}$$

In the diagram, t_1 and t_2 are the associated bilinear maps. τ_1 is the unique linear morphism given by the universal property of $(U \otimes_1 V, t_1)$. τ_2 is the unique linear morphism using the universal property of $(U \otimes_2 V, t_2)$ and τ_3 is the unique linear morphism again given by the universal property of $(U \otimes_1 V, t_1)$. Observe, however, that we can simply take τ_3 to be the identity morphism on $U \otimes_1 V$ - i.e. $\tau_3 = \mathrm{id}_{U \otimes_1 V}$, or we can take $\tau_3 = \tau_2 \circ \tau_1$ and then, by uniqueness

$$\mathrm{id}_{U \otimes_1 V} = \tau_2 \circ \tau_1$$

By a symmetric line of reasoning, if we switch the roles of t_1 and t_2 we also get that

$$\tau_1 \circ \tau_2 = \mathrm{id}_{U \otimes_2 V}$$

This shows that $U \otimes_1 V \cong U \otimes_2 V$ and establishes, more generally, that any two models of $U \otimes V$ will be isomorphic.

Next, we consider some important results concerning certain isomorphisms.

Corollary 2.11. *For any triple (U, V, W) of vector spaces, there is a natural isomorphism of abelian groups*

$$\mathcal{L}(U \otimes V, W) \cong \mathcal{L}(U, \mathcal{L}(V, W))$$

Proof. Recall that

$$\hom^{(2)}(U, V; W) \cong \mathcal{L}(U \otimes V, W)$$

If $\varphi \in \hom^{(2)}(U, V; W)$ and $u \in U$, then $\varphi(u, -) \in \mathcal{L}(V, W)$. Consider, then, the additive mapping

$$\varphi \mapsto \phi$$

where $\phi : U \to \mathcal{L}(V, W)$ is defined by $\phi(u) := \varphi(u, -)$. Since φ is bilinear, $\varphi(u, -)$ will be linear for any $u \in U$, and so, ϕ will be a morphism of abelian groups. This mapping is onto, since if $f \in \mathcal{L}(U, L(V, W))$ then choose $\varphi : U \times V \to W$ to be the function defined by

$$\varphi(u, v) := f(u)(v)$$

which is easily verified to be bilinear. By definition, then, φ gets mapped to the linear map ϕ with

$$\phi(u) := \varphi(u, -) := f(u)$$

Thus, onto is established.

Now suppose φ is in the kernel of the mapping in question. Then $\varphi \mapsto 0 \in \mathcal{L}(U, \mathcal{L}(V, W))$ where $0(u) = 0 \in \mathcal{L}(V, W)$. By the definition of our mapping, however,

$$0(u) := \varphi(u, -) = 0 \in \mathcal{L}(V, W)$$

for all $u \in U$. Thus, for each $u \in U$ we will have that $\varphi(u, v) = 0(v) = 0$ for all $v \in V$. This implies that $\varphi = 0 \in \hom^{(2)}(U, V; W)$ whence our mapping is also 1-1 and therefore an isomorphism. $\square$

Proposition 2.12. *Let $(U_i)_{i \in I}$ be a family of vector spaces and $\bigoplus_{i \in I} U_i$ the direct sum of this family. There exist linear maps $q_j : U_j \to \bigoplus_{i \in I} U_i$, such that for any vector space V, we have that*

$$\hom\Big(\bigoplus_{i \in I} U_i, V\Big) \cong \prod_{i \in I} \hom(U_i, V), \quad f \mapsto (f \circ q_i)_i$$

Proof. Define each q_i in the following way: For all $i \in I$, let q_i be the map such that for $u \in U_i$

$$q_i(u) := (u_j)_{j \in I}, \quad \text{with } u_j = 0 \text{ for } j \neq i \text{ and } u_j = u \text{ for } j = i$$

That these are linear is clear from their construction. Now let V be any vector space and consider the map

$$\Phi : \hom\Big(\bigoplus_{i \in I} U_i, V\Big) \to \prod_{i \in I} \hom(U_i, V)$$

such that $\Phi(f) := (f \circ q_i)_i$. This map is linear, since for any $f, g \in \hom\left(\bigoplus_{i\in I} U_i, V\right)$ and $\lambda \in \kappa$ we have, using properties of composition, that

$$\begin{aligned}\Phi(\lambda f + g) &= \big((\lambda f + g) \circ q_i\big)_i \\ &= (\lambda f \circ q_i + g \circ q_i)_i \\ &= \lambda (f \circ q_i)_i + (g \circ q_i)_i \\ &= \lambda \Phi(f) + \Phi(g)\end{aligned}$$

To show that Φ is an isomorphism we first consider its kernel. Suppose $h \in Ker(\Phi)$. Then by definition $\Phi(h) = (h \circ q_i)_i = (0)$. So, $h \equiv 0$ on $Im(q_i)$ for all $i \in I$. Note that each q_i is essentially a canonical injection of U_i into $\bigoplus_{i\in I} U_i$. Thus

$$\sum_{i\in I} Im(q_i) = \bigoplus_{i\in I} U_i$$

which means that $h \equiv 0$ on its entire domain and is therefore the zero map. This shows that $Ker(\Phi) = \{0\}$ and hence that Φ is 1-1. To finish, let $(g_i)_{i\in I} \in \prod_{i\in I} \hom(U_i, V)$. We want to know if there is $f \in \hom(\bigoplus_{i\in I} U_i, V)$ such that $f \mapsto (g_i)_{i\in I}$. This is the same as asking if there is f such that

$$\begin{tikzcd} U_i \arrow[rr, hook, "q_i"] \arrow[dr, "g_i"'] & & \bigoplus_{i\in I} U_i \arrow[dl, "f"] \\ & V & \end{tikzcd}$$

commutes for all i. Since q_i is just the canonical injection of U_i into $\bigoplus_{i\in I} U_i$, define $f := \bigoplus g_i$ where $(\bigoplus g_i)(v_i)_{i\in I} = \sum g_i(v_i)$. This is the f we need to make the above diagrams commute. This establishes that Φ is onto from which it follows that Φ is an isomorphism. $\square$

We now show that the tensor product distributes over the direct sum of spaces.

Proposition 2.13.

$$\Big(\bigoplus_{i\in I} U_i\Big) \otimes V \cong \bigoplus_{i\in I} (U_i \otimes V)$$

Proof. By Corollary 2.11 and Proposition 2.12 the following chain of natural isomorphisms hold for any vector space W:

$$\begin{aligned}
\hom\Big((\bigoplus_{i\in I} U_i)\otimes V, W\Big) &\cong \hom\Big(\bigoplus_{i\in I} U_i, \hom(V,W)\Big) \\
&\cong \prod_{i\in I} \hom(U_i, \hom(V,W)) \\
&\cong \prod_{i\in I} (U_i\otimes V, W) \\
&\cong \hom\Big(\bigoplus_{i\in I}(U_i\otimes V), W\Big)
\end{aligned}$$

Let α_W represent the isomorphism

$$\hom\Big((\bigoplus_{i\in I} U_i)\otimes V, W\Big) \cong \hom\Big(\bigoplus_{i\in I}(U_i\otimes V), W\Big)$$

where W is considered to be a "variable".

Now, if in particular $W = (\bigoplus_{i\in I} U_i)\otimes V$, then we have

$$\hom\Big((\bigoplus_{i\in I} U_i)\otimes V, (\bigoplus_{i\in I} U_i)\otimes V\Big) \cong \hom\Big(\bigoplus_{i\in I}(U_i\otimes V), (\bigoplus_{i\in I} U_i)\otimes V\Big)$$

For simplicity set $T := (\bigoplus_{i\in I} U_i)\otimes V$. Then the relevant isomorphism is α_T. Define a linear map $\phi : \bigoplus_{i\in I}(U_i\otimes V) \to (\bigoplus_{i\in I} U_i)\otimes V$ by $\phi := \alpha_T(\mathrm{id}_T)$.

Next, if $W = \bigoplus_{i\in I}(U_i\otimes V)$, then

$$\hom\Big((\bigoplus_{i\in I} U_i)\otimes V, \bigoplus_{i\in I}(U_i\otimes V)\Big) \cong \hom\Big(\bigoplus_{i\in I}(U_i\otimes V), \bigoplus_{i\in I}(U_i\otimes V)\Big)$$

and if we set $\bar{T} := \bigoplus_{i\in I}(U_i\otimes V)$, then the relevant isomorphism is $\alpha_{\bar{T}}$. Now define a linear map $\psi : (\bigoplus_{i\in I} U_i)\otimes V \to \bigoplus_{i\in I}(U_i\otimes V)$ by $\psi := \alpha_{\bar{T}}^{-1}(\mathrm{id}_{\bar{T}})$.

Let W' be any vector space and $f : W \to W'$ a linear map. Then because α is a natural isomorphism the following diagram commutes:

$$\begin{tikzcd}
\hom(T,W) \arrow[r, "\alpha_W"] \arrow[d, "\beta_f"'] & \hom(\bar{T},W) \arrow[d, "\beta_f"] \\
\hom(T,W') \arrow[r, "\alpha_{W'}"] & \hom(\bar{T},W')
\end{tikzcd}$$

where β_f is defined by $\beta_f(\omega) := f \circ \omega$ for ω in the relevant domain. The commutativity of this diagram gives us the following relation: for $\omega \in \hom(T, W)$, we have

$$\alpha_{W'}(\beta_f(\omega)) = \beta_f(\alpha_W(\omega))$$

or

$$\alpha_{W'}(f \circ \omega) = f \circ \alpha_W(\omega) \tag{2.4}$$

We now verify that ϕ and ψ are inverses. Using $W = T$ and $W' = \bar{T}$, one order of composition yields

$$\begin{aligned}
\psi \circ \phi &= \psi \circ \alpha_T(\mathrm{id}_T) \\
&= \alpha_{\bar{T}}(\psi \circ \mathrm{id}_T) \quad [\text{by (2.4)}] \\
&= \alpha_{\bar{T}}(\psi) \\
&= \alpha_{\bar{T}}\big(\alpha_{\bar{T}}^{-1}(\mathrm{id}_{\bar{T}})\big) \\
&= \mathrm{id}_{\bar{T}}
\end{aligned}$$

A symmetric argument, using $W = \bar{T}$ and $W' = T$, shows that the composition in reverse order, namely $\phi \circ \psi$, results in id_T. Thus, ϕ and ψ are inverses and hence

$$(\bigoplus_{i \in I} U_i) \otimes V \cong \bigoplus_{i \in I} (U_i \otimes V)$$

□

Now, whenever a product is defined we generally want it to possess the usual niceties such as being associative and commutative. Tensor products do enjoy these properties, but only to a slightly "weaker" degree. That is, instead of strict equality we must settle for isomorphisms.

$$\begin{aligned}
(U \otimes V) \otimes W \cong U \otimes (V \otimes W) &\quad \text{associative isomorphism} \\
V \otimes W \cong W \otimes V &\quad \text{commutative isomorphism}
\end{aligned}$$

We also have, for any vector space V,

$$\kappa \otimes V \cong V \cong V \otimes \kappa$$

Recall, the tensor product of vector spaces is unique (up to isomorphism) and hence these isomorphisms can be easily established by showing that each side is the appropriate tensor product. In particular we have $\kappa \cong \kappa \otimes \kappa$, a fact that will be extensively used later. This follows from the fact that κ is one-dimensional with basis $\{1_\kappa\}$ and so $\kappa \otimes \kappa$, having basis $\{1_\kappa \otimes 1_\kappa\}$, is also one-dimensional. The isomorphism is then provided by the mapping $\lambda \mapsto \lambda \otimes 1 = 1 \otimes \lambda$ with inverse $\lambda \otimes \mu \mapsto \lambda\mu$.

Finally, for convenience we write down some important relations that hold in $U \otimes V$. If $u, u' \in U$ and $v, v' \in V$ and $\lambda \in \kappa$, then bilinearity yields

$$\begin{aligned}(u + u') \otimes v &= u \otimes v + u' \otimes v \quad \text{[right distribution]}\\ u \otimes (v + v') &= u \otimes v + u \otimes v' \quad \text{[left distribution]}\\ \lambda(u \otimes v) = (\lambda u) \otimes v &= u \otimes (\lambda v) \quad \text{[scalar multiplication]}\end{aligned}$$

Besides these, it will also be important to determine just when a tensor product is zero. Note first that

$$0 \otimes u = (0 + 0) \otimes u = 0 \otimes u + 0 \otimes u$$

This implies that $0 \otimes u = 0$ and by similar reasoning $u \otimes 0 = 0$.

Now let $\{u_i\}_{i \in I}$ be a linearly independent set of elements in U and $\{\nu_j\}_{j \in J}$ an arbitrary set of vectors from V. Suppose that

$$\sum_i u_i \otimes \nu_i = 0$$

By the universal property, for any bilinear function $f : U \times V \to W$, there is a unique linear function $\tau : U \otimes V \to W$ such that $\tau \circ t = f$. We therefore have

$$\begin{aligned}0 &= \tau\Big(\sum_i u_i \otimes \nu_i\Big)\\ &= \sum_i \tau\big(t(u_i, \nu_i)\big)\\ &= \sum_i f(u_i, \nu_i)\end{aligned}$$

Hence, $\sum f(u_i, \nu_i) = 0$ must hold for *any* bilinear function $f : U \times V \to W$ whatsoever. Because each u_i and ν_i is fixed, we may choose any bilinear function on $U \times V$ to discover what exactly these elements must be. Let U^* and V^* be the dual spaces of U and V respectively. Take $f : U \times V \to \kappa$ to be the bilinear map defined by

$$f(u, \nu) = \alpha(u)\beta(\nu) \qquad \alpha \in U^*, \beta \in V^*$$

To see that this is indeed bilinear, consider:

$$\begin{aligned} f(\lambda u + \gamma u', \nu) &= \alpha(\lambda u + \gamma u')\beta(\nu) \\ &= \big(\lambda\alpha(u) + \gamma\alpha(u')\big)\beta(\nu) \\ &= \lambda\alpha(u)\beta(\nu) + \gamma\alpha(u')\beta(\nu) \\ &= \lambda f(u,\nu) + \gamma f(u',\nu) \end{aligned}$$

Similar reasoning shows linearity in the second argument. We therefore have

$$\sum_i f(u_i, \nu_i) = \sum_i \alpha(u_i)\beta(\nu_i) = 0$$

We can extend $\{u_i\}_{i\in I}$ to a basis of U, let us say $\{b_k\}_{k\in K}$ where $I \subseteq K$ and $\{u_i\}_{i\in I} \subseteq \{b_k\}_{k\in K}$. Take $\alpha \in U^*$ to be the covector u_i^* defined on the basis by

$$u_i^*(b_j) := \delta_{ij}, \quad \text{[Kronecker map]}$$

We therefore have

$$0 = \sum_i u_k^*(b_i)\beta(\nu_i) = \beta(\nu_k)$$

for all $\beta \in V^*$ and this implies that each $\nu_k = 0$. We have therefore justified the following theorem:

Theorem 2.14. *If $u_1, ..., u_n$ are linearly independent vectors in U and $\nu_1, ..., \nu_n$ are arbitrary vectors in V, then*

$$\sum u_i \otimes v_i = 0 \implies v_i = 0, \text{ for all } i$$

In particular, $u \otimes v = 0$ if and only if $u = 0$ or $v = 0$.

Theorem 2.15. *Let w be a non-zero element of $U \otimes V$ and express*

$$w = \sum_{i=1}^{n} a_i \otimes b_i$$

with n minimal. Then $\{a_i\}$ and $\{b_i\}$ are linearly independent sets.

Proof. Suppose that the result is not true. Without loss of generality, we may assume that $\{a_i\}$ is dependent. Furthermore, by relabeling we may assume that a_n depends on the other a_i's so that

$$a_n = \sum_{i=1}^{n-1} \lambda_i a_i$$

But then

$$\begin{aligned}
w &= \Big(\sum_{i=1}^{n-1} a_i \otimes b_i\Big) + a_n \otimes b_n \\
&= \Big(\sum_{i=1}^{n-1} a_i \otimes b_i\Big) + \Big(\sum_{i=1}^{n-1} \lambda_i a_i\Big) \otimes b_n \\
&= \sum_{i=1}^{n-1} a_i \otimes b_i + \sum_{i=1}^{n-1} \lambda_i a_i \otimes b_n \\
&= \sum_{i=1}^{n-1} (a_i \otimes b_i + \lambda_i a_i \otimes b_n) \\
&= \sum_{i=1}^{n-1} a_i \otimes (b_i + \lambda_i b_n)
\end{aligned}$$

which contradicts the minimality of n. Therefore, $\{a_i\}$ must be linearly independent and the result holds. □

We end this section with two final useful results. The first concerns the tensor product of vector subspaces. The second concerns the tensor product of linear maps and their kernels.

Proposition 2.16. *Let V and W be two κ-vector spaces, and $X \subseteq V$, $Y \subseteq W$ vector subspaces. Then $(V \otimes Y) \cap (X \otimes W) = X \otimes Y$.*

Proof. Since X is a vector subspace of V it has a basis, say $\{x_i\}_{i \in I}$, which is embedded within a basis for V. Thus, we may complete or extend $\{x_i\}_{i \in I}$ to a basis of V, say $\{x_i\}_{i \in I'}$ $(I \subseteq I')$. The same holds true for Y. That is, if $\{y_j\}_{j \in J}$ is a basis of Y, then we may extend it to a basis of W, say $\{y_j\}_{j \in J'}$ $(J \subseteq J')$.

We now know that $X \otimes Y$ has basis $\{x_i \otimes y_j\}_{(i,j) \in I \times J}$. Since both $V \otimes Y$ and $X \otimes W$ are subspaces of $V \otimes W$ we know that $(V \otimes Y) \cap (X \otimes W)$ is a vector space. Note that $V \otimes Y$ has basis $\{x_i \otimes y_j\}_{(i,j) \in I' \times J}$, $X \otimes W$ has basis $\{x_i \otimes y_j\}_{(i,j) \in I \times J'}$ and $V \otimes W$ has basis $\{x_i \otimes y_j\}_{(i,j) \in I' \times J'}$.

Now, it is clear that $X \otimes Y \subseteq (V \otimes Y) \cap (X \otimes W)$. Suppose

$$t \in (V \otimes Y) \cap (X \otimes W)$$

Then since $t \in V \otimes Y$ we have

$$t = \sum_{(i,j') \in I \times J'} \lambda_{ij'} x_i \otimes y_{j'}$$

But also $t \in X \otimes W$ and so

$$t = \sum_{(i',j) \in I' \times J} \gamma_{i'j} x_{i'} \otimes y_j$$

Now consider that

$$\begin{aligned} 0 &= t - t \\ &= \sum_{(i,j') \in I \times J'} \lambda_{ij'} x_i \otimes y_{j'} - \sum_{(i',j) \in I' \times J} \gamma_{i'j} x_{i'} \otimes y_j \end{aligned}$$

Since $\{x_{i'} \otimes y_{j'} : (i', j') \in I' \times J'\}$ is independent, this implies that $\lambda_{ij'} = 0$ for $j' \notin J$, $\gamma_{i'j} = 0$ for $i' \notin I$ and $\lambda_{ij'} = \gamma_{i'j}$ for $(i, j') = (i', j) \in I \times J$. So

$$t = \sum_{(i,j) \in I \times J} \lambda_{ij} x_i \otimes y_j \in X \otimes Y$$

and therefore

$$(V \otimes Y) \cap (X \otimes W) \subseteq X \otimes Y$$

thereby establishing the desired equality. □

Before tackling the final result, we want to establish that the tensor product of two linear maps is again a well-defined linear map. Let U, U', V, V' be vector spaces. Then $\hom(U, U'), \hom(V, V')$ and $\hom(U \otimes V, U' \otimes V')$ are each vector spaces as well. Let the following commutative diagram be our template:

$$\begin{tikzcd} \hom(U, U') \times \hom(V, V') \arrow[rr, "\otimes"] \arrow[dr, "\phi"'] & & \hom(U, U') \otimes \hom(V, V') \arrow[dl, "\theta"] \\ & \hom(U \otimes V, U' \otimes V') & \end{tikzcd}$$

If $f \in \hom(U, U')$ and $g \in \hom(V, V')$ note that the expression $f(u) \otimes g(v)$ makes sense (as opposed to $f(u)g(v)$ in this context). Furthermore, it is clearly bilinear in u and v. There is therefore a unique linear map, say $(f \odot g) \in \hom(U \otimes V, U' \otimes V')$ such that

$$(f \odot g)(u \otimes v) = f(u) \otimes g(v)$$

In the above diagram, then, ϕ is the map: $\phi(f,g) = f \odot g$. This map is bilinear, since

$$\begin{aligned}\big((\lambda f + \gamma g) \odot h\big)(u,v) &= (\lambda f + \gamma g)(u) \otimes h(v)\\ &= (\lambda f(u) + \gamma g(u)) \otimes h(v)\\ &= \lambda f(u) \otimes h(v) + \gamma g(u) \otimes h(v)\\ &= \lambda (f \odot h)(u,v) + \gamma (g \odot h)(u,v)\\ &= \big(\lambda (f \odot h) + \gamma (g \odot h)\big)(u,v)\end{aligned}$$

A symmetric argument shows that linearity also holds in the second coordinate. Thus, by the universal property of tensor products there is a unique linear map

$$\theta : \hom(U,U') \otimes \hom(V,V') \to \hom(U \otimes V, U' \otimes V')$$

with $\theta(f \otimes g) = f \odot g$. In fact, θ is an embedding map. To see this, it suffices to show that θ is injective. So, if $\theta(f \otimes g) = 0$, then $f(u) \otimes g(v) = 0$ for all $u \in U$ and $v \in V$. If $f = 0$, then $f \otimes g = 0$. Suppose, then, that $f \neq 0$. Then there exists a $u \in U$ for which $f(u) \neq 0$. Fix this u. Then since $f(u) \otimes g(v) = 0$ for all $v \in V$, it must be by Theorem 2.14 that $g(v) = 0$ for all $v \in V$ and hence that $g = 0$. It follows that $f \otimes g = 0$. Thus θ is injective.

From here on we shall simply use the notation $f \otimes g$ to refer both to the tensor product of linear maps and the linear map $f \odot g$.

Proposition 2.17. *Let $f : V \to V'$ and $g : W \to W'$ be morphisms of κ-vector spaces. Then*

$$Ker(f \otimes g) = Ker(f) \otimes W + V \otimes Ker(g)$$

Proof. In light of Theorem 2.14, it is clearly true that

$$Ker(f) \otimes W + V \otimes Ker(g) \subseteq Ker(f \otimes g)$$

so we need only show the reverse inclusion. Let $\{v_i\}_{i \in I}$ be a basis for $Ker(f)$ and $\{w_j\}_{j \in J}$ a basis for $Ker(g)$. Extend $\{v_i\}_{i \in I}$ to a basis of V, say $\{v_i\}_{i \in I'}$ and extend $\{w_j\}_{j \in J}$ to a basis for W, say $\{w_j\}_{j \in J'}$. It follows easily that $\{f(v_i)\}_{i \in I' - I}$ is linearly independent in V' and $\{g(w_j)\}_{j \in J' - J}$ is linearly independent in W'.

Now let $t = \sum_{(i,j)\in I'\times J'} \lambda_{ij} v_i \otimes w_j \in Ker(f \otimes g)$. Then

$$\begin{aligned} 0 &= \sum_{(i,j)\in I'\times J'} \lambda_{ij} f(v_i) \otimes g(w_j) \\ &= \sum_{(i,j)\in (I'-I)\times (J'-J)} \lambda_{ij} f(v_i) \otimes g(w_j) \end{aligned}$$

since $f(v_i) = 0$ for $i \in I$ and $g(w_j) = 0$ for $j \in J$. Also, the family

$$\{f(v_i) \otimes g(w_j)\}_{(i,j)\in (I'-I)\times (J'-J)}$$

is linearly independent and therefore $\lambda_{ij} = 0$ whenever $i \in I' - I$ and $j \in J' - J$. Thus $t \in Ker(f) \otimes W + V \otimes Ker(g)$ thereby establishing the equality. $\square$

2.1.4 Duality

Before ending our explicit discussion of vector spaces, let's revisit the notion of duality brought up at the start. Often, duality is conceived in terms of a *pairing*, which is given by a bilinear form between two objects. So, if V is a finite dimensional vector space and V^* its dual, then we can create a pairing

$$\langle , \rangle : V^* \times V \to \kappa$$

defined by $\langle f, v\rangle := f(v)$. Thus, if $\{v_i\}$ is a basis for V and $\{v_i^*\}$ is the set of dual elements, then $\langle v_i^*, v_i\rangle = \delta_{ij}$.

Now, if $f : U \to V$ is a linear map and $f^* : V^* \to U^*$ is the transpose of f, then in terms of our pairing, for any $\alpha \in V^*$ and $u \in U$ we have

$$\langle f^*(\alpha), u\rangle = \langle \alpha, f(u)\rangle$$

Because the map $\langle , \rangle$ is bilinear, it can be represented via the tensor product. That is, we get a linear map, say $\langle\rangle : V^* \otimes V \to \kappa$, such that

$$\begin{tikzcd} V^* \times V \arrow[r, "\otimes"] \arrow[dr, "{\langle , \rangle}"'] & V^* \otimes V \arrow[d, "\langle\rangle"] \\ & \kappa \end{tikzcd}$$

commutes. This approach to duality will be important later when we discuss a special kind of duality between two important Hopf algebras central to our study.

Let us now elaborate a bit on the nature of the map

$$\theta : \hom(U, U') \otimes \hom(V, V') \to \hom(U \otimes V, U' \otimes V')$$

introduced above.

Theorem 2.18. *The map θ is an isomorphism provided at least one of the pairs $(U, U'), (V, V')$ or (U, V) consists of finite-dimensional vector spaces.*

Proof. Begin by supposing that U and U' are finite dimensional. Write

$$U = \bigoplus_{i \in I} \kappa u_i \quad \text{and} \quad U' = \bigoplus_{j \in J} \kappa u'_j$$

with $\{u_i\}_{i \in I}$ a basis for U and $\{u'_j\}_{j \in J}$ a basis for U'. If we repeatedly apply Proposition 2.12 and Proposition 2.13 we end up with θ being the map

$$\theta : \bigoplus_{i,j} \left(\hom(\kappa u_i, \kappa u'_j) \otimes \hom(V, V') \right) \to \bigoplus_{i,j} \hom(\kappa u_i \otimes V, \kappa u'_j \otimes V')$$

which is possible due to the finite dimensionality of U and U'. Because these are finite direct sums and θ acts component wise, we only need to show that

$$\hom(\kappa u_i, \kappa u'_j) \otimes \hom(V, V') \cong \hom(\kappa u_i \otimes V, \kappa u'_j \otimes V')$$

which is just a special application of θ.

We know from the section on tensor products that θ is an embedding map and hence is injective. But it is also quite clearly surjective here, since κu_i and $\kappa u'_j$ are both isomorphic to κ. We then use the fact that

$$\hom(\kappa u_i, \kappa u_j) \cong \hom(\kappa, \kappa) \cong \kappa \quad \text{and} \quad \kappa u_i \otimes V \cong \kappa \otimes V \cong V$$

So θ is indeed an isomorphism. This completes the case for when U and U' are finite dimensional. The other cases are proven via similar arguments. □

The following corollary is a specialization of the above theorem. To see this, it will be important to recall that

$$U^* := \hom(U, \kappa), V^* := \hom(V, \kappa) \quad \text{and} \quad (U \otimes V)^* := \hom(U \otimes V, \kappa)$$

Corollary 2.19. *The map*

$$\theta : U^* \otimes V^* \to (U \otimes V)^*$$

is an isomorphism provided U or V are finite-dimensional.

Thus, the result follows simply from setting

$$U' = V' = \kappa$$

Note that the surjectivity of θ means that for $h \in (U \otimes V)^*$, h can be represented by a finite sum of the form $\sum_i f_i \otimes g_i$ for some f_i, g_i in U^*, V^* respectively.

Corollary 2.20. *The map $\lambda_{U,V} : V \otimes U^* \to \hom(U, V)$ given for $u \in U$, $v \in V$ and $\alpha \in U^*$ by*

$$\lambda_{U,V}(v \otimes \alpha)(u) = \alpha(u)v$$

is an isomorphism if U or V are finite dimensional.

From this corollary we see that if V is a finite dimensional vector space, then

$$V^* \otimes V \cong V \otimes V^* \cong \mathrm{End}(V)$$

Since duality will be a regular theme, θ will be of interest to us again later when we discuss algebras and coalgebras and the important connection between them.

2.2 Modules

Modules are an important generalization of vector spaces. Most of the ideas we considered with vector spaces can be extended to modules, but require some modification. However, since we are primarily interested in vector spaces, this section will be rather brief.

To motivate the idea of a module, let V be a vector space over a field κ and let $\tau \in \mathcal{L}(V)$. Now consider the ring of polynomials $\kappa[x]$. For any $p(x) \in \kappa[x]$ we know that the operator $p(\tau)$ is well-defined and has the form

$$p(\tau) = \lambda_0 \iota + \lambda_1 \tau + \lambda_2 \tau^2 + \ldots + \lambda_n \tau^n$$

where $\lambda_j \in \kappa$ (all j), ι is the identity operator and the notational convention τ^m refers to the iterated composition of τ with itself m times. We can now define a product

$\varsigma : \kappa[x] \times V \to V$ by

$$\varsigma(p(x), v) = p(x)v = p(\tau)(v)$$

If it isn't already obvious, note the similarity of this map to the scalar product for vector spaces. The only difference is that we have allowed the role of the scalar space to be played by a mere ring rather than a field. To see this is, in fact, a generalized scalar product we can check that the usual properties are satisfied. Let $r(x), s(x) \in \kappa[x]$ and $u, v \in V$. Then

$$\begin{aligned} r(x)(u+v) &= r(\tau)[u+v] \\ &= \lambda_0 \iota(u+v) + \ldots + \lambda_n \tau^n(u+v) \\ &= \lambda_0 u + \lambda_0 v + \ldots + \lambda_n \tau^n(u) + \lambda_n \tau^n(v) \quad [\tau \in \mathcal{L}(V)] \\ &= r(\tau)(u) + r(\tau)(v) \quad \text{[after rearranging]} \\ &= r(x)u + r(x)v \end{aligned}$$

By similar reasoning we also find that

$$\begin{aligned} (r(x) + s(x))u &= r(x)u + s(x)u \\ [r(x)s(x)]u &= r(x)[s(x)u] \\ 1u &= u \end{aligned}$$

This example suggests a new structure, which, as mentioned above, generalizes the notion of a vector space in an important way. We define this new structure as follows:

Definition 2.21 (Module)**.** Let R be an arbitrary ring. A left module over R (i.e. left R-module) is an abelian group M under addition together with an operation of R on M ($R \times M \to M$ with $(r, m) \mapsto rm$) such that for all $r, s \in R$ and $x, y \in M$ we have

$$\begin{aligned} r(x+y) &= rx + ry \\ (r+s)x &= rx + sx \\ (rs)x &= r(sx) \\ 1x &= x \end{aligned}$$

Though it might seem, superficially, that weakening the requirement of the scalar space from being a field to just a ring is insignificant, the result is actually quite drastic. For instance, every vector space has a basis, but most modules do not. For example, consider the abelian group $\mathbb{Z}_2$ as a $\mathbb{Z}$-module. It's not hard to see that $\mathbb{Z}_2$ has no linearly independent subsets and hence no basis. Even when a module does have a basis, it need not have a unique rank.

Thankfully, as was mentioned above, many important results about vector spaces do carry over to modules. For instance, one gets quotient modules along with analogous isomorphism theorems, direct sums and products and even tensor products. We won't, however, concern ourselves too much with this. The important thing to keep in mind is that modules embody the notion of an object acting on another object.

As a note, the above definition is actually for a *left* module. Right modules are similarly defined and if we happen to be working with a commutative ring, then the distinction is superfluous. But because everything that can be proved for a left module easily transfers to right modules we will simply refer to left modules as modules. Here are a few common examples.

Example 2.1. *Any ring R is a module over itself.*

Example 2.2. *Any commutative group is a $\mathbb{Z}$-module.*

Example 2.3. *All vector spaces are modules over their respective fields.*

Example 2.4. *Let R be a ring. Then the set $\mathcal{M}_{m,n}(R)$ of all matrices of size $m \times n$ is an R-module under the usual operations of matrix addition and scalar multiplication over R. Since R is a ring, we can even take the product of matrices in $\mathcal{M}_{m,n}(R)$ provided $n = m$.*

2.2.1 Noetherian Rings and Noetherian Modules

One concept from ring and module theory that will be of use is the notion of being Noetherian. This is to be understood as follows.

Theorem 2.22. *Let R be a ring. The following statements are equivalent:*

1. *Any left ideal I of R is finitely generated - i.e. there exist $a_1, ..., a_n$ in I such that*

$$I = Ra_1 + \ldots + Ra_n$$

2. *Any ascending sequence $I_1 \subset I_2 \subset I_3 \subset \ldots \subset R$ of left ideals of R is stationary - i.e. there exists an integer k such that $I_{k+i} = I_k$ for all $i \geq 0$.*

A ring R is said to be Noetherian if it satisfies one of the above equivalent conditions.

Example 2.5. *Consider the polynomial ring $\kappa[x]$ where κ is a field. Let I be a proper non-zero ideal of $\kappa[x]$ and let $f \in I$ be of minimal degree. If $g \in I$, then there exist $q, r \in \kappa[x]$ such that $g = fq + r$ with $r = 0$ or $\deg(r) < \deg(f)$. Now, $r = g - fq$ and* *hence is a member of I. But f has minimal degree, which forces $r = 0$. Thus, $g = fq$ demonstrating that any other member of I is a "multiple" of f. So, any non-zero proper ideal I of $\kappa[x]$ is principal, which satisfies condition 1. Therefore, $\kappa[x]$ is Noetherian.*

As with rings, an R-module M is *Noetherian* if it satisfies the ascending chain condition but with submodules substituted for ideals. That is, if $L_1 \subseteq L_2 \subseteq \ldots$ is any ascending chain of submodules in M, then there is k for which $L_j = L_k$ for all $j \geq k$.

Theorem 2.23. *The following statements regarding R-modules are equivalent:*

1. *M is Noetherian*
2. *Every submodule of M is finitely generated.*
3. *Every nonempty set $\{M_\alpha\}$ of submodules of M has a maximal element with respect to set inclusion.*

For a proof, see [9].

2.2.2 Artinian Rings

As a matter of interest, just as there is an ascending chain condition there is the opposite notion of a descending chain condition. Given a sequence of left ideals $I_1, I_2, I_3, \ldots$, we say it is a descending chain if

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots$$

and we say that the descending chain is *stationary* if it is finite or there exists an integer k such that $I_{k+i} = I_k$ for all $i \in \mathbb{N}$. Now, if R is a ring such that every descending chain of left ideals is stationary, then R is said to be (left) *Artinian.*

Example 2.6. *Let $R = M_n(D)$ where D is a division ring. Any left ideal I of R is a D-subspace and if $I_1 \supseteq I_2 \supseteq \ldots$ is a descending chain, then*

$$\dim_D(I_{k+1}) \leq \dim_D(I_k) \leq n^2$$

for all k and so the chain must be stationary. Therefore $M_n(D)$ is Artinian.

This chapter has addressed some of the key foundational concepts required for delving deeper into the theory of quantum groups. In summary, we explicated a new and important way of creating new vector spaces with the tensor product which is universal for bilinearity. We have also introduced the essential notion of duality and with that the important concept of the transpose of a linear transformation, which we will use next chapter. Finally, we briefly reviewed the notion of a module which will be needed later when we discuss actions and comodules. With these tools in hand, let us proceed to the next level in our pursuit of the fundamentals of quantum groups.

Chapter 3

Algebras and Coalgebras

3.1 Algebras

Intuitively speaking, a κ-Algebra is just a vector space over a field κ equipped with a bilinear *vector product*. This product is essentially a multiplication in that it provides a second way (other than addition) to combine two vectors and obtain a third. This multiplication must satisfy certain compatibility axioms with the vector space structure (e.g. distributivity and scalar multiplication).

Definition 3.1 (Algebra)**.** An algebra A over a field κ is a non-empty set A, together with three operations, called addition (+), multiplication (juxtaposition) and scalar multiplication (also juxtaposition) for which the following properties hold:

(i) A is a vector space over κ under addition and scalar multiplication.

(ii) A is a ring under addition and multiplication.

(iii) If $\lambda \in \kappa$ and $a, b \in A$ then

$$\lambda(ab) = (\lambda a)b = a(\lambda b)$$

Now suppose only that A is a ring. If we let $\eta_A : \kappa \to A$ be a ring map from a field κ to the ring A such that $\mathrm{Im}(\eta_A) \subset Z(A)$ (the center of A), then we can equip A with a vector space structure by defining scalar multiplication, $\varsigma_A : \kappa \times A \to A$, by

$$\varsigma(\lambda, a) = \eta_A(\lambda)a$$

The easiest way to denote η_A is to set $\eta_A(\lambda) := \lambda \cdot 1_A$ for all $\lambda \in \kappa$. It is referred to as the *unit* map. If we denote the multiplication map by $\mu_A : A \times A \to A$, it is easy to see that it is bilinear. Note that for any algebra A, the fact that multiplication is bilinear gives the unique linear transformation $A \otimes A \to A$, which sends $a \otimes b \to ab$. In accord with custom we will also denote this linear map by μ.

In summary, an algebra $(A, \cdot, +; \kappa)$ over a field κ is a ring $(A, \cdot, +)$ and an action of κ on A (i.e. scalar multiplication) which is compatible with both the product and addition. So, $(A, \cdot, +)$ is a ring and $(A, +; \kappa)$ is a vector space such that (iii) above holds.

Really we are saying that algebras can be approached from two points of view. On the one hand, we can obtain an algebra by first starting with a vector space $(A, +; \kappa)$ and equipping it with a vector product (i.e. multiplication). On the other hand, we can begin instead with a ring $(A, \cdot, +)$ and infuse it with a vector space structure via the unit map $\eta_A : \kappa \to A$ as indicated above.

From the perspective of category theory, we are half way to establishing a category of algebras. We have the objects, namely the algebras themselves, and now we need the morphisms between these objects. Below we give two equivalent definitions. The first is a more "standard" definition, while the second is suited for the conversion to coalgebras, which we will come to a bit later.

Definition 3.2 (Algebra Morphism (standard))**.** Let A and B be algebras over a field κ. A morphism of algebras is a map $f : A \to B$ such that

$$f(\lambda ab + c) = \lambda f(a)f(b) + f(c)$$

for all $\lambda \in \kappa$ and all $a, b, c \in A$.

More intuitively, f is a linear map which also preserves multiplication. This is represented by the commuting diagram:

$$\begin{tikzcd}
A \times A \arrow[r, "f \times f"] \arrow[d, "\mu_A"'] & B \times B \arrow[d, "\mu_B"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

Our second definition makes use of the ring map $\eta_A : \kappa \to A$ given above.

Definition 3.3 (Algebra Morphism (alternative))**.** Let A and B be algebras over a field κ. A morphism of algebras or algebra morphism is a ring map $f : A \to B$ such that

$$f \circ \eta_A = \eta_B$$

This relation can be better pictured via the commuting diagram:

$$\begin{tikzcd}
\kappa \arrow[d, "\eta_A"'] \arrow[dr, "\eta_B"] & \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

In this definition, all that is assumed *prima facie* about f is that it is a morphism of rings which means that for all $a, b, c \in A$

$$f(ab + c) = f(a)f(b) + f(c) \quad \text{and} \quad f(1) = 1$$

It is the added relationship with η_A and η_B that gives "scalar slideout", since

$$\begin{aligned}
f(\lambda a) &= f(\lambda \cdot 1_A \cdot a) \\
&= f(\lambda \cdot 1_A) f(a) \\
&= \lambda \cdot 1_B \cdot f(a) \\
&= \lambda f(a)
\end{aligned}$$

This means that our alternative definition implies the satisfaction of the standard definition. That the standard version implies the alternative definition is obvious. We denote the space of all such morphisms by $\hom_{Alg}(A, B)$ (or by $Alg_\kappa(A, B)$) where A and B are any two algebras. We now have a category of algebras, which we denote by Alg from Figure 1.1.

3.1.1 Common Examples of Algebras

Example 3.1 (Opposite Algebra)**.** *Let A be any algebra, then we let A^{op} be the vector space A, but with multiplication defined by*

$$\mu_{A^{op}} := \mu_A \circ \tau_{A,A}$$

where $\tau_{A,A}$ is the transposition map interchanging the order of factors of $A \times A$. More specifically, if $a, b \in A^{op}$, then $\mu_{A^{op}}(ab) = ba$.

Immediately we see that an algebra is commutative if and only if $\mu_{A^{op}} = \mu_A$.

Example 3.2 (Polynomial Algebra). *Let κ be a field. Let $\kappa[x]$ be the space of all polynomials in the indeterminate x. That is*

$$\kappa[x] := \left\{ \sum_{i=0}^{\infty} \lambda_i x^i : \forall i \ \lambda_i \in \kappa, \lambda_i = 0 \textit{ for almost all } i \right\}$$

It is a well known fact that this forms a ring under addition and multiplication of polynomials. But scalar multiplication is also obviously well defined, which makes $\kappa[x]$ into an algebra.

Example 3.3 ($n \times n$ Matrix Algebra). *Let κ be a field. We denote the space of all $n \times n$ matrices with entries in κ by $M_n(\kappa)$. Under the usual operations of matrix addition, multiplication and scaling $M_n(\kappa)$ is an associative algebra, but not, unless $n = 1$, commutative.*

Example 3.4 (Algebra of Endomorphisms). *Let V be a κ-vector space. Then $End(V)$, the space of all endomorphisms on V, is an algebra under the usual function addition and function composition for multiplication. Since endomorphisms on a vector space are linear we get another endomorphism by multiplying by a scalar from κ.*

Example 3.5 (Group Algebra). *The group algebra is reminiscent of the polynomial algebra, which explains the notation $\kappa[G]$ where κ is a field and G is a group with operation $*$. Here the elements of G form a basis for the space. Formally we write*

$$\kappa[G] := \left\{ \sum_{g \in G} \lambda_g g : \lambda_g = 0 \textit{ for almost all } g \in G \right\}$$

Like the polynomial algebra, addition is given by

$$\sum_{g \in G} \lambda_g g + \sum_{g \in G} \gamma_g g = \sum_{g \in G} (\lambda_g + \gamma_g) g$$

and scalar multiplication by

$$\lambda \sum_{g \in G} \gamma_g g = \sum_{g \in G} (\lambda \gamma_g) g$$

The multiplication map makes use of the group operation $*$ and is defined by

$$\left(\sum_{g \in G} \lambda_g g \right) \left(\sum_{g \in G} \gamma_g g \right) = \sum_{g,h \in G} (\lambda_g \gamma_h) g * h$$

This will be an important example in Chapter 4.

Example 3.6 (Quotient Algebra). *Let A be an algebra and I a two-sided ideal of A. We can then create the quotient vector space A/I and endow it with a unique algebra structure. As a quotient vector space addition and scalar multiplication are already well-defined. For multiplication, if $a + I$ and $b + I$ are in A/I, then define their product* *by*

$$(a+I)(b+I) := ab + I$$

This is well-defined since if $a + I = a' + I$ and $b + I = b' + I$, then $a - a' \in I$ and $b - b' \in I$. The fact that I is a two sided ideal tells us $(a - a')b = ab - a'b \in I$ and also that $a'(b - b') = a'b - a'b' \in I$. But then $ab - a'b + a'b - a'b' = ab - a'b' \in I$ thereby implying that $ab + I = a'b' + I$. Therefore this multiplication is independent of the choice of a representative and hence well-defined.

Note, here, that because multiplication is well-defined, the linear canonical projection $\pi : A \to A/I$ extends to an algebra morphism. That is,

$$\begin{aligned} \pi(ab) &= ab + I \\ &= (a+I)(b+I) \\ &= \pi(a)\pi(b) \end{aligned}$$

3.1.2 Setting The Stage: A Preliminary Result

The point of this section is to establish a categorical result which will be instrumental in our study of quantum groups. Specifically, we shall establish the connection between the categories $FinSet$ and $\mathcal{A}$ (see below) in Figure 1.1. Although seemingly isolated, we will return to this and similar investigations throughout the thesis. We start with defining a semi-simple ring.

There are a couple of ways to understand what a semi-simple ring is. For instance, from a homological perspective, a ring R is semi-simple if and only if all of its left modules are projective. For our purposes, we will use a more basic definition.

Definition 3.4 (Semi-simple Ring)**.** A ring R is said to be semi-simple if it is the direct sum of minimal left ideals.

According to the Wedderburn-Artin Theorem, if R is a semi-simple ring, then

$$R \cong \prod_{i=1}^{k} M_{n_i}(D_i) \qquad \text{as rings}$$

where each D_i is a division ring. If R is also a $\mathbb{C}$-algebra, then each division ring D_i is isomorphic to $\mathbb{C}$ as rings so that

$$R \cong \prod_{i=1}^{k} M_{n_i}(\mathbb{C})$$

Furthermore, if R is also commutative, then necessarily $n_i = 1$ for all i yielding:

$$R \cong \prod_{i=1}^{k} \mathbb{C} := \mathbb{C}^k$$

As a $\mathbb{C}$-vector space, then, we get that $\dim_{\mathbb{C}} R = k$.

Recall from basic ring theory that an ideal I of a ring R is *maximal* if whenever

$$I \subseteq J \subseteq R$$

$J = I$ or $J = R$. For $\mathbb{C}^k$ there are exactly k such ideals, say $M_1, M_2, ..., M_k$ where

$$M_i = \{v \in \mathbb{C}^k : i^{th} \text{ component of } v \text{ is } 0\}$$

and $\dim M_i = k - 1$. Since R is commutative, each M_i is also prime, meaning that for any $a, b \in R$ with $ab \in M_i$, either $a \in M_i$ or $b \in M_i$ (or both). In case R is commutative and Artinian, then prime and maximal are the same. But for *any* commutative ring R with identity

$$\bigcap_{I prime} I = nil(R)$$

where $nil(R)$ denotes the set of all nilpotent elements of R. Thus, for any commutative Artinian ring

$$Jac(R) = \bigcap_{M maximal} M = \bigcap_{I prime} I = nil(R)$$

where $Jac(R)$ is the Jacobson radical of R.

Now let A be a finite dimensional commutative $\mathbb{C}$-algebra with no nilpotent elements. As in Example 2.6, since A is of finite dimension, every ideal will be a subspace so that any descending chain must be stationary. This implies that A is Artinian. But A has no nilpotent elements, which means $nil(A) = 0$, implying that $Jac(A) = 0$ and therefore A is semi-simple (as a ring). We therefore find that

$$A \cong \mathbb{C}^k \qquad \text{as rings with } k = \dim_{\mathbb{C}} A$$

Next consider $Alg_{\mathbb{C}}(A, \mathbb{C})$. For any $\alpha \in Alg_{\mathbb{C}}(A, \mathbb{C})$ we have that $\alpha(1_A) = 1_{\mathbb{C}}$ and hence for any $z \in \mathbb{C}$

$$\alpha(z \cdot 1_A) = z\alpha(1_A) = z \cdot 1_{\mathbb{C}} = z$$

This implies that α is onto and therefore, by the first isomorphism theorem for rings

$$A/Ker(\alpha) \cong \mathbb{C}$$

Since $\mathbb{C}$ is a field, this implies that $Ker(\alpha)$ is a maximal ideal. But A is commutative and Artinian, which means $Ker(\alpha)$ is also a prime ideal.

The set of all proper prime ideals of a ring R is called the *spectrum* of R and denoted by $Spec(R)$. Thus, $Ker(\alpha) \in Spec(A)$. What we get is a function

$$\begin{aligned} Alg_{\mathbb{C}}(A, \mathbb{C}) &\to Spec(A) \\ \alpha &\mapsto Ker(\alpha) \end{aligned}$$

Notice that this map is onto, since if M is one of the k maximal ideals of A, then A/M is isomorphic to $\mathbb{C}$.

$$\begin{tikzcd} A \arrow[r, "\pi"] \arrow[rr, bend left, "\alpha"] & A/M \arrow[r, "\sim"] & \mathbb{C} \end{tikzcd}$$

The above diagram shows that this allows us to get an $\alpha \in Alg_{\mathbb{C}}(A, \mathbb{C})$ for which

$$M = Ker(\alpha)$$

Not only is $Alg_{\mathbb{C}}(A, \mathbb{C}) \to Spec(A)$ onto, but it is also one to one, for suppose that $Ker(\alpha_1) = Ker(\alpha_2) = M$. Then, as a subspace, $\dim M = k - 1$ as mentioned above and so

$$A = M \oplus \mathbb{C} \cdot 1_A \qquad \text{as vector spaces}$$

Now, α_1 and α_2 agree on M, since both vanish. Notice too that $\alpha_1 = \alpha_2$ on $\mathbb{C} \cdot 1_A$, since

$$\alpha_1(z \cdot 1_A) = z \cdot 1_{\mathbb{C}} = z$$

and

$$\alpha_2(z \cdot 1_A) = z \cdot 1_{\mathbb{C}} = z$$

Hence

$$Alg_{\mathbb{C}}(A, \mathbb{C}) \leftrightarrow Spec(A)$$

and since there are exactly k maximal/prime ideals of A we have

$$|Spec(A)| = k = \dim_{\mathbb{C}} A$$

Now switch gears and consider an arbitrary set X. Then $\mathbb{C}^X$ represents the set of all functions from X to $\mathbb{C}$. We can think of $\mathbb{C}^X$ as a $\mathbb{C}$-vector space with basis

$$\{\delta_x : x \in X\}, \quad \delta_x(y) = \begin{cases} 1, & \text{if } y = x \\ 0, & \text{if } y \neq x \end{cases}$$

Clearly, the size of the basis is determined by the size of X and hence $\dim \mathbb{C}^X = |X|$.

Now, for each $x \in X$, let $\mathbb{C}_x$ be a copy of $\mathbb{C}$ corresponding to x. We then get a mapping

$$\mathbb{C}^X \to \prod_{x \in X} \mathbb{C}_x$$
$$f \mapsto (f(x))_{x \in X}$$

which is clearly bijective and hence an isomorphism of vector spaces. But $\mathbb{C}^X$ and $\prod \mathbb{C}_x$ can be viewed as $\mathbb{C}$-algebras where the multiplication in $\mathbb{C}^X$ is the ordinary product of functions and the multiplication in $\prod \mathbb{C}_x$ is component wise. It is then clear that the above vector space isomorphism preserves these products and so

$$\mathbb{C}^X \cong \prod_{x \in X} \mathbb{C}_x \qquad \text{as } \mathbb{C}\text{-algebras}$$

Because $\mathbb{C}$ has no (non-zero) nilpotent elements, neither does $\mathbb{C}^X$. Furthermore, $\mathbb{C}^X$ is commutative. So, we have that $\mathbb{C}^X$ is a commutative $\mathbb{C}$-algebra with no (non-zero) nilpotent elements. If X is also finite, then $\dim \mathbb{C}^X = |X|$ and per what we found above we have

$$Alg_{\mathbb{C}}(\mathbb{C}^X, \mathbb{C}) \leftrightarrow Spec(\mathbb{C}^X)$$
$$|Spec(\mathbb{C}^X)| = \dim \mathbb{C}^X = |X|$$

What we get is a contravariant *functor*:

$$FinSet \to \mathcal{A}$$

where $FinSet$ is the category of finite sets and $\mathcal{A}$ is the category of finite dimensional, commutative $\mathbb{C}$-algebras with no nilpotent elements.

If f is an arbitrary element of $\mathbb{C}^X$, then we can express it as

$$f = \sum_{x \in X} \lambda_x \delta_x$$

and so for any $\alpha \in Alg_{\mathbb{C}}(\mathbb{C}^X, \mathbb{C})$ we have

$$\alpha(f) = \alpha\big(\sum_{x \in X} \lambda_x \delta_x\big) \tag{3.1}$$

$$= \sum_{x \in X} \lambda_x \alpha(\delta_x) \quad [\alpha \text{ is linear}] \tag{3.2}$$

For the identity we have

$$1_{\mathbb{C}^X} = \sum_{x \in X} \delta_x$$

and for any $\alpha \in Alg_{\mathbb{C}}(\mathbb{C}^X, \mathbb{C})$, $\alpha(1_{\mathbb{C}^X}) = 1_{\mathbb{C}}$ so therefore

$$\alpha(1_{\mathbb{C}^X}) = \sum_{x \in X} \alpha(\delta_x) = 1_{\mathbb{C}} \tag{3.3}$$

Now, based on the definition of δ_x it is clear it is idempotent for all $x \in X$ - i.e. $\delta_x^2 = \delta_x$. Moreover, $\alpha(\delta_x^2) = (\alpha(\delta_x))^2$, since α is an algebra morphism. Therefore

$$(\alpha(\delta_x))^2 = \alpha(\delta_x)$$

which means that $\alpha(\delta_x)$ is an idempotent in $\mathbb{C}$. However, there are only two elements in $\mathbb{C}$ having this property, namely 0 and 1. Therefore, $\alpha(\delta_x) = 0$ or $\alpha(\delta_x) = 1$ for any $x \in X$. Given (3.3), this implies that there is a unique x such that $\alpha(\delta_x) = 1$ and $\alpha(\delta_y) = 0$ for $y \neq x$. This unique x we label by x_α and so we get a function

$$Alg_{\mathbb{C}}(\mathbb{C}^X, \mathbb{C}) \to X$$
$$\alpha \mapsto x_\alpha$$

This means that for $f \in \mathbb{C}^X$, using (3.2), that

$$\alpha(f) = \sum_{x \in X} \lambda_x \alpha(\delta_x)$$
$$= \lambda_{x_\alpha}$$

But $f(x_\alpha) = \sum_{x \in X} \lambda_x \delta_x(x_\alpha) = \lambda_{x_\alpha}$ and therefore

$$\alpha(f) = f(x_\alpha)$$

so that α is really evaluation at x_α.

Suppose now we fix an element $x_0 \in X$. Then

$$f \mapsto f(x_0)$$

is a function $\mathbb{C}^X \to \mathbb{C}$. But it is also clearly an algebra morphism and so we get

$$X \to Alg_{\mathbb{C}}(\mathbb{C}^X, \mathbb{C})$$
$$x \mapsto E_x$$

where E_x is evaluation at x, $E_x(f) = f(x)$ is a bijection and hence

$$X \equiv Alg_{\mathbb{C}}(\mathbb{C}^X, \mathbb{C}) \quad \text{as sets}$$

So, for any A we have that $A \cong \mathbb{C}^X \cong \mathbb{C}^k$ where $X = Alg_{\mathbb{C}}(A, \mathbb{C})$ or $X = Spec(A)$ and $k = |X|$.

What we get from all this is that our contravariant functor establishes an equivalence of categories, namely

$$FinSet^{op} \cong \mathcal{A}$$

the "inverse" being $A \to Alg_{\mathbb{C}}(A, \mathbb{C})$ or, alternatively $A \to Spec(A)$. We will build off this result throughout this thesis, especially in Chapter 4. In fact, as a bit of foreshadowing, one natural question is to consider what happens when $FinSet^{op}$ is replaced with $FinGp^{op}$ (the opposite category of finite groups). In other words, we shall be interested to know what equivalence comes of this replacement.

$$FinGp^{op} \cong (?)$$

3.1.3 Free Algebras

Definition 3.5 (Free Algebra)**.** Let X be a set. Then the *free algebra* on the set X is the vector space $\kappa\{X\}$ with basis the set of all words $x_{i_1} \dots x_{i_p}$ in the alphabet X, including the empty word $\varnothing$. Multiplication is given by concatenation of words - i.e.

$$(x_{i_1} \dots x_{i_p})(x_{i_{p+1}} \dots x_{i_n}) = x_{i_1} \dots x_{i_p} x_{i_{p+1}} \dots x_{i_n}$$

Here the empty word acts as unit - i.e. $\varnothing = 1$. A word is referred to as a monomial and its degree is given by the length of the word. There is a universal property for free algebras, which is stated in the following theorem.

Theorem 3.6. *Let X be a set. Given an algebra A and a set-theoretic map $f : X \to A$, there exists a unique algebra morphism $\overline{f} : \kappa\{X\} \to A$ such that $\overline{f}(x) = f(x)$ for all $x \in X$.*

For a proof see [7]. Another way of saying this is that

$$\hom_{Alg}(\kappa\{X\}, A) \equiv \hom_{Set}(X, A)$$

so $X \mapsto \kappa\{X\}$ is the left adjoint of the inclusion functor $Alg \hookrightarrow Set$. If X is finite with $|X| = n$, then in particular we get that

$$\hom_{Alg}(\kappa\{X\}, A) \equiv A^n$$

Any algebra A is the quotient of a free algebra $\kappa\{X\}$ and an appropriate ideal I of $\kappa\{X\}$. For example, simply take A and forget that it is an algebra, then create the free algebra $\kappa\{A\}$ and take I to be the two-sided ideal of $\kappa\{A\}$ generated by the elements

$$(a+b)\cdot c - a\cdot c - b\cdot c, \quad a\cdot(b+c) - a\cdot b - a\cdot c$$

$$(\lambda a)\cdot(\lambda' b) - (\lambda\lambda')(a\cdot b)$$

for all $a, b, c \in A$ and $\lambda, \lambda' \in \kappa$. The algebra A is then recovered as the algebra $\kappa\{A\}/I$.

One important example is the polynomial algebra $\kappa[x_1, ..., x_n]$ in n variables. This algebra is isomorphic to the quotient algebra $\kappa\{x_1, ..., x_n\}/I$, where I is the two-sided ideal of $\kappa\{x_1, ..., x_n\}$ generated by all elements of the form $x_i x_j - x_j x_i$, $i, j = 1, ..., n$. These relations give us commutativity.

Proposition 3.7. *For any algebra A' there is a natural bijection*

$$\hom_{Alg}(\kappa\{X\}/I, A') \equiv \{f \in \hom_{Set}(X, A') : \overline{f}(I) = 0\}$$

Proof. This is a consequence of the fact that $\hom_{Alg}(\kappa\{X\}, A) \equiv \hom_{Set}(X, A)$. □

Corollary 3.8. *For any algebra A the following natural bijection holds:*

$$\hom_{Alg}(\kappa[x_1, ..., x_n], A) \equiv \{(a_1, ..., a_n) \in A^n : a_i a_j = a_j a_i \textit{ for all } (i, j)\}$$

Proof. This follows by a straightforward application of the previous Proposition. □

Note that if A is a commutative algebra, then we simply get

$$\hom_{Alg}(\kappa[x_1, ..., x_n], A) \equiv A^n, \quad \text{as sets}$$

From this we get an important functor from the category of commutative κ-algebras to the category of sets where $A \overset{F}{\mapsto} A^n$ can be thought of as

$$A \mapsto \hom_{Alg}(\kappa[x_1, ..., x_n], A) \equiv A^n$$

This is what is known as a *representable functor* being represented by $\kappa[x_1, ..., x_n]$. The polynomial algebra $\kappa[x_1, ..., x_n]$ is itself referred to as the *representing object* for F. Generally, a representable functor $Alg \overset{F}{\mapsto}$ **Set** is called an *affine scheme*. Another example is given by $\kappa\{X\}$ where $A \overset{F}{\mapsto} A^X$ is represented by $\kappa\{X\}$ as a functor $Alg \overset{F}{\mapsto}$ **Set**. It is a well known and famous result that there exists a type of duality between commutative algebra and geometry. For this reason, the elements of $\hom_{Alg}(\kappa[x_1, ..., x_n], A)$ are often referred to as A-points of $\kappa[x_1, ..., x_n]$. We will deal with this again in a more specific context in Chapter 6 when we consider the affine line and the affine plane.

We end this section with one last result which will be of use later. For our purposes, we will omit a proof and simply take it for granted.

Proposition 3.9. *Let A be an algebra with generating set X and a set of defining relations R. Then if B is any algebra and $f : X \to B$ is a function such that f preserves the relations of R, then f extends uniquely to an algebra morphism $f : A \to B$.*

3.1.4 Tensor Products of Algebras

As promised, just as we are able to generate new vector spaces out of old ones using the tensor product, if A and B are algebras over κ, we can define an algebra structure on $A \otimes B$. This can be done in a straightforward way. Let $a \otimes b, a' \otimes b' \in A \otimes B$, where $a, a' \in A$ and $b, b' \in B$. We then take the product of these pure tensors to be given by

$$(a \otimes b)(a' \otimes b') := aa' \otimes bb'$$

and then extend additively to all of $A \otimes B$. Such a product will be well defined since, as a tensor product of vector spaces, $A \otimes B$ has a basis, meaning all elements have unique expressions as linear combinations of this basis. So, suppose $\{a_i\}_{i \in I}$ is a basis for A and $\{b_j\}_{j \in J}$ is a basis for B. Then we know $\{a_i \otimes b_j\}_{(i,j) \in I \times J}$ is a basis for $A \otimes B$ as a vector space. Now suppose $a \otimes b = a' \otimes b'$ and $c \otimes d = c' \otimes d'$. We want to show that

$$ac \otimes bd = a'c' \otimes b'd'$$

If

$$a = \sum_i \lambda_i a_i, \quad a' = \sum_i \lambda'_i a_i, \quad b = \sum_j \gamma_j b_j, \quad b' = \sum_j \gamma'_j b_j$$

$$c = \sum_i \zeta_i a_i, \quad c' = \sum_i \zeta'_i a_i, \quad d = \sum_j \xi_j b_j, \quad d' = \sum_j \xi'_j b_j$$

then

$$a \otimes b = \sum_{i,j} \lambda_i \gamma_j (a_i \otimes b_j) = \sum_{i,j} \lambda'_i \gamma'_j (a_i \otimes b_j) = a' \otimes b'$$

$$c \otimes d = \sum_{i,j} \zeta_i \xi_j (a_i \otimes b_j) = \sum_{i,j} \zeta'_i \xi'_j (a_i \otimes b_j) = c' \otimes d'$$

which implies that $\lambda_i \gamma_j = \lambda'_i \gamma'_j$ for all i, j and $\zeta_i \xi_j = \zeta'_i \xi'_j$ for all i, j. With this we have

$$ac \otimes bd = \sum_{i,j,k,\ell} \lambda_i \gamma_j \zeta_k \xi_\ell (a_i a_j \otimes b_k b_\ell)$$

$$a'c' \otimes b'd' = \sum_{i,j,k,\ell} \lambda'_i \gamma'_j \zeta'_k \xi'_\ell (a_i a_j \otimes b_k b_\ell)$$

Since the coefficients will clearly be equal it follows that $ac \otimes bd = a'c' \otimes b'd'$.

The unit, of course, is $1_A \otimes 1_B$, which we abbreviate to $1 \otimes 1$. $A \otimes B$ also clearly contains isomorphic copies of A and B as sub-algebras. This can be seen by defining embedding maps:

$$i_A : A \to A \otimes B, \quad i_B : B \to A \otimes B$$
$$i_A(a) = a \otimes 1, \quad i_B(b) = 1 \otimes b$$

for all $a \in A$ and $b \in B$. Note further that these maps are algebra morphisms. They will help us establish a universal property for the tensor product of algebras.

Theorem 3.10. *Let $f : A \to C$ and $g : B \to C$ be algebra morphisms such that, for any pair $(a, b) \in A \times B$, the relation $f(a)g(b) = g(b)f(a)$ holds in C. Then there exists a unique morphism of algebras $f \otimes g : A \otimes B \to C$ such that $(f \otimes g) \circ i_A = f$ and $(f \otimes g) \circ i_B = g$.*

This can be pictured in the following commuting digram.

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[d, "i_A"'] & & C \\
A \otimes B \arrow[urr, dashed, "f \otimes g"] & & B \arrow[ll, "i_B"] \arrow[u, "g"']
\end{tikzcd}$$

The essence of this theorem is that

$$\begin{aligned}\hom_{Alg}(A \otimes B, C) &\cong \{(f,g) : \forall a \in A, b \in B, f(a)g(b) = g(b)f(a) \in C\} \\ &\subseteq \hom_{Alg}(A, C) \times \hom_{Alg}(B, C)\end{aligned}$$

with equality holding in case C is commutative.

Proof. Suppose $\alpha : A \otimes B \to C$ is an algebra morphism satisfying the relations $\alpha \circ i_A = f$ and $\alpha \circ i_B = g$. Then

$$\begin{aligned}\alpha(a \otimes b) &= \alpha((a \otimes 1)(1 \otimes b)) \\ &= \alpha(a \otimes 1)\alpha(1 \otimes b) \\ &= (\alpha \circ i_A)(a)(\alpha \circ i_B)(b) \\ &= f(a)g(b)\end{aligned}$$

which means that α is uniquely determined by f and g. Thus, such an algebra morphism is unique.

Let us therefore *define* $\alpha : A \otimes B \to C$ by $\alpha(a \otimes b) := f(a)g(b)$. All that is needed, then, is to verify that this definition entails that α is an algebra morphism.

For $a \otimes b, a' \otimes b' \in A \otimes B$ we have

$$\begin{aligned}\alpha((a \otimes b)(a' \otimes b')) &= \alpha(aa' \otimes bb') \\ &= f(aa')g(bb') \\ &= f(a)f(a')g(b)g(b') \\ &= f(a)g(b)f(a')g(b') \quad \text{[commutativity condition]} \\ &= \alpha(a \otimes b)\alpha(a' \otimes b')\end{aligned}$$

and, hence, α is an algebra morphism. Now, since α is the unique algebra morphism out of $A \otimes B$ determined by f and g we denote α by $f \otimes g$. $\square$

An important result which we will make use of in the sequel is given in the following theorem:

Theorem 3.11. *Let $A = \kappa\{X\}/I$ be a quotient of the free algebra on a set X. Take two copies of X, say X_1 and ${}_1X$. Let I_1 and ${}_1I$ be the corresponding ideals in $\kappa\{X_1\}$ and $\kappa\{{}_1X\}$. Then the tensor product algebra $A \otimes A$ is isomorphic to the algebra*

$$A^{\otimes 2} := \kappa\{X_1 \sqcup {}_1X\}/(I_1, {}_1I, X_1 \cdot {}_1X - {}_1X \cdot X_1)$$

where $X_1 \sqcup {}_1X$ *denotes the disjoint union of the two copies and where* $X_1 \cdot {}_1X - {}_1X \cdot X_1$ *is the two-sided ideal generated by all elements of the form*

$$x_1 \cdot {}_1x - {}_1x \cdot x_1$$

with $x_1 \in X_1$ *and* ${}_1x \in {}_1X$*.*

Proof. For any $x \in X$, take the corresponding copy in X_1 to be x_1 and the corresponding copy in ${}_1X$ to be ${}_1x$. Now define a function $f : X_1 \sqcup {}_1X \to A \otimes A$ by

$$f(x_1) := x \otimes 1 \quad \text{and} \quad f({}_1x) := 1 \otimes x$$

By Theorem 3.6 this extends to a unique morphism of algebras $\overline{f} : \kappa\{X_1 \sqcup {}_1X\} \to A \otimes A$. Now,

$$A \otimes A = \frac{\kappa\{X\}}{I} \otimes \frac{\kappa\{X\}}{I}$$

so it is clearly the case that $\overline{f}(I_1) = \overline{f}({}_1I) = 0$. But we also want

$$X_1{}_1X - {}_1XX_1 \subset Ker(\overline{f})$$

Consider, then, what $\overline{f}$ does to the generators $x_1{}_1x - {}_1xx_1$.

$$\begin{aligned}\overline{f}(x_1 \cdot {}_1x - {}_1x \cdot x_1) &= \overline{f}(x_1)\overline{f}({}_1x) - \overline{f}({}_1x)\overline{f}(x_1)\\ &= (x \otimes 1)(1 \otimes x) - (1 \otimes x)(x \otimes 1)\\ &= (x \otimes x) - (x \otimes x) = 0\end{aligned}$$

We therefore get a uniquely induced algebra morphism $\overline{f} : A^{\otimes 2} \to A \otimes A$.

By Theorem 3.10 there exists an algebra morphism $g : A \otimes A \to A^{\otimes 2}$ such that

$$g(x \otimes y) = x_1 \cdot {}_1y$$

Notice that g is the inverse of $\overline{f}$, since

$$\begin{aligned}\overline{f}(g(x \otimes y)) &= \overline{f}(x_1 \cdot {}_1y)\\ &= \overline{f}(x_1)\overline{f}({}_1y)\\ &= (x \otimes 1)(1 \otimes y)\\ &= x \otimes y\end{aligned}$$

and

$$\begin{aligned} g(\overline{f}({}_1xy_1)) &= g(\overline{f}({}_1x)\overline{f}(y_1)) \\ &= g((1 \otimes x)(y \otimes 1)) \\ &= g(y \otimes x) \\ &= y_1 \cdot {}_1x = {}_1xy_1 \end{aligned}$$

This is sufficient, since $\overline{f}$ and g are algebra morphisms and so this inverse property will extend over all sums and products. Therefore, we may conclude that

$$A \otimes A \cong A^{\otimes 2}$$

□

3.1.4.1 Multilinear Maps and Iterated Tensor Products

Theorem 3.11 is important, at least in part, because it provides motivation for what is known as the *tensor algebra* of a vector space V, which we introduce below. First we do some brief preliminary work that will extend the notion of bilinearity and hence the tensor product of vector spaces.

Definition 3.12. Let $V_1, V_2, ..., V_n$ and W be vector spaces over a field κ. A function

$$f : V_1 \times \ldots \times V_n \to W$$

is said to be *multilinear* if it is linear in each variable separately - i.e.

$$f(u_1, ..., u_{k-1}, \lambda v + \gamma v', u_{k+1}, ..., u_n) = \lambda f(u_1, ..., u_{k-1}, v, u_{k+1}, ..., u_n) + \gamma f(u_1, ..., u_{k-1}, v', u_{k+1}, ..., u_n)$$

for all $k = 1, ..., n$.

Multilinear functions in n variables are also commonly known as *n-linear functions.* We denote the set of all n-linear functions by

$$\hom_\kappa(V_1, ..., V_n; W)$$

A multilinear function $f : V_1 \times \ldots \times V_n \to \kappa$ is called a *multilinear form* or *n-form.*

We can extend the notion of tensor products of vector spaces in two ways, each being respectively similar to how the tensor product was defined above. In light of this, our discussion will be brief.

$\boxed{\text{First Way}}$

Let $\mathcal{B}_i = \{e_{i,j} : j \in J_i\}$ be a basis for V_i $(i = 1, ..., n)$. For each n-tuple, $(e_{1,i_1}, ..., e_{n,i_n})$, we create a new "object" denoted

$$e_{1,i_1} \otimes \ldots \otimes e_{n,i_n}$$

and then define T to be the vector space with basis

$$\mathcal{D} = \{e_{1,i_1} \otimes \ldots \otimes e_{n,i_n} : e_{k,i_k} \in \mathcal{B}_k\}$$

Now define a map $t : V_1 \times \ldots \times V_n \to T$ by setting

$$t(e_{1,i_1}, ..., e_{n,i_n}) = e_{1,i_1} \otimes \ldots \otimes e_{n,i_n}$$

Finally, extend by multilinearity and we have a unique multilinear map, which is as "universal" as possible among multilinear maps.

$\boxed{\text{Second Way}}$

Let $V_1, ..., V_n$ be vector spaces over a field κ and let T be the subspace of the free vector space $\mathcal{F}$ on $V_1 \times \ldots \times V_n$ generated by all vectors of the form

$$\begin{aligned}\lambda(v_1, ..., v_{k-1}, u, v_{k+1}, ..., v_n) + \gamma(v_1, ..., v_{k-1}, u', v_{k+1}, ..., v_n)\\ - (v_1, ..., v_{k-1}, \lambda u + \gamma u', v_{k+1}, ..., v_n)\end{aligned}$$

for all $\lambda, \gamma \in \kappa$ and vectors from their appropriate spaces. We then take the quotient space $\mathcal{F}/T$ to be the tensor product of $V_1, ..., V_n$ and write

$$V_1 \otimes \ldots \otimes V_n$$

As before, a typical element $(v_1, ..., v_n) + T$ is written as $v_1 \otimes \ldots \otimes v_n$. Thus, any element of $V_1 \otimes \ldots \otimes V_n$ is a sum of pure tensors:

$$\sum v_{i_1} \otimes \ldots \otimes v_{i_n}$$

where the vector space operations are linear in each variable.

3.1.4.2 Important Multilinear Maps

Having introduced the notion of mulilinear maps, it will be important here to make special mention of a select subset of these maps.

Definition 3.13 (Symmetric Multilinear Map)**.** A multilinear map $f : V^n \to W$ is called *symmetric* if the image of any element is invariant under any permutation of it's coordinate positions - i.e.

$$f(v_1, ..., v_n) = f(v_{\sigma(1)}, ..., v_{\sigma(n)})$$

for all $\sigma \in S_n$.

Definition 3.14 (Antisymmetric Multilinear Map)**.** A multilinear map $f : V^n \to W$ is called *antisymmetric* (or *skew-symmetric*) if

$$f(v_1, ..., v_n) = (-1)^\sigma f(v_{\sigma(1)}, ..., v_{\sigma(n)})$$

where

$$(-1)^\sigma = \begin{cases} -1 & \text{if } \sigma \text{ is odd} \\ 1 & \text{if } \sigma \text{ is even} \end{cases}$$

Definition 3.15 (Alternating Multilinear Map)**.** A multilinear map $f : V^n \to W$ is called *alternating* if

$$f(v_1, ..., v_n) = 0$$

whenever $v_i = v_j$ and $i \neq j$.

Note, if the characteristic of the underlying vector space is not 2, then every antisymmetric multilinear map is also an alternating map and vice versa. To see this, let $f : V^n \to W$ be an antisymmetric multilinear map. Suppose $v = (x_1, ..., x_n) \in V^n$ is such that $x_i = x_j$ for some $i \neq j$. Then

$$\begin{aligned} f(x_1, ..., x_i, ..., x_j, ...x_n) &= -f(x_1, ..., x_j, ..., x_i, ..., x_n) \\ &= -f(x_1, ..., x_i, ..., x_j, ...x_n) \end{aligned}$$

and hence

$$2f(x_1, ..., x_i, ..., x_j, ...x_n) = 0$$

But the characteristic is not 2 and therefore $f(x_1, ..., x_i, ..., x_j, ...x_n) = 0$ thereby making f alternating. The other direction is actually always true regardless of the characteristic.

Like bilinear maps, multilinear maps possess a universal property. This is expressed in the following theorem, which we shall merely state. The reader not satisfied only with statements may consult [8] for a proof.

Theorem 3.16. *Let $V_1, ..., V_n$ be vector spaces over the field κ. The pair $(V_1 \otimes \ldots \otimes V_n, t)$, where*

$$t : V_1 \times \ldots \times V_n \to V_1 \otimes \ldots \otimes V_n$$

is the multilinear map defined by

$$t(v_1, ..., v_n) = v_1 \otimes \ldots \otimes v_n$$

has the following property. If $f : V_1 \times \ldots \times V_n \to W$ *is any multilinear map to a vector space* W *over* κ*, then there is a unique linear transformation* $\tau : V_1 \otimes \ldots \otimes V_n \to W$ *for which*

$$\tau \circ t = f$$

That is to say, the following diagram commutes.

$$\begin{tikzcd} V_1 \times \ldots \times V_n \arrow[r, "t"] \arrow[dr, "f"'] & V_1 \otimes \ldots \otimes V_n \arrow[d, dashed, "\tau"] \\ & W \end{tikzcd}$$

3.1.5 Graded Algebras

In this section we briefly consider an important classifying property possessed by certain algebras, namely a grading, which essentially amounts to being able to express these algebras as a special kind of direct sum that respects multiplication. In the next section we will see that it is a property possessed by the, very important, tensor algebra.

Definition 3.17 (Graded Algebra)**.** An algebra A over a field κ is said to be *graded* if as a vector space over κ, A can be written as the direct sum of a family of subspaces $(A_i)_{i\in\mathbb{N}}$ - i.e.

$$A = \bigoplus_{i\in\mathbb{N}} A_i$$

and such that multiplication behaves according to

$$A_i \cdot A_j \subseteq A_{i+j} \qquad \text{for all } i, j \in \mathbb{N}$$

The elements of A_i are said to be *homogenous of degree* i, which is essentially a sort of "equivalence". The unit of a graded algebra will belong to A_0.

A simple, but instructive example of a graded algebra is the free algebra on a set X. Such an algebra is graded by the length of the words. This means that the subspace A_i of $\kappa\{X\}$ is defined to be the subspace linearly generated by all monomials (words) of length i.

On a related note, the polynomial algebra $\kappa[x]$ is a prime example of a graded algebra. Simply define $\kappa_i[x]$ to be the subspace of $\kappa[x]$ consisting of all scalar multiples of x^i. Clearly, then, we have that

$$\kappa[x] = \bigoplus_{i=0}^{\infty} \kappa_i[x]$$

This example, of course, can be generalized to the algebra of polynomials in several variables, namely $\kappa[x_1, ..., x_n]$. In this case, each subspace consists of homogeneous polynomials of degree i, where a homogeneous polynomial of degree i is one whose terms are all of degree i. For instance, $x_1x_2^2 + x_1x_2x_3$ is an instance of homogeneous polynomial of degree 3.

3.1.6 The Tensor Algebra

The tensor algebra is one of our most important examples. In fact, it will be instrumental in Chapter 5 when we define something called a universal enveloping algebra, which, in turn, leads to our most important quantum group. We will also consider the tensor algebra again in the next chapter.

To start, we use our antecedent work and define

$$T^0(V) := \kappa, \quad T^1(V) := V \quad \text{and} \quad T^n(V) := V^{\otimes n}$$

for any vector space V, which we can now take to mean the tensor product of n copies of V. Using this idea we define the vector space, $T(V)$ by

$$T(V) := \bigoplus_{i=0}^{\infty} T^i(V)$$

This vector space has an induced associative product coming from the canonical isomorphisms

$$T^n(V) \otimes T^m(V) \cong T^{n+m}(V)$$

which equips it with an algebra structure. So considered, $T(V)$ is called the *tensor algebra* of V. Notice that the tensor algebra is very clearly an example of a graded algebra where $T^n(V)$ is the space of homogeneous elements of degree n, where degree n, like the free algebra, refers to length of "words". For example, if $a, b, c \in V$, then $a \otimes b \otimes b + a \otimes b \otimes c$ is in $T^3(V)$ and is a homogeneous element of degree 3.

Admittedly, operating in $T(V)$ is a bit "clunky", since a product takes the explicit form

$$(x_1 \otimes \ldots \otimes x_n)(x_{n+1} \otimes \ldots \otimes x_{n+m}) = x_1 \otimes \ldots \otimes x_n \otimes x_{n+1} \otimes \ldots \otimes x_{m+n}$$

where $x_1, ..., x_n, x_{n+1}, ..., x_{n+m} \in V$. We can make this more convenient by realizing that the unit for this product is the image of $1 \in \kappa = T^0(V)$. Let i_V be the canonical embedding of $V = T^1(V)$ into $T(V)$. We therefore have that

$$x_1 \otimes \ldots \otimes x_n = i_V(x_1) \ldots i_V(x_n)$$

which permits us to write $x_1 \otimes \ldots \otimes x_n = x_1 \cdots x_n$ whenever $x_1, ..., x_n$ are in V. Note, this is merely a convenience at our disposal and won't necessarily be the norm. Instead, the reader will be notified when it is in use and should be expected only when notation might become convoluted. The reader should also note the clear resemblance to free algebras. In fact, the next result establishes a concrete connection between the two.

Proposition 3.18. *(a) The algebra $T(V)$ is graded such that $T^n(V)$ is the subspace of degree n homogeneous elements.*

(b) [Universal Property] For any algebra A and any linear map $f : V \to A$, there exists a unique algebra morphism $\overline{f} : T(V) \to A$ such that $\overline{f} \circ i_V = f$. Consequently, the map $\overline{f} \mapsto \overline{f} \circ i_V$ is a natural bijection:

$$\hom_{Alg}(T(V), A) \cong \hom(V, A)$$

and so T is the left adjoint of the inclusion functor $Alg \hookrightarrow Vec$.

(c) Let I be an indexing set for a basis B of the vector space V. Then the tensor algebra $T(V)$ is isomorphic to the free algebra $\kappa\{I\}$.

Proof. (a) Since $T(V) = \bigoplus_{i=0}^{\infty} T^i(V)$ by definition and $T^n(V) \otimes T^m(V) \cong T^{n+m}(V)$ it is immediate that $T(V)$ is graded with the subspaces $T^n(V)$ consisting of degree n homogeneous elements.

(b) For any integer $n \geq 0$ define $f_n : V^n \to A$ to be the multilinear map

$$f_n(v_1, ..., v_n) = f(v_1) \cdots f(v_n)$$

By Theorem 3.16 there is a unique linear map $\bar{f}_n : V^{\otimes n} \to A$ with

$$\bar{f}_n(v_1 \otimes \ldots \otimes v_n) = f(v_1) \cdots f(v_n)$$

Putting all these maps together yields a linear, and clearly algebra, morphism

$$\bar{f} : T(V) \to A$$

which will also be unique given that V generates $T(V)$ as an algebra. Now, for any $v \in V$ we find that $(\bar{f} \circ i_V)(v) = \bar{f}(v) = f_1(v) = f(v)$. Thus, (b) is established.

(c) Let $B := \{b_i\}_{i \in I}$ be a basis for V. Since B is indexed by I there is a bijection between them with $i \leftrightarrow b_i$, which we may use to identify I and B. Using this we have

$$\begin{tikzcd}
I \arrow[r, hook, "i"] \arrow[d, leftrightarrow] & \kappa\{I\} \\
B \arrow[ur, "i'"] \arrow[d, hook] & \\
V \arrow[uur, "\exists!\hat{i}"'] &
\end{tikzcd}$$

In the diagram, $\hat{i}$ is the linear transformation extending i' and, via the identification of B with I, also extending i. If we now apply the universal property for $T(V)$ and the universal property for $\kappa\{I\}$ we get

$$\begin{tikzcd}
I \arrow[r, hook, "i"] \arrow[d] & \kappa\{I\} \arrow[ddd, bend left, "\overline{i_V}"] \\
B \arrow[d, hook] & \\
V \arrow[uur, "\hat{i}"] \arrow[d, "i_V"'] & \\
T(V) \arrow[uuur, "\bar{\hat{i}}"] &
\end{tikzcd}$$

where $\bar{\hat{i}}$ is the unique algebra morphism such that $\bar{\hat{i}} \circ i_V = \hat{i}$ and $\overline{i_V}$ is the unique algebra morphism such that $\overline{i_V}(i) = i_V(e_i)$ for all $i \in I$ or $i_V = \overline{i_V} \circ \hat{i}$. From these we find that

$$\hat{i} = (\bar{\hat{i}} \circ \overline{i_V}) \circ \hat{i}, \qquad i_V = (\overline{i_V} \circ \bar{\hat{i}}) \circ i_V$$

But because $\overline{i_V}$ and $\bar{\hat{i}}$ are unique and

$$\hat{i} = \mathrm{id} \circ \hat{i}, \qquad i_V = \mathrm{id} \circ i_V$$

it must be that $\bar{\hat{i}} \circ \overline{i_V} = \mathrm{id} = \overline{i_V} \circ \bar{\hat{i}}$. It follows that $T(V) \cong \kappa\{I\}$.

$\square$

As indicated in (b), T can be thought of as a functor. If $\alpha : V \to W$ is a linear map, then

$$T\alpha : T(V) \to T(W)$$

is the algebra map obtained from setting $A = T(W)$. It is this functor T that gives the connection from Vec to Alg, namely $V \mapsto T(V)$, in Figure 1.1.

