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A Computational Method for Solving a Class of Fractional-Order Non-Linear Singularly Perturbed Volterra Integro-Differential Boundary-Value Problems

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Mohammed M. A. Abuomar

This thesis is submitted in partial fulfilment of the requirements for the degree of Master of Science in Mathematics

Under the Supervision of Professor Muhammed Ibrahim Syam

April 2018

Declaration of Original Work

I, Mohammed M. A. Abuomar, the undersigned, a graduate student at the United Arab Emirates University (UAEU), and the author of this thesis, entitled "*A Computational Method for Solving a Class of Fractional-Order Non-linear Singularly Perturbed Volterra Integro-differential Boundary-value Problems*", hereby, solemnly declare that this thesis is my own original research work that has been done and prepared by me under the supervision of Professor Muhammed Ibrahim Syam, in the College of Science at UAEU. This work has not previously been presented or published, or formed the basis for the award of any academic degree, diploma or a similar title at this or any other university. Any materials borrowed from other sources (whether published or unpublished) and relied upon or included in my thesis have been properly cited and acknowledged in accordance with appropriate academic conventions. I further declare that there is no potential conflict of interest with respect to the research, data collection, authorship, presentation and/or publication of this thesis.

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Abstract

In this thesis, we present a computational method for solving a class of fractional singularly perturbed Volterra integro-differential boundary-value problems with a boundary layer at one end. The implemented technique consists of solving two problems which are a reduced problem and a boundary layer correction problem. The reproducing kernel method is used to the second problem. Pade' approximation technique is used to satisfy the conditions at infinity. Existence and uniformly convergence for the approximate solution are also investigated. Numerical results provided to show the efficiency of the proposed method.

Keywords: Singularly perturbed Volterra integro-differential, Caputo fractional derivative, nonlinear initial value problem

Title and Abstract (in Arabic)

طريقة حسابية لحل مجموعة من مسائل فولتيرا التكاملية-التفاضلية الكسرية المضطربة المعتلة المحيطية غير الخطية

الملخص

في هذه الأطروحة عرضنا طريقة حسابية لحل مجموعة من مسائل فولتيرا التكاملية-التفاضلية الكسرية المضطربة المعتلة المحيطية الخطية و غير الخطية. الطريقة المستخدمة مكونة من مسئلتين وهما المسألة المضطربة المعتلة والثانية المسألة المحيطية غير الخطية. واستخدمنا طريقة توليد كرنيل وايضا تقريب بادي للتحقق من الشرط عند المالانهاية. تم دراست وجود وتقارب الحل التقريبي ونتج عن ذلك نتائج عددية تم عرضها لاثبات دقة وفعالية الطريقة المستخدمة.

مفاهيم البحث الرئيسية: فولتيرا التكاملية-التفاضلية المضطربة، مشتقة الكسور كابوتو، المشاكل غير الخطية للقيمة الأولية.

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Dedication

To my beloved parents and family

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Chapter 1: Introduction

1.1 Fractional Derivative

In 1695, a French mathematician called L'hopital stopped in an important question and decided to send a message to asked a German mathematician named Leibnitz to find the solution of the following question, if the order n $= \frac{1}{2}$, how I can find the derivative for this function;

$$f(x) = x.$$

Leibnitz's answer was "This is an apparent paradox from which, one day, useful consequences will be drawn" [1]. As a result of this, the fractional calculus started to appear in the world by the question of L'hopital. The date September 30, 1695 is considered as the exact birthday of the fractional Calculus. Later, numerous of mathematicians studied the question of L'hopital like Euler in 1738, Lagrange in 1772, Laplace in 1812, Lacroix in 1819, Fourier in 1822, Abel in 1826, Liouville in 1832, Riemann in 1847, Greer in 1859, Holmgren in 1865, Griinwald in 1867, Letnikov in 1868, Sonin in 1869, Laurent in 1884, Nekrassov in 1888, Krug in 1890, and Weyl in 1917. Each mathematician used their own notation and methodology and they found many concepts of the functional integral and derivative [2].

The most important achievements in this regard are, in [3], as follow:

1. In 1822, Fourier proposed an integral representation in order to determine the derivative, and his proposition can be considered as the first definition for the derivative of positive order.
2. In 1826, Abel solved an integral equation related to the tautochrone problem which is count to be the first application of Fractional Calculus.

3. In 1832, Liouville suggested a definition based on the formula for differentiating of the exponential function. The definition considered as the first definition of Liouville. The second definition formulated by Liouville was written in terms of an integral and is now known as the version of the integration of noninteger order.
4. Weyl defined a derivative to circumvent a problem including a particular class of functions, name is the periodic functions.

The story of fractional derivative and integral is more than 300 years old; however in the modern decades the applied scientists and the engineers realized that the fractional derivative and integral supplied better processes to describe the complicated phenomena in nature. For examples, non-Brownian motion, systems identification, control, viscoelastic materials, and polymers. We can use the non-local property of the fractional derivative to describe those complex systems which involve long-memory in time in a better way. Accordingly, the numerical process has become a very required method to analyze the experimental data which is described in a fractional way [4].

Moreover, the applications of fractional derivative and integral are varied and diffuse in engineering and science. For instance, electromagnetics, viscoelasticity, fluid mechanics, electrochemistry, biological population models, optics, signals processing, quantum mechanics, electricity, and ecological systems [5].

In this section, we introduce several definitions for the fractional derivative and integral.

Definition 1.1.1 The Riemann–Liouville fractional derivative of y is defined as

$$_{RL}D_{a,t}^{\propto} y(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^t (t-\tau)^{n-\alpha-1} y(\tau) d\tau, \quad t > a, \ n-1 < \propto < n \in Z^+$$

Definition 1.1.2 The Grünwald–Letnikov fractional derivative of y is defined as

$$D_{a,t}^{\propto}y(t) = \lim_{n\to\infty}\{\frac{(\frac{t-a}{N})^{-\propto}}{\Gamma(-\propto)}\sum_{j=0}^{N-1}\frac{\Gamma(j-\propto)}{\Gamma(j+1)}y(t-j\left(\frac{t-a}{N}\right))\},$$

$$t > a,\ \ n-1 < \propto < n \in Z^{+}.$$

Definition 1.1.3 The Weyl fractional derivative defined as

$${}_{x}D_{\infty}^{\propto}[f(x)] = D_{-}^{\propto}[f(x)] = (-1)^{m}(\frac{d}{d\tau})^{n}\left[{}_{x}W_{\infty}^{\propto}[f(x)]\right].$$

Definition 1.1.4 The Riemann-Liouville definition of fractional integral of a function y reads as

$$J_{a.t}^{\propto}y(t) = \frac{1}{\Gamma(\alpha)}\int_{a}^{t}(t-\tau)^{\alpha-1}y(\tau)d\tau, \quad \alpha > 0.$$

Definition 1.1.5 The Weyl definition of fractional integral

$${}_{x}D_{\infty}^{\propto}[f(x)] = \frac{1}{\Gamma(\alpha)}\int_{x}^{\infty}(\varepsilon-x)^{\alpha-1}f(\varepsilon)d\varepsilon.$$

Definition 1.1.6 The Local fractional Yang integral

$${}_{a}I_{b}^{\propto}[f(x)] = \frac{1}{\Gamma(1+\alpha)}\int_{a}^{b}f(\varepsilon)(d\varepsilon)^{\alpha}.$$

In this thesis, we use the Caputo derivative which is given as follows.

Definition 1.1.7 The Caputo derivative of fractional order of function $x(t)$ is defined as

$${}_{C}D_{0}^{\propto}\mathrm{x(t)} = D_{0,\mathrm{t}}^{-(\mathrm{m}-\propto)}\frac{d^{m}}{dt^{m}}x(t) = \frac{1}{\Gamma(m-\propto)}\int_{0}^{t}(t-\tau)^{m-\propto-1}x^{(m)}(\tau)d\tau,$$

t$< T.n$ which $m-1 <\propto< m\epsilon Z^{+}$.

Caputo derivative has many properties for examples:

1. $l[{}_{\Gamma}D_{0,t}^{\propto}x(t)]\,(s) = s^{\propto}x(s) - \sum_{k=0}^{m-1}s^{\propto-k-1}x^{(k)}(0),$

   $where\ x(s) = l[x](s), m-1 <\propto\leq m\epsilon Z^{+}.$

2. ${}_{C}D_{0,t}^{\propto}c = 0,\ where\ c\ is\ any\ constant.$

Theorem 1.1.1 $If\ x(t)\epsilon\ C^{m}[0,T]\ for\ T>0$ and $m-1\ <\propto<\ m\epsilon Z^{+}$.

Then, ${}_{C}D_{0,t}^{\propto}x(0)=0$.

Proof. By using the definition of Caputo derivative, one has

$${}_{C}D_{0}^{\propto}\mathrm{x(t)}=\frac{1}{\Gamma(m-\propto)}\int_{0}^{t}(t-\tau)^{m-\propto-1}x^{(m)}(\tau)d\tau,\qquad t<T.$$

Put

$$\mathrm{M}\ =max_{t\epsilon[0,t]}\left|x^{(m)}(t)\right|,\text{ where } M \text{ is a positive constant,}$$

Then,

$${}_{C}D_{0}^{\propto}\mathrm{x(t)}\leq\left|\frac{M}{\Gamma(m-\propto)}\int_{0}^{t}(t-\tau)^{m-\propto-1}d\tau=\frac{M}{\Gamma(m-\propto+1)}t^{m-\propto}\right|$$

which follows that ${}_{C}D_{0}^{\propto}\mathrm{x}(0)=0$. ■

Remark 1.1.1

1. $If\ x(t)\epsilon C^{0}[0,T]for\ T>0$ and $\propto>0$, then

   $D_{C}^{-\propto}\mathrm{x}(0)=0$ or $\lim\limits_{t\to0}\frac{1}{\Gamma(\propto)}\int_{0}^{t}(t-\tau)^{\alpha-1}x(\tau)d\tau=0.$

2. Theorem 1.1.1 does not hold for the Riemann-Liouville derivative.

Theorem 1.1.2 $Let\ f\ \epsilon\ C_{-1}^{m}, m\in\ N_{0}$. Then the Caputo fractional derivative ${}_{C}D_{0}^{\mu}f, 0\leq\mu\leq m$, is well defined and the inclusion

$${}_{C}D_{0}^{\mu}f\ \epsilon\begin{cases}C_{-1}, & m-1<\mu\leq m\\ C^{k-1}[0,\infty)\subset\ C_{-1}, & m-k-1<\mu\leq m-k, k=1,\ldots\ldots\ldots, m-1\end{cases}$$

holds true.

Proof. In the case m − 1 < μ ≤ m, the inclusion under consideration follows from the definition of the Caputo derivative ${}_{C}D_{0}^{\mu}$, m ≥ 1, and the corresponding mapping properties of the Riemann-Liouville fractional integral give us the inclusion ${}_{C}D_0^{\mu} \in C^{k-1}[0,\infty)$ for $m-k-1<\mu\leq m-k, k=1,\ldots\ldots\ldots.,m-1$.

The inclusion $C^{k-1}[0,\infty)\subset C_{-1}$. ■

From now on, we use $D^{\alpha}f$ as the Caputo derivative of f.

Example 1.1.1 *Let* $\alpha=\frac{1}{2}$ and $f(t)=t$. Then, for $n=1$, by applying the previous definition of Caputo derivative we get:

$$D^{1/2}t=\frac{1}{\Gamma(\frac{1}{2})}\int_0^t\frac{1}{(t-\tau)^{1/2}}d\tau.$$

Taking into account the properties of the Gamma function and using the substitution $u=t-\tau$, the final result for the Caputo fractional derivative of the function $f(t)=t$ is obtained as**:**

$$D^{1/2}t=\frac{1}{\sqrt{\pi}}\int_0^t\frac{1}{(t-\tau)^{1/2}}d\tau.$$

$$=-\frac{1}{\sqrt{\pi}}\int_{\sqrt{t}}^{0}\frac{du}{\sqrt{u}}$$

$$=\frac{1}{\sqrt{\pi}}\int_0^{\sqrt{t}}\frac{du}{\sqrt{u}}$$

$$=\frac{2}{\sqrt{\pi}}\left(\sqrt{t}-0\right).$$

Thus, it holds

$$D^{1/2}t=\frac{2\sqrt{t}}{\sqrt{\pi}}.$$

Lemma 1.1.1 *Let* $n-1<\alpha<n, n\in\mathbb{N}$,

$\alpha \in \mathbb{R}$ and $f(t)$ be such that $D_0^\alpha f(t)$ exists. Then, the following properties for the Caputo operator hold

$$\lim_{\alpha \to n} D_0^\alpha f(t) = f^{(n)}(t),$$

$$\lim_{\alpha \to n-1} D_0^\alpha f(t) = f^{(n-1)}(t) - f^{(n-1)}(0).$$

Proof. Let's proof it by using integration by parts.

$$D_0^\alpha f(t) = \frac{1}{\Gamma(\mathrm{n}-\alpha)} \int_0^t \frac{f^{(n)}(t)}{(t-\tau)^{\alpha+1-n}} d\tau$$

$$= \frac{1}{\Gamma(\mathrm{n}-\alpha)} \left(-f^n(\tau) \frac{(t-\tau)^{n-\alpha}}{n-\alpha} \Big|_{\tau=0}^{t} - \int_0^t -f^{(n+1)}(\tau) \frac{(t-\tau)^{n-\alpha}}{n-\alpha} d\tau \right)$$

$$= \frac{1}{\Gamma(\mathrm{n}-\alpha+1)} \left(f^n(0) t^{n-\alpha} + \int_0^t f^{(n+1)}(\tau)(t-\tau)^{n-\alpha} d\tau \right).$$

Now, by taking the limit for $\alpha \to n$ and $\alpha \to n-1$, respectively, it follows

$$\lim_{\alpha \to n} D_0^\alpha f(t) = \left(f^{(n)}(0) + f^{(n)}(\tau) \right) \Big|_{\tau = 0}^{t} = f^{(n)}(t)$$

and

$$\lim_{\alpha \to n-1} D_0^\alpha f(t) = \left(f^{(n)}(0)t + f^{(n)}(\tau)(t-\tau) \right) \Big|_{\tau = 0}^{t} - \int_0^t -f^{(n)}(\tau)\, d\tau$$

$$= f^{(n-1)}(\tau) \Big|_{\tau = 0}^{t}$$

$$= f^{(n-1)}(t) - f^{(n-1)}(0). \blacksquare$$

For the Riemann–Liouville fractional differential operator, the corresponding interop-lotion property reads

$$\lim_{\alpha \to n} D^\alpha f(t) = f^{(n)}(t),$$

$$\lim_{\alpha \to n-1} D^\alpha f(t) = f^{(n-1)}(t).$$

Corollary 1.1.1 *Let* $t > 0, \alpha \in \mathbb{R}, n-1 < \propto < n \in \mathbb{N}.$

If $f(\tau)$ and $g(\tau)$ and all its derivative are continuous in $[0,t]$, then the following holds

$$D_*^{\alpha}\big(f(t)g(t)\big)$$
$$=\sum_{k=0}^{\infty}\binom{\alpha}{k}\left(D^{\alpha-k}f(t)\right)g^{(k)}(t)-\sum_{k=0}^{n-1}\frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)}\left((f(t)g(t))^{k}(0)\right).$$

We called this Property as Leibniz Rule.