As we have seen, the tensor algebra is something like a tensor version of a free algebra. Recall, too, that if we start with the free algebra $\kappa\{x_1, ..., x_n\}$ and factor out the ideal I of $\kappa\{x_1, ..., x_n\}$ generated by all elements of the form $x_i x_j - x_j x_i$, where i, j run over $\{1, .., n\}$, then the result is the polynomial algebra $\kappa[x_1, ..., x_n]$. There is an important analogue of this in the case of the tensor algebra called the *symmetric algebra*. It is defined by taking the tensor algebra $T(V)$ of a vector space V and factoring out the ideal $I(V)$ generated by all elements of the form $x \otimes y - y \otimes x$, where x and y run over V. That is,

$$S(V) := T(V)/I(V)$$

The symmetric algebra is also graded and the image of $T^n(V)$ under the projection of $T(V)$ onto $S(V)$ is denoted $S^n(V)$. For this reason, the symmetric algebra can also be expressed as

$$S(V) = \bigoplus_{i=0}^{\infty} S^i(V)$$

The analogue result for the symmetric algebra is as follows.

Proposition 3.19. *(a) The algebra $S(V)$ is commutative, and is graded such that $S^n(V)$ is the subspace of degree n homogeneous elements. (b) For any algebra A and any linear map $f : V \to A$ such that $f(x)f(y) = f(y)f(x)$ for any $x, y \in V$, there exists a unique algebra morphism $\bar{f} : S(V) \to A$ such that $\bar{f} \circ i_v = f$, where i_v is the canonical map from $V = T^1(V)$ to $S(V)$. (c) If I is an indexing set for a basis of V, then the symmetric algebra $S(V)$ is isomorphic to the polynomial algebra $\kappa[I]$ on the set I.*

One interesting result, that does not hold for the tensor algebra, is that for any vector space V', we have an isomorphism

$$S(V \oplus V') \cong S(V) \otimes S(V')$$

Note, too, that (b) implies the bijection

$$\hom_{Alg}(S(V), A) \equiv \hom(V, A)$$

when A is commutative.

3.2 Coalgebras

Now that we have defined the concept of an algebra and the language of tensor products, we introduce the *dual* of an algebra, known as a *coalgebra*. The use of "dual" here is different from how we used it before, but, as we will come to see, there is a connection between the two. Because coalgebras are dual to algebras, we shall begin with a reformulation of the definition and axioms of an algebra in terms of tensor products so as to make this form of duality more transparent.

Before giving the tensor definition of algebra, recall that if A is an algebra over a field κ, then it is also a vector space over κ. Thus, $A \otimes \kappa \cong A \cong \kappa \otimes A$ as vector spaces. But κ is also an algebra in its own right. We can therefore think of $A \otimes \kappa$ as a tensor product of algebras. We then get that $A \otimes \kappa \cong A \cong \kappa \otimes A$ as algebras under the isomorphism $a \otimes 1 \mapsto a$ (resp. $1 \otimes a \mapsto a$).

Definition 3.20. A κ-algebra is a triple (A, μ, η) where A is a vector space and

$$\mu : A \otimes A \to A \quad \text{and} \quad \eta : \kappa \to A$$

are linear maps such that the diagrams:

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "\mu \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes \mu"'] & A \otimes A \arrow[d, "\mu"] \\
A \otimes A \arrow[r, "\mu"] & A
\end{tikzcd}$$

and

$$\begin{tikzcd}
\kappa \otimes A \arrow[r, "\eta \otimes \mathrm{id}"] \arrow[dr, "\sim"] & A \otimes A \arrow[d, "\mu"] & A \otimes \kappa \arrow[l, "\mathrm{id} \otimes \eta"'] \arrow[dl, "\sim"'] \\
& A &
\end{tikzcd}$$

commute.

The first diagram essentially shows that multiplication (μ) is associative and the second shows that $\eta(1_\kappa)$ is the unit for μ. Note that an algebra is commutative if, in addition to the above, the following diagram commutes:

$$\begin{tikzcd} A \otimes A \arrow[rr, "\tau_{A,A}"] \arrow[dr, "\mu"'] & & A \otimes A \arrow[dl, "\mu"] \\ & A & \end{tikzcd}$$

where $\tau_{A,A}$ is the transposition map that swaps the factors of an element $a \otimes b \in A \otimes A$. That is

$$\tau_{A,A}(a \otimes b) = b \otimes a$$

The reader should note that if we represent the product $\mu(a \otimes b)$ by ab, then the distributive property follows from the distributivity of the tensor product over addition along with the linearity of μ. More explicitly, if $a, b, c \in A$, then

$$\begin{aligned} \mu(a \otimes (b + c)) &= \mu(a \otimes b + a \otimes c) \\ &= \mu(a \otimes b) + \mu(a \otimes c) \end{aligned}$$

which translates to

$$a(b + c) = ab + ac$$

Note the use of the tensor product with the linear functions involved in the above diagrams (e.g. $\mu \otimes \mathrm{id}$). Recall (following Proposition 2.16) that this usage is to be understood as follows:

If $f : U \to U'$ and $g : V \to V'$ are linear maps, then their tensor product

$$f \otimes g : U \otimes V \to U' \otimes V'$$

is given by

$$(f \otimes g)(u \otimes v) = f(u) \otimes g(v)$$

for all $u \in U$ and $v \in V$.

To match what we will see below with coalgebra morphisms, we shall give a different version of the definition for an algebra morphism.

Definition 3.21 (Algebra Morphism Revisited)**.** Let (A, μ_A, η_A) and (B, μ_B, η_B) be two κ-algebras. The κ-linear map $f : A \to B$ is a morphism of algebras if the following diagram commutes:

$$\begin{tikzcd}
A \otimes A \arrow[rr, "f \otimes f"] \arrow[dd, "\mu_A"'] & & B \otimes B \arrow[dd, "\mu_B"] \\
& \kappa \arrow[dl, "\eta_A"'] \arrow[dr, "\eta_B"] & \\
A \arrow[rr, "f"] & & B
\end{tikzcd}$$

The outer square is the tensor version of the diagram in Definition 3.2, and the inner triangle says that f respects the units of A and B.

We now proceed to define a coalgebra. As stated in the opening of this section, coalgebras are *dual* to algebras, and so we return to this central concept which was introduced in the previous chapter. Here, however, is where we begin to see some of the nuance. The duality referred to in this context is of a quasi-categorical nature. That is, given an object (e.g. algebra) there is an associated object (here a coalgebra) which is obtained by reversing all the arrows in the diagrams. Where this version of duality departs from the category version is that there is some asymmetry involved because of the linear maps used in defining algebras. In other words, if it were *merely* a matter of reversing arrows, then there would be no need to study coalgebras separately because every result about algebras would have a corresponding result obtained by reversing the direction of arrows. We proceed, then, with the intention of seeing where this "asymmetry" leads.

Definition 3.22 (Coalgebra)**.** A coalgebra is triple (C, Δ, ε) where C is a vector space, while $\Delta : C \to C \otimes C$ and $\varepsilon : C \to \kappa$ are linear maps such that the following diagrams commute:

$$\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\mathrm{id} \otimes \Delta"] \\
C \otimes C \arrow[r, "\Delta \otimes \mathrm{id}"] & C \otimes C \otimes C
\end{tikzcd}$$

and

$$\begin{tikzcd}
\kappa \otimes C & C \otimes C \arrow[l, "\varepsilon \otimes \mathrm{id}"'] \arrow[r, "\mathrm{id} \otimes \varepsilon"] & C \otimes \kappa \\
& C \arrow[ul, "\sim"] \arrow[u, "\Delta"] \arrow[ur, "\sim"'] &
\end{tikzcd}$$

The first diagram represents *coassociativity* and the second the *counit*. The map Δ is called the *coproduct* (or comultiplication) and ε is called the *counit* of the coalgebra. We also say that the coalgebra is cocommutative if

$$\begin{tikzcd} & C \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\ C \otimes C \arrow[rr, "\tau_{C,C}"] & & C \otimes C \end{tikzcd}$$

commutes. Finally, note that the isomorphism $\kappa \otimes C \cong C \cong C \otimes \kappa$ holds under the same mapping as the algebra case, namely $c \mapsto c \otimes 1$ (resp. $c \mapsto 1 \otimes c$).

This is not an altogether unfamiliar concept. Suppose we just have a set X and let $\Delta : X \to X \times X$ be the diagonal function $\Delta(x) = (x, x)$. Let ε be the unique map $X \to 1$ where 1 is the singleton $\{*\}$. It is clearly the case that

$$X \times 1 \cong X \cong 1 \times X$$

Finally, let 1_X be the identity function on X. We then have the following commutative diagrams:

$$\begin{tikzcd} X \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & X \times X \arrow[d, "1_X \times \Delta"] \\ X \times X \arrow[r, "\Delta \times 1_X"'] & X \times X \times X \end{tikzcd}$$

and

$$\begin{tikzcd} 1 \times X & X \times X \arrow[l, "\varepsilon \times 1_X"'] \arrow[r, "1_X \times \varepsilon"] & X \times 1 \\ & X \arrow[ul, "\sim"] \arrow[u, "\Delta"] \arrow[ur, "\sim"'] & \end{tikzcd}$$

which have the same form as the coalgebra axioms.

Example 3.7. *We can also motivate the definition of a coalgebra with our work from Section 3.1.2. Take X to be a finite group G. Then per the definition of a group, G has structure maps*

$$\begin{array}{ccc} m : G \times G \to G & u : \{1\} \to G & S : G \to G \\ (g,h) \mapsto g * h & 1 \mapsto e & g \mapsto g^{-1} \end{array}$$

where $*$ is the group operation and e is the group unit. Now consider the space $\mathbb{C}^G$. The multiplication of the group G induces a map $\mathbb{C}^G \to \mathbb{C}^{G\times G}$ where $f \in \mathbb{C}^G$ is mapped to $\tilde{f}$ defined by $\tilde{f}(g_1, g_2) = f(g_1 * g_2)$. But $\mathbb{C}^{G\times G} \cong \mathbb{C}^G \otimes \mathbb{C}^G$ where $\alpha \in \mathbb{C}^{G\times G}$ corresponds to $\alpha_1 \otimes \alpha_2 \in \mathbb{C}^G \otimes \mathbb{C}^G$ with $\alpha_1(g) = \alpha(g, 1)$ and $\alpha_2(g) = \alpha(1, g)$. This isomorphism will be considered in a more general setting and with greater detail in Chapter 4. But taking this for granted, we actually have an induced map $\Delta : \mathbb{C}^G \to \mathbb{C}^G \otimes \mathbb{C}^G$ where $\Delta(f) = \tilde{f}_1 \otimes \tilde{f}_2$.

Likewise, $u : \{1\} \to G$ induces a map $\mathbb{C}^G \to \mathbb{C}^{\{1\}} \cong \mathbb{C}$ where

$$f \mapsto f \circ u \mapsto f(u(1)) = f(e)$$

Call this induced map ε.

Finally, $S : G \to G$ induces a map $\mathbb{C}^G \to \mathbb{C}^G$ where

$$f \mapsto f \circ S$$

So, for each structure map on the group G we get a corresponding linear structure map. To recap, these are

$$\begin{array}{ccc} \Delta : \mathbb{C}^G \to \mathbb{C}^G \otimes \mathbb{C}^G & \varepsilon : \mathbb{C}^G \to \mathbb{C} & S : \mathbb{C}^G \to \mathbb{C}^G \\ f \mapsto \Delta(f) & f \mapsto f(e) & f \mapsto S(f) \end{array}$$

where Δ, ε and S are the transpose maps of m, u and S respectively. As already indicated, this means, in particular, that

$$\Delta(f)(g,h) := f(gh) \quad \textit{and} \quad S(f)(g) := f(g^{-1})$$

For our current purposes, however, we need not worry about the map S. Instead, consider Δ. Note that since the group multiplication is associative we have the commuting diagram

$$\begin{tikzcd} G \times G \times G \arrow[r, "m \times id"] \arrow[d, "id \times m"'] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"] & G \end{tikzcd}$$

Now, $G \to \mathbb{C}^G$ is a functor (we'll consider this later) and so we immediately get the corresponding commutative diagram

$$\begin{tikzcd}
\mathbb{C}^{G\times G\times G} & \mathbb{C}^{G\times G} \arrow[l] \\
\mathbb{C}^{G\times G} \arrow[u] & \mathbb{C}^{G} \arrow[l] \arrow[u]
\end{tikzcd}$$

and via the above isomorphism, the above diagram yields

$$\begin{tikzcd}
\mathbb{C}^G \otimes \mathbb{C}^G \otimes \mathbb{C}^G & \mathbb{C}^G \otimes \mathbb{C}^G \arrow[l, "\Delta \otimes id"'] \\
\mathbb{C}^G \otimes \mathbb{C}^G \arrow[u, "id \otimes \Delta"] & \mathbb{C}^G \arrow[l, "\Delta"] \arrow[u, "\Delta"']
\end{tikzcd}$$

or more specifically that

$$(\Delta \otimes id) \circ \Delta(f) = (id \otimes \Delta) \circ \Delta(f)$$

which is exactly the coproduct axiom in our definition. A similar procedure yields the counit axiom.

In the next chapter we'll show more completely that κ^G is a coalgebra for κ an arbitrary field and G a finite group.

At this point, let us make a change of notation. In the definition of algebra we used μ for multiplication. Now, there is nothing wrong with using μ, in fact, it is used because it is suggestive. Nevertheless, because we wish to emphasize the duality between algebras and coalgebras we shall denote the product by ∇ to highlight more obviously its connection to the coproduct Δ.

Definition 3.23 (Coalgebra Morphism)**.** Let $(C, \Delta_C, \varepsilon_C)$ and $(D, \Delta_D, \varepsilon_D)$ be two κ-coalgebras. The κ-linear map $g : C \to D$ is a *morphism of coalgebras* if the following diagram is commutative.

$$\begin{tikzcd}
C \arrow[rr, "g"] \arrow[dr, "\varepsilon_C"] \arrow[dd, "\Delta_C"'] & & D \arrow[dl, "\varepsilon_D"'] \arrow[dd, "\Delta_D"] \\
& \kappa & \\
C \otimes C \arrow[rr, "g \otimes g"] & & D \otimes D
\end{tikzcd}$$

We now have a category of coalgebras, which we denote by $CoAlg$ (see Figure 1.1).

Definition 3.24 (Subcoalgebra)**.** Let (C, Δ, ε) be a coalgebra. A κ-subspace D of C is called a subcoalgebra if

$$\Delta(D) \subseteq D \otimes D$$

Example 3.8 (Ground Coalgebra)**.** *A basic first example is the* ground coalgebra*. Just as the ground field κ is naturally an algebra, it also has a canonical coalgebra structure. The coproduct is determined by $\Delta(1) := 1 \otimes 1$ while the counit is simply $\varepsilon(1) := 1$. In this case, the counit is just the identity map, since*

$$\varepsilon(\lambda) = \varepsilon(\lambda \cdot 1) = \lambda\varepsilon(1) = \lambda \cdot 1 = \lambda$$

As for the coproduct, it is just the isomorphism $\kappa \cong \kappa \otimes \kappa$, since

$$\Delta(\lambda) = \lambda\Delta(1) = \lambda(1 \otimes 1) = \lambda \otimes 1 = 1 \otimes \lambda$$

Example 3.9 (Opposite Coalgebra)**.** *Another simple example is the* opposite coalgebra*. This is a more generic example because we can take any coalgebra C and get its opposite coalgebra denoted by C^{cop}. The coproduct for C^{cop} is defined by*

$$\Delta^{op} := \tau_{C,C} \circ \Delta$$

The counit, however, remains unchanged.

Example 3.10 (Trig-Coalgebra)**.** *A more concrete example is the* trigonometric coalgebra*. Let C be the κ-vector space having basis $\{s, c\}$. Define the coproduct by*

$$\Delta(s) := s \otimes c + c \otimes s$$
$$\Delta(c) := c \otimes c - s \otimes s$$

and the counit by

$$\varepsilon(s) := 0$$
$$\varepsilon(c) := 1$$

By way of illustration, let us show that c satisfies the coassociativity axiom. Going one direction we have

$$\begin{aligned}(\Delta \otimes id)(\Delta(c)) &= (\Delta \otimes id)(c \otimes c - s \otimes s) \\ &= \Delta(c) \otimes c - \Delta(s) \otimes s \\ &= (c \otimes c - s \otimes s) \otimes c - (s \otimes c + c \otimes s) \otimes s \\ &= c \otimes c \otimes c - s \otimes s \otimes c - s \otimes c \otimes s - c \otimes s \otimes s\end{aligned}$$

Going the other direction yields

$$\begin{aligned}(id \otimes \Delta)(\Delta(c)) &= (id \otimes \Delta)(c \otimes c - s \otimes s) \\ &= c \otimes \Delta(c) - s \otimes \Delta(s) \\ &= c \otimes (c \otimes c - s \otimes s) - s \otimes (s \otimes c + c \otimes s) \\ &= c \otimes c \otimes c - c \otimes s \otimes s - s \otimes s \otimes c - s \otimes c \otimes s\end{aligned}$$

Hence, we get identical results (save for the order of terms).

The reason for the name "trigonometric coalgebra" becomes clear once one realizes that the use of c and s is suggestive of cosine and sine. The definition of this coalgebra is based on the behavior of sine and cosine, since

$$\begin{aligned}\sin(x + y) &= \sin(x)\cos(y) + \cos(x)\sin(y) \\ \cos(x + y) &= \cos(x)\cos(y) - \sin(x)\sin(y) \\ \sin(0) &= 0 \\ \cos(0) &= 1\end{aligned}$$

We will revisit this example a little later.

Example 3.11 (Polynomial Coalgebra in One Variable)**.** *Next, consider the polynomial ring $\kappa[x]$. This becomes a coalgebra if we set*

$$\begin{aligned}\Delta(x^n) &:= (x \otimes 1 + 1 \otimes x)^n \qquad \varepsilon(x^n) := 0, \quad n \geq 1 \\ \Delta(1) &:= 1 \otimes 1 \qquad \varepsilon(1) := 1\end{aligned}$$

Let us verify that $id \otimes \Delta = \Delta \otimes id$ *on a generic basis element* x^n*. First, we find that*

$$\begin{aligned}
(id \otimes \Delta)\big(\Delta(x^n)\big) &= (id \otimes \Delta)\big((x \otimes 1 + 1 \otimes x)^n\big) \\
&= (id \otimes \Delta)\Big(\sum_{k=0}^{n} \binom{n}{k} x^{n-k} \otimes x^k\Big) \\
&= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} \otimes \Delta(x^k) \\
&= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} \otimes \Big(\sum_{m=0}^{k} \binom{k}{m} x^{k-m} \otimes x^m\Big) \\
&= \sum_{k=0}^{n} \sum_{m=0}^{k} \binom{n}{k} \binom{k}{m} x^{n-k} \otimes x^{k-m} \otimes x^m
\end{aligned}$$

Second, we find that

$$\begin{aligned}
(\Delta \otimes id)\big(\Delta(x^n)\big) &= (\Delta \otimes id)\big((x \otimes 1 + 1 \otimes x)^n\big) \\
&= (\Delta \otimes id)\Big(\sum_{k=0}^{n} \binom{n}{k} x^k \otimes x^{n-k}\Big) \\
&= \sum_{k=0}^{n} \binom{n}{k} \Delta(x^k) \otimes x^{n-k} \\
&= \sum_{k=0}^{n} \binom{n}{k} \Big(\sum_{m=0}^{k} \binom{k}{m} x^m \otimes x^{k-m}\Big) \otimes x^{n-k} \\
&= \sum_{k=0}^{n} \sum_{m=0}^{k} \binom{n}{k} \binom{k}{m} x^m \otimes x^{k-m} \otimes x^{n-k}
\end{aligned}$$

Clearly the two results are equal, since both are

$$\sum_{i+j+k=n} \binom{n}{i+j} \binom{i+j}{i} x^i \otimes x^j \otimes x^k$$

For the counit we will compute $(id \otimes \varepsilon)(\Delta(x^n))$ *only, since the other case is essentially symmetric. We have*

$$\begin{aligned}
(id \otimes \varepsilon)(\Delta(x^n)) &= (id \otimes \varepsilon)\Big(\sum_{k=0}^{n} \binom{n}{k} x^{n-k} \otimes x^k\Big) \\
&= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} \otimes \varepsilon(x^k) \\
&= x^n \otimes 1
\end{aligned}$$

We will return to this example below when we consider the finite dual, where we will see *something of a motivation behind defining the coalgebra structure in this manner. We will also consider it in Chapter 6 where the need for this particular coalgebra structure will be even more apparent.*

3.2.1 Sweedler's sigma notation

At this point we take an important detour in order to consider some convenient notation adopted by those working in this field. If x is an element of the coalgebra (C, Δ, ε), then $\Delta(x) \in C \otimes C$ is a finite sum of the form

$$\Delta(x) = \sum_i x_{1_i} \otimes x_{2_i}, \quad x_{1_i}, x_{2_i} \in C$$

Dealing with multiple subscripts, however, can become quite convoluted so we agree to abbreviate the above to

$$\Delta(x) = \sum_{(x)} x^{(1)} \otimes x^{(2)}$$

Some authors (usually of a physics persuasion) omit the summation sign completely and simply write $x^{(1)} \otimes x^{(2)}$ and take the superscript with parentheses to indicate that summation is intended. What is important to keep in mind is that this version is a purely *formal sum*; it is purely symbolic. More specifically, the $c^{(1)} \otimes c^{(2)}$ are not specific elements, but rather stand for generic elements as a way of separating the first factor from the second factor and are not uniquely determined.

What we can do with this notation is determined by the commuting diagrams that establish the particular structure with which we happen to be working. For instance, with such a convention at our disposal we can express the coassociativity of Δ by

$$\sum_{(x)} \Big(\sum_{(x^{(1)})} (x^{(1)})^{(1)} \otimes (x^{(1)})^{(2)} \Big) \otimes x^{(2)} = \sum_{(x)} x^{(1)} \otimes \Big(\sum_{(x^{(2)})} (x^{(2)})^{(1)} \otimes (x^{(2)})^{(2)} \Big)$$

This becomes

$$\sum_{(x)} \sum_{(x^{(1)})} (x^{(1)})^{(1)} \otimes (x^{(1)})^{(2)} \otimes x^{(2)} = \sum_{(x)} \sum_{(x^{(2)})} x^{(1)} \otimes (x^{(2)})^{(1)} \otimes (x^{(2)})^{(2)}$$

which we can write more succinctly as

$$\sum_{(x),(x^{(1)})} (x^{(1)})^{(1)} \otimes (x^{(1)})^{(2)} \otimes x^{(2)} = \sum_{(x),(x^{(2)})} x^{(1)} \otimes (x^{(2)})^{(1)} \otimes (x^{(2)})^{(2)}$$

This is where the power of Sweedler's convention becomes most evident. Using the convention a second time we identify both sides of this last equation with

$$\sum_{(x)} x^{(1)} \otimes x^{(2)} \otimes x^{(3)}$$

where we have three representative factors. In some sense this means that our choice of labeling is artificial. What is more important is what is implied. If the comultiplication is now applied yet again we find that

$$\sum_{(x)} \Delta(x^{(1)}) \otimes x^{(2)} \otimes x^{(3)} = \sum_{(x)} x^{(1)} \otimes \Delta(x^{(2)}) \otimes x^{(3)} = \sum_{(x)} x^{(1)} \otimes x^{(2)} \otimes \Delta(x^{(3)})$$

which we take to be the element

$$\sum_{(x)} x^{(1)} \otimes x^{(2)} \otimes x^{(3)} \otimes x^{(4)}$$

In other words, the following diagram commutes.

$$\begin{tikzcd}
 & H \otimes H \arrow[dr, "\mathrm{id} \otimes \Delta"] & & \\
H \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] & & H \otimes H \otimes H \arrow[r, bend left, "\Delta \otimes \mathrm{id} \otimes \mathrm{id}"] \arrow[r, "\mathrm{id} \otimes \Delta \otimes \mathrm{id}" description] \arrow[r, bend right, "\mathrm{id} \otimes \mathrm{id} \otimes \Delta"'] & H \otimes H \otimes H \otimes H \\
 & H \otimes H \arrow[ur, "\Delta \otimes \mathrm{id}"'] & &
\end{tikzcd}$$

FIGURE 3.1

In general, define the map $\Delta^{(n)} : C \to C^{\otimes(n+1)}$ inductively on $n \geq 1$ by $\Delta^{(1)} = \Delta$ and

$$\Delta^{(n)} = (\Delta \otimes \mathrm{id}_{C^{\otimes(n-1)}}) \circ \Delta^{(n-1)} = (\mathrm{id}_{C^{\otimes(n-1)}} \otimes \Delta) \circ \Delta^{(n-1)}$$

Following the above conventions we simply write

$$\Delta^{(n)}(x) = \sum_{(x)} x^{(1)} \otimes \ldots \otimes x^{(n+1)}$$

Furthermore, we can express the condition for counitality (see Definition 3.22) by

$$\sum_{(x)} \varepsilon(x^{(1)})x^{(2)} = x = \sum_{(x)} x^{(1)}\varepsilon(x^{(2)}) \tag{3.4}$$

for all $x \in C$. To see this, let $x \in C$. Sweedler's notation says $\Delta(x) = \sum_{(x)} x^{(1)} \otimes x^{(2)}$. If we now apply $\varepsilon \otimes \mathrm{id}$ to this we get $\sum_{(x)} \varepsilon(x^{(1)}) \otimes x^{(2)}$, but recall that $\kappa \otimes C$ is isomorphic to C under the mapping $x \mapsto 1 \otimes x$, so $\varepsilon(x^{(1)}) \otimes x^{(2)} \mapsto \varepsilon(x^{(1)})x^{(2)}$. The other equality is similarly shown.

To get a feel for the Sweedler notation we can express the commutativity of the first diagram in the definition of a coalgebra morphism as

$$\Delta_D(g(x)) = \sum_{(x)} g(x)^{(1)} \otimes g(x)^{(2)} = \sum_{(x)} g(x^{(1)}) \otimes g(x^{(2)})$$

Also, we can now say that a coalgebra is commutative if

$$\sum_{(x)} x^{(1)} \otimes x^{(2)} = \sum_{(x)} x^{(2)} \otimes x^{(1)}$$

for all $x \in C$.

3.2.2 Some Basic Coalgebra Theory

Definition 3.25 (Coideal)**.** Let (C, Δ, ε) be a coalgebra and I a κ- subspace of C. Then I is called:

1. a left (resp. right) coideal if $\Delta(I) \subseteq C \otimes I$ (resp. $\Delta(I) \subseteq I \otimes C$).
2. a coideal if
$$\Delta(I) \subset I \otimes C + C \otimes I$$
and $\varepsilon(I) = 0$.

In the case of an ordinary algebra, A, we say that a subspace I is an ideal if it is both a left and a right ideal. Oddly, this is not the case with a coideal. In fact, if I is a coideal, it may be that I is neither a left nor a right coideal. For instance, consider the polynomial ring $\kappa[x]$ mentioned above. Now consider the subspace spanned by x, namely κx. By definition, for any $\lambda x \in \kappa x$ we have that

$$\Delta(\lambda x) = \lambda(x \otimes 1 + 1 \otimes x) \in \kappa x \otimes \kappa[x] + \kappa[x] \otimes \kappa x$$

Also, it follows straight from the definition of ε that $\varepsilon(\kappa x) = 0$. Thus, κx is a coideal. Notice, though, that $\Delta(\lambda x) = x \otimes \lambda + \lambda \otimes x$ is not a member of $\kappa x \otimes \kappa[x]$ or $\kappa[x] \otimes \kappa x$.

But should I be a left and right coideal, then by Proposition 2.16

$$\Delta(I) \subseteq (C \otimes I) \cap (I \otimes C) = I \otimes I$$

and is thus a subcoalgebra.

We now proceed to establish an important theorem, which reveals a particular sort of finiteness property that is inherent to coalgebras.

Theorem 3.26 (The Fundamental Theorem of Coalgebras). *Every element of a coalgebra C is contained in a finite dimensional subcoalgebra.*

Proof. Let c be an arbitrary element in C. We want to show that $c \in D$ where D is a finite dimensional subcoalgebra (i.e. $\Delta(D) \subseteq D \otimes D$). In this proof it will actually be better to forego using Sweedler's notation. If we apply Δ to c we get a finite sum of the form

$$\Delta(c) = \sum_i a_i \otimes b_i$$

Now consider the element $\Delta^2(c) \in C \otimes C \otimes C$, which is a finite sum, that can be written as follows:

$$\begin{aligned}(\Delta \otimes \mathrm{id})(\Delta(c)) &= \sum_i \Delta(a_i) \otimes b_i &(3.5)\\ &= \sum_i \Big(\sum_j c_j \otimes a_{ij}\Big) \otimes b_i &(3.6)\\ &= \sum_{i,j} c_j \otimes a_{ij} \otimes b_i &(3.7)\end{aligned}$$

Now because C is a vector space, it has a basis, say $\{e_k\}_{k \in K}$. We can therefore express the c_j's and b_i's in terms of these basis elements (also finite sums) to get

$$\Delta^2(c) = \sum_{i,j} \Big(\sum_k \lambda_{jk} e_{jk}\Big) \otimes a_{ij} \otimes \Big(\sum_k \gamma_{ik} e_{ik}\Big)$$

Once we expand this out we can reindex to get something of the form (3.7) again. What this tells us is that, from the beginning, we can assume that we can write

$$\Delta^2(c) = \sum_{i,j} c_j \otimes a_{ij} \otimes b_i$$

where the c_j's and b_i's are linearly independent.

Take D to be the space spanned by the a_{ij}'s. Of course, D will be a finite dimensional subspace since there are only a finite number of a_{ij}'s. Now consider the following commuting diagram:

$$\begin{tikzcd}
C \arrow[r, "\sim"] \arrow[dd, "\Delta"'] & C \otimes \kappa \\
& \kappa \otimes C \otimes \kappa \arrow[u, "\wr"'] \\
C \otimes C \arrow[r, "\Delta \otimes \mathrm{id}"] & C \otimes C \otimes C \arrow[u, "\varepsilon \otimes \mathrm{id} \otimes \varepsilon"']
\end{tikzcd}$$

From this we get that $c = \sum_{i,j} \varepsilon(c_j)\varepsilon(b_i)a_{ij}$, which implies that $c \in D$. It therefore remains to show that D is a subcoalgebra - i.e. that $\Delta(D) \subseteq D \otimes D$. We know that

$$\sum_{i,j} \Delta(c_j) \otimes a_{ij} \otimes b_i = \sum_{i,j} c_j \otimes \Delta(a_{ij}) \otimes b_i$$

and because the b_i's are linearly independent, this implies that

$$\sum_{j} \Delta(c_j) \otimes a_{ij} = \sum_{j} c_j \otimes \Delta(a_{ij}) \quad \text{for all } i$$

It follows that $\sum_j c_j \otimes \Delta(a_{ij}) \in C \otimes C \otimes D$. But the c_j's are also linearly independent, which implies that $\Delta(a_{ij}) \in C \otimes D$ for all i, j. Using a symmetric argument one can also show that $\Delta(a_{i,j}) \in D \otimes C$ for all i, j. So, we have found that

$$\Delta(a_{ij}) \in (C \otimes D) \cap (D \otimes C) = D \otimes D \quad \text{for all } i, j$$

using Proposition 2.16 and hence $\Delta(D) \subseteq D \otimes D$ as desired. □

Theorem 3.27. *Let $f : C \to D$ be a coalgebra morphism. Then $Im(f)$ is a subcoalgebra of D and $Ker(f)$ is a coideal in C.*

Proof. It is a well known fact that $Ker(f)$ is a κ-subspace of C, while $Im(f)$ is a κ-subspace of D. Since f is a coalgebra morphism, the following diagram is commutative.

$$\begin{tikzcd}
C \arrow[r, "f"] \arrow[d, "\Delta_C"'] & D \arrow[d, "\Delta_D"] \\
C \otimes C \arrow[r, "f \otimes f"] & D \otimes D
\end{tikzcd}$$

Using this diagram we see that

$$\begin{aligned}\Delta_D(Im(f)) &= \Delta_D(f(C))\\ &= (f\otimes f)(\Delta_C(C))\\ &\subseteq (f\otimes f)(C\otimes C)\\ &= f(C)\otimes f(C)\\ &= Im(f)\otimes Im(f)\end{aligned}$$

which shows that $Im(f)$ is a subcoalgebra in D.

The diagram also indicates that since

$$\Delta_D(f(Ker(f))) = 0$$

then

$$(f\otimes f)(\Delta_C(Ker(f))) = 0$$

and using Proposition 2.17 it follows that

$$\Delta_C(Ker(f)) \subseteq Ker(f\otimes f) = Ker(f)\otimes C + C\otimes Ker(f)$$

Finally, since f is a coalgebra morphism we also get the following commutative diagram:

$$\begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "\varepsilon_C"'] & & D \arrow[dl, "\varepsilon_D"] \\ & \kappa & \end{tikzcd}$$

which tells us that

$$\varepsilon_C(Ker(f)) = \varepsilon_D\big(f(Ker(f))\big) = 0$$

and thus $Ker(f)$ is a coideal. $\square$

Having developed the idea of a coideal, we can now proceed to construct specific factor objects, which shall be called *factor coalgebras*. We will also see that such objects admit a universal property of their own.

Theorem 3.28. *Let C be a coalgebra, I a coideal and $\pi : C \to C/I$ the canonical projection of κ-vector spaces. Then:*

(i) There exists a unique coalgebra structure on C/I such that π is a morphism of coalgebras.

(ii) If $f : C \to D$ is a morphism of coalgebras with $I \subseteq Ker(f)$, then there exists a unique morphism of coalgebras $\overline{f} : C/I \to D$ for which $\overline{f} \circ \pi = f$.

Proof. (i) From the definition of a coideal, we have that

$$(\pi \otimes \pi)(\Delta(I)) \subseteq (\pi \otimes \pi)(I \otimes C + C \otimes I)$$

Observe that if $i \otimes c + c' \otimes i' \in I \otimes C + C \otimes I$, then

$$\begin{aligned}(\pi \otimes \pi)(i \otimes c + c' \otimes i') &= (\pi \otimes \pi)(i \otimes c) + (\pi \otimes \pi)(c' \otimes i') \\ &= \pi(i) \otimes \pi(c) + \pi(c') \otimes \pi(i') \\ &= 0 \otimes \pi(c) + \pi(c') \otimes 0 \\ &= 0\end{aligned}$$

So, $(\pi \otimes \pi)(I \otimes C + C \otimes I) = 0$ and thus also $(\pi \otimes \pi)(\Delta(I)) = 0$. If we view C as a vector space, then the universal property of the factor vector space implies that there exists a unique linear map

$$\overline{\Delta} : C/I \to C/I \otimes C/I$$

for which the following diagram commutes.

$$\begin{tikzcd} C \arrow[r, "\pi"] \arrow[d, "\Delta"'] & C/I \arrow[d, "\overline{\Delta}"] \\ C \otimes C \arrow[r, "\pi \otimes \pi"] & C/I \otimes C/I \end{tikzcd}$$

This map is defined in the obvious way, namely

$$\overline{\Delta} := (\pi \otimes \pi) \circ \Delta \quad \text{or} \quad \overline{\Delta}(\overline{c}) := \sum_{(\overline{c})} \overline{c^{(1)}} \otimes \overline{c^{(2)}}$$

where $\overline{c} = \pi(c)$ is the coset represented by c. We now construct the following "cube" diagram from the coassociativity diagram for C:

$$\begin{tikzcd}
 & C/I \arrow[rr, "\overline{\Delta}"] \arrow[dd, dotted, "\overline{\Delta}"] & & C/I \otimes C/I \arrow[dd, "\mathrm{id} \otimes \overline{\Delta}"] \\
C \arrow[ur, "\pi"] \arrow[rr, "\Delta"] \arrow[dd, "\Delta"'] & & C \otimes C \arrow[ur, "\pi \otimes \pi"] \arrow[dd, "\mathrm{id} \otimes \Delta"] & \\
 & C/I \otimes C/I \arrow[rr, dotted, "\overline{\Delta} \otimes \mathrm{id}"] & & C/I \otimes C/I \otimes C/I \\
C \otimes C \arrow[ur, "\pi \otimes \pi"] \arrow[rr, "\Delta \otimes \mathrm{id}"] & & C \otimes C \otimes C \arrow[ur, "\pi \otimes \pi \otimes \pi"] &
\end{tikzcd}$$

The top, both sides and the bottom of this diagram commute due to the commutativity of the previous diagram for π. Also, the front commutes by the coassociativity of C and the back commutes since π is onto. Using this diagram we see that

$$(\overline{\Delta} \otimes \mathrm{id})(\overline{\Delta}(\overline{c})) = (\mathrm{id} \otimes \overline{\Delta})(\overline{\Delta}(\overline{c})) = \sum \overline{c^{(1)}} \otimes \overline{c^{(2)}} \otimes \overline{c^{(3)}}$$

and hence $\overline{\Delta}$ is coassociative. Moreover, since I is a coideal, we have that $\varepsilon(I) = 0$ and so can again use the universal property of factor vector spaces to get that there exists a unique linear map $\overline{\varepsilon} : C/I \to \kappa$ such that the following diagram commutes:

$$\begin{tikzcd}
C \arrow[rr, "\pi"] \arrow[dr, "\varepsilon"'] & & C/I \arrow[dl, "\overline{\varepsilon}"] \\
 & \kappa &
\end{tikzcd}$$

Hence, for any $c \in C$ we get that $\overline{\varepsilon}(\overline{c}) = \varepsilon(c)$ and then

$$\sum_{(c)} \overline{\varepsilon}(\overline{c^{(1)}})\overline{c^{(2)}} = \pi\Big(\sum_{(c)} \varepsilon(c^{(1)})c^{(2)}\Big) = \pi(c) = \overline{c}$$

and also

$$\sum_{(c)} \overline{c^{(1)}}\overline{\varepsilon}(\overline{c^{(2)}}) = \pi\Big(\sum_{(c)} c^{(1)}\varepsilon(c^{(2)})\Big) = \pi(c) = \overline{c}$$

Thus

$$\sum_{(c)} \overline{\varepsilon}(\overline{c^{(1)}})\overline{c^{(2)}} = \overline{c} = \sum_{(c)} \overline{c^{(1)}}\overline{\varepsilon}(\overline{c^{(2)}})$$

which is the Sweedler way of saying that the second (counit) diagram in Definition 3.22 commutes for C/I . It follows, then, that $(C/I, \overline{\Delta}, \overline{\varepsilon})$ is a coalgebra and π is a coalgebra morphism. Furthermore, the uniqueness of the coalgebra structure on C/I for which π is a coalgebra morphism follows from the uniqueness of $\overline{\Delta}$ and $\overline{\varepsilon}$.

So, in case I is a coideal the map Δ *factors through* a map

$$\overline{\Delta} : C/I \to C/I \otimes C/I$$

$$\begin{tikzcd}
C \arrow[r, "\overline{\Delta} \circ \pi"] \arrow[d, "\pi"'] & C/I \otimes C/I \\
C/I \arrow[ur, "\overline{\Delta}"'] &
\end{tikzcd}$$

Likewise, the counit map *factors through* a map

$$\overline{\varepsilon} : C/I \to \kappa$$

$$\begin{tikzcd}
C \arrow[r, "\varepsilon"] \arrow[d, "\pi"'] & \kappa \\
C/I \arrow[ur, "\overline{\varepsilon}"'] &
\end{tikzcd}$$

(ii) From the point of view of vector spaces, the universal property of factor vector spaces implies that there exists a unique morphism of κ-vector spaces $\overline{f} : C/I \to D$ such that $\overline{f} \circ \pi = f$, defined by $\overline{f}(\overline{c}) := f(c)$ for all $c \in C$. Now consider the diagram:

$$\begin{tikzcd}
 & C/I \arrow[dr, "\overline{f}"] \arrow[dd, dotted, "\overline{\Delta}"] & \\
C \arrow[ur, "\pi"] \arrow[rr, "f"] \arrow[dd, "\Delta_C"'] & & D \arrow[dd, "\Delta_D"] \\
 & C/I \otimes C/I \arrow[dr, "\overline{f} \otimes \overline{f}"] & \\
C \otimes C \arrow[ur, "\pi \otimes \pi"] \arrow[rr, "f \otimes f"'] & & D \otimes D
\end{tikzcd}$$

Note that

$$\begin{aligned}
\Delta_D(\overline{f}(\overline{c})) &= \Delta_D(f(c)) \\
&= \sum_{(c)} f(c)^{(1)} \otimes f(c)^{(2)} \\
&= \sum_{(c)} f(c^{(1)}) \otimes f(c^{(2)}) \\
&= \sum_{(c)} \overline{f}(\overline{c^{(1)}}) \otimes \overline{f}(\overline{c^{(2)}}) \\
&= (\overline{f} \otimes \overline{f})(\overline{\Delta}(\overline{c}))
\end{aligned}$$

which means that the back right face of the diagram commutes.

$$\begin{tikzcd}
C \arrow[d, "\pi"] \arrow[dr, "\varphi"] \arrow[dd, bend right, "\varepsilon_C"'] & \\
C/I \arrow[r, "\overline{f}"] \arrow[d, "\overline{\varepsilon}"] & D \arrow[dl, "\varepsilon_D"] \\
\kappa &
\end{tikzcd}$$

But we also can easily see, per the above diagram, that

$$\varepsilon_D(\overline{f}(\overline{c})) = \varepsilon_D(f(c)) = \varepsilon_C(c) = \overline{\varepsilon}(\overline{c})$$

and thus we have shown that $\overline{f}$ is a morphism of coalgebras. □

What we can deduce from this is that coalgebras also admit a fundamental isomorphism theorem.

Corollary 3.29 (The Fundamental Isomorphism Theorem for Coalgebras)**.** *Let*

$$f : C \to D$$

be a morphism of coalgebras. Then there exists a canonical isomorphism of coalgebras between $C/Ker(f)$ and $Im(f)$.

Proof. By Proposition 3.27 we know that $Ker(f)$ is a coideal of C and $Im(f)$ is a subcoalgebra of D. By the previous theorem

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\pi"'] & Im(f) \\ C/Ker(f) \arrow[ur, "\overline{f}"'] & \end{tikzcd}$$

we know that $C/Ker(f)$ is a coalgebra and there exists a unique coalgebra morphism $\overline{f} : C/Ker(f) \to Im(f)$ with $\overline{f}(\overline{c}) = f(c)$ for all $c \in C$. Furthermore, we know that f and π are onto, and so, therefore $\overline{f}$ is too. Thus, we need only show that $\overline{f}$ is 1-1. Suppose that $\overline{c} \in Ker(\overline{f})$. Then $\overline{f}(\overline{c}) = 0$. But $\overline{f}(\overline{c}) = f(c)$ and hence $c \in Ker(f)$ thereby implying that $\overline{c} = 0$. Therefore, $Ker(\overline{f}) = \{0\}$ and $\overline{f}$ is 1-1. $\square$

3.2.3 The Tensor Product of Coalgebras

One of the key underlying themes in this thesis is tensor products. We have considered that the tensor product of vector spaces is again a vector space and the tensor product of algebras is again an algebra. We now want to know if the same holds for coalgebras. That is, if C and D are two κ-coalgebras, is $C \otimes D$ a κ-coalgebra? The quick answer is, yes, and we can see this as follows: Recall that $U \otimes V \cong V \otimes U$ as vector spaces and $(U \otimes V) \otimes W \cong U \otimes (V \otimes W)$ as vector spaces. The first isomorphism is specifically given by the "flip" or "transpose" map $\tau_{U,V}$ defined by $\tau_{U,V}(u \otimes v) := v \otimes u$.

Now, since C and D are each coalgebras we have the map

$$\Delta_C \otimes \Delta_D : C \otimes D \to (C \otimes C) \otimes (D \otimes D)$$

since Δ_C and Δ_D are linear. We then have the isomorphism of vector spaces

$$(C \otimes C) \otimes (D \otimes D) \cong C \otimes (C \otimes D) \otimes D$$

But using the twist map we get the isomorphism

$$\mathrm{id} \otimes \tau_{C,D} \otimes \mathrm{id} : C \otimes (C \otimes D) \otimes D \to C \otimes (D \otimes C) \otimes D$$

and then we apply the second isomorphism again to get

$$C \otimes (D \otimes C) \otimes D \cong (C \otimes D) \otimes (C \otimes D)$$

Composing these maps gives

$$(\mathrm{id} \otimes \tau_{C,D} \otimes \mathrm{id}) \circ (\Delta_C \otimes \Delta_D) : C \otimes D \to (C \otimes D) \otimes (C \otimes D)$$

This will be our map Δ for $C \otimes D$. To verify the coassociativity axiom we will show that the required diagram commutes in this context. To accomplish this, we first note that since C and D are coalgebras individually we get the commuting diagram:

$$\begin{tikzcd}
C \otimes D \arrow[r, "\Delta_C \otimes \Delta_D"] \arrow[d, "\Delta_C \otimes \Delta_D"'] & (C \otimes C) \otimes (D \otimes D) \arrow[d, "(\mathrm{id} \otimes \Delta_C) \otimes (\mathrm{id} \otimes \Delta_D)"] \\
(C \otimes C) \otimes (D \otimes D) \arrow[r, "(\Delta_C \otimes \mathrm{id}) \otimes (\Delta_D \otimes \mathrm{id})"] & (C \otimes C \otimes C) \otimes (D \otimes D \otimes D)
\end{tikzcd}$$

which is essentially a tensoring of C and D's coassociativity diagrams. Now, using the twist map we have the isomorphisms

$$(C \otimes C) \otimes (D \otimes D) \cong (C \otimes D) \otimes (C \otimes D)$$

and

$$(C \otimes C \otimes C) \otimes (D \otimes D \otimes D) \cong (C \otimes C) \otimes (D \otimes D) \otimes (C \otimes D)$$

The first isomorphism is specifically given by $\mathrm{id}_C \otimes \tau_{C,D} \otimes \mathrm{id}_D$ and the second by

$$(\mathrm{id}_{C\otimes C} \otimes \tau_{C,D} \otimes \mathrm{id}_{D\otimes D}) \circ (\mathrm{id}_{C\otimes C\otimes D} \otimes \tau_{C,D} \otimes \mathrm{id}_D)$$

We therefore get the commuting square:

$$\begin{tikzcd}
(C \otimes C) \otimes (D \otimes D) \arrow[r, "(\Delta_C \otimes \mathrm{id}) \otimes (\Delta_D \otimes \mathrm{id})"] \arrow[d, "\wr"] & (C \otimes C \otimes C) \otimes (D \otimes D \otimes D) \arrow[d, "\wr"] \\
(C \otimes D) \otimes (C \otimes D) \arrow[r, "(\Delta_C \otimes \Delta_D) \otimes (\mathrm{id} \otimes \mathrm{id})"] & (C \otimes C) \otimes (D \otimes D) \otimes (C \otimes D)
\end{tikzcd}$$

Similarly we have that

$$\begin{tikzcd}
(C \otimes C) \otimes (D \otimes D) \arrow[r, "\sim"] \arrow[d, "(\mathrm{id} \otimes \Delta_C) \otimes (\mathrm{id} \otimes \Delta_D)"'] & (C \otimes D) \otimes (C \otimes D) \arrow[d, "(\mathrm{id} \otimes \mathrm{id}) \otimes (\Delta_C \otimes \Delta_D)"] \\
(C \otimes C \otimes C) \otimes (D \otimes D \otimes D) \arrow[r, "\sim"] & (C \otimes D) \otimes (C \otimes C) \otimes (D \otimes D)
\end{tikzcd}$$

commutes. The final square

$$\begin{tikzcd}
(C \otimes C \otimes C) \otimes (D \otimes D \otimes D) \arrow[r, "\sim"] \arrow[d, "\wr"] & (C \otimes D) \otimes (C \otimes C) \otimes (D \otimes D) \arrow[d, "\wr"] \\
(C \otimes C) \otimes (D \otimes D) \otimes (C \otimes D) \arrow[r, "\sim"] & (C \otimes D) \otimes (C \otimes D) \otimes (C \otimes D)
\end{tikzcd}$$

commutes because the isomorphisms involved are just equivalent permutations achieved via the twist map. That is

$$(I\tau III)\circ(III\tau I)\circ(II\tau II)=(III\tau I)\circ(I\tau III)\circ(II\tau II)$$

where I is the relevant identity map and "$\otimes$" is suppressed for brevity.

Putting all this together implies that

$$\begin{tikzcd}
C\otimes D \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & (C\otimes D)\otimes(C\otimes D) \arrow[d, "\mathrm{id}\otimes\Delta"] \\
(C\otimes D)\otimes(C\otimes D) \arrow[r, "\Delta\otimes\mathrm{id}"] & (C\otimes D)\otimes(C\otimes D)\otimes(C\otimes D)
\end{tikzcd}$$

commutes, thereby showing coassociativity. The counit is defined by $\varepsilon := \varepsilon_C\otimes\varepsilon_D$ (identifying κ with $\kappa\otimes\kappa$) and the counit axiom is proved similarly. Thus, $C\otimes D$ is itself a coalgebra.

3.2.4 The Algebra/Coalgebra Connection

The dual of a vector space determines an important connection between algebras and coalgebras. This is where we will see the relationship between our two uses of the term "dual". More importantly, the following work will uncover the categorical relationship between *Alg* and *CoAlg* in Figure 1.1.

Recall that in the construction of the tensor product of linear maps we obtained the linear map

$$\theta : \hom(U,U')\otimes\hom(V,V')\to\hom(U\otimes V,U'\otimes V')$$

defined by

$$\big(\theta(f\otimes g)\big)(u\otimes v)=f(u)\otimes g(v)$$

We shall also make use of a generalization of Corollary 2.19, extending the specialized version of θ to similar maps involving multiple tensor products.

Lemma 3.30. *For any κ-vector spaces $V_1,...,V_n$ the map*

$$\theta : V_1^*\otimes\ldots\otimes V_n^*\to(V_1\otimes\ldots\otimes V_n)^*$$

defined by

$$\theta(f_1\otimes\ldots\otimes f_n)(v_1\otimes\ldots\otimes v_n):=f_1(v_1)\cdots f_n(v_n)$$

is injective. Moreover, if all the spaces V_i are finite dimensional, then θ is an isomorphism.

For our purposes we will take this for granted (see [8]).

We now make the important connection between algebras and coalgebras, via use of the material in section 2.1.4 involving dual vector spaces.

Theorem 3.31. *The dual vector space of a coalgebra is an algebra.*

Proof. Let (C, Δ, ε) be a coalgebra. Consider, from Corollary 2.19, the (not necessarily isomorphic) map

$$\theta : C^* \otimes C^* \to (C \otimes C)^*$$

defined by

$$\theta(\phi_1 \otimes \phi_2)(c_1 \otimes c_2) := \phi_1(c_1) \otimes \phi_2(c_2) := \phi_1(c_1)\phi_2(c_2)$$

Set $A = C^*$, $\nabla = \Delta^* \circ \theta$ and $\eta = \varepsilon^*$ where the superscript $*$ on the linear maps Δ and ε indicates their transpose. To see that such a setup yields an algebra, we check that the diagrams in the definition of algebra commute.

Since we started with a coalgebra, the coassociativity diagram commutes:

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\mathrm{id} \otimes \Delta"] \\ C \otimes C \arrow[r, "\Delta \otimes \mathrm{id}"] & C \otimes C \otimes C \end{tikzcd}$$

If we now take the transpose of this diagram we get

$$\begin{tikzcd} C^* & (C \otimes C)^* \arrow[l, "\Delta^*"'] \\ (C \otimes C)^* \arrow[u, "(\mathrm{id} \otimes \Delta)^*"] & (C \otimes C \otimes C)^* \arrow[l, "(\Delta \otimes \mathrm{id})^*"] \arrow[u, "\Delta^*"'] \end{tikzcd}$$

which automatically commutes. But since the maps

$$\theta : C^* \otimes C^* \to (C \otimes C)^* \quad \text{and} \quad \theta' : C^* \otimes C^* \otimes C^* \to (C \otimes C \otimes C)^*$$

are embedding maps, we get the commuting diagram

$$\begin{tikzcd}
C^* \otimes C^* \otimes C^* \arrow[r, "\nabla \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes \nabla"'] & C^* \otimes C^* \arrow[d, "\nabla"] \\
C^* \otimes C^* \arrow[r, "\nabla"'] & C^*
\end{tikzcd}$$

which is exactly the associativity diagram we need for showing that C^* is an algebra.

For the unit diagram, we proceed similarly. First, we have that the diagram

$$\begin{tikzcd}
\kappa \otimes C & C \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, dashed, "\Delta"] & C \otimes \kappa \\
 & C \otimes C \arrow[ul, "\varepsilon \otimes \mathrm{id}"] \arrow[ur, "\mathrm{id} \otimes \varepsilon"'] &
\end{tikzcd}$$

commutes. Taking the transpose yields the commuting diagram

$$\begin{tikzcd}
(\kappa \otimes C)^* \arrow[r, "\sim"] \arrow[dr, "(\varepsilon \otimes \mathrm{id})^*"'] & C^* & (C \otimes \kappa)^* \arrow[l, "\sim"'] \arrow[dl, "(\mathrm{id} \otimes \varepsilon)^*"] \\
 & (C \otimes C)^* \arrow[u, dashed, "\Delta^*"] &
\end{tikzcd}$$

But $\theta : C^* \otimes C^* \to (C \otimes C)^*$, $\theta' : \kappa^* \otimes C^* \to (\kappa \otimes C)^*$ and $\theta' \circ \tau_{\kappa^*, C^*} : C^* \otimes \kappa^* \to (C \otimes \kappa)^*$ are embedding maps, which implies that

$$\begin{tikzcd}
\kappa^* \otimes C^* \arrow[r, "\eta \otimes \mathrm{id}"] \arrow[dr, "\sim"] & C^* \otimes C^* \arrow[d, dashed, "\nabla"] & C^* \otimes \kappa^* \arrow[l, "\mathrm{id} \otimes \eta"'] \arrow[dl, "\sim"'] \\
 & C^* &
\end{tikzcd}$$

is a commuting diagram. Since $\kappa^* \cong \kappa$ as vector spaces under the canonical isomorphism $\psi(f) = f(1)$ for all $f \in \kappa^*$, the above diagram tells us that the unit diagram for C^* commutes. Therefore, (C^*, ∇, η) is an algebra.

$\square$

From the above proof, the reader should especially note how taking the dual allows for a canonical way to *reverse arrows* in this context, which gives the connection between each duality. We therefore have a functor $CoAlg \xrightarrow{*} Alg$, where a coalgebra C is sent to the algebra C^* and a coalgebra morphism f is sent to its transpose f^*.

Given a coalgebra C, the actual multiplication in the algebra C^* is to be understood as follows. For $\alpha, \beta \in C^*$, the product $\alpha\beta \in C^*$ is the map given by

$$\begin{aligned}\alpha\beta(c) &= (\alpha \otimes \beta)(\Delta(c)) \\ &= \sum_{(c)} \alpha(c^{(1)})\beta(c^{(2)})\end{aligned}$$

Now, for any κ-algebra A, the multiplication is completely determined by the products of the basis elements. For example, if $\{b_1, b_2, ..., b_n\}$ is a basis for A we define the *structure constants* (or structure coefficients) c_{ijk} by

$$b_i b_j = \sum_{k=1}^{n} c_{ijk} b_k$$

If we now start with a finite dimensional coalgebra C with basis $\{c_1, c_2, ..., c_n\}$ then the dual space C^* has dual basis $\{c_1^*, c_2^*, ..., c_n^*\}$. Since the dimension is finite we may identify $C^* \otimes C^*$ with $(C \otimes C)^*$ and hence the multiplication for C^* is $\nabla = \Delta^*$, which means that

$$\nabla(c_r^* \otimes c_s^*) = (c_r^* \otimes c_s^*) \circ \Delta$$

If applied to a basis element of C, say c_k, one gets $(c_r^* \otimes c_s^*)(\Delta(c_k))$ and so we consider $\Delta(c_k)$ as an element of $C \otimes C$. We write

$$\Delta(c_k) = \sum_{i,j} \lambda_{ijk} c_i \otimes c_j$$

Then we have

$$\begin{aligned}\nabla(c_r^* \otimes c_s^*)(c_k) &= (c_r^* \otimes c_s^*)\Big(\sum_{i,j} \lambda_{ijk} c_i \otimes c_j\Big) \\ &= \sum_{i,j} \lambda_{ijk} c_r^*(c_i) \otimes c_s^*(c_j) \\ &= \sum_{i,j} \lambda_{ijk} c_r^*(c_i) c_s^*(c_j) \\ &= \lambda_{rsk}\end{aligned}$$

This means that the multiplication in C^* of basis elements is given by

$$c_r^* c_s^* = \sum_k \lambda_{rsk} c_k^*$$

where the λ_{rsk}'s are the structure constants for C^*.

Now, the unity of C^* is obtained by

$$1_\kappa \mapsto 1_\kappa^* \mapsto \varepsilon^*(1_\kappa^*) = 1_\kappa^* \circ \varepsilon$$

since $\eta = \varepsilon^*$ once we identify κ^* with κ. Considered as an element of C^*, this unity acts on the basis of C by $(1_\kappa^* \circ \varepsilon)(b_\ell) = \varepsilon(b_\ell)$. This means that the unit of C^*, as an algebra, is the functional which sends $b_i \mapsto \varepsilon(b_i)$ for all i, so $1_{C^*} = \varepsilon$.

Example 3.12 (Trigonometric Coalgebra). *Let's look again at the interesting case of the trigonometric coalgebra, call it C. As a vector space it has basis $\{c, s\}$ and hence is of dimension 2. Recall that the coalgebra structure is given by*

$$\Delta(c) = c \otimes c - s \otimes s$$
$$\Delta(s) = s \otimes c + c \otimes s$$

$$\varepsilon(c) = 1, \qquad \varepsilon(s) = 0$$

Since the dimension is finite, the dual vector space, C^*, is isomorphic to C as vector spaces. Accordingly, C^* has dual basis $\{c^*, s^*\}$ and by the above theorem is an algebra. Note right away that c^* is the element of $C^* := \hom(C, \kappa)$ such that $c^*(c) = 1$ and $c^*(s) = 0$, while s^* is the element such $s^*(s) = 1$ and $s^*(c) = 0$.

Now, $\eta := \varepsilon^*$ and once we identify κ^* with κ we see that

$$\eta(1) = \varepsilon^*(1^*) = 1^* \circ \varepsilon = \varepsilon$$

Thus, $\eta(1)(c) = 1$ and $\eta(1)(s) = 0$ and therefore $\eta(1) = c^*$. This means that c^* is actually the identity element for C^*, which tells us immediately that

$$c^* \cdot c^* = c^*, \qquad c^* \cdot s^* = s^* \cdot c^* = s^*$$
$$c^* \cdot s^* - s^* \cdot c^* = 0, \qquad c^* \cdot s^* + s^* \cdot c^* = 2s^*$$

where we are using "$\cdot$" as short hand for ∇. Let us now see what other relations hold within this algebra. Using our work with the structure constants we find

$$c^* \cdot c^* = \lambda_{ccs} s^* + \lambda_{ccc} c^* = c^*$$

implying that $\lambda_{ccs} = 0$ and $\lambda_{ccc} = 1$. As for $s^* \cdot s^*$ we have

$$s^* \cdot s^* = \lambda_{sss} s^* + \lambda_{ssc} c^*$$

as an element of C^* where

$$\Delta(s) = \lambda_{sss}(s \otimes s) + \lambda_{scs}(s \otimes c) + \lambda_{css}(c \otimes s) + \lambda_{ccs}(c \otimes c)$$

But because $\Delta(s) = s \otimes c + c \otimes s$ it must be that $\lambda_{sss} = \lambda_{ccs} = 0$ and $\lambda_{scs} = \lambda_{css} = 1$. Likewise, because $\Delta(c) = c \otimes c - s \otimes s$ we get that $\lambda_{ssc} = -1$, $\lambda_{ccc} = 1$ and $\lambda_{scc} = \lambda_{csc} = 0$. We therefore find that $s^* \cdot s^* = -c^* = -1_{C^*}$.

Let us agree to write $c^* \cdot c^*$ and $s^* \cdot s^*$ respectively as $(c^*)^2$ and $(s^*)^2$. Then we see that

$$(c^*)^2 + (s^*)^2 = 0 \quad \textit{and} \quad (c^*)^2 - (s^*)^2 = 2c^*$$

We know of something familiar that behaves the same way, namely $i^2 = -1$. Thus, C^*, remarkably, is isomorphic, as an algebra, to $\kappa[i]$ with $i^2 = -1$ or $\kappa[x]/\langle x^2 + 1\rangle$.

Conversely, suppose we start with $\mathbb{C}$, which has $\mathbb{R}$-basis $\{1, i\}$. Then the dual space $\mathbb{C}^*$ has dual basis $\{1^*, i^*\}$. The structure constants are

$$c_{111} = c_{1ii} = c_{i1i} = 1, \qquad c_{ii1} = -1$$

$$c_{11i} = c_{1i1} = c_{i11} = c_{iii} = 0$$

These determine the coalgebra structure. We have

$$\begin{aligned}\Delta(1^*) &= c_{111}1^* \otimes 1^* + c_{1i1}1^* \otimes i^* + c_{i11}i^* \otimes 1^* + c_{ii1}i^* \otimes i^* \\ &= 1^* \otimes 1^* - i^* \otimes i^*\end{aligned}$$

$$\begin{aligned}\Delta(i^*) &= c_{11i}1^* \otimes 1^* + c_{1ii}1^* \otimes i^* + c_{i1i}i^* \otimes 1^* + c_{iii}i^* \otimes i^* \\ &= 1^* \otimes i^* + i^* \otimes 1^*\end{aligned}$$

Furthermore, $\varepsilon(1^*) = 1$ and $\varepsilon(i^*) = 0$ since $1_{\mathbb{C}} = 1 \cdot 1 + 0 \cdot i$. If we now set $1^* := c$ and $i^* := s$ we find that $\mathbb{C}^*$ is the trigonometric coalgebra.

Suppose that C is any coalgebra. Then we now know that its dual $C^* := \hom(C, \kappa)$ is an algebra. Let us now explore the connection between ideals of C^* and sub-coalgebras of C. For any subspace L of C^* define an associated set $L^\perp$ by

$$L^\perp := \{c \in C : \ell(c) = 0 \text{ for all } \ell \in L\}$$

It is easy to see that this set is a subspace of C.

Proposition 3.32. *If L is an ideal of C^*, then $L^\perp$ is a sub-coalgebra of C.*

Proof. It needs to be shown that $\Delta(L^\perp) \subseteq L^\perp \otimes L^\perp$. Let $c \in L^\perp$. Then $c \in C$ and hence $\Delta(c) \in C \otimes C$. Now, every element of the tensor product $C \otimes C$ can be expressed in the form

$$\sum_{i=1}^{n} a_i \otimes b_i$$

where $\{a_i\}$ and $\{b_i\}$ are linearly independent sets (see Theorem 2.15). We can therefore write

$$\Delta(c) = \sum_{i=1}^{n} a_i \otimes b_i$$

Now, $\{a_i\}$ and $\{b_i\}$ can be considered bases of the vector subspaces $Span(a_1, ..., a_n)$ and $Span(b_1, ..., b_n)$ of C respectively. Therefore, we ascertain dual bases $\{a_i^*\}$ and $\{b_i^*\}$ respectively, which we can take to consist of elements of C^* and where the reader should recall that

$$a_i^*(a_j) = \delta_{ij} \quad \text{and} \quad b_i^*(b_j) = \delta_{ij}$$

Let us suppose that $\Delta(c) \notin L^\perp \otimes L^\perp$. Then there exist a_r or b_s such that $\ell(a_r) \neq 0$ or $\ell'(b_s) \neq 0$ for some $\ell, \ell' \in L$. Now, since L is an ideal, it follows that $a_i^* \cdot \ell' \in L$ for all i and $\ell \cdot b_i^* \in L$ for all i. Therefore, it must be that

$$a_i^* \cdot \ell'(c) = 0, \quad \ell \cdot b_i^*(c) = 0 \qquad \text{for all } i$$

Without loss of generality, suppose $\ell'(b_s) \neq 0$. Then, since the multiplication in C^* is given by Δ^* we have

$$\begin{aligned}
0 &= a_s^* \cdot \ell'(c) \\
&= (a_s^* \otimes \ell')(\Delta(c)) \\
&= (a_s^* \otimes \ell')\Big(\sum_{i=1}^{n} a_i \otimes b_i\Big) \\
&= \sum_{i=1}^{n} a_s^*(a_i) \otimes \ell'(b_i) \\
&= 1 \otimes \ell'(b_s) \\
&= 1\ell'(b_s) = \ell'(b_s)
\end{aligned}$$

which is a contradiction, since $\ell'(b_s) \neq 0$. Hence, each b_s is in $L^\perp$, and likewise, each a_r must be in $L^\perp$ and our assumption that $\Delta(c) \notin L^\perp \otimes L^\perp$ is therefore false. Therefore, $L^\perp$ must be a sub-coalgebra of C. ∎

Suppose we now start with a subspace D of C. Then we again construct an associated space $D^{\perp}$ defined by

$$D^{\perp} := \{f \in C^* : f(d) = 0 \text{ for all } d \in D\}$$

In terms of vector spaces, this is also known as the *annihilator* of D.

Proposition 3.33. *For any subspace D of C we have*

$$D^{\perp\perp} = D$$

Proof. By definition we know that

$$D^{\perp\perp} := \{c \in C : f(c) = 0 \text{ for all } f \in D^{\perp}\}$$

Thus, it is clear that $D \subseteq D^{\perp\perp}$. Suppose, however, that $D^{\perp\perp} \neq D$. Then there is $b \in D^{\perp\perp}$ such that $b \notin D$.

From linear algebra we know that any subspace of a vector space has a complement. Thus, as vector spaces we have

$$C = D \oplus T$$

for some subspace T of C. As an element of C, therefore, we can write $b = d + t$ for some $d \in D$ and non-zero $t \in T$. Now there exists $g \in C^*$ such that $g(t) \neq 0$, but $g(D) = 0$. So, $g \in D^{\perp}$, which implies that $g(b) = 0$. But then we also have that $g(b) = g(d + t) = g(d) + g(t) = g(t) \neq 0$, which is a contradiction. Thus, we must have that $D^{\perp\perp} = D$ after all. $\square$

Proposition 3.34. *D is a sub-coalgebra of C if and only if $D^{\perp}$ is an ideal of C^*.*

Proof. Suppose D is a sub-coalgebra of C. Clearly $D^{\perp}$ is a subspace of C^*. Let $f \in D^{\perp}$ and $g \in C^*$ and consider the product map $f \cdot g$. For any $d \in D$ we have

$$\begin{aligned}
(f \cdot g)(d) &= (f \otimes g)(\Delta(d)) \\
&= (f \otimes g)\Big(\sum_{(d)} d^{(1)} \otimes d^{(2)}\Big) \\
&= \sum_{(d)} f(d^{(1)}) \otimes g(d^{(2)}) \\
&= 0 \qquad [\text{since } f(d^{(1)}) = 0]
\end{aligned}$$

Similar reasoning shows that $(g \cdot f)(d) = 0$ for all $d \in D$, and hence, $f \cdot g, g \cdot f \in D^{\perp}$. It follows that $D^{\perp}$ is an ideal of C^*.

The converse holds as a consequence of Proposition 3.32 and Proposition 3.33. □

Proposition 3.35. *We have the following isomorphism of algebras:*

$$\frac{C^*}{D^\perp} \cong D^*$$

Proof. Since D is a sub-coalgebra of C we can think of the coalgebra D as embedded in C. In other words, we have an injective coalgebra map $i : D \to C$. By taking the transpose of i we get a surjective algebra map $i^* : C^* \to D^*$ and hence

$$Im(i^*) = D^*$$

Now, it is clear that $f \in D^\perp$ is equivalent to f restricting to the zero map on D. Thus, $D^\perp \subseteq Ker(i^*)$. Conversely, if $g \in Ker(i^*)$, then $i^*(g) = 0_D$ entailing that $g \circ i = 0_D$. This implies that $g(D) = 0$ and hence that $g \in D^\perp$. So

$$D^\perp = Ker(i^*)$$

Therefore, by the First Isomorphism Theorem and the fact that i^* is an algebra map

$$\frac{C^*}{D^\perp} \cong D^* \qquad \text{as algebras}$$

□

Proposition 3.36. *If C is a finite dimensional coalgebra, then for any subspace L of C^**

$$L^{\perp\perp} = L$$

Proof. Since C is finite dimensional we have that

$$C \cong C^{**} \qquad \text{as vector spaces}$$

If we now identify C with C^{**} we get, by Proposition 3.33, that $L^{\perp\perp} = L$ for any subspace L. □

What we have so far shown is that $\perp$ acts like a map, which we denote by *Perp*. In other words, we have

$$Perp(D) = D^\perp, \quad Perp(L) = L^\perp$$

Notice, too, that *Perp* is its own inverse and hence we get an order reversing (in the sense of reversing inclusion) bijection:

$$\{\text{sub-coalgebras of } C\} \leftrightarrow \{\text{ideals of } C^*\}$$

Interestingly, although algebras and coalgebras are duals of one another, it is not generally true that the dual vector space of an algebra carries a natural coalgebra structure. This is due to the failure of the canonical map $\theta : U^* \otimes V^* \to (U \otimes V)^*$ to always be invertible. To circumvent this issue, we must restrict ourselves to the finite dimensional case, for then, θ *will* be an isomorphism.

Theorem 3.37. *The dual vector space of a finite-dimensional algebra has a coalgebra structure.*

Proof. Let (A, ∇, η) be a finite-dimensional algebra. Then the map

$$\theta : A^* \otimes A^* \to (A \otimes A)^*$$

is an isomorphism, which allows us to define Δ by

$$\Delta := \theta^{-1} \circ \nabla^*$$

Also, set $\varepsilon := \eta^*$. Now because A is an algebra we have the commuting diagram

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "\nabla \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes \nabla"'] & A \otimes A \arrow[d, "\nabla"] \\ A \otimes A \arrow[r, "\nabla"'] & A \end{tikzcd}$$

Like before, we can take the transpose of this diagram to get another commuting diagram, namely

$$\begin{tikzcd} (A \otimes A \otimes A)^* & (A \otimes A)^* \arrow[l, "(\nabla \otimes \mathrm{id})^*"'] \\ (A \otimes A)^* \arrow[u, "(\mathrm{id} \otimes \nabla)^*"] & A^* \arrow[l, "\nabla^*"] \arrow[u, "\nabla^*"'] \end{tikzcd}$$

Since we are working in the finite dimensional case, we have the nice result that

$$(A \otimes A \otimes A)^* \cong A^* \otimes A^* \otimes A^* \quad \text{and} \quad (A \otimes A)^* \cong A^* \otimes A^*$$

as vector spaces. Under these identifications, this last commuting diagram actually becomes the coassociativity diagram for A^*.

The same holds true for establishing the counit axiom. We start with the commuting unit diagram for A and take its transpose. Using the same isomorphisms we get the desired commuting counit diagram for A^*. This shows that $(A^*, \theta^{-1} \circ \nabla^*, \eta^*)$ is a coalgebra.

$\square$

While the proof of the previous theorem is elegant, it doesn't show the structure of specific elements. To see the "nuts and bolts" of what the above theorem means, let us consider two concrete examples. For both the forthcoming examples we should point out again that because we will be working with finite dual vector spaces, the map $\theta : A^* \otimes A^* \to (A \otimes A)^*$ is an isomorphism. As in our proof, the coproduct is:

$$\Delta := \theta^{-1} \circ \nabla^*$$

which we can simply think of as $\Delta = \nabla^*$ after identifying $A^* \otimes A^*$ with $(A \otimes A)^*$. The counit, again, is $\varepsilon := \psi \circ \eta^*$ where ψ is the canonical isomorphism from κ^* to κ, that is, $\psi(f) := f(1)$ for $f \in \kappa^*$.

Example 3.13. *First, take the finite dimensional matrix algebra $A := M_2(\kappa)$. This algebra is 4-dimensional with basis*

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

For ease, let us denote this basis by $\{E_{11}, E_{12}, E_{21}, E_{22}\}$, where the subscripts indicate the position of the 1 entry. The dual vector space A^* is also 4-dimensional, having dual basis $\{f_{11}, f_{12}, f_{21}, f_{22}\}$ (using f instead of E^*). These maps are defined by

$$f_{ij}(E_{k\ell}) := \begin{cases} 1 & \text{if } i = k \text{ and } j = \ell \\ 0 & \text{otherwise} \end{cases} = \delta_{ik}\delta_{j\ell}$$

We would like to understand the coalgebra structure on A^* guaranteed by Theorem 3.37. Note first that the algebra structure of $M_2(\kappa)$ is given by $E_{ij}E_{k\ell} = \delta_{j,k}E_{i\ell}$. Once we identify $A^* \otimes A^*$ with $(A \otimes A)^*$ we get

$$\Delta(f_{ij}) = \nabla^*(f_{ij}) = f_{ij} \circ \nabla$$

Now because of this algebra structure, the structure constants must be of the form

$$\lambda_{ijk\ell,rs} = \begin{cases} 1 & \textit{if } j = k,\ i = r \textit{ and } \ell = s \\ 0 & \textit{otherwise} \end{cases}$$

Thus,

$$\begin{aligned} \Delta(f_{rs}) &= \sum \lambda_{ijk\ell,rs} f_{ij} \otimes f_{k\ell} \\ &= \sum f_{rk} \otimes f_{ks} \end{aligned}$$

In this case, there is no need for the summation; though, it shows that the same argument works for higher dimensions. We therefore define

$$\Delta(f_{ij}) := f_{i1} \otimes f_{1j} + f_{i2} \otimes f_{2j}$$

and with this in hand, one can show, using the counit axiom, that ε must be defined by

$$\varepsilon(f_{ij}) := \delta_{ij}$$

We then extend these by linearity to the rest of A^*.

More generally, when $M_n(A)^*$ is identified with $M_n(A)$, ε becomes the trace map. So, in this case, if $f = \sum_{i,j=1}^{2} \lambda_{ij} f_{ij}$, then

$$\begin{aligned} \varepsilon(f) &= \varepsilon\Big(\sum_{i,j=1}^{2} \lambda_{ij} f_{ij} \Big) \\ &= \sum_{i,j=1}^{2} \lambda_{ij} \varepsilon(f_{ij}) \\ &= \sum_{i,j=1}^{2} \lambda_{ij} \delta_{ij} \\ &= \lambda_{11} + \lambda_{22} \\ &= tr(f) \end{aligned}$$

where f is regarded as the matrix $\begin{bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{bmatrix}$.

As indicated, the example just considered is a special case, which can be extended to the more general algebra $M_n(\kappa)$. That is, since $M_n(\kappa)$ is finite dimensional for any positive integer n, the dual vector space of $M_n(\kappa)$ has a coalgebra structure. This structure is understood according to similar reasoning to that used above.

Example 3.14. *Let* $A := \kappa[x]/\langle g(x)\rangle$ *where* $g(x)$ *is some monic degree* n *polynomial. The dual vector space, then, is* $A^* = \hom(\kappa[x]/\langle g(x)\rangle, \kappa)$ *and* $A \cong A^*$. *Note that* A *has basis* $\{1, x, x^2, ..., x^{n-1}\}$ *and so* A^* *has basis* $\{f_0, f_1, ..., f_{n-1}\}$ *where* $f_i(x^j) := \delta_{ij}$. *We also know that the tensor product* $A^* \otimes A^*$ *will have basis* $\{f_i \otimes f_j\}_{i,j}$. *Thus, for arbitrary* $f \in A^*$ *we will have* $\Delta(f) = \sum_{i,j} \lambda_{ij} f_i \otimes f_j$. *But since we have given the general definition of* Δ *to be* $\theta^{-1} \circ \nabla^*$ *we also have*

$$\begin{aligned}\Delta(f) &= (\theta^{-1} \circ \nabla^*)(f)\\ &= \theta^{-1}(f \circ \nabla)\end{aligned}$$

Next, let us consider what this does to a basis tensor $x^k \otimes x^\ell \in A \otimes A$. *We have*

$$\begin{aligned}\theta^{-1}((f \circ \nabla))(x^k \otimes x^\ell) &= f(\nabla(x^k \otimes x^\ell))\\ &= f(x^{k+\ell})\end{aligned}$$

But also

$$\begin{aligned}\Delta(f)(x^k \otimes x^\ell) &= \sum_{i,j} \lambda_{ij}(f_i \otimes f_j)(x^k \otimes x^\ell)\\ &= \sum_{i,j} \lambda_{ij} f_i(x^k) \otimes f_j(x^\ell)\\ &= \sum_{i,j} \lambda_{ij} f_i(x^k) f_j(x^\ell)\\ &= \sum_{i,j} \lambda_{ij}\delta_{ik}\delta_{j\ell}\\ &= \lambda_{k\ell}\end{aligned}$$

so we can equate $f(x^{k+\ell})$ *and* $\lambda_{k\ell}$. *Putting this all together allows us to write*

$$\Delta(f) = \sum_{i,j} f(x^{i+j}) f_i \otimes f_j$$

At this point we must address an issue, which arises when $i + j$ *becomes too large. To simplify matters, examine* $\Delta(f_k)$. *By the formula for* Δ *we have*

$$\begin{aligned}\Delta(f_k) &= \sum_{i,j} f_k(x^{i+j}) f_i \otimes f_j\\ &= \sum_{i,j} \delta_{k,i+j} f_i \otimes f_j\end{aligned}$$

Since we are factoring out an arbitrary degree n *monic polynomial* g, *we can only use* $\delta_{k,i+j}$ *for* $i + j < n$. *In case* $i + j \geq n$ *we need something that will efficiently reveal what* *is happening. Suppose* $g(x) = a_0 + a_1 x + \ldots + a_{n-1}x^{n-1} + x^n$. *Then the companion matrix of* g *is defined to be*

$$M_g := \begin{bmatrix} 0 & 0 & 0 & \ldots & & -a_0 \\ 1 & 0 & 0 & \ldots & & -a_1 \\ & 1 & 0 & \ldots & & -a_2 \\ & & 1 & & & \vdots \\ & \bigcirc & & \ddots & & \\ & & & & 1 & -a_{n-1} \end{bmatrix}$$

We can associate each of the columns of M_g, *in a canonical way, with a basis element of* A. *First, we make the assignment*

$$1 \leftrightarrow e_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

then

$$x \leftrightarrow e_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad x^2 \leftrightarrow e_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \quad x^n \leftrightarrow e_n = \begin{bmatrix} -a_0 \\ -a_1 \\ -a_2 \\ \vdots \\ -a_{n-1} \end{bmatrix}$$

For any x^ℓ *one can find its coefficients with respect to the basis* $\{1, x, x^2, ..., x^{n-1}\}$ *by examining the first column of* M_g^ℓ. *For example,*

$$x^2 = 0 \cdot 1 + 0 \cdot x + 1 \cdot x^2 + 0 \cdot x^3 + \ldots + 0 \cdots x^{n-1}$$

If we then compute M_g^2 *we will find that* e_2 *has shifted to the first column position and* e_2 *contains the coefficients just mentioned. Note further that if we compute* M_g^n, *then the first column will be* e_n, *which contains the coefficients of* x^n *with respect to the basis of* A, *and is also what we find from solving* $g(x) = 0$ *for* x^n. *We can also compute the specific coefficient for any* $x^k \in \{1, x, x^2, ..., x^{n-1}\}$ *in* x^ℓ. *It is given by*

$$e_k^t M_g^\ell e_0, \qquad [t \text{ indicates transpose}]$$

Thus, for $i + j \geq n$ we have that $f_k(x^{i+j}) = e_k^t M_g^{i+j} e_0$, which means that we should actually write

$$\Delta(f_k) = \sum_{i,j} (e_k^t M_g^{i+j} e_0) f_i \otimes f_j$$

For the counit, since $\varepsilon = \psi \circ \eta^*$, we find

$$\begin{aligned}
\varepsilon(f_k) &= (\psi \circ \eta^*)(f_k) \\
&= \psi(\eta^*(f_k)) \\
&= \psi(f_k \circ \eta) \\
&= (f_k \circ \eta)(1) \\
&= f_k(\eta(1)) \\
&= f_k(1) = \delta_{k0}
\end{aligned}$$

which is what we would find using the counit diagram.

In the generic case, for finite dimensional A^* with basis $\{b_1^*, b_2^*, ..., b_n^*\}$, since $\Delta = \nabla^*$ we have that $\Delta(b_\ell^*) = b_\ell^* \circ \nabla$. Thus

$$\begin{aligned}
\Delta(b_\ell^*)(b_i \otimes b_j) &= b_\ell^*(\nabla(b_i \otimes b_j)) \\
&= b_\ell^*(b_i b_j) \\
&= b_\ell^* \Big(\sum_k c_{ijk} b_k \Big) \\
&= \sum_k c_{ijk} b_\ell^*(b_k) \\
&= c_{ij\ell}
\end{aligned}$$

So the structure constants of the finite dimensional algebra A are also structure constants for the coalgebra A^*, since we can now write

$$\Delta(b_k^*) = \sum_{i,j} c_{ijk} b_i^* \otimes b_j^*$$

In this context, perhaps we can call them co-structure constants or structure co-constants.

Theorem 3.37 raises an interesting question. Essentially, it says that any finite-dimensional algebra can also be given a coalgebra structure via the vector space isomorphism $A \cong A^*$. This means we can think of the space either as an algebra *or* as a coalgebra. But one might then wonder whether these structures are compatible in a natural way. The answer often turns out to be yes and we will talk about such structures in the next chapter.

It would still be nice to be able to associate a coalgebra, in a natural way, to *any* algebra A. Fortunately, we *can* do this as long as we resign ourselves to defining the coalgebra on a particular subspace of A^*. This subspace is

$$A^\circ = \{f \in A^* : Ker(f) \text{ contains an ideal of finite codimension}\}$$

where a subspace W of a vector space V has finite codimension if $\dim(V/W)$ is finite. It would be prudent to at least quickly verify that A° is actually a subspace of A^*. Toward this end, note that if W and U are subspaces of finite codimension in V, then $W \cap U$ will likewise have finite codimension, since there exists an injective morphism $V/(W \cap U) \to V/W \times V/U$. So, if $f, g \in A^\circ$, then $Ker(f) \cap Ker(g) \subseteq Ker(f+g)$ implying that $f + g \in A^\circ$. Likewise, $\alpha f \in A^\circ$, for $\alpha \in \kappa, f \in A^\circ$, since $Ker(f) \subseteq Ker(\alpha f)$.

This subspace is known as the *finite dual* of the algebra A. The finite dual has a natural coalgebra structure, but to show this requires a bit of preliminary work. For a complete treatment of the finite dual see [10]. Here it will be sufficient to simply consider a useful example.

Consider the polynomial algebra in one indeterminate $\kappa[x]$. This algebra is infinite dimensional with standard basis $\{1, x, x^2, ...\}$. The "dual" elements $\hat{1}, \hat{x}, \hat{x}^2, ...$ defined by

$$\hat{x}^i(x^j) = \delta_{ij}$$

are in $\kappa[x]^\circ$, since for each $\hat{x}^i$, $Ker(\hat{x}^i)$ contains the ideal generated by x^{i+1} and $\kappa[x]/\langle x^{i+1}\rangle$ is finite dimensional. The set $\{\hat{1}, \hat{x}, \hat{x}^2, ...\}$ acts as a dual basis for $\kappa[x]^\circ$.

Now, because $x^i x^j = x^{i+j}$ in $\kappa[x]$, then $x^i x^j = \sum_k c_{ijk} x^k$ tells us that the structure constants are

$$c_{ijk} = \delta_{i+j,k}$$

and therefore, the coalgebra structure is given by a coproduct:

$$\begin{aligned}\Delta(\hat{x}^k) &= \sum_{ij} c_{ijk} \hat{x}^i \otimes \hat{x}^j \\ &= \sum_{i+j=k} \hat{x}^i \otimes \hat{x}^j\end{aligned}$$

and counit determined by $(\varepsilon \otimes \mathrm{id})(\Delta(\hat{x}^k))$ or equivalently by $(\mathrm{id} \otimes \varepsilon)(\Delta(\hat{x}^k))$:

$$\begin{aligned}(\varepsilon \otimes \mathrm{id})(\Delta(\hat{x}^k)) &= (\varepsilon \otimes \mathrm{id})\Big(\sum_{i+j=k} \hat{x}^i \otimes \hat{x}^j\Big) \\ &= \sum_{i+j=k} \varepsilon(\hat{x}^i) \otimes \hat{x}^j\end{aligned}$$

This implies that $\varepsilon(\hat{x}^i) = \delta_{i0}$, which is the same as η^*.

We can transfer this coalgebra structure to $\kappa[x]$ by identifying $\hat{x}^i$ with x^i. The coproduct, then, is

$$\Delta(x^k) = \sum_{i+j=k} x^i \otimes x^j$$

and the counit $\varepsilon(x^i) = \delta_{i0}$.

In Example 3.11 we introduced, out of thin air, a coalgebra structure on $\kappa[x]$. Our work here has produced a similar, albeit different, coalgebra structure for $\kappa[x]$. In some sense, though, this construction now motivates the definition used in that example, since we need only add in the correct binomial coefficients to recover the coproduct $\Delta(x^n) = (x \otimes 1 + 1 \otimes x)^n$.

Taking the finite dual now gives us the means to go from *Alg* to *CoAlg*, thereby establishing the connection in Figure 1.1. Next, we shall consider how the dual gives a contravariant equivalence between more specific objects in these categories.

3.2.5 Co-Semi-Simple Coalgebras

Definition 3.38 (Simple Coalgebra)**.** Let C be a coalgebra. Then C is said to be *simple* if it has no proper sub-coalgebras.

An easy, straightforward example of such a coalgebra is afforded by any one dimensional coalgebra.

Definition 3.39 (Co-Semi-Simple Coalgebra)**.** Let C be a coalgebra. Then C is said to be *co-semi-simple* if it is a direct sum of simple sub-coalgebras.

Proposition 3.40. *Any simple coalgebra is finite dimensional.*

Proof. Suppose C is a simple coalgebra. Let $c \in C$ be a non-zero element of C. Then, by the Fundamental Theorem of Coalgebras (see Theorem 3.26) c is contained in a finite dimensional subcoalgebra C' of C. But C is simple and therefore $C' = C$. Thus, C is finite dimensional. □

Theorem 3.41. *If C is a simple coalgebra, then C^* is a simple algebra.*

Proof. In the previous section we showed that sub-coalgebras of C are in bijective correspondence to ideals of C^*. But since C is simple it has no proper sub-coalgebras. Thus, there are no corresponding non-trivial proper ideals of C^* and hence C^* is simple. □

Corollary 3.42. *C is a finite-dimensional co-semi-simple coalgebra if and only if C^* is a finite-dimensional semi-simple algebra.*

Proof. If C is a finite-dimensional co-semi-simple coalgebra, then, by definition

$$C = \bigoplus_i C_i$$

where each C_i is a simple sub-coalgebra. Since C is finite dimensional taking the dual yields

$$C^* = \bigoplus_i C_i^*$$

where each C_i^* is now a simple algebra. Therefore, C^* is semi-simple.

The converse is established in the same manner. Simply start with a finite-dimensional semi-simple algebra A. Then $A = \bigoplus_i A_i$ where each A_i is simple. By taking the dual we get $A^* = \bigoplus_i A_i^*$. Since each A_i has no non-trivial proper ideals, there are no corresponding proper sub-coalgebras in A_i^*. Thus, each A_i^* is a simple coalgebra and hence A^* is co-semi-simple. □

Proposition 3.43. *Any simple, complex co-commutative coalgebra C is one dimensional.*

Proof. Suppose C is a simple complex co-commutative coalgebra. By Proposition 3.40 C must be finite dimensional. Furthermore, by Theorem 3.41 and the fact that C is co-commutative, C^* is a finite dimensional simple commutative algebra. These properties imply that

$$C^* \cong \prod_{i=1}^{1} \mathbb{C} = \mathbb{C} \quad \text{as } \mathbb{C}\text{-algebras}$$

which further implies that $C \cong \mathbb{C}$ as vector spaces. Therefore, C is a one-dimensional complex coalgebra. □

What this section shows is that taking duals yields a contravariant functor from the category of finite-dimensional co-semi-simple coalgebras to the category of finite-dimensional semi-simple algebras. In fact, what we have is a contravariant equivalence

$$\{\text{f.d. co-semi-simple coalgebras}\} \cong \{\text{f.d. semi-simple algebras}\}$$

At the beginning of this chapter we concluded that

$$FinSet^{op} \cong \{\text{f.d., commutative } \mathbb{C}\text{-algebras with no nilpotent elements}\}$$

Note that finite-dimensional $\mathbb{C}$-algebras with no nilpotent elements are the same as finite-dimensional commutative semi-simple $\mathbb{C}$-algebras. So, what we have seen in this section is what happens when we drop the commutativity requirement.

Chapter 4

Bialgebras and Hopf Algebras

While algebras and coalgebras are separately very interesting, the aim of this chapter is to explore a more intimate setting where algebra and coalgebra structures are "married" into a single structure called a bialgebra. Using these as a springboard, we then turn to the most important objects in our study called *Hopf Algebras*. They are the "most" important because, as we shall see later, quantum groups *are* a special kind of Hopf Algebra.

Comparatively speaking, Hopf Algebras are a neoteric phenomenon. Their mathematical origins trace back to the study of algebraic topology and algebraic group theory. The name "Hopf Algebra" (technically *algebre de Hopf*) was coined by Armand Borel in 1953 in honor of Heinz Hopf and his work in the 1940's. The actual definition of a Hopf Algebra went through several revisions as various mathematicians developed the theory. Although an interesting study in its own right, we shall here be concerned only with the contemporary meaning of the term.

4.1 Bialgebras

Even more interesting than being an algebra or coalgebra alone is to possess both structures "simultaneously" - i.e. in a compatible way. This means that a vector space H is at once an algebra (H, ∇, η), and a coalgebra (H, Δ, ε), where some compatibility condition exists on the structure maps $\nabla, \eta, \Delta, \varepsilon$. It is to this compatibility that we now turn.

The following theorem is key to the definition of a bialgebra and expresses the compatibility we require. To prove it, we equip $H \otimes H$ with the structures of a tensor product of algebras and a tensor product of coalgebras.

Theorem 4.1. *The following two statements are equivalent.*

1. *The maps* ∇ *and* η *are morphisms of coalgebras.*

2. *The maps* Δ *and* ε *are morphisms of algebras.*

Proof. With regards to (1), note that ∇ can be a morphism of coalgebras if and only if the following diagrams commute:

$$\begin{tikzcd}
H \otimes H \arrow[rr, "\nabla"] \arrow[d, "\Delta \otimes \Delta"] & & H \arrow[dd, "\Delta"] \\
H \otimes H \otimes H \otimes H \arrow[d, "\mathrm{id} \otimes \tau \otimes \mathrm{id}"] & & \\
H \otimes H \otimes H \otimes H \arrow[rr, "\nabla \otimes \nabla"] & & H \otimes H
\end{tikzcd}
\qquad
\begin{tikzcd}
H \otimes H \arrow[r, "\nabla"] \arrow[d, "\varepsilon \otimes \varepsilon"] & H \arrow[dd, "\varepsilon"] \\
\kappa \otimes \kappa \arrow[d, "\wr"] & \\
\kappa \arrow[r, "\mathrm{id}"] & \kappa
\end{tikzcd}$$

Similarly, for η to be a morphism of coalgebras it is necessary and sufficient that the following two diagrams commute:

$$\begin{tikzcd}
\kappa \arrow[r, "\eta"] \arrow[d, "\wr"] & H \arrow[d, "\Delta"] \\
\kappa \otimes \kappa \arrow[r, "\eta \otimes \eta"] & H \otimes H
\end{tikzcd}
\qquad
\begin{tikzcd}
\kappa \arrow[rr, "\eta"] \arrow[dr, "\mathrm{id}"'] & & H \arrow[dl, "\varepsilon"] \\
& \kappa &
\end{tikzcd}$$

Looking at (2), by definition, Δ is a morphism of algebras if and only if the following diagrams commute:

$$\begin{tikzcd}
H \otimes H \arrow[r, "\Delta \otimes \Delta"] \arrow[d, "\nabla"'] & H \otimes H \otimes H \otimes H \arrow[d, "\nabla_{H \otimes H}"] \\
H \arrow[r, "\Delta"'] & H \otimes H
\end{tikzcd}
\qquad
\begin{tikzcd}
\kappa \arrow[r, "\eta"] \arrow[dr, "\eta_{H \otimes H}"'] & H \arrow[d, "\Delta"] \\
& H \otimes H
\end{tikzcd}$$

Compare the first diagram here to the first diagram for ∇. These will be equivalent provided that $\nabla_{H \otimes H} = (\nabla \otimes \nabla) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id})$. We implicitly showed this when we addressed the tensor product of algebras earlier and defined the product by

$$(a \otimes b) \cdot (c \otimes d) = ac \otimes bd$$

The second diagram for Δ is clearly equivalent to the first diagram for η.

Note, too, that ε is a morphism of algebras if and only if the second diagram for ∇ and the second diagram for η commute ($\eta_\kappa = \mathrm{id}$). The equivalence of (1) and (2), therefore, is clear. $\square$

This result allows us to define our new structure.

Definition 4.2 (Bialgebra)**.** A bialgebra over a field κ is a quintuple $(H, \nabla, \eta, \Delta, \varepsilon)$ where (H, ∇, η) is a κ-algebra and (H, Δ, ε) is a κ-coalgebra verifying the equivalent conditions of theorem 4.1.

Despite having two equivalent ways to check for a bialgebra, the second condition is generally easiest and, therefore, most commonly used. In a more practical format, the diagrams tell us that for Δ and ε to be algebra morphisms requires

$$\Delta(hg) = \Delta(h)\Delta(g), \qquad \varepsilon(hg) = \varepsilon(h)\varepsilon(g) \tag{4.1}$$

$$\Delta(1) = 1 \otimes 1, \qquad \varepsilon(1) = 1 \tag{4.2}$$

for all $h, g \in H$. We can also express the condition that Δ is an algebra morphism in Sweedler notation. Let $h, g \in H$. Then

$$\Delta(hg) = \sum_{(hg)} (hg)^{(1)} \otimes (hg)^{(2)}$$

and

$$\begin{aligned}\Delta(h)\Delta(g) &= \Big(\sum_{(h)} h^{(1)} \otimes h^{(2)}\Big)\Big(\sum_{(g)} g^{(1)} \otimes g^{(2)}\Big) \\ &= \sum_{(h)(g)} h^{(1)}g^{(1)} \otimes h^{(2)}g^{(2)}\end{aligned}$$

Therefore, if Δ is an algebra morphism, then

$$\sum_{(hg)} (hg)^{(1)} \otimes (hg)^{(2)} = \sum_{(h)(g)} h^{(1)}g^{(1)} \otimes h^{(2)}g^{(2)}$$

Definition 4.3 (Bialgebra Morphism)**.** Let B and B' be two κ-bialgebras. A κ-linear map $f : B \to B'$ is called a morphism of bialgebras if it is simultaneously a morphism of algebras and a morphism of coalgebras between the underlying algebras and coalgebras respectively.

We now have our category *BiAlg* from Figure 1.1. As always, whenever we are able to create a new category of objects, we are interested in how to obtain factor objects. For factor bialgebras we have the following theorem:

Theorem 4.4. *Let B be a bialgebra, and I a κ-subspace of B which is an ideal in the underlying algebra and a coideal in the underlying coalgebra. Then the structures of factor algebra and of factor coalgebra on B/I define a bialgebra, and the canonical projection $\pi : B \to B/I$ is a bialgebra morphism. Moreover, if the bialgebra B is commutative (resp. cocommutative), the factor bialgebra B/I is also commutative (resp. cocommutative).*

Proof. Since B is a bialgebra, it is, by definition, compatibly both an algebra and a coalgebra. Because I is an ideal of (B, ∇, η) we know, by Example 3.6, that B/I is a (uniquely determined) algebra and that $\pi : (B, \nabla, \eta) \to B/I$ is an algebra morphism. Likewise, because I is also a co-ideal of (B, Δ, ε) we know, by Theorem 3.28, that B/I is a (uniquely determined) coalgebra with $\pi : (B, \Delta, \varepsilon) \to B/I$ a coalgebra morphism. Since both structures are unique, our task lies in verifying that these two structures are compatible. That is, we must show that Δ and ε (for B/I) are algebra morphisms. Let's start with Δ, which, in the quotient coalgebra case, we denoted by $\overline{\Delta}$. The goal is to show that

$$\begin{tikzcd}
B/I \otimes B/I \arrow[r, "\overline{\Delta}\otimes\overline{\Delta}"] \arrow[d, "\overline{\nabla}"'] & B/I \otimes B/I \otimes B/I \otimes B/I \arrow[d, "\overline{\nabla}_{B/I^{\otimes 2}}"] \\
B/I \arrow[r, "\overline{\Delta}"] & B/I \otimes B/I
\end{tikzcd}$$

commutes. To see that it does, consider the following diagram:

$$\begin{tikzcd}
 & B/I \otimes B/I \arrow[rr, "\overline{\Delta}\otimes\overline{\Delta}"] \arrow[dl, "\pi\otimes\pi"'] \arrow[dd, dotted, "\overline{\nabla}" near start] & & B/I \otimes B/I \otimes B/I \otimes B/I \arrow[dl, "\pi\otimes\pi\otimes\pi\otimes\pi"'] \arrow[dd, "\overline{\nabla}_{B/I^{\otimes 2}}"] \\
B \otimes B \arrow[rr, "\Delta\otimes\Delta"] \arrow[dd, "\nabla"'] & & B \otimes B \otimes B \otimes B \arrow[dd, "\nabla_{B^{\otimes 2}}" near start] & \\
 & B/I \arrow[rr, dotted, "\overline{\Delta}" near end] \arrow[dl, dotted, "\pi"'] & & B/I \otimes B/I \arrow[dl, "\pi\otimes\pi"] \\
B \arrow[rr, "\Delta"] & & B \otimes B &
\end{tikzcd}$$

Note that the front face commutes, since Δ is an algebra morphism. The left and bottom faces commute because π is an algebra morphism and a coalgebra morphism. Also, the right face commutes, since $\pi \otimes \pi$ will also be an algebra morphism. Finally, the top face commutes, since it is essentially tensoring the coalgebra morphism diagram of π (i.e. the bottom face) with itself. Thus, since all these faces commute, the back face is forced to commute as well, which is what we wanted.

To finish showing that $\overline{\Delta}$ is an algebra morphism we also need

$$\begin{tikzcd}
B/I \arrow[rr, "\overline{\Delta}"] & & B/I \otimes B/I \\
& \kappa \arrow[ul, "\overline{\eta}"] \arrow[ur, "\overline{\eta}_{B/I^{\otimes 2}}"'] &
\end{tikzcd}$$

to commute. Again, to see that it does, consider the following diagram:

$$\begin{tikzcd}
B \arrow[rr, "\Delta"] \arrow[dd, "\pi"'] & & B \otimes B \arrow[dd, "\pi \otimes \pi"] \\
& \kappa \arrow[ul, "\eta_B"] \arrow[ur, "\eta_{B^{\otimes 2}}"'] \arrow[dl, "\eta_{B/I}"'] \arrow[dr, "\eta_{B/I^{\otimes 2}}"] & \\
B/I \arrow[rr, "\overline{\Delta}"'] & & B/I \otimes B/I
\end{tikzcd}$$

Now, the upper triangle commutes because Δ is an algebra morphism. Likewise, the left and right triangles commute, since π and $\pi \otimes \pi$ are algebra morphisms. Finally, the overall square commutes on account of π also being a coalgebra morphism. Therefore, the lower triangle is forced to commute and $\overline{\Delta}$ is an algebra morphism.

The case for $\overline{\varepsilon}$ is demonstrated similarly.

□

Due to the connection established between algebras and coalgebras by taking duals, we get the following result for bialgebras.

Proposition 4.5. *Let B be a finite-dimensional bialgebra. Then B^* is a bialgebra with algebra structure which is dual to the coalgebra structure of B and with coalgebra structure which is dual to the algebra structure of B. The bialgebra B^* is called the dual bialgebra of B.*

Proof. The proof here is fairly straightforward. One simply has to apply $*$ to the vector spaces and linear transformations. This reverses all arrows while preserving commutativity of diagrams. The condition of finite dimensionality is needed in order to substitute $(B \otimes B)^*$ and $(B \otimes B \otimes B)^*$ with $B^* \otimes B^*$ and $B^* \otimes B^* \otimes B^*$ respectively. Finally, since

$*$ converts algebra morphisms to coaglebra morphisms and vice versa, the desired result holds. □

We now expand upon our work with the tensor algebra, which, as already mentioned, is one of the most important examples in this thesis. Here we will show that it is also a bialgebra. To motivate this, let V be a vector space and consider the map

$$V \to T(V) \otimes T(V), \quad v \mapsto 1 \otimes v + v \otimes 1$$

By Theorem 3.18 this extends uniquely to an algebra morphism

$$T(V) \to T(V) \otimes T(V)$$

which will be our coproduct Δ. The map $\varepsilon(v) = 0$ extends to $T(V)$ in similar fashion.

Example 4.1 (Tensor Bialgebra)**.** *Given a vector space V, there exists a unique bialgebra structure on the tensor algebra $T(V)$ such that*

$$\Delta(v) = 1 \otimes v + v \otimes 1, \quad \varepsilon(v) = 0$$

for any $v \in V$. This bialgebra structure is cocommutative and for all $v_1, ..., v_n \in V$ we have

$$\varepsilon(v_1 \cdots v_n) = 0$$

and

$$\Delta(v_1 \cdots v_n) = 1 \otimes v_1 \cdots v_n + \sum_{p=1}^{n-1} \sum_{\sigma} v_{\sigma(1)} \cdots v_{\sigma(p)} \otimes v_{\sigma(p+1)} \cdots v_{\sigma(n)} + v_1 \cdots v_n \otimes 1$$

where σ runs over all permutations of the symmetric group S_n such that

$$\sigma(1) < \sigma(2) < ... < \sigma(p) \quad \textit{and} \quad \sigma(p+1) < \sigma(p+2) < ... < \sigma(n)$$

Such a permutation σ is called a $(p, n-p)$-shuffle.

Proof Sketch. We have already shown that the maps Δ and ε are unique. To show the result for $\Delta(v_1 \cdots v_n)$ one can use induction, but we shall simply illustrate the situation by computing a few examples. Consider, for instance, $\Delta(v_1 v_2)$:

$$\begin{aligned}
\Delta(v_1 v_2) &= \Delta(v_1)\Delta(v_2) \\
&= (1 \otimes v_1 + v_1 \otimes 1)(1 \otimes v_2 + v_2 \otimes 1) \\
&= 1 \otimes v_1 v_2 + v_2 \otimes v_1 + v_1 \otimes v_2 + v_1 v_2 \otimes 1
\end{aligned}$$

Since $n = 2$, we are dealing with S_2, which only has two permutations. Note that the end terms are "fixed" in the sense that v_1v_2 occurs on each side of the tensor product. The two middle terms reflect the two permutations in S_2. Let's do one more to get a better idea of these shuffles. For $n = 3$ we have

$$\begin{aligned}\Delta(v_1v_2v_3) &= \Delta(v_1v_2)\Delta(v_3)\\ &= (1 \otimes v_1v_2 + v_2 \otimes v_1 + v_1 \otimes v_2 + v_1v_2 \otimes 1)(1 \otimes v_3 + v_3 \otimes 1)\\ &= 1 \otimes v_1v_2v_3 + v_2 \otimes v_1v_3 + v_1 \otimes v_2v_3 + v_1v_2 \otimes v_3 +\\ &\quad v_3 \otimes v_1v_2 + v_2v_3 \otimes v_1 + v_1v_3 \otimes v_2 + v_1v_2v_3 \otimes 1\end{aligned}$$

Again, we get the end terms with $v_1v_2v_3$ occurring on each side of the tensor product. Now S_3 has six permutations, which are reflected in the middle terms. Notice, however, that indices are always in ascending order on each side of the tensor product. Because of this, the only permutation we don't get from S_3 is 321. Instead, we get another instance of 123, but with the tensor product in a different place. Another way to think of these shuffles is as follows. Besides the end terms, we have six middle terms. Three of them have the form $\_ \otimes \_\_$ where the slot on the left of $\otimes$ represents all the ways of choosing one of v_1, v_2, v_3. Given such a choice, the remaining two must go into the other two slots in ascending index order. The other three terms can be thought of symmetrically, but have the form $\_\_ \otimes \_$. This symmetry is what gives co-commutativity.

The result for ε is trivial, since ε is an algebra morphism. The counit diagram is then easily checked using the formula for ε. Likewise, coassociativity is checked in a straightforward manner using the formula for Δ. □

Since providing a bialgebra structure on $T(V)$ entails providing a coalgebra structure, we see that the functor T also takes us from Vec to $CoAlg$ (see Figure 1.1). In fact, we now see that T goes from Vec straight to $BiAlg$.

4.1.1 The Tensor Product of Bialgebras

Let B and B' be bialgebras. We now wish to consider whether $B \otimes B'$ is a bialgebra. Of course, we already know that $B \otimes B'$ is an algebra and a coalgebra, so the question reduces to whether these structures are compatible. We'll show this by verifying compatibility condition (2). This requires the commutativity of the following diagram:

$$\begin{tikzcd}
B \otimes B' \otimes B \otimes B' \arrow[r, "\Delta_{B\otimes B'} \otimes \Delta_{B\otimes B'}"] \arrow[d, "\nabla_{B\otimes B'}"] & B \otimes B' \otimes B \otimes B' \otimes B \otimes B' \otimes B \otimes B' \arrow[d, "\nabla_{B\otimes B'\otimes B\otimes B'}"] \\
B \otimes B' \arrow[r, "\Delta_{B\otimes B'}"] & B \otimes B' \otimes B \otimes B'
\end{tikzcd}$$

Commutativity can be proved by splitting this diagram into two subdiagrams. Based on the definition of $\Delta_{B\otimes B'} \otimes \Delta_{B\otimes B'}$, the first of these is

$$\begin{tikzcd}
B \otimes B' \otimes B \otimes B' \arrow[r, "\Delta_B \otimes \Delta_{B'} \Delta_B \otimes \Delta_{B'}"] \arrow[d, "\nabla_{B\otimes B'}"] & B \otimes B \otimes B' \otimes B' \otimes B \otimes B \otimes B' \otimes B' \arrow[d, "\nabla_{B\otimes B\otimes B'\otimes B'}"] \\
B \otimes B' \arrow[r, "\Delta_B \otimes \Delta_{B'}"] & B \otimes B \otimes B' \otimes B'
\end{tikzcd}$$

Since B and B' are algebras, so are $B \otimes B$, $B' \otimes B'$, $B \otimes B'$ and $B \otimes B \otimes B' \otimes B'$. The question of the commutativity of this diagram amounts to asking if the map $\Delta_B \otimes \Delta_{B'}$ is an algebra morphism. Surely it is, since Δ_B and $\Delta_{B'}$ are algebra morphisms and the tensor product of two algebra morphisms is an algebra morphism. The second half of the diagram is given by the following subdiagram:

$$\begin{tikzcd}
B \otimes B \otimes B' \otimes B' \otimes B \otimes B \otimes B' \otimes B' \arrow[r, "\mathrm{id} \otimes \tau \otimes \mathrm{id} \otimes \mathrm{id} \otimes \tau \otimes \mathrm{id}"] \arrow[d, "\nabla_{B\otimes B\otimes B'\otimes B'}"] & B \otimes B' \otimes B \otimes B' \otimes B \otimes B' \otimes B \otimes B' \arrow[d, "\nabla_{B\otimes B'\otimes B\otimes B'}"] \\
B \otimes B \otimes B' \otimes B' \arrow[r, "\mathrm{id} \otimes \tau \otimes \mathrm{id}"] & B \otimes B' \otimes B \otimes B'
\end{tikzcd}$$

As in the previous case, the commutativity of this diagram depends on whether $\mathrm{id}\otimes\tau\otimes\mathrm{id}$ is an algebra morphism. It is obvious that id is an algebra morphism and it easy to see that τ is as well (recall $\tau(a \otimes b) = b \otimes a$). Thus, so is the tensor product $\mathrm{id} \otimes \tau \otimes \mathrm{id}$. It follows that the original diagram commutes as well.

The final requirement for showing that $\Delta_{B\otimes B'}$ is an algebra morphism is the commutativity of the following diagram:

$$\begin{tikzcd}
\kappa \arrow[r, "\eta_{B\otimes B'}"] \arrow[dr, "\eta_{B\otimes B'\otimes B\otimes B'}"'] & B \otimes B' \arrow[d, "\Delta_{B\otimes B'}"] \\
& B \otimes B' \otimes B \otimes B'
\end{tikzcd}$$

To show that this commutes we shall again break it into subdiagrams as follows:

$$
\begin{tikzcd}
\kappa \arrow[r, "\sim"] \arrow[drr, "\eta_{B\otimes B\otimes B'\otimes B'}" description] \arrow[ddrr, "\eta_{B\otimes B'\otimes B\otimes B'}" description] & \kappa\otimes\kappa \arrow[r, "\eta_B\otimes\eta_{B'}"] & B\otimes B' \arrow[d, "\Delta_B\otimes\Delta_{B'}"] \\
& & B\otimes B\otimes B'\otimes B' \arrow[d, "\mathrm{id}\otimes\tau\otimes\mathrm{id}"] \\
& & B\otimes B'\otimes B\otimes B'
\end{tikzcd}
$$

The upper subtriangle commutes because $\Delta_B \otimes \Delta_{B'}$ is an algebra morphism and the lower subtriangle commutes because $\mathrm{id} \otimes \tau \otimes \mathrm{id}$ is an algebra morphism. We have therefore shown that $\Delta_{B\otimes B'}$ is an algebra morphism.

Finally, since we may identify κ with $\kappa \otimes \kappa$, we have $\varepsilon_{B\otimes B'} := \varepsilon_B \otimes \varepsilon_{B'}$, which is an algebra morphism, since ε_B and $\varepsilon_{B'}$ are both algebra morphisms. This means that condition (2) is verified and, therefore, that $B \otimes B'$ is a bialgebra.

4.2 Hopf Algebras

Having successfully joined the structures of algebra and coalgebra compatibly into a single bialgebra, we are now in a position to graduate to a higher level of sophistication whereupon we consider the celebrated Hopf algebra. A Hopf algebra is a particularly important and fascinating kind of bialgebra, which comes equipped with some extra, elegant structure. This structure comes from the possession of an additional structure map (joining the ranks of product, coproduct, unit and counit). This "mystery map" emerges quite nicely from an investigation of the linear maps from a coalgebra to an algebra.

Let (A, ∇, η) be an algebra and (C, Δ, ϵ) a coalgebra. Now consider the vector space of all linear maps from C to A, $\mathcal{L}(C, A)$. Suppose we wish to endow $\mathcal{L}(C, A)$ with an algebra structure of its own. We can do this by defining a multiplication $\star$ on $\mathcal{L}(C, A)$. Notice that for $f, g \in \mathcal{L}(C, A)$, $\star$ obviously cannot be a straightforward composition of f and g. We therefore need a way of combining any two linear maps $f, g \in \mathcal{L}(C, A)$ to obtain another, say $h \in \mathcal{L}(C, A)$. An essential strategy in mathematics is to see if one can construct what is needed out of what is already available. Let us see what "resources" are currently at our disposal. Given $f, g \in \mathcal{L}(C, A)$, we want $f \star g \in \mathcal{L}(C, A)$. We therefore need a fixed map defined on C that will give us occasion to make use of both f and g, which are also defined on C. The only promising candidate is the coproduct of

C, namely $\Delta : C \to C \otimes C$. To use f and g, we can take their tensor product, $f \otimes g$, which is a map from $C \otimes C$ to $A \otimes A$. This is a good start. If only we had a map from $A \otimes A \to A$ we could complete the job. Lo and behold, our algebra A has just the map we need, namely $\nabla : A \otimes A \to A$. Putting these together via composition yields the mapping $f \star g$ as:

$$C \xrightarrow{\quad\Delta\quad} C \otimes C \xrightarrow{\quad f \otimes g\quad} A \otimes A \xrightarrow{\quad\nabla\quad} A$$

which is clearly linear, since Δ, ∇, f and g are all linear.

Using Sweedler notation, this translates to:

$$(f \star g)(c) := \sum f(c^{(1)})g(c^{(2)}) \quad \text{for any } c \in C$$

This product map is actually a particular manifestation of one that haunts several diverse branches of mathematics known as *convolution*. For this reason, we retain the name and refer to $\star$ as the convolution product. The term "convolution" comes from the Latin word *convolutus* meaning "to roll together". Intuitively, then, covolution is a way to "meld together" two functions to obtain something of a "hybrid". In an analysis context convolution takes the form of an integral transform. That is, if f and g are two functions, then one form of convolution is given by

$$f \star g \equiv \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$$

In discrete contexts a sum is used instead of an integral or when integration is not defined.

Observe that the convolution product is associative, since if $f, g, h \in \mathcal{L}(C, A)$ and $c \in C$, then we have the following commuting diagram:

$$\begin{tikzcd}
 & C \otimes C \arrow[dr, "\Delta \otimes \mathrm{id}"] & & & A \otimes A \arrow[dr, "\nabla"] & \\
C \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] & & C \otimes C \otimes C \arrow[r, "f \otimes g \otimes h"] & A \otimes A \otimes A \arrow[ur, "\nabla \otimes \mathrm{id}"] \arrow[dr, "\mathrm{id} \otimes \nabla"'] & & A \\
 & C \otimes C \arrow[ur, "\mathrm{id} \otimes \Delta"'] & & & A \otimes A \arrow[ur, "\nabla"'] &
\end{tikzcd}$$

In Sweedler notation, this becomes

$$\begin{aligned}
\big((f \star g) \star h\big)(c) &= \sum (f \star g)(c^{(1)})h(c^{(2)}) \\
&= \sum \left(\sum f(c^{(1)(1)})g(c^{(1)(2)}) \right) h(c^{(2)}) \\
&= \sum f(c^{(1)})g(c^{(2)})h(c^{(3)}) \\
&= \sum f(c^{(1)})(g \star h)(c^{(2)}) \\
&= \big(f \star (g \star h)\big)(c)
\end{aligned}$$

where the upper route is $(f \star g) \star h$ and the lower route is $f \star (g \star h)$. Furthermore, the convolution product is bilinear. Let $f, g, h \in \mathcal{L}(C, A)$ and $k \in \kappa$. Then

$$\begin{aligned}
\big((kf + g) \star h\big)(c) &= \sum (kf + g)(c^{(1)})h(c^{(2)}) \\
&= \sum \big(kf(c^{(1)}) + g(c^{(1)})\big)h(c^{(2)}) \\
&= \sum \Big(kf(c^{(1)})h(c^{(2)}) + g(c^{(1)})h(c^{(2)})\Big) \\
&= k \sum f(c^{(1)})h(c^{(2)}) + \sum g(c^{(1)})h(c^{(2)}) \\
&= k(f \star h)(c) + (g \star h)(c)
\end{aligned}$$

A symmetric argument will show that the convolution product is linear in the second component.

The unit with respect to the convolution product is not immediately obvious, but there is really only one reasonable candidate from $\mathcal{L}(C, A)$, namely, the composition $C \overset{\varepsilon}{\mapsto} \kappa \overset{\eta}{\mapsto} A$. Let us therefore conduct a bit of exploration to ascertain if it is the unit.

$$\begin{tikzcd}
 & C \otimes C \arrow[dr, "\mathrm{id} \otimes \varepsilon"] & & \\
C \arrow[ur, "\Delta"] \arrow[rr, "\sim"] \arrow[drr, "="'] & & C \otimes \kappa \arrow[r, "f \otimes \eta"] \arrow[d, "\wr"] & A \otimes A \arrow[d, "\nabla"] \\
 & & C \arrow[r, "f"] & A
\end{tikzcd}$$

FIGURE 4.1: Revised diagram for $f \star \eta\varepsilon = f$.

Let $f \in \mathcal{L}(C, A)$. Recall that for any $c \in C$ the counit property of C implies

$$c = \sum \varepsilon(c^{(1)})c^{(2)} = \sum c^{(1)}\varepsilon(c^{(2)})$$

Thus

$$\begin{aligned} f(c) &= f\left(\sum c^{(1)}\varepsilon(c^{(2)})\right) \\ &= \sum f(c^{(1)})\varepsilon(c^{(2)}) \qquad \text{[by linearity of } f\text{]} \\ &= (f \star \varepsilon)(c) \\ &= \sum f(c^{(1)}) \otimes \varepsilon(c^{(2)}) \qquad [A \cong A \otimes \kappa] \end{aligned}$$

By Definition 3.20, the diagram

$$\begin{tikzcd} A \otimes \kappa \arrow[r, "\mathrm{id} \otimes \eta"] \arrow[dr, "\sim"'] & A \otimes A \arrow[d, "\nabla"] \\ & A \end{tikzcd}$$

commutes, meaning

$$\sum f(c^{(1)}) \otimes \varepsilon(c^{(2)}) = \sum f(c^{(1)})\eta(\varepsilon(c^{(2)})) = f(c)$$

But notice

$$\sum f(c^{(1)})\eta(\varepsilon(c^{(2)})) = (f \star (\eta \circ \varepsilon))(c)$$

which suggests that $\eta \circ \varepsilon \in \mathcal{L}(C, A)$ is our desired unit. Indeed, a similar argument will show that $\eta \circ \varepsilon$ is a left unit as well. For convenience, let us denote this unit by $1_\star$ ($1_\star := \eta \circ \varepsilon$). Note that if C is a bialgebra, then ε is an algebra morphism and so is $1_\star$, since

$$\begin{aligned} 1_\star(xy) &= \eta(\varepsilon(xy)) \\ &= \varepsilon(xy)\eta(1) \\ &= \varepsilon(x)\varepsilon(y)1 \\ &= \varepsilon(x)1\varepsilon(y)1 \\ &= 1_\star(x)1_\star(y) \end{aligned}$$

We will make use of this result later when working with Hopf algebras.

Having successfully imposed an algebra structure on $\mathcal{L}(C, A)$, imagine the case in which we have a bialgebra H. If H^c is the underlying coalgebra structure and H^a is the underlying algebra, then $\mathcal{L}(H^c, H^a)$ is an algebra under the above construction and $\mathrm{id} : H \to H \in \mathcal{L}(H^c, H^a)$. This last fact is interesting because it leads us to the notion of an *antipode*.

As the name suggests, the antipode will play a role similar to an inverse mapping. It is defined as follows.

Definition 4.6 (Antipode)**.** Let H be a bialgebra. A linear map $S : H \to H$ is called an antipode of the bialgebra H if S is the inverse of the identity map $\mathrm{id} : H \to H$ with respect to the convolution product in $\mathcal{L}(H^c, H^a)$. That is

$$S \star \mathrm{id} = \mathrm{id} \star S = 1_\star$$

or

$$\sum S(h^{(1)})h^{(2)} = \sum h^{(1)} S(h^{(2)}) = 1_\star(h) \qquad \text{for all } h \in H$$

If the reader is familiar with "antipodal" mappings from spherical geometry a word of caution is in order: in this context, it is not required that $S^2 = \mathrm{id}$, and, as a linear map, S may not even have an inverse with respect to composition. It turns out that S only has this kind of inverse in certain circumstances, one of which we will see below.

That a bialgebra has an antipode is not a general property. This leads us to define a special kind of bialgebra, which, the reader may have guessed, is the anticipated Hopf algebra, where the antipode is the "mystery map" we set out to uncover.

Definition 4.7 (Hopf Algebra)**.** A bialgebra H having an antipode is called a Hopf algebra.

It is immediately evident that Hopf algebras have quite a bit of structure. If we wanted to be inefficient we would technically have to say that a Hopf algebra is a sextuple $(H, \nabla, \eta, \Delta, \varepsilon, S)$ such that, in addition to the commuting diagrams given in the definition of bialgebras, the following diagram commutes.

$$\begin{tikzcd}
& H \otimes H \arrow[rr, "S \otimes \mathrm{id}"] & & H \otimes H \arrow[dr, "\nabla"] & \\
H \arrow[ur, "\Delta"] \arrow[rr, "\varepsilon"] \arrow[dr, "\Delta"'] & & \kappa \arrow[rr, "\eta"] & & H \\
& H \otimes H \arrow[rr, "\mathrm{id} \otimes S"] & & H \otimes H \arrow[ur, "\nabla"'] &
\end{tikzcd}$$

Figure 4.2: This diagram encodes the antipode axiom for a Hopf algebra.

Definition 4.8 (Hopf Algebra Morphism)**.** Let H and J be two Hopf algebras. A map $f : H \to J$ is called a morphism of Hopf algebras if it is a morphism of the underlying biagebras.

We now have the means to define the category of Hopf algebras. We shall denote this category by $Hopf$, which is the key to our study and is the central player in Figure 1.1.

Now that we know what a Hopf algebra is and a little of its background, we shall proceed to explore some of the rich structure and beauty of these objects. We begin with a consideration of how morphisms from $Hopf$ interact with the antipodes of Hopf algebras.

Proposition 4.9. *Let H and J be two Hopf algebras with respective antipodes S_H and S_J. Given a linear map $f : H \to J$, then*

(i) $S_J \circ f$ is a left convolution inverse of f if f is a morphism of coalgebras.

(ii) $f \circ S_H$ is a right convolution inverse of f if f is a morphism of algebras.

(iii) $S_J \circ f = f \circ S_H$ if f is a morphism of Hopf algebras. In this case it is the convolution inverse of f.

Part (iii) tells us that Hopf algebra morphisms preserve antipodes.

Proof. From above we know that $\mathcal{L}(H, J)$ is an algebra with respect to the convolution product. Note that $S_J \circ f$ and $f \circ S_H$ are both elements of this algebra. From the definitions of Hopf algebra morphism and antipode we have the following diagram to reference:

$$\begin{tikzcd}
J\otimes J \arrow[r, "\nabla_J"] & J & & J \arrow[r, "\Delta_J"] \arrow[dl, "\varepsilon_J"'] & J\otimes J \arrow[llll, bend right, "\mathrm{id}\otimes S_J"'] \arrow[llll, bend right=20, "S_J\otimes \mathrm{id}"] \\
& & \kappa \arrow[ul, "\eta_J"] \arrow[dl, "\eta_H"'] & & \\
H\otimes H \arrow[r, "\nabla_H"'] \arrow[uu, "f\otimes f"] & H \arrow[uu, "f"] & & H \arrow[r, "\Delta_H"'] \arrow[ul, "\varepsilon_H"] \arrow[uu, "f"'] & H\otimes H \arrow[uu, "f\otimes f"'] \arrow[llll, bend left, "\mathrm{id}\otimes S_H"] \arrow[llll, bend left=20, "S_H\otimes\mathrm{id}"']
\end{tikzcd}$$

FIGURE 4.3: Hopf morphism diagram combined with antipode diagram.

(i) From the above diagram, we single out the following subdiagram:

$$\begin{tikzcd}
& \kappa \arrow[dr, "\eta_J"] & \\
H \arrow[ur, "\varepsilon_H"] \arrow[r, "f"] \arrow[d, "\Delta_H"'] \arrow[dr, phantom, "1"] & J \arrow[u, "\varepsilon_J"'] \arrow[d, "\Delta_J"] \arrow[r, phantom, "3"] & J \\
H \otimes H \arrow[r, "f \otimes f"'] & J \otimes J \arrow[r, "S_J \otimes \mathrm{id}"'] & J \otimes J \arrow[u, "\nabla"']
\end{tikzcd}$$

Note that blocks (1) and (2) commute on account of f being a coalgebra morphism, while block (3) commutes since $S_J \star \mathrm{id} = 1_\star$. Algebraically, this says

$$\begin{aligned}
((S_J \circ f) \star f)(h) &= \sum (S_J \circ f)(h^{(1)}) f(h^{(2)}) \\
&= \sum S_J(f(h^{(1)})) f(h^{(2)}) \\
&= \sum S_J(f(h)^{(1)}) f(h)^{(2)} \qquad [f \text{ is coalgebra morphism}] \\
&= (S_J \star \mathrm{id})(f(h)) \\
&= (\eta_J \circ \varepsilon_J)(f(h)) \\
&= (\eta_J \circ \varepsilon_H)(h) = 1_\star(h) \quad [\text{see diagram}]
\end{aligned}$$

So $S_J \circ f$ is a left inverse for f with respect to the convolution product when f is a coalgebra morphism. Thus, (i) is established.

(ii) Now single out the following subdiagram from Figure 4.3.

$$\begin{tikzcd}
H \otimes H \arrow[r, "\mathrm{id} \otimes S_H"] & H \otimes H \arrow[r, "f \otimes f"] \arrow[d, "\nabla_H"'] \arrow[dr, phantom, "1"] & J \otimes J \arrow[d, "\nabla_J"] \\
H \arrow[u, "\Delta_H"] \arrow[dr, "\varepsilon_H"'] \arrow[ur, phantom, "3"] & H \arrow[r, "f"] \arrow[dr, phantom, "2"] & J \\
& \kappa \arrow[u, "\eta_H"] \arrow[ur, "\eta_J"'] &
\end{tikzcd}$$

This time, blocks (1) and (2) commute because f is an algebra morphism and block (3) commutes because $\mathrm{id} \star S_H = 1_\star$. Algebraically, we have

$$\begin{aligned}
(f \star (f \circ S_H))(h) &= \sum f(h^{(1)}) f(S_H(h^{(2)})) \\
&= \sum f(h^{(1)} S_H(h^{(2)})) \quad [f \text{ is algebra morphism}] \\
&= f\left(\sum h^{(1)} S_H(h^{(2)})\right) \\
&= f(\mathrm{id} \star S_H(h)) \\
&= f(\eta_H(\varepsilon_H(h))) \\
&= \eta_J(\varepsilon_H(h)) = 1_\star(h) \quad [\text{see diagram}]
\end{aligned}$$

so that $f \circ S_H$ is a right inverse for f with respect to the convolution product, which establishes (ii).

(iii) If f is now taken to be a Hopf algebra morphism, then by (i) and (ii) f has both a left and a right inverse. It is therefore *convolution invertible*, whence $S_J \circ f = f \circ S_H$. □

This last part means that every Hopf morphism is invertible in the algebra $\mathcal{L}(H, J)$ with convolution product. Thus, the set of Hopf morphisms is contained in the group of units of $\mathcal{L}(H, J)$.

We now turn to some important properties of the antipode. Our first result says that a bialgebra can have *at most* one antipode.

Proposition 4.10. *Let H be a Hopf Algebra. If S is the antipode of H, then S is unique.*

Proof. Suppose that S and S' are antipodes for H. Then

$$\begin{aligned}
S &= S \star (\eta \circ \varepsilon) \\
&= S \star (\mathrm{id} \star S') \\
&= (S \star \mathrm{id}) \star S' \\
&= (\eta \circ \varepsilon) \star S' = S'
\end{aligned}$$

□

Before proceeding to the next result, let us recall the opposite algebra and the opposite coalgebra. For any algebra A, there is an opposite algebra, denoted by A^{op}, which has the same underlying vector space as A, but with a multiplication defined by

$$\nabla_{A^{op}} := \nabla_A \circ \tau_{A,A}, \quad \nabla_{A^{op}}(a \otimes a') = a'a$$

Similarly, for any coalgebra C, there is an opposite coalgebra C^{cop} having the same underlying vector space, but with comultiplication

$$\Delta^{op} := \tau_{C,C} \circ \Delta, \quad \Delta^{op}(c) = \sum c^{(2)} \otimes c^{(1)}$$

Proposition 4.11. *If $B = (B, \nabla, \eta, \Delta, \varepsilon)$ is a bialgebra, then*

$$B^{op} = (B, \nabla^{op}, \eta, \Delta, \varepsilon), B^{cop} = (B, \nabla, \eta, \Delta^{op}, \varepsilon) \quad \textit{and} \quad B^{opcop} = (B, \nabla^{op}, \eta, \Delta^{op}, \varepsilon)$$

are also bialgebras.

Proof. To show that B^{op} is a bialgebra we need (B, Δ, ε) to be a coalgebra, (B, ∇^{op}, η) to be an algebra and Theorem 4.1 to be satisfied. Since B is an algebra, it is immediate that B^{op} is an algebra.

$$b_1 \otimes b_2 \otimes b_3 \xmapsto{\mathrm{id} \otimes \nabla^{op}} b_1 \otimes (b_3 b_2) \xmapsto{\nabla^{op}} b_3 b_2 b_1$$

and

$$b_1 \otimes b_2 \otimes b_3 \xmapsto{\nabla^{op} \otimes \mathrm{id}} (b_2 b_1) \otimes b_3 \xmapsto{\nabla^{op}} b_3 b_2 b_1$$

But since B is a coalgebra, B^{op} must be a coalgebra too. For the bialgebra conditions note that Δ and ε are algebra morphisms by hypothesis and η is a coalgebra map by hypothesis. We therefore only need to verify that ∇^{op} is a coalgebra map, which is done by showing that the bialgebra diagrams commute. Let $b_1 \otimes b_2 \in B \otimes B$. Then

$$\begin{aligned} b_1 \otimes b_2 &\xmapsto{\Delta \otimes \Delta} \sum b_1^{(1)} \otimes b_1^{(2)} \otimes b_2^{(1)} \otimes b_2^{(2)} \\ &\xmapsto{\mathrm{id} \otimes \tau \otimes \mathrm{id}} \sum b_1^{(1)} \otimes b_2^{(1)} \otimes b_1^{(2)} \otimes b_2^{(2)} \\ &\xmapsto{\nabla^{op} \otimes \nabla^{op}} \sum b_2^{(1)} b_1^{(1)} \otimes b_2^{(2)} b_1^{(2)} \end{aligned}$$

Going the other way

$$\begin{aligned} b_1 \otimes b_2 &\xmapsto{\nabla^{op}} b_2 b_1 \\ &\xmapsto{\Delta} \sum (b_2 b_1)^{(1)} \otimes (b_2 b_1)^{(1)} \\ &= \sum b_2^{(1)} b_1^{(1)} \otimes b_2^{(2)} b_1^{(2)} \qquad [\Delta \text{ is an algebra morphism}] \end{aligned}$$

Finally

$$b_1 \otimes b_2 \xmapsto{\varepsilon \otimes \varepsilon} \varepsilon(b_1) \otimes \varepsilon(b_2) \xmapsto{\sim} \varepsilon(b_1)\varepsilon(b_2)$$

and

$$b_1 \otimes b_2 \xmapsto{\nabla^{op}} b_2 b_1 \xmapsto{\varepsilon} \varepsilon(b_2 b_1) = \varepsilon(b_2)\varepsilon(b_1)$$

Since κ is a field, we have that $\varepsilon(b_1)\varepsilon(b_2) = \varepsilon(b_2)\varepsilon(b_1)$.

The other two are shown in similar fashion. □

Definition 4.12 (Antialgebra morphism)**.** Let A and A' be algebras. A κ-linear map $f : A \to A'$ of the underlying vector spaces is said to be an antialgebra morphism if $f(ab) = f(b)f(a)$ for all $a, b \in A$.

Proposition 4.13. *Let H be a Hopf algebra with antipode S. Then:*

1. *$S(hg) = S(g)S(h)$ for all $h, g \in H$ - i.e. $S \circ \nabla = \nabla^{op} \circ (S \otimes S)$. Also, ∇ has convolution inverse $S \circ \nabla$.*
2. *$S(1_H) = 1_H$*
3. *$\Delta(S(h)) = \sum S(h^{(2)}) \otimes S(h^{(1)})$ - i.e. $\Delta \circ S = (S \otimes S) \circ \Delta^{op}$. Also, Δ has convolution inverse $\Delta \circ S$.*
4. *$\varepsilon(S(h)) = \varepsilon(h)$*

Notice that (1) and (2) say that the antipode is an antialgebra morphism, while (3) and (4) say that the antipode is also an anticoalgebra morphism.

Proof. Since H is a Hopf algebra we can consider $H \otimes H$ with the structure of the tensor product of coalgebras and H with its algebra structure. Using the convolution product we therefore have that $\mathcal{L}(H \otimes H, H)$ is an algebra. From this algebra, single out the maps ∇, α, and γ defined by

$$\nabla(h \otimes g) = hg, \quad \alpha(h \otimes g) = S(g)S(h), \quad \gamma(h \otimes g) = S(hg)$$

for any $h, g \in H$. Note that ∇ is just the product map of H, γ is the composite map $S \circ \nabla$ and α is the composition $\nabla^{op} \circ (S \otimes S)$.

The strategy here will be to show that ∇ is a left convolution inverse for α and a right convolution inverse for γ. Since we are working in an algebra, this will tell us that $\alpha = \gamma$. Let $h, g \in H$. Then

$$\begin{aligned}
(\nabla \star \alpha)(h \otimes g) &= \sum \nabla\big((h \otimes g)^{(1)}\big)\alpha\big((h \otimes g)^{(2)}\big) \\
&= \sum \nabla(h^{(1)} \otimes g^{(1)})\alpha(h^{(2)} \otimes g^{(2)}) \quad \text{[Using (4.1) and def. of convolution]} \\
&= \sum h^{(1)} g^{(1)} S(g^{(2)}) S(h^{(2)}) \\
&= \sum_{(h)} h^{(1)} \Big(\sum_{(g)} g^{(1)} S(g^{(2)})\Big) S(h^{(2)}) \\
&= \sum h^{(1)} 1_\star(g) S(h^{(2)}) \qquad \text{[by def 4.6]}
\end{aligned}$$

Recall that $1_\star := \eta \circ \varepsilon$. From Definition 3.1 $Im(\eta) \subset Z(H)$ yielding:

$$\begin{aligned}\sum h^{(1)} 1_\star(g) S(h^{(2)}) &= \left(\sum h^{(1)} S(h^{(2)})\right) 1_\star(g)\\ &= 1_\star(h) 1_\star(g) \qquad \text{[by def 4.6]}\\ &= 1_\star(hg) \qquad \text{[by (4.1)]}\\ &= 1_\star(h \otimes g)\end{aligned}$$

where this last $1_\star$ is $\eta_H \circ \varepsilon_{H\otimes H} \in \mathcal{L}(H \otimes H, H)$. Thus, ∇ is a left convolution inverse for α.

That ∇ is a right convolution inverse of γ follows from Proposition 4.9, since ∇ is a coalgebra morphism. Thus, (1) is established.

For (2) we simply apply the definition of the antipode to $1 \in H$. Following the commutative antipode diagram (see Figure 4.2) we have

$$1_H \overset{\Delta}{\longmapsto} 1_H \otimes 1_H \overset{S\otimes \mathrm{id}}{\longmapsto} S(1_H) \otimes 1_H \overset{\nabla}{\longmapsto} S(1_H), \qquad [\Delta(1_H) = 1_H \otimes 1_H \text{ by (4.2)}]$$

and

$$1_H \overset{\varepsilon}{\longmapsto} \varepsilon(1_H) \overset{\eta}{\longmapsto} \eta(\varepsilon(1_H)) = 1_\star(1_H) = 1_H \quad [\eta,\, \varepsilon \text{ are alg. morphisms}]$$

where the commutativity gives $S(1_H) = 1_H$.

The technique for (3) is similar to that used for (1) except that we now consider H with its coalgebra structure and view $H \otimes H$ as a tensor product of algebras. Using the convolution product, we have that $\mathcal{L}(H, H \otimes H)$ is an algebra with unit

$$1_{\star_{H\otimes H}} = \eta_{H\otimes H}(\varepsilon(h))$$

where $\eta_{H\otimes H} = \eta_H \otimes \eta_H$ (under the identification of κ with $\kappa \otimes \kappa$).

It is equivalent to show that $\Delta \circ S = (S \otimes S) \circ \Delta^{op}$. Single out the maps $\delta, \beta \in \mathcal{L}(H, H \otimes H)$ defined by

$$\delta(h) = \Delta(S(h)), \quad \beta(h) = \sum S(h^{(2)}) \otimes S(h^{(1)})$$

for any $h \in H$. Note that δ is obviously just the composite map $\Delta \circ S$, while β is the composite map $(S \otimes S) \circ \Delta^{op}$. Immediately we get that Δ is a left convolution inverse for δ by Proposition 4.9, since Δ is an algebra morphism.

We now show that Δ is a right convolution inverse for β. The following computation relies heavily on the coassociativity of Δ expressed in Sweedler notation. Details will follow the computation.

$$\begin{aligned}
(\beta \star \Delta)(h) &= \sum \beta(h^{(1)})\Delta(h^{(2)}) \\
&= \sum \left(\sum S(h^{(1,2)}) \otimes S(h^{(1,1)})\right) \Delta(h^{(2)}) \\
&= \sum \left(\sum S(h^{(1,2)}) \otimes S(h^{(1,1)})\right) \left(\sum h^{(2,1)} \otimes h^{(2,2)}\right) \\
&= \sum \left(S(h^{(1,2)}) \otimes S(h^{(1,1)})\right)(h^{(2,1)} \otimes h^{(2,2)}) \\
&= \sum (S(h^{(2)}) \otimes S(h^{(1)}))(h^{(3)} \otimes h^{(4)}) \qquad \text{[a]} \\
&= \sum S(h^{(2)})h^{(3)} \otimes S(h^{(1)})h^{(4)} \\
&= \sum S(h^{(2,1)})h^{(2,2)} \otimes S(h^{(1)})h^{(3)} \qquad \text{[b]} \\
&= \sum \varepsilon(h^{(2)})1 \otimes S(h^{(1)})h^{(3)} \\
&= \sum 1 \otimes S(h^{(1)})\varepsilon(h^{(2)})h^{(3)} \qquad [\varepsilon(h^{(2)}) \text{ is a scalar}] \\
&= \sum 1 \otimes S(h^{(1)})\varepsilon(h^{(2,1)})h^{(2,2)} \qquad \text{[c]} \\
&= \sum 1 \otimes S(h^{(1)})h^{(2)} \\
&= 1 \otimes \varepsilon(h)1 \\
&= (\eta \otimes \eta)(\varepsilon(h))
\end{aligned}$$

To understand why this chain of equalities holds, we need to revisit Sweedler notation and the coassociativity of Δ. Recall that we determined the coassociativity to be expressed by

$$\sum c^{(1,1)} \otimes c^{(1,2)} \otimes c^{(2)} = \sum c^{(1)} \otimes c^{(2,1)} \otimes c^{(2,2)} = \sum c^{(1)} \otimes c^{(2)} \otimes c^{(3)}$$

Following Figure 3.1, if Δ is applied again we get that

$$\sum \Delta(c^{(1)}) \otimes c^{(2)} \otimes c^{(3)} = \sum c^{(1)} \otimes \Delta(c^{(2)}) \otimes c^{(3)} = \sum c^{(1)} \otimes c^{(2)} \otimes \Delta(c^{(3)})$$

We can also write these respectively as

$$\sum c^{(1,1)} \otimes c^{(1,2)} \otimes c^{(2)} \otimes c^{(3)} = \sum c^{(1)} \otimes c^{(2,1)} \otimes c^{(2,2)} \otimes c^{(3)} = \sum c^{(1)} \otimes c^{(2)} \otimes c^{(3,1)} \otimes c^{(3,2)}$$

and the first one can be alternatively expressed by $\sum c^{(1,1)} \otimes c^{(1,2)} \otimes c^{(2,1)} \otimes c^{(2,2)}$; all of these are identified with

$$\sum c^{(1)} \otimes c^{(2)} \otimes c^{(3)} \otimes c^{(4)}$$

So, [a] uses the fact that $\sum h^{(1,1)} \otimes h^{(1,2)} \otimes h^{(2,1)} \otimes h^{(2,2)} = \sum h^{(1)} \otimes h^{(2)} \otimes h^{(3)} \otimes h^{(4)}$. [b] uses the fact that $\sum h^{(1)} \otimes h^{(2)} \otimes h^{(3)} \otimes h^{(4)} = \sum h^{(1)} \otimes h^{(2,1)} \otimes h^{(2,2)} \otimes h^{(3)}$ and [c] uses the fact that $\sum h^{(1)} \otimes h^{(2)} \otimes h^{(3)} = \sum h^{(1)} \otimes h^{(2,1)} \otimes h^{(2,2)}$.

Finally, for (4) we use the fact that $\sum h^{(1)}S(h^{(2)}) = \eta(\varepsilon(h))$. Then

$$\begin{aligned}
\varepsilon(\eta(\varepsilon(h))) &= \sum \varepsilon(h^{(1)})\varepsilon(S(h^{(2)})) \\
&= (\varepsilon \star (\varepsilon \circ S))(h) \\
&= \eta_\kappa(\varepsilon(h)) \quad \text{[by Proposition 4.9]} \\
&= \varepsilon(h) \quad [\text{since } \eta_\kappa = \mathrm{id}_\kappa]
\end{aligned}$$

We use this result to finish the following chain of equalities.

$$\begin{aligned}
\varepsilon(S(h)) &= \varepsilon\left(S\left(\sum \varepsilon(h^{(1)})h^{(2)}\right)\right) \quad \text{[by (3.4), the counit condition]} \\
&= \varepsilon\left(\sum \varepsilon(h^{(1)})S(h^{(2)})\right) \quad [S \text{ is linear}] \\
&= \sum \varepsilon(h^{(1)})\varepsilon(S(h^{(2)})) \quad [\varepsilon \text{ is linear}] \\
&= \varepsilon(h) \quad \text{[using above result]}
\end{aligned}$$

Therefore, $\varepsilon(S(h)) = \varepsilon(h)$. This completes the proof. □

Proposition 4.14. *Let H be a Hopf algebra with antipode S. Then the following statements are equivalent:*

(i) $\sum S(h^{(2)})h^{(1)} = 1_\star(h)$ *for any* $h \in H$.

(ii) $\sum h^{(2)}S(h^{(1)}) = 1_\star(h)$ *for any* $h \in H$.

(iii) $S \circ S = id$ *(or* $S^2 = id$*).*

Proof. [$(i) \implies (iii)$] We are given that $\sum S(h^{(2)})h^{(1)} = 1_\star(h)$ for all $h \in H$. But we also know that $\sum h^{(1)}S(h^{(2)}) = 1_\star(h)$ and so we may say that

$$(\mathrm{id} \star S)(h) = \sum S(h^{(2)})h^{(1)}$$

The reader may take this as a reminder that id is the left (and right) convolution inverse of S. We now show that S^2 is a right convolution inverse of S. Consider

$$\begin{aligned}
(S \star S^2)(h) &= \sum S(h^{(1)})S^2(h^{(2)}) \\
&= \sum S(S(h^{(2)})h^{(1)}) \qquad [S \text{ an anti-alg. map}] \\
&= S\left(\sum S(h^{(2)})h^{(1)}\right) \\
&= S(1_\star(h)) \\
&= S(\eta(\varepsilon(h))) \\
&= S(\varepsilon(h)1) \\
&= \varepsilon(h)S(1) \quad [S \text{ is linear}] \\
&= \varepsilon(h)1 \quad [S(1) = 1] \\
&= 1_\star(h)
\end{aligned}$$

Because the inverse is unique in an algebra we have the desired equality, namely

$$S^2 = \mathrm{id}$$

$[(iii) \implies (ii)]$ Again, we know that $\sum h^{(1)}S(h^{(2)}) = 1_\star(h)$ and therefore

$$\begin{aligned}
S(1_\star(h)) &= S\left(\sum h^{(1)}S(h^{(2)})\right) \\
&= \sum S(h^{(1)}S(h^{(2)})) \\
&= \sum S^2(h^{(2)})S(h^{(1)}) \quad [S \text{ anti-alg. morphism}] \\
&= \sum h^{(2)}S(h^{(1)}) \quad [S^2 = \mathrm{id}]
\end{aligned}$$

But $S(1_\star(h)) = 1_\star(h)$ as shown above and therefore $\sum h^{(2)}S(h^{(1)}) = 1_\star(h)$.

$[(ii) \implies (iii)]$ See the case $(i) \implies (iii)$, but with S^2 shown to be the left convolution inverse of S.

$[(iii) \implies (i)]$ See the case $(iii) \implies (ii)$, but with S applied to

$$\sum S(h^{(1)})h^{(2)} = 1_\star(h)$$

□

Corollary 4.15. *Let H be a commutative or cocommutative Hopf algebra. Then*

$$S^2 = id$$

Proof. If H is commutative, then obviously $\sum S(h^{(1)})h^{(2)} = \sum h^{(2)}S(h^{(1)}) = 1_\star(h)$. Similarly, if H is cocommutative, then $\sum h^{(1)} \otimes h^{(2)} = \sum h^{(2)} \otimes h^{(1)}$ and thus, by applying $\nabla \circ (S \otimes \mathrm{id})$ one obtains

$$\sum S(h^{(1)})h^{(2)} = 1_\star(h) \implies \sum S(h^{(2)})h^{(1)} = 1_\star(h)$$

□

The next result is a useful sufficient condition for determining if a particular algebra morphism is an antipode for a bialgebra. Essentially, it says that we only need to verify the antipode axiom on a generating set for the algebra structure of the bialgebra.

Lemma 4.16. *Let H be a bialgebra and $S : H \to H^{op}$ be an algebra morphism. Assume that H is generated as an algebra by a subset X such that*

$$\sum_{(x)} x^{(1)}S(x^{(2)}) = 1_\star(x) = \sum_{(x)} S(x^{(1)})x^{(2)} \tag{4.3}$$

for all $x \in X$. Then S is an antipode for H.

Proof. Let $x, y \in X$. Since X generates H as an algebra, it will be sufficient to show that the defining antipodal property (4.1) holds for the product xy, since the property will then extend to the rest of H.

$$\begin{aligned}
\sum_{(xy)}(xy)^{(1)}S((xy)^{(2)}) &= \sum_{(x),(y)} x^{(1)}y^{(1)}S(x^{(2)}y^{(2)}) \quad \text{[since } \Delta \text{ is an algebra morphism]} \\
&= \sum_{(x),(y)} x^{(1)}y^{(1)}S(y^{(2)})S(x^{(2)}) \qquad [S \text{ an alg. morphism to } H^{op}] \\
&= \sum_{(x)} x^{(1)}\Big(\sum_{(y)} y^{(1)}S(y^{(2)})\Big)S(x^{(2)}) \\
&= \sum_{(x)} x^{(1)}1_\star(y)S(x^{(2)}) \\
&= \Big(\sum_{(x)} x^{(1)}S(x^{(2)})\Big)1_\star(y) \qquad [1_\star(y) \in Z(H)] \\
&= 1_\star(x)1_\star(y) = 1_\star(xy) \quad \text{[since } \varepsilon \text{ is alg. morphism]}
\end{aligned}$$

That $\sum S((xy)^{(1)})(xy)^{(2)} = 1_\star(xy)$, follows by similar reasoning. □

Definition 4.17 (Hopf Subalgebra)**.** Let H be a Hopf algebra. A subspace A of H is called a *Hopf subalgebra* if

(i) A is a subalgebra of H^a.

(ii) A is a subcoalgebra of H^c.

(iii) $S(A) \subseteq A$.

Definition 4.18 (Hopf Ideal)**.** $I \subseteq H$ is called a *Hopf Ideal* of H if I is an ideal of the algebra H^a, a coideal of the coalgebra H^c, and $S(I) \subseteq I$.

Proposition 4.19. *Let H be a Hopf algebra, and I a Hopf ideal of H. Then we can impose a Hopf algebra structure on the quotient space H/I. When this structure is defined, the canonical projection $\pi : H \to H/I$ is a morphism of Hopf algebras.*

Proof. Earlier in this chapter, namely Theorem 4.4, we showed that H/I has a bialgebra structure. To get a Hopf algebra, however, we need an antipode. By hypothesis, $S(I) \subseteq I$ and so, the map $S : H \to H$ gives rise to the map $\overline{S} : H/I \to H/I$ with

$$\overline{S}(\overline{h}) = \overline{S(h)}$$

Since S is the antipode for H, it is reasonable to suspect that $\overline{S}$ is our desired antipode for H/I. Indeed, if we check the defining relation for an antipode we find

$$\begin{aligned}
\sum \overline{S}(\overline{h^{(1)}})\overline{h^{(2)}} &= \sum \overline{S(h^{(1)})h^{(2)}} \\
&= \overline{\sum S(h^{(1)})h^{(2)}} \\
&= \overline{\eta(\varepsilon(h))} \\
&= \overline{\varepsilon(h)\eta(1)} \\
&= \overline{\varepsilon(h)}\overline{1} \\
&= \overline{\varepsilon}(\overline{h})\overline{1} \\
&= \overline{1}_\star(\overline{h})
\end{aligned}$$

It is similarly shown that $\sum \overline{h^{(1)}}\overline{S}(\overline{h^{(2)}}) = \overline{\varepsilon}(\overline{h})\overline{1}$. □

Example 4.2 (The Tensor Algebra)**.** *Let $H = T(V)$, the tensor algebra of the vector space V. Earlier (see Example 4.1) we showed that $T(V)$ is a bialgebra. We now show that it is also a Hopf algebra with antipode determined by $S(1) = 1$, and for all $v_1, ..., v_n \in V$ by*

$$S(v_1 \cdots v_n) = (-1)^n v_n \cdots v_1$$

Let us begin by first determining what S would have to do to a single $v \in V$. Since $\varepsilon(v) = 0$ (all $v \in V$), we have that $1_\star(v) = 0$ (all $v \in V$). Thus

$$\begin{aligned} 0 &= (\nabla \circ (S \otimes id) \circ \Delta)(v) \\ &= (\nabla \circ (S \otimes id))(1 \otimes v + v \otimes 1) \\ &= \nabla(S(1) \otimes v + S(v) \otimes 1) \\ &= v + S(v) \end{aligned}$$

which shows that $S(v) = -v$. From here the rest is easy, since S is an anti-algebra morphism we get

$$\begin{aligned} S(v_1 \cdots v_n) &= S(v_n) \cdots S(v_1) \\ &= (-v_n) \cdots (-v_1) \\ &= (-1)^n v_n \cdots v_1 \end{aligned}$$

for all $v_1, ..., v_n \in V$.

This shows that the functor T is quite important, because it takes us from Vec to $Hopf$.

We end this section by again considering duals.

Proposition 4.20. *Let H be a finite-dimensional Hopf algebra with antipode S. Then the bialgebra H^* is a Hopf algebra with antipode S^*.*

Proof. Since H^* is already known to be a bialgebra, the only thing before us is to establish S^*, the transpose of S, as the antipode for H^*. Since H is a Hopf algebra, the antipode diagram (see Figure 4.2) commutes. If we take the dual, we get the commuting diagram

$$\begin{tikzcd} & (H \otimes H)^* \arrow[dl, "\Delta^*"'] & & (H \otimes H)^* \arrow[ll, "(S \otimes \mathrm{id})^*"'] & \\ H^* & & \kappa \arrow[ll, "\varepsilon^*"] & & H^* \arrow[ll, "\eta^*"] \arrow[ul, "\nabla^*"'] \arrow[dl, "\nabla^*"] \\ & (H \otimes H)^* \arrow[ul, "\Delta^*"] & & (H \otimes H)^* \arrow[ll, "(\mathrm{id} \otimes S)^*"] & \end{tikzcd}$$

But because H is finite-dimensional we may identify $(H \otimes H)^*$ with $H^* \otimes H^*$ and therefore $(S \otimes \mathrm{id})^*$ and $(\mathrm{id} \otimes S)^*$ with $S^* \otimes \mathrm{id}^*$ and $\mathrm{id}^* \otimes S^*$ respectively. This makes S^* an antipode for H^*. $\square$

4.2.1 The Tensor Product of Hopf Algebras

Once more we seek to test the effectiveness of the tensor product to create new structures, of the same kind, out of old ones. In particular we shall endeavor to establish that the tensor product of Hopf algebras is also a Hopf algebra. Toward this end, suppose H and J are Hopf algebras. Building from our previous work, it is immediate that $H \otimes J$ is a bialgebra. It remains only to show that $H \otimes J$ has an antipode.

An obvious candidate for the antipode of $H \otimes J$ is $S_H \otimes S_J$.

$$\begin{aligned}
\big((S_H \otimes S_J) \star \mathrm{id}\big)(h \otimes j) &= \sum S_H(h^{(1)})h^{(2)} \otimes S_J(j^{(1)})j^{(2)} \\
&= 1_{\star_H}(h) \otimes 1_{\star_J}(j) \\
&= \eta_H(\varepsilon_H(h)) \otimes \eta_J(\varepsilon_J(j)) \\
&= (\eta_H \otimes \eta_J)(\varepsilon_H(h) \otimes \varepsilon_J(j)) \\
&= \big((\eta_H \otimes \eta_J) \circ (\varepsilon_H \otimes \varepsilon_J)\big)(h \otimes j) \\
&= 1_{\star_{H\otimes J}}(h \otimes j)
\end{aligned}$$

It can similarly be shown that $\big(\mathrm{id} \star (S_H \otimes S_J)\big) = 1_{\star_{H\otimes J}}$. Therefore, $H \otimes J$ is a Hopf algebra with antipode $S_H \otimes S_J$.

4.3 Comodules and Hopf Modules

In Chapter 2 we reviewed some of the basics about modules over a ring R. We now extend this to modules over an algebra, which will open the door to *comodules* and *Hopf modules*. This section will aim only to give a cursory overview of this area of study as it won't be directly used later. Nevertheless, it gives an idea of the usefulness of Hopf algebras and their ability to act on a variety of objects. For proofs, see [10].

Definition 4.21 (A-Module)**.** Let A be a κ-algebra. A left A-module is a pair (V, μ_V), where V is a κ-vector space and $\mu_V : A \otimes V \to V$ is a morphism of κ-vector spaces such that the following diagram commutes:

$$\begin{tikzcd}
A \otimes A \otimes V \arrow[rr, "\mathrm{id}_A \otimes \mu_V"] \arrow[dd, "\nabla \otimes \mathrm{id}_V"'] & & A \otimes V \arrow[dd, "\mu_V"] \\
& \kappa \otimes V \arrow[ur, "\eta_A \otimes \mathrm{id}_V"] \arrow[dr, "\sim"] & \\
A \otimes V \arrow[rr, "\mu_V"] & & V
\end{tikzcd}$$

This diagram expresses the usual module axioms of distribution and associativity. A right A-module is defined similarly.

It turns out that the tensor product of A-modules is an $A \otimes A$-module. More specifically, if U and V are two A-modules, then $U \otimes V$ is an $A \otimes A$-module where

$$(a \otimes a')(u \otimes v) := au \otimes a'v, \qquad a, a' \in A, u \in U, v \in V$$

This result can be strengthened, however, if A has a bialgebra structure. In this case, Δ is an algebra morphism and allows for $U \otimes V$ to be given an A-module structure by

$$a \cdot (u \otimes v) = \Delta(a)(u \otimes v) = \sum_{(a)} a^{(1)} u \otimes a^{(2)} v$$

Definition 4.22 (A-Module Morphism)**.** Let A be a κ-algebra, and $(M, \mu_M), (N, \mu_N)$ two left A-modules. The κ-linear map $f : M \to N$ is called a morphism of left A-modules if the following diagram commutes:

$$\begin{tikzcd}
A \otimes M \arrow[r, "\mathrm{id} \otimes f"] \arrow[d, "\mu_M"'] & A \otimes N \arrow[d, "\mu_N"] \\
M \arrow[r, "f"] & N
\end{tikzcd}$$

This depicts, in diagram form, the idea of scalar slide-out. We therefore have that algebras *act* on modules. Naturally, then, coalgebras will act on objects called comodules. The idea behind such objects is arrived at in the same way we came to coalgebras. That is, we reverse arrows to obtain the notion of a comodule over a coalgebra C.

Definition 4.23 (Comodule)**.** Let C be a κ-coalgebra. A right C-comodule is a pair (M, δ), where M is a κ-vector space and $\delta : M \to M \otimes C$ is a morphism of κ-vector spaces such that the following diagram is commutative:

$$\begin{tikzcd}
M \arrow[rr, "\delta"] \arrow[dr, "\sim"] \arrow[dd, "\delta"'] & & M \otimes C \arrow[dd, "\mathrm{id}_M \otimes \Delta"] \\
& M \otimes \kappa & \\
M \otimes C \arrow[ur, "\mathrm{id}_M \otimes \varepsilon"] \arrow[rr, "\delta \otimes \mathrm{id}_C"'] & & M \otimes C \otimes C
\end{tikzcd}$$

A left C-comodule is defined similarly.

Definition 4.24 (C-Comodule Morphism)**.** Let C be a κ-coalgebra and $(J, \delta_J), (L, \delta_L)$ two right C-comodules. The κ-linear map $g : J \to L$ is called a morphism of right C-comodules if the following diagram commutes:

$$\begin{tikzcd} J \arrow[r, "g"] \arrow[d, "\delta_J"'] & L \arrow[d, "\delta_L"] \\ J \otimes C \arrow[r, "g \otimes \mathrm{id}"] & L \otimes C \end{tikzcd}$$

Just as Sweedler's summation notation for coalgebras is the convention of choice, comodules enjoy a similar convention. Let M be a right C-comodule. If $x \in M$, then we write

$$\delta(x) = \sum_{(x)} x_M \otimes x_C$$

where each $x_M \in M$ and each $x_C \in C$. Again, these should be taken as formal representatives as opposed to specific elements.

Using the above comodule diagram and our summation convention we have that

$$\sum (x_M)_M \otimes (x_M)_C \otimes x_C = \sum x_M \otimes x_C^{(1)} \otimes x_C^{(2)} \tag{4.4}$$

$$\sum x_M \otimes \varepsilon(x_C) = x \iff \sum x_M \varepsilon(x_C) = x \tag{4.5}$$

Also, the commutativity of the C-comodule morphism diagram is represented by

$$\sum g(x_L) \otimes x_C = \sum g(x)_L \otimes g(x)_C \tag{4.6}$$

In the language of categories we denote the category of right C-comodules by $\mathcal{M}^C$. The morphisms of this category are the C-comodule morphisms and we write $Com_C(M, N)$ for all the C-comodule morphisms from M to N. Finally, the category of left A-modules will be represented by ${}_A\mathcal{M}$.

It is natural to ask if comodules possess an analogue to The Fundamental Theorem of Coalgebras (see theorem 3.26). It turns out that this is indeed the case. It requires the following definition.

Definition 4.25 (Right C-subcomodule)**.** Let (N, δ) be a right C-comodule. A κ-vector subspace L of N is called a right C-subcomodule if $\delta(L) \subseteq L \otimes C$.

With this, The Fundamental Theorem of Comodules reads just like its coalgebra counterpart.

Theorem 4.26 (The Fundamental Theorem of Comodules)**.** *Let N be a right C-comodule. Any element $\nu \in N$ belongs to a finite dimensional subcomodule.*

So far we have only mentioned right comodules. There will be no need to mention left comodules, since any result concerning right comodules has an analogue for left comodules. The following proposition from category theory establishes this:

Proposition 4.27. *Let C be a coalgebra. Then the categories ${}^{C}\mathcal{M}$ and $\mathcal{M}^{C^{cop}}$ are isomorphic.*

Having established the notion of a C-subcomodule one can, per usual, construct a factor object that, unsurprisingly, is termed a *factor comodule.* First, suppose that (M, δ) is a C-comodule and that L is a C-subcomodule of M. Now consider the factor vector space M/L and the canonical projection map $\pi : M \to M/L$.

Proposition 4.28. *There exists a unique structure of a right C-comodule on M/L for which $\pi : M \to M/L$ is a morphism of C-comodules.*

This is a useful proposition, but it doesn't say how to find such factor comodules. The next result gives some direction to this endeavor.

Proposition 4.29. *Let M and N be two right C-comodules and $f : M \to N$ a morphism of C-comodules. Then $Im(f)$ and $Ker(f)$ are C-subcomodules of N and M respectively.*

This result is an excellent segue into the next theorem, which establishes a fundamental isomorphism theorem for comodules.

Theorem 4.30 (Fundamental Isomorphism Theorem for Comodules)**.** *Let M and N be two right C-comodules, $f : M \to N$ a morphism of right C-comodules, and $\pi : M \to M/Ker(f)$ the canonical projection. Then there exists a unique isomorphism $\overline{f} : M/Ker(f) \to Im(f)$ of C-comodules for which the following diagram commutes*

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\pi"'] & N \\ M/Ker(f) \arrow[r, "\overline{f}"] & Im(f) \arrow[u, hook] \end{tikzcd}$$

We now bring Hopf algebras back into the picture.

Proposition 4.31. *Let H be a Hopf algebra with antipode S. Then the following hold:*

(i) If M is a left H-module (with action denoted by hm for $h \in H$, $m \in M$), then M has a structure of a right H-module given by $mh = S(h)m$ for any $m \in M, h \in H$.

(ii) If M is a right H-comodule, then M has a left H-comodule structure with structure map $\delta' : M \to H \otimes M$ given by

$$\delta'(m) = \sum S(m_H) \otimes m_M$$

Bringing together the notions of H-modules and H-comodules, we get the fascinating structure of a Hopf module.

Definition 4.32 (Hopf Module)**.** A κ-vector space M is called a right H-Hopf module if H has a right H-module structure and a right H-comodule structure given by the map $\delta : M \to M \otimes H$ such that

$$\delta(m) := \sum m_M \otimes m_H$$

and for any $m \in M$, $h \in H$

$$\delta(mh) = \sum m_M h^{(1)} \otimes m_H h^{(2)}$$

This last equality is a compatibility condition to ensure that the module and comodule structures do not conflict. It is similar to the compatibility required in the development of bialgebras (see beginning of this chapter). It can be expressed diagrammatically as

$$\begin{tikzcd}
M \otimes H \arrow[r, "\mu_M"] \arrow[d, "\delta_M \otimes \Delta"'] & M \arrow[r, "\delta_M"] & M \otimes H \\
M \otimes H \otimes H \otimes H \arrow[rr, "\mathrm{id}_M \otimes \tau \otimes \mathrm{id}_H"] & & M \otimes H \otimes H \otimes H \arrow[u, "\mu_M \otimes \mu_H"']
\end{tikzcd}$$

Definition 4.33 (H-Hopf Module Morphism)**.** A map f is called an H-Hopf module morphism if it is a morphism of right H-modules and a morphism of right H-comodules.

We now consider an important kind of Hopf module. Suppose that V is a κ-vector space and define on $V \otimes H$ a right H-module structure given by

$$\mu_{V\otimes H} = \mathrm{id} \otimes \nabla, \quad (v \otimes h)g = v \otimes hg \qquad \text{for any } v \in V \text{ and } h, g \in H$$

Now define a right H-comodule structure given by the map

$$\delta = \mathrm{id} \otimes \Delta : V \otimes H \to V \otimes H \otimes H, \qquad \delta(v \otimes h) = \sum v \otimes h^{(1)} \otimes h^{(2)}$$

for any $v \in V$ and $h \in H$. Equipped with these structures, $V \otimes H$ becomes a right H-Hopf module. That is, provided the compatibility condition is satisfied. Let $v \otimes h \in V \otimes H$, $g \in H$ and consider

$$\begin{aligned}
\delta\big((v \otimes h)g\big) &= \delta(v \otimes hg) \\
&= \sum v \otimes (hg)^{(1)} \otimes (hg)^{(2)} \\
&= \sum v \otimes h^{(1)}g^{(1)} \otimes h^{(2)}g^{(2)} \\
&= \sum (v \otimes h^{(1)})g^{(1)} \otimes h^{(2)}g^{(2)} \\
&= \sum m_{V\otimes H}g^{(1)} \otimes m_H g^{(2)}
\end{aligned}$$

Thus, we can now be sure that the structures are compatible and we have a right H-Hopf module.

Definition 4.34 (Subspace of Coinvariants)**.** Let M be a right H-comodule, with comodule structure given by the map $\delta : M \to M \otimes H$. The set

$$M^{coH} := \{m \in M : \delta(m) = m \otimes 1\}$$

is a vector subspace of M called the subspace of coinvariants of M.

What is interesting about the H-Hopf modules $V \otimes H$ is that these are the only H-Hopf modules (up to isomorphism)! This is expressed as a fundamental theorem of Hopf modules.

Theorem 4.35 (The Fundamental Theorem of Hopf Modules)**.** *Let H be a Hopf algebra and M a right H-Hopf module. Then the map*

$$f : M^{coH} \otimes H \to M, \qquad f(m \otimes h) = mh$$

for any $m \in M^{coH}$ and $h \in H$, is an isomorphism of Hopf modules, where $M^{coH} \otimes H$ has the H-Hopf module structure from above.

4.4 Actions and Coactions

One of the most important features of Hopf algebras is their ability to act on other objects. We build up to this by first examining what it means for an algebra to act on a vector space.

4.4.1 Actions

Definition 4.36 (Left Action)**.** A left action (or representation) of an algebra H is a pair (α, V) where V is a vector space and α is a linear map $\alpha : H \otimes V \to V$, say

$$\alpha(h \otimes v) = \alpha_h(v)$$

such that

$$\alpha_{hg}(v) = \alpha_h(\alpha_g(v)), \quad \alpha(1 \otimes v) = v$$

Right actions are similarly defined. As a matter of notation, it is convenient to write $h \triangleright v$ for the action so that the above becomes

$$h \triangleright v \in V, \quad (hg) \triangleright v = h \triangleright (g \triangleright v), \quad 1 \triangleright v = v$$

The attentive reader might object that this is nothing more than a restatement of the definition of an H-module. While this is true, the focus is different. In the case of H-modules, the focus is on the vector space V, which is being acted upon. In this case, the focus is on the action itself, namely α.

Definition 4.37 (Pull Back)**.** Let (α, V) be a left action of an algebra H. Let A be another algebra and $f : A \to H$ an algebra morphism. We say that the action α *pulls back* to an action α' of A on V given by $\alpha' = \alpha \circ (f \otimes \mathrm{id})$.

The point of this section is not to redo everything from the previous section concerning modules, but to segue into an even more interesting generalization. That is, more than just acting on vector spaces, Hopf algebras can act on algebras, coalgebras and even other Hopf algebras.

Definition 4.38 (H-Module Algebra)**.** Let H be a Hopf algebra. Then an algebra A is called an *H-module algebra* if A is a left H-module and

$$h \triangleright (ab) = \sum (h^{(1)} \triangleright a)(h^{(2)} \triangleright b), \quad h \triangleright 1 = 1_\star(h)$$

for any $h \in H$ and $a, b \in A$.

This is depicted by the following commuting diagram:

$$\begin{tikzcd}
H \otimes A \otimes A \arrow[rr, "\mathrm{id} \otimes \nabla"] \arrow[d, "\Delta \otimes \mathrm{id} \otimes \mathrm{id}"'] & & H \otimes A \arrow[dd, "\alpha"] \\
H \otimes H \otimes A \otimes A \arrow[d, "\mathrm{id} \otimes \tau \otimes \mathrm{id}"'] & H \arrow[ur, "\mathrm{id} \otimes \eta"] \arrow[dr, "1_\star"] & \\
H \otimes A \otimes H \otimes A \arrow[r, "\alpha \otimes \alpha"] & A \otimes A \arrow[r, "\nabla"] & A
\end{tikzcd}$$

where we are identifying H with $H \otimes \kappa$. Similarly, we have

Definition 4.39 (H-Module Coalgebra)**.** A coalgebra C is called a left H-module coalgebra if

$$\Delta(h \triangleright c) = \sum h^{(1)} \triangleright c^{(1)} \otimes h^{(2)} \triangleright c^{(2)}, \qquad \varepsilon(h \triangleright c) = \varepsilon(h)\varepsilon(c)$$

The corresponding commutative diagram is

$$\begin{tikzcd}
H \otimes C \arrow[rr, "\alpha"] \arrow[d, "\Delta_H \otimes \Delta_C"'] \arrow[dr, "\varepsilon_H \otimes \varepsilon_C"] & & C \arrow[dl, "\varepsilon"'] \arrow[dd, "\Delta_C"] \\
H \otimes H \otimes C \otimes C \arrow[d, "\mathrm{id} \otimes \tau \otimes \mathrm{id}"'] & \kappa & \\
H \otimes C \otimes H \otimes C \arrow[rr, "\alpha \otimes \alpha"] & & C \otimes C
\end{tikzcd}$$

Note that in the diagram we are identifying $\kappa \otimes \kappa$ with κ. Upon inspection we see that this diagram also reveals $\triangleright : H \otimes C \to C$ to be a coalgebra map, where $H \otimes C$ is considered as a tensor product of coalgebras.