Proof. Applying the Leibniz Rule for the Riemann-Liouville

$$D^{\alpha}\big(f(t)g(t)\big)=\sum_{k=0}^{\infty}\binom{\alpha}{k}\left(D^{\alpha-k}f(t)\right)g^{(k)}(t),$$

Then, the Leibniz rule for the Caputo derivative is obtained

$$D_0^{\alpha}\big(f(t)g(t)\big)=D^{\alpha}\big(f(t)g(t)\big)-\sum_{k=0}^{n-1}\frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)}\left(\big(f(t)g(t)\big)^{k}(0)\right)$$
$$=\sum_{k=0}^{\infty}\binom{\alpha}{k}\left(D^{\alpha-k}f(t)\right)g^{(k)}(t)-\sum_{k=0}^{n-1}\frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)}\left((f(t)g(t))^{k}(0)\right). \blacksquare$$

At the end of this section, some important properties of fractional integral operators should be mentioned [6]:

1. Semi-Group Property L: ${}_aD_x^{-\sigma}\,{}_aD_x^{-\tilde{\sigma}}u={}_aD_x^{-\sigma-\tilde{\sigma}}u.$
2. Semi-Group Property R: ${}_xD_b^{-\sigma}\,{}_xD_b^{-\tilde{\sigma}}u={}_xD_b^{-\sigma-\tilde{\sigma}}u.$
3. Adjoint Property: $({}_aD_x^{-\sigma}u,v)_{L^2(a,b)}=(u,{}_xD_b^{-\sigma}\,v)_{L^2(a,b)}.$
4. Commutative Property L: ${}_aD_x^{-\sigma}Du=D\,{}_aD_x^{-\sigma}u.$
5. Commutative Property R: ${}_xD_b^{-\sigma}Du=D\,{}_xD_b^{-\sigma}u.$

1.2 Volterra Integro-Differential Equations

Volterra integral equations considered as type of integral equations. In 1913, Volterra published the first book talk about Volterra integral equations with title “Leçons sur les équations intégrales et les équations intégro-différentielles”. In 1884, Volterra began working on integral equations, but his important study began in 1896. However, the name Volterra integral equation was first called by Lalesco in 1908. Since then, Volterra integral equations have been a major source of research work.

Many application in science and engineering that used Volterra integral equations such as elasticity, semi-conductors, scattering theory, seismology, heat conduction, metallurgy, fluid flow, chemical reactions, population dynamics, and spread of epidemics [7]

Volterra integral equations have growingly been recognized as useful tools for problems in science and engineering. In [8], they proposed and examined a spectral Jacobi-collocation approximation for fractional order integro-differential equations. According to Suha and Ray [9], they used Legendre wavelet method to find the solutions of system of nonlinear Volterra integro-differential equations. In [10], they used Laguerre polynomials which depended on the collocation method to solve the pantograph-type Volterra integro-differential equations under the initial conditions.

Yang, Tang, and Zhang [11], discussed about the blow-up of Volterra integro-differential equations with a dissipative linear term to beat the differences of the solutions. In [12], they solved a non-linear system of higher order Volterra integro-differential equations by using the Single Term Walsh Series (STWS) method. Also in [13], they solved the fractional Fredholem–Volterra integro-differential equations by defining the new fractional-order functions based on the Bernoulli polynomials. We also indicate the interested reader to [14, 15, 16, 17, 18, and 19] for more research works on Volterra integro-differential equations.

Volterra integro-differential equations divided into two groups referred to as the first and the second kind.

The first kind, [20], is

$$f(x) = \int_0^x K(x,t)u(t)\, dt$$

where $u(x)$ is the unknown function and it occurs only under the integral sign.

The second kind, [21],

$$u(x) = f(x) + \lambda \int_0^x K(x,t)u(t)\, dt.$$

where $u(x)$ is the unknown function and it is occurs inside and outside the integral sign. $K(x,t)$ is the kernel and the function $f(x)$ are given real-valued functions, and λ is a parameter.

In this section, we will present some example of Volterra integro-differential equations.

Example 1.2.1 $u''(x) = -x + \int_0^x (x-t)u(t)dt, \quad u(0) = 0, \;\; u'(o) = 1,$

Example 1.2.2 $u'(x) = -\sin x - 1 + \int_0^x u(t)dt, \quad u(0) = 1,$

Example 1.2.3 Show that $u(x) = e^x$ is the solution of the Volterra integral equation

$$u(x) = 1 + \int_0^x u(t)\, dt.$$

Substituting $u(x) = e^x$ in the left hand side to get

$$1 + \int_0^x e^t\, dt$$

$$= 1 + [e^t]_0^x$$

$$= e^x = u(x).$$

1.3 Non-linear Singularly Perturbed

In 1904, A German physicist called Ludwig Prandtl revolutionized fluid dynamics with his concept. He noted that "the influence of friction is experienced only very near an object moving through a fluid". In one of his paper [22], he presented, in the Third International Mathematics Congress in Heidelberg, the idea of the boundary layer and its significance for drag and streamlining and the title of his paper was "Fluid Flow in Very Little Friction". In his paper, Ludwig Prandtl assumed that the impact of friction was to cause the fluid instantly adjacent to the surface to stick to the surface. This boundary-layer notion has been the basis stone for the new fluid dynamics. Schlichting was one of the most famous books on boundary layer theory [23]. The scientific justification of boundary layer theory gave us a more general hypothesis to determine asymptotic expansions of the solutions to the complete equations of the motion. Singular perturbation problem was the result of reduced the problem which is then solved by the method of matched asymptotic expansions. In 1946, Friedrichs and Wasow were the first time used the expression ‘‘singular perturbation” [24].

The differential equations of singularly perturbed indicate to the study of a group of differential equations including an asymptotically small parameter where the character of the limiting solution was totally various than the solutions acquired at finite values of the parameter. The singularly perturbed problem is very important to both applied and pure mathematicians, physicists and engineers because of the fact that the solutions exhibit some interesting behavior, for example, boundary layer, interior layer, and resonance phenomena [25].

There are a lot of applications of singularly perturbed problem such as Chou Huan-wen discussed the nonlinear problems of plates and shells by means of the singular perturbation method [26], Petar discussed typical applications of singular perturbation techniques to control problems in the last fifteen years [27], Kokotovic, O'malley and Sannuti, showed recent results on singular perturbations surveyed as a tool for model order reduction and separation of time scales in control system design [28], Ghorbel and Spong, reviewed results of integral manifolds of singularly perturbed non-linear differential equations and outlined the basic elements of the integral manifold method in the context of control system design [29], Fridman, studied the H_∞ control problem for an affine nonlinear singularly perturbed system [30], Fridman, studied the infinite horizon nonlinear quadratic optimal control problem for a singularly perturbed system [31].

1.4 Perturbed Problem

Definition 1.4.1 When the problem does not include any small parameter is defined as unperturbed problem [26].

Example 1.4.1 $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 2x^2 - 8x + 4, \; y(0) = 3, \quad \frac{dy}{dx}(0) = 3.$

Definition 1.4.2 When the problem include a small parameter is defined as perturbed problem [26].

Example 1.4.2 $\frac{\mathrm{dy}}{\mathrm{dx}} + \mathrm{y} = \varepsilon y^2, \ \mathrm{y}(0) = 1.$

The perturbed problem can be divided into two groups and that depending onto the nature of the perturbed problem. The two groups are

1. Regularly perturbed.
2. Singularly perturbed.

Definition 1.4.3 A regular perturbation problem $P_\varepsilon(y_\varepsilon) = 0$ depends on its small parameter ε in such a way that its solution $y_\varepsilon(x)$ converges as $\varepsilon \to 0$ (uniformly with respect to the independent variable x in the relevant domain) to the solution $y_0(x)$ of the limiting problem $P_0(y_0) = 0$. In general, the parameter presented at lower order terms [32].

Example 1.4.3 $\frac{\mathrm{d^2y}}{\mathrm{dx^2}} + \mathrm{y} = \varepsilon y^2, \ \mathrm{y}(0) = 1, \quad \frac{\mathrm{dy}}{\mathrm{dx}}(0) = -1.$

Example 1.4.4 $\frac{\mathrm{dy}}{\mathrm{dx}} + \mathrm{y} = \varepsilon y^2, \ \mathrm{y}(0) = 1.$

Definition 1.4.4 A singular perturbation is said to be occur whenever the order of the problem is reduced when we set $\varepsilon = 0$. In general, the parameter presented at higher order terms and the lower order terms start to dominate [33].

Example 1.4.5 $\varepsilon\frac{\mathrm{d^2y}}{\mathrm{dx^2}} + \frac{\mathrm{dy}}{\mathrm{dx}} = 2\mathrm{x} + 1, \ \mathrm{y}(0) = 1, \quad \mathrm{y}(0) = 4.$

Example 1.4.6 $\varepsilon\frac{\mathrm{d^2y}}{\mathrm{dx^2}} + \frac{\mathrm{dy}}{\mathrm{dx}} - \mathrm{y} = 0, \ \mathrm{y}(0) = 0, \quad \mathrm{y}(1) = 1.$

Example 1.4.7 $\varepsilon x^2 - x + 1 = 0.$

Example 1.4.8 $\varepsilon \frac{d^2y}{dx^2} + \frac{dy}{dx} = 1 + 2x, \quad y(0) = 0, \quad y(1) = 1.$

1.5 Fractional Perturbed Problem

It is clear that the fractional-order models of the integration and the derivative are more satisfactory than formerly integer-order models. Specially, they have been confirmed that fractional integrals and derivatives give a phenomenal instrument for the depiction of memory and hereditary properties of different materials and procedures, impacts neglected in traditional integer-order models. In 1998, Podlubny discussed the history of the Fractional differential equations, applications, and a scanning of a literature of fractional integrals and derivative models [34].

One of the uses of singular perturbation techniques is to find the solution of the problems of numerous sections of applied sciences and to have a successful approximation. Excessive use of fractional order models in physical processes impacts the necessity to have appropriate corresponding singular perturbation techniques available. The reason for this fundamentally because in the process of modeling, one is properly to end up with a singularly perturbed problem. In [35], [36], and [37], the method of additive decomposition was used successfully to build asymptotic solutions of nonlinear singularly perturbed Volterra integral equations with smooth kernels, to the main and higher order terms.

One of the significant points of singularly perturbed problems is to obtain asymptotic solutions of the problem to all orders since in most problems the singularness of the problem is reveal only in the higher order adjustment terms of the perturbation extension. Furthermore, the higher order of the asymptotic solutions are given as far as the linear equations and it's solvable.

There are some examples of singularly perturbed fractional order models and this motivated the current research:

1) Problem with order $\frac{1}{2}$ explaining the process of cooling of a semi-infinite body by radiation

$$\varepsilon\,{}_0^1D_t^{\frac{1}{2}}y(t) = \{a_0 - y(t)\}^4, t > 0,\ 0 < \varepsilon \ll 1, y(0) = 0,$$

and here a_0 is a given constant such that $x(t) = a_0 - y(t)$, where $x(t)$ is the surface temperature to be determined.

2) In [38], the author considered a class of fractional singularly perturbed two boundary-value problems with Dirichlet boundary conditions of the form

$$-\varepsilon D^{\alpha}y(x) + u(x,y)y'(x) + v(x.y)y(x) = 0,$$

$$x \in I\ := [0,1], \qquad 1 < \propto \leq 2,$$

subject to

$$y(0) = \beta_1,\ \ y(1) = \beta_2,$$

where $\varepsilon > 0$ is a small positive parameter, β_1, β_2 are given constant, $u(x,y)$, $v(x,y)$ are sufficiently smooth function such that

$u(x,y(x)) \neq 0$ for all $x \in I$, and $y \in L_1[a,b]\ := \Big\{z :\ [a,b]\ \rightarrow \mathbb{R}\ \Big|\int_a^b z(t)dt\ < \infty\Big\}$. Here, $D^{\propto}$ denoted the left-sided Caputo fractional derivative, defined as follows

$$D^{\propto}y(x) = \frac{1}{\Gamma(k-\propto)}\int_0^x (x-t)^{k-\propto-1}y^{(k)}(\tau)d\tau,\ \ where\ k\epsilon\mathbb{N}$$

where the definition left-sided Caputo fractional derivative is

$${}_{*}D_{a^+}^{a}[f(x)] = \frac{1}{\tau(n-\alpha)} \int_a^x (x-\xi)^{n-\alpha-1} \frac{d^n}{d\xi^n} [f(\xi)] d\xi, \qquad x \geq a$$

3) In [39], the author presents analysis and computational experiments for the singularly perturbed fractional advection–dispersion equation in one spatial dimension:

$$-\varepsilon D(p\,{}_a^1D_x^{\alpha-2} + q\,{}_x^1D_b^{\alpha-2})Du - u_x = f, \qquad in\ \Omega$$

$$u = 0, \qquad on\ d\Omega,$$

where Ω is the real interval $(a, b), 1 < \propto \leq 2$ is the order of the fractional dispersion operator. With skewness parameters define by p, q satisfying

$p + q = 1$, and $\varepsilon \ll 1$.

Chapter 2: First Order Fractional Initial Value Problems

In this chapter, we study the first order fractional initial value problems. In the next section, we presented Kernel method for first order initial value problems.