Example 4.3. *Recall from group theory the notion of an inner automorphism. If G is a group, then an inner automorphism corresponds to the conjugation map $\alpha_g : G \to G$ given by*

$$\alpha_g(x) = gxg^{-1}$$

for some fixed $g \in G$. This gives a left action of G on itself and is sometimes called the left adjoint action. This idea can be generalized to the Hopf algebra case where "conjugation" becomes

$$\alpha_h(g) = \sum_{(h)} h^{(1)} g S(h^{(1)})$$

for fixed $h \in H$. Recall that in a Hopf algebra the role of S is similar to an inverse. Here too we call this a left adjoint action and it makes H into an H-module algebra.

Definition 4.40 (H-Comodule-Algebra)**.** Let $(H, \nabla_H, \eta_H, \Delta_H, \varepsilon_H)$ be a bialgebra and (A, ∇_A, η_A) an algebra. We say A is an H-comodule-algebra if

(i) the vector space A has an H-comodule structure given by a map

$$\delta_A : A \to H \otimes A$$

and

(ii) the structure maps $\nabla_A : A \otimes A \to A$ and $\eta_A : \kappa \to A$ are morphisms of H-comodules with $A \otimes A$ and κ being endowed with H-comodule structures.

Proposition 4.41. *Let H be a bialgebra and A an algebra. Then A is an H-comodule-algebra if and only if*

(i) the vector space A has an H-comodule structure given by a map

$$\delta_A : A \to H \otimes A$$

and

(ii) the map $\delta_A : A \to H \otimes A$ is a morphism of algebras.

For a proof, see [7].

4.5 The Group Algebra

So far, we have focused primarily on the *theory* of bialgebras and Hopf algebras. For clarity, we now endeavor to embody the abstract in more concrete form via an in depth look at some typical, but important examples. The first example we will consider is known as *the group algebra*, which is essential in establishing the connection between groups and quantum groups. We briefly introduced this algebra in Chapter 3, Example 3.5. The others are related and so we consider them together. Specifically, we will consider the polynomial algebra $M(2) := \kappa[a, b, c, d]$, which will lead us to the examples $GL(2)$ and $SL(2)$ which are related to the general linear group and the special linear group respectively.

As always, let κ denote a field and suppose G is any group with group operation $*$. The group algebra, $\kappa[G]$, is the vector space with G as a basis. Thus, a generic element has the form:

$$\sum_{g \in G} \lambda_g g$$

where $\lambda_g \in \kappa$, $g \in G$ and $\lambda_g = 0$ for almost all $g \in G$. In other words, $\kappa[G]$ is the set of all finite linear combinations of elements in G with coefficients from κ. Essentially, we will see, that this is just a polynomial algebra in G. To get an algebra we define multiplication by

$$\Big(\sum_{g\in G}\lambda_g g\Big)\Big(\sum_{g\in G}\gamma_g g\Big) := \sum_{g,h\in G}(\lambda_g\gamma_h)g*h$$

Let us now verify that the relevant diagrams commute:

$$\begin{tikzcd}
\kappa[G]\otimes\kappa[G]\otimes\kappa[G] \arrow[r, "\nabla\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\nabla"'] & \kappa[G]\otimes\kappa[G] \arrow[d, "\nabla"] \\
\kappa[G]\otimes\kappa[G] \arrow[r, "\nabla"] & \kappa[G]
\end{tikzcd}$$

FIGURE 4.4: Algebra diagram for $\kappa[G]$.

and

$$\begin{tikzcd}
\kappa\otimes\kappa[G] \arrow[r, "\eta\otimes\mathrm{id}"] \arrow[dr, "\sim"] & \kappa[G]\otimes\kappa[G] \arrow[d, "\mu"] & \kappa[G]\otimes\kappa \arrow[l, "\mathrm{id}\otimes\eta"'] \arrow[dl, "\sim"'] \\
& \kappa[G] &
\end{tikzcd}$$

FIGURE 4.5: Unit diagram for $\kappa[G]$

Toward this end, let $\sum_{g\in G}a_g g, \sum_{g\in G}b_g g, \sum_{g\in G}c_g g \in \kappa[G]$. Then for the first diagram

$$\Big(\sum_{g\in G}a_g g\Big)\otimes\Big(\sum_{g\in G}b_g g\Big)\otimes\Big(\sum_{g\in G}c_g g\Big) \xmapsto{\nabla\otimes\mathrm{id}} \Big(\sum_{g,h\in G}(a_g b_h)g*h\Big)\otimes\Big(\sum_{g\in G}c_g g\Big)$$

$$\xmapsto{\nabla} \sum_{g,h,j\in G}\big((a_g b_h)c_j\big)(g*h)*j$$

Compare this to

$$\Big(\sum_{g\in G}a_g g\Big)\otimes\Big(\sum_{g\in G}b_g g\Big)\otimes\Big(\sum_{g\in G}c_g g\Big) \xmapsto{\mathrm{id}\otimes\nabla} \Big(\sum_{g\in G}(a_g)g\Big)\otimes\Big(\sum_{h,j\in G}(b_h c_j)h*j\Big)$$

$$\xmapsto{\nabla} \sum_{g,h,j\in G}\big(a_g(b_h c_j)\big)g*(h*j)$$

Because the group operation $*$ is associative and multiplication in κ is associative, we get that

$$\sum_{g,h,j\in G}\big((a_g b_h)c_j\big)(g*h)*j=\sum_{g,h,j\in G}\big(a_g(b_h c_j)\big)g*(h*j)$$

which means that Diagram 4.4 commutes. For Diagram 4.5, recall that η is a linear morphism, so $\eta(1):=1_G$ where 1_G is the identity element of G. Then, if $k\in\kappa$ we have

$$k\otimes\Big(\sum_{g\in G}a_g g\Big)\xmapsto{\eta\otimes\mathrm{id}}k1_G\otimes\Big(\sum_{g\in G}a_g g\Big)$$

$$\xmapsto{\nabla}\sum_{g\in G}(ka_g)1_G*g=\sum_{g\in G}(ka_g)g$$

and

$$k\otimes\Big(\sum_{g\in G}a_g g\Big)\xmapsto{\sim}\sum_{g\in G}(ka_g)g$$

The other half of the diagram is commutative by similar reasoning. So $\kappa[G]$ is indeed an algebra. Note, however, that $\kappa[G]$ will be commutative if and only if G is an abelian group.

Because $\kappa[G]$ is an algebra, there is a natural coalgebra structure on $\kappa[G]^*$ (use finite dual if G is infinite). This has dual basis $\{g^*:g\in G\}$. Now, the structure constants for $\kappa[G]$ can be ascertained as follows: let $g,h\in\kappa[G]$. Then

$$gh=\sum_{f\in G}c_{ghf}f$$

which implies that

$$c_{ghf}=\begin{cases}1 & \text{if } f=gh\\ 0 & \text{if } f\neq gh\end{cases}$$

These determine the coalgebra structure since

$$\Delta(f^*)=\sum_{g,h\in G}c_{ghf}g^*\otimes h^*$$

and hence

$$\Delta(f^*)=\sum_{\substack{g,h\\ f=gh}}g^*\otimes h^*=\sum_{g\in G}g^*\otimes(g^{-1}f)^*$$

For the counit, we use $\varepsilon=\eta^t$ where t is used to denote the transpose to avoid confusion with notation used for dual basis elements. Note again that $\eta(1)=1_G$. Now consider that

$$\begin{aligned}\varepsilon(1_G^*) &= \eta^t(1_G^*)\\ &= 1_G^* \circ \eta\end{aligned}$$

and for any $\lambda \in \kappa$

$$\begin{aligned}(1_G^* \circ \eta)(\lambda) &= 1_G^*(\lambda\eta(1))\\ &= 1_G^*(\lambda 1_G)\\ &= \lambda 1_G^*(1_G)\\ &= \lambda\end{aligned}$$

Thus, $\varepsilon(1_G^*)$ is the identity map on κ, and since $\kappa^* \cong \kappa$, the identity map corresponds to 1. We therefore say that $\varepsilon(1_G^*) = 1$. For $g \neq 1_G$ we apply the same reasoning to find that

$$\begin{aligned}(g^* \circ \eta)(\lambda) &= g^*(\lambda\eta(1))\\ &= \lambda g^*(1_G)\\ &= \lambda 0\\ &= 0\end{aligned}$$

and hence, $\varepsilon(g^*) = 0$ for $g \neq 1_g$.

Now recall the algebra κ^G, which is the space of all functions $G \to \kappa$ with pointwise addition and multiplication. The standard basis for κ^G is $\{\hat{g} : g \in G\}$ where

$$\hat{g}(x) = \delta_{gx} = \begin{cases} 1 & \text{if } g = x \\ 0 & \text{if } g \neq x \end{cases}$$

Since every linear transformation $\kappa[G] \to \kappa$ is uniquely determined by its effect on the basis G, $\kappa[G]^* := \hom(\kappa[G], \kappa)$ can be identified with κ^G. Under this identification g^* becomes $\hat{g}$.

For a generic element $f \in \kappa^G$ we have

$$f = \sum_{g \in G} \lambda_g \hat{g}$$

Evaluating f at $h \in G$ yields

$$f(h) = \sum_{g \in G} \lambda_g \hat{g}(h) = \lambda_h$$

and therefore

$$f = \sum_{g\in G} f(g)\hat{g}$$

This generic element corresponds to $\sum_{g\in G} f(g)g^*$ in $\kappa[G]^*$ and so $\Delta(f)$ corresponds to

$$\begin{aligned}
\Delta\Big(\sum_{g\in G} f(g)g^*\Big) &= \sum_{g\in G} f(g)\Delta(g^*) \\
&= \sum_{g\in G} f(g)\Big(\sum_{h\in G} h^*\otimes(h^{-1}g)^*\Big) \\
&= \sum_{g,h\in G} f(g)h^*\otimes(h^{-1}g)^*
\end{aligned}$$

Converting the pieces back to elements of κ^G yields

$$\Delta(f) = \sum_{g,h\in G} f(g)\hat{h}\otimes\widehat{h^{-1}g}$$

For the counit

$$\begin{aligned}
\varepsilon(f) &= \varepsilon\Big(\sum_{g\in G} f(g)g^*\Big) \\
&= \sum_{g\in G} f(g)\varepsilon(g^*) \\
&= f(1_G)
\end{aligned}$$

Next on the agenda, we want to ascertain an appropriate coalgebra structure on $\kappa[G]$ which will be compatible with its algebra structure. To do this, we shall use κ^G beginning with determining its structure constants. So, for $g, h \in G$ we have

$$\begin{aligned}
(\hat{g}\hat{h})(x) &= \hat{g}(x)\hat{h}(x) \\
&= \delta_{gx}\delta_{hx} \\
&= \begin{cases} 0 & \text{if } g \neq h \\ \delta_{gx} & \text{if } g = h \end{cases}
\end{aligned}$$

and hence

$$\hat{g}\hat{h} = \begin{cases} 0 & \text{if } g \neq h \\ \hat{g} & \text{if } g = h \end{cases}$$

But we also have

$$\hat{g}\hat{h} = \sum_{f\in G} c_{ghf}\hat{f}$$

which implies that the structure constants for κ^G are

$$c_{ghf} = \begin{cases} 1 & \text{if } g = h = f \\ 0 & \text{if } g \neq h \text{ or } g = h, \text{but } f \neq g \end{cases}$$

If we now move to $(\kappa^G)^*$ (where we use the finite dual if G should be infinite), then the same structure constants are used. Hence

$$\begin{aligned} \Delta(\hat{f}^*) &= \sum_{g,h \in G} c_{ghf} \hat{g}^* \otimes \hat{h}^* \\ &= \hat{f}^* \otimes \hat{f}^* \end{aligned}$$

So, under the bijection of sets $(\kappa^G)^* \leftrightarrow \kappa[G]$, with $\hat{g}^* \leftrightarrow g$, we get that $\Delta(g) = g \otimes g$.

To determine the counit for $\kappa[G]$ we begin with the unit of κ^G. Since multiplication is pointwise, it must be that $\eta(1)(g) = 1$ for all $g \in G$. In terms of the basis elements, this implies that

$$\eta(1) = \sum_{g \in G} \hat{g}$$

The counit for $(\kappa^G)^*$, then, is $\varepsilon = \eta^t$, where we again use t to indicate the transpose to avoid confusion with the notation used for dual basis elements g^*. So

$$\begin{aligned} \varepsilon(\hat{g}^*) &= \eta^t(\hat{g}^*) \\ &= \hat{g}^* \circ \eta \end{aligned}$$

But for any $\lambda \in \kappa$ we have

$$\begin{aligned} (\hat{g}^* \circ \eta)(\lambda) &= \hat{g}^*(\eta(\lambda)) \\ &= \hat{g}^*(\lambda\eta(1)) \\ &= \lambda\hat{g}^*(\eta(1)) \\ &= \lambda \end{aligned}$$

So, $\hat{g}^* \circ \eta$ is the identity map on κ. But $\kappa^* \cong \kappa$ and hence the identity map corresponds to 1. Thus, under the identification of $(\kappa^G)^*$ with $\kappa[G]$ we get that $\varepsilon(g) = 1$ for all $g \in G$.

So, the coproduct and counit maps are now to be defined by

$$\Delta(g) := g \otimes g, \quad \varepsilon(g) := 1$$

for all $g \in G$. Since we have identified $\kappa[G]$ with the finite dual of κ^G we immediately get that it is a coalgebra and hence the following diagrams commute:

$$\begin{tikzcd}
\kappa[G] \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & \kappa[G] \otimes \kappa[G] \arrow[d, "\mathrm{id} \otimes \Delta"] \\
\kappa[G] \otimes \kappa[G] \arrow[r, "\Delta \otimes \mathrm{id}"] & \kappa[G] \otimes \kappa[G] \otimes \kappa[G]
\end{tikzcd}$$

FIGURE 4.6: The coalgebra diagram for $\kappa[G]$.

and

$$\begin{tikzcd}
\kappa \otimes \kappa[G] & \kappa[G] \otimes \kappa[G] \arrow[l, "\varepsilon \otimes \mathrm{id}"'] \arrow[r, "\mathrm{id} \otimes \varepsilon"] & \kappa[G] \otimes \kappa \\
& \kappa[G] \arrow[ul, "\sim"'] \arrow[u, "\Delta"] \arrow[ur, "\sim"] &
\end{tikzcd}$$

FIGURE 4.7: The counit diagram for $\kappa[G]$.

What we have found, so far, is that $\kappa[G]$ can be thought separately as an algebra or a coalgebra. It is therefore natural to ask if these two structures are compatible so as to hold simultaneously. In other words, can $\kappa[G]$ be thought of as a bialgebra? To check this, Theorem 4.1 says it is sufficient to show that Δ and ε are algebra morphisms. This amounts to showing that

$$\Delta(gh) = \Delta(g)\Delta(h) = gh \otimes gh, \quad \varepsilon(gh) = \varepsilon(g)\varepsilon(h)$$
$$\Delta(1_G) = 1_G \otimes 1_G \quad \varepsilon(1_G) = 1$$

This is not difficult since three of the conditions hold by definition and $\varepsilon(gh) = \varepsilon(g)\varepsilon(h)$ is a straightforward consequence of the fact that ε sends everything to 1. Thus, as the very existence of this section suggests, $\kappa[G]$ is not just an algebra or coalgebra separately, but a bialgebra.

It remains only to show that $\kappa[G]$ has an antipode. Examine $\mathrm{End}(\kappa[G])$. If there is $S \in \mathrm{End}(\kappa[G])$ which acts as an antipode for $\kappa[G]$, then S must be the convolution inverse of $\mathrm{id}_{\kappa[G]}$. More precisely, for $g \in G$ it is requisite that

$$S(g)g = \varepsilon(g)1_G = gS(g)$$

since for $\alpha, \beta \in \mathrm{End}(\kappa[G])$ the convolution product $\alpha \star \beta$ is given by $\nabla \circ (\alpha \otimes \beta) \circ \Delta$ and hence for any $g \in G$

$$\begin{aligned}(\alpha \star \beta)(g) &= (\nabla \circ (\alpha \otimes \beta) \circ \Delta)(g) \\ &= (\nabla \circ (\alpha \otimes \beta))(g \otimes g) \\ &= \nabla(\alpha(g) \otimes \beta(g)) \\ &= \alpha(g)\beta(g)\end{aligned}$$

But $\varepsilon(g) = 1$ for all $g \in G$ indicating that $S : \kappa[G] \to \kappa[G]$ is to be determined by $S(g) = g^{-1}$ for all $g \in G$, where g^{-1} denotes the group inverse of g with respect to $*$. This, then, is our desired antipode and hence $\kappa[G]$ is a Hopf algebra.

4.5.1 Grouplike Elements

Our consideration of the group algebra thus far has introduced an important kind of element of coalgebras in general, and, by inheritance, for Hopf algebras. For our purposes let H be a Hopf algebra. A nonzero element $x \in H$ is called *grouplike* if $\Delta(x) = x \otimes x$. The set of all grouplike elements of H is denoted by $\mathcal{G}(H)$. Notice, then, that for the group algebra we obviously have

$$G \subseteq \mathcal{G}(\kappa[G])$$

In a moment, however, we will see that this must be an equality.

Proposition 4.42. *Let H be a Hopf algebra with antipode S. Then any grouplike element x has an inverse in $\mathcal{G}(H)$ which is $S(x)$. Consequently, $\mathcal{G}(H)$ is a group.*

Proof. We need to show that $S(x) \in \mathcal{G}(H)$ whenever $x \in \mathcal{G}(H)$. Consider that

$$\begin{aligned}\Delta(S(x)) &= (S \otimes S)(\Delta^{op}(x)) \quad \text{[see Prop. 4.13]} \\ &= (S \otimes S)(x \otimes x) \\ &= S(x) \otimes S(x) \quad [\text{since } x^{(1)} = x = x^{(2)}]\end{aligned}$$

Thus, $S(x) \in \mathcal{G}(H)$. Next, for $x \in \mathcal{G}(H)$ we shall show that $S(x)$ is the inverse of x in $\mathcal{G}(H)$. To do this, we simply follow the antipode diagram (see Figure 4.2):

$$x \xmapsto{\Delta} x \otimes x \xmapsto{S \otimes \mathrm{id}} S(x) \otimes x \xmapsto{\nabla} S(x)x$$

Taking the other “route” we have

$$x \xmapsto{\nabla \circ (\mathrm{id} \otimes S) \circ \Delta} xS(x)$$

Finally, if we take the "direct route" we get that $x \mapsto \eta(\varepsilon(x))$. Since diagram 4.2 commutes we end up with

$$\eta(\varepsilon(x)) = S(x)x = xS(x)$$

This just says that $S(x)x = xS(x) = 1_\star(x)$. All that remains is to show that $1_\star(x) = 1$ when x is grouplike. For this we use the counit diagram for H,

$$x \overset{\Delta}{\longmapsto} x \otimes x \overset{\varepsilon\otimes\mathrm{id}}{\longmapsto} \varepsilon(x) \otimes x \overset{\sim}{\longmapsto} x$$

and from $\varepsilon(x) \otimes x \overset{\sim}{\longmapsto} x$, it is necessary that $\varepsilon(x) = 1$. Thus, each $x \in \mathcal{G}(H)$ is invertible and hence $\mathcal{G}(H)$ is a group. □

So, besides the defining property $\Delta(g) = g \otimes g$ for grouplike elements, the last part of this proof shows that for $g \in \mathcal{G}(H)$ it must be that $\varepsilon(g) = 1$.

Proposition 4.43. *Any set of grouplike elements of H is linearly independent.*

Proof. Suppose $g_1, ..., g_n$ is an independent set of grouplike elements. Let g be another grouplike element, which is not a member of the aforementioned independent set. Now consider the expanded set $\{g_1, ..., g_n, g\}$. If the expanded set were not linearly independent, then we would have

$$g = \sum_{i=1}^{n} \lambda_i g_i, \qquad \lambda_i \in \kappa$$

Taking the coproduct of both sides yields

$$\sum_{ij} \lambda_i \lambda_j g_i \otimes g_j = \sum_{i=1}^{n} \lambda_i g_i \otimes g_i$$

Now because g is grouplike there must exist at least one coefficient λ_k which is non-zero. Via the matching of terms, which can be done since $\{g_i \otimes g_j\}_{ij}$ will be independent, we find that $\lambda_i \lambda_j = 0$ for all $i \neq j$. In particular, then, $\lambda_k \lambda_j = 0$ for all $j \neq k$ implying that $\lambda_j = 0$ for all $j \neq k$. Thus $g = \lambda_k g_k$. But

$$\begin{aligned} 1 &= \varepsilon(g) \\ &= \varepsilon(\lambda_k g_k) \\ &= \lambda_k \varepsilon(g_k) \\ &= \lambda_k 1 \\ &= \lambda_k \end{aligned}$$

Note that this result could also have been obtained from the fact that comparing coefficients would yield $\lambda_k = \lambda_k^2$ implying that $\lambda_k = 1$ or 0. But since λ_k was assumed to be non-zero, it must be 1. From this, however, we acquire the contradiction that $g = g_k$. It therefore follows that the expanded set must be linearly independent and hence, any set of grouplike elements will be linearly independent. $\square$

Since $\dim(\kappa[G]) = |G|$, this last result implies that there are at most $|G|$ grouplike elements. We may therefore conclude that

$$\mathcal{G}(\kappa[G]) = G$$

Proposition 4.44. *If $g \in \mathcal{G}(H)$, then the one dimensional subspace κg is a sub-coalgebra of H and hence is a simple coalgebra.*

Proof. This is easily established, since

$$\Delta(\kappa g) = \kappa(g \otimes g) \subseteq \kappa g \otimes \kappa g$$

and any one dimensional coalgebra must be simple. $\square$

More interesting, however, is the converse.

Proposition 4.45. *If κg is a one-dimensional sub-coalgebra of H, then κg is spanned by a grouplike element.*

Proof. Since $\Delta(\kappa g) \subseteq \kappa g \otimes \kappa g$ we have that

$$\Delta(g) = \lambda_1 g \otimes \lambda_2 g, \quad \lambda_1, \lambda_2 \in \kappa$$

But $\lambda_1 g \otimes \lambda_2 g = \lambda_1\lambda_2 g \otimes g$ and so, letting $\lambda = \lambda_1\lambda_2$, we have that $\Delta(g) = \lambda g \otimes g$.

Now consider the element λg. We have

$$\Delta(\lambda g) = \lambda\Delta(g) = \lambda\lambda g \otimes g = \lambda g \otimes \lambda g$$

and hence λg is grouplike. Replacing g with this element gives us that $\kappa g = \kappa(\lambda g)$ is spanned by a grouplike element. $\square$

Proposition 4.46. *The subspace spanned by $\mathcal{G}(H)$ is a sub-coalgebra and is co-semi-simple.*

Proof. By Proposition 4.43, $\mathcal{G}(H)$ is linearly independent. We therefore have that

$$Span(\mathcal{G}(H)) = \bigoplus_{\alpha} \kappa g_\alpha, \quad g_\alpha \in \mathcal{G}(H)$$

Since each κg_α is a sub-coalgebra of H then so is $Span(\mathcal{G}(H))$. Furthermore, since each κg_α is simple, it follows, by definition, that $Span(\mathcal{G}(H))$ is co-semi-simple. □

Corollary 4.47. *$Span(\mathcal{G}(H))$ is a sub-Hopf algebra of H isomorphic to the group algebra $\kappa\mathcal{G}(H)$.*

Proof. That $Span(\mathcal{G}(H))$ is a sub-Hopf algebra of H is immediate from the fact that it is a sub-coalgebra of the Hopf algebra H. Also

$$Span(\mathcal{G}(H)) \cong \kappa\mathcal{G}(H)$$

follows from the fact that $\mathcal{G}(H)$ is a group. □

Proposition 4.48. *If G is a finite group, then the group algebra $\kappa[G]$ is a finite-dimensional co-commutative, co-semi-simple Hopf algebra and the assignment*

$$G \to \kappa[G]$$

is functorial.

Proof. If G is a finite group, then it is obvious that $\kappa[G]$ is finite-dimensional, since G is a basis for $\kappa[G]$. Co-commutativity holds on account of the elements of G being grouplike. It is co-semi-simple as a consequence of Proposition 4.46 and Corollary 4.47. Functoriality follows easily from the fact that any group homomorphism $\phi : G \to G'$ extends to a linear transformation $\hat{\phi} : \kappa[G] \to \kappa[G']$, which will also be a Hopf algebra morphism. We can see this by first noting that ϕ and $\hat{\phi}$ will agree on G. Because of this, $\hat{\phi}$ is an algebra morphism since ϕ is a group homomorphism and hence $\hat{\phi}(gh) = \hat{\phi}(g)\hat{\phi}(h)$ for all $g, h \in G$.

To be a coalgebra morphism we need the equality $\Delta_{\kappa[G']} \circ \hat{\phi} = (\hat{\phi} \otimes \hat{\phi}) \circ \Delta_{\kappa[G]}$. Let $g \in G$. Then the right hand side yields

$$\begin{aligned}((\hat{\phi} \otimes \hat{\phi}) \circ \Delta_{\kappa[G]})(g) &= (\hat{\phi} \otimes \hat{\phi})(g \otimes g) \\ &= \hat{\phi}(g) \otimes \hat{\phi}(g)\end{aligned}$$

and from the left hand side we get

$$\begin{aligned}(\Delta_{\kappa[G']} \circ \hat{\phi})(g) &= \Delta_{\kappa[G']}(\hat{\phi}(g)) \\ &= \hat{\phi}(g) \otimes \hat{\phi}(g)\end{aligned}$$

So far, this only means that $\hat{\phi}$ is a bialgebra morphism. The last step is to verify that

$$\hat{\phi}(S_{\kappa[G]}(g)) = S_{\kappa[G']}(\hat{\phi}(g)) \quad \text{all } g \in \kappa[G]$$

which, based on the antipodes, reduces to the statement

$$\hat{\phi}(g^{-1}) = \hat{\phi}(g)^{-1}$$

But this is plainly true, since group homomorphisms preserve inverses. Therefore, $\hat{\phi}$ is a Hopf morphism. □

Proposition 4.49. *If $\alpha : H_1 \to H_2$ is a morphism of Hopf algebras, then α restricts to a group homomorphism*

$$\alpha : \mathcal{G}(H_1) \to \mathcal{G}(H_2)$$

Proof. Let $h \in \mathcal{G}(H_1)$. Since α is a Hopf algebra morphism, it is, in particular, a coalgebra morphism which means that

$$(\alpha \otimes \alpha) \circ \Delta_{H_1} = \Delta_{H_2} \circ \alpha$$

Thus $\Delta_{H_2}(\alpha(h)) = \alpha(h) \otimes \alpha(h)$ and hence $\alpha(h) \in \mathcal{G}(H_2)$.

Now, the group operations for $\mathcal{G}(H_1)$ and $\mathcal{G}(H_2)$ are simply the inherited algebra products from H_1 and H_2 respectively. Since being a Hopf algebra morphism entails being an algebra morphism, it follows that α respects the group operations. Therefore, $\alpha : \mathcal{G}(H_1) \to \mathcal{G}(H_2)$ is a group homomorphism. □

The implication here is that we get a functor from Hopf algebras to groups:

$$H \mapsto \mathcal{G}(H), \qquad (\alpha : H_1 \to H_2) \mapsto (\alpha : \mathcal{G}(H_1) \to \mathcal{G}(H_2))$$

Note further that if H is finite-dimensional, then $\mathcal{G}(H)$ must be a finite group. To see why, suppose that $\mathcal{G}(H)$ were an infinite group, then by Proposition 4.46, $Span(\mathcal{G}(H))$ is an infinite-dimensional sub-coalgebra of H. But this is impossible, since H is finite-dimensional. Thus, $\mathcal{G}(H)$ must be a finite group.

Now, if H is any Hopf algebra, then H^* is also a Hopf algebra, where again, the finite dual is intended if H is not finite-dimensional. For any $f \in H^*$ it was found (in our work with duals) that

$$\Delta(f)(x \otimes y) = f(xy), \quad \text{since } \Delta = \nabla^*$$

and f is an algebra map if and only if $f(xy) = f(x)f(y)$ in which case we would have

$$\begin{aligned}\Delta(f)(x \otimes y) &= f(x)f(y)\\ &= (f \otimes f)(x \otimes y)\end{aligned}$$

for all x, y. This is equivalent to $\Delta(f) = f \otimes f$ and hence that f is grouplike. The result here is that

$$\mathcal{G}(H^*) = Alg(H, \kappa)$$

When H is finite dimensional, however, $H \cong H^{**}$ and therefore

$$\mathcal{G}(H) = \mathcal{G}(H^{**}) = Alg(H^*, \kappa) \tag{4.7}$$

Continuing with finite-dimensional H, the assignment

$$H \mapsto H^*$$

gives a contravariant duality on the category of finite-dimensional Hopf algebras. More specifically, we get a contravariant equivalence between finite-dimensional semi-simple commutative Hopf algebras and finite-dimensional co-semi-simple co-commutative Hopf algebras.

So, in light of (4.7), the functor $\mathcal{G}$ (restricted to finite dimensional Hopf algebras) is the composition of the contravariant functors

$$H \mapsto H^* \mapsto Alg(H^*, \kappa)$$

where the functor $L \mapsto Alg(L, \kappa)$ is the representable functor $Alg(-, \kappa)$ restricted to the category of finite-dimensional Hopf algebras. In general, however, this is a functor from the category of κ-algebras to the category of sets. So, how can we be assured that we will land in groups?

Consider the natural map $H \mapsto H^{**}$ with $x \mapsto \hat{x}$ where

$$\hat{x}(f) = f(x) \quad \text{for } f \in H^*$$

Note that $\hat{x}$ is an algebra morphism if and only if $\hat{x}(fg) = \hat{x}(f)\hat{x}(g)$ for all $f, g \in H^*$, which implies that

$$fg(x) = f(x)g(x) \quad \text{for all } f, g \in H^*$$

Since the product for H^* is convolution, we therefore have

$$(\nabla \circ (f \otimes g) \circ \Delta)(x) = f(x)g(x)$$

Note, in particular, that this will be the case whenever x is grouplike ($\Delta(x) = x \otimes x$) so each grouplike element does give an algebra morphism $\hat{x} \in Alg(H^*, \kappa)$.

Conversely, Suppose $\hat{x}$ is an algebra morphism. By Theorem 3.26, x resides in a finite-dimensional sub-coalgebra (and hence Hopf algebra) H' of H. Thus, $\Delta(x)$ lies in some finite-dimensional sub-space of $H \otimes H$.

Now, from linear algebra we have the following: $Span(x)$ is a sub-space of H'. Thus, there exists a sub-space T for which

$$H' = Span(x) \oplus T$$

We already know that $\{x\}$ is a basis for $Span(x)$, so suppose B is a basis for T. Then it follows that $\{x\} \cup B$ is a basis for H'. Let us write this basis as

$$\{h_1, ..., h_n\}$$

where $h_1 = x$. So

$$\Delta(x) = \sum_{ij} \lambda_{ij} h_i \otimes h_j$$

and

$$fg(x) = \sum_{ij} \lambda_{ij} f(h_i) g(h_j)$$

Let the dual basis for $(H')^*$ be $\{h_1^*, ..., h_n^*\}$ and extend each h_i^* to be an element of H^*. Then

$$\begin{aligned} h_r^* h_s^*(x) &= \sum_{ij} \lambda_{ij} h_r^*(h_i) h_s^*(h_j) \\ &= \lambda_{rs} \end{aligned}$$

But because $\hat{x}$ is an algebra morphism, we also have that

$$\begin{aligned} h_r^* h_s^*(x) &= h_r^*(x) h_s^*(x) \\ &= \delta_{r1} \delta_{s1} \end{aligned}$$

since $x = h_1$. Therefore, when $\hat{x}$ is an algebra morphism we get that

$$\lambda_{rs} = \delta_{r1} \delta_{s1}$$

This allows us to say that

$$\begin{aligned}\Delta(x) &= \sum_{ij} \lambda_{ij} h_i \otimes h_j \\ &= \sum_{ij} \delta_{i1}\delta_{j1} h_i \otimes h_j \\ &= h_1 \otimes h_1 \\ &= x \otimes x\end{aligned}$$

and hence $\hat{x}$ is an algebra morphism if and only if x is grouplike.

If A is any commutative algebra, then $Alg(H, A)$ is a sub-group of the group of units of the algebra $\hom(H, A)$ where the group operation is convolution. In particular, when $A = \kappa$ as in the case above and H is finite-dimensional, then $Alg(H, \kappa)$ is a finite group. This is because H being finite-dimensional implies that H^* is finite-dimensional and hence that $\mathcal{G}(H^*) = Alg(H, \kappa)$ is a finite group.

Next, let us denote the category of finite groups by **FinGp**. Then the assignment

$$G \mapsto \kappa[G]$$

yields a functor from **FinGp** to the category of Hopf algebras, $Hopf$. But when G is a finite group, $\kappa[G]$ is finite-dimensional, co-commutative and co-semi-simple. So, let $\mathcal{H}$ represent the subcategory of all Hopf algebras consisting of those which are finite-dimensional, co-commutative and co-semi-simple. Then our functor is

$$\begin{aligned}\mathbf{FinGp} &\to \mathcal{H} \\ G &\mapsto \kappa[G]\end{aligned}$$

Going the other way, we have seen that if H is finite-dimensional, then $\mathcal{G}(H)$ is a finite group. If H is also co-commutative and co-semi-simple, then $\kappa[\mathcal{G}(H)]$ is a sub-Hopf algebra of H which is finite-dimensional, co-commutative and co-semi-simple, which means that $\kappa[\mathcal{G}(H)]$ itself is a member of $\mathcal{H}$. Now, since H is co-semi-simple, it is a direct sum of simple sub-coalgebras. If $\kappa = \mathbb{C}$, then these simple sub-coalgebras are one-dimensional and hence are spanned by a grouplike element (see Proposition 4.45) . This implies that

$$H = \mathbb{C}[\mathcal{G}(H)]$$

If we revise our conception of $\mathcal{H}$ to now be the sub-category of all complex finite-dimensional, co-commutative, co-semi-simple Hopf algebras, one gets an equivalence of categories:

$$\begin{aligned}\mathbf{FinGp} &\leftrightarrow \mathcal{H}\\ G &\rightarrow \mathbb{C}[G]\\ \mathcal{G}(H) &\leftarrow H\end{aligned}$$

By now we are used to the idea of taking a dual. In this case, if we restrict ourselves to $\mathcal{H}$, then the assignment $H \mapsto H^*$ yields another contravariant equivalence with "inverse" $H^* \mapsto H^{**} \cong H$. Recall that this was explored in Section (3.2.5). Let $\mathcal{H}^*$ represent the category of these dual Hopf algebras and note that each H^* is finite-dimensional, commutative and semi-simple. The contravariant equivalence

$$\mathcal{H} \rightarrow \mathcal{H}^*$$

is called a duality.

Now, because **FinGp** and $\mathcal{H}$ are equivalent categories we also have that

$$\mathcal{H}^* \rightarrow \mathcal{H} \rightarrow \mathbf{FinGp}$$

is a duality. Let $\mathcal{A}$ be the category of all finite-dimensional, commutative, semi-simple $\mathbb{C}$-algebras. In Chapter 3 it was established that the functor

$$\begin{aligned}\mathbf{FinSet} &\rightarrow \mathcal{A}\\ X &\rightarrow \mathbb{C}^X\end{aligned}$$

is a duality as well where the "inverse" is $A \mapsto Alg(A, \mathbb{C})$. We can summarize everything in the following diagram:

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, bend left=5, "Alg(-{,}\mathbb{C})"] & & \mathbf{FinSet} \arrow[ll, bend left=5, "\mathbb{C}^-"] \\
\mathcal{H}^* \arrow[u, hook] \arrow[r, leftrightarrow, "*"] \arrow[rr, bend right, "Alg(-{,}\mathbb{C})"'] & \mathcal{H} \arrow[r, bend left=10, "\mathcal{G}(-)"] & \mathbf{FinGp} \arrow[u, hook] \arrow[l, bend left=10, "\mathbb{C}[-]"]
\end{tikzcd}$$

where the "inclusion" arrows should be taken to indicate the forgetful functor.

The above diagram explains a substantial portion of Figure 1.1 and motivates the notion of a "quantum group". We have seen that when our (finite-dimensional) Hopf algebras are commutative, the corresponding objects are finite groups. But what happens when the commutativity requirement is dropped? We will see later that the corresponding objects are no longer groups, but what we have been calling “quantum groups”. In other words, quantum groups evolve from an attempt to understand $Alg_{\mathbb{C}}(H, \mathbb{C})$ for a finite-dimensional non-commutative Hopf algebra H. Special types of groups (in the commutative case) and quantum groups (in the non-commutative case) arise when finite-dimensionality is dropped. A prominent philosophy, espoused by Drinfel’d, was that one should *quantize* classical objects like $\kappa[G]$ by deforming them to non-cocommutative Hopf algebras. We will explore this concept of deformation in Chapter 6. This is where the name “quantum group” comes from.

Let us now investigate the Hopf algebras of $\mathcal{H}^*$ in a little more detail. Since $\mathcal{H}$ contains Hopf algebras of the form $\mathbb{C}[G]$, $\mathcal{H}^*$ consists of the Hopf algebras $\mathbb{C}[G]^*$. As we already have seen, $\mathbb{C}[G]^*$ has basis $\{g^* : g \in G\}$. If we multiply two basis elements we find that

$$\begin{aligned} g^*h^*(x) &= g^*(x)h^*(x) \\ &= \delta_{gx}\delta_{hx} \quad \text{all } x \in G \end{aligned}$$

and therefore

$$g^*h^* = \begin{cases} g^* & \text{if } g = h \\ 0 & \text{if } g \neq h \end{cases} = \delta_{gh}g^*$$

More explicitly we see that $g^*g^* = g^*$ meaning that the basis elements are idempotent. Since $g^*h^* = 0$ whenever $g \neq h$, the basis elements are also orthogonal. Moreover, since the identity element of $\mathbb{C}[G]^*$ is the sum of all the basis elements: $\sum_{g \in G} g^*$ we say that the basis forms a *complete* set.

As for the co-multiplication, we saw above that

$$\Delta(g^*) = \sum_{h \in G} h^* \otimes (h^{-1}g)^*$$

Thus, if one identifies $\mathbb{C}[G]$ with $\mathbb{C}[G]^*$ the resulting Hopf algebra has basis G with

$$\begin{aligned} gh &= \delta_{gh}g \\ \Delta(g) &= \sum_{h \in G} h \otimes h^{-1}g \end{aligned}$$

The antipode for $\mathbb{C}[G]^*$ is simply S^t (transpose) where we recall that $S(g) = g^{-1}$. Hence, for all $g \in G$, we have

$$S^t(g^*) = g^* \circ S$$

and when applied to any $x \in G$ we get

$$\begin{aligned} S^t(g^*)(x) &= (g^* \circ S)(x) \\ &= g^*(x^{-1}) \\ &= \begin{cases} 1 & \text{if } g = x^{-1} \\ 0 & \text{if } g \neq x^{-1} \end{cases} \\ &= (g^{-1})^*(x) \end{aligned}$$

So, when G^* is identified with G we get the same result: $S(g) = g^{-1}$. Note that as an algebra $\mathbb{C}[G]$ is isomorphic to $\mathbb{C}^G$ and so the coalgebra structure on $(\mathbb{C}^G)^*$ gives $\mathbb{C}[G]$.

Let us now switch our focus to the duality

$$\begin{aligned} \mathbf{FinSet} &\leftrightarrow \hat{\mathcal{A}} \\ X &\rightarrow \kappa^X \\ Alg(A, \kappa) &\leftarrow A \end{aligned}$$

where $\hat{\mathcal{A}}$ is the category of all finite-dimensional, commutative and semi-simple κ-algebras. We have already done some work with $\mathbb{C}^X$ and the situation for κ^X is similar. It has basis $\{\delta_x : x \in X\}$ where

$$\delta_x(z) = \delta_{xz}$$

We began this section on grouplike elements by noting that $\mathcal{G}(\kappa[G]) = G$ and prior to that established an intimate connection between $\kappa[G]$ and κ^G. It seems appropriate, then, to end this section by disclosing one more interesting connection between these two algebras, namely that

$$\mathcal{G}(\kappa^G) = \hat{G}$$

where $\hat{G}$ indicates the character group of G. By way of reminder, a group character is a group homomorphism $G \rightarrow \kappa - \{0\}$.

First, let us motivate a natural coalgebra structure on κ^G. Let us begin by motivating $\Delta : \kappa^G \rightarrow \kappa^G \otimes \kappa^G$.

In the above duality, note that the cartesian product $X \times Y$ gets sent to $\kappa^{X \times Y}$.

Proposition 4.50. *For finite sets X and Y, we have the following isomorphism of algebras:*

$$\kappa^{X \times Y} \cong \kappa^X \otimes \kappa^Y$$

Proof. Suppose $X = \{x_1, ..., x_n\}$ and $Y = \{y_1, ..., y_m\}$. Then κ^X and κ^Y have bases

$$\{\delta_{x_i} : 1 \leq i \leq n\}, \qquad \{\delta_{y_j} : 1 \leq j \leq m\}$$

respectively and therefore $\kappa^X \otimes \kappa^Y$ has basis $\{\delta_{x_i} \otimes \delta_{y_j}\}_{(i,j)}$.

Also, since $X \times Y = \{(x_i, y_j) : x_i \in X, y_j \in Y\}$ we have that $\{\delta_{(x_i,y_j)}\}_{(i,j)}$ is a basis for $\kappa^{X\times Y}$. Define $\alpha : \kappa^{X\times Y} \to \kappa^X \otimes \kappa^Y$ on the basis elements $\delta_{(x_i,y_j)}$ by

$$\alpha(\delta_{(x_i,y_j)}) = \delta_{x_i} \otimes \delta_{y_j}$$

and extend by linearity. So, if $f \in \kappa^{X\times Y}$, then $f = \sum_{i,j} f(x_i, y_j)\delta_{(x_i,y_j)}$ and

$$\begin{aligned}
\alpha(f) &= \alpha\Big(\sum_{i,j} f(x_i, y_j)\delta_{(x_i,y_j)}\Big) \\
&= \sum_{i,j} f(x_i, y_j)\alpha(\delta_{(x_i,y_j)}) \\
&= \sum_{i,j} f(x_i, y_j)\delta_{x_i} \otimes \delta_{y_j}
\end{aligned}$$

This is clearly an algebra morphism on account of the basis elements of $\kappa^{X\times Y}$ being idempotent and orthogonal, which is preserved by α. It is clearly surjective and since

$$\dim \kappa^{X\times Y} = \dim \kappa^X \otimes \dim \kappa^Y = nm$$

it is also injective. Therefore it is our desired isomorphism of algebras. $\square$

Now let $G = \{g_1, ..., g_n\}$ be a finite group with operation $m : G\times G \to G$. By functorality, one obtains

$$\kappa^G \to \kappa^{G\times G} \xrightarrow{\sim} \kappa^G \otimes \kappa^G$$

where

$$f \mapsto f \circ m \mapsto \sum_{i,j} \big(f \circ m\big)(g_i, g_j)\hat{g}_i \otimes \hat{g}_j = \sum_{g,h\in G} f(gh)\hat{g} \otimes \hat{h}$$

Note, in particular, that if $f = \hat{x}$ with $x \in G$, then it gets mapped to the element

$$\begin{aligned}
\sum_{g,h\in G} \hat{x}(gh)\hat{g} \otimes \hat{h} &= \sum_{\substack{g,h\in G \\ gh=x}} \hat{g} \otimes \hat{h} \\
&= \sum_{g\in G} \hat{g} \otimes \widehat{g^{-1}x}
\end{aligned}$$

This motivates our coproduct Δ for κ^G.

For the counit, consider the map $i : \{1\} \to G$ where $i(1) = 1_G$. This gives a map

$$\kappa^G \to \kappa^{\{1\}} = \kappa$$

with

$$f \mapsto f \circ i = f(1_G)$$

So, in particular, $\hat{g} \mapsto \hat{g}(1_G) = \delta_{g1_G}$. In other words, this tells us that we should define $\varepsilon : \kappa^G \to \kappa$ by $\varepsilon(\hat{g}) = \delta_{g1_G}$, which is the same thing we would get if instead we used the counit axiom to determine ε.

We can also define an antipode on κ^G. Consider the map $inv : G \to G$ where $inv(g) = g^{-1}$. Then we get the map

$$\kappa^G \to \kappa^G$$

with

$$f \mapsto f \circ inv$$

and in particular

$$\hat{g} \mapsto \hat{g} \circ inv = \widehat{g^{-1}}$$

Hence, the antipode is $S : G \to G$ with $S(\hat{g}) = \widehat{g^{-1}}$.

To recap, notice that a group is essentially a set that comes with three associated structure maps: (1) a group operation or "multiplication" $m : G \times G \to G$, (2) a unit map $i : \{1\} \to G$ and (3) an inverse map $inv : G \to G$. Each of these resulted in a corresponding structure map for κ^G, namely a co-multiplication, counit, and antipode respectively, which allows for κ^G to be considered a Hopf algebra.

Now for our last result. Let $f \in \kappa^G$ and suppose f is grouplike so that

$$\begin{aligned}\Delta(f) &= f \otimes f \\ &= \Big(\sum_{g \in G} f(g)\hat{g}\Big) \otimes \Big(\sum_{g \in G} f(g)\hat{g}\Big) \\ &= \sum_{g,h \in G} f(g)f(h)\hat{g} \otimes \hat{h}\end{aligned}$$

But based on the coalgebra structure we determined that

$$\Delta(f) = \sum_{g,h \in G} f(gh)\hat{g} \otimes \hat{h}$$

and therefore f is grouplike if and only if $f(gh) = f(g)f(h)$ (and $f \neq 0$), which is equivalent to saying that f is grouplike if and only if f is a group character. Thus

$$\mathcal{G}(\kappa^G) = \hat{G}$$

4.6 $M(2)$, $GL(2)$ and $SL(2)$

Suppose A is a commutative algebra. Let

$$M_2(A) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in A \right\}$$

As a set, $M_2(A) \equiv A^4$. Now, if $M(2) := \kappa[a, b, c, d]$, then we know, from Corollary 3.8, that

$$\hom_{Alg}(M(2), A) \equiv M_2(A) \tag{4.8}$$

with

$$f \mapsto \begin{bmatrix} f(a) & f(b) \\ f(c) & f(d) \end{bmatrix}$$

Since this holds for any commutative algebra we can represent the matrix multiplication of $M_2(A)$ universally on $M(2)$. Let us unpack what this means in more detail. First, we get a natural transformation

$$\hom_{Alg}(M(2) \otimes M(2), A) \to \hom_{Alg}(M(2), A)$$

where A is regarded as a "variable", from the following natural isomorphism of abelian groups:

$$\begin{aligned}
\hom_{Alg}(M(2) \otimes M(2), A) &\cong \hom_{Alg}(M(2), A) \times \hom_{Alg}(M(2), A) \quad \text{[Proposition 3.10]} \\
&\cong M_2(A) \times M_2(A) \quad [(4.8)] \\
&\overset{\nabla}{\longmapsto} M_2(A) \\
&\cong \hom_{Alg}(M(2), A) \quad [(4.8)]
\end{aligned}$$

By basic results from category theory, this natural transformation must be induced by a morphism $M(2) \to M(2) \otimes M(2)$ which will be our coproduct Δ. To find it explicitly, set $A = M(2) \otimes M(2)$ and follow the distinguished element $\mathrm{id}_{M(2)\otimes M(2)}$ through the above isomorphisms with ∇ (this is a standard category technique).

First, we need to determine what $\mathrm{id}_{M(2)\otimes M(2)}$ corresponds to in

$$\hom_{Alg}(M(2), A) \times \hom_{Alg}(M(2), A)$$

Let $i_1 : M(2) \to M(2) \otimes M(2)$ be the algebra map $i_1(a) = a \otimes 1$ and $i_2 : M(2) \to M(2) \otimes M(2)$ the algebra map $i_2(a) = 1 \otimes a$. By Theorem 3.10 we get a unique morphism of algebras $i_1 \otimes i_2$ such that

$$\begin{aligned}(i_1 \otimes i_2)(a \otimes b) &= i_1(a)i_2(b) \\ &= (a \otimes 1)(1 \otimes b) \\ &= a \otimes b\end{aligned}$$

So, $i_1 \otimes i_2$ is the identity map $\mathrm{id}_{M(2)\otimes M(2)}$ and, by Theorem 3.10, corresponds to (i_1, i_2). We therefore have

$$\begin{aligned}\mathrm{id}_{M(2)\otimes M(2)} &\overset{\sim}{\longmapsto} (i_1, i_2) \\ &\overset{\sim}{\longmapsto} \left(\begin{bmatrix} i_1(a) & i_1(b) \\ i_1(c) & i_1(d) \end{bmatrix}, \begin{bmatrix} i_2(a) & i_2(b) \\ i_2(c) & i_2(d) \end{bmatrix} \right) \\ &= \left(\begin{bmatrix} a \otimes 1 & b \otimes 1 \\ c \otimes 1 & d \otimes 1 \end{bmatrix}, \begin{bmatrix} 1 \otimes a & 1 \otimes b \\ 1 \otimes c & 1 \otimes d \end{bmatrix} \right) \\ &\overset{\nabla}{\longmapsto} \begin{bmatrix} a \otimes 1 & b \otimes 1 \\ c \otimes 1 & d \otimes 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \otimes a & 1 \otimes b \\ 1 \otimes c & 1 \otimes d \end{bmatrix} \\ &= \begin{bmatrix} a \otimes a + b \otimes c & a \otimes b + b \otimes d \\ c \otimes a + d \otimes c & c \otimes b + d \otimes d \end{bmatrix} \\ &\overset{\sim}{\longmapsto} \Delta\end{aligned}$$

where

$$\begin{aligned}\Delta(a) &= a \otimes a + b \otimes c & \Delta(b) &= a \otimes b + b \otimes d \\ \Delta(c) &= c \otimes a + d \otimes c & \Delta(d) &= c \otimes b + d \otimes d\end{aligned}$$

Note that since $\Delta \in \hom_{Alg}(M(2), M(2) \otimes M(2))$ it is an algebra morphism.

If we now let $A = \kappa$, then under the isomorphism $\hom_{Alg}(M(2), \kappa) \cong M_2(\kappa)$, the identity of $M_2(\kappa)$ corresponds to ε so that $\varepsilon(a) = \varepsilon(d) = 1$ and $\varepsilon(b) = \varepsilon(c) = 0$. Note, then, that ε is automatically an algebra morphism. Thus, $M(2)$ has a coalgebra structure where the coproduct and counit are algebra morphisms and therefore $M(2)$ is a bialgebra.

We can represent Δ in terms of a kind of matrix product:

$$\Delta \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

This is merely a symbolic matrix product which encodes the above relations of Δ in one simple matrix equation.

Example 4.4 ($GL_2(A)$ and $SL_2(A)$)**.** *For any commutative algebra A we know that a matrix in $M_2(A)$ is invertible if and only if its determinant is invertible in A. Let $A^\times$ be the group of all invertible elements of the algebra A. The set of all invertible matrices of $M_2(A)$ forms a group known as the general linear group of 2×2 matrices:*

$$GL_2(A) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(A) : ad - bc \in A^\times \right\}$$

There is a distinguished subgroup of $GL_2(A)$ called the special linear group of 2×2 matrices. It is defined as

$$SL_2(A) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(A) : ad - bc = 1 \right\}$$

Now define the commutative algebras $GL(2)$ and $SL(2)$ by

$$GL(2) := M(2)[t]/((ad - bc)t - 1), \quad SL(2) := GL(2)/(t - 1)$$

Note that factoring out by $(ad - bc)t - 1$ causes $ad - bc$ to be invertible. For $SL(2)$, factoring out by $t - 1$ just says that $t = 1$ so that $SL(2)$ reduces to $M(2)/(ad - bc - 1)$ which corresponds to the determinant being 1. Notice that $\{a, b, c, d, t\}$ is a generating set for both $GL(2)$ and $SL(2)$.

Now, given a commutative algebra A we know that

$$\hom_{Alg}(\kappa[x_1, ..., x_n], A) \equiv A^n$$

If L is an ideal of $\kappa[x_1, ..., x_n]$ and $\phi : \kappa[x_1, ..., x_n]/L \to A$ is an algebra morphism, then we can always precompose ϕ with the projection map $\pi : \kappa[x_1, ..., x_n] \to \kappa[x_1, ..., x_n]/L$ to get $\phi \circ \pi \in \hom_{Alg}(\kappa[x_1, ..., x_n], A)$. So, $\phi \circ \pi$ is a "point" and, therefore, corresponds to an element $(a_1, ..., a_n) \in A^n$. The set of all such points as ϕ varies over $\hom_{Alg}(\kappa[x_1, ..., x_n]/L, A)$ forms a subset V_L of A^n. Such subsets are known as varieties.

Now, an arbitrary point $(b_1, ..., b_n) \in A^n$ corresponds to an algebra morphism
$\psi : \kappa[x_1, ..., x_n] \to A$, where $\psi(x_i) = b_i$ and induces a morphism $\kappa[x_1, ..., x_n]/L \to A$ if

and only if $\psi(f) = 0$ for all $f \in L$. Equivalently, we get an induced morphism if and only if $\psi(X) = 0$, where X is a generating set for L. If $f = f(x_1, ..., x_n)$ - i.e. a polynomial in $x_1, ..., x_n$, then $\psi(f) = f(b_1, ..., b_n)$. So, $(b_1, ..., b_n) \in V_L$ if and only if $f(b_1, ..., b_n) = 0$ for all $f \in L$. Thus,

$$\hom_{Alg}(\kappa[x_1, ..., x_n]/L, A) \equiv V_L$$

and $V_L = \{(a_1, ..., a_n) : f(a_1, ..., a_n) = 0 \ \text{ for all } f \in L\}$.

Proposition 4.51. *For any commutative algebra A there are natural equivalences*

$$\hom_{Alg}(M(2), A) \equiv M_2(A)$$

$$\hom_{Alg}(GL(2), A) \equiv GL_2(A) \quad \textit{and} \equiv \hom_{Alg}(SL(2), A) \cong SL_2(A)$$

sending an algebra morphism f to the matrix

$$\begin{bmatrix} f(a) & f(b) \\ f(c) & f(d) \end{bmatrix}$$

Proof. By definition, $M(2) := \kappa[a, b, c, d]$ and therefore

$$\hom_{Alg}(M(2), A) \equiv A^4 \equiv M_2(A)$$

Next, $\hom_{Alg}(GL(2), A) = \hom_{Alg}(\kappa[a, b, c, d, t]/((ad - bc)t - 1), A)$ and, by our work above, this is equivalent to

$$\{\left(\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, a_5\right) : (a_1a_4 - a_2a_3)a_5 - 1 = 0\}$$

But this set is equivalent to $GL_2(A)$ under the mapping

$$\left(\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, a_5\right) \to \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$$

and the inverse mapping is obtained since $a_5 = (a_1a_4 - a_2a_3)^{-1}$. Thus

$$\hom_{Alg}(GL(2), A) \equiv GL_2(A)$$

Similarly, $\hom_{Alg}(SL(2), A) = \hom_{Alg}(\kappa[a, b, c, d]/(ad - bc - 1), A)$, which is therefore equivalent to the set

$$\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} : a_1a_4 - a_2a_3 - 1 = 0\}$$

which is equivalent to $SL_2(A)$. Therefore

$$\hom_{Alg}(SL(2), A) \equiv SL_2(A)$$

$\square$

Lemma 4.52. *The algebra morphism Δ of $M(2)$ satisfies*

$$\Delta(ad - bc) = (ad - bc) \otimes (ad - bc)$$

and hence $ad - bc$ is grouplike.

Proof. By straightforward computation:

$$\begin{aligned}
\Delta(ad - bc) &= \Delta(a)\Delta(d) - \Delta(b)\Delta(c) \\
&= (a \otimes a + b \otimes c)(c \otimes b + d \otimes d) - (a \otimes b + b \otimes d)(c \otimes a + d \otimes c) \\
&= ac \otimes ab + ad \otimes ad + bc \otimes cb + bd \otimes cd - ac \otimes ba - ad \otimes bc - bc \otimes da - bd \otimes dc \\
&= ad \otimes ad - bc \otimes ad + bc \otimes bc - ad \otimes bc + ac \otimes ab - ac \otimes ab + bd \otimes dc - bd \otimes dc \\
&= (ad - bc) \otimes ad + (bc - ad) \otimes bc \\
&= (ad - bc) \otimes (ad - bc)
\end{aligned}$$

$\square$

Note how the above computations are crucially dependent on A being commutative.

Now, for any commutative bi-algebra B and grouplike element g

$$B(1 - g) = \{b(1 - g) : b \in B\}$$

is a bi-ideal. It is quite clearly an ideal, since B is commutative. It is less obvious, however, that $B(1 - g)$ is a co-ideal. For this, it is requisite that

$$\Delta B(1 - g) \subseteq B(1 - g) \otimes B + B \otimes B(1 - g)$$

and

$$\varepsilon B(1 - g) = 0$$

Observe, then, that

$$\begin{aligned}\Delta(1-g) &= \Delta(1)-\Delta(g)\\ &= 1\otimes 1 - g\otimes g\\ &= 1\otimes 1 - 1\otimes g + 1\otimes g - g\otimes g\\ &= 1\otimes(1-g) + (1-g)\otimes g\end{aligned}$$

Therefore,

$$\begin{aligned}\Delta B(1-g) &= \Delta B\Delta(1-g)\\ &\subseteq (B\otimes B)(1\otimes(1-g)+(1-g)\otimes g)\\ &\subseteq B\otimes B(1-g) + B(1-g)\otimes Bg\\ &\subseteq B\otimes B(1-g) + B(1-g)\otimes B\end{aligned}$$

Next, since

$$\varepsilon(1-g) = \varepsilon(1)-\varepsilon(g) = 1-1 = 0$$

we have that

$$\begin{aligned}\varepsilon B(1-g) &= \varepsilon B\varepsilon(1-g)\\ &= 0\end{aligned}$$

and hence $B(1-g)$ is a co-ideal and thus a bi-ideal. By Theorem 4.4, then, $B/B(1-g)$ is a commutative bi-algebra.

Taking $B = M(2)$ we see immediately that $SL(2) = M(2)/(1-(ad-bc))$ is a commutative bialgebra. Also, if we set $\Delta(t) = t\otimes t$, then $(ad-bc)t$ is grouplike (since Δ is an algebra morphism) and therefore $GL(2) = M(2)[t]/(1-(ad-bc)t)$ is a commutative bialgebra as well.

The remaining question is whether these are Hopf algebras. By definition, the requirement is the existence of an antipode. Now, since $GL(2)$ and $SL(2)$ are derived from $M(2)$, let us begin by considering $\mathcal{L}(M(2), M(2))$, which, we know, is an algebra under convolution. If there is to be an antipode $S \in \mathcal{L}(M(2), M(2))$, then it must commute with $\mathrm{id}_{M(2)}$. Now, for any commutative algebra A, $\hom_{Alg}(M(2), A) \subseteq \mathcal{L}(M(2), A)$. For $\alpha, \beta \in \hom_{Alg}(M(2), A)$ we have that $\alpha \star \beta := \nabla \circ (\alpha\otimes\beta)\circ\Delta$, which is a composition of three algebra morphisms, since $M(2)$ and A are commutative. Thus, $\alpha\star\beta \in \hom_{Alg}(M(2), A)$ and so corresponds to a matrix. We can determine this as follows:

$$\begin{aligned}
\begin{bmatrix} a & b \\ c & d \end{bmatrix} &\overset{\Delta}{\longmapsto} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\
&= \begin{bmatrix} a\otimes a + b\otimes c & a\otimes b + b\otimes d \\ c\otimes a + d\otimes c & c\otimes b + d\otimes d \end{bmatrix} \\
&\overset{\alpha\otimes\beta}{\longmapsto} \begin{bmatrix} \alpha(a)\otimes\beta(a) + \alpha(b)\otimes\beta(c) & \alpha(a)\otimes\beta(b) + \alpha(b)\otimes\beta(d) \\ \alpha(c)\otimes\beta(a) + \alpha(d)\otimes\beta(c) & \alpha(c)\otimes\beta(b) + \alpha(d)\otimes\beta(d) \end{bmatrix} \\
&\overset{\nabla}{\longmapsto} \begin{bmatrix} \alpha(a)\beta(a) + \alpha(b)\beta(c) & \alpha(a)\beta(b) + \alpha(b)\beta(d) \\ \alpha(c)\beta(a) + \alpha(d)\beta(c) & \alpha(c)\beta(b) + \alpha(d)\beta(d) \end{bmatrix} \\
&= \begin{bmatrix} \alpha(a) & \alpha(b) \\ \alpha(c) & \alpha(d) \end{bmatrix} \cdot \begin{bmatrix} \beta(a) & \beta(b) \\ \beta(c) & \beta(d) \end{bmatrix} \\
&= \alpha \begin{bmatrix} a & b \\ c & d \end{bmatrix} \beta \begin{bmatrix} a & b \\ c & d \end{bmatrix}
\end{aligned}$$

It follows that the correspondence $\alpha \mapsto \begin{bmatrix} \alpha(a) & \alpha(b) \\ \alpha(c) & \alpha(d) \end{bmatrix}$ preserves the binary structure of multiplication so that $\alpha \star \beta \mapsto \alpha\beta$ and

$$\hom_{Alg}(M(2), A) \equiv M_2(A)$$

So, to determine whether such an S exists we use that

$$\hom_{Alg}(M(2), M(2)) \equiv M_2(M(2))$$

via

$$f \mapsto \begin{bmatrix} f(a) & f(b) \\ f(c) & f(d) \end{bmatrix} =: F$$

so that $f \star g \mapsto FG$ where FG is matrix multiplication. Since finding an inverse for $\mathrm{id}_{M(2)}$ requires finding S such that

$$S \star \mathrm{id}_{M(2)} = \varepsilon \circ \eta = \mathrm{id}_{M(2)} \star S$$

and since $\varepsilon \circ \eta \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, this translates to needing

$$\begin{bmatrix} S(a) & S(b) \\ S(c) & S(d) \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} S(a) & S(b) \\ S(c) & S(d) \end{bmatrix}$$

But this is equivalent to requiring that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be invertible in $M_2(M(2))$, which is not, in general, possible. What this tells us is that $M(2)$ is not a Hopf algebra. However, this requirement *does* become possible when we pass to the factor bialgebras $GL(2)$ and $SL(2)$. Therefore, there is an antipode S for $GL(2)$ and $SL(2)$ thereby making them both commutative (but not co-commutative) Hopf algebras. For $SL(2)$ we will have

$$\begin{aligned} S(a) &= d(ad - bc)^{-1} \\ S(b) &= -b(ad - bc)^{-1} \\ S(c) &= -c(ad - bc)^{-1} \\ S(d) &= a(ad - bc)^{-1} \end{aligned}$$

The same will hold for $GL(2)$ with the added assignment $S(t) = t^{-1}$, since t is grouplike.

There is no doubt that this chapter is pivotal in our study. Hopf algebras are the key foundational objects required for understanding quantum groups. Again, this is because quantum groups can be considered as special kinds of Hopf algebras. We also found that the name "quantum group" itself arises out of the connection between Hopf algebras and groups. It was briefly mentioned that the general idea is to deform certain Hopf algebras into non-commuting or non-cocommuting structures. We will investigate this in Chapter 6 and Chapter 7 when we explore some specific quantum groups. The next chapter, however, provides another key connection to Hopf algebras and, by extension, quantum groups. It will also be instrumental in understanding the example presented in Chapter 7.

Chapter 5

Lie Algebras

5.1 Background and Importance to the Theory of Quantum Groups

The term "Lie Algebra" (pronounced "Lee") was coined by the German mathematician Hermann Weyl in the 1930's in honor of Sophus Lie. Lie was a Norwegian mathematician famous for pretty well developing the theory of *continuous symmetry.* Lie algebras in particular arose as a means of studying *Lie groups* and infinitesimal transformations. One can describe a Lie group, rather crudely, as a topological group which is also a differential manifold, where the group operations respect the manifold's smooth structure. Because of this, one can employ methods of analysis, such as differential calculus, to study the properties of such objects. In particular, Lie's insight was to study the local properties of these structures rather than look at them globally. Due to the smoothness of the manifold, one can think of the local structure as being linear. Lie called these linearized spaces "infinitesimal groups". Today, they are known as the Lie algebra of the group. In other words, because Lie groups are differential manifolds, there is an associated tangent space at each point of the manifold. Lie algebras can be thought of as *tangent spaces* of Lie groups at the identity element of the Lie group, which arises from the so called Lie bracket.

In this chapter a broad overview of Lie theory will be given with the aim of highlighting the features important to the study of quantum groups. While some of the material will concern Lie algebras in general, the main focus will be on finite-dimensional Lie algebras over $\mathbb{C}$. This is to prepare the way for a detailed study of the important finite-dimensional Lie algebra $\mathfrak{sl}(2)$ with ground field $\mathbb{C}$.

5.2 The Basics

5.2.1 Introducing Lie Algebras

Definition 5.1 (Lie Algebra)**.** A Lie algebra L is a vector space over a field κ with a bilinear map $[,] : L \times L \to L$, called the Lie bracket, satisfying the following two conditions for all $x, y, z \in L$:

(i) (Alternating):

$$[x, x] = 0$$

(ii) (Jacobi Identity):

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

The Lie bracket is also antisymmetric since

$$\begin{aligned} &[x+y, x+y] = 0 \quad \text{[alternating]} \\ &\implies [x, x] + [x, y] + [y, x] + [y, y] = 0 \quad \text{[bilinearity]} \\ &\implies [x, y] + [y, x] = 0 \end{aligned}$$

which is the definition of antisymmetry. The converse holds provided that κ does not have characteristic 2.

Notice that the alternating property and bilinearity also imply that $[x, 0] = [0, x] = 0$ for all $x \in L$. Indeed

$$\begin{aligned} 0 &= [x, x] \\ &= [x + 0, x] \\ &= [x, x] + [0, x] = [0, x] \end{aligned}$$

By antisymmetry, it immediately follows that $[x, 0] = 0$ as well.

We define the *center* of a Lie algebra analogously to the center of an algebra. That is,

$$Z(L) := \{x \in L : [x, y] = 0, \text{ for all } y \in L\}$$

The reason for using of the term “center” will become apparent when we introduce a particular version of the Lie bracket called the commutator. We also say that a Lie algebra L is *abelian* if $L = Z(L)$. Again, we'll see below that this notion of abelian corresponds to the usual usage of the term under the commutator. Note, then, that any vector space V can be viewed as a Lie algebra by taking the Lie bracket to be $[u, v] = 0$ for all $u, v \in V$, which is the abelian Lie structure on V. Now, it is clearly the case that $Z(L) = L$ for any one-dimensional L, since, if $\{x\}$ is a basis of L, then

$$[\lambda x, \gamma x] = \lambda\gamma[x, x] = 0$$

Thus, if $Z(L) \neq L$, then it must be that $\dim(L) > 1$ and so L must contain at least two linearly independent elements.

As hinted at above, despite the name "Lie *algebra*", one should be careful to note that this is not an algebra according to the usual definition (see Chapter 3). In particular, the Lie bracket of a Lie algebra need not be associative. That is,

$$[x, [y, z]] = [[x, y], z]$$

need not hold. This naturally raises the question as to when the Lie bracket *is* associative. Suppose that L is an associative Lie algebra. Then

$$[x, [y, z]] + [z, [x, y]] = 0 \quad \text{all } x, y, z \in L$$

Along with the Jacobi identity this implies that $[y, [z, x]] = 0$ or, equivalently, that $[[z, x], y] = 0$. Since this will hold for all $x, y, z \in L$ we see that $[a, b] \in Z(L)$ for all $a, b \in L$. Conversely, if $[a, b] \in Z(L)$ for all $a, b \in L$, then

$$[x, [y, z]] = [[x, y], z]$$

since both are zero. So the Lie bracket is associative precisely when the *derived algebra* (see below) $[L, L] \subseteq Z(L)$.

Recall that being associative means that the order in which an operation is applied is irrelevant. The Jacobi identity can be interpreted as "measuring" how much the order of evaluation matters for the operation in question.

Definition 5.2 (Ideal)**.** Let L be a Lie algebra. A subspace $I \subseteq L$ is called an ideal if

$$[x, y] \in I \quad \text{for any } x \in L \text{ and } y \in I$$

We shall soon see that these play the role for Lie algebra theory what normal subgroups and two-sided ideals play for group theory and ring theory respectively.

Definition 5.3 (Lie Subalgebra)**.** Let L be a Lie algebra. A vector subspace $K \subseteq L$ is called a Lie subalgebra if

$$[x, y] \in K \quad \text{for all } x, y \in K$$

It is easy to see from these definitions that every ideal is also a Lie subalgebra. Of course, the converse is not true in general. We shall see quite a few examples of Lie subalgebras below, especially in Section 5.2.3.

5.2.2 Adjoints and the Commutator

Let $\mathfrak{L}$ be a Lie algebra and take $\mathrm{End}(\mathfrak{L})$ to be the set of linear transformations of $\mathfrak{L}$. Define a map

$$\mathrm{ad} : \mathfrak{L} \to \mathrm{End}(\mathfrak{L}), \quad x \mapsto \mathrm{ad}_x : \mathfrak{L} \to \mathfrak{L}$$

such that $\mathrm{ad}_x(y) := [x, y]$ for all $y \in \mathfrak{L}$. The map ad is called the *adjoint representation* of $\mathfrak{L}$, while each ad_x is called an *adjoint endomorphism* or *adjoint action.* We can express the Jacobi identity using this map by

$$\mathrm{ad}_{[x,y]}(z) = (\mathrm{ad}_x \circ \mathrm{ad}_y - \mathrm{ad}_y \circ \mathrm{ad}_x)(z)$$

The expression on the right is the *commutator* of the elements $\mathrm{ad}_x, \mathrm{ad}_y \in \mathrm{End}(\mathfrak{L})$.

Suppose we begin with an arbitrary associative algebra A. Define a product map on A by

$$[a, b] := ab - ba \qquad \text{for any } a, b \in A$$

This is commonly referred to as the *commutator* of a and b. One can think of it as "measuring" how "close" an algebra is to being commutative.

Proposition 5.4. *Given an associative algebra A and $a, b, c \in A$, then*

$$[a, bc] = [a, b]c + b[a, c]$$

Proof. This only requires a bit of algebraic manipulation.

$$\begin{aligned} [a, bc] &= abc - bca \\ &= abc - bac + bac - bca \\ &= (ab - ba)c + b(ac - ca) \\ &= [a, b]c + b[a, c] \end{aligned}$$

□

Now, A will be commutative (i.e. abelian) if and only if $[a, b] = 0$ for all $a, b \in A$. Another way of saying this is that A is abelian if and only if $[A, A] = 0$, where $[A, A]$ denotes the space generated by all elements of the form $[a, b]$ with $a, b \in A$. If $\mathfrak{g}$ is a

Lie algebra, then $[\mathfrak{g}, \mathfrak{g}]$ is known as the *derived Lie algebra* of $\mathfrak{g}$ and is analogous to the commutator subgroup of a group.

We now turn to a simple, but important result in Lie theory which says that *any* associative algebra can be endowed with a Lie algebra structure.

Proposition 5.5. *If A is an associative algebra, then the commutator is a Lie bracket and hence $(A, [,]) =: L(A)$ is a Lie algebra.*

Proof. It is obvious that $[,]$ is alternating, since for any $a \in A$ we have that

$$[a, a] = aa - aa = 0$$

To verify that the Jacobi identity is satisfied, let $a, b, c \in A$. Then

$$\begin{aligned}
&[a,[b,c]] + [b,[c,a]] + [c,[a,b]] = a[b,c] - [b,c]a + b[c,a] - [c,a]b + c[a,b] - [a,b]c \\
&= a(bc - cb) - (bc - cb)a + b(ca - ac) - (ca - ac)b + c(ab - ba) - (ab - ba)c \\
&= a(bc) - a(cb) - (bc)a + (cb)a + b(ca) - b(ac) - (ca)b + (ac)b + c(ab) - c(ba) - (ab)c + (ba)c \\
&= a(bc) - (ab)c - a(cb) + (ac)b - (bc)a + b(ca) + (cb)a - c(ba) + (ba)c - b(ac) + c(ab) - (ca)b \\
&= 0 \qquad \text{[by associativity of } A\text{]}
\end{aligned}$$

Finally, let $a, b, c \in A$ and $\lambda \in \kappa$. We show that $[,]$ is bilinear:

$$\begin{aligned}
[\lambda a + b, c] &= (\lambda a + b)c - c(\lambda a + b) \\
&= \lambda ac + bc - \lambda ca - cb \\
&= \lambda(ac - ca) + bc - cb \\
&= \lambda[a, c] + [b, c]
\end{aligned}$$

A symmetric argument shows linearity in the second coordinate. □

Though nice to see, the bilinearity part is superfluous, since it follows directly from the bilinearity of multiplication in A. In fact, the entire proof is nearly trivial, but the result is rather interesting, since it means that *any* associative algebra can be turned into a Lie algebra. Even more remarkable, however, is the fact we can go the other way. That is, given any Lie algebra, there is an associated associative algebra with the important property that it preserves the representation theory. So, given a Lie algebra $\mathfrak{L}$ we want the "most general" κ-algebra A such that $L(A)$ contains $\mathfrak{L}$. But before we elaborate on this any further let us consider some examples as well as supply some required definitions.

Example 5.1. *Let H be a Hopf algebra. Then, by Proposition 5.5, H becomes a Lie algebra $L(H)$. For any Hopf algebra, a* primitive element *is an element $h \in H$ such that*

$$\Delta(h) = h \otimes 1 + 1 \otimes h$$

Denote the set of all such elements for H by $Prim(H)$. If $a, b \in Prim(H)$, then $a + b \in Prim(H)$ since

$$\begin{aligned} \Delta(a+b) &= \Delta(a) + \Delta(b) \quad [\Delta \textit{ is linear}] \\ &= a \otimes 1 + 1 \otimes a + b \otimes 1 + 1 \otimes b \\ &= a \otimes 1 + b \otimes 1 + 1 \otimes a + 1 \otimes b \\ &= (a+b) \otimes 1 + 1 \otimes (a+b) \end{aligned}$$

Note, too, that if $a \in Prim(H)$ and $\lambda \in \kappa$, then $\lambda a \in Prim(H)$ since

$$\begin{aligned} \Delta(\lambda a) &= \lambda \Delta(a) \\ &= \lambda(a \otimes 1 + 1 \otimes a) \\ &= \lambda(a \otimes 1) + \lambda(1 \otimes a) \\ &= \lambda a \otimes 1 + 1 \otimes \lambda a \end{aligned}$$

This establishes that $Prim(H)$ is a subspace of H. Finally, if $a, b \in Prim(H)$, then consider that

$$\begin{aligned} \Delta([a,b]) &= \Delta(ab - ba) \\ &= \Delta(a)\Delta(b) - \Delta(b)\Delta(a) \quad [\Delta \textit{ is an algebra morphism}] \\ &= (a \otimes 1 + 1 \otimes a)(b \otimes 1 + 1 \otimes b) - (b \otimes 1 + 1 \otimes b)(a \otimes 1 + 1 \otimes a) \\ &= ab \otimes 1 - ba \otimes 1 + 1 \otimes ab - 1 \otimes ba \quad [\textit{distribute and simplify}] \\ &= (ab - ba) \otimes 1 + 1 \otimes (ab - ba) \\ &= [a,b] \otimes 1 + 1 \otimes [a,b] \end{aligned}$$

which shows that $[a, b] \in Prim(H)$ and hence, $Prim(H)$ is a Lie subalgebra of H.

We should also mention that for $a \in Prim(H)$ it is necessarily the case that $\varepsilon(a) = 0$. This follows from the interaction between being primitive and the counit axiom. The counit axiom entails the commutativity of the diagram

$$\begin{tikzcd}
H \otimes H \arrow[r, "id \otimes \varepsilon"] & H \otimes \kappa \\
H \arrow[u, "\Delta"] \arrow[ur, "\sim"] &
\end{tikzcd}$$

This tells us that

$$\begin{aligned}
a \otimes 1 &= (id \otimes \varepsilon)(\Delta(a)) \\
&= (id \otimes \varepsilon)(a \otimes 1 + 1 \otimes a) \\
&= a \otimes \varepsilon(1) + 1 \otimes \varepsilon(a) \\
&= a \otimes 1 + 1 \otimes \varepsilon(a)
\end{aligned}$$

which implies that $1 \otimes \varepsilon(a) = 0$ *and hence that* $\varepsilon(a) = 0$. *We conclude, then, that*

$$Prim(H) \subseteq Ker(\varepsilon)$$

As always, we are interested in those maps which preserve or respect the pertinent structure between two objects, in this case the Lie structure.

Definition 5.6 (Morphism of Lie Algebras)**.** Let $\mathfrak{L}$ and $\mathfrak{L}'$ be two Lie algebras over a field κ. A linear map $f : \mathfrak{L} \to \mathfrak{L}'$ is a morphism of Lie algebras if

$$f([x, y]) = [f(x), f(y)]$$

for all $x, y \in \mathfrak{L}$.

With Lie morphisms defined, we have our category of Lie algebras denoted by *Lie* (see Figure 1.1). In light of Proposition 5.5, L is a functor from *Alg* to *Lie* where algebra morphisms double as Lie morphisms given the definition of the commutator bracket.

We have already seen an example of a Lie morphism, namely the adjoint representation map. That is, when $\mathrm{End}(\mathfrak{L})$ becomes a Lie algebra with the commutator map, then we showed above that

$$\mathrm{ad}([x, y]) = \mathrm{ad}_{[x,y]} = \mathrm{ad}_x \circ \mathrm{ad}_y - \mathrm{ad}_y \circ \mathrm{ad}_x = [\mathrm{ad}_x, \mathrm{ad}_y]$$

Next, consider the associative algebras $M_n(\kappa)$ and κ thought of as Lie algebras. Besides the determinant, one of the more well known features of square matrices is the trace. We can think of the trace as a map that assigns to each matrix the value of its trace in the ground field. That is,

$$\mathrm{tr} : M_n(\kappa) \to \kappa, \qquad \begin{bmatrix} \lambda_{11} & \lambda_{12} & \dots & \lambda_{1n} \\ \lambda_{21} & \lambda_{22} & \dots & \lambda_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{n1} & \lambda_{n2} & \dots & \lambda_{nn} \end{bmatrix} \longmapsto \sum_{i=1}^{n} \lambda_{ii}$$

This is a morphism of Lie algebras since, if $N, M \in M_n(\kappa)$, then by properties of the trace

$$\mathrm{tr}([M, N]) = \mathrm{tr}(MN - NM) = \mathrm{tr}(MN) - \mathrm{tr}(NM) = 0$$

while $[\mathrm{tr}(M), \mathrm{tr}(N)] = 0$, since κ is commutative.

Definition 5.7 (Derivation)**.** Let $\mathfrak{L}$ be a Lie algebra. A derivation on $\mathfrak{L}$ is an endomorphism of $\mathfrak{L}$, $d : \mathfrak{L} \to \mathfrak{L}$ such that

$$d([x, y]) = [d(x), y] + [x, d(y)], \quad \text{for all } x, y \in \mathfrak{L}$$

The condition in this definition is a particular type of Leibniz law, which the calculus student will recognize is just a generalized version of the so called product rule for derivatives. Once more, the adjoint action provides a nice example of a derivation. Let ad_x be an adjoint action on $\mathfrak{L}$ and $a, b \in \mathfrak{L}$. Then by manipulating the Jacobi identity we get

$$\mathrm{ad}_x([a, b]) = [x, [a, b]] = [[x, a], b] + [a, [x, b]] = [\mathrm{ad}_x(a), b] + [a, \mathrm{ad}_x(b)]$$

The last part of this section introduces several more important analogues to algebras.

5.2.3 The General Linear Group and The General Linear Algebra

Let V be a finite-dimensional κ-vector space. Then $\mathrm{End}(V)$ is an associative algebra under composition. By applying Proposition 5.5, however, we get that $\mathrm{End}(V)$ is a Lie algebra, namely $L(\mathrm{End}(V))$, which is generally denoted by $\mathfrak{gl}(V)$. This is referred to as the *general linear algebra* and is closely associated with an important example, which we shall frequent often: the *general linear group* $GL(V)$. This is the group of all invertible endomorphisms of V. For $\kappa = \mathbb{C}$ or $\mathbb{R}$, this provides the connection between a Lie group and Lie algebra. In this case, $\mathfrak{gl}(V)$ is the tangent space at zero for $GL(V)$.

There is a very convenient relationship between the general linear algebra and matrices. Let $M_n(\kappa)$ be the space of all $n \times n$ matrices with entries in κ. Then this forms an associative algebra and thus, by Proposition 5.5, $M_n(\kappa)$ becomes a Lie algebra. From linear algebra we then get the following nice result.

Proposition 5.8. *If deg*$(V) = n < \infty$*, then End*$(V) \cong M_n(\kappa)$ *as algebras.*

and the immediate consequence:

Corollary 5.9. *If End*(V) *and* $M_n(\kappa)$ *are as in the previous proposition, then*

$$L(\mathit{End}(V)) \cong L(M_n(\kappa))$$

When we want to think in terms of matrices we shall write $\mathfrak{gl}_n(\kappa)$ in place of $\mathfrak{gl}(V)$. The standard basis for $\mathfrak{gl}_n(\kappa)$ consists of all matrices E_{ij} having 1 in the (i,j) position and 0 elsewhere. Since

$$E_{ij}E_{k\ell} = \delta_{jk}E_{i\ell}$$

The commutator, relative to this basis, is given by

$$[E_{ij}, E_{k\ell}] = \delta_{jk}E_{i\ell} - \delta_{\ell i}E_{kj}$$

The general linear algebra is important for many reasons. One reason, of current interest, is that many examples of Lie algebras are born out of $\mathfrak{gl}_n(\kappa)$ as subalgebras.

Example 5.2 ($\mathfrak{sl}_n(\kappa)$)**.** *One of the most important examples is the* special linear algebra $\mathfrak{sl}_n(\kappa)$*, which consists of all* $n \times n$ *matrices in* $\mathfrak{gl}_n(\kappa)$ *having trace zero. In fact, as we will later see, this example with* $n = 2$ *will command almost our exclusive attention. To see that it is a Lie subalgebra, let* $X, Y \in \mathfrak{sl}_n(\kappa)$*. We require that* $tr([X,Y]) = 0$*, which we know to be true by properties of the trace map. Thus,* $\mathfrak{sl}_n(\kappa)$ *is a Lie subalgebra of* $\mathfrak{gl}_n(\kappa)$ *and a Lie algebra in its own right with respect to the inherited commutator product.*

Example 5.3 (Skew-symmetric Matrices)**.** *Let* $\mathfrak{ss}_n(\kappa)$ *denote the space of all* $n \times n$ *skew-symmetric matrices. This means that for any* $M \in \mathfrak{ss}_n(\kappa)$*,* $M + M^t = 0$*. Let* $A, B \in \mathfrak{ss}_n(\kappa)$*. Then*

$$\begin{aligned}
[A,B] &= AB - BA \\
&= (-A^t)(-B^t) - (-B^t)(-A^t) \\
&= A^tB^t - B^tA^t \\
&= -(B^tA^t - A^tB^t) \\
&= -((AB)^t - (BA)^t) \\
&= -(AB - BA)^t \\
&= -[A,B]^t
\end{aligned}$$

implying that $[A, B] \in \mathfrak{ss}_n(\kappa)$. *Thus,* $\mathfrak{ss}_n(\kappa)$ *is a Lie subalgebra of* $\mathfrak{gl}_n(\kappa)$.