2.1 Reproducing Kernel Method for First Order Initial Value Problems

Definition 2.1.1 Let E be a nonempty abstract set. A function $M: E \times E \rightarrow C$ is a reproducing Kernel of the Hilbert space H if and inly if

- $M(.,x) \in H$ for all $x \in E$,
- $\left(\phi(.), M(.,x)\right) = \phi(x)$ for all $x \in E$ and $\phi \in H$.

The second condition is called the reproducing property and a Hilbert space which possesses a reproducing kernel is called a reproducing kernel Hilbert space.

Consider the first order nonlinear fractional equation of the form

$$D^{\alpha}y + g(y) = c,\ \ x \in [0,1], 0 < \alpha \leq 1 \qquad (2.1.1)$$

subject to

$$y(0) = \theta \qquad (2.1.2)$$

where c and θ are constants. First, we study the linear case where $g(y) = a(x)y.$ To homogenize the initial condition, we assume $u = y - \theta$. Thus, Problem 2.1.1-2.1.2 can be written as

$$D^{\alpha}u + h(u) = c, x \in [0,1], 0 < \alpha \leq 1 \qquad (2.1.3)$$

subject to

$$u(0) = 0. \qquad (2.1.4)$$

In order to solve the linear Problem 2.1.3-2.1.4, we construct the kernel Hilbert spaces $W_2^1[0,1]$ and $W_2^2[0,1]$ in which every function satisfy the initial condition 2.1.4.

Let $W_2^1[0,1] = \{u(s): u$ is absolutely continuous real value function, $u' \in L^2[0,1]\}$.

The inner product in $W_2^1[0,1]$ is defined as

$$(u(y), v(y))_{W_2^1[0,1]} = u(0)v(0) + \int_0^1 u'(y)v'(y)dy,$$

and the norm $\|u\|_{W_2^1[0,1]}$ is given by

$$\|u\|_{W_2^1[0,1]} = \sqrt{(u(y), u(y))_{W_2^1[0,1]}}$$

where $u, v \in W_2^1[0,1]$.

Theorem 2.1.1 The space $W_2^1[0,1]$ is a reproducing Kernel Hilbert space, $i, e.$; there exist $R(s, y) \in W_2^1[0,1]$ and its second partial derivative with respect to y exists such that for any $u \in W_2^1[0,1]$ and each fixed $y, s \in [0,1]$, we have

$$(u(y), R(s, y))_{W_2^1[0,1]} = u(s).$$

In this case, $R(s, y)$ is given by

$$R(s, y) = \begin{Bmatrix} 1 + y, & y \le s \\ 1 + s, & y > s \end{Bmatrix}.$$

Proof. Using integration by parts, one can get

$$(u(y), R(s, y))_{W_2^1[0,1]} = u(0)R(s, 0) + \int_0^1 u'(y)\frac{\partial R}{\partial y}(s, y)dy$$

$$= u(0)R(s, 0) + u(1)\frac{\partial R}{\partial y}(s, 1) - u(0)\frac{\partial R}{\partial y}(s, 0) - \int_0^1 u(y)\frac{\partial^2 R}{\partial y^2}(s, y)dy.$$

Since $R(s, y)$ is a reproducing kernel of $W_2^1[0,1]$,

$$(u(y), R(s, y))_{W_2^1[0,1]} = u(s)$$

which implies that

$$-\frac{\partial^2 R}{\partial y^2}(s, y) = \delta(y - s), \quad (2.1.5)$$

$$R(s, 0) - \frac{\partial R}{\partial y}(s, 0) = 0, \quad (2.1.6)$$

and

$$\frac{\partial R}{\partial y}(s,1) = 0, \quad (2.1.7)$$

Since the characteristic equation of $-\frac{\partial^2 R}{\partial y^2}(s,y) = \delta(y-s)$ is $\lambda^2 = 0$ and its characteristic value is $\lambda = 0$ with 2, multiplicity roots, we write $R(s,y)$ as

$$R(s,y) = \begin{cases} c_0(s) + c_1(s)y, \ \ y \le s \\ d_0(s) + d_1(s)y, \ \ y > s \end{cases}.$$

Since $\frac{\partial^2 R}{\partial y^2}(s,y) = -\delta(y-s)$, we have

$$R(s,s+0) - R(s,s+0) = 0, \quad (2.1.8)$$

$$\frac{\partial R}{\partial y}(s,s+0) - \frac{\partial R}{\partial y}(s,s+0) = -1. \quad (2.1.9)$$

Using the conditions 2.1.6-2.1.9, we get the following system of equations

$$c_0(s) - c_1(s) = 0, \quad (2.1.10)$$

$$d_1(s) = 0,$$

$$c_0(s) + c_1(s)s = d_0(s) + d_1(s)s,$$

$$d_1(s) - c_1(s) = -1,$$

which implies that

$$c_0(s) = 1,\ c_1(s) = 1, d_0(s) = 1 + s, d_1(s) = 0$$

which completes the proof of the theorem. Next, we study the space $W_2^2[0,1]$.

Let

$$W_2^2[0,1] = \{f(s): f \text{ is absolutely continuous real value function}, f, f', f'' \in L^2[0,1], f(0) = 0\}.$$

The inner product in $W_2^2[0,1]$ is defined as

$$(u(y), v(y))_{W_2^2[0,1]} = u(0)v(0) + u(1)v(1) + \int_0^1 u^{(2)}(y)v^{(2)}(y)dy$$

and the norm $\|u\|_{W_2^2[0,1]}$ is given by

$$\|u\|_{W_2^2[0,1]} = \sqrt{(u(y), u(y))_{W_2^2[0,1]}}$$

where $u, v \in W_2^2[0,1]$.

Theorem 2.1.2 The space $W_2^2[0,1]$ is a reproducing Kernel Hilbert space, $i, e.$; there exist $K(s,y) \in W_2^2[0,1]$ which has its six partial derivative with respect to y such that for any $u \in W_2^2[0,1]$ and each fixed $y, s \in [0,1]$, we have

$$(u(y), K(s,y))_{W_2^2[0,1]} = u(s).$$

In this case, $K(s,y)$ is given by

$$K(s,y) = \begin{Bmatrix} \sum_{i=0}^{3} c_i(s) y^i, & y \le s \\ \sum_{i=0}^{3} d_i(s) y^i, & y > s \end{Bmatrix}$$

where

$$c_0 = 0,\ c_1 = \frac{1}{6}(8s - 3s^2 + s^2), c_2 = 0,\ c_3 = \frac{1}{6}(s-1),$$

$$d_0 = -\frac{s^3}{6},\ d_1 = \frac{1}{6}(8s + s^3), d_2 = -\frac{s}{2},\ d_3 = -\frac{s}{6}.$$

Proof: Using integration by parts, one can get

$$(u(y), K(s,y))_{W_2^2[0,1]} = u(0)K(s,0) + u(1)K(s,1) + u'(1)K_{yy}(s,1)$$

$$- u'(0)K_{yy}(s,0) - u(1)\frac{\partial^3 K}{\partial y^3}(s,1) + u(0)\frac{\partial^3 K}{\partial y^3}(s,0) + \int_0^1 u(y)\frac{\partial^4 K}{\partial y^4}(s,y)dy.$$

Since $u(y)$ and $K(s,y) \in W_2^2[0,1]$,

$$u(0) = 0$$

and

$$K(s,0) = 0. \tag{2.1.11}$$

Thus,

$$(u(y), K(s,y))_{W_2^2[0,1]} = u(1)K(s,1) + \ u'(1)K_{yy}(s,1) - \ u'(0)K_{yy}(s,0)$$

$$-u(1)\frac{\partial^3 K}{\partial y^3}(s,1) + \int_0^1 u(y)\frac{\partial^4 K}{\partial y^4}(s,y)dy.$$

Since $K(s,y)$ is a reproducing kernel of $W_2^2[0,1]$

$$(u(y), K(s,y))_{W_2^2[0,1]} = u(s)$$

which implies that

$$\frac{\partial^4 K}{\partial y^4}(s,y) = \delta(y-s) \qquad (2.1.12)$$

where δ is the Dirac-delta function and

$$K(s,1) - \frac{\partial^3 K}{\partial y^3}(s,1) = 0, \qquad (2.1.13)$$

$$K_{yy}(s,1) = 0, \qquad (2.1.14)$$

$$K_{yy}(s,0) = 0. \qquad (2.1.15)$$

Since the characteristic equation of $\frac{\partial^3 K}{\partial y^3}(s,y) = \delta(s-y)$ is $\lambda^4 = 0$ and its characteristic value is $\lambda = 0$ with 4 multiplicity roots, we write $K(s,y)$ as

$$K(s,y) = \begin{cases} \sum_{i=0}^{3} c_i(s)y^i\,, & y \le s \\ \sum_{i=0}^{3} d_i(s)y^i\,, & y > s \end{cases}$$

Since $\frac{\partial^3 K}{\partial y^3}(s,y) = \delta(s-y)$, we have

$$\frac{\partial^m K}{\partial y^m}(s,s+0) = \frac{\partial^m K}{\partial y^m}(s,s-0), m = 0,1,2. \qquad (2.1.16)$$

On the other hand, integrating $\frac{\partial^6 K}{\partial y^8}(s,y) = \ \delta(s-y)$ from $s-\epsilon$ to $s+\epsilon$ with respect to y and letting $\epsilon \to 0$ to get

$$\frac{\partial^3 K}{\partial y^3}(s,s+0) - \frac{\partial^3 K}{\partial y^3}(s,s-0) = 1. \qquad (2.1.17)$$

Using the conditions 2.1.11 and 2.1.13-2.1.17, we get the following system of equations

$$c_0 = 0, \sum_{i=0}^{3} d_i(s) - 6d_3(s) = 0,$$

$$6d_3(s) + 2d_2(s) = 0, c_2(s) = 0,$$

$$\sum_{i=0}^{3} c_i(s)s^i = \sum_{i=0}^{3} d_i(s)s^i,$$

$$\sum_{i=1}^{3} ic_i(s)s^{i-1} = \sum_{i=1}^{3} id_i(s)s^{i-1},$$

$$\sum_{i=1}^{3} i(i-1)c_i(s)s^{i-2} = \sum_{i=1}^{3} i(i-1)d_i(s)s^{i-2},$$

$$3!\,d_3(s) - 3!\,c_3(s) = 1.$$

We solved the last system using Mathematica to get

$$c_0 = 0,\ c_1 = \frac{1}{6}(8s - 3s^2 + s^2), c_2 = 0,\ c_3 = \frac{1}{6}(s-1),$$

$$d_0 = -\frac{s^3}{6},\ d_1 = \frac{1}{6}(8s + s^3), d_2 = -\frac{s}{2},\ d_3 = -\frac{s}{6}$$

which completes the proof of the theorem.

Now, we present how to solve Problem 2.1.3-2.1.4

$$\sigma_i(s) = R(s_i, s)$$

For $i = 1,2,\cdots$ where $\{s_i\}_{i=1}^{\infty}$ is dense on $[0,1]$. Let $L(\sigma_i(s)) = D^{\alpha}\sigma_i(s) + a(s)\sigma_i(s)$. It is clear that $L: W_2^2[0,1] \to W_2^1[0,1]$ is bounded linear operator. Let

$$\psi_i(s) = L^*\sigma_i(s)$$

where L^* is the adjoint operator of L. Using Gram-Schmidt orthonormalization to generate orthonormal set of function $\{\overline{\psi}_i(s)\}_{i=1}^{\infty}$ where

$$\overline{\psi}_i(s) = \sum_{j=1}^{i} \alpha_{ij}\psi_i(s) \qquad (2.1.18)$$

and α_{ij} are coefficients of Gram-Schmidt orthonormalization. In the next theorem, we show the existence of the solution of Problem (2.1.3-2.1.4).

Theorem 2.1.3 If $\{s_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, then

$$u(s) = c\sum_{i=1}^{\infty}\sum_{j=1}^{i} \alpha_{ij}\overline{\psi}_i(s) \qquad (2.1.19)$$

Proof: First , we want to prove that $\{\psi_i(s)\}_{i=1}^{\infty}$ is complete system of $W_2^2[0,1]$ and $\psi_i(s) = L\big(k(s,s_i)\big)$. It is clear that $\psi_i(s) \in W_2^2[0,1]$ for $i = 1,2,\cdots$ Simple calculations imply that

$$\psi_i(s) = L^*\sigma_i(s) = (L^*\sigma_i(s), K(s,y))_{W_2^2[0,1]}$$

$$= (\sigma_i(s), L(K(s,y)))_{W_2^2[0,1]} = L\big(K(s,s_i)\big).$$

For each fixed $u(s) \in W_2^2[0,1]$, let

$$(u(s), \psi_i(s))_{W_2^2[0,1]} = 0, i = 1,2,\cdots$$

Then

$$(u(s), \psi_i(s))_{W_2^2[0,1]} = \big(u(s), L^*\sigma_i(s)\big)_{W_2^2[0,1]}$$

$$= \big(Lf(s), \sigma_i(s)\big)_{W_2^2[0,1]}$$

$$= Lu(s_i) = 0.$$

Since $\{s_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, $Lu(s) = 0$. Since L^{-1} exists, $u(s) = 0$. Thus, $\{\psi_i(s)\}_{i=1}^{\infty}$ is the complete system of $W_2^2[0,1]$.

Second, we prove Equation 2.1.19. Simple calculations implies that

$$u(s) = \sum_{i=1}^{\infty} (u(s), \bar{\psi}_i(s))_{W_2^2[0,1]}\bar{\psi}_i(s)$$

$$= \sum_{i=1}^{\infty}\sum_{j=1}^{i}\alpha_{ij}\left(u(s), L^*\left(K(s,s_j)\right)\right)_{W_2^2[0,1]}\bar{\psi}_i(s)$$

$$= \sum_{i=1}^{\infty}\sum_{j=1}^{i}\alpha_{ij}\left(Lf(s), K(s,s_j)\right)_{W_2^2[0,1]}\bar{\psi}_i(s)$$

$$= \sum_{i=1}^{\infty}\sum_{j=1}^{i}\alpha_{ij}\left(c\,, K(s,s_j)\right)_{W_2^2[0,1]}\bar{\psi}_i(s)$$

$$= c\sum_{i=1}^{\infty}\sum_{j=1}^{i}\alpha_{ij}\,\bar{\psi}_i(s)$$

and the proof is complete.

Let the approximation solution of Problem 2.1.3-2.1.4 be given by

$$u_N(s) = c\ \textstyle\sum_{i=1}^{N}\sum_{j=1}^{i}\alpha_{ij}\,\bar{\psi}_i(s). \tag{2.1.20}$$

In the next theorem, we show the uniformly convergence of the $\{\frac{d^m f_N(s)}{ds^m}\}_{N=1}^{\infty}$ to $\frac{df(s)}{ds}$ for $m = 0,1,2$.

Theorem 2.1.4 If $u(s)$ and $u_N(s)$ are given as in (2.1.19) and (2.1.20), then $\{\frac{d^m f_N(s)}{ds^m}\}_{N=1}^{\infty}$ converges uniformly to $\frac{d^m u(s)}{ds^m}$ for $m = 0,1$.

Proof: First, we prove the theorem for $m = 0$. For any $s \in [0,1]$,

$\| u(s) - u_N(s) \|^2_{w_2^2[0,1]} = \left(u(s) - u_N(s), u(s) - u_N(s)\right)_{W_2^2[0,1]}$

$$= \sum_{i=N+1}^{\infty}\begin{pmatrix}\left(u(s),\bar{\psi}_i(s)\right)_{W_2^2[0,1]}\bar{\psi}_i(s)\\ ,\left(u(s),\bar{\psi}_i(s)\right)_{W_2^2[0,1]}\bar{\psi}_i(s)\end{pmatrix}_{W_2^2[0,1]}$$

$$= \sum_{i=N+1}^{\infty} (u(s), \bar{\psi}_i(s))^{\,2}_{W_2^2[0,1]}.$$

Thus,

$$Sub_{s\in[0,1]} \|u(s) - u_N(s)\|^{\,2}_{W_2^2[0,1]} = Sup_{s\in[0,1]} \sum_{i=N+1}^{\infty} (u(s), \bar{\psi}_i(s))^{\,2}_{W_2^2[0,1]}.$$

From Theorem (2.1.3), one can see that $\sum_{i=1}^{\infty} (u(s), \bar{\psi}_i(s))_{W_2^2[0,1]} \bar{\psi}_i(s)$ converges uniformly to $u(s)$. Thus,

$$\lim_{N\to\infty} Sup_{s\in[0,1]} \|u(s) - u_N(s)\|_{W_2^2[0,1]} = 0$$

which implies that $\{u_N(s)\}_{N=1}^{\infty}$ converges uniformly to $u(s)$.

Second, we prove the uniformly convergence for $m = 1$. Since $\frac{d^m K(s,y)}{ds^m}$ is bounded function on $[0,1] \times [0,1]$,

$$\left\| \frac{d^m K(s,y)}{ds^m} \right\|_{W_2^2[0,1]} \leq X_m \,, \;\; m = 1.$$

Thus, for any $s \in [0,1]$,

$$\left| u^{(m)}(s) - u_N^{(m)}(s) \right| = \left| (u(s) - u_N(s), \tfrac{d^m K(s,y)}{ds^m})_{W_2^2[0,1]} \right|$$

$$\leq \|u(s) - u_N(s)\|_{W_2^2[0,1]} \left\| \frac{d^m K(s,y)}{ds^m} \right\|_{W_2^2[0,1]}$$

$$\leq \chi_m \|u(s) - u_N(s)\|_{W_2^2[0,1]}$$

$$\leq \chi_m Sup_{s\in[0,1]} \|u(s) - u_N(s)\|_{W_2^2[0,1]}.$$

Hence,

$$Sup_{s\in[0,1]} \left\| u^{(m)}(s) - u_N^{(m)}(s) \right\|_{W_2^2[0,1]}$$

$$\leq \chi_{m_m} Sup_{s\in[0,1]} \|u(s) - u_N(s)\|_{W_2^2[0,1]}$$

which implies that

$$\lim_{N\to\infty} Sup_{s\in[0,1]} \left\| u^{(m)}(s) - u_N^{(m)}(s) \right\|_{W_2^2[0,1]} = 0.$$

Therefore, $\left\{\frac{d^m u_N(s)}{ds^m}\right\}_{N=1}^{\infty}$ converges uniformly to $\frac{d^m u(s)}{ds^m}$ for $m = 1$.

Now, we discuss how to solve Problem (2.1.1) – (2.1.2). Let $\mathcal{L}\big(y(x)\big) = D^\alpha y(x) - c$ and $N\big(y(x)\big) = g(y)$ are the linear and nonlinear parts of Problem 2.1.1, respectively. We construct the homotopy as follows:

$$H(y,\lambda) = \mathcal{L}\big(y(x)\big) + \lambda N\big(y(x)\big) = 0 \tag{2.1.21}$$

where $\lambda \in [0,1]$ is an embedding parameter. If $\lambda = 0$, we get a linear equation

$$D^\alpha y(x) - c = 0$$

which implies that $y(x) = c\,\frac{x^\alpha}{\Gamma(1+\alpha)}$. If $\lambda = 1$, we turn out to be Problem 2.1.1.

Following the Homotopy Perturbation method [40], we expand the solution in term of the Homotopy parameter λ as

$$y = y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3 + \cdots \tag{2.1.22}$$

Substitute Equation 2.1.22 into Equation 2.1.21 and equating the coefficient of the identical power of λ to get the following system

$$\lambda^0 :\ D^\alpha y_0(x) = c, y_0(0) = \theta,$$

$$\lambda^1 :\ D^\alpha y_1(x) = -N\big(\textstyle\sum_{i=0}^{\infty} \lambda^i y_i(x)\big)\big|_{\lambda=0}, y_1(0) = 0,$$

$$\lambda^2 :\ D^\alpha y_2(x) = -\frac{dN\left(\sum_{i=0}^{\infty} \lambda^i y_i(x)\right)}{d\lambda}\Big|_{\lambda=0}, y_2(0) = 0,$$

$$\lambda^3 :\ D^\alpha y_3(x) = -\frac{d^2N\left(\sum_{i=0}^{\infty} \lambda^i y_i(x)\right)}{d\lambda^2}\Big|_{\lambda=0}, y_3(0) = 0,$$

$$\vdots$$

$$\lambda^k :\ D^\alpha y_k(x) = -\frac{d^{k-1}N\left(\sum_{i=0}^{\infty} \lambda^i y_i(x)\right)}{d\lambda^{k-1}}\Big|_{\lambda=0}, y_k(0) = 0.$$

To solve the above equations, we use the RKM which is described above and we obtain

$$y_k(x) = \sum_{i=1}^{\infty} \sum_{j=1}^{i} \alpha_{ij}\, h_k(x_j)\bar{\psi}_i(s), k = 0,1,\cdots \qquad (2.1.23)$$

where

$$h_0(s) = c$$

$$h_1(s) = -N\left(\sum_{i=0}^{\infty} \lambda^i y_i(x)\right)\Big|_{\lambda=0}$$

$$\vdots$$

$$h_k(s) = -\frac{d^{k-1}N\left(\sum_{i=0}^{\infty} \lambda^i y_i(x)\right)}{d\lambda^{k-1}}\Big|_{\lambda=0}, k > 1.$$

From Equation 2.1.23, it is easy to see the solution to Problem 2.1.1-2.1.2 is giving by

$$y(s) = \sum_{0}^{\infty} y_k(x) = \sum_{k=0}^{\infty} \left(\sum_{i=1}^{\infty} \sum_{j=1}^{i} \alpha_{ij}\, h_k(x_j)\bar{\psi}_i(x)\right). \qquad (2.1.24)$$

We approximate the solution of Problem 2.1.1-2.1.2 by

$$y_{n.m}(s) = \sum_{k=0}^{\infty} \left(\sum_{i=1}^{\infty} \sum_{j=1}^{i} \alpha_{ij}\, h_k(x_j)\bar{\psi}_i(x)\right). \qquad (2.1.25)$$

2.2 Analytical Results

In this section, three important theorems are presented which are the maximum principle, the stability theorem, and the uniqueness theorem. Firstly Eqs. 2.2.1-2.2.2 are transformed into an equivalent problem as follows:

$$Py:\ \epsilon D^{\alpha} y + u(x,y) + \int_0^x K(x,t)v(t,y)dt = f(x), x \in (0,1), 0 < \alpha \le 1, \qquad (2.2.1)$$

subject to

$$y(0) = y_0 \qquad (2.2.2)$$

The following conditions are needed in order to guarantee that Eqs. 2.2.1-2.2.2 does not have turning-point problem;

$$-k_2 \geq u(x,y) \geq -k_1, \quad (2.2.3)$$

$$0 \geq v(x,y) \geq -k_3, \quad (2.2.4)$$

$$K(x,t) \geq k_4 \geq 0, \quad (2.2.5)$$

for all $x \in [0,1]$, where $k_1, k_2, k_3, and\ k_4$ are positive constants and $y \in C^1(0,1) \cup C[0,1]$.

Theorem 2.2.1 (Maximum Principle). Consider the initial value problem 2.2.1-2.2.2 with conditions 2.2.3-2.2.5. Assume that $P\phi \geq 0$ and $\phi(0) \geq 0$. Then $\phi(x) \geq 0$ in [0,1].

Proof: Assume that the conclusion is false, then $\phi(x) < 0$ for some $x \in [0,1]$. Then, $\phi(x)$ has a local minimum at x_0 for some $x_0 \in (0,1]$. Simple calculations and using (2.2.5) implies that

$$P\phi(x_0) = \epsilon D^\alpha \phi(x_0) + u(x_0,\phi) + \int_0^{x_0} K(x_0,t)v(t,\phi)dt$$

$$\leq \epsilon \frac{x_0^{-\alpha}}{\Gamma(1-\alpha)}\big(\phi(x_0) - \phi(0)\big) + u(x_0,\phi) + \int_0^{x_0} K(x_0,t)v(t,\phi)dt$$

$$\leq 0.$$

This a contradiction. Therefore, $\phi(x) \geq 0$ in $[0,1]$. ■

In the next theorem, the stability result is presented.

Theorem 2.2.2 (Stability Result). Consider Eqs. 2.2.1-2.2.2 with conditions $u = u(x)$ and $v = v(x)$. If $y(x)$ is a smooth function, then

$$\|y\| = \frac{1}{\epsilon} max\{|y(x)|: x \in [0,1]\} \leq \frac{1}{\epsilon} max\{|y_0|, max_{x\in[0,1]}|Py|\}.$$

Proof: Let

$$K_0 = max\{|y_0|, max_{x\in[0,1]}|Py|\} = max\{|y_0|, max_{x\in[0,1]}|f(x)|\}$$

and let

$$s^{\pm}(x) = \frac{K_0}{\epsilon}\left(1 + \frac{x^{\alpha}}{\Gamma(1+\alpha)}\right) \pm y(x), x \in [0,1].$$

Then,

$$Ps^{\pm}(x) = \epsilon D^{\alpha}\left(\frac{K_0}{\epsilon}\left(1 + \frac{x^{\alpha}}{\Gamma(1+\alpha)}\right) \pm y(x)\right) + u(x) + \int_0^x K(x,t)v(t)dt$$

$$= \epsilon \frac{K_0}{\epsilon} \pm Py(x) = K_0 \pm Py(x) \geq 0$$

for all $x \in [0,1]$ Also,

$$s^{\pm}(0) = \frac{K_0}{\epsilon} \pm y(0) > K_0 \pm y_0 \geq 0$$

since $0 < \epsilon \ll 1$. From Theorem 3.2.1, we can see that $s^{\pm}(x) \geq 0$ for all $x \in [0,1]$. Therefore,

$$\|y\| \leq max_{x\in[0,1]}\left\{\frac{K_0}{\epsilon}\left(1 - \frac{x^{\alpha}}{\Gamma(1+\alpha)}\right)\right\} \leq \frac{K_0}{\epsilon} = \frac{1}{\epsilon} max\{|y_0|, max_{x\in[0,1]}|Py|\}. \blacksquare$$

Theorem 2.2.3 (Uniqueness Theorem). Consider Eqs. 2.2.1-2.2.2 under the conditions 2.2.3-2.2.5 with conditions $u = u(x)$ and $v = v(x)$. If y_1 and y_2 are two solutions to Eqs. 2.2.1-2.2.2, then $y_1(x) = y_2(x)$ for all $x \in [0,1]$.

Proof: Let $w(x) = y_1(x) - y_2(x)$. Then,

$$Pw = 0, \qquad w(0) = 0,$$

$$P(-w) = 0, -w(0) = 0.$$

Using Theorem 2.2.2, it follows that $w(x) \geq 0$ and $w(x) \leq 0$ for all $x \in [0,1]$ which implies that $y_1(x) = y_2(x)$ for all $x \in [0,1]$. ■

2.3 Method of Solution

Consider the following of class of fractional nonlinear Volterra integro-differential type of singularly perturbed problems of the form

$$\epsilon D^{\alpha} y + u(x,y) + \int_0^x K(x,t)v(t,y)dt = f(x), x \in (0,1), 0 < \alpha \leq 1, \quad (2.3.1)$$

subject to

$$y(0) = y_0 \quad (2.3.2)$$

where $\epsilon > 0$ is a small positive parameter, y_0 is constant, and $K(x,t)$ and $f(x)$ are smooth functions. To solve Eqs. 2.3.1-2.3.2, we use the following steps.

Step 1: A reduced problem is obtained by setting $\epsilon = 0$ in Eqs. 2.3.1 to get

$$u(x,y_1) + \int_0^x K(x,t)v(t,y_1)dt = f(x), x \in [0,1]. \quad (2.3.3)$$

On most of the interval, the solution of Eq. 2.3.3 behaves like the solution of Eqs. 2.3.1-2.3.2. However, there is small interval around $x = 0$ in which the solution of problem 2.3.1-2.3.2 does not agree with the solution of Problem 2.3.1-2.3.2 to handle this situation, the boundary layer correction problem is introduced in step 2.