Notice the implication of the skew-symmetric property. Since $(a_{ij}) + (a_{ij})^t = 0$, *we see that* $a_{ij} = -a_{ji}$. *In particular, this means that* $a_{kk} = -a_{kk}$ *implying that the diagonal entries are all zero. It follows that the trace of a skew-symmetric matrix is zero and hence* $\mathfrak{ss}_n(\kappa)$ *is also a Lie subalgebra of* $\mathfrak{sl}_n(\kappa)$.

Example 5.4 (Symplectic Algebra)**.** *Closely related to the previous example is the symplectic algebra. For reasons unnecessary to the example, this algebra requires even dimensionality. It is denoted by* $\mathfrak{sp}_{2n}(\kappa)$. *The matrix elements of the symplectic algebra have the form* $\begin{bmatrix} m & n \\ p & q \end{bmatrix}$, *where* $m, n, p, q \in \mathfrak{gl}_n(\kappa)$ *are such that* $n^t = n$, $p^t = p$ *and* $m^t = -q$. *Let* $\begin{bmatrix} m & n \\ p & q \end{bmatrix}, \begin{bmatrix} m' & n' \\ p' & q' \end{bmatrix} \in \mathfrak{sp}_{2n}(\kappa)$. *Then*

$$[\begin{bmatrix} m & n \\ p & q \end{bmatrix}, \begin{bmatrix} m' & n' \\ p' & q' \end{bmatrix}] = \begin{bmatrix} mm' + np' - m'm - n'p & mn' + nq' - m'n - n'q \\ pm' + qp' - p'm - q'p & pn' + qq' - p'n - q'q \end{bmatrix}$$

Now let

$$\begin{aligned} M &= mm' + np' - m'm - n'p \\ N &= mn' + nq' - m'n - n'q \\ P &= pm' + qp' - p'm - q'p \\ Q &= pn' + qq' - p'n - q'q \end{aligned}$$

we then have that

$$\begin{aligned} N^t &= (mn' + nq' - m'n - n'q)^t \\ &= n'^t m^t + q'^t n^t - n^t m'^t - q^t n'^t \\ &= -n'q - m'n + nq' + mn' \\ &= mn' + nq' - m'n - n'q \\ &= N \end{aligned}$$

$$\begin{aligned} P^t &= (pm' + qp' - p'm - q'p)^t \\ &= m'^t p^t + p'^t q^t - m^t p'^t - p^t q'^t \\ &= -q'p - p'm + qp' + pm' \\ &= pm' + qp' - p'm - q'p \\ &= P \end{aligned}$$

$$\begin{aligned}
M^t &= (mm' + np' - m'm - n'p)^t \\
&= m'^t m^t + p'^t n^t - m^t m'^t - p^t n'^t \\
&= q'q + q'n - qq' - pn' \\
&= -pn' - qq' + p'n + q'q \\
&= -(pn' + qq' - p'n - q'q) \\
&= -Q
\end{aligned}$$

This shows that $[\begin{bmatrix} m & n \\ p & q \end{bmatrix}, \begin{bmatrix} m' & n' \\ p' & q' \end{bmatrix}] \in \mathfrak{sp}_{2n}(\kappa)$. *Thus,* $\mathfrak{sp}_{2n}(\kappa)$ *is a Lie subalgebra of* $\mathfrak{gl}_{2n}(\kappa)$. *Furthermore, the condition that* $m^t = -q$ *implies that the elements of the symplectic algebra are trace zero and hence, the symplectic algebra is also a Lie subalgebra of* $\mathfrak{sl}_{2n}(\kappa)$.

Example 5.5 ($\mathfrak{u}_n(\mathbb{C})$)**.** *Consider the space of all skew-Hermitian matrices denoted* $\mathfrak{u}_n(\mathbb{C})$. *These have the property that for any* $M \in \mathfrak{u}_n(\mathbb{C})$, $M^\dagger + M = 0$, *where* $M^\dagger$ *represents the conjugate transpose of* M. *Note that the conjugate transpose satisfies*

$$(A + B)^\dagger = A^\dagger + B^\dagger, \quad (AB)^\dagger = B^\dagger A^\dagger$$

and, hence, the same argument as in Example 5.3 applies. Thus, $\mathfrak{u}_n(\mathbb{C})$ *is a Lie subalgebra of* $\mathfrak{gl}_n(\mathbb{C})$.

Example 5.6 ($\mathfrak{t}_n(\kappa)$, $\mathfrak{n}_n(\kappa)$ and $\mathfrak{d}_n(\kappa)$)**.** *The following three examples are closely related. Let* $\mathfrak{t}_n(\kappa)$ *represent the space of all* $n \times n$ *upper-triangular matrices* ($a_{ij} = 0$ *if* $i > j$), $\mathfrak{n}_n(\kappa)$ *the space of all* $n \times n$ *strictly upper-triangular matrices* ($a_{ij} = 0$ *if* $i \geq j$), *and* $\mathfrak{d}_n(\kappa)$ *the space of all* $n \times n$ *diagonal matrices. Since the product and sum of two triangular matrices is again triangular, it follows that each of these is a Lie subalgebra of* $\mathfrak{gl}_n(\kappa)$. *Of the three, however, only* $\mathfrak{n}_n(\kappa)$ *is additionally a Lie subalgebra of* $\mathfrak{sl}_n(\kappa)$. *Notice, however, that both* $\mathfrak{n}_n(\kappa)$ *and* $\mathfrak{d}_n(\kappa)$ *are Lie subalgebras of* $\mathfrak{t}_n(\kappa)$.

Together, these examples reveal the remarkable fact that many matrix properties are preserved by the commutator bracket.

5.2.4 New Lie Algebras

The Opposite Lie Algebra

Probably the easiest way to obtain a new Lie algebra out of an old one is by creating the *opposite* Lie algebra. Given a Lie algebra $\mathfrak{L}$, the opposite Lie algebra, $\mathfrak{L}^{op}$, is defined to be the vector space $\mathfrak{L}$ with Lie bracket $[x,y]^{op}$ defined by

$$[x,y]^{op} := [y,x] = -[x,y]$$

It should be noted that one only obtains a "new" Lie algebra in a superficial way by taking the opposite Lie algebra since $\mathfrak{L}^{op} \cong \mathfrak{L}$ under the Lie algebra isomorphism $\mathrm{op} : \mathfrak{L} \to \mathfrak{L}^{op}$ defined by

$$\mathrm{op}(x) := -x$$

Factor Lie Algebras

Since we have defined the concept of an ideal for a Lie algebra, it is natural to query as to whether new Lie algebras can be obtained via construction of a quotient space. Toward this end, let $\mathfrak{L}$ be a Lie algebra and I an ideal of $\mathfrak{L}$. Then we can certainly create the quotient vector space

$$\mathfrak{L}/I := \{x + I : x \in \mathfrak{L}\}$$

We can turn this into a Lie algebra by defining

$$[v+I, w+I] := [v,w] + I, \quad \text{for all } v, w \in \mathfrak{L}$$

The crucial question, which arises in all cases such as these, is: Is this well-defined?

Proposition 5.10. *The correspondence* $[,] : \mathfrak{L}/I \times \mathfrak{L}/I \to \mathfrak{L}/I$ *given above is well-defined and* $\mathfrak{L}/I$ *is a Lie algebra.*

Proof. Suppose that $v + I = v' + I$ and $w + I = w' + I$. Then $v - v' \in I$ and $w - w' \in I$ and because we are factoring out by I, these elements act like 0. We therefore have

$$\begin{aligned}
[v,w] &= [v' + (v - v'), w' + (w - w')] \\
&= [v', w' + (w - w')] + [v - v', w + (w - w')] \\
&= [v', w'] + [v', w - w'] + [v - v', w] + [v - v', w - w'] \\
&= [v', w'] \quad [\text{as elements of } \mathfrak{L}/I]
\end{aligned}$$

since the last three terms will be in I. This establishes that the correspondence is well-defined.

Next, notice that the bracket for the quotient space is defined in terms of the Lie bracket for $\mathfrak{L}$. It follows, then, that our new bracket will be alternating and satisfy the Jacobi identity. Thus, it is a Lie bracket for $\mathfrak{L}/I$ thereby bestowing upon $\mathfrak{L}/I$ a Lie algebra structure. □

Example 5.7. *Let $\mathfrak{L}$ be a Lie algebra. Then clearly $[\mathfrak{L}, \mathfrak{L}]$ is an ideal of $\mathfrak{L}$. Let $\mathfrak{L}^{ab}$ denote the quotient Lie algebra $\mathfrak{L}/[\mathfrak{L}, \mathfrak{L}]$. It is clear that any Lie algebra created in this way is abelian.*

Just as one would suspect, we refer to $\mathfrak{L}/I$ as a factor or quotient Lie algebra. It may also come as no surprise to the reader that there are isomorphism theorems for factor Lie algebras. For the sake of brevity we shall merely provide the statement of the theorems (see [11]).

Theorem 5.11 (Isomorphism Theorems)**.** *We state all three theorems together:*

(i) Let $\phi : \mathfrak{L} \to \mathfrak{L}'$ be a morphism of Lie algebras. Then $Ker(\phi)$ is an ideal of $\mathfrak{L}$, $Im(\phi)$ is a Lie subalgebra of $\mathfrak{L}'$ and

$$\mathfrak{L}/Ker(\phi) \cong Im(\phi)$$

(ii) If I and J are ideals of a Lie algebra, then

$$(I + J)/J \cong I/(I \cap J)$$

(iii) If I and J are ideals of a Lie algebra $\mathfrak{L}$ such that $I \subset J$, then J/I is an ideal of $\mathfrak{L}/I$ and

$$(\mathfrak{L}/I)/(J/I) \cong \mathfrak{L}/J$$

Direct Sum of Lie Algebras

Another means of creating new Lie algebras out of old ones is to take their direct sum, so that, if $\mathfrak{L}$ and $\mathfrak{L}'$ are Lie algebras, then so is $\mathfrak{L} \oplus \mathfrak{L}'$. Clearly $\mathfrak{L} \oplus \mathfrak{L}'$ is a vector space. Define a Lie bracket on $\mathfrak{L} \oplus \mathfrak{L}'$ by

$$[(x, x'), (y, y')] := ([x, y], [x', y'])$$

for all $x, y \in \mathfrak{L}$ and $x', y' \in \mathfrak{L}'$. It is alternating since

$$\begin{aligned}[(x, x'), (x, x')] &= ([x, x], [x', x']) \\ &= (0, 0)\end{aligned}$$

It satisfies the Jacobi identity since

$$\begin{aligned}
&[(x,x'),[(y,y'),(z,z')]] + [(y,y'),[(z,z'),(x,x')]] + [(z,z')[(x,x'),(y,y')]] \\
&= [(x,x'),([y,z],[y',z'])] + [(y,y'),([z,x],[z',x'])] + [(z,z'),([x,y],[x',y'])] \\
&= ([x,[y,z]],[x',[y',z']]) + ([y,[z,x]],[y',[z',x']]) + ([z,[x,y]],[z',[x',y']]) \\
&= (0,0)
\end{aligned}$$

That this bracket it bilinear is clear. Thus $\mathfrak{L} \oplus \mathfrak{L}'$ with $[,]$ as defined above is a Lie algebra.

Example 5.8. *Consider again the space $\mathfrak{t}_n(\kappa)$ having upper-triangular matrix elements. In Example 5.6 we considered how it becomes a Lie algebra under the inherited operation of $\mathfrak{gl}_n(\kappa)$. However, this space admits another Lie algebra structure as a direct sum. If we call upon the other two participants of Example 5.6, then clearly*

$$\mathfrak{t}_n(\kappa) = \mathfrak{d}_n(\kappa) + \mathfrak{n}_n(\kappa) \quad \textit{as vector spaces}$$

By their very definitions, it is equally clear that

$$\mathfrak{d}_n(\kappa) \cap \mathfrak{n}_n(\kappa) = \{0\}$$

Thus

$$\mathfrak{t}_n(\kappa) = \mathfrak{d}_n(\kappa) \oplus \mathfrak{n}_n(\kappa)$$

The Lie bracket in this case would operate as follows: let $T, T' \in \mathfrak{t}_n(\kappa)$. Then each can be decomposed to $T = D + N$ and $T' = D' + N'$ where $D, D' \in \mathfrak{d}_n(\kappa)$ and $N, N' \in \mathfrak{n}_n(\kappa)$. Computing the Lie bracket yields

$$\begin{aligned}
[T,T'] &= [D+N, D'+N'] \\
&= [D,D'] + [N,N']
\end{aligned}$$

where $[D,D']$ and $[N,N']$ are the usual Lie brackets of $\mathfrak{d}_n(\kappa)$ and $\mathfrak{n}_n(\kappa)$ respectively.

Definition 5.12 (Simple Lie Algebra)**.** We say that a Lie algebra $\mathfrak{L}$ is non-trivially *simple* if

1. $\mathfrak{L}$ has no ideals except 0 and itself.

2. $[\mathfrak{L}, \mathfrak{L}] \neq 0$.

Condition (2) is what prevents a Lie algebra from being trivially simple. These are the one-dimensional abelian Lie algebras. A Lie algebra is said to be *semisimple* if it is a direct sum of simple Lie algebras.

Having acquired some basic theory, let us proceed to explore how one can obtain an associative algebra from any Lie algebra. This is a very important part of our study because it will tie into Hopf algebras and, more importantly, is a "primary" source of quantum groups.

5.3 Enveloping Algebras

As mentioned above, we can assign to any Lie algebra a corresponding associative algebra. These associative algebras are called *Enveloping algebras.* The idea is to represent a Lie algebra $\mathfrak{L}$ with an associative algebra that captures the important properties of $\mathfrak{L}$. The usefulness of this comes from the fact that it is usually nicer to work in an associative algebra. The term "enveloping" means "to wrap up in" or "surround entirely". Hence, enveloping algebras can be thought of as "wrapping up" or "enclosing" the essential features of a Lie algebra.

The formal definition uses the tensor algebra considered last chapter.

Definition 5.13 (Enveloping Algebra)**.** Let $\mathfrak{L}$ be a Lie algebra. The enveloping algebra, $U(\mathfrak{L})$, for $\mathfrak{L}$ is defined to be the quotient space

$$U(\mathfrak{L}) := \frac{T(\mathfrak{L})}{I(\mathfrak{L})}$$

where $T(\mathfrak{L})$ is the tensor algebra of $\mathfrak{L}$ and $I(\mathfrak{L})$ is the two sided ideal of $T(\mathfrak{L})$ generated by all elements of the form

$$x \otimes y - y \otimes x - [x, y], \qquad x, y \in \mathfrak{L}$$

This allows one to import the commutation relations of $\mathfrak{L}$ into $U(\mathfrak{L})$. Along with assigning $U(\mathfrak{L})$ to $\mathfrak{L}$ we also assign a morphism of Lie algebras

$$i_{\mathfrak{L}} : \mathfrak{L} \to L(U(\mathfrak{L}))$$

which we define to be the composition of the canonical injection of $\mathfrak{L}$ into $T(\mathfrak{L})$ and the canonical surjection of $T(\mathfrak{L})$ onto $U(\mathfrak{L})$. It is immediate, then, that

$$i_{\mathfrak{L}}([x, y]) = x \otimes y - y \otimes x$$

where we are "ignoring" the coset, and that $i_{\mathfrak{L}}$ is a Lie morphism, since the Lie bracket in $L(U(\mathfrak{L}))$ is the commutator. That is,

$$i_{\mathfrak{L}}([x,y]) = x \otimes y - y \otimes x = [i_{\mathfrak{L}}(x), i_{\mathfrak{L}}(y)]$$

An important part of Lie theory is studying the product properties within a particular representation of a Lie algebra. Enveloping algebras are a key tool in representation theory. To understand better what this means we take a quick detour into some representation theory.

5.3.1 Representations of Lie Algebras

The reader may recall that we introduced a map $\mathrm{ad} : \mathfrak{L} \to \mathrm{End}(\mathfrak{L})$ above called the adjoint *representation*. This section will elucidate the reason for this name.

Definition 5.14 (Lie Algebra Representation)**.** Let $\mathfrak{g}$ be a Lie algebra and V a κ-vector space. A representation of $\mathfrak{g}$ is a morphism of Lie algebras

$$\rho : \mathfrak{g} \to \mathfrak{gl}(V), \quad x \mapsto \rho_x$$

In particular

$$\rho([x,y]) = [\rho_x, \rho_y] = \rho_x \circ \rho_y - \rho_y \circ \rho_x$$

for all $x, y \in \mathfrak{g}$.

An n-dimensional *matrix* representation, then, is just a linear morphism of $\mathfrak{g}$ into $L\big(M_n(\kappa)\big)$ and, as mentioned above, one can obtain isomorphic representations by choosing a basis in V.

Continuing in this vein, if ρ and ρ' are two representations of degree n for $\mathfrak{g}$, then we say that these two representations are equivalent if there exists a non-singular $n \times n$ matrix T over κ such that

$$\rho' = T^{-1}\rho T$$

The reader may have noticed that there is an essential similarity between the above description of representations and the notion of a module. In fact, the language of module theory is perfectly suited to characterizing the representation of a Lie algebra. In this context we refer to a Lie algebra module, or $\mathfrak{g}$-module. This is analogous to the result that a group representation is equivalent to a module over the corresponding group algebra. We define a *left* $\mathfrak{g}$-module as follows.

Definition 5.15 (Left $\mathfrak{g}$-Module)**.** A left $\mathfrak{g}$-module is a κ-vector space V with a bilinear scalar multiplication $\mathfrak{g} \times V \to V$ with $(x, v) \mapsto xv$ such that

$$[x, y]v = x(yv) - y(xv)$$

for all $x, y \in \mathfrak{g}$ and $v \in V$.

The dimension of a $\mathfrak{g}$-module is identical to that of the underlying vector space V.

Definition 5.16 ($\mathfrak{g}$-Module Morphism)**.** Let $\mathfrak{g}$ be a Lie algebra and let V, W be $\mathfrak{g}$-modules. A $\mathfrak{g}$-module morphism from V to W is a linear map $\theta : V \to W$ such that

$$\theta(g \cdot v) = g \cdot \theta(w), \quad \text{for all } v \in V, w \in W \text{ and } g \in \mathfrak{g}$$

We now show that representations and $\mathfrak{g}$-modules are essentially equivalent ways of studying Lie algebras.

Proposition 5.17. *Every $\mathfrak{g}$-module gives a representation of $\mathfrak{g}$ and every representation gives a $\mathfrak{g}$-module.*

Proof. Let V be a finite-dimensional $\mathfrak{g}$-module. Define a map $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ by

$$x \mapsto \rho_x(-)$$

where $\rho_x(v) := xv$. This is a linear map, since if $x, y \in \mathfrak{g}$ and $\lambda \in \kappa$, then

$$\lambda x + y \mapsto \rho_{\lambda x + y}(-)$$

and

$$\begin{aligned} \rho_{\lambda x+y}(v) &= (\lambda x + y)v \\ &= \lambda xv + yv \\ &= \lambda \rho_x(v) + \rho_y(v) \end{aligned}$$

We now show that this map is a representation of $\mathfrak{g}$. By definition

$$[x, y] \mapsto \rho_{[x.y]}(-)$$

Now let $v \in V$. Then

$$\begin{aligned}
\rho_{[x,y]}(v) &= [x,y]v \\
&= x(yv) - y(xv) \qquad [V \text{ a } \mathfrak{g}\text{-module}] \\
&= x\rho_y(v) - y\rho_x(v) \\
&= (\rho_x \circ \rho_y)(v) - (\rho_y \circ \rho_x)(v)
\end{aligned}$$

which satisfies the condition of being a representation.

Conversely, given a representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$, the vector space V becomes a $\mathfrak{g}$-module if we define

$$xv := \rho_x(v)$$

Checking bilinearity we see that

$$\begin{aligned}
(\lambda x + \gamma y)v &= \rho_{\lambda x + \gamma y}(v) \\
&= \rho_{\lambda x}(v) + \rho_{\gamma y}(v) \\
&= \lambda \rho_x(v) + \gamma \rho_y(v) \\
&= \lambda(xv) + \gamma(yw)
\end{aligned}$$

Also,

$$\begin{aligned}
x(\lambda v + w) &= \rho_x(\lambda v + w) \\
&= \rho_x(\lambda v) + \rho_x(w) \\
&= \lambda \rho_x(v) + \rho_x(w) \\
&= \lambda(xv) + xw
\end{aligned}$$

Finally, we check that $[x,y]v = x(yv) - y(xv)$.

$$\begin{aligned}
[x,y]v &= \rho_{[x,y]}(v) \\
&= (\rho_x \circ \rho_y)(v) - (\rho_y \circ \rho_x)(v) \\
&= x(yv) - y(xv)
\end{aligned}$$

□

What this really says is that the category of representations of $\mathfrak{g}$ is equivalent to the category of $\mathfrak{g}$-modules.

Representations allow one to study products and series within $\mathfrak{gl}(V)$, which reveal various properties of the Lie algebra in question. This is because there is, in general, no defined multiplication in a Lie algebra, which is not the case for $\mathfrak{gl}(V)$. Unfortunately, certain properties can be sensitive to the particular representation used. Nevertheless, there are properties which appear to be "universal" in that they hold simultaneously for all representations. This leads us to the notion of a *universal enveloping algebra* which captures exactly these properties, and only these properties.

5.3.2 The *Universal* Enveloping Algebra

For any Lie algebra $\mathfrak{g}$, the enveloping algebra $U(\mathfrak{g})$ enjoys a universal property.

Theorem 5.18. *Let $\mathfrak{g}$ be a Lie algebra. Given any associative algebra A and any morphism of Lie algebras $f : \mathfrak{g} \to L(A)$, there exists a unique morphism of algebras $\phi : U(\mathfrak{g}) \to A$ such that $\phi \circ i_{\mathfrak{g}} = f$.*

This can be summed up as saying there is a natural bijection

$$\hom_{Lie}(\mathfrak{g}, L(A)) \cong \hom_{Alg}(U(\mathfrak{g}), A) \qquad \text{as sets}$$

So, U is the left adjoint of the inclusion functor L, that is, the inclusion of the category of associative algebras into the category of Lie algebras.

Proof. By Proposition 3.18, regarding the tensor algebra, we have that the map f extends to a unique morphism of algebras $f' : T(\mathfrak{g}) \to A$ such that

$$f'(x_1 \dots x_n) = f(x_1) \dots f(x_n), \quad x_1, ..., x_n \in \mathfrak{g}$$

Notice, here, that "$\otimes$" is being suppressed in the input per the convenience mentioned in Chapter 3 regarding the tensor algebra. So far, this only means that the diagram

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "f"] \arrow[d, "i_{\mathfrak{g}}"'] & A \\ T(\mathfrak{g}) \arrow[ur, "f'"'] & \end{tikzcd}$$

commutes. We can extend this to get the commuting triangle

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "f"] \arrow[d, "i_{\mathfrak{g}}"'] & A \\ T(\mathfrak{g})/I(\mathfrak{g}) \arrow[ur, "\overline{f}"'] & \end{tikzcd}$$

by showing that $f'(I(\mathfrak{g})) = \{0\}$. It will suffice to show that $f'(xy - yx - [x, y]) = 0$ for $x, y \in \mathfrak{g}$. Consider that

$$\begin{aligned} f'(xy - yx - [x, y]) &= f(xy) - f(yx) - f([x, y]) \\ &= f(x)f(y) - f(y)f(x) - f([x, y]) \\ &= f(x)f(y) - f(y)f(x) - [f(x), f(y)] \quad [f \text{ is Lie morphism on } \mathfrak{g}] \\ &= f(x)f(y) - f(y)f(x) - (f(x)f(y) - f(y)f(x)) \\ &= 0 \end{aligned}$$

The uniqueness of $\overline{f}$ also comes from Proposition 3.18. We therefore take $\overline{f}$ to be our ϕ and this completes the proof since $L(A)$ is just $(A, [,])$. □

Corollary 5.19. *(a) For any morphism of Lie algebras $f : \mathfrak{L} \to \mathfrak{L}'$, there exists a unique morphism of algebras $U(f) : U(\mathfrak{L}) \to U(\mathfrak{L}')$ such that*

$$U(f) \circ i_{\mathfrak{L}} = i_{\mathfrak{L}'} \circ f$$

and $U(id_{\mathfrak{L}}) = id_{U(\mathfrak{L})}$.

(b) If $f' : \mathfrak{L}' \to \mathfrak{L}''$ is another morphism of Lie algebras, then

$$U(f' \circ f) = U(f') \circ U(f)$$

Proof. (a) Consider the diagram

$$\begin{tikzcd} \mathfrak{L} \arrow[r, "f"] \arrow[d, "i_{\mathfrak{L}}"'] & \mathfrak{L}' \arrow[d, "i_{\mathfrak{L}'}"] \\ U(\mathfrak{L}) \arrow[r, dashed, "\phi"] & U(\mathfrak{L}') \end{tikzcd}$$

Applying Theorem 5.18 to the composition

$$i_{\mathfrak{L}'} \circ f : \mathfrak{L} :\to L(U(\mathfrak{L}'))$$

there exists a unique morphism of algebras $\phi : U(\mathfrak{L}) \to U(\mathfrak{L}')$ making the above diagram commute - i.e. $\phi \circ i_{\mathfrak{L}} = i_{\mathfrak{L}'} \circ f$. Thus, $\phi = U(f)$ as desired.

(b) Since the composition of Lie algebra morphisms is again a morphism of Lie algebras we can use f and f' to get

$$f' \circ f : \mathfrak{L} \to \mathfrak{L}''$$

By part (a) there is a unique morphism of algebras $U(f' \circ f) : U(\mathfrak{L}) \to U(\mathfrak{L}'')$ such that

$$U(f' \circ f) \circ i_{\mathfrak{L}} = i_{\mathfrak{L}''} \circ f' \circ f \tag{5.1}$$

Now consider that

$$\begin{aligned} U(f') \circ U(f) \circ i_{\mathfrak{L}} &= U(f') \circ \big(U(f) \circ i_{\mathfrak{L}}\big) \quad \text{[associativity of composition]} \\ &= U(f') \circ (i_{\mathfrak{L}'} \circ f) \quad \text{[by (a)]} \\ &= (U(f') \circ i_{\mathfrak{L}'}) \circ f \quad \text{[associativity]} \\ &= (i_{\mathfrak{L}''} \circ f') \circ f \quad \text{[by (a)]} \\ &= U(f' \circ f) \circ i_{\mathfrak{L}} \quad \text{[by (5.1)]} \end{aligned}$$

Because of uniqueness, this implies that $U(f' \circ f) = U(f') \circ U(f)$. $\square$

This corollary allows us to think of U as a functor from the category of Lie algebras to the category of algebras (see Figure 1.1) with

$$\mathfrak{L} \mapsto U(\mathfrak{L}) \quad \text{and} \quad f \mapsto U(f)$$

Theorem 5.20. *Let $\mathfrak{L}$ and $\mathfrak{L}'$ be Lie algebras and $\mathfrak{L} \oplus \mathfrak{L}'$ their direct sum. Then*

$$U(\mathfrak{L} \oplus \mathfrak{L}') \cong U(\mathfrak{L}) \otimes U(\mathfrak{L}') \quad \textit{as algebras}$$

Proof. We have already shown above that $\mathfrak{L} \oplus \mathfrak{L}'$ is a Lie algebra so we proceed to construct a linear map $f : \mathfrak{L} \oplus \mathfrak{L}' \to U(\mathfrak{L}) \otimes U(\mathfrak{L}')$ defined by

$$f(x, x') := i_{\mathfrak{L}}(x) \otimes 1 + 1 \otimes i_{\mathfrak{L}'}(x')$$

which we will simply write as

$$f(x, x') = x \otimes 1 + 1 \otimes x'$$

The reason for defining f in this manner is as follows: If

$$U(\mathfrak{L}) \to U(\mathfrak{L}) \otimes U(\mathfrak{L}')$$

is the embedding map of Lie algebras where $x \mapsto x \otimes 1$ and

$$U(\mathfrak{L}') \to U(\mathfrak{L}) \otimes U(\mathfrak{L}')$$

is the embedding map where $y \mapsto 1 \otimes y$, then for $(x, y) \in U(\mathfrak{L}) \oplus U(\mathfrak{L}')$, we get an induced map on the direct sum given by

$$(x, y) = (x, 0) + (0, y) \mapsto x \otimes 1 + 1 \otimes y$$

Now, $\mathfrak{L} \oplus \mathfrak{L}'$ is the direct sum in the category of Lie algebras. So, if $\mathfrak{g}$ is any Lie algebra and $f : \mathfrak{L} \to \mathfrak{g}$ and $f' : \mathfrak{L}' \to \mathfrak{g}$ are Lie maps, then there exists a unique Lie map $f \oplus f' : \mathfrak{L} \oplus \mathfrak{L}' \to \mathfrak{g}$ where

$$(f \oplus f')(x, x') = f(x) + f'(x')$$

In our case $\mathfrak{g} = L(U(\mathfrak{L}) \otimes U(\mathfrak{L}'))$, while f and f' are the maps

$$x \mapsto x \otimes 1, \quad y \mapsto 1 \otimes y$$

respectively. Therefore, f is a morphism from the Lie algebra $\mathfrak{L} \oplus \mathfrak{L}'$ into the Lie algebra

$$L\big(U(\mathfrak{L}) \otimes U(\mathfrak{L}')\big)$$

Therefore, Theorem 5.18 says that there is a unique morphism of algebras

$$\phi : U(\mathfrak{L} \oplus \mathfrak{L}') \to U(\mathfrak{L}) \otimes U(\mathfrak{L}')$$

such that $\phi \circ i_{\mathfrak{L} \oplus \mathfrak{L}'} = f$.

Our next goal is to show that ϕ is an isomorphism. Since our information concerning ϕ itself is limited, we shall employ the universal property of the tensor product of algebras (i.e. Theorem 3.10) to construct an inverse for ϕ. First, though, let $\iota_{\mathfrak{L}} : \mathfrak{L} \to \mathfrak{L} \oplus \mathfrak{L}'$ and $\iota_{\mathfrak{L}'} : \mathfrak{L}' \to \mathfrak{L} \oplus \mathfrak{L}'$ be the canonical injections

$$\iota_{\mathfrak{L}}(x) = (x, 0), \quad \iota_{\mathfrak{L}'}(x') = (0, x')$$

These are clearly morphisms of Lie algebras and hence, their respective compositions with $i_{\mathfrak{L} \oplus \mathfrak{L}'}$ are also morphisms of Lie algebras. By Theorem 5.18 there exists

$$\begin{tikzcd}
\mathfrak{L} \arrow[r, "i_{\mathfrak{L} \oplus \mathfrak{L}'} \circ \iota_{\mathfrak{L}}"] \arrow[d, "i_{\mathfrak{L}}"'] & U(\mathfrak{L} \oplus \mathfrak{L}') \\
U(\mathfrak{L}) \arrow[ur, "\psi_{\mathfrak{L}}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathfrak{L}' \arrow[r, "i_{\mathfrak{L} \oplus \mathfrak{L}'} \circ \iota_{\mathfrak{L}'}"] \arrow[d, "i_{\mathfrak{L}'}"'] & U(\mathfrak{L} \oplus \mathfrak{L}') \\
U(\mathfrak{L}') \arrow[ur, "\psi_{\mathfrak{L}'}"'] &
\end{tikzcd}$$

morphisms of algebras $\psi_{\mathfrak{L}} : U(\mathfrak{L}) \to U(\mathfrak{L} \oplus \mathfrak{L}')$ and $\psi_{\mathfrak{L}'} : U(\mathfrak{L}') \to U(\mathfrak{L} \oplus \mathfrak{L}')$ such that for each $x \in \mathfrak{L}$ and $x' \in \mathfrak{L}'$ we have

$$\psi_{\mathfrak{L}}(x) = i_{\mathfrak{L}\oplus\mathfrak{L}'}(x, 0) \quad \text{and} \quad \psi_{\mathfrak{L}'}(x') = i_{\mathfrak{L}\oplus\mathfrak{L}'}(0, x')$$

At this point, we wish to apply Theorem 3.10, which will give us the existence of a unique morphism of algebras

$$\psi := \psi_{\mathfrak{L}} \otimes \psi_{\mathfrak{L}'} : U(\mathfrak{L}) \otimes U(\mathfrak{L}') \to U(\mathfrak{L} \oplus \mathfrak{L}')$$

such that $\psi(a \otimes a') = \psi_{\mathfrak{L}}(a)\psi_{\mathfrak{L}'}(a')$ for all $a \in U(\mathfrak{L})$ and $a' \in U(\mathfrak{L}')$. First, however, we must ensure that $\psi_{\mathfrak{L}}(a)\psi_{\mathfrak{L}'}(a') = \psi_{\mathfrak{L}'}(a')\psi_{\mathfrak{L}}(a)$ always holds. It is enough to show that the equation holds when $a = x \in \mathfrak{L}$ and $a' = x' \in \mathfrak{L}'$. In particular, this will be true if $[\psi_{\mathfrak{L}}(x), \psi_{\mathfrak{L}'}(x')] = 0$. We check this now.

$$\begin{aligned}
[\psi_{\mathfrak{L}}(x), \psi_{\mathfrak{L}'}(x')] &= [i_{\mathfrak{L}\oplus\mathfrak{L}'}(x, 0), i_{\mathfrak{L}\oplus\mathfrak{L}'}(0, x')] \\
&= i_{\mathfrak{L}\oplus\mathfrak{L}'}([(x, 0), (0, x')]) \quad [i_{\mathfrak{L}\oplus\mathfrak{L}'} \text{ a Lie alg. morphism}] \\
&= i_{\mathfrak{L}\oplus\mathfrak{L}'}([x, 0], [0, x']) \\
&= i_{\mathfrak{L}\oplus\mathfrak{L}'}(0, 0) = 0
\end{aligned}$$

Thus, we may apply the universal property for the tensor product of algebras. The map ψ is our candidate for ϕ^{-1}. Consider that for all $x \in \mathfrak{L}$ and $x' \in \mathfrak{L}'$

$$\begin{aligned}
\psi(\phi(x, x')) &= \psi(f(x, x')) \\
&= \psi(i_{\mathfrak{L}}(x) \otimes 1 + 1 \otimes i_{\mathfrak{L}'}(x')) \\
&= \psi(i_{\mathfrak{L}}(x) \otimes 1) + \psi(1 \otimes i_{\mathfrak{L}'}(x')) \\
&= \psi(x \otimes 1) + \psi(1 \otimes x') \\
&= \psi_{\mathfrak{L}}(x)\psi_{\mathfrak{L}'}(1) + \psi_{\mathfrak{L}}(1)\psi_{\mathfrak{L}'}(x') \\
&= \psi_{\mathfrak{L}}(x) + \psi_{\mathfrak{L}'}(x') \\
&= i_{\mathfrak{L}\oplus\mathfrak{L}'}(x, 0) + i_{\mathfrak{L}\oplus\mathfrak{L}'}(0, x') \\
&= i_{\mathfrak{L}\oplus\mathfrak{L}'}((x, 0) + (0, x')) \\
&= i_{\mathfrak{L}\oplus\mathfrak{L}'}(x, x')
\end{aligned}$$

This implies that $\psi \circ \phi = \text{id}$ (see diagram below).

$$\begin{tikzcd} \mathfrak{L} \oplus \mathfrak{L}' \arrow[r, "f"] \arrow[d, "i_{\mathfrak{L}\oplus\mathfrak{L}'}"'] & U(\mathfrak{L}) \otimes U(\mathfrak{L}') \\ U(\mathfrak{L} \oplus \mathfrak{L}') \arrow[ur, "\phi"'] & \end{tikzcd}$$

We now want to show that $\phi \circ \psi = \mathrm{id}$. Note that the algebra $U(\mathfrak{L})$ is generated by $\mathfrak{L}$ and hence

$$\{x \otimes 1 : x \in \mathfrak{L}\} \cup \{1 \otimes x' : x' \in \mathfrak{L}'\}$$

is a generating set for $U(\mathfrak{L}) \otimes U(\mathfrak{L}')$ as an algebra. Thus

$$\begin{aligned} \phi(\psi(x \otimes 1)) &= \phi(\psi_{\mathfrak{L}}(x)\psi_{\mathfrak{L}'}(1)) \\ &= \phi(x, 0) \\ &= i_{\mathfrak{L}}(x) \otimes 1 + 1 \otimes i_{\mathfrak{L}'}(0) \\ &= x \otimes 1 + 1 \otimes 0 \\ &= x \otimes 1 \end{aligned}$$

A symmetric argument shows that the same holds for elements $1 \otimes x'$. We have therefore shown that $\psi = \phi^{-1}$ and therefore that ϕ is an isomorphism, which completes the proof. $\square$

It is here that we are now able to connect our work with Lie algebras to the heart of the thesis. That is, the functor U provides us with the most important type of cocommutative Hopf algebra.

Theorem 5.21. *The enveloping algebra $U(\mathfrak{L})$ is a cocommutative Hopf algebra with*

$$\Delta := \phi \circ U([\diagdown]), \quad \varepsilon := U(0), \quad S := U(\mathrm{op})$$

where $[\diagdown] : \mathfrak{L} \to \mathfrak{L} \oplus \mathfrak{L}$ is the diagonal relation $[\diagdown](x) = (x, x)$, ϕ is the isomorphism $U(\mathfrak{L}\oplus\mathfrak{L}) \to U(\mathfrak{L})\otimes U(\mathfrak{L})$ from the previous corollary, $0 : \mathfrak{L} \to \{0\}$, op is the isomorphism from $\mathfrak{L}$ onto $\mathfrak{L}^{op}$ and U is the functor from Corollary 5.19. Also, for $x_1, ..., x_n \in \mathfrak{L}$, we have

$$\Delta(x_1 \dots x_n) = 1 \otimes x_1 \dots x_n + \sum_{p=1}^{n-1} \sum_{\sigma} x_{\sigma(1)} \dots x_{\sigma(p)} \otimes x_{\sigma(p+1)} \dots x_{\sigma(n)} + x_1 \dots x_n \otimes 1$$

where σ runs over all (p, q)-shuffles of the symmetric group S_n, and

$$S(x_1 \dots x_n) = (-1)^n x_n \dots x_1$$

Proof. We begin by checking the coassociativity axiom. It is obvious that the following diagram commutes.

$$\begin{tikzcd}
\mathfrak{L} \arrow[r, "\lceil\diagdown\rceil"] \arrow[d, "\lceil\diagdown\rceil"'] & \mathfrak{L}\oplus\mathfrak{L} \arrow[d, "\mathrm{id}\oplus\lceil\diagdown\rceil"] \\
\mathfrak{L}\oplus\mathfrak{L} \arrow[r, "\lceil\diagdown\rceil\oplus\mathrm{id}"'] & \mathfrak{L}\oplus\mathfrak{L}\oplus\mathfrak{L}
\end{tikzcd}$$

If we now apply the functor U, we get the commuting diagram

$$\begin{tikzcd}
U(\mathfrak{L}) \arrow[r, "U(\lceil\diagdown\rceil)"] \arrow[d, "U(\lceil\diagdown\rceil)"'] & U(\mathfrak{L}\oplus\mathfrak{L}) \arrow[d, "U(\mathrm{id}\oplus\lceil\diagdown\rceil)"] \\
U(\mathfrak{L}\oplus\mathfrak{L}) \arrow[r, "U(\lceil\diagdown\rceil\oplus\mathrm{id})"'] & U(\mathfrak{L}\oplus\mathfrak{L}\oplus\mathfrak{L})
\end{tikzcd}$$

But $U(\mathfrak{L}\oplus\mathfrak{L}) \cong U(\mathfrak{L})\otimes U(\mathfrak{L})$ under ϕ and $U(\mathfrak{L}\oplus\mathfrak{L}\oplus\mathfrak{L}) \cong U(\mathfrak{L})\otimes U(\mathfrak{L})\otimes U(\mathfrak{L})$ by applying ϕ twice, so, we get the commuting diagram

$$\begin{tikzcd}
U(\mathfrak{L}) \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & U(\mathfrak{L})\otimes U(\mathfrak{L}) \arrow[d, "\mathrm{id}\otimes\Delta"] \\
U(\mathfrak{L})\otimes U(\mathfrak{L}) \arrow[r, "\Delta\otimes\mathrm{id}"'] & U(\mathfrak{L})\otimes U(\mathfrak{L})\otimes U(\mathfrak{L})
\end{tikzcd}$$

which establishes coassociativity (recall that $\Delta := \phi\circ U(\lceil\diagdown\rceil)$). A more precise reason for $U(\mathrm{id}\oplus\lceil\diagdown\rceil)$ turning into $\mathrm{id}\otimes\Delta$ is indicated in the following diagram:

$$\begin{tikzcd}
\mathfrak{L} \arrow[r, "\lceil\diagdown\rceil"] \arrow[d, hook, "i_{\mathfrak{L}}"'] \arrow[dr, phantom, "2"] & \mathfrak{L}\oplus\mathfrak{L} \arrow[r, "\mathrm{id}\oplus\lceil\diagdown\rceil"] \arrow[d, hook, "i_{\mathfrak{L}^2}"] \arrow[dr, phantom, "3"] & \mathfrak{L}\oplus\mathfrak{L}\oplus\mathfrak{L} \arrow[d, hook, "i_{\mathfrak{L}^3}"] \\
U(\mathfrak{L}) \arrow[r, "U(\lceil\diagdown\rceil)"] \arrow[dr, "\Delta"'] \arrow[dr, phantom, bend left, "1"] & U(\mathfrak{L}\oplus\mathfrak{L}) \arrow[r, "U(\mathrm{id}\oplus\lceil\diagdown\rceil)"] \arrow[d, "\phi"] & U(\mathfrak{L}\oplus\mathfrak{L}\oplus\mathfrak{L}) \arrow[d, "(\mathrm{id}\otimes\phi)\circ\phi"] \\
 & U(\mathfrak{L})\otimes U(\mathfrak{L}) \arrow[r, dashed, "\mathrm{id}\otimes\Delta"] & U(\mathfrak{L})\otimes U(\mathfrak{L})\otimes U(\mathfrak{L})
\end{tikzcd}$$

Subdiagram (1) commutes by definition of Δ, while subdiagrams (2) and (3) commute by Corollary 5.19. The remaining subdiagram is then forced to commute due to the definitions of the maps involved. In short, $U(\mathrm{id}\oplus\lceil\diagdown\rceil)$ becomes $\mathrm{id}\otimes\Delta$ due to the functorial nature of U in that it essentially changes direct sums into tensor products.

The counit axiom is verified by a similar application of the functor U to the commuting diagram

$$\begin{tikzcd} 0 \oplus \mathfrak{L} & \mathfrak{L} \oplus \mathfrak{L} \arrow[l, "0 \oplus \mathrm{id}"'] \arrow[r, "\mathrm{id} \oplus 0"] & \mathfrak{L} \oplus 0 \\ & \mathfrak{L} \arrow[ul, "\sim"] \arrow[u, "\Delta"] \arrow[ur, "\sim"'] & \end{tikzcd}$$

Cocommutativity also comes from application of U to the commuting diagram

$$\begin{tikzcd} & \mathfrak{L} \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\ \mathfrak{L} \oplus \mathfrak{L} \arrow[rr, "\tau_{\mathfrak{L},\mathfrak{L}}"'] & & \mathfrak{L} \oplus \mathfrak{L} \end{tikzcd}$$

Finally, S is an antipode for $U(\mathfrak{L})$ by Lemma 4.16 since $U(\mathfrak{L})$ is generated by $\mathfrak{L}$ and, if $x \in \mathfrak{L}$, then

$$\begin{aligned} (S \star \mathrm{id})(x) &= (\nabla \circ (S \otimes \mathrm{id}) \circ \Delta)(x) \\ &= \nabla\big((S \otimes \mathrm{id})(1 \otimes x + x \otimes 1)\big) \\ &= \nabla(S(1) \otimes x + S(x) \otimes 1) \\ &= \nabla(1 \otimes x - x \otimes 1) \\ &= \nabla(1 \otimes x) - \nabla(x \otimes 1) \\ &= x - x \\ &= 0 \\ &= U(0)(x) \\ &= \varepsilon(x) \end{aligned}$$

A similar argument shows we get the same thing for $(\mathrm{id} \star S)(x)$.

The result for $\Delta(x_1 \ldots x_n)$ follows from the work we did with the tensor algebra so let us focus on the antipode.

First, since the inclusion map $i_{\mathfrak{L}} : \mathfrak{L} \to U(\mathfrak{L})$ is a Lie algebra morphism, so is the map $i_{\mathfrak{L}}^{op} : \mathfrak{L}^{op} \to U(\mathfrak{L})^{op}$. We therefore get a unique extension of $i_{\mathfrak{L}}^{op}$ to an algebra morphism φ.

$$\begin{tikzcd}
\mathfrak{L}^{op} \arrow[r, "i_{\mathfrak{L}}^{op}"] \arrow[d, hook, "i_{\mathfrak{L}^{op}}"'] & U(\mathfrak{L})^{op} \\
U(\mathfrak{L}^{op}) \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

Now replace $\mathfrak{L}$ with $\mathfrak{L}^{op}$ in the above diagram:

$$\begin{tikzcd}
\mathfrak{L} \arrow[r] \arrow[d, hook, "i_{\mathfrak{L}}"'] & U(\mathfrak{L}^{op})^{op} \\
U(\mathfrak{L}) \arrow[ur, "\psi"'] &
\end{tikzcd}$$

Then ψ^{op} is an algebra morphism $U(\mathfrak{L})^{op} \to U(\mathfrak{L}^{op})$ which is inverse to φ. Therefore, $U(\mathfrak{L}^{op}) \cong U(\mathfrak{L})^{op}$ as algebras.

Second, we already know that $\mathfrak{L} \cong \mathfrak{L}^{op}$ via the Lie isomorphism $\mathrm{op} : \mathfrak{L} \to \mathfrak{L}^{op}$ defined by

$$\mathrm{op}(x) := -x$$

which lifts to an algebra isomorphism $U(\mathrm{op}) : U(\mathfrak{L}) \to U(\mathfrak{L}^{op})$. If we string our isomorphisms together we see that

$$U(\mathfrak{L}) \cong U(\mathfrak{L})^{op}$$

as algebras. However, regarded as a morphism $U(\mathfrak{L}) \to U(\mathfrak{L})$ it is an anti-automorphism and since $x \mapsto -x$ for $x \in \mathfrak{L}$, it follows that

$$x_1 \dots x_n \mapsto (-1)^n x_n \dots x_1$$

which is our antipode S. □

The next and last result of this section is important. However, because it is rather involved it will merely be stated and henceforth taken for granted.

Theorem 5.22. *Let $\mathfrak{L}$ be a Lie algebra.*

(a) (Poincaré-Birkhoff-Witt Theorem) The algebra $U(\mathfrak{L})$ is filtered as a quotient of the tensor algebra $T(\mathfrak{L})$ and the corresponding graded algebra is isomorphic to the symmetric algebra on $\mathfrak{L}$:

$$grU(\mathfrak{L}) \cong S(\mathfrak{L})$$

Hence, if $\{v_i\}_{i \in I}$ is a totally ordered basis of $\mathfrak{L}$, $\{v_{i_1}, ..., v_{i_n}\}_{i_1 \leq \ldots \leq i_n \in I, n \in \mathbb{N}}$ is a basis of $U(\mathfrak{L})$.

(b) *When the characteristic of the field κ is zero, the symmetrization map $\eta : S(\mathfrak{L}) \to U(\mathfrak{L})$ defined by*

$$\eta(v_1 \dots v_n) = \frac{1}{n!} \sum_{\sigma \in S_n} v_{\sigma(1)} \dots v_{\sigma(n)}$$

for $v_1, ..., v_n \in \mathfrak{L}$, is an isomorphism of coalgebras.

For details, see [11].

Let us now move from our general discussion to one of a more particular nature. In the section that follows we shall endeavor to explore the very significant Lie algebra $\mathfrak{sl}(2)$ in some detail.

5.4 The Lie Algebra $\mathfrak{sl}(2)$

The Lie algebra $\mathfrak{sl}(2)$ is of significant importance to the theory of semisimple Lie algebras. For simplicity, it is common to take the ground field κ to be $\mathbb{C}$. We too shall embrace this custom. Now, recall that $M_2(\kappa)$ can be viewed as a Lie algebra by taking the commutator map as a Lie bracket. We denote this particular Lie algebra by $\mathfrak{gl}(2)$ (i.e. $\mathfrak{gl}(2) := L(M_2(\kappa))$). Choose the following four matrices to serve as a basis:

$$X := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad Y := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad H := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad I := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Simple calculations yield

$$[X, Y] = H, \quad [Y, X] = -H \tag{5.2}$$

$$[X, H] = -2X, \quad [H, X] = 2X \tag{5.3}$$

$$[Y, H] = 2Y, \quad [H, Y] = -2Y \tag{5.4}$$

$$[X, I] = [I, X] = [Y, I] = [I, Y] = [H, I] = [I, H] = 0 \tag{5.5}$$

Notice that X, Y and H each share the property of vanishing trace. This indicates that there is a *special* subspace of $\mathfrak{gl}(2)$. This special subspace consists of all the trace zero matrices and is denoted by $\mathfrak{sl}(2)$. As a vector space $\{X, Y, H\}$ forms a basis for $\mathfrak{sl}(2)$.

Proposition 5.23. $\mathfrak{sl}(2)$ *is an ideal of* $\mathfrak{gl}(2)$ *and hence a Lie-subalgebra of* $\mathfrak{gl}(2)$.

Proof. We already know that $\mathfrak{sl}(2)$ is a subspace of $\mathfrak{gl}(2)$. So, take arbitrary $x \in \mathfrak{gl}(2)$ and $y \in \mathfrak{sl}(2)$. Then, by properties of the trace, we have that

$$\begin{aligned}\operatorname{tr}([x, y]) &= \operatorname{tr}(xy - yx) \\ &= \operatorname{tr}(xy) - \operatorname{tr}(yx) \\ &= 0\end{aligned}$$

and hence $[x, y] \in \mathfrak{sl}(2)$. □

Notice that nothing in the proof depended on y being an element of $\mathfrak{sl}(2)$. The same argument would work for any two arbitrary elements of $\mathfrak{gl}(2)$. This leads to the conclusion that $[\mathfrak{gl}(2), \mathfrak{gl}(2)] \subseteq \mathfrak{sl}(2)$. Another way of saying this is that the derived algebra of $\mathfrak{gl}(2)$ is a Lie subalgebra of $\mathfrak{sl}(2)$. But because $H = [X, Y]$, $X = 1/2[H, X]$ and $Y = 1/2[Y, H]$, we can say more than this, namely that $[\mathfrak{gl}(2), \mathfrak{gl}(2)] = \mathfrak{sl}(2)$.

The next proposition shows that $\mathfrak{sl}(2)$ is "special" in another sense. That is, if one knows everything about $\mathfrak{sl}(2)$, then one knows everything about $\mathfrak{gl}(2)$. Hence, the study of $\mathfrak{gl}(2)$ reduces to the study of $\mathfrak{sl}(2)$.

Proposition 5.24. *There is an isomorphism of Lie algebras*

$$\mathfrak{gl}(2) \cong \mathfrak{sl}(2) \oplus \kappa I$$

Proof. Since $\mathfrak{gl}(2)$ has basis $\{X, Y, H, I\}$ and $\mathfrak{sl}(2)$ has basis $\{X, Y, H\}$ it is clear that

$$\mathfrak{gl}(2) = \mathfrak{sl}(2) + \kappa I$$

Moreover, it is clearly the case that

$$\mathfrak{sl}(2) \cap \kappa I = \{0\}$$

and thus $\mathfrak{gl}(2) = \mathfrak{sl}(2) \oplus \kappa I$ as a vector space. Now, the Lie bracket in $\mathfrak{sl}(2) \oplus \kappa I$ is defined by (see section "New Lie Algebras")

$$[S + \lambda I, S' + \gamma I] := [S, S'] + [\lambda I, \gamma I]$$

But $[\lambda I, \gamma I] = 0$ and so $[S + \lambda I, S' + \gamma I] = [S, S']$.

On the other hand, if $G, G' \in \mathfrak{gl}(2)$, then we can write

$$G = S + \lambda I \quad \text{and} \quad G' = S' + \gamma I$$

where $S, S' \in \mathfrak{sl}(2)$. We then have

$$\begin{aligned}
[G, G'] &= [S + \lambda I, S' + \gamma I] \\
&= (S + \lambda I)(S' + \gamma I) - (S' + \gamma I)(S + \lambda I) \\
&= SS' + \gamma S + \lambda S' + \lambda\gamma I - S'S - \lambda S' - \gamma S - \gamma\lambda I \\
&= SS' - S'S \\
&= [S, S']
\end{aligned}$$

and hence we get that the Lie algebras are isomorphic. □

The Lie algebra $\mathfrak{sl}(2)$ also has important connections to quantum mechanics given its relationship to the real Lie algebra $\mathfrak{su}(2)$, which consists of 2×2 trace zero skew-hermitian matrices. It has basis

$$U_1 := \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad U_2 := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad U_3 := \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

and is used to represent *spin* of elementary particles. We can express these in terms of the basis elements of $\mathfrak{sl}(2)$.

$$U_1 = i(X + Y), \quad U_2 = Y - X, \quad U_3 = iH \tag{5.6}$$

Now, for $\mathfrak{su}(2)$, the ground field is $\mathbb{R}$ and so a generic element has the form $\begin{bmatrix} ia & -\overline{\alpha} \\ \alpha & -ia \end{bmatrix}$ where $a \in \mathbb{R}$ and $\alpha \in \mathbb{C}$. If the ground field were $\mathbb{C}$, then $\mathfrak{su}(2)$ would be identical to $\mathfrak{sl}(2)$. This is evident from (5.6) along with the fact that $H = -iU_3$, $X = -i/2U_1 - 1/2U_2$ and $Y = 1/2U_2 - i/2U_1$.

The same thing occurs with the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$. Obviously, this would become $\mathfrak{sl}(2)$ if the base field were changed to $\mathbb{C}$. However, $\mathfrak{sl}_2(\mathbb{R})$ and $\mathfrak{su}(2)$ are not isomorphic. Instead, these two distinct real Lie algebras are called the *real forms* for $\mathfrak{sl}(2)$. One reason for calling these real forms of $\mathfrak{sl}(2)$ is the following:

Proposition 5.25.

$$\mathfrak{sl}_2(\mathbb{R}) \oplus i\mathfrak{sl}_2(\mathbb{R}) = \mathfrak{sl}(2) = \mathfrak{su}(2) \oplus i\mathfrak{su}(2)$$

as vector spaces.

Proof. Beginning with the left equality, it is clearly the case that

$$\mathfrak{sl}(2) = \mathfrak{sl}_2(\mathbb{R}) \oplus i\mathfrak{sl}(\mathbb{R})$$

as vector spaces, since for any $M \in \mathfrak{sl}(2)$ we have

$$M = z_1 X + z_2 Y + z_3 H, \quad z_1, z_2, z_3 \in \mathbb{C}$$

which can be expanded and re-ordered as

$$M = \mathfrak{Re}(z_1)X + \mathfrak{Re}(z_2)Y + \mathfrak{Re}(z_3)H + i(\mathfrak{Im}(z_1)X + \mathfrak{Im}(z_2)Y + \mathfrak{Im}(z_3)H)$$

The right equality is a little less obvious, but follows by the same reasoning. First,

$$\mathfrak{sl}(2) = \mathfrak{su}(2) + i\mathfrak{su}(2)$$

since for $z_1X + z_2Y + z_3H \in \mathfrak{sl}(2)$ we have

$$\begin{aligned} z_1X + z_2Y + z_3H &= z_1(-\frac{i}{2}U_1 - \frac{1}{2}U_2) + z_2(-\frac{i}{2}U_1 + \frac{1}{2}U_2) + z_3(-iU_3) \\ &= -\frac{i}{2}z_1U_1 - \frac{1}{2}z_1U_2 - \frac{i}{2}z_2U_1 + \frac{1}{2}z_2U_2 - iz_3U_3 \\ &= -\frac{i}{2}(z_1 + z_2)U_1 + \frac{1}{2}(z_2 - z_1)U_2 - iz_3U_3 \end{aligned}$$

If we now let $\zeta_1 = -\frac{i}{2}(z_1 + z_2), \zeta_2 = \frac{1}{2}(z_2 - z_1)$ and $\zeta_3 = -iz_3$, then the above can be written as

$$\mathfrak{Re}(\zeta_1)U_1 + \mathfrak{Re}(\zeta_2)U_2 + \mathfrak{Re}(\zeta_3)U_3 + i(\mathfrak{Im}(\zeta_1)U_1 + \mathfrak{Im}(\zeta_2)U_2 + \mathfrak{Im}(\zeta_3)U_3)$$

Going the other way is similar.

Finally, $\mathfrak{su}(2) \cap i\mathfrak{su}(2) = \{0\}$ based on the form of the generic elements of $\mathfrak{su}(2)$. Thus,

$$\mathfrak{sl}(2) = \mathfrak{su}(2) \oplus i\mathfrak{su}(2)$$

as vector spaces. $\square$

Notice that under the usual notion of a direct sum of Lie algebras $\mathfrak{sl}_2(\mathbb{R}) \oplus i\mathfrak{sl}_2(\mathbb{R})$ and $\mathfrak{su}(2) \oplus i\mathfrak{sl}(2)$ are not the same Lie algebras as $\mathfrak{sl}(2)$. For instance, if $U, V \in \mathfrak{sl}(2)$, then

$$[U, V] = UV - VU$$

If we write $U = N + iM$ and $V = N' + iM'$, then this becomes

$$\begin{aligned} [N + iM, N' + iM'] &= [N, N'] + [N, iM'] + [iM, N'] + [iM, iM'] \\ &= [N, N'] + i[N, M'] + i[M, N'] - [M, M'] \end{aligned}$$

However, as elements of $\mathfrak{sl}_2(\mathbb{R}) \oplus i\mathfrak{sl}_2(\mathbb{R})$ we would have

$$\begin{aligned}[N + iM, N' + iM'] &= [N, N'] + [iM, iM'] \\ &= [N, N'] - [M, M']\end{aligned}$$

We can remedy this by redefining the Lie bracket on $\mathfrak{sl}_2(\mathbb{R}) \oplus i\mathfrak{sl}_2(\mathbb{R})$ to be the same as that of $\mathfrak{sl}(2)$, which makes sense since they are actually equal as vector spaces and

$$[N + iM, N' + iM'] = [N, N'] - [M, M'] + i([N, M'] + [M, N'])$$

has the right form.

Proposition 5.26. $\mathfrak{sl}(2)$ *is a simple Lie algebra.*

Proof. A straightforward, albeit pedestrian, proof proceeds as follows. Suppose J is a non-zero ideal of $\mathfrak{sl}(2)$. The goal is to show that J must be $\mathfrak{sl}(2)$. Recall that being a Lie algebra ideal means that $[T, S] \in J$ for all $T \in \mathfrak{sl}(2)$ and $S \in J$.

Let S be a non-zero element of J. Since $S \in \mathfrak{sl}(2)$ too, it must have a vanishing trace, and hence, must be of the form $\begin{bmatrix} a & b \\ c & -a \end{bmatrix}$. If $c \neq 0$, then compute $[X, S]$.

$$\begin{aligned}[X, S] &= XS - SX \\ &= \begin{bmatrix} c & -2a \\ 0 & -c \end{bmatrix}\end{aligned}$$

Call the resulting matrix S'. Now compute $[X, S']$.

$$\begin{aligned}[X, S'] &= XS' - S'X \\ &= \begin{bmatrix} 0 & -2c \\ 0 & 0 \end{bmatrix}\end{aligned}$$

Since $[X, S'] \in J$, so is $\frac{1}{-2c}[X, S'] = X$. But $X \in J$ implies that $[Y, X] = -H \in J$ and therefore $H \in J$. Finally, since $H \in J$, then $[Y, H] = 2Y \in J$ and therefore $Y \in J$. We therefore get that $J = \mathfrak{sl}(2)$.

By similar reasoning we also find that if $b \neq 0$, then

$$[Y, [Y, S]] = \begin{bmatrix} 0 & 0 \\ -2b & 0 \end{bmatrix}$$

implies that $Y \in J$. Using the commutator relations, we again get that $J = \mathfrak{sl}(2)$.

Finally, if $a \neq 0$, then computing $[H, [X, S]]$ gives

$$[H, [X, S]] = \begin{bmatrix} 0 & -4a \\ 0 & 0 \end{bmatrix}$$

which again implies that $X \in J$ and hence that $J = \mathfrak{sl}(2)$. It follows that J contains the basis of $\mathfrak{sl}(2)$ and hence must be $\mathfrak{sl}(2)$. $\square$

Above we showed that the derived algebra $[\mathfrak{gl}(2), \mathfrak{gl}(2)]$ is a Lie subalgebra of $\mathfrak{sl}(2)$. But it is also an ideal of $\mathfrak{sl}(2)$, since it is an ideal of $\mathfrak{gl}(2)$. But we have just found that $\mathfrak{sl}(2)$ is simple. Thus, because the derived algebra of $\mathfrak{gl}(2)$ is not zero we have another way of seeing that

$$[\mathfrak{gl}(2), \mathfrak{gl}(2)] = \mathfrak{sl}(2)$$

5.5 Representations of $\mathfrak{sl}(2)$

Let us begin with an example using the now familiar adjoint representation. Note that in this case the adjoint representation of $\mathfrak{sl}(2)$ will be the map

$$\mathrm{ad} : \mathfrak{sl}(2) \to \mathfrak{gl}(3)$$

We already understand that

$$\begin{aligned} \mathrm{ad}(X) &= \mathrm{ad}_X = [X, -] \\ \mathrm{ad}(Y) &= \mathrm{ad}_Y = [Y, -] \\ \mathrm{ad}(H) &= \mathrm{ad}_H = [H, -] \end{aligned}$$

so let us now determine each one's corresponding matrix. To be consistent, let us agree to order our basis as $\{X, Y, H\}$. We start with ad_X and compute its columns as follows:

$$\begin{aligned} \mathrm{ad}_X(X) &= 0X + 0Y + 0H, \quad \text{[column one]} \\ \mathrm{ad}_X(Y) &= 0X + 0Y + H, \quad \text{[column two]} \\ \mathrm{ad}_X(H) &= -2X + 0Y + 0H, \quad \text{[column three]} \end{aligned}$$

Thus we can say that

$$\mathrm{ad}_X = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Doing the same for the other adjoint actions yields

$$\mathrm{ad}_Y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 0 \end{bmatrix}, \quad \mathrm{ad}_H = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Notice that X, Y and H are eigenvectors for ad_H with respective eigenvalues 2, -2 and 0.

5.5.1 Weight Space

Definition 5.27 (Weight Vector)**.** Let V be a representation of $\mathfrak{sl}(2)$ (i.e. an $\mathfrak{sl}(2)$-module). A vector $v \in V$ is said to be a vector of weight $\lambda \in \mathbb{C}$ if it is an eigenvector for H with eigenvalue λ:

$$Hv = \lambda v$$

If, in addition, v is such that $Xv = 0$, then we say that v is a highest weight vector of weight λ. The subspace of V consisting of all vectors of weight λ is denoted by V_λ - i.e.

$$V_\lambda := \{v \in V : Hv = \lambda v\}$$

So, V_λ is actually the corresponding eigenspace of H for the eigenvalue λ and is called a *weight space.*

By way of reminder, recall that the above notation Hv, Xv, etc. is technically shorthand for $\rho(H)v, \rho(X)v$, where ρ *is* the representation (homomorphism) associated with the representation space V. We write these in the form of matrix multiplication, since $\rho(H)$ and $\rho(X)$ can be thought of as matrices.

Lemma 5.28.

$$XV_\lambda \subset V_{\lambda+2}$$
$$YV_\lambda \subset V_{\lambda-2}$$

Proof. Let $v \in V_\lambda$. Then $Xv \in V$ and we would like to know (a) if it has a weight and (b) what that weight might be. Consider that

$$\begin{aligned} H(Xv) &= (HX)v \\ &= (2X + XH)v, \qquad [H, X] = 2X \\ &= 2Xv + (XH)v \\ &= 2Xv + X(Hv) \\ &= 2Xv + \lambda Xv \\ &= (\lambda + 2)Xv \end{aligned}$$

which shows that $Xv \in V_{\lambda+2}$. This answers both (a) and (b). The second inclusion is shown via a similar argument. □

This Lemma shows how X and Y act on the weight spaces of V. Note that each V_λ is invariant under H.

Now that we have introduced weight spaces and know how $\mathfrak{sl}(2)$ acts on them, our next task will be to classify irreducible finite-dimensional representations.

Proposition 5.29. *Any non-zero finite-dimensional $\mathfrak{sl}(2)$-module has a highest weight vector.*

Proof. Since V is finite-dimensional and $\mathbb{C}$ is algebraically closed, the operator H must have at least one eigenvector and hence an eigenvalue. Let w be our eigenvector and let α be the associated eigenvalue. If $Xw = 0$, then obviously we are done, so suppose that $Xw \neq 0$. Consider, then, the sequence $\{X^n w\}_{n\in\mathbb{N}}$. It can easily be shown, by induction on n, that

$$H(X^n w) = (\alpha + 2n)X^n w$$

and hence that this gives a sequence of eigenvectors for H with distinct eigenvalues. But V is finite-dimensional, which implies that H can have only a finite number of eigenvalues. It follows that there must exist an n for which $X^n w \neq 0$, but $X^{n+1}w = 0$. This tells us that $X^n w$ is our desired highest weight vector. □

Lemma 5.30. *Let V be a finite-dimensional irreducible $\mathfrak{sl}(2)$-module. Let w be a highest weight vector with weight λ. Let $k \geq 0$ be such that $Y^k w \neq 0$, but $Y^{k+1}w = 0$. Then $\{w, Yw, ..., Y^k w\}$ is a basis of V.*

Before giving a proof of this lemma, let's consider a familiar example. Let $V = \mathbb{R}^2$. Then one possibility is that $\mathfrak{sl}(2)$ acts on $\mathbb{R}^2$ via regular matrix multiplication. If we want a highest weight vector we need

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix}$$

and

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

These equations imply that $\lambda = 1$ and that any vector of the form $\begin{pmatrix} a \\ 0 \end{pmatrix}$ is a highest weight vector. If we now apply Y to vectors of this form we get $\begin{pmatrix} 0 \\ a \end{pmatrix}$ and the zero vector if applied again. Of course, we know that

$$\left\{ \begin{pmatrix} a \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ a \end{pmatrix} \right\}$$

forms a basis for $\mathbb{R}^2$ for any $a \in \mathbb{R}$. In particular, we get the standard basis when $a = 1$.

Proof of Lemma 5.30. Using the same argument as in Proposition 5.29 it follows that there is such a k with $Y^k w \neq 0$ and $Y^{k+1} w = 0$. Let $W := \text{Span}\{w, Yw, ..., Y^k w\}$. Now, since this spanning set consists of eigenvectors for H with distinct eigenvalues, it is immediate that they are linearly independent and so form a basis for W. It is clear that W is invariant under the action of Y and likewise under the action of H since the spanning set consists of eigenvectors for H. We now check for invariance under X. We shall use induction. To get an idea of what's going on, however, let's do the first three cases.

$$Xw = 0$$

$$\begin{aligned} X(Yw) &= (XY)w \\ &= (H + YX)w \\ &= Hw + YXw \\ &= \lambda w \end{aligned}$$

$$\begin{aligned} X(Y^2w) &= (H + YX)Yw \\ &= HYw + YXYw \\ &= HYw + \lambda Yw, \quad \text{[using previous case]} \\ &= (\lambda - 2)Yw + \lambda Yw, \quad \text{[Lemma 5.28]} \\ &= 2(\lambda - 1)Yw \end{aligned}$$

These first three cases suggest that X sends Y^jw to a multiple of $Y^{j-1}w$. More specifically,

$$X(Y^jw) = j\big(\lambda - (j-1)\big)Y^{j-1}w$$

Suppose that this last statement is true and consider the $j+1$ case. We have

$$\begin{aligned} X(Y^{j+1}w) &= (XY)Y^jw \\ &= (H + YX)Y^jw \\ &= HY^jw + YXY^jw \\ &= HY^jw + j(\lambda - j + 1)Y^jw, \quad \text{[induction hypothesis]} \\ &= (\lambda - 2j)Y^jw + j(\lambda - j + 1)Y^jw, \quad \text{[by Lemma 5.28]} \\ &= (\lambda j + \lambda - j^2 - j)Y^jw \\ &= (j+1)(\lambda - j)Y^jw \end{aligned}$$

By induction, we have therefore shown that W is invariant under the action of X. By definition, this means that W is a non-zero submodule of V. But V is irreducible by hypothesis, which means that $W = V$. Therefore, $\{w, Yw, ..., Y^kw\}$ is a basis for V. $\square$

Corollary 5.31. *If the conditions of Lemma 5.30 hold, then $k = \lambda$ and hence $dim(V) = \lambda + 1$ and H has distinct eigenvalues $\{\lambda, \lambda - 2, \lambda - 4, ..., -\lambda\}$ implying that every highest weight of H is a non-negative integer equal to*

$$\lambda = dim(V) - 1$$

Proof. Since $Y^{k+1}w = 0$ we have that

$$0 = X(Y^{k+1}w) = (k+1)(\lambda - k)Y^k w$$

But $Y^k w \neq 0$ and $k+1 \neq 0$. Thus, it must be that $\lambda - k = 0$ and therefore, $\lambda = k$.

Now, since $\{w, Yw, ..., Y^\lambda w\}$ will be a basis of distinct eigenvectors for V with distinct eigenvalues $\{\lambda, \lambda-2, \lambda-4, ..., -\lambda\}$, the matrix representation of H with respect to this basis is the diagonal matrix

$$\begin{bmatrix} \lambda & & & \\ & \lambda - 2 & & \\ & & \ddots & \\ & & & -\lambda \end{bmatrix}$$

and so has characteristic polynomial

$$(x-\lambda)(x-(\lambda-2))\cdots(x-(-\lambda))$$

Thus, $\{\lambda, \lambda-2, \lambda-4, ..., -\lambda\}$ is the complete set of eigenvalues of H. □

Corollary 5.32. *Let V be a finite-dimensional irreducible $\mathfrak{sl}(2)$-module. Suppose γ is a weight with respect to H with corresponding weight space V_γ. Then, as a vector space, V_γ is of dimension one.*

Proof. From the previous corollary $\gamma = \lambda - 2j$ for some $j \in \{0, 1, ..., \lambda\}$ and any vector in V_γ is a scalar multiple of $Y^j w$. □

We can now deduce the following theorem, which says that any finite-dimensional representation V of $\mathfrak{sl}(2)$ is decomposable in terms of its weight spaces. This is called the weight decomposition of V.

Theorem 5.33. *Every finite-dimensional representation V of $\mathfrak{sl}(2)$ can be expressed in the form*

$$V = \bigoplus_\lambda V_\lambda$$

Proof. First, V can be expressed as a direct sum of indecomposable $\mathfrak{sl}(2)$-modules:

$$V = \bigoplus_{i=1}^{n} V_i$$

By Lemma 5.30, we can reorder the sum and collect all $(V_i)_\lambda$ having the same weight and then

$$V = \bigoplus_\lambda V_\lambda$$

□

Note that our work above implies that if V is an irreducible $\mathfrak{sl}(2)$-module of dimension k, then V has basis $\{w, Yw, ..., Y^\lambda w\}$ where $\lambda = k - 1$ so that

$$V = \bigoplus_\lambda V_\lambda$$

where each V_λ is one-dimensional. This is called the irreducible representation of highest weight λ and is denoted by $V[\lambda]$. Furthermore, we have that there is at most one irreducible $\mathfrak{sl}(2)$-module up to isomorphism for each of the $\lambda + 1$ dimensions ($\lambda \geq 0$). Next, we'll see that given k there does, in fact, exist an irreducible $\mathfrak{sl}(2)$-module of that dimension.

5.5.1.1 Constructing Irreducible $\mathfrak{sl}(2)$-modules

At the moment, this is all very general. But there is a nice way to construct these irreducible $\mathfrak{sl}(2)$-modules. Let $V = \mathbb{C}[x, y]$ and for each $\lambda \geq 0$ take $V[\lambda]$ to be the vector subspace of all homogeneous polynomials of degree λ. For each λ, the vector space $V[\lambda]$ has basis $\{x^\lambda, x^{\lambda-1}y, ..., xy^{\lambda-1}, y^\lambda\}$. Our claim is that this $V[\lambda]$ is a model for the $V[\lambda]$ just considered above. Now, define a map $\rho : \mathfrak{sl}(2) \to \mathfrak{gl}(V[\lambda])$ by

$$\rho(X) := x\frac{\partial}{\partial y}, \qquad \rho(Y) := y\frac{\partial}{\partial x}$$

$$\rho(H) := x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$$

By construction, ρ will be a linear map, so we need only show that it preserves the Lie bracket. First, we establish that

$$\rho([X, Y]) = \rho(H) = [\rho(X), \rho(Y)] \tag{5.7}$$

The first equality is trivial, since $[X, Y] = H$. Now let $x^a y^b$ be such that $a, b \geq 0$ and $a + b = \lambda$. Then

$$\begin{aligned}
\rho(H)(x^a y^b) &= \Big(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\Big)x^a y^b \\
&= x\frac{\partial}{\partial x}(x^a y^b) - y\frac{\partial}{\partial y}(x^a y^b) \\
&= a x^a y^b - b x^a y^b \\
&= (a-b)x^a y^b
\end{aligned}$$

Notice that this means each $x^a y^b$ is an eigenvector for $\rho(H)$ with eigenvalue $a - b$. Let us now see what $[\rho(X), \rho(Y)]$ does to $x^a y^b$.

$$\begin{aligned}
[\rho(X), \rho(Y)](x^a y^b) &= x\frac{\partial}{\partial y}\Big(y\frac{\partial}{\partial x}(x^a y^b)\Big) - y\frac{\partial}{\partial x}\Big(x\frac{\partial}{\partial y}(x^a y^b)\Big) \\
&= x\frac{\partial}{\partial y}(a x^{a-1} y^{b+1}) - y\frac{\partial}{\partial x}(b x^{a+1} y^{b-1}) \\
&= a(b+1)x^a y^b - b(a+1)x^a y^b \\
&= (ab + a - ba - b)x^a y^b \\
&= (a-b)x^a y^b
\end{aligned}$$

Thus (5.7) is established.

Next, we show

$$\rho([H, X]) = 2\rho(X) = [\rho(H), \rho(X)] \tag{5.8}$$

Again, the first equality is trivial since $[H, X] = 2X$. Once more, if $a, b \geq 0$ and $a + b = \lambda$ we have

$$\begin{aligned}
2\rho(X)(x^a y^b) &= 2x\frac{\partial}{\partial y}(x^a y^b) \\
&= 2b x^{a+1} y^{b-1}
\end{aligned}$$

and

$$\begin{aligned}
[\rho(H), \rho(X)] &= \left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right)\Big(x\frac{\partial}{\partial y}(x^a y^b)\Big) - x\frac{\partial}{\partial y}\Big(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\Big)(x^a y^b) \\
&= \Big(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\Big)(b x^{a+1} y^{b-1}) - x\frac{\partial}{\partial y}\big((a-b)x^a y^b\big) \\
&= (ab + 2b - b^2)x^{a+1} y^{b-1} - (ab - b^2)x^{a+1} y^{b-1} \\
&= 2b x^{a+1} y^{b-1}
\end{aligned}$$

A similar argument to this last case shows that $\rho([H,Y]) = [\rho(H),\rho(Y)]$. It follows that ρ is a Lie algebra morphism and hence is a representation of $\mathfrak{sl}(2)$. The matrices corresponding to $\rho(X),\rho(Y)$ and $\rho(H)$ can be ascertained in a manner similar to the adjoint representation above.

$$\begin{aligned}
\rho(X)x^\lambda &= 0x^\lambda + 0x^{\lambda-1}y + \ldots + 0xy^{\lambda-1} + 0y^\lambda, \quad \text{[column one]}\\
\rho(X)x^{\lambda-1}y &= x^\lambda + 0x^{\lambda-1}y + \ldots + 0xy^{\lambda-1} + 0y^\lambda, \quad \text{[column two]}\\
\rho(X)x^{\lambda-2}y^2 &= 0x^\lambda + 2x^{\lambda-1}y + \ldots + 0xy^{\lambda-1} + 0y^\lambda, \quad \text{[column three]}\\
&\vdots\\
\rho(X)xy^{\lambda-1} &= 0x^\lambda + \ldots + (\lambda-1)x^2y^{\lambda-2} + 0xy^{\lambda-1} + 0y^\lambda, \quad \text{[column } \lambda\text{]}\\
\rho(X)y^\lambda &= 0x^\lambda + \ldots + \lambda xy^{\lambda-1} + 0y^\lambda, \quad \text{[column } \lambda+1\text{]}
\end{aligned}$$

The associated matrix for $\rho(X)$ is therefore

$$\rho(X) = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0\\ 0 & 0 & 2 & \ldots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \ldots & \lambda\\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix} \tag{5.9}$$

Via similar computations one will find that

$$\rho(Y) = \begin{bmatrix} 0 & 0 & \ldots & 0 & 0\\ \lambda & 0 & \ldots & 0 & 0\\ 0 & \lambda-1 & \ldots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \ldots & 1 & 0 \end{bmatrix}, \quad \rho(H) = \begin{bmatrix} \lambda & 0 & \ldots & 0 & 0\\ 0 & \lambda-2 & \ldots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \ldots & 2-\lambda & 0\\ 0 & 0 & \ldots & 0 & -\lambda \end{bmatrix} \tag{5.10}$$

Notice that $\rho(X)$ is superdiagonal and $\rho(Y)$ is subdiagonal, while $\rho(H)$ is a diagonal matrix. Furthermore, from computing these matrices we have also discovered that x^λ is a highest weight vector, since $\rho(X)x^\lambda = 0$.

Now, because $\{x^\lambda, x^{\lambda-1}y, ..., xy^{\lambda-1}, y^\lambda\}$ is a basis of homogeneous polynomials for $V[\lambda]$, which are also eigenvectors for $\rho(H)$ with corresponding eigenvalues found on the diagonal of $\rho(H)$, we may conclude that $V[\lambda]$ is an irreducible representation of highest weight λ. Thus, any finite-dimensional irreducible $\mathfrak{sl}(2)$-module is isomorphic to a space of homogeneous polynomials of some fixed degree. But not only have we given a concrete realization of the $V[\lambda]$, we have also specified an action of $\mathfrak{sl}(2)$ on the affine plane (see next chapter) $\mathbb{C}[x,y]$. We will revisit this in the next chapter where $\mathbb{C}$ is replaced by an arbitrary field κ.

5.5.2 The Universal Enveloping Algebra of $\mathfrak{sl}(2)$

Since the universal enveloping algebra is defined in terms of the tensor algebra, let's start there. Of course, $V = \mathfrak{sl}(2)$ and so we have

$$T^0(\mathfrak{sl}(2)) = \mathbb{C}, \quad T^1(\mathfrak{sl}(2)) = \mathfrak{sl}(2), \quad T^2(\mathfrak{sl}(2)) = \mathfrak{sl}(2) \otimes \mathfrak{sl}(2)$$

and in general

$$T^n(\mathfrak{sl}(2)) = \mathfrak{sl}(2) \otimes \mathfrak{sl}(2) \otimes \ldots \otimes \mathfrak{sl}(2) \quad [n \text{ times}]$$

The tensor algebra of $\mathfrak{sl}(2)$, then, is

$$T(\mathfrak{sl}(2)) = \bigoplus_{n=0}^{\infty} T^n(\mathfrak{sl}(2))$$

Obviously, $\{1\}$ is basis for $T^0(\mathfrak{sl}(2))$ and $\{X, Y, H\}$ is a basis for $T^1(\mathfrak{sl}(2))$. From our work with tensor products we find that

$$\{X \otimes X, X \otimes Y, X \otimes H, Y \otimes X, Y \otimes Y, Y \otimes H, H \otimes X, H \otimes Y, H \otimes H\}$$

is a basis for $T^2(\mathfrak{sl}(2))$. To condense this notation we can suppress the "$\otimes$" per the convention described in Chapter 3 concerning the tensor algebra. So doing, the above becomes

$$\{XX, XY, XH, YX, YY, YH, HX, HY, HH\}$$

Recall that the product in the tensor algebra is:

$$(v_1 \otimes \ldots \otimes v_n)(v_{n+1} \otimes \ldots \otimes v_{n+m}) = v_1 \otimes \ldots \otimes v_n \otimes v_{n+1} \otimes \ldots \otimes v_{n+m}$$

where, in this case, $v_1, ..., v_n, v_{n+1}, ..., v_{n+m} \in \mathfrak{sl}(2)$. In our more concise notation, this becomes

$$(v_1 \cdots v_n)(v_{n+1} \cdots v_{n+m}) = v_1 \cdots v_n v_{n+1} \cdots v_{n+m}$$

So written, it is easy to see that

$$T(\mathfrak{sl}(2)) \cong \kappa\{X, Y, H\} \quad [\text{as algebras}]$$

which follows from Proposition 3.18.

Proposition 5.34. *The set $\{X^i Y^j H^k\}_{i,j,k \in \mathbb{N}}$ is a basis of $U(\mathfrak{sl}(2))$.*

Proof. This follows from the Poincaré-Birkhoff-Witt Theorem (see Theorem 5.22). $\square$

Using Theorem 5.22 is handy, but it doesn't give us a feel for *why* this is true, so let us embark on a brief, albeit indicative, exploration. Recall from Section 5.3 that the universal enveloping algebra of a Lie algebra $\mathfrak{L}$ is defined to be the quotient space

$$U(\mathfrak{L}) := T(\mathfrak{L})/I(\mathfrak{L})$$

where $I(\mathfrak{L})$ is the two-sided ideal generated by all elements

$$x \otimes y - y \otimes x - [x, y]$$

with $x, y \in \mathfrak{L}$. For $\mathfrak{L} = \mathfrak{sl}(2)$ we can use the quotient relation, along with the commutator relations of $\mathfrak{sl}(2)$, to order the basis of $U(\mathfrak{sl}(2))$ in the manner given in the above proposition. For instance, take the basis of $T^2(\mathfrak{sl}(2))$ given above. The only elements of the basis that don't fit the required ordering are YX, HX and HY. But in $U(\mathfrak{sl}(2))$ we have

$$YX = XY + [Y, X] = XY - H \tag{5.11}$$

$$HX = XH + [H, X] = XH + 2X \tag{5.12}$$

$$HY = YH + [H, Y] = YH - 2Y \tag{5.13}$$

So, each improperly ordered element can be written as a linear combination of properly ordered elements. These results can then be used to show that the same holds for all $T^n(\mathfrak{sl}(2))$. For instance, in the basis of $T^3(\mathfrak{sl}(2))$ one will find the element XHX, which is improperly ordered in $U(\mathfrak{sl}(2))$. However, using (5.11), in $U(\mathfrak{sl}(2))$ we have that

$$XHX = X(HX) = X(XH + 2X) = X^2H + 2X^2$$

Another consequence of the Poincaré-Birkhoff-Witt Theorem (Theorem 5.22) is the following:

Proposition 5.35. *The canonical map $i_{\mathfrak{sl}(2)} : \mathfrak{sl}(2) \to U(\mathfrak{sl}(2))$ is injective.*

This means that $\mathfrak{sl}(2)$ can be considered as a subspace of $U(\mathfrak{sl}(2))$. It is important to note, however, that in $U(\mathfrak{sl}(2))$ we are not thinking of X, Y, H as matrices anymore. For instance, as a matrix, $X^2 = 0$, but in $U(\mathfrak{sl}(2))$, $X^2 \neq 0$ (since we are actually taking the tensor product). This is essential, since there are numerous representations of $\mathfrak{sl}(2)$ in which $\rho(X)^2 \neq 0$. Besides this, the commutation relations for X, Y, H remain unchanged. We can also establish some other relations in $U(\mathfrak{sl}(2))$.