Step 2: Choose $x = \epsilon^{\frac{1}{\alpha}} s^{\frac{1}{\alpha}}$ to get

$$D^{\alpha} y(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-t)^{-\alpha}\, y'(t)dt$$

$$= \frac{1}{\Gamma(1-\alpha)} \int_0^{\epsilon^{\frac{1}{\alpha}} s^{\frac{1}{\alpha}}} \left(\epsilon^{\frac{1}{\alpha}} s^{\frac{1}{\alpha}} - t\right)^{-\alpha}\, y'(t)dt$$

$$= \frac{1}{\epsilon\Gamma(1-\alpha)} \int_0^{\epsilon^{\frac{1}{\alpha}} s^{\frac{1}{\alpha}}} \left(s^{\frac{1}{\alpha}} - \frac{t}{\epsilon^{\frac{1}{\alpha}}}\right)^{-\alpha}\, y'(t)dt.$$

Let $r = \frac{t}{\epsilon^{\frac{1}{\alpha}}}$. Then, $dt = \epsilon^{\frac{1}{\alpha}} dr$ and

$$\frac{dy}{dt} = \frac{dy}{dr}\frac{dr}{dt} = \frac{1}{\epsilon^{\frac{1}{\alpha}}}\frac{dy}{dr}.$$

Thus,

$$D^{\alpha}y(x) = \frac{1}{\epsilon\Gamma(1-\alpha)}\int_0^{s^{\frac{1}{\alpha}}}\left(s^{\frac{1}{\alpha}}-r\right)^{-\alpha}\frac{1}{\epsilon^{\frac{1}{\alpha}}}\frac{dy}{dr}\epsilon^{\frac{1}{\alpha}}dr$$

$$= \frac{1}{\epsilon\Gamma(1-\alpha)}\int_0^{s^{\frac{1}{\alpha}}}\left(s^{\frac{1}{\alpha}}-r\right)^{-\alpha}\frac{dy}{dr}dr$$

$$= \frac{1}{\epsilon}D^{\alpha}y\left(s^{\frac{1}{\alpha}}\right). \tag{2.3.4}$$

Hence, Eq. 2.3.1 becomes

$$D^{\alpha}y + u\left(\epsilon^{\frac{1}{\alpha}}s^{\alpha}, y\right) + \int_0^{\epsilon^{\frac{1}{\alpha}}s^{\alpha}} K\left(\epsilon^{\frac{1}{\alpha}}s^{\alpha}s, t\right)v(t,y)dt = f\left(\epsilon^{\frac{1}{\alpha}}s^{\alpha}s\right). \tag{2.3.5}$$

Setting $\epsilon = 0$ in Eqs. 2.3.5 implies that

$$D^{\alpha}y + u(0,y) = f(0). \tag{2.3.6}$$

Since the solution of the reduced problem in step 1 does not satisfy the initial condition at $x = 0$, then the solution of the above equation should satisfy it. This means, its solution has the form $y_1(0) + y_2(x)$. Substitute

$$y(x) = y_1(0) + y_2(x)$$

in Eq. 2.3.6 to get the boundary layer correction equation

$$D^{\alpha}y_2\left(s^{\frac{1}{\alpha}}\right) + u\left(0, y_1(0) + y_2\left(s^{\frac{1}{\alpha}}\right)\right) = f(0). \tag{2.3.7}$$

The solution of Eq. 2.3.1 will be expressed in the form as

$$y(x) = y_1(x) + y_2\left(\frac{x^{\frac{1}{\alpha}}}{\epsilon}\right), \tag{2.3.8}$$

and the initial condition 2.3.2 must be satisfied by expression 2.3.8. When $x = 0$, the condition will be

$$y_0 = y(0) = y_1(0) + y_2(0)$$

or

$$y_2(0) = y_0 - y_1(0), \tag{2.3.9}$$

The solution of Eqs. 2.3.1-2.3.2 can be produced using the RKM as described in the previous section. More details can be found in [41]-[43].

2.3 Numerical Results

In this section, we present two of our examples to show the efficiency of the proposed method.

Example 2.3.1: Consider the following problem

$$\epsilon D^{\frac{1}{2}}y(x) + y(x) + \int_0^x y(t)dt = f(x), 0 \leq x \leq 1, 0 < \epsilon \ll 1, \tag{2.3.1}$$

subject to

$$y(0) = 2 \tag{2.3.2}$$

where

$$f(x) = \frac{2}{\sqrt{\pi}}x^{1/2} - x^{1/2}E_{1,3/2}\left(\frac{-x}{\epsilon}\right) + \frac{x^2}{2} + 2x + (2-\epsilon)e^{-x/\epsilon} + (1+\epsilon)$$

and $E_{a,b}(x)$ is the Mittag-Leffler function. When $\epsilon \to 0$,

$$y_1(x) + \int_0^x y_1(t)dt = \frac{x^2}{2} + 2x + 1 \tag{2.3.3}$$

since $\lim_{\epsilon \to 0} E_{1,3/2}\left(\frac{-x}{\epsilon}\right) = 0$. Thus,

$$y_1'(x) + y_1(x) = x + 2.$$

Hence,

$$y_1(x) = 1 + x + ce^{-x}. \tag{2.3.4}$$

Substitute Eq. 2.3.4 into Eq. 2.3.3 to get

$$D^{1/2}y_2(s^2) + 1 + y_2(s^2) = 1$$

or

$$D^{1/2}y_2(s^2) + y_2(s^2) = 0$$

subject to

$$1 + x + ce^{-x} + \frac{x^2}{2} + x - ce^{-x} + c = \frac{x^2}{2} + 2x + 1$$

which implies that $c = 0$ and

$$y_1(x) = x + 1.$$

Using the change of variable $x = \epsilon^2 s^2$, we get

$$y_2(0) = y_0 - y_1(0) = 1.$$

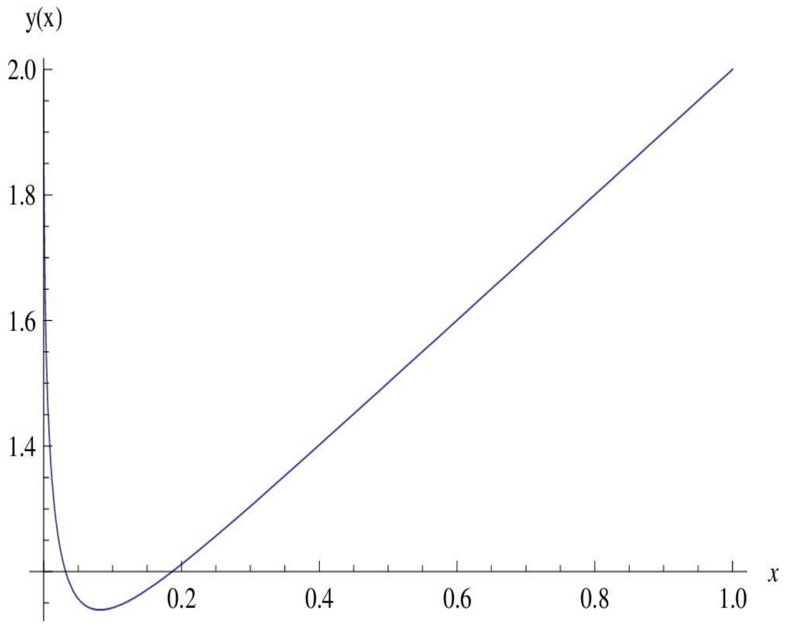

Figure 2.1: Approximate solution of Example 2.3.1 for $\epsilon = 0.1$

Using the RKM, we get

$$y_2(s^\alpha) = 1 - \frac{s}{1} + \frac{s^2}{2!} - \frac{s^3}{3!} + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k s^k}{k!} = e^{-s}.$$

Thus,

$$y(x) = y_1(x) + y_2\left(\frac{\sqrt{x}}{\epsilon}\right)$$

$$= x + 1 + e^{-\frac{\sqrt{x}}{\epsilon}}.$$

In Figure 2.1-2.3, we plot the approximate solution for $\epsilon = 0.1, 0.01$ and 0.001, respectively.

Example 2.3.2: Consider the following problem

$$\epsilon D^{\frac{1}{4}} y(x) - \frac{1}{2} y^2 + \int_0^x y(t)dt = 0,\ \ 0 \le x \le 1,\ \ 0 < \epsilon \ll 1, \tag{2.3.5}$$

subject to

$$y(0) = 1. \tag{2.3.6}$$

When $\epsilon \to 0$,

$$-\frac{1}{2} y_1^2(x) + \int_0^x y_1(t)dt = 0 \tag{2.3.7}$$

and $E_{a,b}(x)$ is the Mittag-Leffler function. When $\epsilon \to 0$,

$$y_1(x) + \int_0^x y_1(t)dt = \frac{x^2}{2} + 2x + 1 \tag{2.3.8}$$

since $\lim_{\epsilon \to 0} E_{1,3/2}\left(\frac{-x}{\epsilon}\right) = 0$. Thus,

$$y_1'(x) + y_1(x) = x + 2.$$

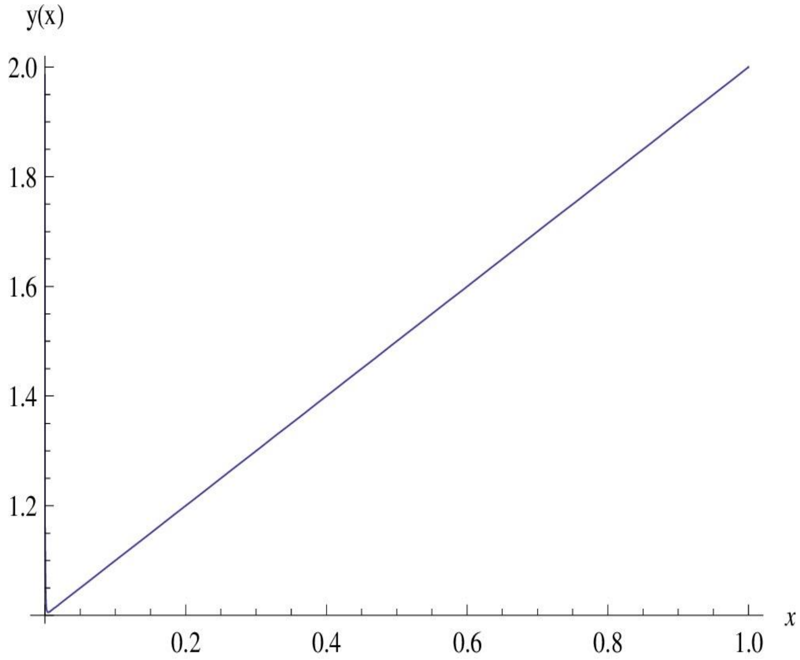

Figure 2.2: Approximate solution of Example 2.3.2 for $\epsilon = 0.01$

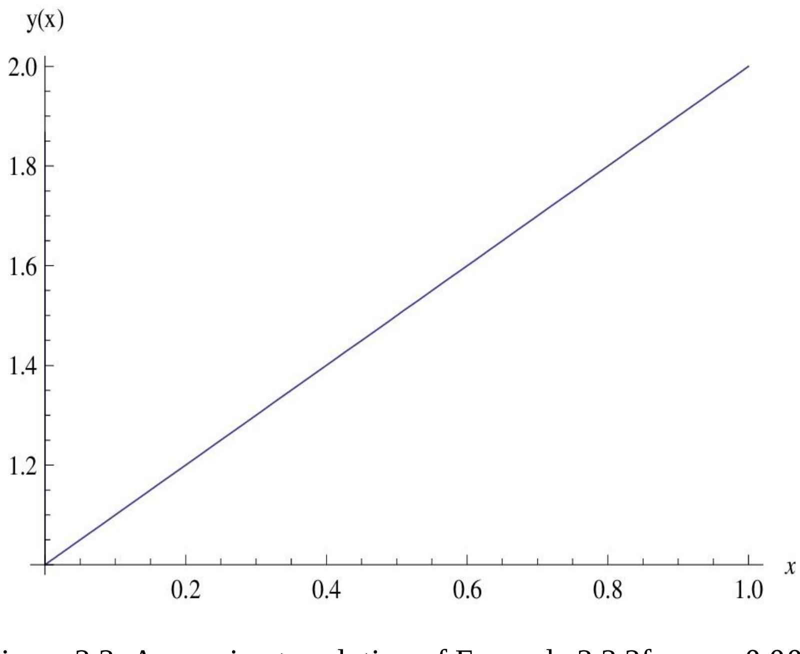

Figure 2.3: Approximate solution of Example 2.3.2for $\epsilon = 0.001$

Hence

$$y_1(x) = c + x. \tag{2.3.9}$$

Substitute Eq. 2.3.9 into Eq. 2.3.8 to get

$$-\frac{1}{2}(c+x)^2+\frac{1}{2}(c+x)^2-\frac{1}{2}c^2=0$$

which implies that $c=0$ and

$$y_1(x)=x.$$

Using the change of variable $x=\epsilon^4 s^4$, we get

$$D^{1/4}y_2(s^4)-\frac{1}{2}y_2^2(s^4)=0$$

subject to

$$y_2(0)=y_0-y_1(0)=1.$$

Using the RKM, we get

$$y_2(s^4)=1+\frac{s}{2}+\frac{s^2}{4}+\frac{s^3}{8}+\cdots$$

$$=\sum_{k=0}^{\infty}\frac{s^k}{2^k}=\frac{1}{1-\frac{x}{2}}=\frac{2}{2-x}.$$

Thus,

$$y(x)=y_1(x)+y_2\left(\frac{\sqrt[4]{x}}{\epsilon}\right)$$

$$=x+\frac{2\epsilon}{2\epsilon-\sqrt[4]{x}}.$$

In figure 2.4-2.6, we plot the approximate solution for $\epsilon=0.1, 0.01, and\ 0.001$, respectively.

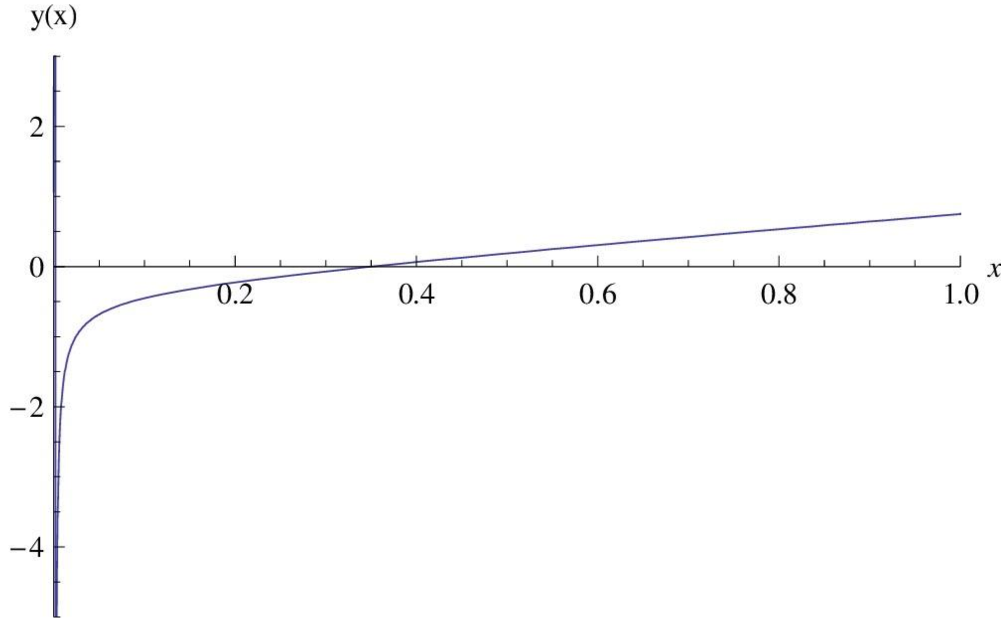

Figure 2.4: Approximate solution of Example 2.3.2 for $\epsilon = 0.1$

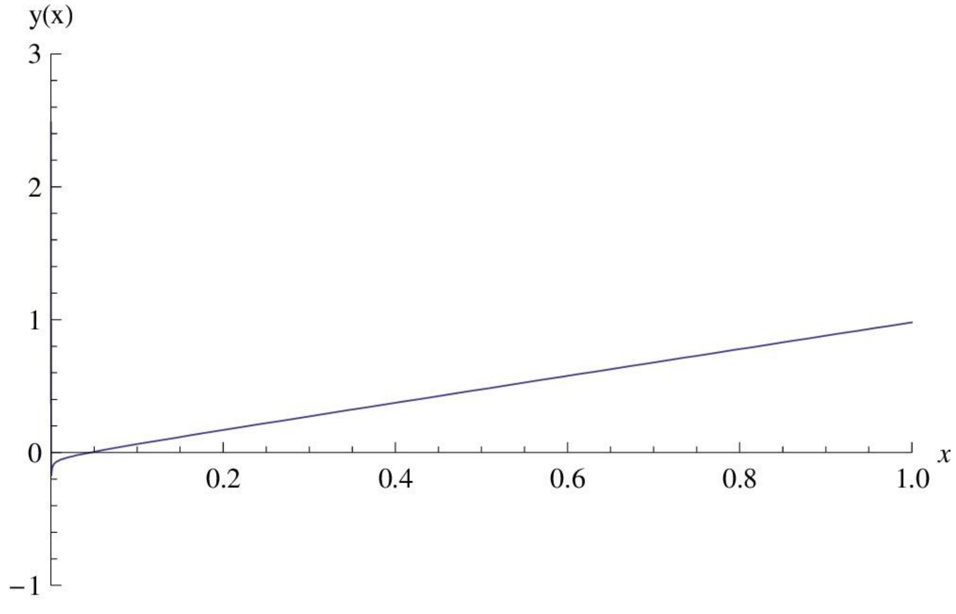

Figure 2.5: Approximate solution of Example 2.3.2 for $\epsilon = 0.01$

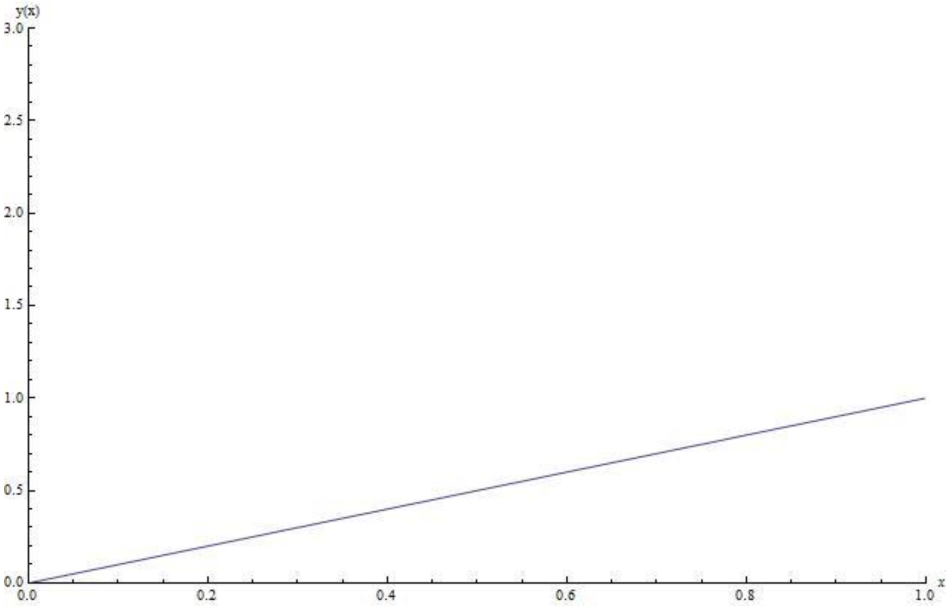

Figure 2.6: Approximate solution of Example 2.3.2 for $\epsilon = 0.001$

Chapter 3: Second Order Fractional Initial Value Problems

In this chapter, we study the second order fractional initial value problems. In the next section, we presented Kernel method for fractional second order initial value problems.

3.1 Reproducing Kernel Method for Fractional Second Order Initial Value Problems

Consider the second order nonlinear fractional equation of the form

$$D^{\alpha}y + g(x,y)y' = 0,\ x \in [0,1], 1 < \alpha \leq 2 \tag{3.1.1}$$

subject to

$$y(0) = \theta, y(1) = \phi \tag{3.1.2}$$

where θ and ϕ are constants. First, we study the linear case where $g(y) = a(x)$. To homogenize the initial condition, we assume $u = y - \phi x - \theta(1-x)$. Thus, Problems 3.1.1-3.1.2 can be written as

$$D^{\alpha}u + a(x)y' = (-\phi + \theta)a(x) = h(x), x \in [0,1], 0 < \alpha \leq 1 \tag{3.1.3}$$

subject to

$$u(0) = 0, u(1) = 0. \tag{3.1.4}$$

In order to solve the linear Problem 3.1.3-3.1.4, we construct the kernel Hilbert spaces $W_2^1[0,1]$ and $W_2^3[0,1]$ in which every function satisfies the initial condition 3.1.4.

Let $W_2^1[0,1] = \{u(s): u \text{ is absolutely continuous real value function, } u' \in L^2[0,1]\}$.

The inner product in $W_2^1[0,1]$ is defined as

$$(u(y), v(y))_{W_2^1[0,1]} = u(0)v(0) + \int_0^1 u'(y)v'(y)dy,$$

and the norm $\|u\|_{W_2^1[0,1]}$ is given by

$$\|u\|_{W_2^1[0,1]} = \sqrt{(u(y), u(y))_{W_2^1[0,1]}}$$

where $u, v \in W_2^1[0,1]$.

Theorem 3.1.1 The space $W_2^1[0,1]$ is a reproducing Kernel Hilbert space, $i, e.$; there exist $R(s, y) \in W_2^1[0,1]$ and its second partial derivative with respect to y exists such that for any $u \in W_2^1[0,1]$ and each fixed $y, s \in [0,1]$, we have

$$(u(y), R(s, y))_{W_2^1[0,1]} = u(s).$$

In this case, $R(s, y)$ is given by

$$R(s, y) = \begin{Bmatrix} 1 + y, & y \leq s \\ 1 + s, & y > s \end{Bmatrix}.$$

Proof. Using integration by parts, one can get

$$(u(y), R(s, y))_{W_2^1[0,1]} = u(0)R(s, 0) + \int_0^1 u'(y)\frac{\partial R}{\partial y}(s, y)dy$$

$$= u(0)R(s, 0) + u(1)\frac{\partial R}{\partial y}(s, 1) - u(0)\frac{\partial R}{\partial y}(s, 0) - \int_0^1 u(y)\frac{\partial^2 R}{\partial y^2}(s, y)dy.$$

Since $R(s, y)$ is a reproducing kernel of $W_2^1[0,1]$,

$$(u(y), R(s, y))_{W_2^1[0,1]} = u(s)$$

which implies that

$$-\frac{\partial^2 R}{\partial y^2}(s, y) = \delta(y - s), \tag{3.1.5}$$

$$R(s, 0) - \frac{\partial R}{\partial y}(s, 0) = 0 \tag{3.1.6}$$

and

$$\frac{\partial R}{\partial y}(s,1) = 0, \quad (3.1.7)$$

Since the characteristic equation of $-\frac{\partial^2 R}{\partial y^2}(s,y) = \delta(y-s)$ is $\lambda^2 = 0$ and its characteristic value is $\lambda = 0$ with 2, multiplicity roots, we write $R(s,y)$ as

$$R(s,y) = \begin{cases} c_0(s) + c_1(s)y, \ \ y \le s \\ d_0(s) + d_1(s)y, \ \ y > s \end{cases}.$$

Since $\frac{\partial^2 R}{\partial y^2}(s,y) = -\delta(y-s)$, we have

$$R(s,s+0) - R(s,s+0) = 0, \quad (3.1.8)$$

$$\frac{\partial R}{\partial y}(s,s+0) - \frac{\partial R}{\partial y}(s,s+0) = -1. \quad (3.1.9)$$

Using the conditions 3.1.6-3.1.9, we get the following system of equation

$$c_0(s) - c_1(s) = 0, \quad (3.1.10)$$

$$d_1(s) = 0,$$

$$c_0(s) + c_1(s)s = d_0(s) + d_1(s)s,$$

$$d_1(s) - c_1(s) = -1,$$

which implies that

$$c_0(s) = 1, \ c_1(s) = 1, d_0(s) = 1 + s, d_1(s) = 0$$

which completes the proof of the theorem. Next, we study the space $W_2^3[0,1]$.

Let

$$W_2^3[0,1] = \{f(s) \colon f \text{ is absolutely continuous real value function}, f, f', f'', f''' \in L^2[0,1], f(0) = 0, f(1) = 0\}.$$

The inner product in $W_2^3[0,1]$ is defined as

$$\big(u(y), v(y)\big)_{W_2^3[0,1]} = u(0)v(0) + u'(0)v'(0) + u(1)v(1) + u'(1)v'(1) + \int_0^1 u^{(3)}(y)v^{(3)}(y)dy$$

and the norm $\|u\|_{W_2^3[0,1]}$ is given by

$$\|u\|_{W_2^3[0,1]} = \sqrt{(u(y),u(y))_{W_2^3[0,1]}}$$

Where $u, v \in W_2^3[0,1]$.

Theorem 3.1.2 The space $W_2^3[0,1]$ is a reproducing Kernel Hilbert space, $i, e.$; there exist $K(s,y) \in W_2^3[0,1]$ which has its six partial derivative with respect to y such that for any $u \in W_2^3[0,1]$ and each fixed $y, s \in [0,1]$, we have

$$(u(y), K(s,y))_{W_2^3[0,1]} = u(s).$$

In this case, $K(s,y)$ is given by

$$K(s,y) = \begin{Bmatrix} \sum_{i=0}^{5} c_i(s)y^i\,, & y \leq s \\ \sum_{i=0}^{5} d_i(s)y^i\,, & y > s \end{Bmatrix}$$

where

$$c_0 = 0,\ c_1 = 0,\ c_2 = \frac{1}{120}(5s^4 - 111s^2 - 10s^3 - s^5), c_3 = 0,\ c_4 = -\frac{s}{24},$$

$$c_5 = \frac{1}{120}(1+s^5)\,,$$

$$d_0 = \frac{s^5}{120},\ d_1 = -\frac{s^4}{24}, d_2 = \frac{1}{120}(5s^4 - 111s^2 - s^5),\ d_3 = -\frac{s^2}{12}, d_4 = 0,$$

$$d_5 = \frac{s^2}{120}.$$

Proof: Using integration by parts, one can get

$$
\begin{aligned}
&(u(y), K(s,y))_{W_2^3[0,1]} \\
&\qquad = u(0)K(s,0) + u(1)K(s,1) + u'(0)K_y(s,0) + u'(1)K_y(s,1) \\
&\qquad + u''(1)K_{yyy}(s,1) - u''(0)K_{yyy}(s,0) - u'(1)\frac{\partial^4 K}{\partial y^4}(s,1) \\
&\qquad + u'(0)\frac{\partial^4 K}{\partial y^4}(s,0) + u(1)\frac{\partial^5 K}{\partial y^5}(s,1) - u(0)\frac{\partial^5 K}{\partial y^5}(s,0) \\
&\qquad + \int_0^1 u(y)\frac{\partial^6 K}{\partial y^6}(s,y)dy.
\end{aligned}
$$

Since $u(y)$ and $K(s,y) \in W_2^3[0,1]$,

$$u(0) = 0, u(1) = 0$$

and

$$K(s,0) = 0, K(s,1) = 0. \tag{3.1.11}$$

Thus,

$$
\begin{aligned}
&(u(y), K(s,y))_{W_2^3[0,1]} \\
&\qquad = u'(0)K_y(s,0) + u'(1)K_y(s,1) \\
&\qquad + u''(1)K_{yyy}(s,1) - u''(0)K_{yyy}(s,0) - u'(1)\frac{\partial^4 K}{\partial y^4}(s,1) \\
&\qquad + u'(0)\frac{\partial^4 K}{\partial y^4}(s,0) + \int_0^1 u(y)\frac{\partial^6 K}{\partial y^6}(s,y)dy.
\end{aligned}
$$