Proposition 5.36. *The following relations hold in $U(\mathfrak{sl}(2))$ for all $p, q \geq 0$:*

$$X^pH^q = (H-2p)^qX^p \tag{5.14}$$

$$Y^pH^q = (H+2p)^qY^p \tag{5.15}$$

$$[X, Y^p] = pY^{p-1}(H-p+1) = p(H+p-1)Y^{p-1} \tag{5.16}$$

$$[X^p, Y] = pX^{p-1}(H+p-1) = p(H-p+1)X^{p-1} \tag{5.17}$$

Proof. For brevity we'll show (5.13) and (5.15), since the other two will be similar. Beginning with (5.13), we first note that

$$\begin{aligned} XH &= HX + [X, H] \\ &= HX - 2X \\ &= (H-2)X \end{aligned}$$

which establishes a base case. Suppose, then, that

$$X^{p-1}H = (H - 2(p-1))X^{p-1}$$

Now consider

$$\begin{aligned} X^pH &= XX^{p-1}H \\ &= X(H-2(p-1))X^{p-1} \\ &= XHX^{p-1} - 2(p-1)X^p \\ &= (HX-2X)X^{p-1} - 2(p-1)X^p \\ &= HX^p - 2X^p - 2(p-1)X^p \\ &= (H-2-2(p-1))X^p \\ &= (H-2p)X^p \end{aligned}$$

So, by induction

$$X^pH = (H-2p)X^p \quad \text{for all } p$$

This provides the base case for our next induction. Suppose now that

$$X^pH^{q-1} = (H-2p)^{q-1}X^p$$

and consider

$$\begin{aligned}
X^pH^q &= X^pH^{q-1}H \\
&= (H-2p)^{q-1}X^pH \\
&= (H-2p)^{q-1}(H-2p)X^p \\
&= (H-2p)^qX^p
\end{aligned}$$

Thus, by induction (5.13) holds.

For (5.15) we have the base case

$$[X,Y] = H = 1Y^0(H-1+1) = 1(H+1-1)Y^0$$

so suppose that

$$[X,Y^{p-1}] = (p-1)Y^{p-2}(H-(p-1)+1) = (p-1)(H+(p-1)-1)Y^{p-2}$$

We'll show that the first equality holds, since, again, the right will follow by similar reasoning. For practical purposes, we'll express this as

$$XY^{p-1} - Y^{p-1}X = (p-1)Y^{p-2}(H-p+2)$$

We then see that

$$\begin{aligned}
XY^p - Y^pX &= XYY^{p-1} - Y^pX && (5.18)\\
&= (H+YX)Y^{p-1} - Y^pX && (5.19)\\
&= HY^{p-1} + YXY^{p-1} - Y^pX && (5.20)\\
& HY^{p-1} + Y((p-1)Y^{p-2}(H-p+2) + Y^{p-1}X) - Y^pX && (5.21)\\
&= HY^{p-1} + (p-1)Y^{p-1}(H-p+2) + Y^pX - Y^pX && (5.22)\\
&= Y^{p-1}H - 2(p-1)Y^{p-1} + (p-1)Y^{p-1}(H-p+2) && (5.23)\\
&= Y^{p-1}(H-2p+2+pH-p^2+2p-H+p-2) && (5.24)\\
&= pY^{p-1}(H-p+1) && (5.25)
\end{aligned}$$

Step (5.18) makes use of the fact that $XY - YX = H$. Step (5.20) uses the induction hypothesis and step (5.22) uses (5.14). Thus, the desired result holds by induction. □

Notice that the equations of Proposition 5.36 are reminiscent of derivatives. Indeed, these equations hold for $\mathbb{C}[x,y]$ when X, Y and H are replaced by (or act by) $x\frac{\partial}{\partial y}, y\frac{\partial}{\partial x}$

and $x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$ respectively. In fact, we will see this again in Chapter 6 in the slightly more general setting of $\kappa[x, y]$.

5.5.3 Duality

If H is a finite-dimensional Hopf algebra, then we have essentially seen (especially in Chapter 4) that there is a corresponding dual Hopf algebra H^*. We can think of H and H^* as being "symmetric" in the sense that $H \cong H^{**}$ in a natural way and the algebra structure of H determines a coalgebra structure on H^* while the coalgebra structure of H determines an algebra structure on H^*. Because of this symmetry, we can express the dual Hopf structure of H^* in terms of a pairing. That is, instead of writing $f(x)$ for evaluation of a map $f \in H^*$ at $x \in H$ we express it as $\langle f, x\rangle$. So, since $\nabla = \Delta^*$ for H^* we get the relation

$$\langle \nabla(f \otimes g), x\rangle = \langle f \otimes g, \Delta(x)\rangle$$

Also, since $\Delta = \nabla^*$ for H^* we get

$$\langle \Delta(f), x \otimes y\rangle = \langle f, \nabla(x \otimes y)\rangle$$

Now, since $\eta(1)$ is the unit in H^* we also get

$$\langle 1, x\rangle = \varepsilon(x)$$

and $\varepsilon = \eta^*$ for H^* so

$$\varepsilon(f) = \langle f, 1\rangle$$

Finally, the antipode for H^* is S^* and therefore

$$\langle S(f), x\rangle = \langle f, S(x)\rangle$$

We now generalize from H and H^* to an arbitrary pair of bialgebras (or Hopf algebras) U and W, where we will require that these same relations be satisfied in order to define a more general kind of duality.

Firstly, given a bilinear form $\langle , \rangle : U \times W \to \kappa$ we get an induced bilinear form $\langle , \rangle : U \otimes U \times W \otimes W \to \kappa$ defined by

$$\langle u \otimes v, w \otimes x\rangle := \langle u, w\rangle\langle v, x\rangle$$

With this, we proceed to investigate a key duality relationship between $U(\mathfrak{sl}(2))$ and $SL(2)$ considered as Hopf algebras. We start with a definition of our more general duality.

Definition 5.37. Given bialgebras $(U, \nabla, \eta, \Delta, \varepsilon)$ and $(W, \nabla, \eta, \Delta, \varepsilon)$ and a non-degenerate bilinear form $\langle , \rangle : U \times W \to \kappa$, we say that the bilinear form realizes a duality between U and W if we have

$$\langle uv, x\rangle = \sum_{(x)} \langle u, x^{(1)}\rangle\langle v, x^{(2)}\rangle = \langle u \otimes v, \Delta(x)\rangle \tag{5.26}$$

$$\langle u, xy\rangle = \sum_{(u)} \langle u^{(1)}, x\rangle\langle u^{(2)}, y\rangle = \langle \Delta(u), x \otimes y\rangle \tag{5.27}$$

$$\langle 1, x\rangle = \varepsilon(x) \tag{5.28}$$

$$\langle u, 1\rangle = \varepsilon(u) \tag{5.29}$$

for all $u, v \in U$ and $x, y \in W$ and where we are using Sweedler's convention in (5.26) and (5.27). This bilinear form can be turned into a linear functional $\langle\rangle : U \otimes V \to \kappa$ where all the same relations hold, but we are now using the tensor product.

If it happens that U and W are also Hopf algebras with antipode S, then, additionally, the bilinear form must satisfy

$$\langle S(u), x\rangle = \langle u, S(x)\rangle \tag{5.30}$$

for all $u \in U$ and $x \in W$.

By way of reminder, we can understand this bilinear form in terms of certain linear maps. If U^* and W^* are the dual spaces of U and W respectively, then let $\phi : U \to W^*$ be the linear map such that $\phi(u) = \langle u, -\rangle$. Similarly, let $\psi : W \to U^*$ be the linear map such that $\psi(x) = \langle -, x\rangle$. In other words, $\langle u, -\rangle$ is a linear functional on U and $\langle -, x\rangle$ is a linear functional on W. If it should happen that both ϕ and ψ are injective, then we shall say that the duality between U and W is *perfect*. Furthermore, if U and W are finite-dimensional, then a perfect duality between U and W entails that

$$U \cong W^* \quad \text{and} \quad W \cong U^* \quad \text{as bialgebras}$$

We characterize these ideas in the following proposition.

Proposition 5.38. *Given bialgebras U and W and a bilinear form $\langle , \rangle$ on $U \times W$, the bilinear form realizes a duality between U and W if and only if the linear maps ϕ and ψ are morphisms of algebras. In case W is finite-dimensional, the bilinear form realizes a duality if and only if ϕ is a morphism of bialgebras.*

Proof. Suppose that ϕ and ψ are algebra morphisms. This makes sense since U^* and W^* carry natural algebra structures. By way of reminder, this is expressed in the form of commuting diagrams by

$$\begin{tikzcd}
U \otimes U \arrow[rr, "\phi \otimes \phi"] \arrow[dd, "\mu_U"'] & & W^* \otimes W^* \arrow[dd, "\mu_{W^*}"] \\
& \kappa \arrow[dl, "\eta_U"'] \arrow[dr, "\eta_{W^*}"] & \\
U \arrow[rr, "\phi"] & & W^*
\end{tikzcd}
\qquad
\begin{tikzcd}
W \otimes W \arrow[rr, "\psi \otimes \psi"] \arrow[dd, "\mu_W"'] & & U^* \otimes U^* \arrow[dd, "\mu_{U^*}"] \\
& \kappa \arrow[dl, "\eta_W"'] \arrow[dr, "\eta_{U^*}"] & \\
W \arrow[rr, "\psi"] & & U^*
\end{tikzcd}$$

Using this property we find that

$$\begin{aligned}
\langle uv, x\rangle &= \phi(uv)(x) \\
&= \big(\phi(u)\phi(v)\big)(x) \quad [\phi \text{ is algebra morphism}] \\
&= \sum_{(x)} \phi(u)(x^{(1)})\phi(v)(x^{(2)}) \quad [\text{product in } H^*] \\
&= \sum_{(x)} \langle u, x^{(1)}\rangle\langle v, x^{(2)}\rangle
\end{aligned}$$

showing that (5.26) holds. Also, $\langle 1, x\rangle = \phi(1)(x) = 1(x) = \varepsilon(x)$, since the unit of W^* is the counit of W, so (5.28) holds. A symmetric argument shows that (5.27) and (5.29) hold. Thus $\langle , \rangle$ realizes a duality between U and W. Notice, too, that the above argument is exactly reversible so that the converse is immediately verified.

Now suppose that W is finite-dimensional. Then, by Proposition 4.5, W^* is a bialgebra as well. Let us assume that ϕ is a morphism of bialgebras. By definition, this means that ϕ is simultaneously a morphism of algebras and a morphism of coalgebras. It suffices to show that the property of being a coalgebra morphism gives (5.27) and (5.29), since we have already shown that being an algebra morphism gives (5.26) and (5.28). Again, we make use of the commuting diagram for a coalgebra morphism (see Definition 3.23).

$$\begin{tikzcd}
U \otimes U \arrow[rr, "\phi \otimes \phi"] & & W^* \otimes W^* \\
& \kappa & \\
U \arrow[rr, "\phi"] \arrow[uu, "\Delta_U"] \arrow[ur, "\varepsilon_U"] & & W^* \arrow[uu, "\Delta_{W^*}"'] \arrow[ul, "\varepsilon_{W^*}"']
\end{tikzcd}$$

Following the diagram yields

$$\begin{aligned}
\langle u, xy\rangle &= \phi(u)(xy) \\
&= \Delta\big(\phi(u)\big)(x \otimes y) \\
&= \sum_{(\phi(u))} \phi(u)^{(1)}(x)\phi(u)^{(2)}(y) \\
&= \sum_{(u)} \phi(u^{(1)})(x)\phi(u^{(2)})(y) \quad [\phi \text{ is coalgebra morphism}] \\
&= \sum_{(u)} \langle u^{(1)}, x\rangle\langle u^{(2)}, y\rangle
\end{aligned}$$

which is (5.27). For (5.29) we see that

$$\varepsilon_U(u) = \varepsilon_{W^*}(\phi(u)) = (\varepsilon_{W^*} \circ \phi)(u) = \phi(u)(1) = \langle u, 1\rangle$$

Again, this argument is directly reversible so that the converse holds. This finishes the proof. □

We showed, in Proposition 5.21, that the enveloping algebra $U(\mathfrak{L})$ is a cocommutative Hopf algebra. From this it follows that $U(\mathfrak{sl}(2))$ is a Hopf algebra which entails being a bialgebra. We now invite the polynomial algebra $M(2) := \kappa[a, b, c, d]$ back to the scene. This is the bialgebra we used to derive the bialgebra structures of $GL(2)$ and $SL(2)$. Recall that we were able to represent Δ for $M(2)$ symbolically by the matrix relation

$$\Delta \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

which encodes the relations

$$\begin{aligned}
\Delta(a) &= a \otimes a + b \otimes c \\
\Delta(b) &= a \otimes b + b \otimes d \\
\Delta(c) &= c \otimes a + d \otimes c \\
\Delta(d) &= c \otimes b + d \otimes d
\end{aligned}$$

So, for $f \in M(2)$, f is a polynomial in a, b, c, d, where we shall write

$$f \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

in place of $f(a,b,c,d)$. With Δ as an algebra morphism we then have

$$\Delta f = f(\Delta(a),\Delta(b),\Delta(c),\Delta(d)) = f\begin{bmatrix}\Delta(a) & \Delta(b)\\ \Delta(c) & \Delta(d)\end{bmatrix}$$

In order to obtain the desired duality between $U(\mathfrak{sl}(2))$ and $SL(2)$ we shall endeavor to devise an algebra morphism $\psi : M(2) \to U(\mathfrak{sl}(2))^*$ from which will be deduced a bilinear form on $U(\mathfrak{sl}(2)) \times M(2)$ defined by $\langle u, f\rangle := \psi(f)(u)$ satisfying (5.27) and (5.29).

Because $U(\mathfrak{sl}(2))$ is cocommutative, $U(\mathfrak{sl}(2))^*$ is commutative and so, from Chapter 3

$$\hom_{Alg}(M(2), U(\mathfrak{sl}(2))^*) \equiv \big(U(\mathfrak{sl}(2))^*\big)^4 \equiv M_2\big(U(\mathfrak{sl}(2))^*\big)$$

Each $\psi \in \hom_{Alg}\big(M(2), U(\mathfrak{sl}(2))^*\big)$ is equivalent to giving a "point" (A,B,C,D) where $A,B,C,D \in U(\mathfrak{sl}(2))^*$. Alternatively, we have that ψ corresponds to the matrix

$$\begin{bmatrix}\psi(a) & \psi(b)\\ \psi(c) & \psi(d)\end{bmatrix} = \begin{bmatrix}A & B\\ C & D\end{bmatrix}$$

The goal, then, is to find appropriate A,B,C,D so that the associated ψ gives the desired bilinear form. In other words, for $f \in M(2)$ we will have $\psi(f) \in U(\mathfrak{sl}(2))^*$ so that $\psi(f)(u) \in \kappa$ allowing us to set $\langle u, f\rangle := \psi(f)(u)$.

To aid us in defining A,B,C,D we make use of the representations given in (5.9) and (5.10). More specifically, we are interested in $\rho(1)$ which is just the natural embedding of $\mathfrak{sl}(2)$ into $\mathfrak{gl}(2)$. Hence

$$\rho(1)(X) = X, \quad \rho(1)(Y) = Y \quad \text{and} \quad \rho(1)(H) = H$$

This embedding has a unique extension to an algebra morphism $\rho(1) : U(\mathfrak{sl}(2)) \to \mathfrak{gl}(2)$:

$$\begin{tikzcd}
\mathfrak{sl}(2) \arrow[r, hook, "\rho(1)"] \arrow[d, hook] & \mathfrak{gl}(2) \\
U(\mathfrak{sl}(2)) \arrow[ur, "\rho(1)"'] &
\end{tikzcd}$$

So, $\rho(1)$ sends $u \in U(\mathfrak{sl}(2))$ to a matrix $\rho(1)(u) \in \mathfrak{gl}(2)$ called

$$\begin{bmatrix}A(u) & B(u)\\ C(u) & D(u)\end{bmatrix}$$

This defines four linear forms A, B, C and D on $U(\mathfrak{sl}(2))$ and so we get a corresponding algebra morphism which assigns

$$\begin{aligned} a &\mapsto A \quad & b &\mapsto B \\ c &\mapsto C \quad & d &\mapsto D \end{aligned}$$

This will be our ψ, and hence, for $f \in M(2)$ we have

$$f(a,b,c,d) \mapsto f\big(A(u), B(u), C(u), D(u)\big) = \psi(f)(u) =: \langle u, f\rangle$$

Now, since ψ is an algebra morphism, it is also linear, and because ψ sends elements of $M(2)$ to linear functionals, we see that the result is indeed a *bilinear* form.

Proposition 5.39. *The bilinear form $\langle u, f\rangle = \psi(f)(u)$ realizes an imperfect duality between the bialgebras $U(\mathfrak{sl}(2))$ and $M(2)$.*

Proof. Our first task is to show that

$$\langle uv, f\rangle = \langle u \otimes v, \Delta f\rangle$$

For the left hand side we have

$$\begin{aligned} \langle uv, f\rangle &= \psi(f)(uv) \\ &= f\begin{bmatrix} A(uv) & B(uv) \\ C(uv) & D(uv) \end{bmatrix} \end{aligned}$$

Now,

$$\begin{aligned} \begin{bmatrix} A(uv) & B(uv) \\ C(uv) & D(uv) \end{bmatrix} &= \rho(1)(uv) \\ &= \rho(1)(u)\rho(1)(v) \quad [\rho(1) \text{ an alg. morphism}] \\ &= \begin{bmatrix} A(u) & B(u) \\ C(u) & D(u) \end{bmatrix} \cdot \begin{bmatrix} A(v) & B(v) \\ C(v) & D(v) \end{bmatrix} \end{aligned}$$

So

$$\langle uv, f\rangle = f\begin{bmatrix} A(u) & B(u) \\ C(u) & D(u) \end{bmatrix} \cdot \begin{bmatrix} A(v) & B(v) \\ C(v) & D(v) \end{bmatrix}$$

On the right hand side we get

$$\begin{aligned}
\langle u \otimes v, \Delta f \rangle &= \langle u \otimes v, \Delta f \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rangle \\
&= \langle u \otimes v, f \begin{bmatrix} \Delta(a) & \Delta(b) \\ \Delta(c) & \Delta(d) \end{bmatrix} \rangle \\
&= f \begin{bmatrix} \langle u \otimes v, \Delta(a) \rangle & \langle u \otimes v, \Delta(b) \rangle \\ \langle u \otimes v, \Delta(c) \rangle & \langle u \otimes v, \Delta(d) \rangle \end{bmatrix}
\end{aligned}$$

This last equality holds because $f(\Delta(a), \Delta(b), \Delta(c), \Delta(d))$ is a polynomial in $\Delta(a), \Delta(b), \Delta(c)$ and $\Delta(d)$ and $\langle , \rangle$ is bilinear.

Let us now examine the entries. For instance,

$$\begin{aligned}
\langle u \otimes v, \Delta(a) \rangle &= \langle u \otimes v, a \otimes a + b \otimes c \rangle \\
&= \langle u \otimes v, a \otimes a \rangle + \langle u \otimes v, b \otimes c \rangle \\
&= \langle u, a \rangle \langle v, a \rangle + \langle u, b \rangle \langle v, c \rangle \\
&= \psi(a)(u)\psi(a)(v) + \psi(b)(u)\psi(c)(v) \\
&= A(u)A(v) + B(u)C(v)
\end{aligned}$$

Similar computations yield

$$\begin{aligned}
\langle u \otimes v, \Delta f \rangle &= f \begin{bmatrix} A(u)A(v) + B(u)C(v) & A(u)B(v) + B(u)D(v) \\ C(u)A(v) + D(u)C(v) & C(u)B(v) + D(u)D(v) \end{bmatrix} \\
&= f \begin{bmatrix} A(u) & B(u) \\ C(u) & D(u) \end{bmatrix} \cdot \begin{bmatrix} A(v) & B(v) \\ C(v) & D(v) \end{bmatrix}
\end{aligned}$$

thereby establishing that $\langle uv, f \rangle = \langle u \otimes v, \Delta f \rangle$.

Now, because $\rho(1)(1) = 1$ we have

$$\begin{aligned}
\langle 1, f \rangle &= \psi(f)(1) \\
&= f \begin{bmatrix} A(1) & B(1) \\ C(1) & D(1) \end{bmatrix} \\
&= f \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\end{aligned}$$

But also

$$\begin{aligned}
\varepsilon(f) &= \varepsilon f \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\
&= f \begin{bmatrix} \varepsilon(a) & \varepsilon(b) \\ \varepsilon(c) & \varepsilon(d) \end{bmatrix} \\
&= f \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\end{aligned}$$

So we also have $\langle 1, f\rangle = \varepsilon(f)$.

The other required duality relations: $\langle u, fg\rangle = \langle \Delta(u), f \otimes g\rangle$ and $\langle u, 1\rangle = \varepsilon(u)$ follow from Proposition 5.38, since $\langle , \rangle$ was defined using the algebra morphism ψ.

$\square$

The reason the duality established above is not perfect is because ψ is not injective. This is not immediately obvious, but we will show that $\psi(ad-bc) = 1$ in addition to $\psi(1) = 1$. This isn't so bad because it means that ψ factors through $SL(2) = M(2)/(ad - bc - 1)$.

On our way to establishing this, let us begin with some more general considerations. Let x be a grouplike element. Then

$$\begin{aligned}
\langle uv, x\rangle &= \langle u \otimes v, \Delta(x)\rangle \\
&= \langle u \otimes v, x \otimes x\rangle \\
&= \langle u, x\rangle\langle v, x\rangle
\end{aligned}$$

But also, $\langle 1, x\rangle = \varepsilon(x) = 1$, since x is grouplike, and therefore, $\langle -, x\rangle$ is an algebra morphism whenever x is grouplike. Likewise, $\langle u, -\rangle$ is an algebra morphism whenever u is grouplike.

Now, to say that $\psi(x) = 1$ really means that ψ is the unit of the algebra $U(\mathfrak{sl}(2))^*$, namely ε, since the algebra structure comes from the coalgebra structure on $U(\mathfrak{sl}(2))$. Therefore, $\psi(x) = 1$ is equivalent to $\langle u, x\rangle = \varepsilon(u)$ for all $u \in U(\mathfrak{sl}(2))$. When x is grouplike it suffices to verify this for the generating set $1, X, Y$ and H of the algebra $U(\mathfrak{sl}(2))$.

In this particular case we have that $x = ad - bc$, which we know to be grouplike. Thus, because ε is the unique algebra morphism extending the zero map - i.e.

$$\begin{tikzcd}
\mathfrak{sl}(2) \arrow[rr, "0"] \arrow[d, hook] & & \kappa \\
U(\mathfrak{sl}(2)) \arrow[urr, "\varepsilon"'] & &
\end{tikzcd}$$

we need

$$\begin{aligned}
\langle 1, ad - bc\rangle &= \varepsilon(1) = 1 \\
\langle X, ad - bc\rangle &= \varepsilon(X) = 0 \\
\langle Y, ad - bc\rangle &= \varepsilon(Y) = 0 \\
\langle H, ad - bc\rangle &= \varepsilon(H) = 0
\end{aligned}$$

For computational purposes, recall that

$$\begin{aligned}
\rho(1)(X) = X &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A(X) & B(X) \\ C(X) & D(X) \end{bmatrix} \\
\rho(1)(Y) = Y &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} A(Y) & B(Y) \\ C(Y) & D(Y) \end{bmatrix} \\
\rho(1)(H) = H &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} A(H) & B(H) \\ C(H) & D(H) \end{bmatrix}
\end{aligned}$$

Then

$$\begin{aligned}
\psi(ad - bc)(X) &= \langle X, ad - bc\rangle \\
&= \langle X, ad\rangle - \langle X, bc\rangle \\
&= \langle \Delta(X), a \otimes d\rangle - \langle \Delta(X), b \otimes c\rangle \\
&= \langle X \otimes 1 + 1 \otimes X, a \otimes d\rangle - \langle X \otimes 1 + 1 \otimes X, b \otimes c\rangle \\
&= \langle X \otimes 1, a \otimes d\rangle + \langle 1 \otimes X, a \otimes d\rangle - \langle X \otimes 1, b \otimes c\rangle - \langle 1 \otimes X, b \otimes c\rangle \\
&= \langle X, a\rangle\langle 1, d\rangle + \langle 1, a\rangle\langle X, d\rangle - \langle X, b\rangle\langle 1, c\rangle - \langle 1, b\rangle\langle X, c\rangle \\
&= A(X)D(1) + A(1)D(X) - B(X)C(1) - B(1)C(X) \\
&= 0 \cdot 1 + 1 \cdot 0 - 1 \cdot 0 - 0 \cdot 0 \\
&= 0 = \varepsilon(X)1
\end{aligned}$$

Via similar calculations we also have

$$\begin{aligned}\psi(ad-bc)(Y) &= A(Y)D(1)+A(1)D(Y)-B(Y)C(1)-B(1)C(Y)\\ &= 0\cdot 1+1\cdot 0-0\cdot 0-0\cdot 1\\ &= 0=\varepsilon(Y)1\end{aligned}$$

and

$$\begin{aligned}\psi(ad-bc)(H) &= A(H)D(1)+A(1)D(H)-B(H)C(1)-B(1)C(H)\\ &= 1\cdot 1+1\cdot(-1)-0\cdot 0-0\cdot 0\\ &= 0=\varepsilon(H)1\end{aligned}$$

Finally,

$$\begin{aligned}\psi(ad-bc)(1) &= A(1)D(1)-B(1)C(1)\\ &= 1\cdot 1-0\cdot 0\\ &= 1=\varepsilon(1)1\end{aligned}$$

The last one we could simply have stated, since $ad-bc$ is grouplike, but the computation is interesting to see.

So, we have succeeded in showing that $\psi(ad-bc)=1$ and therefore it cannot be injective. Nevertheless, we can use this fact to establish a duality between $U(\mathfrak{sl}(2))$ and $SL(2)$. As a reminder, the antipode S for $SL(2)$ is

$$S\begin{bmatrix}a & b\\ c & d\end{bmatrix}=\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$$

and the antipode S for $U(\mathfrak{sl}(2))$ is given by Proposition 5.21.

Theorem 5.40. *The bilinear form $\langle u,x\rangle=\psi(x)(u)$ realizes a duality between the Hopf algebras $U(\mathfrak{sl}(2))$ and $SL(2)$.*

Proof. We use ψ here as the induced morphism of algebras $SL(2)\to U(\mathfrak{sl}(2))^*$ from the previous proposition. Being a morphism of algebras means that conditions (5.27) and (5.29) already hold. Also, from the previous proposition we get that

$$\phi: U(\mathfrak{sl}(2))\to M(2)^*$$

defined by $\phi(u)(f)=\langle u,f\rangle$, is an algebra morphism. Now, the dual of the natural projection $\pi: M(2)\to SL(2)$ is the natural injection $\iota: SL(2)^*\to M(2)^*$, which is a morphism of algebras. We therefore get an induced algebra morphism

$$\phi' : U(\mathfrak{sl}(2)) \to SL(2)^*$$

such that $\phi = \iota \circ \phi'$. We therefore have a duality of bialgebras.

Because we want a duality between Hopf algebras, we need the extra condition expressed in (5.30). We begin by showing that (5.30) holds for the generators $1, X, Y, H$ of $U(\mathfrak{sl}(2))$. Since the reasoning for X, Y and H is essentially the same, we will concern our selves with 1 and X only.

$$\begin{aligned} \langle S(1), f \rangle &= \langle 1, f \rangle \\ &= \varepsilon(f) \end{aligned}$$

and

$$\begin{aligned} \langle 1, S(f) \rangle &= \varepsilon(S(f)) \\ &= \varepsilon(f) \quad \text{[by Proposition 4.13]} \end{aligned}$$

Now, for X we have (and similarly for Y and H)

$$\begin{aligned} \langle S(X), f \rangle &= \langle -X, f \rangle \\ &= -\langle X, f \rangle \\ &= -f \begin{bmatrix} A(X) & B(X) \\ C(X) & D(X) \end{bmatrix} \\ &= -f \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned}
\langle X, S(f)\rangle &= \langle X, Sf \begin{bmatrix} a & b \\ c & d \end{bmatrix}\rangle \\
&= \langle X, f \begin{bmatrix} S(a) & S(b) \\ S(c) & S(d) \end{bmatrix}\rangle \\
&= \langle X, f \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\rangle \\
&= f \begin{bmatrix} D(X) & -B(X) \\ -C(X) & A(X) \end{bmatrix} \\
&= f \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \\
&= -f \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}
\end{aligned}$$

Next, suppose $u, v \in U(\mathfrak{sl}(2))$ are such that (5.30) holds for all $f \in SL(2)$. Then

$$\begin{aligned}
\langle S(uv), f\rangle &= \langle S(v)S(u), f\rangle \\
&= \langle S(v) \otimes S(u), \Delta f \begin{bmatrix} a & b \\ c & d \end{bmatrix}\rangle \\
&= \langle S(v) \otimes S(u), f \begin{bmatrix} \Delta(a) & \Delta(b) \\ \Delta(c) & \Delta(d) \end{bmatrix}\rangle \\
&= f \begin{bmatrix} \langle S(v) \otimes S(u), \Delta(a)\rangle & \langle S(v) \otimes S(u), \Delta(b)\rangle \\ \langle S(v) \otimes S(u), \Delta(c)\rangle & \langle S(v) \otimes S(u), \Delta(d)\rangle \end{bmatrix}
\end{aligned}$$

Consider, for instance, $\langle S(v) \otimes S(u), \Delta(a)\rangle$. We have

$$\begin{aligned}
\langle S(v) \otimes S(u), \Delta(a)\rangle &= \langle S(v) \otimes S(u), a \otimes a + b \otimes c\rangle \\
&= \langle S(v) \otimes S(u), a \otimes a\rangle + \langle S(v) \otimes S(u), b \otimes c\rangle \\
&= \langle S(v), a\rangle\langle S(u), a\rangle + \langle S(v), b\rangle\langle S(u), c\rangle \\
&= \langle v, S(a)\rangle\langle u, S(a)\rangle + \langle v, S(b)\rangle\langle u, S(c)\rangle \\
&= \langle v, d\rangle\langle u, d\rangle + \langle v, -b\rangle\langle u, -c\rangle \\
&= D(v)D(u) + B(v)C(u)
\end{aligned}$$

Thus, via similar computations we get

$$\langle S(uv), f\rangle = f \begin{bmatrix} D(v)D(u) + B(v)C(u) & -D(v)B(u) - B(v)A(u) \\ -C(v)D(u) - A(v)C(u) & C(v)B(u) + A(v)A(u) \end{bmatrix}$$

Compare this to

$$\begin{aligned}
\langle uv, S(f)\rangle &= \langle uv, f\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\rangle \\
&= \langle u\otimes v, f\begin{bmatrix} \Delta(d) & -\Delta(b) \\ -\Delta(c) & \Delta(a) \end{bmatrix}\rangle \\
&= f\begin{bmatrix} \langle u\otimes v, \Delta(d)\rangle & -\langle u\otimes v, \Delta(b)\rangle \\ -\langle u\otimes v, \Delta(c)\rangle & \langle u\otimes v, \Delta(a)\rangle \end{bmatrix} \\
&= f\begin{bmatrix} C(u)B(v)+D(u)D(v) & -A(u)B(v)-B(u)D(v) \\ -C(u)A(v)-D(u)C(v) & A(u)A(v)+B(u)C(v) \end{bmatrix}
\end{aligned}$$

and hence

$$\langle S(uv), f\rangle = \langle uv, S(f)\rangle$$

Therefore, since we have shown that (5.30) holds for the generators of $U(\mathfrak{sl}(2))$ we now have that (5.30) holds for all products of these generators as well, implying that (5.30) holds in general and we therefore have our Hopf algebra duality. $\square$

Most of what we have here done is fairly abstract, even the computations. Let us, therefore, consider an easy example. For instance,

$$\begin{aligned}
\langle XY, \begin{bmatrix} a & b \\ c & d \end{bmatrix}\rangle &= \begin{bmatrix} \langle XY, a\rangle & \langle XY, b\rangle \\ \langle XY, c\rangle & \langle XY, d\rangle \end{bmatrix} \\
&= \begin{bmatrix} \langle X\otimes Y, \Delta(a)\rangle & \langle X\otimes Y, \Delta(b)\rangle \\ \langle X\otimes Y, \Delta(c)\rangle & \langle X\otimes Y, \Delta(d)\rangle \end{bmatrix} \\
&= \langle X\otimes Y, \begin{bmatrix} \Delta(a) & \Delta(b) \\ \Delta(c) & \Delta(d) \end{bmatrix}\rangle \\
&= \langle X\otimes Y, \begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{bmatrix} a & b \\ c & d \end{bmatrix}\rangle \\
&= \langle X, \begin{bmatrix} a & b \\ c & d \end{bmatrix}\rangle\langle Y, \begin{bmatrix} a & b \\ c & d \end{bmatrix}\rangle \\
&= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}
\end{aligned}$$

This is the same result we get computing the entries manually. For instance, we have

$$\begin{aligned}\langle X \otimes Y, \Delta(a)\rangle &= \langle X \otimes Y, a \otimes a + b \otimes c\rangle \\ &= \langle X \otimes Y, a \otimes a\rangle + \langle X \otimes Y, b \otimes c\rangle \\ &= \langle X, a\rangle\langle Y, a\rangle + \langle X, b\rangle\langle Y, c\rangle \\ &= A(X)A(Y) + B(X)C(Y) \\ &= 0 \cdot 0 + 1 \cdot 1 = 1\end{aligned}$$

One performs the other computations similarly.

This concludes our brief tour of the theory of Lie algebras, a rich and deep subject with important connections to the topic at hand. We have here highlighted one the most important connections, namely to Hopf algebras. One cannot overstate the importance of the universal enveloping algebra, since this provides a functor from the category of Lie algebras into the category of Hopf algebras. Not only this, but the most important Hopf algebras arise in this way. Furthermore, we also explored an interesting duality between $U(\mathfrak{sl}(2))$ and $SL(2)$, which will be relevant to Chapter 7. At this point, we now proceed to enter the "rabbit hole" of the quantum realm as we next study deformations of classical objects, which will lead us to our first examples of quantum groups.

Chapter 6

Deformation Quantization: The Quantum Plane and Other Deformed Spaces

6.1 Introduction

The first five chapters of this thesis have laid the groundwork for understanding what we shall do here and in the next chapter. This gives an idea of just how much background is needed to really engage the topic of quantum groups. The present chapter will ease us into the subject by first considering an important example of deformation quantization of a classical object into a quantum one, namely the quantum plane. More than just an illustration, however, the quantum plane is important in its own right. It is heavily studied in physics and is acted upon by certain interesting quantum groups. The remainder of the chapter will be devoted to two examples of such quantum groups.

6.2 The Affine Line and Plane

Perhaps the two most familiar notions in Euclidean geometry and calculus, as far as spaces go, are the line and the plane. At least part of their familiarity is due to the ease for which they can be visualized. Of course, almost nothing in mathematics is safe from generalization and in this chapter we begin with an interesting generalization of the line and plane called the *affine line* and *affine plane*. The adjective "affine" refers to a more general connection to affine spaces. In short, an affine space is an abstract structure which generalizes certain ("affine") geometric properties of Euclidean space. To get an idea of what this means, an affine geometry is one involving no notions of length, angle or an origin. In other words, there are no "preferred" points in an affine space. Note, then, that a general affine space, unlike its Euclidean counterpart, ceases to be a metric space.

6.2.1 The Affine Line

It is now time that we revisit the material introduced in the section on free algebras in Chapter 3. Specifically, we wish to utilize the notion of "points" in the context of two key examples. This section is devoted to the affine line, a name highlighting the duality between algebra and geomtry. We begin with a proposition connecting back to the universal property of free algebras.

Proposition 6.1. *Let A be a commutative algebra and f a function from the finite set $\{x_1, ..., x_n\}$ to A. Then there exists a unique morphism of algebras $\overline{f} : \kappa[x_1, ..., x_n] \to A$ such that $\overline{f}(x_i) = f(x_i)$ for all i.*

Proof. From the universal property of free algebras (see Theorem 3.6) there is a unique algebra morphism $\underline{f} : \kappa\{x_1, ..., x_n\} \to A$ such that $\underline{f}(x_i) = f(x_i)$ for all i. Recall that

$$\kappa[x_1, ..., x_n] \cong \kappa\{x_1, ..., x_n\}/I$$

where I is the two-sided ideal of $\kappa\{x_1, ..., x_n\}$ generated by all elements of the form $x_i x_j - x_j x_i$. Observe that

$$\begin{aligned} \underline{f}(x_i x_j - x_j x_i) &= \underline{f}(x_i)\underline{f}(x_j) - \underline{f}(x_j)\underline{f}(x_i) \\ &= 0 \quad [\text{since } A \text{ is commutative}] \end{aligned}$$

Therefore $I \subseteq Ker(\underline{f})$ and hence $\underline{f}$ induces an algebra morphism $\overline{f} : \kappa[x_1, ..., x_n] \to A$ with $\overline{f}(x_i) = f(x_i)$ for all i, which is clearly unique. □

In Chapter 3, the universal property allowed us to say that

$$\hom_{Alg}(\kappa[x_1, ..., x_n], A) \equiv \{(a_1, ..., a_n) \in A^n : a_i a_j - a_j a_i \text{ for all } (i, j)\}$$

for *any* algebra A. The above proposition, then, implies that

$$\hom_{Alg}(\kappa[x_1, ..., x_n], A) \equiv A^n$$

where A is commutative. What this means is that giving a morphism of algebras from the polynomial algebra $\kappa[x_1, ..., x_n]$ to the commutative algebra A is equivalent to giving an n-point $(a_1, ..., a_n)$ of A^n. In particular, if $n = 1$, then we have

$$\hom_{Alg}(\kappa[x], A) \equiv A$$

In general, the functor $\hom_{Alg}(\kappa[x_1, ..., x_n], -)$ is from the category of commutative algebras to the category of sets, where $A \mapsto A^n$ and a morphism of algebras, $f : A \to B$, gets sent to a componentwise function $A^n \to B^n$. This functor is said to be represented by the algebra $\kappa[x_1, ..., x_n]$ or that $\kappa[x_1, ..., x_n]$ is the representing object for $\hom_{Alg}(\kappa[x_1, ..., x_n], -)$. When $n = 1$, the functor is called a forgetful functor because, for an algebra A, it "forgets" the algebra structure of A and simply treats it as a set. Likewise, algebra morphisms are considered only as functions. The representing object in this case is $\kappa[x]$ and we refer to it as the affine line.

The morphisms of $\hom_{Alg}(\kappa[x], A)$ are called A-points of the affine line. They are points "of the line" in the sense that each morphism is out of $\kappa[x]$ and the designation "A-point" is due to the bijective correspondence between the morphisms of $\hom_{Alg}(\kappa[x], A)$ and the elements (or points) of A.

6.2.2 The Affine Plane

As the reader may have guessed, the affine plane comes from the case where $n = 2$ - i.e.

$$\hom_{Alg}(\kappa[x, y], A) \equiv A^2$$

Here, each algebra morphism out of $\kappa[x, y]$ corresponds to a point $(a, b) \in A^2$. Like before, we call $\kappa[x, y]$ the affine plane and the algebra morphisms A-points of the affine plane. Making use of the affine plane we can turn the affine line into a cocommutative Hopf algebra.

By Theorem 3.11, the affine plane, $\kappa[x_1, x_2]$, is isomorphic to $\kappa[x] \otimes \kappa[x]$. Under this isomorphism we get the same coalgebra structure as before, namely

$$\Delta(x^k) = \sum_{i+j=k} x^i \otimes x^j, \quad \varepsilon(x^k) = \delta_{k0}$$

As is, however, this coalgebra structure will not be compatible with the algebra structure, since Δ needs to be an algebra morphism, which means

$$\begin{aligned}\Delta(x^k) &= \Delta(x)^k \\ &= (x \otimes 1 + 1 \otimes x)^k \\ &= \sum_{n=0}^{k} \binom{k}{n} x^{k-n} \otimes x^n\end{aligned}$$

Setting $i = k - n$ and $j = n$ we get our modified Δ:

$$\Delta(x^k) = \sum_{i+j=k} \binom{i+j}{j} x^i \otimes x^j$$

and now we have a bialgebra with modified product $x^i x^j = \binom{i+j}{j} x^{i+j}$. In fact, this particular coalgebra structure is the same one introduced in Example 3.11 and, as promised, it is now apparent why this particular structure is favored, namely because it allows for a fairly "natural" bialgebra structure. As we will see, however, it also allows for an antipode, which will give our desired Hopf algebra structure.

Per usual, we verify this by determining an anti-algebra morphism $S \in End(\kappa[x])$ which satisfies

$$S \star \mathrm{id} = \varepsilon \circ \eta = \mathrm{id} \star S$$

Remember that $S \star \mathrm{id} := \nabla \circ (S \otimes \mathrm{id}) \circ \Delta$. We need

$$\begin{aligned}0 &= \varepsilon(x)1 \\ &= (S \star \mathrm{id})(x) \\ &= \nabla\big((S \otimes \mathrm{id})(x \otimes 1 + 1 \otimes x)\big) \\ &= \nabla(S(x) \otimes 1 + S(1) \otimes x) \\ &= S(x) + x\end{aligned}$$

So, we must have $S(x) := -x$ and the resulting anti-algebra morphism is the sought after antipode. Thus, the affine line is shown to be a Hopf algebra. But we also said that it is cocommutative, which requires the commutativity of the following diagram:

$$\begin{tikzcd} & \kappa[x] \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\ \kappa[x] \otimes \kappa[x] \arrow[rr, "\tau"] & & \kappa[x] \otimes \kappa[x] \end{tikzcd}$$

where τ is the transposition map. This is clearly the case, since

$$\begin{aligned} x\otimes 1+1\otimes x &\mapsto \tau(x\otimes 1+1\otimes x)\\ &=\tau(x\otimes 1)+\tau(1\otimes x)\\ &=1\otimes x+x\otimes 1\\ &=x\otimes 1+1\otimes x \end{aligned}$$

While we are working with commutative A, recall the matrix algebra $M_2(A)$. In particular, we are interested in the general linear group $GL_2(A)$ and special linear group $SL_2(A)$. These act on the affine plane in the usual linear algebra fashion. That is, if $(p,q)\in A^2$ and $\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}\in GL_2(A)$ (resp. $SL_2(A)$), then

$$\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}\begin{pmatrix}p\\ q\end{pmatrix}=\begin{pmatrix}\alpha p+\beta q\\ \gamma p+\delta q\end{pmatrix}$$

In Proposition 4.51 we determined that

$$\hom_{Alg}(GL(2),A)\cong GL_2(A)\quad\text{and}\quad \hom_{Alg}(SL(2),A)\cong SL_2(A)$$

Thus if

$$F\leftrightarrow\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}=\begin{bmatrix}F(a) & F(b)\\ F(c) & F(d)\end{bmatrix}\quad\text{and}\quad f\leftrightarrow\begin{pmatrix}p\\ q\end{pmatrix}=\begin{pmatrix}f(x)\\ f(y)\end{pmatrix}$$

then we find that F acts on f by $F\triangleright f=g$ where

$$\begin{aligned} g(x)&=F(a)f(x)+F(b)f(y)=\alpha p+\beta q\\ g(y)&=F(c)f(x)+F(d)f(y)=\gamma p+\delta q \end{aligned}$$

We shall now see that the affine plane is related to $\mathfrak{sl}(2)$ in that it becomes a module-algebra over $U(\mathfrak{sl}(2))$ (see Definition 4.38). We begin by establishing a more convenient means of determining if an algebra is a module-algebra over some bialgebra.

Lemma 6.2. *Let H be a bialgebra and A be an algebra with a structure of H-module such that $h\cdot 1=\varepsilon(h)1$. Assume that H is generated as an algebra by a subset X whose elements satisfy the relation*

$$h(ab)=\sum_{(h)}(h^{(1)}a)(h^{(2)}b) \tag{6.1}$$

for all $a,b\in A$. Then A is a module-algebra over H (see Definition 4.38).

Proof. Since H is generated by X as an algebra, then for $h, g \in X$ it is enough to show that hg satisfies relation (6.1), since every element of H will then inherit this property.

Let's begin by understanding, more fully, the relation (6.1). Since A is an H-module, every $h \in H$ gives an action $h\triangleright : A \to A$ where $h \triangleright a = ha$. This extends to an action of $H \otimes H$ on $A \otimes A$ where for $h, g \in H$ we have $(h \otimes g)\triangleright : A \otimes A \to A \otimes A$ defined by

$$(h \otimes g) \triangleright a \otimes b = h \triangleright a \otimes g \triangleright b = ha \otimes gb$$

Relation (6.1), then, means that the following diagram commutes:

$$\begin{tikzcd}
A \otimes A \arrow[r, "\nabla"] \arrow[d, "\Delta(h)\triangleright"'] & A \arrow[d, "h\triangleright"] \\
A \otimes A \arrow[r, "\nabla"] & A
\end{tikzcd}$$

Now because H is a bialgebra, Δ is an algebra morphism and so if $h, g \in X$ we have that

$$\begin{aligned}
\Delta(hg)\triangleright &= (\Delta(h)\Delta(g))\triangleright \\
&= (\Delta(h)\triangleright) \circ (\Delta(g)\triangleright)
\end{aligned}$$

and, hence, the following diagram commutes:

$$\begin{tikzcd}
A \otimes A \arrow[r, "\nabla"] \arrow[d, "\Delta(g)\triangleright"] \arrow[dd, bend right=60, "\Delta(hg)\triangleright"'] & A \arrow[d, "g\triangleright"] \\
A \otimes A \arrow[r, "\nabla"] \arrow[d, "\Delta(h)\triangleright"] & A \arrow[d, "h\triangleright"] \\
A \otimes A \arrow[r, "\nabla"] & A
\end{tikzcd}$$

Therefore, the product hg satisfies (6.1) and the result is proved. □

Definition 6.3 (Derivation)**.** Let A be a κ-algebra. A κ-derivation on A is a κ-linear map $D : A \to A$ satisfying the Leibniz law:

$$D(ab) = aD(b) + D(a)b$$

Derivations generalize the idea of the derivative operator. In fact, the astute reader may have already noticed a resemblance to the so called "product rule" of calculus, which happens to be a special case since

$$\frac{d}{dx}(fg) = f\frac{d}{dx}(g) + \frac{d}{dx}(f)g$$

where f, g are differentiable functions of x.

The alert reader will also recall that a definition of derivation was given in Chapter 5 in the setting of Lie algebras. The definition given here is more general, the Lie derivation being a specific example. For the sake of comparison, recall that a Lie derivation on a Lie algebra $\mathfrak{L}$ is an endomorphism d of $\mathfrak{L}$ such that $d([x, y]) = [x, d(y)] + [d(x), y]$ for all $x, y \in \mathfrak{L}$.

Notice, too, that for a derivation D, one has

$$\begin{aligned} D(1) &= D(1 \cdot 1) \\ &= D(1)1 + 1D(1) \\ &= D(1) + D(1) \end{aligned}$$

which can only hold if $D(1) = 0$. We'll make use of this fact in the following Lemma.

Lemma 6.4. *Let $\mathfrak{L}$ be a Lie algebra. An algebra A is a module-algebra over $U(\mathfrak{L})$ if and only if A has an $\mathfrak{L}$-module structure such that the elements of $\mathfrak{L}$ act on A as derivations.*

Proof. Suppose that A is an algebra which is a module-algebra over $U(\mathfrak{L})$. Then, by definition, A (as a vector space) is a $U(\mathfrak{L})$-module and for all $u \in U(\mathfrak{L})$ and $a, b \in A$

$$u(ab) = \sum_{(u)} (u^{(1)}a)(u^{(2)}b) \tag{6.2}$$

$$u1 = \varepsilon(u)1 \tag{6.3}$$

Now, being a $U(\mathfrak{L})$-module is equivalent to there being an algebra morphism $\rho : U(\mathfrak{L}) \to End(A)$. By the universal property of U there is a unique corresponding Lie morphism $\rho : \mathfrak{L} \to \mathfrak{gl}(A)$ thereby making A an $\mathfrak{L}$-module.

If $x \in \mathfrak{L}$, then $\Delta(x) = x \otimes 1 + 1 \otimes x$, and (6.2) becomes

$$\begin{aligned} \rho(x)(ab) &= (xa)b + a(xb) \\ &= \rho(x)(a)b + a\rho(x)(b) \end{aligned}$$

which says that x acts as a derivation.

Conversely, suppose that A is an $\mathfrak{L}$-module such that the elements of $\mathfrak{L}$ act on A as derivations. Again, being an $\mathfrak{L}$-module is equivalent to there being a representation $\rho : \mathfrak{L} \to \mathfrak{gl}(A)$ and by the universal property of U, this extends to a unique morphism of algebras $\rho : U(\mathfrak{L}) \to End(A)$, which makes A a $U(\mathfrak{L})$-module.

Because $U(\mathfrak{L})$ is generated, as an algebra, by $\mathfrak{L}$ (technically 1 and $\mathfrak{L}$, but we really get 1 for free), then by Lemma 6.2 we need only show that the result holds on $\mathfrak{L}$. So, if $x \in \mathfrak{L}$, then $\Delta(x) = x \otimes 1 + 1 \otimes x$ and since x acts on A as a derivation we have

$$x(ab) = (xa)b + a(xb)$$

and therefore relation (6.1) is again satisfied for all $a, b \in A$.

Finally, for $x \in \mathfrak{L}$, since x acts as a derivation on A we must have that $x \cdot 1 = 0$. But we also know that $\varepsilon(x) = 0$, since ε is the algebra morphism extension of the zero morphism $\mathfrak{L} \to \kappa$. Furthermore, because ε is an algebra morphism we get that $u \cdot 1 = \varepsilon(u)1$ for all $u \in U(\mathfrak{L})$. Therefore, by Lemma 6.2, A is a module-algebra over $U(\mathfrak{L})$. □

In Chapter 5 we provided a convenient way to construct irreducible $\mathfrak{sl}(2)$-modules. We will use the same idea here to show that $\mathfrak{sl}(2)$ acts on the (general) affine plane and that the affine plane is actually a module-algebra over $U(\mathfrak{sl}(2))$.

Theorem 6.5. *Define an action of the Lie algebra* $\mathfrak{sl}(2)$ *on the affine plane by*

$$XP = x\frac{\partial P}{\partial y}, \quad YP = y\frac{\partial P}{\partial x}, HP = x\frac{\partial P}{\partial x} - y\frac{\partial P}{\partial y}$$

where P *denotes any polynomial of* $\kappa[x, y]$ *and* X*,* Y *and* H *are the basis elements for* $\mathfrak{sl}(2)$*. Then*

(i) $\kappa[x, y]$ *becomes a module-algebra over* $U(\mathfrak{sl}(2))$*.*

(ii) The subspace $\kappa[x, y]_n$ *of homogeneous polynomials of degree* n *is a submodule of* $\kappa[x, y]$ *isomorphic to the simple* $\mathfrak{sl}(2)$*-module* $V[n]$*.*

Proof. (i) Per definition, we need that $\kappa[x, y]$, as a vector space, is a $\mathfrak{sl}(2)$-module. Thankfully, we already know that it is from our previous work (see Section 5.5.1.1).

Nevertheless, let's show that the action is well-defined as a matter of review. For instance, since $[X,Y]=H$ it better be that $[X,Y]P=HP$. Indeed,

$$\begin{aligned}
[X,Y]P &= (XY-YX)P \\
&= (XY)P-(YX)P \\
&= X(YP)-Y(XP) \\
&= x\frac{\partial}{\partial y}(y\frac{\partial P}{\partial x})-y\frac{\partial}{\partial x}(x\frac{\partial P}{\partial y}) \\
&= x\frac{\partial P}{\partial x}+xy\frac{\partial^2 P}{\partial y\partial x}-y\frac{\partial P}{\partial y}-yx\frac{\partial^2 P}{\partial x\partial y} \\
&= x\frac{\partial P}{\partial x}-y\frac{\partial P}{\partial y} \\
&= HP
\end{aligned}$$

That $[H,X]P=2XP$ and $[H,Y]P=-2YP$ are similarly verified.

If we can now show that the elements of $\mathfrak{sl}(2)$ act as derivations on $\kappa[x,y]$, then we may deduce that the affine plane really is a module-algebra over $U(\mathfrak{sl}(2))$. Now,

$$\begin{aligned}
X(PQ) &= x\frac{\partial PQ}{\partial y} \\
&= x(P\frac{\partial Q}{\partial y}+\frac{\partial P}{\partial y}Q) \\
&= Px\frac{\partial Q}{\partial y}+x\frac{\partial P}{\partial y}Q \\
&= PX(Q)+X(P)Q
\end{aligned}$$

Similarly, $Y(PQ)=PY(Q)+Y(P)Q$. Lastly,

$$\begin{aligned}
H(PQ) &= x\frac{\partial PQ}{\partial x}-y\frac{\partial PQ}{\partial y} \\
&= Px\frac{\partial Q}{\partial x}+x\frac{\partial P}{\partial x}Q-Py\frac{\partial Q}{\partial y}-y\frac{\partial P}{\partial y}Q \\
&= P\left(x\frac{\partial Q}{\partial x}-y\frac{\partial Q}{\partial y}\right)+\left(x\frac{\partial P}{\partial x}-y\frac{\partial P}{\partial y}\right)Q \\
&= PH(Q)+H(P)Q
\end{aligned}$$

Since X,Y and H act as derivations and respect the defining relations of $\mathfrak{sl}(2)$, this can be extended so that all actions by $\mathfrak{sl}(2)$ are derivations. Therefore, by Lemma 6.4, $\kappa[x,y]$ is a module-algebra over $U(\mathfrak{sl}(2))$.

(ii) Again, this follows from our previous work, but let's give a quick review. Let n be a non-negative integer and consider the monomial $x^n \in \kappa[x,y]_n$. Since

$$\begin{aligned} Hx^n &= x\frac{\partial x^n}{\partial x} - y\frac{\partial x^n}{\partial y} \\ &= xnx^{n-1} - 0 \\ &= nx^n \end{aligned}$$

it is of weight n. Furthermore, since $Xx^n = x\frac{\partial x^n}{\partial y} = 0$, it is a highest weight vector. For the sake of ease, set $v := x^n$. Also, for all $p \geq 0$ set

$$v_p := \frac{1}{p!}Y^p v$$

Then, if $p \leq n$ we get

$$v_p = \binom{n}{p} x^{n-p}y^p$$

and if $p > n$, then $v_p = 0$. As a vector space, these v_p generate $\kappa[x,y]_n$ and therefore this subspace is a submodule of $\kappa[x,y]$ generated by a highest weight vector of weight n. Since the v_p are eigenvectors for H, it is clear that

$$\kappa[x,y]_n \cong V[n]$$

□

In addition to there being an action on the affine plane we define next a coaction of $SL(2)$ (and $GL(2)$) on the affine plane. The following theorem establishes that the affine plane possesses a comodule-algebra structure over the bialgebras $M(2)$ and $SL(2)$.

Here we obtain a natural transformation in similar fashion to our work in Section 4.6. In this case we have

$$\begin{aligned} &\hom_{Alg}(M(2) \otimes \kappa[x,y], A) \\ &\equiv \hom_{Alg}(M(2), A) \times \hom_{Alg}(\kappa[x,y], A) \\ &\equiv M_2(A) \times (A)^2 \\ &\overset{\nabla}{\longmapsto} (A)^2 \\ &\equiv \hom_{Alg}(\kappa[x,y], A) \end{aligned}$$

where again A is a "variable" representing some commutative algebra. This natural transformation is then induced by some algebra morphism

$$\kappa[x,y] \to M(2) \otimes \kappa[x,y]$$

As we did before, let us follow $\mathrm{id}_{M(2)\otimes\kappa[x,y]}$ where $A = M(2)\otimes\kappa[x,y]$. We have

$$\begin{aligned}
\mathrm{id}_{M(2)\otimes\kappa[x,y]} &\overset{\sim}{\longmapsto} (i_{M(2)}, i_{\kappa[x,y]}) \\
&\overset{\sim}{\longmapsto} (\begin{bmatrix} i_{M(2)}(a) & i_{M(2)}(b) \\ i_{M(2)}(c) & i_{M(2)}(d) \end{bmatrix}, \begin{pmatrix} i_{\kappa[x,y]}(x) \\ i_{\kappa[x,y]}(y) \end{pmatrix}) \\
&= (\begin{bmatrix} a\otimes 1 & b\otimes 1 \\ c\otimes 1 & d\otimes 1 \end{bmatrix}, \begin{pmatrix} 1\otimes x \\ 1\otimes y \end{pmatrix}) \\
&\overset{\nabla}{\longmapsto} \begin{bmatrix} a\otimes 1 & b\otimes 1 \\ c\otimes 1 & d\otimes 1 \end{bmatrix} \cdot \begin{pmatrix} 1\otimes x \\ 1\otimes y \end{pmatrix} \\
&= \begin{pmatrix} a\otimes x + b\otimes y \\ c\otimes x + d\otimes y \end{pmatrix} \\
&\overset{\sim}{\longmapsto} \delta_{\kappa[x,y]}
\end{aligned}$$

where

$$\delta_{\kappa[x,y]}(x) = a\otimes x + b\otimes y \quad \text{and} \quad \delta_{\kappa[x,y]}(y) = c\otimes x + d\otimes y \tag{6.4}$$

Symbolically, we represent this as the matrix product:

$$\delta_{\kappa[x,y]}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix}$$

Theorem 6.6. *There exists a unique $M(2)$-comodule-algebra structure and a unique $SL(2)$-comodule-algebra structure on $\kappa[x,y]$ such that*

$$\delta_{\kappa[x,y]}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix}$$

Proof. From our motivating work above $\delta_{\kappa[x,y]} : \kappa[x,y] \to M(2)\otimes\kappa[x,y]$ is a morphism of algebras. Note, too, that the projection of $M(2)$ onto $SL(2)$ is an algebra morphism and therefore the composite map $\kappa[x,y] \to SL(2)\otimes\kappa[x,y]$ is an algebra morphism. So $\delta_{\kappa[x,y]}$ satisfies condition (ii) of Proposition 4.41. To satisfy the remaining condition of Proposition 4.41 requires showing that

$$(\mathrm{id}\otimes\delta_{\kappa[x,y]})\circ\delta_{\kappa[x,y]}(z) = (\Delta\otimes\mathrm{id})\circ\delta_{\kappa[x,y]}(z) \quad \text{and} \quad (\varepsilon\otimes\mathrm{id})\circ\delta_{\kappa[x,y]}(z) = 1\otimes z$$

where Δ and ε are the respective coproduct and counit maps for $M(2)$ and $SL(2)$. These are the conditions for being a left H-comodule.

To check the above equations it is sufficient to verify that they hold for x and y. This is due to the fact that x and y generate $\kappa[x,y]$ as an algebra and the only maps we are working with are algebra morphisms. Here too the convenience of our matrix notation is especially useful, since it allows us to check x and y simultaneously.

$$\begin{aligned}\left((\mathrm{id}\otimes\delta_{\kappa[x,y]})\circ\delta_{\kappa[x,y]}\right)\begin{pmatrix}x\\y\end{pmatrix} &= (\mathrm{id}\otimes\delta_{\kappa[x,y]})\begin{bmatrix}a & b\\c & d\end{bmatrix}\otimes\begin{pmatrix}x\\y\end{pmatrix}\\ &= \mathrm{id}\begin{bmatrix}a & b\\c & d\end{bmatrix}\otimes\delta_{\kappa[x,y]}\begin{pmatrix}x\\y\end{pmatrix}\\ &= \begin{bmatrix}a & b\\c & d\end{bmatrix}\otimes\left(\begin{bmatrix}a & b\\c & d\end{bmatrix}\otimes\begin{pmatrix}x\\y\end{pmatrix}\right)\end{aligned}$$

Also,

$$\begin{aligned}\left((\Delta\otimes\mathrm{id})\circ\delta_{\kappa[x,y]}\right)\begin{pmatrix}x\\y\end{pmatrix} &= (\Delta\otimes\mathrm{id})\begin{bmatrix}a & b\\c & d\end{bmatrix}\otimes\begin{pmatrix}x\\y\end{pmatrix}\\ &= \left(\begin{bmatrix}a & b\\c & d\end{bmatrix}\otimes\begin{bmatrix}a & b\\c & d\end{bmatrix}\right)\otimes\begin{pmatrix}x\\y\end{pmatrix}\end{aligned}$$

Since the two results are the same, this establishes the first equation. For the second equation, consider that

$$\begin{aligned}\left((\varepsilon\otimes\mathrm{id})\circ\delta_{\kappa[x,y]}\right)\begin{pmatrix}x\\y\end{pmatrix} &= (\varepsilon\otimes\mathrm{id})\begin{bmatrix}a & b\\c & d\end{bmatrix}\otimes\begin{pmatrix}x\\y\end{pmatrix}\\ &= \varepsilon\begin{bmatrix}a & b\\c & d\end{bmatrix}\otimes\mathrm{id}\begin{pmatrix}x\\y\end{pmatrix}\\ &= \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\otimes\begin{pmatrix}x\\y\end{pmatrix}\end{aligned}$$

We have now established that $\kappa[x,y]$ has a comodule structure whence it follows, by Proposition 4.41, that $\kappa[x,y]$ has the desired comodule-algebra structures. $\square$

Lemma 6.7. *For all $i, j \geq 0$ we have*

$$\delta_{\kappa[x,y]}(x^i y^j) = \sum_{r=0}^{i} \sum_{s=0}^{j} \binom{i}{r} \binom{j}{s} a^r b^{i-r} c^s d^{j-s} \otimes x^{r+s} y^{i+j-r-s}$$

Proof. Since $\delta_{\kappa[x,y]}$ is an algebra morphism we have

$$\begin{aligned}
\delta_{\kappa[x,y]}(x^i y^j) &= \delta_{\kappa[x,y]}(x^i)\delta_{\kappa[x,y]}(y^j) \\
&= \delta_{\kappa[x,y]}(x)^i \delta_{\kappa[x,y]}(y)^j \\
&= (a \otimes x + b \otimes y)^i (c \otimes x + d \otimes y)^j
\end{aligned}$$

The desired result now follows from applying the binomial formula. □

As a consequence of this lemma, $\kappa[x, y]_n$ is a sub-comodule of the affine plane, since

$$\delta_{\kappa[x,y]}(\kappa[x, y]_n) \subset M(2) \otimes \kappa[x, y]_n$$

and, in fact,

$$\kappa[x, y] = \bigoplus_n \kappa[x, y]_n$$

as comodules.

Now that we have a coaction of $M(2)$, $SL(2)$ and $GL(2)$ on the affine plane we can actually obtain a coaction of the affine plane on itself. This is done by realizing that a copy of the affine plane is "sitting" inside $M(2)$ in the form of affine transformations of the line. These transformations have the form $\begin{bmatrix} a & 0 \\ c & 1 \end{bmatrix}$. Note, then, that each point of the affine plane can be identified with an affine transformation by specializing to $\begin{bmatrix} x & 0 \\ y & 1 \end{bmatrix}$. In this case, the matrix formula for the coaction becomes

$$\begin{bmatrix} x & 0 \\ y & 1 \end{bmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \otimes x \\ y \otimes x + 1 \otimes y \end{pmatrix}$$

More precisely, one can embed $\kappa[x, y]$ into $M(2) = \kappa[a, b, c, d]$. There is more than one way to do this, but to be consistent with what has been said already one can assign

$x \mapsto a$ and $y \mapsto c$. By then "factoring out" from $\kappa[a, b, c, d]$ one can recover $\kappa[x, y]$ with

$$a \mapsto x$$
$$b \mapsto 0$$
$$c \mapsto y$$
$$d \mapsto 1$$

Now, if one treats $\begin{pmatrix} x \\ y \end{pmatrix}$ as $\begin{bmatrix} x & 0 \\ y & 1 \end{bmatrix}$ then

$$\begin{bmatrix} x & 0 \\ y & 1 \end{bmatrix} \otimes \begin{bmatrix} x & 0 \\ y & 1 \end{bmatrix} = \begin{bmatrix} x \otimes x & 0 \\ y \otimes x + 1 \otimes y & 1 \otimes 1 \end{bmatrix}$$

and the coaction actually turns into a colagebra structure for $\kappa[x, y]$ where

$$\Delta(x) = x \otimes x$$
$$\Delta(y) = y \otimes x + 1 \otimes y$$

$$\varepsilon(x) = 1$$
$$\varepsilon(y) = 0$$

This is a special coalgebra structure that, as we will see below, passes to the quantum plane and, by extending Δ and ε to algebra morphisms, allows for a bialgebra structure on both the affine and quantum plane.

6.3 The Quantum Plane

The quantum plane is a well known example in quantum group theory; however, it is not a quantum group. One could say that it is "close", but, as we will see, it fails to be a Hopf algebra. It is obtained from the affine plane via a method known as *deformation quantization*. This is a term well known in algebraic and differential geometry and the quantum plane will be an important illustration for understanding quantum groups. Recall that the affine plane is a more intuitive object retaining that familiar Euclidean property of being commutative which allows for easy visualization. The quantum plane, by contrast, is a more bizarre object having the peculiar property of *not* being commutative. Because of this, it is perhaps best to forego any attempts to picture this "plane". The term "plane" is more of a formal title based upon its construction.

Roughly speaking, a deformation of a mathematical object is a family of "similar" objects, which depend on some parameter or parameters such that the original object corresponds to a particular chosen initial value for the parameter (see [12]). In our case, the quantum plane is a one-parameter deformation of the affine plane. To understand what this means, recall that the affine plane is freely generated by the two variables x, y subject to the trivial commutation relation

$$yx = xy$$

Using the commutator, this is equivalent to $[y, x] = 0$. Suppose we modify or *deform* this relation so that we obtain a new (deformed) commutation relation

$$yx = qxy$$

where q is a non-zero element from the ground field κ. The element q serves as our parameter giving us a family of similar objects. Note, then, that the affine plane corresponds to an initial value of $q = 1$. So, in this context one can think of a deformation as "deforming" the commuting relations of an algebra. In this case, we are taking an algebra that was originally commutative and "deforming" it into similar structures which no longer commute. As an analogue to the commutator $[y, x]$ we can define a deformed commutator by

$$[y, x]_q := yx - qxy$$

which allows us to express the new commuting relation by $[y, x]_q = 0$.

More formally, the quantum plane is defined to be the algebra

$$\kappa_q[x, y] := \kappa\{x, y\}/I_q$$

where I_q is the two-sided ideal of the free algebra $\kappa\{x, y\}$ generated by the element $yx - qxy$.

Let us now show that the coaction of the affine plane on itself, discovered above, allows for a bialgebra structure on the quantum plane.

Proposition 6.8. *Define a coproduct Δ and counit ε for the quantum plane by*

$$\Delta(x) := x \otimes x, \quad \Delta(y) := y \otimes x + 1 \otimes y \qquad \varepsilon(x) := 1, \quad \varepsilon(y) := 0$$

These equip $\kappa_q[x, y]$ with a bialgebra structure.

Proof. Let us begin by showing that the coproduct respects the algebra structure of the quantum plane.

$$\begin{aligned}\Delta(y)\Delta(x) &= (y\otimes x + 1\otimes y)(x\otimes x)\\ &= yx\otimes x^2 + x\otimes yx\\ &= qxy\otimes x^2 + qx\otimes xy\\ &= q(xy\otimes x^2 + x\otimes xy\\ &= q(x\otimes x)(y\otimes x + 1\otimes y)\\ &= q\Delta(x)\Delta(y)\end{aligned}$$

Thus, Δ can be extended to an algebra morphism. Clearly ε can be so extended as well based on its definition. We therefore need only verify the coproduct and counit axioms using the generating elements x and y. But because x is grouplike, it clearly satisfies the coproduct axiom. Let us therefore check y.

$$\begin{aligned}(\Delta\otimes\mathrm{id})(\Delta(y)) &= (\Delta\otimes\mathrm{id})(y\otimes x + 1\otimes y)\\ &= \Delta(y)\otimes x + \Delta(1)\otimes y\\ &= (y\otimes x + 1\otimes y)\otimes x + 1\otimes 1\otimes y\\ &= y\otimes x\otimes x + 1\otimes y\otimes x + 1\otimes 1\otimes y\end{aligned}$$

and

$$\begin{aligned}(\mathrm{id}\otimes\Delta)(\Delta(y)) &= (\mathrm{id}\otimes\Delta)(y\otimes x + 1\otimes y)\\ &= y\otimes\Delta(x) + 1\otimes\Delta(y)\\ &= y\otimes x\otimes x + 1\otimes(y\otimes x + 1\otimes y)\\ &= y\otimes x\otimes x + 1\otimes y\otimes x + 1\otimes 1\otimes y\end{aligned}$$

So the coproduct axiom is satisfied.

For the counit, it is certainly satisfied for x, since it is grouplike and $\varepsilon(x) = 1$. Let us therefore verify it for y.

$$\begin{aligned}(\varepsilon\otimes\mathrm{id})(\Delta(y)) &= (\varepsilon\otimes\mathrm{id})(y\otimes x + 1\otimes y)\\ &= \varepsilon(y)\otimes x + \varepsilon(1)\otimes y\\ &= 0\otimes x + 1\otimes y\\ &= 1\otimes y\end{aligned}$$

Also,

$$
\begin{aligned}
(\mathrm{id} \otimes \varepsilon)(\Delta(y)) &= (\mathrm{id} \otimes \varepsilon)(y \otimes x + 1 \otimes y) \\
&= y \otimes \varepsilon(x) + 1 \otimes \varepsilon(y) \\
&= y \otimes 1 + 1 \otimes 0 \\
&= y \otimes 1
\end{aligned}
$$

So, because Δ and ε are algebra morphisms, which satisfy the coproduct and counit axioms, it follows that the quantum plane is indeed a bialgebra. □

As mentioned before, the quantum plane is not a quantum group. This is because it fails to be a Hopf algebra. Since it is a bialgebra, the failure must be with regards to having an antipode. Suppose that $\kappa_q[x, y]$ did have an antipode. Then we would have an anti-algebra morphism $S : \kappa_q[x, y] \to \kappa_q[x, y]$ such that $S \star \mathrm{id} = 1_\star = \mathrm{id} \star S$. In particular, since x is grouplike, $S(x) = x^{-1}$. But x^{-1} is not a member of $\kappa_q[x, y]$; so, there cannot be an antipode after all.

6.3.1 Ore Extensions

We begin with a generalization of a derivation.

Definition 6.9 (α-derivation)**.** Let A be an algebra and α an algebra endomorphism of A. An α-derivation of A is a linear endomorphism $\mathscr{D}$ of A such that

$$\mathscr{D}(ab) = \alpha(a)\mathscr{D}(b) + \mathscr{D}(a)b$$

for all $a, b \in A$.

Notice how Defintion 6.3 is the special case where $\alpha = \mathrm{id}$.

From this definition we see again that $\mathscr{D}$ has the property that $\mathscr{D}(1) = 0$ since

$$
\begin{aligned}
\mathscr{D}(1) &= \mathscr{D}(1 \cdot 1) \\
&= \alpha(1)\mathscr{D}(1) + \mathscr{D}(1) \\
&= \mathscr{D}(1) + \mathscr{D}(1)
\end{aligned}
$$

Definition 6.10 (Ore Extension)**.** Let A be an algebra, α an algebra endomorphism of A and $\mathscr{D}$ an α-derivation of A. Then the Ore extension $A[\lambda; \alpha, \mathscr{D}]$ is the algebra obtained by giving the polynomial algebra $A[\lambda]$ a new multiplication, subject to the identity

$$\lambda a = \alpha(a)\lambda + \mathscr{D}(a)$$

Now take $A[t]$ to be the free (left) A-module containing all polynomials of the form

$$P = a_n t^n + a_{n-1}t^{n-1} + \ldots + a_0 t^0$$

where $a_i \in A$ for all i. We say that the degree of P is $\deg(P) = n$ whenever $a_n \neq 0$. Note that it is a typical convention to set $\deg(0) := -\infty$. We will use Ore extensions to find all algebra structures on $A[t]$ which are compatible with the algebra structure of A and the degree.

Theorem 6.11. *(i) Assume that $A[t]$ has an algebra structure such that the natural inclusion of A into $A[t]$ is a morphism of algebras, and we have*

$$\deg(PQ) = \deg(P) + \deg(Q)$$

for any pair (P, Q) of elements of $A[t]$. Then A has no zero-divisors and there exist a unique injective algebra endomorphism α of A and a unique α-derivation $\mathscr{D}$ of A such that

$$ta = \alpha(a)t + \mathscr{D}(a)$$

for all $a \in A$.

(ii) Conversely, let A be an algebra having no zero-divisors. Given an injective algebra endomorphism α of A and an α-derivation $\mathscr{D}$ of A, there exists a unique algebra structure on $A[t]$ such that the inclusion of A into $A[t]$ is an algebra morphism and $ta = \alpha(a)t + \mathscr{D}(a)$ for all $a \in A$.

Proof. (i) That A has no zero-divisors is a direct consequence of the degree of elements of A considered as embedded in $A[t]$. That is, for non-zero $a, b \in A$ we have that $\deg(a) = \deg(b) = 0$ in $A[t]$ and hence $\deg(ab) = 0 \neq -\infty$ thereby implying that $ab \neq 0$. Thus, A has no zero-divisors.

For the next part, let a any non-zero element of A. Then $a \in A[t]$ as well. Now consider *left* multiplication by t, which gives the product $ta \in A[t]$. By hypothesis we have $\deg(ta) = \deg(t) + \det(a) = 1$. Thus, the product ta corresponds to a first degree polynomial in $A[t]$. Specifically, there are uniquely determined elements $\alpha(a) \neq 0$ and $\mathscr{D}(a)$ in A such that $ta = \alpha(a)t + \mathscr{D}(a)$. By letting a vary we get uniquely defined maps α and $\mathscr{D}$ on A. We now need to show that α is an algebra endomorphism and $\mathscr{D}$ is a linear endomorphism that satisfies the relation in Definition 6.9. Since $A[t]$ is an associative algebra, we have

$$(ta)b = t(ab)$$

for $a, b \in A$. This equality will give us a way to explore some of the properties of α and $\mathscr{D}$ when applied to a product. We find that

$$\begin{aligned}(ta)b &= t(ab)\\ (\alpha(a)t + \mathscr{D}(a))b &= \alpha(ab)t + \mathscr{D}(ab)\\ \alpha(a)tb + \mathscr{D}(a)b &= \alpha(ab)t + \mathscr{D}(ab)\\ \alpha(a)(\alpha(b)t + \mathscr{D}(b)) + \mathscr{D}(a)b &= \alpha(ab)t + \mathscr{D}(ab)\\ \alpha(a)\alpha(b)t + \alpha(a)\mathscr{D}(b) + \mathscr{D}(a)b &= \alpha(ab)t + \mathscr{D}(ab)\end{aligned}$$

Since the coefficients are unique, this implies that

$$\alpha(ab) = \alpha(a)\alpha(b) \quad \text{and} \quad \mathscr{D}(ab) = \alpha(a)\mathscr{D}(b) + \mathscr{D}(a)b \tag{6.5}$$

Furthermore, left multiplication by t is a linear operation which means that

$$\begin{aligned}t(a+b) &= ta + tb\\ \alpha(a+b)t + \mathscr{D}(a+b) &= \alpha(a)t + \mathscr{D}(a) + \alpha(b)t + \mathscr{D}(b)\\ &= (\alpha(a) + \alpha(b))t + \mathscr{D}(a) + \mathscr{D}(b)\end{aligned}$$

and hence

$$\alpha(a+b) = \alpha(a) + \alpha(b) \quad \text{and} \quad \mathscr{D}(a+b) = \mathscr{D}(a) + \mathscr{D}(b) \tag{6.6}$$

Finally, we have that

$$\begin{aligned}t1 &= t\\ &= \alpha(1)t + \mathscr{D}(1)\end{aligned}$$

implying that

$$\alpha(1) = 1 \quad \text{and} \quad \mathscr{D}(1) = 0 \tag{6.7}$$

Together, (6.5), (6.6) and (6.7) imply that α and $\mathscr{D}$ have the desired properties.

(ii) Firstly, we need to obtain an appropriate algebra structure. Since $A[t]$ is infinite dimensional, the basic strategy for proving (ii) is to embed $A[t]$ into the associative algebra of all infinite matrices $(f_{ij})_{i,j\geq 1}$ with $f_{ij} \in \mathrm{End}(A)$ (linear endomorphisms) such that each column has only finitely many non-zero entries. We'll denote this algebra by $M_\infty(\mathrm{End}(A))$. More specifically, the reason for doing this is because given an algebra

A, then, as a vector space

$$A[t] = \bigoplus_{i=0}^{\infty} At^i = \bigoplus_{i=0}^{\infty} A_i$$

where each A_i is an isomorphic copy of A ($A_i = At^i$). What we get is that

$$\hom_\kappa(A[t], A[t]) \cong M_\infty(End(A))$$

Now, for any algebra A there is a left representation $A \to \hom_\kappa(A, A)$ given by

$$a \mapsto \widehat{a}$$

where $\widehat{a}$ represent the endomorphism of A that is left multiplication by a for any $a \in A$. So, when $A[t]$ is endowed with its algebra structure we get

$$A[t] \to \hom_\kappa(A[t], A[t]) \to M_\infty(End(A))$$

Once we successfully embed $A[t]$ into $M_\infty(\mathrm{End}(A))$ we will show that the image of the embedding is a subalgebra of $M_\infty(\mathrm{End}(A))$ thereby allowing us to lift this structure to $A[t]$. Note, too, that once we show the product ta to be $\alpha(a)t + \mathscr{D}(a)$ for any $a \in A$, then the lifted algebra structure must be unique.

Let us now construct our embedding map. Using the left multiplication, we can express the property of α that $\alpha(ab) = \alpha(a)\alpha(b)$ by

$$\alpha \circ \widehat{a} = \widehat{\alpha(a)} \circ \alpha \tag{6.8}$$

in $\mathrm{End}(A)$. Likewise, we can express the condition for being an α-derivation by

$$\mathscr{D} \circ \widehat{a} = \widehat{\alpha(a)} \circ \mathscr{D} + \widehat{\mathscr{D}(a)} \tag{6.9}$$

Next, based on the algebra structure of $A[t]$, we need to determine $T \in M_\infty(\mathrm{End}(A))$ for which $t \mapsto T$. To do this, we use the fact that $A[t] = \bigoplus_{i=0}^{\infty} A_i$. Since $A[t]$ has basis $1, t, t^2, t^3, ...$ we have that

$$\begin{aligned} a &\leftrightarrow (a, 0, 0, ...) \\ at &\leftrightarrow (0, a, 0, ...) \\ at^2 &\leftrightarrow (0, 0, a, ...) \\ &\vdots \end{aligned}$$

Let us now see what left multiplication by t achieves. For example, consider at^2. If we multiply on the left by t we get

$$\begin{aligned} tat^2 &= (\alpha(a)t + \mathscr{D}(a))t^2 \\ &= \alpha(a)t^3 + \mathscr{D}(a)t^2 \end{aligned}$$

So, under left multiplication by t, one gets that

$$(0, 0, a, 0, ...) \to (0, 0, \mathscr{D}(a), \alpha(a), 0, ...)$$

and in general $tat^n = \alpha(a)t^{n+1} + \mathscr{D}(a)t^n$ so that

$$(0, 0, ..., 0, a, 0, ...) \to (0, 0, ..., 0, \mathscr{D}(a), \alpha(a), 0, ...)$$

Using these as columns we can construct the desired matrix, namely

$$T = \begin{bmatrix} \mathscr{D} & 0 & 0 & 0 & \dots \\ \alpha & \mathscr{D} & 0 & 0 & \dots \\ 0 & \alpha & \mathscr{D} & 0 & \dots \\ 0 & 0 & \alpha & \mathscr{D} & \dots \\ 0 & 0 & 0 & \alpha & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

So, $t \mapsto T$ and from this we get a linear map $\varphi : A[t] \to M_\infty(End(A))$ with

$$\varphi\Big(\sum_{i=0}^{n} a_i t^i\Big) = \sum_{i=0}^{n} (\widehat{a_i} I)T^i$$

The claim is that φ is an injective map. To see why, let e_i be the infinite column vector with 1_A in the i-th entry and zeros for the rest. If we apply T to such a vector we find that $Te_i = e_{i+1}$ on account of the fact that $\alpha(1) = 1$ and $\mathscr{D}(1) = 0$. Now, suppose $P = \sum_{i=0}^{n} a_i t^i \in A[t]$ is such that $\varphi(P) = 0$. In other words, $P \in Ker(\varphi)$. The goal will be to demonstrate that $P = 0$, which entails showing that the coefficients $a_0, ..., a_n$ are all zero. Applying $\varphi(P)$ to e_1 we find that

$$\begin{aligned} 0 &= \varphi(P)e_1 \\ &= \sum_{i=0}^{n} (\widehat{a_i} I)T^i e_1 \\ &= \sum_{i=0}^{n} \widehat{a_i} e_{i+1} \end{aligned}$$

But because $\{e_i\}_{i\geq 0}$ is a linearly independent set, it must be that the $\widehat{a_i}$ are all zero. This means that left multiplication by a_i (all i) is always zero for any $a \in A$. But A has no zero divisors and specifically $\widehat{a_i}(1) = 0$, which implies that $a_i = 0$ for all i. Therefore, $P = 0$ and so φ is injective.

Let S be the subalgebra of $M_\infty(End(A))$ generated by the elements T and $\widehat{a}I$. Now, $\varphi(\alpha(a)t + \mathscr{D}(a)) = (\widehat{\alpha(a)}I)T + (\widehat{\mathscr{D}(a)}I)$, which is the matrix

$$\begin{bmatrix} \widehat{\alpha(a)}\mathscr{D} + \widehat{\mathscr{D}(a)} & & & \\ \widehat{\alpha(a)}\alpha & \widehat{\alpha(a)}\mathscr{D} + \widehat{\mathscr{D}(a)} & & \\ & \widehat{\alpha(a)}\alpha & & \\ & & & \ddots \end{bmatrix}$$

By (6.8) and (6.9) we can rewrite this matrix as

$$\begin{bmatrix} \mathscr{D}\widehat{a} & & & \\ \alpha\widehat{a} & \mathscr{D}\widehat{a} & & \\ & \alpha\widehat{a} & \mathscr{D}\widehat{a} & \\ & & \alpha\widehat{a} & \\ & & & \ddots \end{bmatrix} = T(\widehat{a}I)$$

From this it follows that $Im(\varphi) = S$ and since φ is injective we get an induced linear isomorphism from $A[t]$ to the algebra S thereby allowing us to lift the algebra structure of S to $A[t]$. Specifically, since

$$\begin{aligned} \varphi(\alpha(a)t + \mathscr{D}(a)) &= (\widehat{\alpha(a)}I)T + (\widehat{\mathscr{D}(a)}I) \\ &= T(\widehat{a}I) \end{aligned}$$

we get the assignment $\varphi(ta) = T(\widehat{a}I)$ and therefore $ta = \alpha(a)t + \mathscr{D}(a)$ for all $a \in A$. ☐

Corollary 6.12. *Let A be an algebra without zero-divisors, α an injective algebra endomorphism of A and $\mathscr{D}$ an α-derivation of A. Then the algebra $A[t;\alpha,\mathscr{D}]$ has no zero-divisors. As a left A-module, it is free with basis $\{t^i\}_{i\in\mathbb{N}}$. Furthermore, if α is an automorphism, then $A[t;\alpha,\mathscr{D}]$ is also a right free A-module with the same basis $\{t^i\}_{i\in\mathbb{N}}$.*

Proof. Since A has no zero-divisors and all elements of $A[t;\alpha,\mathscr{D}]$ are finite polynomials in t, we get a well defined concept of degree. Now,

$$\deg(PQ) = \deg(P) + \deg(Q)$$

hence, if $P, Q \neq 0$, then $\deg(P) = n$ and $\deg(Q) = m$ $(n, m \in \mathbb{N})$ implying that $\deg(PQ) = n + m$ and therefore PQ cannot be 0. Thus, $A[t; \alpha, \mathscr{D}]$ has no zero divisors.