Since $K(s,y)$ is a reproducing kernel of $W_2^3[0,1]$

$$(u(y), K(s,y))_{W_2^3[0,1]} = u(s)$$

which implies that

$$\frac{\partial^6 K}{\partial y^6}(s,y) = \delta(y-s) \tag{3.1.12}$$

where δ is the dirac-delta function and

$$K(s,1)-\frac{\partial^5 K}{\partial y^5}(s,0)=0, \tag{3.1.13}$$

$$K_y(s,1)-\frac{\partial^4 K}{\partial y^4}(s,1)=0, \tag{3.1.14}$$

$$K_{yyy}(s,1)=0, \tag{3.1.15}$$

$$K_{yyyy}(s,0)=0. \tag{3.1.16}$$

Since the characteristic equation of $\frac{\partial^6 K}{\partial y^6}(s,y)=\delta(s-y)$ is $\lambda^6=0$ and its characteristic value is $\lambda=0$ with 6 multiplicity roots, we write $K(s,y)$ as

$$K(s,y)=\begin{Bmatrix}\sum_{i=0}^{5} c_i(s)y^i\,,\ y\le s\\ \sum_{i=0}^{5} d_i(s)y^i\,,\ y>s\end{Bmatrix}.$$

Since $\frac{\partial^5 K}{\partial y^5}(s,y)=\delta(s-y)$, we have

$$\frac{\partial^m K}{\partial y^m}(s,s+0)=\frac{\partial^m K}{\partial y^m}(s,s-0), m=0,1,\cdots,4. \tag{3.1.17}$$

On the other hand, integrating $\frac{\partial^5 K}{\partial y^5}(s,y)=\ \delta(s-y)$ from $s-\epsilon$ to $s+\epsilon$ with respect to y and letting $\epsilon\to 0$ to get

$$\frac{\partial^5 K}{\partial y^5}(s,s+0)-\frac{\partial^5 K}{\partial y^5}(s,s-0)=-1. \tag{3.1.18}$$

Using the conditions 3.1.11 and 3.1.13-3.1.18, we get the following system of equations

$$c_0(s)=0, c_1(s)=0\,, c_3(s)=0,$$

$$6d_3(s)+24d_4(s)+60d_5(s)=0, \sum_{i=0}^{5} d_i(s)-120d_5(s)=0,$$

$$\sum_{i=0}^{5} c_i(s)s^i=\sum_{i=0}^{5} d_i(s)s^i\,,$$

$$\sum_{i=1}^{5} ic_i(s)s^{i-1}=\sum_{i=1}^{5} id_i(s)s^{i-1}\,,$$

$$\sum_{i=1}^{5} i(i-1)c_i(s)s^{i-2} = \sum_{i=1}^{5} i(i-1)d_i(s)s^{i-2},$$

$$\sum_{i=1}^{5} i(i-1)(i-2)c_i(s)s^{i-3} = \sum_{i=1}^{5} i(i-1)(i-2)d_i(s)s^{i-3},$$

$$\sum_{i=1}^{5} i(i-1)(i-2)(i-3)c_i(s)s^{i-4} = \sum_{i=1}^{5} i(i-1)(i-2)(i-3)d_i(s)s^{i-4},$$

$$5!\, d_5(s) - 5!\, c_5(s) = -1.$$

We solved the last system using Mathematica to get

$$c_0 = 0,\ c_1 = 0,\ c_2 = \frac{1}{120}(5s^4 - 111s^2 - 10s^3 - s^5), c_3 = 0,\ c_4 = -\frac{s}{24},$$

$$c_5 = \frac{1}{120}(1+s^5),$$

$$d_0 = \frac{s^5}{120},\ d_1 = -\frac{s^4}{24}, d_2 = \frac{1}{120}(5s^4 - 111s^2 - s^5),\ d_3 = -\frac{s^2}{12}, d_4 = 0,$$

$$d_5 = \frac{s^2}{120}$$

which completes the proof of the theorem.

Now, we present how to solve Problem 3.1.3-3.1.4

$$\sigma_i(s) = R(s_i, s).$$

For $i = 1,2,\cdots$ where $\{s_i\}_{i=1}^{\infty}$ is dense on $[0,1]$. Let $L(\sigma_i(s)) = D^{\alpha}\sigma_i(s) + a(s)\sigma_i(s)$. It is clear that $L: W_2^3[0,1] \to W_2^1[0,1]$ is bounded linear operator. Let

$$\psi_i(s) = L^* \sigma_i(s)$$

where L^* is the adjoint operator of L. Using Gram-Schmidt orthonormalization to generate orthonormal set of function $\{\overline{\psi}_i(s)\}_{i=1}^{\infty}$ where

$$\overline{\psi}_i(s) = \sum_{j=1}^{i} \alpha_{ij}\psi_i(s) \tag{3.1.19}$$

and α_{ij} are coefficients of Gram-Schmidt orthonormalization. In the next theorem, we show the existence of the solution of Problem (3.1.3-3.1.4).

Theorem 3.1.3 If $\{s_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, then

$$u(s) = \sum_{i=1}^{\infty} \sum_{j=1}^{i} \alpha_{ij} h(s_i) \overline{\psi}_i(s) \tag{3.1.20}$$

Proof: First, we want to prove that $\{\psi_i(s)\}_{i=1}^{\infty}$ is complete system of $W_2^3[0,1]$ and $\psi_i(s) = L\big(k(s,s_i)\big)$. It is clear that $\psi_i(s) \in W_2^3[0,1]$ for $i = 1,2,\cdots$ Simple calculations imply that

$$\psi_i(s) = L^* \sigma_i(s) = (L^* \sigma_i(s), K(s,y))_{W_2^3[0,1]}$$

$$= (\sigma_i(s), L(K(s,y)))_{W_2^3[0,1]} = L\big(K(s,s_i)\big).$$

For each fixed $u(s) \in W_2^3[0,1]$, let

$$(u(s), \psi_i(s))_{W_2^3[0,1]} = 0, i = 1,2,\cdots.$$

Then,

$$(u(s), \psi_i(s))_{W_2^3[0,1]} = \big(u(s), L^* \sigma_i(s)\big)_{W_2^3[0,1]}$$

$$= \big(Lf(s), \sigma_i(s)\big)_{W_2^3[0,1]}$$

$$= Lu(s_i) = 0.$$

Since $\{s_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, $Lu(s) = 0$. Since L^{-1} exists, $u(s) = 0$. Thus, $\{\psi_i(s)\}_{i=1}^{\infty}$ is the complete system of $W_2^3[0,1]$.

Second, we prove Equation 3.1.20. Simple calculations implies that

$$u(s) = \sum_{i=1}^{\infty} (u(s), \bar{\psi}_i(s))_{W_2^3[0,1]} \bar{\psi}_i(s)$$

$$= \sum_{i=1}^{\infty} \sum_{j=1}^{i} \alpha_{ij} \Big(u(s), L^* \Big(K\big(s,s_j\big)\Big)\Big)_{W_2^3[0,1]} \bar{\psi}_i(s)$$

$$= \sum_{i=1}^{\infty}\sum_{j=1}^{i}\alpha_{ij}\left(Lf(s), K(s,s_j)\right)_{W_2^3[0,1]}\bar{\psi}_i(s)$$

$$= \sum_{i=1}^{\infty}\sum_{j=1}^{i}\alpha_{ij}\left(c\,, K(s,s_j)\right)_{W_2^3[0,1]}\bar{\psi}_i(s)$$

$$= \sum_{i=1}^{\infty}\sum_{j=1}^{i}\alpha_{ij}\,\bar{\psi}_i(s)$$

and the proof is complete.

Let the approximation solution of Problem 3.1.3-3.1.4 be given by

$$u_N(s) = \sum_{i=1}^{N}\sum_{j=1}^{i}\alpha_{ij}h(s_i)\,\bar{\psi}_i(s). \qquad (3.1.21)$$

In the next theorem, we show the uniformly convergence of the $\{\frac{d^m f_N(s)}{ds^m}\}_{N=1}^{\infty}$ to $\frac{df(s)}{ds}$ for $m = 0,1,2$.

Theorem 3.1.4 If $u(s)$ and $u_N(s)$ are given as in (3.1.20) and (3.1.21), then $\{\frac{d^m f_N(s)}{ds^m}\}_{N=1}^{\infty}$ converges uniformly to $\frac{d^m u(s)}{ds^m}$ for $m = 0,1,2$.

Proof: First, we prove the theorem for $m = 0$. For any $s \in [0,1]$,

$$\| u(s) - u_N(s) \|_{W_2^3[0,1]}^2 = \left(u(s) - u_N(s), u(s) - u_N(s)\right)_{W_2^3[0,1]}$$

$$= \sum_{i=N+1}^{\infty}\begin{pmatrix}\left(u(s),\bar{\psi}_i(s)\right)_{W_2^3[0,1]}\bar{\psi}_i(s), \\ \left(u(s),\bar{\psi}_i(s)\right)_{W_2^3[0,1]}\bar{\psi}_i(s)\end{pmatrix}_{W_2^3[0,1]}$$

$$= \sum_{i=N+1}^{\infty}(u(s),\bar{\psi}_i(s))_{W_2^3[0,1]}^2.$$

Thus,

$$Sub_{s\in[0,1]}\|u(s)-u_N(s)\|^{\,2}_{W_2^3[0,1]} = Sup_{s\in[0,1]}\sum_{i=N+1}^{\infty}(u(s),\bar{\psi}_i(s))^{\,2}_{W_2^3[0,1]}.$$

From Theorem (3.1.3), one can see that $\sum_{i=1}^{\infty}(u(s),\bar{\psi}_i(s))_{W_2^3[0,1]}\bar{\psi}_i(s)$ converges uniformly to $u(s)$. Thus,

$$\lim_{N\to\infty} Sup_{s\in[0,1]}\|u(s)-u_N(s)\|_{W_2^3[0,1]} = 0$$

which implies that $\{u_N(s)\}_{N=1}^{\infty}$ converges uniformly to $u(s)$.

Second, we prove the uniformly convergence for $m = 1,2$. Since $\frac{d^m K(s,y)}{ds^m}$ is bounded function on $[0,1]\times[0,1]$,

$$\left\|\frac{d^m\, K(s,y)}{ds^m}\right\|_{W_2^3[0,1]} \le X_m\,,\ \ m = 1.$$

Thus, for any $s\ \in[0,1]$,

$$\begin{aligned}\left|u^{(m)}(s)-\ u_N^{(m)}(s)\right| &= \left|(u(s)-u_N(s),\frac{d^m K(s,y)}{ds^m})_{W_2^3[0,1]}\right| \\ &\le\ \|u(s)-\ u_N(s)\|_{W_2^3[0,1]}\left\|\frac{d^m\, K(s,y)}{ds^m}\right\|_{W_2^3[0,1]} \\ &\le\ \chi_m\,\|u(s)-\ u_N(s)\|_{W_2^3[0,1]} \\ &\le\ \chi_m Sup_{s\in[0,1]}\,\|u(s)-\ u_N(s)\|_{W_2^3[0,1]}.\end{aligned}$$

Hence,

$$Sup_{s\in[0,1]}\left\|u^{(m)}(s)-\ u_N^{(m)}(s)\right\|_{W_2^3[0,1]}\ \le\ \chi_{m_m}\,Sup_{s\in[0,1]}\|u(s)-\ u_N(s)\|_{W_2^3[0,1]}$$

which implies that

$$\lim_{N\to\infty}\ Sup_{s\in[0,1]}\ \left\|u^{(m)}(s)-\ u_N^{(m)}(s)\right\|_{W_2^3[0,1]} = 0.$$

Therefore, $\left\{\frac{d^m u_N(s)}{ds^m}\right\}_{N=1}^{\infty}$ converges uniformly to $\frac{d^m\, u(s)}{ds^m}$ for $m = 1,2$.

Now, we discuss how to solve Problem (3.1.1) – (3.1.2). Let $\mathcal{L}\big(y(x)\big) = D^{\alpha}\,y(x)$ and $N\big(y(x)\big) = g(x,y)y'$ are the linear and nonlinear parts of Problem 3.1.1, respectively.

We construct the homotopy as follows:

$$H(y,\lambda) = \mathcal{L}\big(y(x)\big) + \lambda N\big(y(x)\big) = 0 \tag{3.1.22}$$

where $\lambda \in [0,1]$ is an embedding parameter. If $\lambda = 0$, we get a linear equation

$$D^{\alpha} y(x) = 0$$

which implies that $y(x) = 0$. If $\lambda = 1$, we turn out to be Problem 3.1.1. Following the Homotopy Perturbation method [40], we expand the solution in term of the Homotopy parameter λ as

$$y = y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3 + \cdots \tag{3.1.23}$$

Substitute Equation 3.1.23 into Equation 3.1.22 and equating the coefficient of the identical power of λ to get the following system

$$\lambda^0 : \ D^{\alpha} y_0(x) = 0, y_0(0) = \theta,$$

$$\lambda^1 : \ D^{\alpha} y_1(x) = -N\big(\textstyle\sum_{i=0}^{\infty} \lambda^i y_i(x)\big)\big|_{\lambda=0}, y_1(0) = 0,$$

$$\lambda^2 : \ D^{\alpha} y_2(x) = -\frac{dN(\sum_{i=0}^{\infty} \lambda^i y_i(x))}{d\lambda}\Big|_{\lambda=0}, y_2(0) = 0,$$

$$\lambda^3 : \ D^{\alpha} y_3(x) = -\frac{d^2N(\sum_{i=0}^{\infty} \lambda^i y_i(x))}{d\lambda^2}\Big|_{\lambda=0}, y_3(0) = 0,$$

$$\vdots$$

$$\lambda^k : \ D^{\alpha} y_k(x) = -\frac{d^{k-1}N(\sum_{i=0}^{\infty} \lambda^i y_i(x))}{d\lambda^{k-1}}\Big|_{\lambda=0}, y_k(0) = 0.$$

To solve the above equations, we use the RKM which is described above and we obtain

$$y_k(x) = \textstyle\sum_{i=1}^{\infty} \sum_{j=1}^{i} \alpha_{ij}\, h_k\big(x_j\big)\bar{\psi}_i(s), k = 0,1,\cdots \tag{3.1.24}$$

where

$$h_0(x) = c$$

$$h_1(x) = -N\big(\textstyle\sum_{i=0}^{\infty} \lambda^i y_i(x)\big)\big|_{\lambda=0}$$

$$\vdots$$

$$h_k(x) = -\frac{d^{k-1}N(\sum_{i=0}^{\infty}\lambda^i y_i(x))}{d\lambda^{k-1}}\Big|_{\lambda=0}, k > 1.$$

From Equation 3.1.24, it is easy to see the solution to Problem 3.1.1-3.1.2 is giving by

$$y(s) = \sum_0^{\infty} y_k(x) = \sum_{k=0}^{\infty}\left(\sum_{i=1}^{\infty}\sum_{j=1}^{i}\alpha_{ij}\, h_k(x_j)\bar{\psi}_i(x)\right). \qquad (3.1.25)$$

We approximate the solution of Problem 3.1.1-3.1.2 by

$$y_{n.m}(x) = \sum_{k=0}^{\infty}\left(\sum_{i=1}^{\infty}\sum_{j=1}^{i}\alpha_{ij}\, h_k(x_j)\bar{\psi}_i(x)\right). \qquad (3.1.26)$$

3.2 Analytical Results

In this section, three important theorems are presented which are the maximum principle, the stability theorem, and the uniqueness theorem. Firstly Eqs. 3.2.1-3.2.2 are transformed into an equivalent problem as follows

$$Py: -\epsilon D^{\alpha}y + u(x,y)y' + \int_0^x K(x,t)v(t,y)dt = f(x), x \in (0,1), 0 < \alpha \leq 1, \qquad (3.2.1)$$

$$y(0) = y_0, y(1) = y_1 \qquad (3.2.2)$$

The following conditions are needed in order to guarantee that Eqs. 3.2.1-3.2.2 does not have turning-point problem;

$$-k_2 \geq u(x,y) \geq -k_1, \qquad (3.2.3)$$

$$0 \geq v(x,y) \geq -k_3, \qquad (3.2.4)$$

$$K(x,t) \geq k_4 \geq 0, \qquad (3.2.5)$$

for all $x \in [0,1]$, where $k_1, k_2, k_3, and\ k_4$ are positive constants and $y \in C^2(0,1) \cup C[0,1]$.