Now, $A[t]$ is a free left A-module, since it consists of all polynomials of the form

$$P = a_n t^n + a_{n-1} t^{n-1} + \ldots + a_0 t^0$$

and so $\{t^i\}_{i \in \mathbb{N}}$ is a basis. As a left A-module $A[t; \alpha, \mathscr{D}]$ is the same as $A[t]$ and therefore has the same basis.

But we can also express every element of $A[t; \alpha, \mathscr{D}]$ in the form $P = \sum_{i=0}^{n} t^i a_i$ when α is an automorphism. To see this, note that since $ta = \alpha(a)t + \mathscr{D}(a)$ and α is invertible, we have

$$\begin{aligned} t\alpha^{-1}(a) &= \alpha(\alpha^{-1}(a))t + \mathscr{D}(\alpha^{-1}(a)) \\ &= at + \mathscr{D}(\alpha^{-1}(a)) \end{aligned}$$

so $at = t\alpha^{-1}(a) - \mathscr{D}(\alpha^{-1}(a))$. It is also clearly the case that $at^0 = t^0\alpha^{-0}(a)$, so suppose that

$$at^n = t^n \alpha^{-n}(a) + \text{lower-degree terms} \tag{6.10}$$

up to some $n \in \mathbb{N}$. Then

$$\begin{aligned} at^{n+1} &= (at^n)t \\ &= (t^n\alpha^{-n}(a) + \text{L.D.T})t \quad \text{[induction hypothesis]} \\ &= t^n\alpha^{-n}(a)t + (\text{L.D.T})t \\ &= t^n(t\alpha^{-1}(\alpha^{-n}(a)) - (\alpha^{-1}(\alpha^{-n}(a)))) + \text{L.D.T} \quad \text{[induction hypothesis]} \\ &= t^{n+1}\alpha^{-(n+1)}(a) + \text{L.D.T} \end{aligned}$$

It can similarly be shown that

$$t^n a = \alpha^n(a)t^n + \text{lower-degree terms} \tag{6.11}$$

so that we are able to go back and forth from the right side to the left side.

So, when α is an automorphism, the set $\{t^i\}_{i \in \mathbb{N}}$ generates $A[t; \alpha, \mathscr{D}]$ as a right A-module. We now need to show that, in this context, $\{t^i\}_{i \in \mathbb{N}}$ is independent. Suppose, however, that it is not. Then there exists a relation of the form

$$t^n a_n + t^{n-1} a_{n-1} + \ldots + t a_1 + a_0 = 0$$

where at least $a_n \neq 0$. But then we would also have that

$$\alpha^n(a_n)t^n + \text{lower-degree terms} = 0$$

by (6.11). In this form, however, $\{t^i\}_{i\in\mathbb{N}}$ is a basis and hence $\alpha^n(a_n) = 0$. But α is an isomorphism and so it must be that $a_n = 0$, which is a contradiction. Therefore, $\{t^i\}_{i\in\mathbb{N}}$ must also be independent when the multiplication is on the right. Hence, $A[t;\alpha,\mathscr{D}]$ is also a free right A-module with basis $\{t^i\}_{i\in\mathbb{N}}$. □

Theorem 6.13. *Let R be an algebra, α an algebra automorphism and δ an α-derivation of R. If R is left-(resp. right) Noetherian, then so is the Ore extension $R[t;\alpha,\delta]$.*

This is an extension of the Hilbert Basis Theorem (see [8]), which states:

> If R is a left-(resp. right) Noetherian ring, then the polynomial ring $R[x]$ is also a left-(resp. right) Noetherian ring.

Proposition 6.14. *(i) If α is the automorphism of the polynomial ring $\kappa[x]$ determined by $\alpha(x) = qx$, then the algebra $\kappa_q[x,y]$ is isomorphic to the Ore extension $\kappa[x][y;\alpha,0]$. Thus, $\kappa_q[x,y]$ is Noetherian, has no zero-divisors and the set of monomials $\{x^iy^j\}_{i,j\geq 0}$ is a basis of the underlying vector space.*

(ii) For any pair (i,j) of nonnegative integers, we have

$$y^jx^i = q^{ij}x^iy^j$$

(iii) Given any κ-algebra A, there is a natural bijection

$$\hom_{Alg}(\kappa_q[x,y], A) \equiv \{(X,Y) \in A\times A : YX = qXY\}$$

Proof. (i) Note, firstly, that it is clear that $\kappa_q[x,y]$ and $\kappa[x][y;\alpha,0]$ are the same vector space with basis $\{x^iy^j\}_{i,j\in\mathbb{N}}$. All that is now required is that they possess the same algebra structure. For the quantum plane we already know that the defining algebra relation is: $yx = qxy$. Indeed, for the Ore extension $\kappa[x][y;\alpha,0]$ we also have

$$\begin{aligned} yx &= \alpha(x)y + 0(x) \\ &= qxy \end{aligned}$$

Thus, $\kappa_q[x,y] \cong \kappa[x][y;\alpha,0]$ as algebras.

Now, it is a well known fact that any field is Noetherian; hence, so is $\kappa[x]$. By Theorem 6.13, the Ore extension $\kappa[x][y;\alpha,0]$ is therefore Noetherian and, by isomorphism, so must be the quantum plane $\kappa_q[x,y]$. Finally, since $\kappa[x]$ has no zero-divisors, it follows by Corollary 6.12 that $\kappa[x][y;\alpha,0]$, and therefore $\kappa_q[x,y]$, has no zero-divisors.

(ii) Since $yx = qxy$, suppose that $y^n x = q^n xy^n$ up to some $n \in \mathbb{N}$. Now consider that

$$\begin{aligned}
y^{n+1}x &= yy^n x \\
&= yq^n xy^n \quad \text{[induction hypothesis]} \\
&= q^n yxy^n \\
&= q^n qxyy^n \quad \text{[base case]} \\
&= q^{n+1}xy^{n+1}
\end{aligned}$$

So, by mathematical induction, we have that $y^k x = q^k xy^k$ for any $k \in \mathbb{N}$. Next, suppose that

$$y^k x^\ell = q^{k\ell} x^\ell y^k$$

up to some $\ell \in \mathbb{N}$ and for all k. Consider now that

$$\begin{aligned}
y^k x^{\ell+1} &= y^k x^\ell x \\
&= q^{k\ell} x^\ell y^k x \quad \text{[induction hypothesis]} \\
&= q^{k\ell} x^\ell q^k xy^k \\
&= q^{k(\ell+1)} x^{\ell+1} y^k
\end{aligned}$$

and so, by induction, we also get that $y^j x^i = q^{ij} x^i y^j$ for all $i, j \in \mathbb{N}$.

(iii) By Proposition 3.7 we have the natural bijection

$$\hom_{Alg}(\kappa\{X\}/I, A) \equiv \{f \in \hom_{Set}(X, A) : \overline{f}(I) = 0\}$$

where $\overline{f}$ is the unique algebra morphism from $\kappa\{X\}$ to A induced by f. In particular, then, we have

$$\hom_{Alg}(\kappa\{x,y\}/I_q, A) \equiv \{f \in \hom_{Set}(\{x,y\}, A) : \overline{f}(I_q) = 0\}$$

which implies the result

$$\hom_{Alg}(\kappa\{x,y\}/I_q, A) \equiv \{(X,Y) \in A^2 : YX = qXY\}$$

□

6.3.2 q-Analysis

Suppose we now wish to see what kind of algebra can be done in the quantum plane. This section will give a glimpse into what it is like to calculate within a deformed structure and will ease us into, as well as motivate, the types of computations done within quantum groups. The quantum plane is a nice place to start because it is fairly simple to work with, since it only has one deformed relation. As an interesting example, we'll do a little quantum pre-calculus and consider an analogue of the Binomial Theorem.

For the affine plane the Binomial Theorem says,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \in \kappa[x,y]$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 1} = \prod_{m=1}^{k} \frac{n-m+1}{m}$$

We want to be able to compute powers of $x+y$ in the quantum plane too. It would even be nice if we could produce something similar to the result of the normal Binomial Theorem. Indeed, this can be done, but we need to first procure some basic tools of q-analysis. We begin with the notion of a q-deformation of a positive integer.

Definition 6.15. For any $n \in \mathbb{Z}^+$ set

$$(n)_q := 1 + q + q^2 + \ldots + q^{n-1} = \frac{q^n - 1}{q - 1}$$

Being a deformation we expect to recover the original integer in the limit $q \to 1$. Let's make sure this is the case.

$$\begin{aligned} \lim_{q\to 1} (n)_q &= \lim_{q\to 1} \frac{q^n - 1}{q - 1} \\ &= \lim_{q\to 1} \frac{\frac{d}{dq}(q^n - 1)}{\frac{d}{dq}(q - 1)} \qquad \text{[L'Hospital's Rule]} \\ &= \lim_{q\to 1} nq^{n-1} = n \end{aligned}$$

Note that we can actually do all integers, where $(0)_q = 0$ and for $n \in \mathbb{Z}^+$

$$(-n)_q = \frac{q^{-n} - 1}{q - 1} = \frac{1 - q^n}{q^n(q - 1)}$$

We still recover $-n$ in the limit $q \to 1$ since

$$\begin{aligned}\lim_{q\to 1}(-n)_q &= \lim_{q\to 1}\frac{1-q^n}{q^n(q-1)}\\ &= \lim_{q\to 1}\frac{-nq^{n-1}}{nq^{n-1}(q-1)+q^n}\\ &= -n\end{aligned}$$

Now, although n and $-n$ are additive inverses, it is not the case that $(n)_q$ and $(-n)_q$ are additive inverses. Instead,

$$\begin{aligned}-(n)_q &= -\frac{q^n-1}{q-1}\\ &= \frac{1-q^n}{q-1}\\ &= q^n\frac{1-q^n}{q^n(q-1)}\\ &= q^n(-n)_q\end{aligned}$$

So that

$$(n)_q + q^n(-n)_q = 0$$

Let's also consider how to understand the deformation of a sum. For integers n and m,

$$\begin{aligned}(n+m)_q &= \frac{q^{n+m}-1}{q-1}\\ &= \frac{q^nq^m-1}{q-1}\\ &= \frac{q^nq^m-q^n+q^n-1}{q-1}\\ &= \frac{q^n(q^m-1)+q^n-1}{q-1}\\ &= q^n\frac{q^m-1}{q-1}+\frac{q^n-1}{q-1}\\ &= q^n(m)_q+(n)_q\end{aligned}$$

By similar reasoning, we also get the alternative: $(n+m)_q = (m)_q + q^m(n)_q$.

From the idea of a deformed integer we can construct the idea of a q-factorial.

Definition 6.16 (q-factorial)**.**

$$(0)!_q := 1 \quad\text{and}\quad (n)!_q := (1)_q(2)_q\cdots(n)_q = \frac{(q-1)(q^2-1)\cdots(q^n-1)}{(q-1)^n}$$

Since $\lim_{q\to 1}(n)_q = n$, it is easy to see that the usual factorial is also recovered in the limit $q \to 1$.

Next we construct the q-analogue of the binomial coefficient, which is called a Gauss polynomial.

Definition 6.17 (Gauss Polynomial)**.** The Gauss polynomials for $0 \leq k \leq n$ are given by

$$\binom{n}{k}_q := \frac{(n)!_q}{(k)!_q(n-k)!_q}$$

For $k > n$ we take $\binom{n}{k}_q = 0$.

Again, via basic calculus, we see that the usual binomial coefficient is recovered in the limit $q \to 1$. Note, too, that we always have

$$\binom{n}{n}_q = \binom{n}{0}_q = 1 \quad \text{all } n$$

just like the normal binomial coefficient. The next result will justify the name Gauss *polynomial*.

Proposition 6.18. *Let $0 \leq k \leq n$. Then*

(i) $\binom{n}{k}_q$ *is a polynomial in q with integral coefficients such that*

$$\binom{n}{k}_1 = \binom{n}{k}$$

(ii) We have

$$\binom{n}{k}_q = \binom{n}{n-k}_q$$

(iii) We have

$$\binom{n}{k}_q = \binom{n-1}{k-1}_q + q^k\binom{n-1}{k}_q = \binom{n-1}{k}_q + q^{n-k}\binom{n-1}{k-1}_q$$

Proof. Let's start with (ii) since it is the easiest. By definition we have that

$$\begin{aligned}
\binom{n}{n-k}_q &= \frac{(n)!_q}{(n-k)!_q(n-(n-k))!_q} \\
&= \frac{(n)!_q}{(n-k)!_q(k)!_q} \\
&= \frac{(n)!_q}{(k)!_q(n-k)!_q} \\
&= \binom{n}{k}_q
\end{aligned}$$

Next, let's tackle (iii).

$$\begin{aligned}
\binom{n-1}{k-1}_q + q^k\binom{n-1}{k}_q &= \frac{(n-1)!_q}{(k-1)!_q(n-k)!_q} + q^k\frac{(n-1)!_q}{(k)!_q(n-k-1)!_q} \\
&= (n-1)!_q\left(\frac{1}{(k-1)!_q(n-k)!_q} + \frac{q^k}{(k)!_q(n-k-1)!_q}\right) \\
&= (n-1)!_q\frac{(k)_q + q^k(n-k)_q}{(k)!_q(n-k)!_q}
\end{aligned}$$

Examine $(n-1)!_q((k)_q + q^k(n-k)_q)$. By definition, this expression is equal to

$$\begin{aligned}
(n-1)!_q\left(\frac{q^k-1}{q-1} + q^k\frac{q^{n-k}-1}{q-1}\right) &= (n-1)!_q\frac{q^k-1+q^n-q^k}{q-1} \\
&= (n-1)!_q\frac{q^n-1}{q-1} \\
&= (n-1)!_q(n)_q \\
&= (n)!_q
\end{aligned}$$

We therefore see that

$$\begin{aligned}
\binom{n-1}{k-1}_q + q^k\binom{n-1}{k}_q &= \frac{(n)!_q}{(k)!_q(n-k)!_q} \\
&= \binom{n}{k}_q
\end{aligned}$$

as desired. For the other equality we use (ii) along with the result we just established.

$$\begin{aligned}
\binom{n}{k}_q &= \binom{n}{n-k}_q \quad \text{[by (ii)]} \\
&= \binom{n-1}{n-k-1}_q + q^{n-k}\binom{n-1}{n-k}_q \\
&= \binom{n-1}{(n-1)-k}_q + q^{n-k}\binom{n-1}{n-k}_q \\
&= \binom{n-1}{k}_q + q^{n-k}\binom{n-1}{n-1-n+k}_q \quad \text{[by (ii)]} \\
&= \binom{n-1}{k}_q + q^{n-k}\binom{n-1}{k-1}_q
\end{aligned}$$

As for (i), we have already mentioned, and it is not hard to see, that $\binom{n}{k}_1 = \binom{n}{k}$. To prove the rest, we shall use induction on n. Clearly

$$\binom{1}{0}_q = \binom{1}{1}_q = 1$$

For $n = 2$ we can use result (iii).

$$\binom{2}{k}_q = \binom{1}{k-1}_q + q^k\binom{1}{k}_q$$

But we only need to worry about $k = 1$; so, we get

$$\begin{aligned}
\binom{2}{1}_q &= \binom{1}{0}_q + q\binom{1}{1}_q \\
&= 1 + q
\end{aligned}$$

Now, suppose $\binom{n}{k}_q$ is a polynomial in q with integral coefficients up to some n. Then

$$\binom{n+1}{k}_q = \binom{n}{k-1}_q + q^k\binom{n}{k}_q \quad \text{[by (iii)]}$$

By our induction hypothesis both $\binom{n}{k-1}_q$ and $\binom{n}{k}_q$ are polynomials in q with integral coefficients. Therefore $\binom{n+1}{k}_q$ is a polynomial in q with integral coefficients. Thus, by induction (i) is proved. □

Proposition 6.19. *Let x and y be variables subject to the quantum plane relation $[y, x]_q = 0$. Then for all $n > 0$ we have*

$$(x + y)^n = \sum_{0\leq k\leq n} \binom{n}{k}_q x^k y^{n-k}$$

Proof. To get a feel for what is going on, let us compute the first few powers.

$$(x+y)^1 = x+y$$

$$\begin{aligned}(x+y)^2 &= x^2+xy+yx+y^2\\ &= x^2+xy+qxy+y^2\\ &= x^2+(1+q)xy+y^2\end{aligned}$$

$$\begin{aligned}(x+y)^3 &= (x+y)(x^2+(1+q)xy+y^2)\\ &= x^3+(1+q)x^2y+xy^2+yx^2+(1+q)yxy+y^3\\ &= x^3+(1+q)x^2y+xy^2+q^2x^2y+q(1+q)xy^2+y^3\\ &= x^3+(1+q+q^2)x^2y+(1+q+q^2)xy^2+y^3\end{aligned}$$

So, each of the above powers of $x+y$ satisfy the desired equation, where the polynomial coefficients are the appropriate Gauss polynomials of Proposition 6.18, (i). Suppose, then, that

$$(x+y)^n = \sum_{0\le k\le n} \binom{n}{k}_q x^k y^{n-k}$$

up to some n and consider the case for $n+1$.

$$\begin{aligned}(x+y)^{n+1} &= (x+y)(x+y)^n\\ &= (x+y)\Big(\sum_{0\le k\le n}\binom{n}{k}_q x^k y^{n-k}\Big) \quad \text{[induction hypothesis]}\\ &= x\sum_{0\le k\le n}\binom{n}{k}_q x^k y^{n-k} + y\sum_{0\le k\le n}\binom{n}{k}_q x^k y^{n-k}\\ &= \sum_{0\le k\le n}\binom{n}{k}_q x^{k+1} y^{n-k} + \sum_{0\le k\le n}\binom{n}{k}_q y x^k y^{n-k}\\ &= \sum_{1\le k\le n+1}\binom{n}{k-1}_q x^k y^{n-k+1} + \sum_{0\le k\le n} q^k\binom{n}{k}_q x^k y^{n-k+1} \quad [y^j x^i = q^{ij}x^i y^j]\\ &= \left(\sum_{1\le k\le n}\left(\binom{n}{k-1}_q + q^k\binom{n}{k}_q\right) x^k y^{n+1-k}\right) + x^{n+1} + y^{n+1}\\ &= \sum_{0\le k\le n+1}\binom{n+1}{k}_q x^k y^{n+1-k} \quad \text{[by (iii) of prev. prop.]}\end{aligned}$$

Therefore, by induction, the result is proved. □

Having had our appetizer, let us now proceed to consider some actual quantum groups.

6.4 The Quantum Groups $GL_q(2)$ and $SL_q(2)$

6.4.1 $M_q(2)$

The quantum plane is a very well known example of a deformed space. It is also a natural segue to the idea of a quantum matrix algebra which we now explore. This section directly paves the way for two of the three quantum groups we will examine in this thesis. We have already spent a bit of time studying the bialgebra $M(2)$ and so we here introduce its family of deformations $M_q(2)$.

For reasons that will hopefully become clear, we assume that $q^2 \neq -1$. Let x, y be elements of an algebra subject to the quantum plane relation and let a, b, c, d be four variables which commute with x and y. Now consider the following matrix equations:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}, \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}^T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x'' \\ y'' \end{pmatrix} \tag{6.12}$$

Given that $y'x' = qx'y'$ and $y''x'' = qx''y''$, we get six equivalent relations describing how a, b, c, d must relate to each other in order to be consistent with matrix multiplication. In fact, if we proceed with normal matrix multiplication we get

$$ax + by = x', \quad cx + dy = y', \quad ax + cy = x'', \quad bx + dy = y''$$

Substituting these into the quantum plane relations yields

$$(cx + dy)(ax + by) = q(ax + by)(cx + dy) \tag{6.13}$$

$$(bx + dy)(ax + cy) = q(ax + cy)(bx + dy) \tag{6.14}$$

Expanding these equations and identifying coefficients (assuming x^2, xy and y^2 are independent) gives

$$\begin{aligned} ca &= qac, \quad & cb + qda &= qad + q^2bc, \quad & db &= qbd \qquad &\text{[from (6.13)]} \\ ba &= qab, \quad & bc + qda &= qad + q^2cb, \quad & dc &= qcd \qquad &\text{[from (6.14)]} \end{aligned}$$

Using the equations in the middle column allows us to deduce the further relations

$$bc - cb = q^2(cb - bc), \quad ad - da = q^{-1}cb - qbc \quad \text{and} \quad ad - da = q^{-1}bc - qcb$$

If we examine the first equation we see that it is equivalent to $(1 + q^2)(bc - cb) = 0$ and since $q^2 \neq -1$ this entails that $cb = bc$. So, the six relations become

1. $ba = qab$
2. $db = qbd$
3. $ca = qac$
4. $dc = qcd$
5. $bc = cb$
6. $ad - da = (q^{-1} - q)bc$

Conversely, it can be shown that these six relations imply that $y'x' = qx'y'$ and $y''x'' = qx''y''$ so we have an equivalence. This is our motivation for the definition of $M_q(2)$.

Definition 6.20. Let J_q be the two-sided ideal of the free algebra $\kappa\{a, b, c, d\}$ generated by the six relations above. Then the algebra $M_q(2)$ is the quotient of $\kappa\{a, b, c, d\}$ by J_q - i.e.

$$M_q(2) := \frac{\kappa\{a, b, c, d\}}{J_q}$$

This rigorous definition of $M_q(2)$ is consistent with the intuitive idea of a deformation given at the start of this chapter. It has five deformed commuting relations (since $bc = cb$) as opposed to just one. Also, just as the affine plane was recovered when $q = 1$ so we see that $M(2)$ is recovered when $q = 1$. Remember, too, that given a commutative algebra $\mathcal{A}$ we have the bijection

$$\hom_{Alg}(M(2), \mathcal{A}) \equiv M_2(\mathcal{A})$$

where either set is referred to as the space of $\mathcal{A}$-points of $M(2)$. Denote the set of $\mathcal{A}$-points for $M_q(2)$ by $M_2^q(\mathcal{A})$, which consists of all 2×2 matrices

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad \text{with } A, B, C, D \in \mathcal{A} \text{ subject to the designated six relations above.}$$

Then we still have that

$$\hom_{Alg}(M_q(2), \mathcal{A}) \equiv M_2^q(\mathcal{A})$$

Proposition 6.21. *Let*

$$\mathcal{A}' := \mathcal{A} \otimes \kappa_q[x, y] = \mathcal{A}\{x, y\}/(yx - qxy)$$

then $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$*, with entries in* $\mathcal{A}$*, is an* $\mathcal{A}$*-point of* $M_q(2)$ *if and only if*

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X' \\ Y' \end{pmatrix} \quad \text{and} \quad \begin{bmatrix} A & B \\ C & D \end{bmatrix}^T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X'' \\ Y'' \end{pmatrix}$$

where (X', Y') *and* (X'', Y'') *are* $\mathcal{A}'$*-points of the quantum plane.*

Proof. Begin by supposing that $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is an $\mathcal{A}$-point of $M_q(2)$. By Proposition 6.14

$$\hom_{Alg}(\kappa_q[x,y],\mathcal{A}') \equiv \{(X,Y) \in \mathcal{A}'^2 : YX = qXY\}$$

So, to be $\mathcal{A}'$-points of the quantum plane, (X',Y') and (X'',Y'') must be members of the set on the right. To check this, let us first determine what X',Y',X'' and Y'' are specifically. First, X' and Y':

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A\otimes x + B\otimes y \\ C\otimes x + D\otimes y \end{pmatrix}$$

So, $X' = A\otimes x + B\otimes y$ and $Y' = C\otimes x + D\otimes y$ are elements of $\mathcal{A}'$. Now we see if they satisfy the appropriate commutation relation.

$$\begin{aligned}
Y'X' &= (C\otimes x + D\otimes y)(A\otimes x + B\otimes y) \\
&= CA\otimes x^2 + CB\otimes xy + DA\otimes yx + DB\otimes y^2 \\
&= qAC\otimes x^2 + q^{-1}BC\otimes yx + (qAD - (1-q^2)BC)\otimes xy + qBD\otimes y^2 \\
&= qAC\otimes x^2 + q^{-1}BC\otimes yx + qAD\otimes xy - (1-q^2)BC\otimes xy + qBD\otimes y^2 \\
&= qAC\otimes x^2 + q^{-1}BC\otimes yx - (q^{-1}-q)BC\otimes yx + qAD\otimes xy + qBD\otimes y^2 \\
&= qAC\otimes x^2 + qBC\otimes yx + qAD\otimes xy + qBD\otimes y^2 \\
&= q(AC\otimes x^2 + AD\otimes xy + BC\otimes yx + BD\otimes y^2) \\
&= q(A\otimes x + B\otimes y)(C\otimes x + D\otimes y) \\
&= qX'Y'
\end{aligned}$$

Therefore, (X',Y') is shown to be an $\mathcal{A}'$-point of the quantum plane.

We proceed similarly for X'' and Y''.

$$\begin{bmatrix} A & C \\ B & D \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A\otimes x + C\otimes y \\ B\otimes x + D\otimes y \end{pmatrix}$$

So $X'' = A \otimes x + C \otimes y \in \mathcal{A}'$ and $Y'' = B \otimes x + D \otimes y \in \mathcal{A}'$. Now

$$\begin{aligned}
Y''X'' &= (B \otimes x + D \otimes y)(A \otimes x + C \otimes y) \\
&= BA \otimes x^2 + BC \otimes xy + DA \otimes yx + DC \otimes y^2 \\
&= qAB \otimes x^2 + q^{-1}CB \otimes yx + (AD - (q^{-1} - q)BC) \otimes yx + qCD \otimes y^2 \\
&= qAB \otimes x^2 + q^{-1}CB \otimes yx + AD \otimes yx - (q^{-1} - q)BC \otimes yx + qCD \otimes y^2 \\
&= qAB \otimes x^2 + qCB \otimes yx + qAD \otimes xy + qCD \otimes y^2 \\
&= q(AB \otimes x^2 + AD \otimes xy + CB \otimes yx + CD \otimes y^2) \\
&= q(A \otimes x + C \otimes y)(B \otimes x + D \otimes y) \\
&= qX''Y''
\end{aligned}$$

Therefore, (X'', Y'') is also an $\mathcal{A}'$-point of the quantum plane.

Now let us suppose that we have $\mathcal{A}'$-points of the quantum plane (X', Y') and (X'', Y'') and a matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ with entries in $\mathcal{A}$ such that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X' \\ Y' \end{pmatrix} \quad \text{and} \quad \begin{bmatrix} A & B \\ C & D \end{bmatrix}^T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X'' \\ Y'' \end{pmatrix}$$

The goal is to show that $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in M_2^q(\mathcal{A})$, which amounts to showing that A, B, C and D obey the six relations given above. Since we have $Y'X' = qX'Y'$ and $Y''X'' = qX''Y''$ along with the above matrix equations, this is done using the same argument that we used above to arrive at the six relations in the first place. □

Note that this proposition is just another way of understanding the equivalence we explored in the motivation of $M_q(2)$ above.

6.4.2 Quantum Determinant

Because of the important role of the determinant in ordinary matrix theory, we take a moment to explore the quantum version. Let's begin by looking at the determinant in its usual form: $ad - bc$. Using relations 5 and 6 we find that

$$\begin{aligned}
ad - bc &= (q^{-1} - q)bc + da - bc \\
&= q^{-1}bc - qbc + da - bc
\end{aligned}$$

which implies that

$$ad = q^{-1}bc - qbc + da \quad \Longleftrightarrow \quad ad - q^{-1}bc = da - qbc$$

The latter resulting equation is what we take to be the quantum determinant. That is, we define

$$\det_q := ad - q^{-1}bc = da - qbc \in M_q(2)$$

Once more we note that the usual determinant is recovered if $q = 1$. It can also be shown that $\det_q$ is in the center of $M_q(2)$ due to the fact that it commutes with the generators a, b, c, d. For example,

$$\begin{aligned} a\det_q &= a(da - qbc) \\ &= ada - qabc \\ &= ada - bac \\ &= ada - q^{-1}bca \\ &= (ad - q^{-1}bc)a \\ &= \det_q a \end{aligned}$$

When referring to an $\mathcal{A}$-point of $M_q(2)$, say $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, we shall denote its determinant by

$$\mathrm{Det}_q(M) = AD - q^{-1}BC$$

Proposition 6.22. *The matrix product of* $\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \in M_2^q(\mathcal{A})$*, where* A, B, C, D *commute with* A', B', C', D'*, is also a member of* $M_2^q(\mathcal{A})$*.*

Proof. Here we make use of Proposition 6.21. Let us agree to write

$$\mathcal{A} \otimes \kappa_q[X, Y] = \mathcal{A}\{X, Y\}/(YX - qXY)$$

as $\mathcal{A}_q[X, Y]$. Let M and M' respectively denote the hypothesized $\mathcal{A}$-points of $M_q(2)$ and take $v = \begin{pmatrix} X \\ Y \end{pmatrix}$. Since M is an $\mathcal{A}$-point of $M_q(2)$ we have that $Mv = u$ and $M^t v = u'$, where u, u' are $\mathcal{A}_q[X, Y]$-points of the quantum plane. Likewise, since M' is an $\mathcal{A}$-point of $M_q(2)$, we have that $M'v = w$ and $M'^t v = w'$, where w, w' are $\mathcal{A}_q[X, Y]$-points of the quantum plane. Thus

$$\begin{aligned}(MM')v &= M(M'v)\\ &= Mw\end{aligned}$$

Now, w is an $\mathcal{A}_q[X,Y]$-point of the quantum plane since $M' \in M_2^q(\mathcal{A})$. By hypothesis, the entries in w will commute with the entries of M; so, Mw is an $\mathcal{A}_q[X,Y]$-point of the quantum plane. Also,

$$\begin{aligned}(MM')^t v &= (M'^t M^t)v\\ &= M'^t(M^t v)\\ &= M'^t u'\end{aligned}$$

Again, the entries of u' will commute with the entries of M'^t and so $M'^t u'$ is an $\mathcal{A}_q[X,Y]$-point of the quantum plane. Therefore, $MM' \in M_2^q(\mathcal{A})$. $\square$

Proposition 6.23. *If M and M' are as in the previous proposition, then*

$$Det_q(MM') = Det_q(M)Det_q(M')$$

Proof. Since

$$MM' = \begin{bmatrix} AA' + BC' & AB' + BD' \\ CA' + DC' & CB' + DD' \end{bmatrix}$$

we have

$$Det_q(MM') = (AA' + BC')(CB' + DD') - q^{-1}(AB' + BD')(CA' + DC')$$

Expanding the right hand side yields

$$AA'CB' + AA'DD' + BC'CB' + BC'DD' - q^{-1}AB'CA' - q^{-1}AB'DC' - q^{-1}BD'CA' - q^{-1}BD'DC'$$

Now rearrange the terms to get

$$AA'CB' - q^{-1}AB'CA' + BC'DD' - q^{-1}BD'DC' + AA'DD' - q^{-1}AB'DC' - q^{-1}BD'CA' + BC'CB'$$

Consider the first two terms

$$\begin{aligned} AA'CB' - q^{-1}AB'CA' &= ACA'B' - q^{-1}AB'CA' \quad [A', C \text{ commute}] \\ &= q^{-1}ACB'A' - q^{-1}AB'CA' \quad [A'B' = q^{-1}B'A'] \\ &= q^{-1}AB'CA' - q^{-1}AB'CA' \quad [B', C \text{ commute}] \\ &= 0 \end{aligned}$$

We get the same result for the next two terms. That is,

$$BC'DD' - q^{-1}BD'DC' = 0$$

and so $Det_q(MM')$ reduces to

$$\begin{aligned} &AA'DD' - q^{-1}AB'DC' - q^{-1}BD'CA' + BC'CB' \\ =&ADA'D' - q^{-1}ADB'C' - q^{-1}BCD'A' + BCB'C' \\ =&ADA'D' - q^{-1}ADB'C' - q^{-1}BC(A'D' - (q^{-1} - q)B'C') + BCB'C' \\ =&ADA'D' - q^{-1}ADB'C' - q^{-1}BCA'D' + q^{-2}BCB'C' - BCB'C' + BCB'C' \\ =&ADA'D' - q^{-1}ADB'C' - q^{-1}BCA'D' + q^{-2}BCB'C' \\ =&(AD - q^{-1}BC)(A'D' - q^{-1}B'C') \\ =&Det_q(M)Det_q(M') \end{aligned}$$

□

Proposition 6.24. *Let* $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ *be an* $\mathcal{A}$*-point of* $M_q(2)$*. Then the matrix*

$$\begin{bmatrix} D & -qB \\ -q^{-1}C & A \end{bmatrix}$$

is an $\mathcal{A}$*-point of* $M_{q^{-1}}(2)$ *(i.e. in* $M_2^{q^{-1}}(\mathcal{A})$*).*

Proof. Define $A' := D, B' := -qB, C' := -q^{-1}C$ and $D' := A$. Then using the six relations which hold for A, B, C and D we see that

$$\begin{aligned} B'A' &= -qBD \\ &= -DB \\ &= q^{-1}A'B' \end{aligned}$$

$$\begin{aligned} C'A' &= -q^{-1}CD \\ &= -q^{-2}DC \\ &= q^{-1}A'C' \end{aligned}$$

$$\begin{aligned} B'C' &= -q(-q^{-1})BC \\ &= CB \\ &= -q(-q^{-1})C'B' \\ &= C'B' \end{aligned}$$

$$\begin{aligned} D'B' &= -qAB \\ &= -BA \\ &= q^{-1}B'D' \end{aligned}$$

$$\begin{aligned} D'C' &= -q^{-1}AC \\ &= -q^{-2}CA \\ &= q^{-1}C'D' \end{aligned}$$

$$\begin{aligned} A'D' - D'A' &= DA - AD \\ &= (q - q^{-1})BC \\ &= (q - q^{-1})(-q^{-1})(-q)B'C' \\ &= (q - q^{-1})B'C' \end{aligned}$$

By definition, these six relations mean that (A', B', C', D') is an $\mathcal{A}$-point of $M_{q^{-1}}(2)$. $\square$

Interestingly, being an $\mathcal{A}$-point of $M_{q^{-1}}(2)$ is equivalent to being an $\mathcal{A}^{op}$-point of $M_q(2)$, since if (A, B, C, D) is an $\mathcal{A}$-point of $M_{q^{-1}}(2)$, then the six relations obtained in the previous proof can be rewritten as

$$AB = qBA$$
$$AC = qCA$$
$$CB = BC$$
$$BD = qDB$$
$$CD = qDC$$
$$DA - AD = (q^{-1} - 1)BC$$

Note that these are the usual six relations for a point of $M_q(2)$, but with multiplication reversed, that is, the multiplication given in $\mathcal{A}^{op}$. Thus, (A, B, C, D) is an $\mathcal{A}^{op}$-point of $M_q(2)$.

Although $M_q(2)$ is a deformed version of $M(2)$ and is no longer commutative, it nevertheless retains some key properties of $M(2)$. For instance, $M(2)$ is Noetherian and has no zero-divisors. This holds true for $M_q(2)$ as well. The common means of showing this is via iterated Ore extensions. In other words, one builds a tower of algebras

$$A_0 \subset A_1 \subset A_2 \subset A_3 \subset A_4$$

where each A_i is an Ore extension of A_{i-1}. One then applies Theorem 6.13.

To make this work for $M_q(2)$ one starts with $A_0 := \kappa$, since all fields are Noetherian. The goal, then, is to construct Ore extensions so that one ends up with $A_4 = M_q(2)$. This is accomplished by defining the intermediate algebras by

$$A_1 := \kappa[a], \qquad A_2 := \frac{\kappa\{a, b\}}{(ba - qab)}$$

$$A_3 := \frac{\kappa\{a, b, c\}}{(ba - qab, ca - qac, cb - bc)}$$

It is trivial that A_1 is an Ore extension of A_0. Further, note that A_2 is just the quantum plane in the variables a, b, which we already know is isomorphic to the Ore extension $A_1[b, \alpha_1, 0]$ where α_1 is defined by $\alpha_1(a) = qa$. We therefore get that $\{a^i b^j\}_{i,j\geq 0}$ is a basis for A_2.

For the next case, one defines $\alpha_2(a) := qa$ and $\alpha_2(b) := b$, which is easily seen to be an automorphism of A_2, since $\alpha_2(b)\alpha_2(a) = q\alpha_2(a)\alpha_2(b)$. Now in $A_2[c, \alpha_2, 0]$, $ba = qab$

from the product in A_2, but we also have

$$\begin{aligned} ca &= \alpha_2(a)c + 0(a) \\ &= qac \end{aligned}$$

$$\begin{aligned} cb &= \alpha_2(b)c + 0(b) \\ &= bc \end{aligned}$$

With this we get that $A_3 \cong A_2[c, \alpha_2, 0]$ as algebras, since they are the same vector space and have the same defining multiplication relations. The set $\{a^i b^j c^k\}_{i,j,k \geq 0}$ is a basis of A_3.

We now pass to the last and most interesting case. That is, we want to show that $M_q(2)$ is isomorphic to the Ore extension $A_4 = A_3[d, \alpha_3, \mathscr{D}]$, where α_3 is defined by

$$\alpha_3(a) := a, \quad \alpha_3(b) := qb, \quad \alpha_3(c) := qc$$

and $\mathscr{D}$ is defined on the generators of A_3 by

$$\mathscr{D}(a) := (q - q^{-1})bc, \quad \mathscr{D}(b) := 0, \quad \mathscr{D}(c) := 0$$

Note that α_3 is an algebra morphism since it preserves the relations of A_3.

$$\begin{aligned} \alpha_3(b)\alpha_3(a) &= qba \\ &= q^2 ab \\ &= q\alpha_3(a)\alpha_3(b) \end{aligned}$$

The other two are similarly verified.

Now, $\mathscr{D}$ can certainly be extended by linearity. But we can also extend $\mathscr{D}$ by the derivation relation, since it will depend uniquely on α and what $\mathscr{D}$ does to the generating elements. It only remains to show that if we do extend $\mathscr{D}$ in this way, that it preserves the defining relations of A_3. Consider first the case $ba = qab$.

$$\begin{aligned} \mathscr{D}(ba) &= \alpha_3(b)\mathscr{D}(a) + \mathscr{D}(b)a \\ &= qb(q - q^{-1})bc + 0 \cdot a \\ &= (q^2 - 1)b^2 c \end{aligned}$$

and also

$$\begin{aligned}\mathscr{D}(qab) &= q(\alpha_3(a)\mathscr{D}(b) + \mathscr{D}(a)b) \\ &= q(a \cdot 0 + (q - q^{-1})bcb) \\ &= (q^2 - 1)b^2c\end{aligned}$$

For $ca = qac$ we have:

$$\begin{aligned}\mathscr{D}(ca) &= \alpha_3(c)\mathscr{D}(a) + \mathscr{D}(c)a \\ &= qc(q - q^{-1})bc + 0 \cdot a \\ &= (q^2 - 1)bc^2\end{aligned}$$

and

$$\begin{aligned}\mathscr{D}(qac) &= q(\alpha_3(a)\mathscr{D}(c) + \mathscr{D}(a)c) \\ &= q(a \cdot 0 + (q - q^{-1})bcc) \\ &= (q^2 - 1)bc^2\end{aligned}$$

Finally, it is obvious that $\mathscr{D}(bc) = \mathscr{D}(cb)$, since both will be zero.

We can now proceed to see how this α_3-derivation will act on any basis element. From the definition of $\mathscr{D}$ one can easily deduce that $\mathscr{D}(b^j) = 0$ and $\mathscr{D}(c^k) = 0$ for all $j, k \in \mathbb{N}$. For example,

$$\begin{aligned}\mathscr{D}(b^2) &= \alpha_3(b)\mathscr{D}(b) + \mathscr{D}(b)b \\ &= \alpha_3(b) \cdot 0 + 0 \cdot b \\ &= 0\end{aligned}$$

Therefore,

$$\begin{aligned}\mathscr{D}(b^jc^k) &= \alpha_3(b^j)\mathscr{D}(c^k) + \mathscr{D}(b^j)c^k \\ &= \alpha_3(b^j) \cdot 0 + 0 \cdot c^k \\ &= 0\end{aligned}$$

Now let's do the more difficult one, namely $\mathscr{D}(a^i)$. We use induction on i to show that

$$\mathscr{D}(a^i) = (q - q^{-1})\frac{1 - q^{2i}}{1 - q^2}a^{i-1}bc \tag{6.15}$$

Let's do the first two base cases for insight

$$\begin{aligned}\mathscr{D}(a) &= (q-q^{-1})bc \\ &= (q-q^{-1})\frac{1-q^2}{1-q^2}a^0bc\end{aligned}$$

$$\begin{aligned}\mathscr{D}(a^2) &= \alpha_3(a)\mathscr{D}(a) + \mathscr{D}(a)a \\ &= a(q-q^{-1})bc + (q-q^{-1})bca \\ &= (q-q^{-1})(abc+bca) \\ &= (q-q^{-1})(abc+q^2abc) \\ &= (q-q^{-1})(1+q^2)abc \\ &= (q-q^{-1})\frac{1-q^4}{1-q^2}abc\end{aligned}$$

Assume that the result holds up to some $m \in \mathbb{N}$ and consider the case for $m+1$.

$$\begin{aligned}\mathscr{D}(a^{m+1}) &= \alpha_3(a)\mathscr{D}(a^m) + \mathscr{D}(a)a^m \\ &= a(q-q^{-1})\frac{1-q^{2m}}{1-q^2}a^{m-1}bc + (q-q^{-1})bca^m \\ &= (q-q^{-1})\left(\frac{1-q^{2m}}{1-q^2}a^mbc + bca^m\right) \\ &= (q-q^{-1})\left(\frac{1-q^{2m}}{1-q^2}a^mbc + q^{2m}a^mbc\right) \\ &= (q-q^{-1})\left(\frac{1-q^{2m}}{1-q^2} + q^{2m}\right)a^mbc \\ &= (q-q^{-1})\left(\frac{1-q^{2m}+q^{2m}(1-q^2)}{1-q^2}\right)a^mbc \\ &= (q-q^{-1})\frac{1-q^{2(m+1)}}{1-q^2}a^mbc\end{aligned}$$

Therefore, by mathematical induction (6.15) is established for all i. Putting this all together yields

$$\begin{aligned}\mathscr{D}(a^ib^jc^k) &= \alpha_3(a^i)\mathscr{D}(b^jc^k) + \mathscr{D}(a^i)b^jc^k \\ &= \mathscr{D}(a^i)b^jc^k \\ &= (q-q^{-1})\frac{1-q^{2i}}{1-q^2}a^{i-1}bcb^jc^k \\ &= (q-q^{-1})\frac{1-q^{2i}}{1-q^2}a^{i-1}b^{j+1}c^{k+1}\end{aligned}$$

We can express this more succinctly as $\mathscr{D}(a^ib^jc^k) = (q-q^{-1})(i)_{q^2}a^{i-1}b^{j+1}c^{k+1}$.

Let us now show that $M_q(2) \cong A_3[d, \alpha_3, \mathscr{D}]$. Since they are the same vector space, we need only show that they have the same algebra structure. Observe that three of the relations hold by definition of A_3. Let us show that the other three hold as well.

$$\begin{aligned} db &= \alpha_3(b)d + \mathscr{D}(b) \\ &= qbd \end{aligned}$$

$$\begin{aligned} dc &= \alpha_3(c)d + \mathscr{D}(c) \\ &= qcd \end{aligned}$$

$$\begin{aligned} da &= \alpha_3(a)d + \mathscr{D}(a) \\ &= ad + (q - q^{-1})bc \end{aligned}$$

The last one implies that $ad - da = (q^{-1} - q)bc$. Therefore, all six defining relations for $M_q(2)$ are also the defining relations for $A_3[d, \alpha_3, \mathscr{D}]$; hence, they must be the same algebra. By now applying the extended Hilbert Basis Theorem (Theorem 6.13) one gets a Noetherian $M_q(2)$.

But we also claimed that $M_q(2)$ has no zero divisors and this follows directly from Corollary 6.12.

Another property of $M(2)$ not lost in deformation is its bialgebra structure. In fact, we can endow $M_q(2)$ with a bialgebra structure without having to change the coproduct or counit maps defined on $M(2)$. Recall that these were defined according to the matrix relations

$$\begin{aligned} \Delta \begin{bmatrix} a & b \\ c & d \end{bmatrix} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ \varepsilon \begin{bmatrix} a & b \\ c & d \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Of course, while the bialgebra structure is retained, it is nevertheless "deformed" in that it is no longer commutative or cocommutative.

To verify this, the maps Δ and ε must still be morphisms of algebras. So we need

$$\Delta \in \hom_{Alg}(M_q(2), M_q(2) \otimes M_q(2)) \equiv M_2^q(M_q(2) \otimes M_q(2))$$

and

$$\varepsilon \in \hom_{Alg}(M_q(2), \kappa) \equiv M_2^q(\kappa)$$

which means that $\Delta \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is an $M_q(2) \otimes M_q(2)$-point of $M_q(2)$ and $\varepsilon \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a κ-point of $M_q(2)$. The case for the counit is clear due to its definition. The coproduct case is a consequence of Proposition 6.22, since $\begin{bmatrix} a \otimes 1 & b \otimes 1 \\ c \otimes 1 & d \otimes 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \otimes a & 1 \otimes b \\ 1 \otimes c & 1 \otimes d \end{bmatrix}$ are clearly $M_q(2) \otimes M_q(2)$-points and so therefore their product, $\begin{bmatrix} a \otimes a + b \otimes c & a \otimes b + b \otimes d \\ c \otimes a + d \otimes c & c \otimes b + d \otimes d \end{bmatrix}$, must be also. Thus, both the coproduct and the counit are algebra morphisms.

It remains to show that the coassociativity and counit axioms are still satisfied. Once more, the counit axioms are clear from the matrix relation for ε. Coassociativity is nearly just as transparent. The matrix form commends itself here too, since it allows one to simply use the associativity of the matrix product. That is,

$$\begin{aligned}
\big((\Delta \otimes \mathrm{id}) \circ \Delta\big) \begin{bmatrix} a & b \\ c & d \end{bmatrix} &= \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) \otimes \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\
&= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) \\
&= \big((\mathrm{id} \otimes \Delta) \circ \Delta\big) \begin{bmatrix} a & b \\ c & d \end{bmatrix}
\end{aligned}$$

The result is a non-commuting and non-cocommuting bialgebra. We should also take note of what happens to $\det_q$ under Δ and ε. It can computationally be shown that

$$\Delta(\det_q) = \det_q \otimes \det_q \quad \text{and} \quad \varepsilon(\det_q) = 1$$

So, $\det_q$ remains grouplike for $q \neq 1$. Finally, just as the affine plane $\kappa[x, y]$ is a comodule-algebra over $M(2)$, there is a unique $M_q(2)$-comodule-algebra structure on the quantum plane $\kappa_q[x, y]$ where we write

$$\delta_{\kappa_q[x,y]} \begin{pmatrix} x \\ y \end{pmatrix} := \begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix}$$

Now recall Proposition 4.41 which specifies two conditions that need to be satisfied: (1) $\delta_{\kappa_q[x,y]}$ needs to define an $M_q(2)$-comodule structure on the quantum plane; (2) the map $\delta_{\kappa_q[x,y]}$ must be a morphism of algebras. Let's start with condition (2). It suffices to show that

$$\delta_{\kappa_q[x,y]}(y)\delta_{\kappa_q[x,y]}(x) = q\delta_{\kappa_q[x,y]}(x)\delta_{\kappa_q[x,y]}(y)$$

since then $(\delta_{\kappa_q[x,y]}(x), \delta_{\kappa_q[x,y]}(y))$ will be a $M_q(2) \otimes \kappa_q[x,y]$-point of the quantum plane. Begin with the left hand side.

$$\begin{aligned}
\delta_{\kappa_q[x,y]}(y)\delta_{\kappa_q[x,y]}(x) &= (c \otimes x + d \otimes y)(a \otimes x + b \otimes y) \\
&= ca \otimes x^2 + cb \otimes xy + da \otimes yx + db \otimes y^2 \\
&= qac \otimes x^2 + (bc + qda) \otimes xy + qbd \otimes y^2 \\
&= q(ac \otimes x^2 + (q^{-1}bc + da) \otimes xy + bd \otimes y^2) \\
&= q(ac \otimes x^2 + (ad + qbc) \otimes xy + bd \otimes y^2) \\
&= q(ac \otimes x^2 + ad \otimes xy + bc \otimes yx + bd \otimes y^2) \\
&= q(a \otimes x + b \otimes y)(c \otimes x + d \otimes y) \\
&= q\delta_{\kappa_q[x,y]}(x)\delta_{\kappa_q[x,y]}(y)
\end{aligned}$$

So, $\delta_{\kappa_q[x,y]}$ is a morphism of algebras.

Next, by Definition 4.23, (1) holds provided

$$(\mathrm{id} \otimes \delta_{\kappa_q[x,y]}) \circ \delta_{\kappa_q[x,y]}(z) = (\Delta \otimes \mathrm{id}) \circ \delta_{\kappa_q[x,y]}(z)$$

$$(\varepsilon \otimes \mathrm{id}) \circ \delta_{\kappa_q[x,y]}(z) = 1 \otimes z$$

for all $z \in \kappa_q[x,y]$. Thankfully, because the quantum plane is generated by x and y and all maps involved are algebra morphisms we can restrict our considerations to $z = x$ and $z = y$. Thanks again to matrix notation, we check these simultaneously.

$$\begin{aligned}
(\mathrm{id} \otimes \delta_{\kappa_q[x,y]}) \circ \delta_{\kappa_q[x,y]} \begin{pmatrix} x \\ y \end{pmatrix} &= (\mathrm{id} \otimes \delta_{\kappa_q[x,y]}) \begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix} \\
&= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix}
\end{aligned}$$

But also

$$\begin{aligned}
(\Delta \otimes \mathrm{id}) \circ \delta_{\kappa_q[x,y]} \begin{pmatrix} x \\ y \end{pmatrix} &= (\Delta \otimes \mathrm{id}) \begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix} \\
&= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix}
\end{aligned}$$

Next

$$(\varepsilon \otimes \mathrm{id}) \circ \Delta_{\kappa_q[x,y]} \begin{pmatrix} x \\ y \end{pmatrix} = (\varepsilon \otimes \mathrm{id}) \begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix}$$

While $M_q(2)$ is a quantum *algebra* it fails to be a quantum group. This is because it is not a Hopf algebra, which, in this case means that it lacks an antipode. Even so we now have a good framework for introducing the two quantum groups promised at the beginning of this section, namely $GL_q(2)$ and $SL_q(2)$.

6.4.3 The Quantum Groups $GL_q(2)$ and $SL_q(2)$

Both are defined analogously to their non-deformed counterparts, the former being

$$GL_q(2) := \frac{M_q(2)[t]}{(t\det_q - 1)}$$

and the latter

$$SL_q(2) := \frac{M_q(2)}{(\det_q - 1)}$$

For any algebra $\mathcal{A}$, then, an $\mathcal{A}$-point for $GL_q(2)$ is simply an $\mathcal{A}$-point of $M_q(2)$, say $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, but with the added condition that

$$\mathrm{Det}_q(M) = AD - q^{-1}BC$$

be invertible in $\mathcal{A}$. The same holds for $SL_q(2)$ except $\mathrm{Det}_q(M)$ must be 1.

Theorem 6.25. *The coproduct and counit of $M_q(2)$ equip the algebras $GL_q(2)$ and $SL_q(2)$ with Hopf algebra structures such that the antipode S is given in matrix form by*

$$S \begin{bmatrix} a & b \\ c & d \end{bmatrix} := det_q^{-1} \begin{bmatrix} d & -qb \\ -q^{-1}c & a \end{bmatrix}$$

Proof. Because $GL_q(2)$ and $SL_q(2)$ are quotient spaces, it is necessary to ensure that Δ and ε are still well-defined. Consider that

$$\begin{aligned}\Delta(\det_q - 1) &= \det_q \otimes \det_q - 1 \otimes 1 \\ &= \det_q \otimes \det_q - 1 \otimes \det_q + 1 \otimes \det_q - 1 \otimes 1 \\ &= (\det_q - 1) \otimes \det_q + 1 \otimes (\det_q - 1) \\ &= 0 \quad [\text{as an element of } SL_q(2) \otimes SL_q(2)]\end{aligned}$$

and

$$\begin{aligned}\varepsilon(\det_q - 1) &= \varepsilon(\det_q) - 1 \\ &= 1 - 1 = 0\end{aligned}$$

This implies that Δ and ε are well-defined for $SL_q(2)$ and the same will be true of $GL_q(2)$ provided we set $\Delta(t) := t \otimes t$ and $\varepsilon(t) := 1$. The computations are essentially the same:

$$\begin{aligned}\Delta(t\det_q - 1) &= (t \otimes t)(\det_q \otimes \det_q) - 1 \otimes 1 \\ &= t\det_q \otimes t\det_q - 1 \otimes 1 \\ &= t\det_q \otimes t\det_q - 1 \otimes t\det_q + 1 \otimes t\det_q - 1 \otimes 1 \\ &= (t\det_q - 1) \otimes t\det_q + 1 \otimes (t\det_q - 1) \\ &= 0 \quad [\text{as an element of } GL_q(2) \otimes GL_q(2)]\end{aligned}$$

and

$$\begin{aligned}\varepsilon(t\det_q - 1) &= \varepsilon(t)\varepsilon(\det_q) - \varepsilon(1) \\ &= 1 \cdot 1 - 1 \\ &= 0\end{aligned}$$

Given that the coproduct and counit are well-defined, we immediately get that the coassociativity and counit axioms hold on account of their holding for $M_q(2)$. We therefore get to skip straight to showing that $GL_q(2)$ and $SL_q(2)$ have an antipode.

Begin by setting

$$S' \begin{bmatrix} a & b \\ c & d \end{bmatrix} := \begin{bmatrix} d & -qb \\ -q^{-1}c & a \end{bmatrix}$$

This is just the q-analogue of the 2×2 adjoint matrix. By Proposition 6.24, $S' \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

is an $M_q(2)^{op}$-point of $M_q(2)$ thereby implying that S' is a morphism of algebras from $M_q(2)$ to $M_q(2)^{op}$. If we now set $S'(t) = t$, then S' extends to $GL_q(2)$ and $SL_q(2)$. It is well-defined for $GL_q(2)$ because

$$\begin{aligned} S'(t\det_q - 1) &= S'(t)S'(\det_q) - 1 \\ &= t\big(S'(d)S'(a) - q^{-1}S'(c)S'(b)\big) - 1 \\ &= t(ad - q^{-1}bc) - 1 \\ &= t\det_q - 1 = 0 \end{aligned}$$

This reasoning also shows that $S'(\det_q - 1) = \det_q - 1$ and therefore S' is well-defined for $SL_q(2)$ as well. Now, since $\det_q$ is invertible and is in the center of both $GL_q(2)$ and $SL_q(2)$ we are able to define a morphism of algebras

$$S : GL_q(2)(resp.SL_q(2)) \to GL_q(2)^{op}(resp.SL_q(2)^{op})$$

by $S(t) := t^{-1}$ and

$$S\begin{bmatrix} a & b \\ c & d \end{bmatrix} := \det_q^{-1} S' \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

By Lemma 4.16, S is an antipode provided

$$\sum_{(x)} x^{(1)}S(x^{(2)}) = 1_\star(x) = \sum_{(x)} S(x^{(1)})x^{(2)}$$

for all $x \in \{a, b, c, d\}$. In matrix form

$$\begin{aligned} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -qb \\ -q^{-1}c & a \end{bmatrix} &= \begin{bmatrix} ad - q^{-1}bc & -qab + ba \\ cd - q^{-1}dc & -qcb + da \end{bmatrix} \\ &= \begin{bmatrix} \det_q & 0 \\ 0 & \det_q \end{bmatrix} = \det_q \varepsilon \begin{bmatrix} a & b \\ c & d \end{bmatrix} \end{aligned}$$

But also

$$\begin{aligned} \begin{bmatrix} d & -qb \\ -q^{-1}c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} &= \begin{bmatrix} da - qbc & db - qbd \\ -q^{-1}ca + ac & -q^{-1}cb + ad \end{bmatrix} \\ &= \begin{bmatrix} \det_q & 0 \\ 0 & \det_q \end{bmatrix} = \det_q \varepsilon \begin{bmatrix} a & b \\ c & d \end{bmatrix} \end{aligned}$$

Therefore, S is an antipode for $GL_q(2)$ and $SL_q(2)$ thereby making them Hopf algebras and hence *quantum groups*. $\square$

Given that $GL_q(2) := M_q(2)[t]/(t\det_q - 1)$ and $SL_q(2) := M_q(2)/(\det_q - 1)$, it is not hard to see that the classical objects $GL(2)$ and $SL(2)$ are recovered when $q = 1$. Since $M_q(2)$ is the quotient of the free algebra $\kappa\{a, b, c, d\}$ by the ideal J_q generated by the relations $ba = qab$, $db = qbd$, $ca = qac$, $dc = qcd$, $bc = cb$ and $ad - da = (q - q^{-1})bc$, we see that when $q = 1$, these respectively become

$$ba = ab, \quad db = bd, \quad ca = ac, \quad dc = cd, \quad bc = cb, \quad ad - da = 0$$

which are the usual commutation relations yielding $M(2) = \kappa[a, b, c, d]$. Also,

$$\det_1 = ad - 1^{-1}bc = ad - bc$$

which is the usual determinant. Thus, $GL_1(2)$ and $SL_1(2)$ are the desired classical objects.

So, deformations can give us non-commutative, non-cocommutative Hopf algebras, as is the case here, which correspond to quantum groups. In other words, these were the sort of Hopf algebras alluded to in Chapter 4, which led to the concept of a quantum group by studying $Alg_{\mathbb{C}}(H, \mathbb{C})$. So, in this case one tries to understand $Alg_{\mathbb{C}}(GL_q(2), \mathbb{C})$ and $Alg_{\mathbb{C}}(SL_q(2), \mathbb{C})$, but we simply identify the quantum groups with the representing objects, namely $GL_q(2)$ and $SL_q(2)$.

It turns out that $SL_q(2)$ is the "simplest" of quantum groups, but unless one is thoroughly acquainted with the subject, this fact does not appear very meaningful. There is still much that could be said about $SL_q(2)$ (and $GL_q(2)$) and one should not mistake this to mean that these are not important quantum groups. For our purposes, these have served as a sample, which is why we have treated them with brevity. Now that the reader has a taste, we shall proceed to consider, in depth, one of the most important quantum groups, namely $U_q(\mathfrak{sl}(2))$, which is one of those special types arising from dropping finite-dimensionality of the Hopf algebra.

Chapter 7

The Quantum Enveloping Algebra $U_q(\mathfrak{sl}(2))$

7.1 Introduction

This chapter is concerned with familiarizing the reader with perhaps the most well known quantum group, namely $U_q(\mathfrak{sl}(2))$, which is a one parameter deformation of the universal enveloping algebra of the simple Lie algebra $\mathfrak{sl}(2)$. This quantum group is actually but one in a family of quantum groups said to be of enveloping algebra type. These are denoted by $U_q(\mathfrak{g})$, where $\mathfrak{g}$ is a general complex simple Lie algebra and were essentially developed in the mid-1980's by Drinfeld and Jimbo. The specific case of $U_q(\mathfrak{sl}(2))$, however, is of primary importance because much of the theory for the general case is an extension from $U_q(\mathfrak{sl}(2))$.

7.2 Some Basic Properties of $U_q(\mathfrak{sl}(2))$

7.2.1 q-Analysis Revisited

We set the stage for this section by developing some notation, which should elicit fond memories of basic combinatorics. Let κ be a fixed field. To simplify matters, it is common to work with $\kappa = \mathbb{C}$. This will be our assumption throughout this chapter. Suppose q is an invertible element of κ. For any $n \in \mathbb{Z}$ we define a new deformation (compare to previous chapter) $[n]$ by

$$[n] := \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \ldots + q^{3-n} + q^{1-n}$$

which is an example of what is called a Laurent polynomial. The algebra of Laurent polynomials in one variable, in this case q, is denoted by $\kappa[q, q^{-1}]$ and is isomorphic to $\kappa[q, r]/I$, where I is the ideal of $\kappa[q, r]$ generated by $qr - 1$. This provides an alternative way of defining a q-deformation of an integer from that seen in the previous chapter. Again, one can show via limits that n is reacquired in the limit $q \to 1$.

An advantage of this version is that it is more symmetric. For instance, the additive inverse of $[n]$ is

$$\begin{aligned} -[n] &= -\frac{q^n - q^{-n}}{q - q^{-1}} \\ &= \frac{q^{-n} - q^n}{q - q^{-1}} \\ &= [-n] \end{aligned}$$

So, the additive inverse of the deformation of n is the deformation of the additive inverse of n. We also find that

$$\begin{aligned} [n + m] &= \frac{q^{n+m} - q^{-(n+m)}}{q - q^{-1}} \\ &= \frac{q^n q^m - q^{-n} q^{-m}}{q - q^{-1}} \\ &= \frac{q^n q^m - q^n q^{-m} + q^n q^{-m} - q^{-n} q^{-m}}{q - q^{-1}} \\ &= \frac{q^n (q^m - q^{-m}) + q^{-m}(q^n - q^{-n})}{q - q^{-1}} \\ &= q^n \frac{q^m - q^{-m}}{q - q^{-1}} + q^{-m} \frac{q^n - q^{-n}}{q - q^{-1}} \\ &= q^n [m] + q^{-m}[n] \end{aligned}$$

For $0 \le k \le n$ we define another analogue to the usual factorial by

$$[0]! := 1, \qquad [k]! := [1][2]\cdots[k] \quad (k > 0)$$

Again, this allows us to construct another analogue to the binomial coefficient.

$$\begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n]!}{[k]![n-k]!}$$

Comparing this to the version defined in the previous chapter we find the following connections:

$$[n] := \frac{q^n - q^{-n}}{q - q^{-1}}$$

$$= q^{1-n} \frac{(q^2)^n - 1}{q^2 - 1}$$

$$= q^{1-n} (n)_{q^2}$$

and

$$[n]! := [1][2] \cdots [n]$$

$$= (1)_{q^2} q^{-1} (2)_{q^2} \cdots q^{1-n} (n)_{q^2}$$

$$= q^{-(1+2+\ldots+n-1)} (n)!_{q^2}$$

$$= q^{(n-n^2)/2} (n)!_{q^2}$$

Together these entail that

$$\begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n]!}{[k]![n-k]!}$$

$$= \frac{q^{(n-n^2)/2}(n)!_{q^2}}{q^{(k-k^2)/2}(k)!_{q^2} \cdot q^{(n-k-(n-k)^2)/2}(n-k)!_{q^2}}$$

$$= q^{k^2-nk} \frac{(n)!_{q^2}}{(k)!_{q^2}(n-k)!_{q^2}}$$

$$= q^{k^2-nk} \binom{n}{k}_{q^2}$$

Thus, if x and y are variables subject to the relation $yx = q^2 xy$, then the deformed binomial theorem becomes

$$(x+y)^n = \sum_{k=0}^{n} q^{k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix} x^k y^{n-k}$$

7.3 Motivating $U_q(\mathfrak{sl}(2))$

Having developed this particular sort of deformation suggests a means of deforming $U(\mathfrak{sl}(2))$. Recall that the generators of this algebra are X, Y and H with relations

$$[X,Y] = H, \quad [H,X] = 2X, \quad [H,Y] = -2Y$$

Suppose, then, that we deform this structure by setting

$$\begin{aligned}[X,Y] &= XY - YX \\ &= [H] \\ &= \frac{q^H - q^{-H}}{q - q^{-1}}\end{aligned}$$

To make this work, however, we need to assign some meaning to the symbols q^H and q^{-H}. The easiest thing to do is to make them formal generators. To make things easier, let us set $K := q^H$ and $K^{-1} := q^{-H}$. Thus, we get

$$XY - YX = \frac{K - K^{-1}}{q - q^{-1}}$$

and, as is suggested by the notation, we set $KK^{-1} = K^{-1}K = 1$.

We now want to see how this K (i.e. q^H) should interact with X and Y. If we interpret q^H formally as having the usual meaning (working over power series instead of the base field), then we write

$$q^H = \sum_{n=0}^{\infty} \frac{H^n(\ln(q))^n}{n!}$$

Now consider the other relations of $U(\mathfrak{sl}(2))$, for instance, $HX - XH = 2X$. We can rewrite this equation as

$$HX = X(2 + H)$$

An easy induction shows that $H^n X = X(2 + H)^n$ for any integer $n \geq 0$. For example,

$$\begin{aligned}H^2X &= H(HX) \\ &= H(X(2 + H)) \\ &= (HX)(2 + H) \\ &= X(2 + H)(2 + H) \\ &= X(2 + H)^2\end{aligned}$$

From this we have

$$\begin{aligned} KX &= q^H X \\ &= \sum_{n=0}^{\infty} \frac{H^n (\ln(q))^n}{n!} X \\ &= \sum_{n=0}^{\infty} \frac{H^n X (\ln(q))^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{X(2+H)^n (\ln(q))^n}{n!} \\ &= X \sum_{n=0}^{\infty} \frac{(2+H)^n (\ln(q))^n}{n!} \\ &= X q^{(2+H)} \\ &= q^2 X q^H \\ &= q^2 X K \end{aligned}$$

Similarly, we find that $KY = q^{-2} YK$; hence, our other generating relations are

$$KX = q^2 XK, \qquad KY = q^{-2} YK$$

This is the motivation for our definition.

Definition 7.1 ($U_q(\mathfrak{sl}(2))$)**.** The quantum group $U_q(\mathfrak{sl}(2))$ is defined to be the algebra generated by the four variables E, F, K, K^{-1} obeying the relations

$$KK^{-1} = K^{-1}K = 1 \tag{7.1}$$

$$KEK^{-1} = q^2 E, \qquad KFK^{-1} = q^{-2} F \tag{7.2}$$

$$[E, F] = \frac{K - K^{-1}}{q - q^{-1}} \tag{7.3}$$

Notice that we cannot directly set $q = 1$. However, formally we can still take it that $U(\mathfrak{sl}(2))$ is recovered in the limit $q \to 1$, since

$$\lim_{q \to 1} \frac{K - K^{-1}}{q - q^{-1}} = \lim_{q \to 1} \frac{q^H - q^{-H}}{q - q^{-1}} = H$$

The relations $q^H E = q^2 E q^H$ and $q^H F = q^{-2} F q^H$ also degenerate into the usual relations $HE - EH = 2E$ and $HF - FH = 2F$ respectively if we consider a formal derivative with respect to q. For instance

$$\begin{aligned} \frac{d}{dq} q^H E &= \frac{d}{dq} q^2 E q^H \\ H q^{-1} q^H E &= 2qEq^H + q^2 q^{-1} EHq^H \end{aligned}$$

If we now set $q = 1$ we get $HE = 2E + EH$ or $HE - EH = 2E$ as desired. We'll discuss another way of constructing $U_q(\mathfrak{sl}(2))$ below that allows for directly setting $q = 1$.

At this point, to call $U_q(\mathfrak{sl}(2))$ a quantum group is a bit premature since we have not established that it possesses a Hopf algebra structure. We remedy the situation in the following manner.

First, we would like an algebra morphism Δ that will be our coproduct. Since $KK^{-1} = K^{-1}K = 1$ we need

$$\begin{aligned} \Delta(KK^{-1}) &= \Delta(K)\Delta(K^{-1}) \\ &= 1 \otimes 1 \end{aligned}$$

and likewise for $\Delta(K^{-1}K)$. This seems to naturally suggest the assignments

$$\Delta(K) = K \otimes K, \qquad \Delta(K^{-1}) = K^{-1} \otimes K^{-1}$$

Thus, it must be that

$$\begin{aligned} \Delta(EF - FE) &= \Delta\left(\frac{K - K^{-1}}{q - q^{-1}}\right) \\ &= \frac{K \otimes K - K^{-1} \otimes K^{-1}}{q - q^{-1}} \end{aligned}$$

Now,

$$\begin{aligned} \frac{K \otimes K - K^{-1} \otimes K^{-1}}{q - q^{-1}} &= \frac{K \otimes K - K^{-1} \otimes K + K^{-1} \otimes K - K^{-1} \otimes K^{-1}}{q - q^{-1}} \\ &= \frac{(K - K^{-1}) \otimes K + K^{-1} \otimes (K - K^{-1})}{q - q^{-1}} \\ &= \frac{K - K^{-1}}{q - q^{-1}} \otimes K + K^{-1} \otimes \frac{K - K^{-1}}{q - q^{-1}} \\ &= (EF - FE) \otimes K + K^{-1} \otimes (EF - FE) \\ &= EF \otimes K - FE \otimes K + K^{-1} \otimes EF - K^{-1} \otimes FE \end{aligned}$$

If we now add and substract $F \otimes E$ we have

$$
\begin{aligned}
&EF \otimes K - FE \otimes K - F \otimes E + K^{-1} \otimes EF + F \otimes E - K^{-1} \otimes FE \\
=&EF \otimes K - (F \otimes 1)(E \otimes K + 1 \otimes E) + (1 \otimes E)(K^{-1} \otimes F + F \otimes 1) - K^{-1} \otimes FE \\
=&(E \otimes K)(F \otimes 1) - (F \otimes 1)(E \otimes K + 1 \otimes E) + (1 \otimes E)(K^{-1} \otimes F + F \otimes 1) - (K^{-1} \otimes F)(1 \otimes E)
\end{aligned}
$$

At this point, we can begin to see some interesting form emerging. If we finally add and subtract $EK^{-1} \otimes KF = (E \otimes K)(K^{-1} \otimes F)$ we end up with

$$(E \otimes K + 1 \otimes E)(F \otimes 1 + K^{-1} \otimes F) - (F \otimes 1 + K^{-1} \otimes F)(E \otimes K + 1 \otimes E)$$

and this suggests that we set

$$\Delta(E) = E \otimes K + 1 \otimes E \qquad \Delta(F) = F \otimes 1 + K^{-1} \otimes F$$

which, one can see, is very similar to E and F being primitive as they would be in $U(\mathfrak{sl}(2))$. Indeed, if $q = 1$, then $K = 1$ and we get the usual coproduct for $U(\mathfrak{sl}(2))$. In fact, E and F, in this context, are called *skew*-primitive. In general, an element c of a coalgebra is called skew-primitive if $\Delta(c) = c \otimes g + h \otimes c$ where h and g are grouplike elements.

Define linear maps Δ, ε and S by

$$\Delta(E) := 1 \otimes E + E \otimes K, \quad \Delta(F) := K^{-1} \otimes F + F \otimes 1 \tag{7.4}$$

$$\Delta(K) := K \otimes K, \quad \Delta(K^{-1}) := K^{-1} \otimes K^{-1} \tag{7.5}$$

$$\varepsilon(E) := \varepsilon(F) := 0, \quad \varepsilon(K) := \varepsilon(K^{-1}) := 1 \tag{7.6}$$

$$S(E) := -EK^{-1}, \quad S(F) := -KF, \quad S(K) := K^{-1}, \quad S(K^{-1}) := K \tag{7.7}$$

Proposition 7.2. *The algebra $U_q(\mathfrak{sl}(2))$ is a Hopf algebra under the relations (7.4-7.7).*

Proof. Our first goal is to show that Δ is an algebra morphism, which entails showing that it respects the defining relations (7.1-7.3). Beginning with (7.1) we have

$$\begin{aligned}
\Delta(K)\Delta(K^{-1}) &= (K \otimes K)(K^{-1} \otimes K^{-1}) \\
&= KK^{-1} \otimes KK^{-1} \\
&= 1 \otimes 1
\end{aligned}$$

The same result is obtained for $K^{-1}K$.

Now (7.2):

$$\begin{aligned}
\Delta(K)\Delta(E)\Delta(K^{-1}) &= (K \otimes K)(1 \otimes E + E \otimes K)(K^{-1} \otimes K^{-1}) \\
&= (K \otimes KE + KE \otimes K^2)(K^{-1} \otimes K^{-1}) \\
&= KK^{-1} \otimes KEK^{-1} + KEK^{-1} \otimes K^2K^{-1} \\
&= 1 \otimes q^2E + q^2E \otimes K \\
&= q^2(1 \otimes E + E \otimes K) \\
&= q^2\Delta(E)
\end{aligned}$$

The second relation of (7.2) is proved similarly. Finally, we check (7.3).

$$\begin{aligned}
[\Delta(E), \Delta(F)] &= (1 \otimes E + E \otimes K)(K^{-1} \otimes F + F \otimes 1) - (K^{-1} \otimes F + F \otimes 1)(1 \otimes E + E \otimes K) \\
&= K^{-1} \otimes EF - K^{-1} \otimes FE + EF \otimes K - FE \otimes K + EK^{-1} \otimes KF - K^{-1}E \otimes FK \\
&= K^{-1} \otimes (EF - FE) + (EF - FE) \otimes K + q^2K^{-1}E \otimes q^{-2}FK - K^{-1}E \otimes FK \\
&= K^{-1} \otimes [E, F] + [E, F] \otimes K \\
&= K^{-1} \otimes \frac{K - K^{-1}}{q - q^{-1}} + \frac{K - K^{-1}}{q - q^{-1}} \otimes K \\
&= \frac{K^{-1} \otimes K - K^{-1} \otimes K^{-1} + K \otimes K - K^{-1} \otimes K}{q - q^{-1}} \\
&= \frac{K \otimes K - K^{-1} \otimes K^{-1}}{q - q^{-1}} \\
&= \frac{\Delta(K) - \Delta(K^{-1})}{q - q^{-1}}
\end{aligned}$$

So Δ is a morphism of algebras. But we also need Δ to satisfy the coassociativity axiom, which will be satisfied if it holds on the generators. Consider for instance

$$\begin{aligned}(\Delta \otimes \mathrm{id})\Delta(E) &= (\Delta \otimes \mathrm{id})(1 \otimes E + E \otimes K) \\ &= 1 \otimes 1 \otimes E + \Delta(E) \otimes K \\ &= 1 \otimes 1 \otimes E + (1 \otimes E + E \otimes K) \otimes K \\ &= 1 \otimes 1 \otimes E + 1 \otimes E \otimes K + E \otimes K \otimes K \\ &= 1 \otimes (1 \otimes E + E \otimes K) + E \otimes \Delta(K) \\ &= 1 \otimes \Delta(E) + E \otimes \Delta(K) \\ &= (\mathrm{id} \otimes \Delta)(1 \otimes E + E \otimes K) \\ &= (\mathrm{id} \otimes \Delta)\Delta(E)\end{aligned}$$

The case for F follows similarly. Since K and K^{-1} are also similar, we'll finish by verifying the result for K.

$$\begin{aligned}(\Delta \otimes \mathrm{id})\Delta(K) &= (\Delta \otimes \mathrm{id})(K \otimes K) \\ &= \Delta(K) \otimes K \\ &= K \otimes K \otimes K \\ &= K \otimes \Delta(K) \\ &= (\mathrm{id} \otimes \Delta)(K \otimes K) \\ &= (\mathrm{id} \otimes \Delta)\Delta(K)\end{aligned}$$

Second, we need to show that ε is also a morphism of algebras and that the counit axiom is satisfied. Again, we start with (7.1).

$$\varepsilon(K)\varepsilon(K^{-1}) = 1 \cdot 1 = 1$$

(7.2):

$$\begin{aligned}\varepsilon(K)\varepsilon(E)\varepsilon(K^{-1}) &= 1 \cdot 0 \cdot 1 \\ &= 0 \\ &= q^2\varepsilon(E)\end{aligned}$$

Again, the second relation is similar so we move on to (7.3).

$$\begin{aligned}[][\varepsilon(E), \varepsilon(F)] &= 0 \\ &= \frac{1-1}{q-q^{-1}} \\ &= \frac{\varepsilon(K)-\varepsilon(K^{-1})}{q-q^{-1}}\end{aligned}$$

Finally, we check that the counit axiom is satisfied. As before, this is done on the generators. For brevity, we will again consider only E and K.

$$\begin{aligned}(\varepsilon \otimes \mathrm{id})\Delta(E) &= (\varepsilon \otimes \mathrm{id})(1 \otimes E + E \otimes K) \\ &= 1 \otimes E + \varepsilon(E) \otimes K \\ &= 1 \otimes E + 0 \otimes K \\ &= 1 \otimes E\end{aligned}$$

and

$$\begin{aligned}(\varepsilon \otimes \mathrm{id})\Delta(K) &= (\varepsilon \otimes \mathrm{id})(K \otimes K) \\ &= \varepsilon(K) \otimes K \\ &= 1 \otimes K\end{aligned}$$

Our last objective is to verify that S is an antipode for $U_q(\mathfrak{sl}(2))$. Note that S is an antimorphism of algebras since:

$$S(K^{-1})S(K) = KK^{-1} = 1$$

$$\begin{aligned}S(K^{-1})S(E)S(K) &= K(-EK^{-1})K^{-1} \\ &= -(KEK^{-1})K^{-1} \\ &= -q^2EK^{-1} \\ &= q^2(-EK^{-1}) \\ &= q^2S(E)\end{aligned}$$

That $S(K^{-1})S(F)S(K) = q^{-2}S(F)$ is similarly verified. Finally we have

$$\begin{aligned}
[S(F), S(E)] &= S(F)S(E) - S(E)S(F) \\
&= -KF(-EK^{-1}) - (-EK^{-1})(-KF) \\
&= KFEK^{-1} - EK^{-1}KF \\
&= KFEK^{-1} - EF \\
&= (q^{-2}FK)(q^2K^{-1}E) - EF \\
&= FE - EF \\
&= \frac{K^{-1} - K}{q - q^{-1}} \\
&= \frac{S(K) - S(K^{-1})}{q - q^{-1}}
\end{aligned}$$

This shows that S is an antimorphism of algebras on $U_q(\mathfrak{sl}(2))$, but we must now make sure that

$$S \star \mathrm{id} = \mathrm{id} \star S = 1_\star$$

if it is to be an antipode. Once more, since verification of this is largely computational, we'll show that it holds with E and K. For E:

$$\begin{aligned}
\big(\nabla \circ (S \otimes \mathrm{id}) \circ \Delta\big)(E) &= (\nabla \circ (S \otimes \mathrm{id}))(1 \otimes E + E \otimes K) \\
&= \nabla(1 \otimes E - EK^{-1} \otimes K) \\
&= E - EK^{-1}K = 0
\end{aligned}$$

Also

$$\begin{aligned}
1_\star(E) &= \eta(\varepsilon(E)) \\
&= \eta(0) = 0
\end{aligned}$$

and finally

$$\begin{aligned}
\big(\nabla \circ (\mathrm{id} \otimes S) \circ \Delta\big)(E) &= (\nabla \circ (\mathrm{id} \otimes S))(1 \otimes E + E \otimes K) \\
&= \nabla(1 \otimes (-EK^{-1}) + E \otimes K^{-1}) \\
&= -EK^{-1} + EK^{-1} = 0
\end{aligned}$$

Now for K:

$$\begin{aligned}(\nabla \circ (S \otimes \mathrm{id}) \circ \Delta)(K) &= (\nabla \circ (S \otimes \mathrm{id}))(K \otimes K) \\ &= \nabla(S(K) \otimes K) \\ &= S(K)K \\ &= K^{-1}K = 1\end{aligned}$$

and

$$\begin{aligned}(\nabla \circ (\mathrm{id} \otimes S) \circ \Delta)(K) &= (\nabla \circ (\mathrm{id} \otimes S))(K \otimes K) \\ &= \nabla(K \otimes S(K)) \\ &= KS(K) \\ &= KK^{-1} = 1\end{aligned}$$

and lastly

$$\begin{aligned}1_\star(K) &= \eta(\varepsilon(K)) \\ &= \eta(1) = 1\end{aligned}$$

Since the other cases hold on account of similar reasoning, we have that S is an antipode for $U_q(\mathfrak{sl}(2))$ as desired. Therefore, $U_q(\mathfrak{sl}(2))$ is a Hopf algebra and hence a quantum group. $\square$

This quantum group is one of the most well-known examples of a noncommutative/non-cocommutative Hopf algebra. In this example the antipode is not an involution, though, S^2 is an inner automorphism. For reference, an *inner automorphism* of an algebra A is a map $f_x : A \to A$ defined by $f_x(y) := xyx^{-1}$ for some unit x.

Proposition 7.3. *For any* $u \in U_q(\mathfrak{sl}(2))$ *we have* $S^2(u) = KuK^{-1}$.

Proof. Since S^2 is a morphism of algebras we need only establish that this holds on the generators.

$$\begin{aligned} S^2(E) &= S(-EK^{-1}) \\ &= -S(K^{-1})S(E) = KEK^{-1} \end{aligned}$$

$$\begin{aligned} S^2(F) &= S(-KF) \\ &= -S(F)S(K) = KFK^{-1} \end{aligned}$$

$$\begin{aligned} S^2(K) &= S(K^{-1}) \\ &= K = KKK^{-1} \end{aligned}$$

□

Lemma 7.4. *(i) There is a unique automorphism ω of $U_q(\mathfrak{sl}(2))$ with*

$$\omega(E) = F, \quad \omega(F) = E, \quad \omega(K) = K^{-1}, \qquad \textit{furthermore } \omega^2 = id$$

(ii) There is a unique antiautomorphism τ of $U_q(\mathfrak{sl}(2))$ with

$$\tau(E) = E, \quad \tau(F) = F, \quad \tau(K) = K^{-1}, \qquad \textit{furthermore } \tau^2 = id$$

Proof. (i) We must show that ω preserves the defining relations of $U_q(\mathfrak{sl}(2))$. If we consider them in the order shown above we have

$$\omega(K)\omega(K^{-1}) = K^{-1}K = 1$$

$$\begin{aligned} \omega(K)\omega(E)\omega(K^{-1}) &= K^{-1}FK \\ &= q^2F \\ &= q^2\omega(E) \end{aligned}$$

$$\begin{aligned}\omega(K)\omega(F)\omega(K^{-1}) &= K^{-1}EK \\ &= q^{-2}E \\ &= q^{-2}\omega(F)\end{aligned}$$

$$\begin{aligned}[\omega(E),\omega(F)] &= [F,E] \\ &= \frac{K^{-1}-K}{q-q^{-1}} \\ &= \frac{\omega(K)-\omega(K^{-1})}{q-q^{-1}}\end{aligned}$$

So, ω preserves the defining relations. But it is also clearly unique given that E, F, K, K^{-1} generate $U_q(\mathfrak{sl}(2))$. By definition, it is equally clear that each of the generators is fixed under ω^2 and hence $\omega^2 = \text{id}$.

(ii) We proceed similarly, but in this case require that $\tau(ab) = \tau(b)\tau(a)$ for all $a, b \in U_q(\mathfrak{sl}(2))$. First, $KK^{-1} = K^{-1}K = 1$ and

$$\begin{aligned}\tau(K^{-1})\tau(K) &= KK^{-1} \\ &= 1 \\ &= K^{-1}K \\ &= \tau(K)\tau(K^{-1})\end{aligned}$$

Next, $KEK^{-1} = q^2E$ and

$$\begin{aligned}\tau(K^{-1})\tau(E)\tau(K) &= KEK^{-1} \\ &= q^2E \\ &= q^2\tau(E)\end{aligned}$$

Finally, $[E,F] = \frac{K-K^{-1}}{q-q^{-1}}$ and

$$\begin{aligned}\tau(F)\tau(E)-\tau(E)\tau(F) &= FE-EF \\ &= \frac{K^{-1}-K}{q-q^{-1}} \\ &= \frac{\tau(K)-\tau(K^{-1})}{q-q^{-1}}\end{aligned}$$

As before the uniqueness is obvious, as is the fact that $\tau^2 = \text{id}$. □

Note that ω is an involution while τ is an anti-involution.

Lemma 7.5. *Let* $m \geq 0$ *and* $n \in \mathbb{Z}$. *The following relations hold in* $U_q(\mathfrak{sl}(2))$*:*

$$E^m K^n = q^{-2mn} K^n E^m, \quad F^m K^n = q^{2mn} K^n F^m$$

$$\begin{aligned}[E, F^m] &= [m] F^{m-1} \frac{q^{1-m}K - q^{m-1}K^{-1}}{q - q^{-1}} \\ &= [m] \frac{q^{m-1}K - q^{1-m}K^{-1}}{q - q^{-1}} F^{m-1}\end{aligned}$$

$$\begin{aligned}[E^m, F] &= [m] \frac{q^{1-m}K - q^{m-1}K^{-1}}{q - q^{-1}} E^{m-1} \\ &= [m] E^{m-1} \frac{q^{m-1}K - q^{1-m}K^{-1}}{q - q^{-1}}\end{aligned}$$

Proof. Relation (7.2) implies that $EK = q^{-2}KE$. Suppose

$$E^n K = q^{-2n} K E^n \quad \text{for some } n$$

and now consider the next case.

$$\begin{aligned}E^{n+1}K &= EE^nK \\ &= Eq^{-2n}KE^n \quad \text{[induction hypothesis]} \\ &= q^{-2n}EKE^n \\ &= q^{-2n}q^{-2}KEE^n \quad \text{[base case]} \\ &= q^{-2(n+1)}KE^{n+1}\end{aligned}$$

Thus, by mathematical induction, $E^nK = q^{-2n}KE^n$ for all $n \in \mathbb{Z}^+$. We now have our base case for the next induction. Suppose

$$E^n K^m = q^{-2nm} K^m E^n$$

for all n and some $m \in \mathbb{Z}$. Now consider the next case.

$$\begin{aligned}E^nK^{m+1} &= E^nK^mK \\ &= q^{-2nm}K^mE^nK \quad \text{[induction hypothesis]} \\ &= q^{-2nm}q^{-2n}K^mKE^n \quad \text{[base case]} \\ &= q^{-2n(m+1)}K^{m+1}E^n\end{aligned}$$

Therefore, the first relation holds via mathematical induction.

For the second relation we can apply ω from the previous lemma. Since

$$E^n K^{-m} = q^{2nm} K^{-m} E^n$$

we have

$$\begin{aligned}
\omega(E^n K^{-m}) &= \omega(q^{2nm} K^{-m} E^n) \\
\omega(E^n)\omega(K^{-m}) &= q^{2nm}\omega(K^{-m})\omega(E^n) \\
\omega(E)^n\omega(K)^{-m} &= q^{2nm}\omega(K)^{-m}\omega(E)^n \\
F^n(K^{-1})^{-m} &= q^{2nm}(K^{-1})^{-m}F^n \\
F^nK^m &= q^{2nm}K^mF^n
\end{aligned}$$

For the third relation, note that

$$\begin{aligned}
[E,F] &= \frac{K-K^{-1}}{q-q^{-1}} \\
&= [1]F^0\frac{q^0K-q^0K^{-1}}{q-q^{-1}}
\end{aligned}$$

Now suppose

$$[E,F^m] = [m]F^{m-1}\frac{q^{1-m}K-q^{m-1}K^{-1}}{q-q^{-1}}$$

and consider the case of $[E,F^{m+1}]$. We have

$$\begin{aligned}
[E,F^{m+1}] &= [E,F^mF] \\
&= [E,F^m]F + F^m[E,F] \quad \text{[by Proposition 5.4]} \\
&= [m]F^{m-1}\frac{q^{1-m}K-q^{m-1}K^{-1}}{q-q^{-1}}F + F^m\frac{K-K^{-1}}{q-q^{-1}} \\
&= \frac{[m]q^{1-m}F^{m-1}KF - [m]q^{m-1}F^{m-1}K^{-1}F + F^mK - F^mK^{-1}}{q-q^{-1}} \\
&= \frac{[m]q^{-(m+1)}F^mK - [m]q^{m+1}F^mK^{-1} + F^mK - F^mK^{-1}}{q-q^{-1}} \quad \text{[using 7.2]} \\
&= F^m\frac{([m]q^{-(m+1)}+1)K - ([m]q^{m+1}+1)K^{-1}}{q-q^{-1}}
\end{aligned}$$

At this point, examine $[m]q^{-(m+1)}+1$ and $[m]q^{m+1}+1$.

$$\begin{aligned}[m]q^{-(m+1)}+1 &= \frac{q^m-q^{-m}}{q-q^{-1}}q^{-(m+1)}+1\\ &= \frac{q^{-1}-q^{-2m-1}}{q-q^{-1}}+1\\ &= \frac{q^{-1}-q^{-2m-1}+q-q^{-1}}{q-q^{-1}}\\ &= \frac{q-q^{-2m-1}}{q-q^{-1}}\end{aligned}$$

and

$$\begin{aligned}[m]q^{m+1}+1 &= \frac{q^m-q^{-m}}{q-q^{-1}}q^{m+1}+1\\ &= \frac{q^{2m+1}-q}{q-q^{-1}}+1\\ &= \frac{q^{2m+1}-q+q-q^{-1}}{q-q^{-1}}\\ &= \frac{q^{2m+1}-q^{-1}}{q-q^{-1}}\end{aligned}$$

Using these results we get

$$\begin{aligned}F^m\frac{\frac{q-q^{-2m-1}}{q-q^{-1}}K-\frac{q^{2m+1}-q^{-1}}{q-q^{-1}}K^{-1}}{q-q^{-1}} &= \frac{q^{m+1}-q^{-(m+1)}}{q-q^{-1}}F^m\frac{q^{-m}K-q^mK^{-1}}{q-q^{-1}}\\ &= [m+1]F^m\frac{q^{-m}K-q^mK^{-1}}{q-q^{-1}}\end{aligned}$$

So, by induction

$$[E,F^m]=[m]F^{m-1}\frac{q^{1-m}K-q^{m-1}K^{-1}}{q-q^{-1}}$$

for all $m\in\mathbb{Z}^+$. As for the second part of this relation, note that

$$\begin{aligned}[m]F^{m-1}\frac{q^{1-m}K-q^{m-1}K^{-1}}{q-q^{-1}} &= [m]\frac{q^{1-m}F^{m-1}K-q^{m-1}F^{m-1}K^{-1}}{q-q^{-1}}\\ &= [m]\frac{q^{1-m}q^{2(m-1)}KF^{m-1}-q^{m-1}q^{-2(m-1)}K^{-1}F^{m-1}}{q-q^{-1}}\\ &= [m]\frac{q^{m-1}K-q^{1-m}K^{-1}}{q-q^{-1}}F^{m-1}\end{aligned}$$

Finally, to get the fourth relation, simply apply the involution ω from Lemma 7.4 to the third relation we just considered. $\square$

Proposition 7.6. *The algebra $U_q(\mathfrak{sl}(2))$ is Noetherian and has no zero-divisors. The set $\{E^iF^jK^\ell\}_{i,j\in\mathbb{N};\ell\in\mathbb{Z}}$ is a basis of $U_q(\mathfrak{sl}(2))$.*

Proof. Our procedure will be to show that $U_q(\mathfrak{sl}(2))$ is an iterated Ore extension (see Chapter 6). We will build a "three-story" tower

$$A_0 \subset A_1 \subset A_2$$

such that $A_2 \cong U_q(\mathfrak{sl}(2))$. Starting at the base, define

$$A_0 := \kappa[K, K^{-1}]$$

This is the algebra of Laurent polynomials in one variable. Formally, $\kappa[K, K^{-1}]$ is the quotient algebra $\kappa[K, J]/I$, where I is the ideal of $\kappa[K, J]$ generated by $KJ - 1$. As such, it is Noetherian, has no zero divisors and has basis $\{K^\ell\}_{\ell \in \mathbb{Z}}$. Now let $\alpha_0 : A_0 \to A_0$ be the automorphism given by $\alpha_0(K) := q^2 K$. We then take A_1 to be the Ore extension $A_0[F, \alpha_0, 0]$, which has basis $\{F^j K^\ell\}_{j \in \mathbb{N}; \ell \in \mathbb{Z}}$. Then

$$A_1 \cong \frac{\kappa\{F, K, K^{-1}\}}{(FK - q^2 KF)}$$

Finally, we construct an Ore extension $A_2 = A_1[E, \alpha_1, \mathscr{D}]$ which we want to be isomorphic to $U_q(\mathfrak{sl}(2))$. First, define

$$\alpha_1(F) := F, \quad \alpha_1(K) := q^{-2} K$$

This extends to a unique algebra morphism since

$$\begin{aligned} \alpha_1(F)\alpha_1(K) &= q^{-2} FK \\ &= KF \\ &= q^2 q^{-2} KF \\ &= q^2 \alpha(K)\alpha(F) \end{aligned}$$

Next, let $\mathscr{D}$ be defined on the generators by

$$\mathscr{D}(F) := \frac{K - K^{-1}}{q - q^{-1}} \quad \text{and} \quad \mathscr{D}(K) := 0$$

We now need to extend $\mathscr{D}$ to a uniquely determined α_1-derivation. First, we check that, under this extension, the defining relation for A_1 is satisfied.

$$\begin{aligned}\mathscr{D}(FK) &= \alpha_1(F)\mathscr{D}(K) + \mathscr{D}(F)K \\ &= F \cdot 0 + \frac{K-K^{-1}}{q-q^{-1}}K \\ &= \frac{K^2-1}{q-q^{-1}}\end{aligned}$$

and

$$\begin{aligned}\mathscr{D}(q^2KF) &= q^2(\alpha_1(K)\mathscr{D}(F) + \mathscr{D}(K)F) \\ &= q^2(q^{-2}K\frac{K-K^{-1}}{q-q^{-1}} + 0 \cdot F) \\ &= \frac{K^2-1}{q-q^{-1}}\end{aligned}$$

So, $\mathscr{D}$ preserves the defining relation $FK = q^2KF$. Let us now consider how $\mathscr{D}$ acts on an arbitrary basis element. By induction it can easily be shown that

$$\mathscr{D}(K^\ell) = 0$$

For example,

$$\begin{aligned}\mathscr{D}(K^2) &= \mathscr{D}(KK) \\ &= \alpha_1(K)\mathscr{D}(K) + \mathscr{D}(K)K \\ &= \alpha_1(K) \cdot 0 + 0 \cdot K \\ &= 0\end{aligned}$$

Thus,

$$\begin{aligned}\mathscr{D}(F^mK^\ell) &= \alpha_1(F^m)\mathscr{D}(K^\ell) + \mathscr{D}(F^m)K^\ell \\ &= \alpha_1 \cdot 0 + \mathscr{D}(F^m)K^\ell \\ &= \mathscr{D}(F^m)K^\ell\end{aligned}$$

To go any further we need a formula for $\mathscr{D}(F^m)$. Observe that

$$\begin{aligned}\mathscr{D}(F^{m+1}) &= \mathscr{D}(F^mF) \\ &= \alpha_1(F^m)\mathscr{D}(F) + \mathscr{D}(F^m)F \\ &= F^m\mathscr{D}(F) + \mathscr{D}(F^m)F\end{aligned}$$

which, by induction, implies that

$$\mathscr{D}(F^{m+1}) = \sum_{i=o}^{m} F^{m-i}\mathscr{D}(F)F^i$$

But

$$\begin{aligned} F^{m-i}\mathscr{D}(F)F^i &= F^{m-i}\frac{K-K^{-1}}{q-q^{-1}}F^i \\ &= F^{m-i}\frac{KF^i - K^{-1}F^i}{q-q^{-1}} \\ &= F^{m-i}\frac{q^{-2i}F^iK - q^{2i}F^iK^{-1}}{q-q^{-1}} \\ &= F^m\frac{q^{-2i}K - q^{2i}K^{-1}}{q-q^{-1}} \end{aligned}$$

and so

$$\mathscr{D}(F^{m+1}) = \sum_{i=0}^{m} F^m\frac{q^{-2i}K - q^{2i}K^{-1}}{q-q^{-1}}$$

Therefore

$$\begin{aligned} \mathscr{D}(F^mK^\ell) &= \mathscr{D}(F^m)K^\ell \\ &= \sum_{i=0}^{m-1} F^{m-1}\frac{q^{-2i}K - q^{2i}K^{-1}}{q-q^{-1}}K^\ell \end{aligned}$$

Now, according to Definition 6.10, E interacts with the other generators by

$$EF = \alpha_1(F)E + \mathscr{D}(F) \tag{7.8}$$

$$EK = \alpha_1(K)E + \mathscr{D}(K) \tag{7.9}$$

$$EK^{-1} = \alpha_1(K^{-1})E + \mathscr{D}(K^{-1}) \tag{7.10}$$

Relation (7.8) yields $EF = FE + \frac{K-K^{-1}}{q-q^{-1}}$ or

$$[E,F] = \frac{K-K^{-1}}{q-q^{-1}}$$

Relation (7.9) yields $EK = q^{-2}KE$ and (7.10) that $EK^{-1} = q^2KE$, which implies the same relation. Thus

$$KE = q^2EK$$

and since we already have the relation $KF = q^{-2}FK$ from A_1, it follows that A_2 has generators E, F, K, K^{-1} satisfying the same relations as $U_q(\mathfrak{sl}(2))$ and hence

$$A_2 \cong U_q(\mathfrak{sl}(2))$$

□

7.4 An Alternative Presentation of $U_q(\mathfrak{sl}(2))$

Although the way we have defined $U_q(\mathfrak{sl}(2))$ is typical, it has the disadvantage of not reflecting deformation in the way in which it was described in Chapter 6. That is, one cannot simply set $q = 1$. Instead, one must take a sort of "limit" to recover $U(\mathfrak{sl}(2))$. To remedy this, we can give a slightly modified presentation of $U_q(\mathfrak{sl}(2))$ which is equivalent to the original version.

Notice that the problem in the first presentation lies with $[E, F] = \frac{K-K^{-1}}{q-q^{-1}}$. But this is easily circumvented via basic algebra. Simply write

$$(q - q^{-1})[E, F] = K - K^{-1}$$

We can then introduce a "new" generator L, setting $[E, F] = L$. So, this modified presentation can be thought of as having five generators: E, F, K, K^{-1}, L. We still have the relations

$$KK^{-1} = K^{-1}K = 1, \quad KE = q^2EK, \quad KF = q^{-2}FK$$

but immediately obtain the two "new" relations

$$[E, F] = L, \quad (q - q^{-1})L = K - K^{-1}$$

From these we can also establish that

$$\begin{aligned}
[L, E] &= LE - EL \\
&= \frac{K - K^{-1}}{q - q^{-1}}E - E\frac{K - K^{-1}}{q - q^{-1}} \\
&= \frac{KE - K^{-1}E}{q - q^{-1}} - \frac{EK - EK^{-1}}{q - q^{-1}} \\
&= \frac{q^2EK - K^{-1}E}{q - q^{-1}} - \frac{EK - q^2K^{-1}E}{q - q^{-1}} \\
&= \frac{q^2EK - EK + q^2K^{-1}E - K^{-1}E}{q - q^{-1}} \\
&= \frac{(q^2 - 1)EK + (q^2 - 1)K^{-1}E}{q - q^{-1}} \\
&= \frac{(q^2 - 1)(EK + K^{-1}E)}{q - q^{-1}} \\
&= q(EK + K^{-1}E)
\end{aligned}$$

and similarly that

$$[L, F] = -q^{-1}(FK + K^{-1}F)$$

With these relations at hand, we can present $U_q(\mathfrak{sl}(2))$ as follows.

Definition 7.7 (Alternative Presentation for $U_q(\mathfrak{sl}(2))$)**.** The algebra $U_q(\mathfrak{sl}(2))$ can be thought of as the algebra generated by the five variables E, F, K, K^{-1}, L subject to the relations

$$KK^{-1} = K^{-1}K = 1 \tag{7.11}$$

$$KEK^{-1} = q^2E, \qquad KFK^{-1} = q^{-2}F \tag{7.12}$$

$$[E, F] = L, \qquad (q - q^{-1})L = K - K^{-1} \tag{7.13}$$

$$[L, E] = q(EK + K^{-1}E), \quad [L, F] = -q^{-1}(FK + K^{-1}F) \tag{7.14}$$

In this form, the parameter q is allowed to take on any value. In particular, when $q = 1$ relations (7.11) and (7.13) imply that $K = K^{-1}$ and hence, $K^2 = 1$. We are then left with the generators E, F, L having relations identical to X, Y, H in $U(\mathfrak{sl}(2))$. We therefore recover $U(\mathfrak{sl}(2))$ in case $q = 1$ thereby legitimizing the deformation originally used.

This can be made more rigorous as follows.

Proposition 7.8. *If* $q = 1$*, then*

$$U_1(\mathfrak{sl}(2)) \cong \frac{U(\mathfrak{sl}(2))[K]}{(K^2 - 1)} \quad \textit{and} \quad U(\mathfrak{sl}(2)) \cong \frac{U_1(\mathfrak{sl}(2))}{(K - 1)}$$

as algebras.

Proof. The presentation of $U_1(\mathfrak{sl}(2))$ is characterized as the algebra generated by E, F, K, K^{-1}, L subject to the relations

$$KK^{-1} = K^{-1}K = 1$$

$$KEK^{-1} = E, \quad KFK^{-1} = F$$

$$[E, F] = L, \quad K - K^{-1} = 0$$

$$[L, E] = EK + K^{-1}E, \quad [L, F] = -(FK + K^{-1}F)$$

Notice that these imply that K is in the center of $U(\mathfrak{sl}(2))$ and that $K^2 = 1$. We can therefore rewrite the last two relations as

$$[L, E] = 2EK, \quad [L, F] = -2FK$$

If we now define a map $\phi : U_1(\mathfrak{sl}(2)) \to U(\mathfrak{sl}(2))[K]/(K^2 - 1)$ by

$$\phi(E) := XK, \quad \phi(F) := Y, \quad \phi(K) := K, \quad \phi(L) := HK$$

then ϕ extends to an algebra morphism so long as the defining relations of $U_1(\mathfrak{sl}(2))$ are preserved in $U(\mathfrak{sl}(2))[K]/(K^2 - 1)$ under ϕ. First,

$$\begin{aligned}
\phi(K)\phi(E)\phi(K^{-1}) &= KXKK \\
&= KX \\
&= XK = \phi(E)
\end{aligned}$$

Second,

$$\begin{aligned}
\phi(K)\phi(F)\phi(K^{-1}) &= KYK \\
&= KKY \\
&= Y = \phi(F)
\end{aligned}$$

Third,

$$\begin{aligned}
[\phi(E), \phi(F)] &= \phi(E)\phi(F) - \phi(F)\phi(E) \\
&= XKY - YXK \\
&= XYK - YXK \\
&= (XY - YX)K \\
&= HK = \phi(L)
\end{aligned}$$

Fourth,

$$\begin{aligned}
[\phi(L), \phi(E)] &= \phi(L)\phi(E) - \phi(E)\phi(L) \\
&= HKXK - XKHK \\
&= HXKK - XHKK \\
&= HX - XH \\
&= 2X \\
&= 2XKK = 2\phi(E)\phi(K)
\end{aligned}$$

Fifth,

$$\begin{aligned}
[\phi(L), \phi(F)] &= \phi(L)\phi(F) - \phi(F)\phi(L) \\
&= HKY - YHK \\
&= HYK - YHK \\
&= (HY - YH)K \\
&= -2YK = -2\phi(F)\phi(K)
\end{aligned}$$

The relations involving K are obviously satisfied. Thus, ϕ is a well-defined algebra morphism.

Now define $\psi : U(\mathfrak{sl}(2))[K]/(K^2 - 1) \to U_1(\mathfrak{sl}(2))$ by

$$\psi(X) := EK, \quad \psi(Y) := F, \quad \psi(H) := LK, \quad \psi(K) := K$$

Again, this extends to an algebra morphism provided the defining relations are respected. In this case, there are four relations to check. First,

$$\begin{aligned}
[\psi(X), \psi(Y)] &= \psi(X)\psi(Y) - \psi(Y)\psi(X) \\
&= EKF - FEK \\
&= (EF - FE)K \\
&= LK = \psi(H)
\end{aligned}$$

Second,

$$\begin{aligned}
[\psi(H), \psi(X)] &= \psi(H)\psi(X) - \psi(X)\psi(H) \\
&= LKEK - EKLK \\
&= LE - EL \\
&= 2EK = 2\psi(X)
\end{aligned}$$

Third,

$$\begin{aligned}
[\psi(H), \psi(Y)] &= \psi(H)\psi(Y) - \psi(Y)\psi(H) \\
&= LKF - FLK \\
&= (LF - FL)K \\
&= -2FKK \\
&= -2F = -2\psi(Y)
\end{aligned}$$

Finally,

$$\begin{aligned}\psi(K)^2 &= \psi(K)\psi(K) \\ &= KK \\ &= 1 = \psi(1)\end{aligned}$$

Thus, it is a well-defined algebra morphism.

It remains to show that ϕ and ψ are inverses. This is again checked on the generators.

$$\begin{aligned}(\psi \circ \phi)(E) &= \psi(\phi(E)) \\ &= \psi(XK) \\ &= \psi(X)\psi(K) \\ &= EKK = E\end{aligned}$$

Next

$$\begin{aligned}(\psi \circ \phi)(F) &= \psi(\phi(F)) \\ &= \psi(Y) = F\end{aligned}$$

Since K is obvious, we'll skip to L.

$$\begin{aligned}(\psi \circ \phi)(L) &= \psi(\phi(L)) \\ &= \psi(HK) \\ &= \psi(H)\psi(K) \\ &= LKK = L\end{aligned}$$

Showing that $\phi \circ \psi = \mathrm{id}$ is similarly verified. Thus, $\psi = \phi^{-1}$ and we have our isomorphism.

Note, then, that the projection of $U_1(\mathfrak{sl}(2))$ onto $U(\mathfrak{sl}(2))$ is obtained by

$$E \mapsto X, \quad F \mapsto Y, \quad L \mapsto H, \quad K \mapsto 1$$

□

But we can actually say more than this, namely, that $U(\mathfrak{sl}(2)) \cong U_1(\mathfrak{sl}(2))/(K-1)$ as Hopf algebras. To see why, we first need to know what $\Delta(L), \varepsilon(L)$ and $S(L)$ are. Since

$L = [E, F]$ we find that

$$\begin{aligned}
\Delta(L) &= \Delta(EF - FE) \\
&= (E \otimes K + 1 \otimes E)(F \otimes 1 + K^{-1} \otimes F) - (F \otimes 1 + K^{-1} \otimes F)(E \otimes K + 1 \otimes E) \\
&= EF \otimes K + EK^{-1} \otimes KF + F \otimes E + K^{-1} \otimes EF \\
&\quad - FE \otimes K - F \otimes E - K^{-1}E \otimes FK - K^{-1} \otimes FE \\
&= EF \otimes K - FE \otimes K + K^{-1} \otimes EF - K^{-1} \otimes FE + EK^{-1} \otimes KF - K^{-1}E \otimes FK \\
&= (EF - FE) \otimes K + K^{-1} \otimes (EF - FE) + q^2K^{-1}E \otimes q^{-2}FK - K^{-1}E \otimes FK \\
&= L \otimes K + K^{-1} \otimes L
\end{aligned}$$

and

$$\varepsilon(L) = \varepsilon(E)\varepsilon(F) - \varepsilon(F)\varepsilon(E) = 0$$

Finally

$$\begin{aligned}
S(L) &= S(EF - FE) \\
&= S(EF) - S(FE) \\
&= S(F)S(E) - S(E)S(F) \\
&= -KF(-EK^{-1}) + EK^{-1}(-KF) \\
&= KFEK^{-1} - EK^{-1}KF \\
&= q^2q^{-2}KK^{-1}FE - EF \\
&= FE - EF = -L
\end{aligned}$$

Now, for $U_1(\mathfrak{sl}(2))/(K-1)$ the coalgebra structure becomes

$$\begin{aligned}
\Delta_1(E) &= E \otimes 1 + 1 \otimes E \\
\Delta_1(F) &= F \otimes 1 + 1 \otimes F \\
\Delta_1(L) &= L \otimes 1 + 1 \otimes L
\end{aligned}$$

To see that $\phi : U_1(\mathfrak{sl}(2))/(K-1) \to U(\mathfrak{sl}(2))$, as defined at the end of the above proof, will be a morphism of coalgebras in addition to being a morphism of algebras, it must be shown that

$$\begin{tikzcd}
\frac{U_1(\mathfrak{sl}(2))}{(K-1)} \arrow[r, "\phi"] \arrow[d, "\Delta_1"'] & U(\mathfrak{sl}(2)) \arrow[d, "\Delta"] \\
\frac{U_1(\mathfrak{sl}(2))}{(K-1)}^{\otimes 2} \arrow[r, "\phi \otimes \phi"'] & U(\mathfrak{sl}(2))^{\otimes 2}
\end{tikzcd}$$

commutes for each generator. Let's do E:

$$\begin{aligned}(\Delta \circ \phi)(E) &= \Delta(\phi(E)) \\ &= \Delta(X) \\ &= X \otimes 1 + 1 \otimes X\end{aligned}$$

and

$$\begin{aligned}((\phi \otimes \phi) \circ \Delta_1)(E) &= (\phi \otimes \phi)(E \otimes 1 + 1 \otimes E) \\ &= \phi(E) \otimes \phi(1) + \phi(1) \otimes \phi(E) \\ &= X \otimes 1 + 1 \otimes X\end{aligned}$$

The rest hold similarly. So, since ϕ is a coalgebra morphism, so is its inverse. Finally, we show that ϕ respects the antipodes, which requires that

$$\begin{tikzcd} \frac{U_1(\mathfrak{sl}(2))}{(K-1)} \arrow[r, "S_1"] \arrow[d, "\phi"'] & \frac{U_1(\mathfrak{sl}(2))}{(K-1)} \arrow[d, "\phi"] \\ U(\mathfrak{sl}(2)) \arrow[r, "S"'] & U(\mathfrak{sl}(2)) \end{tikzcd}$$

commute for the generators. Again, let us check E. First, $S(\phi(E)) = S(X) = -X$. But also, $\phi(S_1(E)) = \phi(-E) = -\phi(E) = -X$. The rest are similarly shown. This shows ϕ to be a Hopf morphism and thus, we have our isomorphism of Hopf algebras.

7.5 Representations of $U_q(\mathfrak{sl}(2))$

When talking about representations of $U_q(\mathfrak{sl}(2))$ it matters whether or not q is a root of unity. In case q is not a root of unity, then $U_q(\mathfrak{sl}(2))$ behaves like $U(\mathfrak{sl}(2))$ over a field of characteristic 0. However, if q is a root of unity, then $U_q(\mathfrak{sl}(2))$ behaves like $U(\mathfrak{sl}(2))$ over a field of prime characteristic. In fact, when the field is the complex numbers and q is a prime p^{th} root of unity bigger or equal to 3, then the representation theory looks like that of $\mathfrak{sl}(2)$ over an algebraically closed field of characteristic p. We shall consider both options for q starting with q as not a root of unity.

7.5.1 When q is not a Root of Unity

What we will do is determine all finite-dimensional simple $U_q(\mathfrak{sl}(2))$-modules when q is not a root of unity. In case V is such a $U_q(\mathfrak{sl}(2))$-module, this time denote by V^λ ($\lambda \neq 0$) the weight space of weight λ of V. That is, the space of all $v \in V$ such that $Kv = \lambda v$.

Definition 7.9. Let V be a $U_q(\mathfrak{sl}(2))$-module and λ a scalar. An element $v \neq 0$ of V is a highest weight vector of weight λ if $Ev = 0$ and $Kv = \lambda v$. A $U_q(\mathfrak{sl}(2))$-module is a highest weight module of highest weight λ if it is generated (as a module) by a highest weight vector of weight λ.

Lemma 7.10.

$$EV^\lambda \subset V^{q^2\lambda} \quad \textit{and} \quad FV^\lambda \subset V^{q^{-2}\lambda}$$

Proof. Let $v \in V^\lambda$ and consider the vector Ev. We have

$$\begin{aligned} K(Ev) &= (KE)v \\ &= (q^2 EK)v \\ &= q^2 E(Kv) \\ &= q^2 E(\lambda v) \\ &= q^2 \lambda E v \end{aligned}$$

Thus, by definition, $Ev \in V^{q^2\lambda}$. Similar calculations show that $Fv \in V^{q^{-2}\lambda}$. □

Proposition 7.11. *Any non-zero finite-dimensional* $U_q(\mathfrak{sl}(2))$*-module* V *contains a highest weight vector. Also, the endomorphisms induced by* E *and* F *on* V *are nilpotent.*

Proof. The proof of the first part proceeds identically to the proof of Proposition 5.29. The second part we can show by contradiction. First, we note that E and F will be nilpotent in case their only eigenvalue is 0. Suppose, then, that v is a non-zero eigenvector for E with $\lambda \neq 0$ as its eigenvalue. Then

$$\begin{aligned} E(Kv) &= (EK)v \\ &= (q^{-2} KE)v \\ &= q^{-2} K(Ev) \\ &= q^{-2} K\lambda v \\ &= q^{-2} \lambda K v \end{aligned}$$

which shows that Kv is also an eigenvector of E. By induction, $K^n v$ is likewise an eigenvector of E for every $n \in \mathbb{N}$ with eigenvalue $q^{-2n}\lambda$. This implies that E has infinitely many distinct eigenvalues, which is not possible since V is finite-dimensional. Thus, our assumption that $\lambda \neq 0$ is false implying that E is nilpotent. The same reasoning holds for F. $\square$

Lemma 7.12. *Let v be a highest weight vector of weight λ. Set $v_0 = v$ and for $p > 0$ let $v_p := \frac{1}{[p]!}F^p v$. Then*

$$Kv_p = \lambda q^{-2p} v_p, \quad Ev_p = \frac{q^{-(p-1)}\lambda - q^{p-1}\lambda^{-1}}{q - q^{-1}} v_{p-1}, \quad Fv_{p-1} = [p]v_p$$

Proof. Take each in turn.

$$\begin{aligned} Kv_p &= \frac{1}{[p]!}KF^p v \\ &= q^{-2p}\frac{1}{[p]!}F^p K v \quad \text{[using Lemma 7.5]} \\ &= \lambda q^{-2p}\frac{1}{[p]!}F^p v \\ &= \lambda q^{-2p} v_p \end{aligned}$$

$$\begin{aligned} Ev_p &= \frac{1}{[p]!}EF^p v \\ &= \frac{1}{[p]!}\Big([p]F^{p-1}\frac{q^{1-p}K - q^{p-1}K^{-1}}{q - q^{-1}} + F^p E\Big)v \quad \text{[by Lemma 7.5]} \\ &= \frac{1}{[p-1]!}F^{p-1}\frac{q^{1-p}K - q^{p-1}K^{-1}}{q - q^{-1}}v + \frac{1}{[p]!}F^p E v \\ &= \frac{1}{[p-1]!}F^{p-1}\frac{q^{1-p}Kv - q^{p-1}K^{-1}v}{q - q^{-1}} + 0 \quad [v \text{ is highest weight}] \\ &= \frac{1}{[p-1]!}F^{p-1}\frac{q^{1-p}\lambda q^0 v - q^{p-1}\lambda^{-1}q^0 v}{q - q^{-1}} \quad [\text{by previous case with } v_0 = v] \\ &= \frac{q^{1-p}\lambda - q^{p-1}\lambda^{-1}}{q - q^{-1}} \cdot \frac{1}{[p-1]!}F^{p-1}v \\ &= \frac{q^{1-p}\lambda - q^{p-1}\lambda^{-1}}{q - q^{-1}} v_{p-1} \end{aligned}$$

$$\begin{aligned} Fv_{p-1} &= F\Big(\frac{1}{[p-1]!}F^{p-1}v\Big) \\ &= \frac{1}{[p-1]!}F^p v \\ &= \frac{[p]}{[p]!}F^p v \\ &= [p]v_p \end{aligned}$$

$\square$

Theorem 7.13. *(a) Let V be a finite-dimensional $U_q(\mathfrak{sl}(2))$-module generated (as a module) by a highest weight vector v of weight λ. Then*

(i) The scalar λ is of the form $\lambda = \pm q^n$ where $n := \dim(V) - 1$.

(ii) The set $\{v_0, v_1, ..., v_n\}$ is a basis of V and $v_p = 0$ for $p > n$.

(iii) The operator K acting on V is diagonalizable with the $(n+1)$ distinct eigenvalues

$$\{\pm q^n, \pm q^{n-2}, ..., \pm q^{2-n}, \pm q^{-n}\}$$

(iv) Any other highest weight vector in V is a scalar multiple of v and is of weight λ.

(v) The module V is simple.

(b) Any simple finite-dimensional $U_q(\mathfrak{sl}(2))$-module is generated by a highest weight vector. Any two such finite-dimensional modules generated by highest weight vectors of the same weight are isomorphic.

Proof. By Lemma 7.12 we know that $Kv_p = \lambda q^{-2p} v_p$, which implies that $\{v_p\}_{p \geq 0}$ is a sequence of eigenvectors for K with distinct eigenvalues. But this sequence cannot be infinite on account of V being finite-dimensional. It follows that there must be an integer n such that $v_n \neq 0$, but $v_{n+1} = 0$. But if $v_n \neq 0$, then the fact that $Fv_{p-1} = [p]v_p$ implies that $v_m \neq 0$ for $0 \leq m \leq n$. Similarly, if $v_{n+1} = 0$, then $v_m = 0$ for $m > n$. We therefore have

$$Ev_{n+1} = \frac{q^{-n}\lambda - q^n\lambda^{-1}}{q - q^{-1}} v_n = 0$$

which implies that $q^{-n}\lambda = q^n\lambda^{-1}$ and, in turn, that $\lambda = \pm q^n$.

Now, since $\{v_0, ..., v_n\}$ consists of non-zero eigenvectors with distinct eigenvalues, this set must be linearly independent. It is also the case that $\mathrm{Span}(v_0, ..., v_n) = V$. This follows from the fact that v generates V as a $U_q(\mathfrak{sl}(2))$-module along with the equations of Lemma 7.12. Thus, $\dim(V) = n + 1$. With respect to this basis, the relation

$$Kv_p = \lambda q^{-2p} v_p$$

together with the fact that $\lambda = \pm q^n$ implies that K, acting on V, has $n + 1$ distinct eigenvalue, namely

$$\pm q^n, \pm q^{n-2}, ..., \pm q^{2-n}, \pm q^{-n}$$

And because $\dim(V) = n + 1$, it follows that K is diagonalizable. We have thus established (i)-(iii).

For (iv), suppose v' is another highest weight vector. It must therefore be an eigenvector for the action of K and hence a scalar multiple of one of the v_i. By Lemma 7.12, E annihilates v_i if and only if $i = 0$. But $v_0 = v$; thus, v' is a scalar multiple of v and therefore has the same weight.

Suppose V' is a non-zero $U_q(\mathfrak{sl}(2))$-sub-module of V and v' is a highest weight vector of V'. Then v' is also a highest weight vector of V. By (iv) v' must be a non-zero scalar multiple of v and hence $v \in V'$. But v generates V and therefore $V \subset V'$ implying that $V = V'$. This establishes (v), that V is simple.

Finally, for (b), let v be a highest weight vector of V. From what we just established, if V is simple, then the submodule generated by v is necessarily identical to V and hence V is generated by a highest weight vector. It is now easy to see that given finite-dimensional simple $U_q(\mathfrak{sl}(2))$-modules V and V', respectively generated by highest weight vectors v and v' of weight λ, the map $v_i \mapsto v_i'$ for all i is an isomorphism of modules. $\square$

Note in particular that the above theorem implies that $V = \bigoplus_\lambda V^\lambda$. It also implies that there is a unique (up to isomorphism) simple $U_q(\mathfrak{sl}(2))$-module of dimension $n+1$ which is generated by a highest weight vector of weight $\pm q^n$. Let's denote this module by $V_{\epsilon,n}$, where $\epsilon = \pm 1$. Denote the corresponding algebra morphism by

$$\rho_{\epsilon,n} : U_q(\mathfrak{sl}(2)) \longrightarrow \mathrm{End}(V_{\epsilon,n})$$

With respect to the basis $\{v_0, ..., v_n\}$, we now give the representations for the action of generators E, F and K on $V_{\epsilon,n}$. Since K is diagonalizable we have

$$\rho_{\epsilon,n}(K) = \pm \begin{bmatrix} q^n & 0 & \dots & 0 & 0 \\ 0 & q^{n-2} & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & q^{2-n} & 0 \\ 0 & 0 & \dots & 0 & q^{-n} \end{bmatrix}$$

For E and F we have

$$\rho_{\epsilon,n}(E) = \pm \begin{bmatrix} 0 & [n] & 0 & \dots & 0 \\ 0 & 0 & [n-1] & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & [1] \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}, \quad \rho_{\epsilon,n}(F) = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ [1] & 0 & \dots & 0 & 0 \\ 0 & [2] & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & [n] & 0 \end{bmatrix}$$

7.5.1.1 Verma Modules

Suppose we now fix an *arbitrary* non-zero value for λ. Let $V(\lambda)$ be the infinite-dimensional vector space having denumerable basis $\{v_i\}_{i\in\mathbb{N}}$. For $p \geq 0$, set

$$Kv_p := \lambda q^{-2p} v_p, \quad K^{-1}v_p := \lambda^{-1} q^{2p} v_p, \tag{7.15}$$

$$Ev_{p+1} := \frac{q^{-p}\lambda - q^p\lambda^{-1}}{q - q^{-1}} v_p, \quad Fv_p := [p+1]v_{p+1}, \quad Ev_0 := 0 \tag{7.16}$$

Lemma 7.14. *The infinite-dimensional vector space $V(\lambda)$ is a $U_q(\mathfrak{sl}(2))$-module under the relations (7.15) and (7.16). Furthermore, $V(\lambda)$, as a module, is generated by v_0 which is a highest weight vector of weight λ.*

Proof. First, note that

$$\begin{aligned} KK^{-1}v_p &= K(\lambda^{-1}q^{2p}v_p) \\ &= \lambda^{-1}q^{2p}Kv_p \\ &= \lambda^{-1}q^{2p}\lambda q^{-2p}v_p \\ &= v_p \end{aligned}$$

Similar reasoning results in $K^{-1}Kv_p = v_p$. Next, we have

$$\begin{aligned} KEK^{-1}v_p &= KE(\lambda^{-1}q^{2p}v_p) \\ &= \lambda^{-1}q^{2p}K\Big(\frac{q^{1-p}\lambda - q^{p-1}\lambda^{-1}}{q-q^{-1}}v_{p-1}\Big) \\ &= \lambda^{-1}q^{2p}\frac{q^{1-p}\lambda - q^{p-1}\lambda^{-1}}{q-q^{-1}}Kv_{p-1} \\ &= \lambda^{-1}q^{2p}\frac{q^{1-p}\lambda - q^{p-1}\lambda^{-1}}{q-q^{-1}}\lambda q^{2-2p}v_{p-1} \\ &= q^2\frac{q^{1-p}\lambda - q^{p-1}\lambda^{-1}}{q-q^{-1}}v_{p-1} \\ &= q^2Ev_p \end{aligned}$$

Similar reasoning produces $KFK^{-1} = q^{-2}Fv_p$. Finally, consider that

$$\begin{aligned}
[E,F]v_p &= EFv_p - FEv_p \\
&= E([p+1]v_{p+1}) - F\Big(\frac{q^{1-p}\lambda - q^{p-1}\lambda^{-1}}{q-q^{-1}}v_{p-1}\Big) \\
&= [p+1]\frac{q^{-p}\lambda - q^{p}\lambda^{-1}}{q-q^{-1}}v_p - [p]\frac{q^{1-p}\lambda - q^{p-1}\lambda^{-1}}{q-q^{-1}}v_p \\
&= \frac{q^{p+1}-q^{-(p+1)}}{q-q^{-1}}\cdot\frac{q^{-p}\lambda - q^{p}\lambda^{-1}}{q-q^{-1}}v_p - \frac{q^{p}-q^{-p}}{q-q^{-1}}\cdot\frac{q^{1-p}\lambda - q^{p-1}\lambda^{-1}}{q-q^{-1}}v_p \\
&= \frac{q^{1-2p}\lambda - q^{-(2p+1)}\lambda + q^{2p-1}\lambda^{-1} - q^{2p+1}\lambda^{-1}}{(q-q^{-1})^2}v_p \\
&= \frac{(q-q^{-1})(q^{-2p}\lambda - q^{2p}\lambda^{-1})}{(q-q^{-1})^2}v_p \\
&= \frac{K-K^{-1}}{q-q^{-1}}v_p
\end{aligned}$$

This shows that $V(\lambda)$ is a $U_q(\mathfrak{sl}(2))$-module. Now, it is easily seen that $Kv_0 = \lambda v_0$ and, by definition, $Ev_0 = 0$, so v_0 is a highest weight vector of weight λ. To show that v_0 generates $V(\lambda)$, note that $Fv_p = [p+1]v_{p+1}$ implies that $v_{p+1} = \frac{1}{[p+1]}Fv_p$ and hence that $v_p = \frac{1}{[p]}Fv_{p-1}$. Applying F to both sides yields $Fv_p = \frac{1}{[p]}F^2v_{p-1}$. By substitution, this turns into $[p+1]v_{p+1} = \frac{1}{[p]}F^2v_{p-1}$, which then becomes $v_{p+1} = \frac{1}{[p+1][p]}F^2v_{p-1}$. In turn, this implies that $v_p = \frac{1}{[p][p-1]}F^2v_{p-2}$. Iterating this procedure eventually yields that $v_p = \frac{1}{[p]!}F^pv_0$ for all p. But $v_0 = v$, so it generates all of $V(\lambda)$. □

This infinite-dimensional $U_q(\mathfrak{sl}(2))$-module is what we call a *Verma module* of highest weight λ. The technical definition of a Verma module need not concern us here, but one may consult [13] for a more thorough treatment of the topic.

Verma modules are important because they generate highest weight modules.

Proposition 7.15. *Any highest weight $U_q(\mathfrak{sl}(2))$-module of highest weight λ is a quotient of the Verma module $V(\lambda)$.*

Proof. Let V be an arbitrary highest weight $U_q(\mathfrak{sl}(2))$-module generated by highest weight vector v. Let $f : V(\lambda) \to V$ be the linear map defined by

$$f(v_p) := \frac{1}{[p]!}F^pv$$

By Lemma 7.12, f is a $U_q(\mathfrak{sl}(2))$ - morphism, since

$$\begin{aligned} f(Kv_p) &= f(\lambda q^{-2p} v_p) \\ &= \lambda q^{-2p} f(v_p) \\ &= \lambda q^{-2p} \frac{1}{[p]!} F^p v \\ &= \lambda q^{-2p} \frac{1}{[p]!} F^p \lambda^{-1} K v \\ &= q^{-2p} \frac{1}{[p]!} F^p K v \end{aligned}$$

and

$$\begin{aligned} Kf(v_p) &= K \frac{1}{[p]!} F^p v \\ &= \frac{1}{[p]!} K F^p v \\ &= q^{-2p} \frac{1}{[p]!} F^p K v \end{aligned}$$

Also

$$\begin{aligned} f(Ev_p) &= f\left(\frac{q^{1-p}\lambda - q^{p-1}\lambda^{-1}}{q - q^{-1}} v_{p-1} \right) \\ &= \frac{q^{1-p}\lambda - q^{p-1}\lambda^{-1}}{q - q^{-1}} f(v_{p-1}) \\ &= \frac{q^{1-p}\lambda - q^{p-1}\lambda^{-1}}{q - q^{-1}} \cdot \frac{1}{[p-1]!} F^{p-1} v \end{aligned}$$

and

$$\begin{aligned} Ef(v_p) &= E \frac{1}{[p]!} F^p v \\ &= \frac{1}{[p]!} E F^p v \\ &= \frac{1}{[p]!} \left([p] F^{p-1} \frac{q^{1-p}K - q^{p-1}K^{-1}}{q - q^{-1}} + F^p E \right) v \quad \text{[by Lemma 7.5]} \\ &= \frac{1}{[p-1]!} F^{p-1} \frac{q^{1-p}Kv - q^{p-1}K^{-1}v}{q - q^{-1}} \quad [Ev = 0] \\ &= \frac{q^{1-p}\lambda - q^{p-1}\lambda^{-1}}{q - q^{-1}} \cdot \frac{1}{[p-1]!} F^{p-1} v \end{aligned}$$

Finally,

$$\begin{aligned} f(Fv_p) &= f([p+1]v_{p+1}) \\ &= [p+1]f(v_{p+1}) \\ &= [p+1]\frac{1}{[p+1]!}F^{p+1}v \\ &= \frac{1}{[p]!}F^{p+1}v \end{aligned}$$

and

$$\begin{aligned} Ff(v_p) &= F\frac{1}{[p]!}F^{p}v \\ &= \frac{1}{[p]!}F^{p+1}v \end{aligned}$$

Furthermore, f is surjective since $f(v_0) = v$ generates V. The First Isomorphism Theorem therefore yields

$$\begin{tikzcd} V(\lambda) \arrow[r, "f"] \arrow[d] & V \\ \frac{V(\lambda)}{Ker(f)} \arrow[ur, "\sim"'] & \end{tikzcd}$$

□

As an example we have that the simple finite-dimensional module $V_{\epsilon,n}$ is a quotient of the Verma module $V(\pm q^n)$.

7.5.2 When q is a Root of Unity

When q happens to be a root of unity, the resulting theory becomes less tame. In other words, matters become more convoluted. Nevertheless, let us dive in and see what becomes of it. Suppose q is a root of unity, though not ± 1 so that $q^2 \neq 1$. Let $d > 2$ represent the order of q. Next, define

$$d_{e,o} := \begin{cases} d/2, & \text{if } d \text{ is even} \\ d, & \text{if } d \text{ is odd} \end{cases}$$

Note that $[d_{e,o}] = 0$, since

$$[d_{e,o}] = \frac{q^{d_{e,o}} - q^{-d_{e,o}}}{q - q^{-1}}$$

If d is odd, then we have

$$\frac{q^{d_{e,o}} - q^{-d_{e,o}}}{q - q^{-1}} = \frac{q^d - q^{-d}}{q - q^{-1}} = 0$$

and if d is even we get

$$\begin{aligned} \frac{q^{d_{e,o}} - q^{-d_{e,o}}}{q - q^{-1}} &= \frac{q^{d/2} - q^{-d/2}}{q - q^{-1}} \\ &= \frac{q^{d/2}}{q^{d/2}} \cdot \frac{q^{d/2} - q^{-d/2}}{q - q^{-1}} \\ &= \frac{q^d - q^0}{q^{d/2}(q - q^{-1})} \\ &= \frac{1 - 1}{q^{d/2}(q - q^{-1})} = 0 \end{aligned}$$

Things aren't so bad if the dimension of a simple $U_q(\mathfrak{sl}(2))$-module is sufficiently low. When this is the case, we get essentially the same modules as when q is not a root of unity. But how low is sufficiently low?

Proposition 7.16. *Any simple non-zero $U_q(\mathfrak{sl}(2))$-module of dimension $< d_{e,o}$ is isomorphic to a module of the form $V_{\epsilon,n}$ where $0 \leq n < d_{e,o}$.*

Proof. Observe that when $n < d_{e,o}$ we get distinct scalars $1, q^2, ..., q^{2n}$. The rest of the proof proceeds as in Theorem 7.13. □

Things are quite different when the dimension of the module exceeds $d_{e,o}$. We unveil this difference after establishing a few lemmas.

Lemma 7.17. *The elements $E^{d_{e,o}}, F^{d_{e,o}}$ and $K^{d_{e,o}}$ are members of Z_q (i.e. the center of $U_q(\mathfrak{sl}(2))$).*

Proof. Because each follows a similar line of reasoning using the results of Lemma 7.5, we'll consider it sufficient to demonstrate the centrality of $E^{d_{e,o}}$. We have

$$\begin{aligned} E^{d_{e,o}} K &= q^{-2d_{e,o}} K E^{d_{e.o}} \\ &= 1 K E^{d_{e,o}} \\ &= K E^{d_{e,o}} \end{aligned}$$

The result holds similarly for K^{-1} and E is obvious. We therefore proceed to check F.

$$\begin{aligned} E^{d_{e,o}}F &= [d_{e,o}]E^{d_{e,o}-1}\frac{q^{d_{e,o}-1}K - q^{1-d_{e,o}}K^{-1}}{q-q^{-1}} + FE^{d_{e,o}} \\ &= 0 + FE^{d_{e,o}} \quad [[d_{e,o}] = 0] \\ &= FE^{d_{e,o}} \end{aligned}$$

□

Lemma 7.18. *If $z \in Z_q$, then z acts on any finite-dimensional simple $U_q(\mathfrak{sl}(2))$-module V by scalar multiplication.*

Proof. Let ρ_z be the endomorphism of V induced by the action of z. Note that this will be a $U_q(\mathfrak{s}(2))$-morphism, since z is central. That is, for any $u \in U_q(\mathfrak{sl}(2))$ and $v \in V$

$$\rho_z(uv) = zuv = uzv = u\rho_z(v)$$

Now, because V is finite-dimensional and we are working over $\mathbb{C}$, the endomorphism ρ_z has an eigenvalue λ. This means that $Ker(\rho_z - \lambda \mathrm{id}_V)$ is a non-trivial submodule of V. But V is simple. Therefore, $Ker(\rho_z - \lambda \mathrm{id}_V) = V$. It follows that z acts on V by scalar multiplication. □

Proposition 7.19. *There is no simple finite-dimensional $U_q(\mathfrak{sl}(2))$-module of dimension greater than $d_{e,o}$.*

Proof. Let's suppose that there does exist a simple finite-dimensional $U_q(\mathfrak{sl}(2))$-module V of dimension $> d_{e,o}$.

Case 1

Suppose there exists a non-zero eigenvector $v \in V$ for the action of K such that $Fv = 0$. Consider, then, the subspace V' of V generated by $v, Ev, ..., E^{d_{e,o}-1}v$. As a subspace it has dimension at most $d_{e,o}$ and if we can show that it is also a sub-module of V, then we will have a contradiction. To show that V' is indeed a submodule we will demonstrate that it is stable under the actions of the generators K, E and F. Clearly, V' is stable under the action of K, since

$$\begin{aligned} K(E^p v) &= KE^p v \\ &= q^{2p}E^p K v \quad [\text{by Lemma 7.5}] \\ &= q^{2p}\lambda E^p v \end{aligned}$$

For E we have $E(E^p v) = E^{p+1}v$, which is clearly in V' if $p+1 < d_{e,o}$. If, however, $p+1 = d_{e,o}$, then we will have $E^{d_{e,o}}v$, which, by the previous two lemmas, is equal to cv for some constant c. Thus, V' is stable under E.

Finally, for F we have

$$\begin{aligned}
F(E^p v) &= FE^p v \\
&= \left(E^p F - [p]E^{p-1} \frac{q^{p-1}K - q^{1-p}K^{-1}}{q-q^{-1}} \right) v \quad \text{[Lemma 7.5]} \\
&= E^p F v - [p]E^{p-1} \frac{q^{p-1}Kv - q^{1-p}K^{-1}v}{q-q^{-1}} \\
&= 0 - [p]E^{p-1} \frac{q^{p-1}\lambda v - q^{1-p}\lambda^{-1}v}{q-q^{-1}} \\
&= -[p] \frac{q^{p-1}\lambda - q^{1-p}\lambda^{-1}}{q-q^{-1}} E^{p-1}v
\end{aligned}$$

So, V' is also stable under F. This shows that V' is a non-zero submodule of V of dimension $\leq d_{e,o}$, which contradicts the fact that V is simple. Thus, we cannot suppose that there is a non-zero eigenvector $v \in V$ for the action of K such that $Fv = 0$.

$\boxed{\text{Case 2}}$

Despite there being no non-zero eigenvector $v \in V$ for the action of K such $Fv = 0$, there is still a non-zero eigenvector $v \in V$ for the action of K (but $Fv \neq 0$). Let V'' be the subspace of V generated by $v, Fv, ..., F^{d_{e,o}-1}v$. As in the case above, V'' is stable under the action of K. It is also stable under F by similar reasoning to the previous case with stability under E. The more interesting case is stability under E in this context. First, if $p > 0$, then

$$\begin{aligned}
E(F^p v) &= EF(F^{p-1}v) \\
&= \left(FE - \frac{K^{-1} - K}{q - q^{-1}} \right) F^{p-1}v \\
&= \left(FE + \frac{qK + q^{-1}K^{-1}}{(q-q^{-1})^2} - \left(\frac{qK + q^{-1}K^{-1}}{(q-q^{-1})^2} + \frac{K^{-1} - K}{q-q^{-1}} \right) \right) F^{p-1}v
\end{aligned}$$

Let $C_q = FE + \frac{qK + q^{-1}K^{-1}}{(q-q^{-1})^2}$. The reason for introducing this C_q is that it is actually a very important element called the *quantum Casimir element*. While interesting, the only thing we need to know about C_q is that it belongs to the center of $U_q(\mathfrak{sl}(2))$ (see

[7]). Thus, by Lemma 7.18, C_q acts by scalar multiplication on V. We therefore have

$$\begin{aligned} E(F^p v) &= \left(C_q - \frac{qK + q^{-1}K^{-1} + (q-q^{-1})(K^{-1}-K)}{(q-q^{-1})^2} \right) F^{p-1}v \\ &= \left(C_q - \frac{qK^{-1} + q^{-1}K}{(q-q^{-1})^2} \right) F^{p-1}v \\ &= C_q F^{p-1}v - \frac{qK^{-1} + q^{-1}K}{(q-q^{-1})^2} F^{p-1}v \\ &= cF^{p-1}v - \frac{qK^{-1} + q^{-1}K}{(q-q^{-1})^2} F^{p-1}v \quad [c \text{ a scalar}] \end{aligned}$$

and this is an element of V'', since K stabilizes V''. Now, if $p = 0$, then we may employ the same argument, but with the substitution $v = c'^{-1}F^{d_{e,o}}v$ for some scalar c'. This holds by Lemma 7.17 and Lemma 7.18. In other words, $F^{d_{e,o}}$ is central and so acts by scalar multiplication.

Therefore, V'' is a non-zero submodule of V, which again is a contradiction. It must therefore be that there are no simple finite-dimensional $U_q(\mathfrak{sl}(2))$-modules of dimension greater than $d_{e,o}$. □

So, for sufficiently low dimension (less that $d_{e,o}$) we recover the modules obtained when q is not a root of unity. On the other hand, there is a point (greater than $d_{e,o}$) at which no such modules exist. But what about the dividing line itself? In other words, what happens if the dimension is $d_{e,o}$? In this case, it turns out that there are only three "kinds" of modules up to isomorphism. These are given in the following list:

1. $V(\lambda, a, b)$ with $b \neq 0$
2. $V(\lambda, a, 0)$ where λ is not of the form $\pm q^{j-1}$ for any $1 \leq j \leq d_{e,o} - 1$
3. $\widetilde{V}(\pm q^{1-j}, c)$ with $c \neq 0$ and $1 \leq j \leq d_{e,o} - 1$

The notation $V(\lambda, a, b)$ is meant to indicate that modules of this kind depend on the three complex numbers λ, a and b, where λ is assumed non-zero. These modules have basis $\{v_0, ..., v_{d_{e,o}-1}\}$ and for $0 \leq p < d_{e,o} - 1$

$$\begin{aligned} Kv_p &:= \lambda q^{-2p} v_p \\ Ev_{p+1} &:= \left(\frac{q^{-p}\lambda - q^p \lambda^{-1}}{q - q^{-1}} [p+1] + ab \right) v_p \\ Fv_p &:= v_{p+1} \end{aligned}$$

and $Ev_0 := av_{d_{e,o}-1}$, $Fv_{d_{e,o}-1} := bv_0$ and $Kv_{d_{e,o}-1} := \lambda q^{-2(d_{e,o}-1)} v_{d_{e,o}-1}$ otherwise.

The modules denoted by $\widetilde{V}(\mu, c)$ depend on two scalars μ and c where $\mu \neq 0$. These too have basis $\{v_0, ..., v_{d_{e,o}-1}\}$, but E, F and K, in this case, act by

$$\begin{aligned} Kv_p &:= \mu q^{2p} v_p \\ Fv_{p+1} &:= \frac{q^{-p}\mu^{-1} - q^p\mu}{q - q^{-1}}[p+1]v_p \\ Ev_p &= v_{p+1} \end{aligned}$$

if $0 \leq p < d_{e,o} - 1$ and $Fv_0 := 0$, $Ev_{d_{e,o}-1} := cv_0$ and $Kv_{d_{e,o}-1} := \mu q^{-2} v_{d_{e,o}-1}$ otherwise. These modules are all indecomposable (see [7]).

7.5.3 Action on the Quantum Plane

Enter once more the quantum plane. We now describe how $U_q(\mathfrak{sl}(2))$ acts on this mysterious object.

Let A be an algebra. For any $a \in A$, let a_r represent right multiplication by a. Similarly, let a_ℓ represent left multiplication by a. Now, if α is an automorphism of A, then we have

$$\alpha \circ a_\ell = \alpha(a)_\ell \circ \alpha \quad \text{and} \quad \alpha \circ a_r = \alpha(a)_r \circ \alpha$$

We now define an even more generalized kind of derivation as follows:

Definition 7.20. Given any two automorphisms α and σ of an algebra A, we say that a linear endomorphism $\mathscr{D}_{\alpha,\sigma}$ is a (α, σ)-derivation if

$$\mathscr{D}_{\alpha,\sigma}(ab) = \alpha(a)\mathscr{D}_{\alpha,\sigma}(b) + \mathscr{D}_{\alpha,\sigma}(a)\sigma(b)$$

for all $a, b \in A$.

Note that $\mathscr{D}_{\alpha,\sigma} \circ a_\ell(b) = \mathscr{D}_{\alpha,\sigma}(ab)$, which allows us to write

$$\mathscr{D}_{\alpha,\sigma} \circ a_\ell = \alpha(a)_\ell \circ \mathscr{D}_{\alpha,\sigma} + \mathscr{D}_{\alpha,\sigma}(a)_\ell \circ \sigma$$

Using right multiplication we also have

$$\mathscr{D}_{\alpha,\sigma} \circ a_r = \sigma(b)_r \circ \mathscr{D}_{\alpha,\sigma} + \mathscr{D}_{\alpha,\sigma}(b)_r \circ \alpha$$

Lemma 7.21. *Let $\mathscr{D}_{\alpha,\sigma}$ be a (α, σ)-derivation of A and $a \in A$. If there exist algebra automorphisms α' and σ' such that*

$$a_r \circ \alpha' = a_\ell \circ \alpha \quad \textit{and} \quad a_\ell \circ \sigma' = a_r \circ \sigma$$

then the linear endomorphism $a_\ell \circ \mathscr{D}_{\alpha,\sigma}$ *is a* (α', σ)*-derivation and* $a_r \circ \mathscr{D}_{\alpha,\sigma}$ *is a* (α, σ')*-derivation.*

Proof. For the first,

$$\begin{aligned}
(a_\ell \circ \mathscr{D}_{\alpha,\sigma})(ab) &= a_\ell(\alpha(a)\mathscr{D}_{\alpha,\sigma}(b) + \mathscr{D}_{\alpha,\sigma}(a)\sigma(b)) \\
&= a(\alpha(a)\mathscr{D}_{\alpha,\sigma}(b) + \mathscr{D}_{\alpha,\sigma}(a)\sigma(b)) \\
&= a\alpha(a)\mathscr{D}_{\alpha,\sigma}(b) + a\mathscr{D}_{\alpha,\sigma}(a)\sigma(b) \\
&= (a_\ell \circ \alpha)(a)\mathscr{D}_{\alpha,\sigma}(b) + (a_\ell \circ \mathscr{D}_{\alpha,\sigma})(a)\sigma(b) \\
&= (a_r \circ \alpha')(a)\mathscr{D}_{\alpha,\sigma}(b) + (a_\ell \circ \mathscr{D}_{\alpha,\sigma})(a)\sigma(b) \\
&= \alpha'(a)a\mathscr{D}_{\alpha,\sigma}(b) + (a_\ell \circ \mathscr{D}_{\alpha,\sigma})(a)\sigma(b) \\
&= \alpha'(a)(a_\ell \circ \mathscr{D}_{\alpha,\sigma})(b) + (a_\ell \circ \mathscr{D}_{\alpha,\sigma})(a)\sigma(b)
\end{aligned}$$

For the second,

$$\begin{aligned}
(a_r \circ \mathscr{D}_{\alpha,\sigma})(ab) &= a_r(\alpha(a)\mathscr{D}_{\alpha,\sigma}(b) + \mathscr{D}_{\alpha,\sigma}(a)\sigma(b)) \\
&= \alpha(a)\mathscr{D}_{\alpha,\sigma}(b)a + \mathscr{D}_{\alpha,\sigma}(a)\sigma(b)a \\
&= \alpha(a)(a_r \circ \mathscr{D}_{\alpha,\sigma})(b) + \mathscr{D}_{\alpha,\sigma}(a)(a_r \circ \sigma)(b) \\
&= \alpha(a)(a_r \circ \mathscr{D}_{\alpha,\sigma})(b) + \mathscr{D}_{\alpha,\sigma}(a)(a_\ell \circ \sigma')(b) \\
&= \alpha(a)(a_r \circ \mathscr{D}_{\alpha,\sigma})(b) + \mathscr{D}_{\alpha,\sigma}(a)a\sigma'(b) \\
&= \alpha(a)(a_r \circ \mathscr{D}_{\alpha,\sigma})(b) + (a_r \circ \mathscr{D}_{\alpha,\sigma})(a)\sigma'(b)
\end{aligned}$$

□

From here on, let us assume that the algebra A is the quantum plane $\kappa_q[x, y]$. Define algebra automorphisms σ_x, σ_y of $\kappa_q[x, y]$ by

$$\sigma_x(x) := qx, \quad \sigma_x(y) := y \qquad\qquad \sigma_y(y) := qy, \quad \sigma_y(x) := x$$

Clearly these automorphisms reduce to the identity morphism when $q = 1$. Now just as there are partial derivatives in the classical case, we also get q-analogues for the quantum plane, which, according to our pattern thus far, we denote by $\partial_q/\partial x$ and $\partial_q/\partial y$. Define these as follows: For all $m, n \geq 0$ set

$$\frac{\partial_q(x^m y^n)}{\partial x} := [m]x^{m-1}y^n \quad \text{and} \quad \frac{\partial_q(x^m y^n)}{\partial y} := [n]x^m y^{n-1}$$

That this is an *analogue* should be stressed, since one might find it troubling that the classical partial derivative is not recovered when $q = 1$. Nevertheless, the imitation of the power rule keeps things rather tame or well behaved as in the classical case.

Through patient calculation it can be shown that the following commutation relations hold amongst the maps $x_\ell, x_r, y_\ell, y_r, \sigma_x, \sigma_y, \partial_q/\partial x$ and $\partial_q/\partial y$ (see [7]):

$$\begin{aligned} y_\ell x_\ell &= q x_\ell y_\ell, & x_r y_r &= q y_r x_r \\ \sigma_x x_{\ell,r} &= q x_{\ell,r}\sigma_x, & \sigma_y y_{\ell,r} &= q y_{\ell,r}\sigma_y \\ \frac{\partial_q}{\partial x}\sigma_x &= q\sigma_x\frac{\partial_q}{\partial x}, & \frac{\partial_q}{\partial y}\sigma_y &= q\sigma_y\frac{\partial_q}{\partial y} \\ \frac{\partial_q}{\partial x}y_\ell &= q y_\ell\frac{\partial_q}{\partial x}, & \frac{\partial_q}{\partial y}x_r &= q x_r\frac{\partial_q}{\partial y} \end{aligned}$$

$$\begin{aligned} \frac{\partial_q}{\partial x}x_\ell &= q^{-1}x_\ell\frac{\partial_q}{\partial x} + \sigma_x = q x_\ell\frac{\partial_q}{\partial x} + \sigma_x^{-1} \\ \frac{\partial_q}{\partial y}y_r &= q^{-1}y_r\frac{\partial_q}{\partial y} + \sigma_y = q y_r\frac{\partial_q}{\partial y} + \sigma_y^{-1} \end{aligned}$$

Besides this, one can also show that

$$x_\ell\frac{\partial_q}{\partial x} = \frac{\sigma_x - \sigma_x^{-1}}{q - q^{-1}} \quad \text{and} \quad y_r\frac{\partial_q}{\partial y} = \frac{\sigma_y - \sigma_y^{-1}}{q - q^{-1}}$$

All other commutations between these are trivial in the sense that they commute perfectly. We now connect these q-partial derivatives to the modified derivations we defined above.

Proposition 7.22. *The endomorphism $\partial_q/\partial x$ is a $(\sigma_x^{-1}\circ\sigma_y, \sigma_x)$-derivation, while $\partial_q/\partial y$ is a $(\sigma_y, \sigma_x\circ\sigma_y^{-1})$-derivation.*

Proof. Let $a, b \in A$ and D a linear endomorphism of A be such that

$$D \circ a_\ell = \alpha(a)_\ell \circ D + D(a)_\ell \circ \sigma$$

and

$$D \circ b_\ell = \alpha(b)_\ell \circ D + D(b)_\ell \circ \sigma$$

Then we have

$$\begin{aligned}
D \circ (ab)_\ell(c) &= D((ab)c)\\
&= D(a_\ell \circ b_\ell(c))\\
&= (D \circ a_\ell \circ b_\ell)(c)\\
&= \big((\alpha(a)_\ell \circ D + D(a)_\ell \circ \sigma) \circ b_\ell\big)(c)\\
&= (\alpha(a)_\ell \circ D \circ b_\ell)(c) + (D(a)_\ell \circ \sigma \circ b_\ell)(c)\\
&= \big(\alpha(a)_\ell \circ (\alpha(b)_\ell \circ D + D(b)_\ell \circ \sigma)\big)(c) + (D(a)_\ell \circ \sigma \circ b_\ell)(c)\\
&= (\alpha(a)_\ell \circ \alpha(b)_\ell \circ D)(c) + (\alpha(a)_\ell \circ D(b)_\ell \circ \sigma)(c) + (D(a)_\ell \circ \sigma \circ b_\ell)(c)\\
&= (\alpha(ab)_\ell \circ D)(c) + \alpha(a)D(b)\sigma(c) + D(a)\sigma(b)\sigma(c)\\
&= (\alpha(ab)_\ell \circ D)(c) + (\alpha(a)D(b) + D(a)\sigma(b))\sigma(c)\\
&= (\alpha(ab)_\ell \circ D)(c) + (D(ab)_\ell \circ \sigma)(c)
\end{aligned}$$

Since c was arbitrary we get the same relation for the product ab. Because of this, we now check to see if the (α,σ)-derivation relations hold for our q-partial derivatives when $a = x$ and $b = y$. If so, then it will hold for the product xy and since x and y generate the quantum plane, it will follow that the q-partial derivatives are the desired derivations. Consider $\partial_q/\partial x$. Using the commutation relations above we find that

$$(\sigma_x^{-1}\sigma_y)(x)_\ell \frac{\partial_q}{\partial x} + \left(\frac{\partial_q x}{\partial x}\right)_\ell \sigma_x = q^{-1}x_\ell \frac{\partial_q}{\partial x} + \sigma_x = \frac{\partial_q}{\partial x} x_\ell$$

and

$$(\sigma_x^{-1}\sigma_y)(y)_\ell \frac{\partial_q}{\partial x} + \left(\frac{\partial_q y}{\partial x}\right)_\ell \sigma_x = q y_\ell \frac{\partial_q}{\partial x} = \frac{\partial_q}{\partial x} y_\ell$$

The case for $\partial_q/\partial y$ holds similarly. □

Theorem 7.23. *For any* $P \in \kappa_q[x,y]$*, set*

$$EP := x\frac{\partial_q P}{\partial y}, \quad FP := \frac{\partial_q P}{\partial x} y \tag{7.17}$$

$$KP := (\sigma_x \circ \sigma_y^{-1})(P), \quad K^{-1}P := (\sigma_y \circ \sigma_x^{-1})(P) \tag{7.18}$$

(i) Under the above formulas $\kappa_q[x,y]$ *is a* $U_q(\mathfrak{sl}(2))$*-module-algebra.*

(ii) The subspace $\kappa_q[x,y]_n$ *of homogeneous elements of degree* n *is a* $U_q(\mathfrak{sl}(2))$*-submodule of the quantum plane. It is generated by the highest weight vector* x^n *and is isomorphic to the simple module* $V_{1,n}$*.*

Proof. (i) First we need to check that the action is well-defined. For computational purposes, we shall suppress the "$\circ$" symbol. Now, clearly $(KK^{-1})P = (K^{-1}K)P = P$.

Next, using the commuting relations listed above we have

$$\begin{aligned}
(KEK^{-1})P &= (\sigma_x\sigma_y^{-1})(x\frac{\partial_q}{\partial y})(\sigma_y\sigma_x^{-1})P \\
&= (\sigma_x\sigma_y^{-1}x_\ell\frac{\partial_q}{\partial y}\sigma_y\sigma_x^{-1})P \\
&= (\sigma_x\sigma_y^{-1}x_\ell\sigma_x^{-1}\frac{\partial_q}{\partial y}\sigma_y)P \\
&= (q\sigma_x\sigma_y^{-1}\sigma_x^{-1}x_\ell\frac{\partial_q}{\partial y}\sigma_y)P \\
&= (q\sigma_y^{-1}x_\ell\frac{\partial_q}{\partial y}\sigma_y)P \\
&= (q^2x_\ell\frac{\partial_q}{\partial y})P \\
&= q^2EP
\end{aligned}$$

That $(KFK^{-1})P = q^{-2}FP$ is similarly proved.

Finally, we again use the above relations to show that

$$\begin{aligned}
[E,F]P &= \left(x_\ell\frac{\partial_q}{\partial y}y_r\frac{\partial_q}{\partial x} - y_r\frac{\partial_q}{\partial x}x_\ell\frac{\partial_q}{\partial y}\right)P \\
&= \left(x_\ell\left(q^{-1}y_r\frac{\partial_q}{\partial y}+\sigma_y\right)\frac{\partial_q}{\partial x} - y_r\left(q^{-1}x_\ell\frac{\partial_q}{\partial x}+\sigma_x\right)\frac{\partial_q}{\partial y}\right)P \\
&= \left(q^{-1}x_\ell y_r\frac{\partial_q}{\partial y}\frac{\partial_q}{\partial x} + x_\ell\sigma_y\frac{\partial_q}{\partial x} - q^{-1}y_rx_\ell\frac{\partial_q}{\partial x}\frac{\partial_q}{\partial y} - y_r\sigma_x\frac{\partial_q}{\partial y}\right)P \\
&= \left(x_\ell\sigma_y\frac{\partial_q}{\partial x} - y_r\sigma_x\frac{\partial_q}{\partial y}\right)P \\
&= \left(\sigma_yx_\ell\frac{\partial_q}{\partial x} - \sigma_xy_r\frac{\partial_q}{\partial y}\right)P \\
&= \left(\sigma_y\frac{\sigma_x-\sigma_x^{-1}}{q-q^{-1}} - \sigma_x\frac{\sigma_y-\sigma_y^{-1}}{q-q^{-1}}\right)P \\
&= \frac{\sigma_y\sigma_x-\sigma_y\sigma_x^{-1}-\sigma_x\sigma_y+\sigma_x\sigma_y^{-1}}{q-q^{-1}}P \\
&= \frac{\sigma_x\sigma_y^{-1}-\sigma_y\sigma_x^{-1}}{q-q^{-1}}P \\
&= \frac{K-K^{-1}}{q-q^{-1}}P
\end{aligned}$$

So $\kappa_q[x,y]$ is a $U_q(\mathfrak{sl}(2))$-module. To show that it is a $U_q(\mathfrak{sl}(2))$-module-algebra we shall make use of Lemma 6.2, which requires that for any $u \in U_q(\mathfrak{sl}(2))$

$$u1 = \varepsilon(u)1 \tag{7.19}$$

as well as

$$K(PQ) = K(P)K(Q) \tag{7.20}$$
$$E(PQ) = PE(Q) + E(P)K(Q) \tag{7.21}$$
$$F(PQ) = K^{-1}(P)F(Q) + F(P)Q \tag{7.22}$$

for any $P, Q \in \kappa_q[x, y]$.

For (7.19), since $\{E^i F^j K^\ell\}_{i,j\in\mathbb{N};\ell\in\mathbb{Z}}$ is a basis for $U_q(\mathfrak{sl}(2))$ and ε is an algebra morphism, $\varepsilon(u) = 0$ for all $u \in U_q(\mathfrak{sl}(2))$ involving non-zero powers of E and/or F given that $\varepsilon(E) = \varepsilon(F) = 0$. Now $K \cdot 1 = (\sigma_x \sigma_y^{-1})(1) = 1$ and likewise $K^{-1} \cdot 1 = 1$. For E we have

$$\begin{aligned} E \cdot 1 &= x\frac{\partial_q 1}{\partial y} \\ &= x[0]y^{-1} = 0 \end{aligned}$$

and likewise $F \cdot 1 = 0$. From these it follows that, for u containing non-zero powers of E and/or F, $u \cdot 1 = 0$ and hence $u \cdot 1 = \varepsilon(u) \cdot 1$.

If, however, u is in terms of powers of K only, then because $\varepsilon(K) = \varepsilon(K^{-1}) = 1$ and $K \cdot 1 = K^{-1} \cdot 1 = 1$, we again have that $u \cdot 1 = \varepsilon(u) \cdot 1$. Thus, (7.19) holds.

Relation (7.20) is obvious since K acts as an algebra automorphism. The remaining cases of (7.21) and (7.22) follow from Lemma 7.21 and Proposition 7.22.

(ii) First note that $Ex^n = x\frac{\partial_q x^n}{\partial y} = 0$. Next,

$$\begin{aligned} Kx^n &= (\sigma_x \sigma_y^{-1})(x^n) \\ &= \sigma_x(\sigma_y^{-1}(x)^n) \quad [\sigma_y^{-1} \text{ an algebra automorphism}] \\ &= \sigma_x(x^n) \\ &= \sigma_x(x)^n \quad [\sigma_x \text{ an algebra automorphism}] \\ &= (qx)^n \\ &= q^n x^n \end{aligned}$$

So, x^n is a highest weight vector of weight q^n. Finally, note that

$$\begin{aligned} Fx^n &= \frac{\partial_q x^n}{\partial x} y \\ &= [n]x^{n-1}y \end{aligned}$$

and then

$$\begin{aligned} F^2 x^n &= F([n]x^{n-1}y) \\ &= [n]F(x^{n-1}y) \\ &= [n]\frac{\partial_q x^{n-1}y}{\partial x}y \\ &= [n][n-1]x^{n-2}y^2 \end{aligned}$$

Iterating this process yields $F^p x^n = [n][n-1]\cdots[n-(p-1)]x^{n-p}y^p$. From this it follows that

$$\frac{1}{[p]!}F^p x^n = \begin{bmatrix} n \\ p \end{bmatrix} x^{n-p}y^p$$

Thus, $\kappa_q[x,y]_n$ is a submodule of the quantum plane generated by the highest weight vector x^n and is therefore isomorphic to $V_{1,n}$. □

7.5.4 Duality between $U_q(\mathfrak{sl}(2))$ and $SL_q(2)$

In this section we explore the quantum analogue of the duality between $U(\mathfrak{sl}(2))$ and $SL(2)$ established in Chapter 5. Again, our goal is to construct an appropriate algebra automorphism ψ, first from $M_q(2)$ into $U_q^*(\mathfrak{sl}(2))$. Having such a morphism would be equivalent to possessing a matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$, with $A, B, C, D \in U_q^*(\mathfrak{sl}(2))$.

In this case we shall make use of the simple $U_q(\mathfrak{sl}(2))$-module $V_{1,1}$ of highest weight q, with basis $\{v_0, v_1\}$ and representation $\rho_{1,1}$. Computing the matrix representations of the generators yields

$$\rho_{1,1}(E) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \rho_{1,1}(F) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \rho_{1,1}(K) = \begin{bmatrix} q & 0 \\ 0 & q^{-1} \end{bmatrix}$$

For an arbitrary element $u \in U_q(\mathfrak{sl}(2))$ we have that $\rho_{1,1}(u)$ is some 2×2 matrix and, as before, we shall write this as

$$\rho_{1,1}(u) = \begin{bmatrix} A(u) & B(u) \\ C(u) & D(u) \end{bmatrix}$$

We shall now start from the above matrix with entries A, B, C, D and work backwards to see if we get the desired morphism ψ that will give us the duality inducing bilinear form $\langle u, x \rangle = \psi(x)(u)$.

Lemma 7.24. $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ *is a* $U_q^*(\mathfrak{sl}(2))$*-point of* $M_q(2)$.

Proof. The proof of this is direct, albeit tedious. So, to obviate unnecessary length to an already lengthy thesis, the reader is directed to [7] for a complete proof. □

Proposition 7.25. *The bilinear form* $\langle u, x\rangle = \psi(x)(u)$ *realizes an imperfect duality between the bialgebras* $U_q(\mathfrak{sl}(2))$ *and* $M_q(2)$*.*

Proof. The proof follows the same line of reasoning used in the classical case. □

Lemma 7.26. *For the quantum determinant* det_q *of* $M_q(2)$*, we have that* $\psi(det_q) = 1$*.*

Proof. As in the classical case, $\det_q$ is a grouplike element. Thus, $\langle -, \det_q\rangle$ is an algebra morphism. Our task, again, is to show that for $u \in U_q(\mathfrak{sl}(2))$, $\langle u, \det_q\rangle = \varepsilon(u)$, which is true if it holds for the generators of $U_q(\mathfrak{sl}(2))$. For our purposes, however, we'll content ourselves with showing that it holds for E and K.

$$\begin{aligned}
\psi(\det_q)(E) &= \langle E, ad - q^{-1}bc\rangle \\
&= \langle E, ad\rangle - q^{-1}\langle E, bc\rangle \\
&= \langle \Delta(E), a \otimes d\rangle - q^{-1}\langle \Delta(E), b \otimes c\rangle \\
&= \langle E \otimes K + 1 \otimes E, a \otimes d\rangle - q^{-1}\langle E \otimes K + 1 \otimes E, b \otimes c\rangle \\
&= \langle E \otimes K, a \otimes d\rangle + \langle 1 \otimes E, a \otimes d\rangle - q^{-1}\langle E \otimes K, b \otimes c\rangle - q^{-1}\langle 1 \otimes E, b \otimes c\rangle \\
&= \langle E, a\rangle\langle K, d\rangle + \langle 1, a\rangle\langle E, d\rangle - q^{-1}\langle E, b\rangle\langle K, c\rangle - q^{-1}\langle 1, b\rangle\langle E, c\rangle \\
&= A(E)D(K) + A(1)D(E) - q^{-1}B(E)C(K) - q^{-1}B(1)C(E) \\
&= 0 \cdot q^{-1} + 1 \cdot 0 - q^{-1} \cdot 1 \cdot 0 - q^{-1} \cdot 0 \cdot 0 \\
&= 0 = \varepsilon(E)
\end{aligned}$$

For K we have

$$\begin{aligned}
\psi(\det_q)(K) &= \langle K, ad - q^{-1}bc\rangle \\
&= \langle K, ad\rangle - q^{-1}\langle K, bc\rangle \\
&= \langle \Delta(K), a \otimes d\rangle - q^{-1}\langle \Delta(K), b \otimes c\rangle \\
&= \langle K \otimes K, a \otimes d\rangle - q^{-1}\langle K \otimes K, b \otimes c\rangle \\
&= \langle K, a\rangle\langle K, d\rangle - q^{-1}\langle K, b\rangle\langle K, c\rangle \\
&= A(K)D(K) - q^{-1}B(K)C(K) \\
&= qq^{-1} - q^{-1} \cdot 0 \cdot 0 \\
&= 1 = \varepsilon(K)
\end{aligned}$$

□

Now, because $\psi(\det_q) = 1$, it factors through $SL_q(2) = M_q(2)/(\det_q - 1)$ and so we get an induced morphism of algebras $\psi : SL_q(2) \to U_q^*(\mathfrak{sl}(2))$.

Theorem 7.27. *The bilinear form* $\langle u, x\rangle = \psi(x)(u)$ *realizes a duality between the Hopf algebras* $U_q(\mathfrak{sl}(2))$ *and* $SL_q(2)$.

Proof. Again, the proof here is nearly the same as in the classical case. There is, however, some novelty concerning the antipodes.

$$\begin{aligned}
\langle S(E), f\rangle &= \langle -EK^{-1}, f\rangle \\
&= -\langle EK^{-1}, f\begin{bmatrix} a & b \\ c & d\end{bmatrix}\rangle \\
&- f\begin{bmatrix} \langle EK^{-1}, a\rangle & \langle EK^{-1}, b\rangle \\ \langle EK^{-1}, c\rangle & \langle EK^{-1}, d\rangle\end{bmatrix} \\
&= -f\begin{bmatrix} A(EK^{-1}) & B(EK^{-1}) \\ C(EK^{-1}) & D(EK^{-1})\end{bmatrix}
\end{aligned}$$

Now, $\rho_{1,1}(-EK^{-1}) = -\begin{bmatrix} A(EK^{-1}) & B(EK^{-1}) \\ C(EK^{-1}) & D(EK^{-1})\end{bmatrix}$. But also

$$\begin{aligned}
\rho_{1,1}(-EK^{-1}) &= -\rho_{1,1}(E)\rho_{1,1}(K^{-1}) \\
&= -\begin{bmatrix} 0 & 1 \\ 0 & 0\end{bmatrix} \cdot \begin{bmatrix} q^{-1} & 0 \\ 0 & q\end{bmatrix} \\
&= \begin{bmatrix} 0 & -q \\ 0 & 0\end{bmatrix}
\end{aligned}$$

So, we end up with

$$\langle S(E), f\rangle = f\begin{bmatrix} 0 & -q \\ 0 & 0\end{bmatrix}$$

Consider now that

$$\begin{aligned}
\langle E, f\begin{bmatrix} S(a) & S(b) \\ S(c) & S(d)\end{bmatrix}\rangle &= \langle E, f\begin{bmatrix} d & -qb \\ -q^{-1}c & a\end{bmatrix}\rangle \\
&= f\begin{bmatrix} \langle E, a\rangle & \langle E, -qb\rangle \\ \langle E, -q^{-1}c\rangle & \langle E, a\rangle\end{bmatrix} \\
&= f\begin{bmatrix} D(E) & -qB(E) \\ -q^{-1}C(E) & D(E)\end{bmatrix} \\
&= f\begin{bmatrix} 0 & -q \\ 0 & 0\end{bmatrix}
\end{aligned}$$

Thus, $\langle S(E), f\rangle = \langle E, S(f)\rangle$. The result for F is checked similarly. Also, since K and K^{-1} are sufficiently similar, we'll finish by checking K.

On the one hand we have

$$\begin{aligned}\langle S(K), f\rangle &= \langle K^{-1}, f\rangle \\ &= f\begin{bmatrix} A(K^{-1}) & B(K^{-1}) \\ C(K^{-1}) & D(K^{-1})\end{bmatrix} \\ &= f\begin{bmatrix} q^{-1} & 0 \\ 0 & q\end{bmatrix}\end{aligned}$$

and on the other hand

$$\begin{aligned}\langle K, S(f)\rangle &= \langle K, \begin{bmatrix} d & -qb \\ -q^{-1}c & a\end{bmatrix}\rangle \\ &= f\begin{bmatrix} D(K) & -qB(K) \\ -q^{-1}C(K) & A(K)\end{bmatrix} \\ &= f\begin{bmatrix} q^{-1} & 0 \\ 0 & q\end{bmatrix}\end{aligned}$$

Thus, $\langle S(K), f\rangle = \langle K, S(f)\rangle$. $\square$

The above theorem is a nice symmetric way to finish off our discussion, given our work in Chapter 5. There is a great deal more that could be said and explored, but as we have seen, it takes a lot of work and it only gets more difficult and abstract. Hopefully the reader has gained some understanding from playing around in the shallow end of the quantum group "pool". As stated at the beginning of this chapter, the quantum group explored here is a significant one. However, there are other very interesting kinds of quantum groups as well. For instance, there are bicrossproduct quantum groups, which are directly relevant to quantum mechanics and Planck-scale physics.

With the basis we have established in this thesis, one can now proceed to study the famous Yang-Baxter equation. This involves notions of universal R-matrices, braided and cobraided bialgebras as well as braided categories. In fact, we have barely scratched the surface. Nevertheless, I hope this thesis was enlightening and spurs the reader on to study this exciting and fascinating world in greater and greater detail.

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