Lemma 3.2.1 [44] Let $y \in C^2[0,1]$ attains its minimum at $x_0 \in (0,1)$. Then, $y'(x_0) \leq 0$ and $D^{\propto}y(x_0) \geq 0$ for $1 < \propto \leq 2$.

Theorem 3.2.2 (Maximum Principle). Consider the initial value problem 3.2.6-3.2.7 with conditions 3.2.3-3.2.5. Assume that $Py \geq 0$ and $y(0) \geq 0$. Then $y(1) \geq 0$ in [0,1].

$$\epsilon D^{\alpha} y + u(x,y)y' + \int_0^x K(x,t)v(t,y)dt = f(x), x \in (0,1), 1 < \alpha \leq 2, \qquad (3.2.6)$$

subject to

$$y(0) = y_0, y(1) = y_1 \qquad (3.2.7)$$

Proof: Assume that the conclusion is false, then $\phi(x) < 0$ for some $x \in [0,1]$. Then, $y(x)$ has a local minimum at x_0 for some $x_0 \in (0,1]$. Simple calculations and using Lemma (3.2.1) implies that

$$Py(x_0) = \epsilon D^{\alpha} y(x_0) + u(x_0,y)y'(x_0) + \int_0^{x_0} K(x_0,t)v(t,y)dt$$

$$\leq 0.$$

This a contradiction. Therefore, $y(x) \geq 0$ in [0,1]. ■

In the next theorem, the stability result is presented.

Theorem 3.2.3 (Stability Result). Consider Eqs. 3.2.6-3.2.7 with conditions $u = u(x)$ and $v = v(x)$. If $y(x)$ is a smooth function, then

$$\|y\| = max\{|y(x)|: x \in [0,1]\} \leq 2\varsigma\, max\{|y_0|, |y_1|, max_{x\in[0,1]}|Py|\}.$$

Where $\varsigma = 1 + \frac{1}{k_2}$.

Proof: Let

$$K_0 = max\{|y_0|, |y_1|, max_{x\in[0,1]}|Py|\} = max\{|y_0|, |y_1|, max_{x\in[0,1]}|f(x)|\}$$

and let

$$s^{\pm}(x) = 2\varsigma\, K_0\left(1 - \frac{x}{2}\right) \pm y(x), x \in [0,1].$$

Then,

$$Ps^{\pm}(x) = -\epsilon D^{\alpha}\left(2\varsigma\, K_0\left(1-\frac{x}{2}\right) \pm y(x)\right) + u(x)\left(2\varsigma\, K_0\left(1-\frac{x}{2}\right) \pm y(x)\right)'$$

$$+\int_0^x K(x,t)v(t)dt = 2\varsigma\, K_0 u(x) \pm Py(x) > K_0 \pm Py(x) \geq 0.$$

for all $x \in [0,1]$. Also,

$$s^{\pm}(0) = 2\varsigma\, K_0 \pm y(0) > K_0 \pm y_0 \geq 0, x \in [0,1]$$

and

$$s^{\pm}(1) = \varsigma\, K_0 \pm y_1 > K_0 \pm y_1 \geq 0, x \in [0,1].$$

From Theorem 3.2.2, we can see that $s^{\pm}(x) \geq 0$ for all $x \in [0,1]$.

Therefore,

$$\|y\| = \max\{|y(x)| : x \in [0,1] \leq 2\varsigma\, max\{|y_0|, |y_1|, max_{x\in[0,1]}|Py|\}. \blacksquare$$

Theorem 3.2.4 (Uniqueness Theorem). Consider Eqs. 3.2.6-3.2.7 under the conditions 3.2.3-3.2.5 with conditions $u = u(x)$ and $v = v(x)$. If y_1 and y_2 are two solutions to Eqs. 3.2.6-3.2.7, then $y_1(x) = y_2(x)$ for all $x \in [0,1]$.

Proof: Let $w(x) = y_1(x) - y_2(x)$. Then,

$$Pw = 0, \; w(0) = 0, \mathrm{w(1)=0}$$

$$P(-w) = 0, -w(0) = 0, -w(1) = 0.$$

Using Theorem 3.2.2, it follows that $w(x) \geq 0$ and $w(x) \leq 0$ for all $x \in [0,1]$ which implies that $y_1(x) = y_2(x)$ for all $x \in [0,1]$. ■

3.3 Method of Solution

Consider the following of class of fractional nonlinear Volterra integro-differential type of singularly perturbed problems of the form

$$-\epsilon D^{\alpha}y + u(x,y)y' + \int_0^x K(x,t)v(t,y)dt = f(x), x \in (0,1), 1 < \alpha \leq 2$$

subject to

$$y(0) = \ y_0, y(1) = y_1$$

where $\epsilon > 0$ is a small positive parameter, y_0 and y_1 are constant, and $K(x,t)$ and $f(x)$ are smooth functions. To solve Eqs. 3.2.6-3.2.7, we use the following steps.

Step 1: A reduced problem is obtained by setting $\epsilon = 0$ in Eqs. 3.3.6 to get

$$u(x,y_1)y' + \int_0^x K(x,t)v(t,y_1)dt = f(x), x \in [0,1]. \qquad (3.3.1)$$

On most of the interval, the solution of Eq. 3.3.1 behaves like the solution of Eqs. 3.2.6-3.2.7. However, there is small interval around $x = 0$ in which the solution of problem 3.2.6-3.2.7 does not agree with the solution of Problem 3.2.6-3.2.7 to handle this situation, the boundary layer correction problem is introduced in step 2.

Step 2: Choose $x = \epsilon^{\frac{1}{\alpha-1}}s$ to get

$$D^{\alpha}y(x) = \frac{1}{\Gamma(1-\alpha)}\int_0^x (x-t)^{1-\alpha}\, y''(t)dt$$

$$= \frac{1}{\Gamma(1-\alpha)}\int_0^{\epsilon^{\frac{1}{\alpha-1}}s} \left(\epsilon^{\frac{1}{\alpha-1}}s - t\right)^{1-\alpha}\, y''(t)dt$$

$$= \frac{\epsilon^{\frac{1-\alpha}{\alpha-1}}}{\Gamma(1-\alpha)}\int_0^{\epsilon^{\frac{1}{\alpha}}s^{\frac{1}{\alpha}}} \left(s - \frac{t}{\epsilon^{\frac{1}{\alpha-1}}}\right)^{-\alpha}\, y''(t)dt.$$

Let $r = \frac{t}{\epsilon^{\frac{1}{\alpha-1}}}$. Then, $dt = \epsilon^{\frac{1}{\alpha-1}}dr$ and

$$\frac{dy}{dt} = \frac{dy}{dr}\frac{dr}{dt} = \frac{1}{\epsilon^{\frac{1}{\alpha-1}}}\frac{dy}{dr}.$$

$$\frac{d^2y}{dt^2} = \frac{d\left(\frac{dy}{dt}\right)}{dr}\frac{dr}{dt} = \left(\frac{1}{\epsilon^{\frac{1}{\alpha-1}}}\right)^2\frac{dy}{dr}.$$

Thus,

$$D^{\alpha}y(x) = \frac{\epsilon^{\frac{1-\alpha}{\alpha-1}}}{\Gamma(1-\alpha)}\int_0^s (s-r)^{-\alpha}\frac{1}{\left(\epsilon^{\frac{1}{\alpha-1}}\right)^2}\frac{dy}{dr}\epsilon^{\frac{1}{\alpha-1}}dr$$

$$= \frac{\epsilon^{\frac{-\alpha}{\alpha-1}}}{\Gamma(1-\alpha)}\int_0^s (s-r)^{-\alpha}\frac{dy}{dr}dr$$

$$= \epsilon^{\frac{-\alpha}{\alpha-1}}D^{\alpha}y(s). \tag{3.3.2}$$

Hence, Eq. 3.2.6 becomes

$$-\epsilon\epsilon^{\frac{-\alpha}{\alpha-1}}D^{\alpha}y(s) + \frac{1}{\epsilon^{\frac{1}{\alpha-1}}}u\left(\epsilon^{\frac{1}{\alpha-1}}s, y\right)\frac{dy}{ds} + \int_0^{\epsilon^{\frac{1}{\alpha-1}}s} K\left(\epsilon^{\frac{1}{\alpha-1}}s, t\right)v(t,y)dt = f\left(\epsilon^{\frac{1}{\alpha-1}}s\right) \tag{3.3.3}$$

or

$$-D^{\alpha}y + u\left(\epsilon^{\frac{1}{\alpha-1}}s, y\right)\frac{dy}{ds} + \epsilon^{\frac{1}{\alpha-1}}\int_0^{\epsilon^{\frac{1}{\alpha-1}}s} K\left(\epsilon^{\frac{1}{\alpha-1}}s, t\right)v(t,y)dt = \epsilon^{\frac{1}{\alpha-1}}f\left(\epsilon^{\frac{1}{\alpha-1}}s\right) \tag{3.3.4}$$

Setting $\epsilon = 0$ in Eqs. 3.3.3 implies that

$$-D^{\alpha}y(s) + u(0,y)\frac{dy}{ds} = 0. \tag{3.3.5}$$

Since the solution of the reduced problem in step 1 does not satisfy the initial condition at $x = 0$, then the solution of the above equation should satisfy it. This means, its solution has the form $y_1(0) + y_2(x)$. Substitute

$$y(x) = y_1(0) + y_2(x)$$

in Eq. 3.3.5 to get the boundary layer correction equation

$$-D^{\alpha}y_2(s) + u\big(0, y_1(0) + y_2(s)\big)\frac{dy2}{ds} = f(0). \tag{3.3.6}$$

The solution of Eq. 3.2.6 will be expressed in the form as

$$y(x) = y_1(x) + y_2\left(\frac{x}{\epsilon^{\frac{1}{\alpha-1}}}\right), \tag{3.3.7}$$

and the initial condition must be satisfied by expression 3.3.7. When $x = 0$, the condition will be

$$y_0 = y(0) = y_1(0) + y_2(0)$$

or

$$y_2(0) = y_0 - y_1(0), \quad (3.3.8)$$

The solution of Eqs. 3.2.6-3.2.7 can be produced using the RKM as described in the previous section. More details can be found in [41]-[43].

3.4 Numerical Results

In this section, we present two of our examples to show the efficiency of the proposed method.

Example 3.4.1: Consider the following problem

$$-\epsilon D^{\frac{3}{2}} y(x) - 2y'(x) - \int_0^x e^{y(t)} dt = x^2 - 2x - \frac{2}{x-2}, 0 \le x \le 1, 0 < \epsilon \ll 1, \quad (3.4.1)$$

subject to

$$y(0) = 0, y(1) = 0. \quad (3.4.2)$$

When $\epsilon = 0$,

$$-2y'(x) - 2\int_0^x e^{y(t)} dt = -2\ln(x+1) + \frac{2}{x+1}, y(1) = 0. \quad (3.4.3)$$

We discretized the interval [0,1] by $x_i = ih, h = \frac{1}{n}, n \in N$. Let $y_k \approx y(x_k)$ for $k = 0 : n$. Using the backward finite difference method to approximate $y'(x_k)$ and the trapezoidal quadrature to approximate the integral $\int_0^{x_k} e^{y(t)} dt$, we get

$$-2\frac{y_k - y_{k-1}}{h} - h\sum_{j=0}^{k-1}(e^{y_i} + e^{y_{j+1}}) = -2\ln(x_k+1) + \frac{2}{x_k+1}, x_n = 0.$$

Thus, we get the following system

$$AY + Be^Y = F$$

Where

$$A = -\frac{2}{h}\begin{pmatrix} -1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & 0 & -1 & 1 & 0 \\ \vdots & \cdots & \cdots & 0 & -1 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & -1 \end{pmatrix}, B = -h\begin{pmatrix} 1 & 1 & 0 & \ldots 0 & \cdots & 0 \\ 1 & 2 & 1 & \ddots 0 & \cdots & 0 \\ 1 & 2 & 1 & \ddots 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \cdots 2 & 1 & 0 \\ \vdots & \vdots & \vdots & \cdots 2 & 2 & 1 \\ 2 & 2 & 2 & \cdots 2 & 2 & 2 \end{pmatrix}$$

$$F = \begin{pmatrix} f(x_1) \\ f(x_1) \\ \vdots \\ f(x_{n-1}) \\ f(x_n) \end{pmatrix}, Y = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-2} \\ y_n \end{pmatrix}.$$

Using Mathematica, one can see that the solution of the above system for $n = 12$ is giving in Figure 3.1. Using the change of variable $x = \epsilon^2 s$, we get

$$-D^{\frac{3}{2}} y_2(s) - 2\frac{dy_2}{ds} = 0$$

subject to

$$y_2(0) = y_0 - y_1(0) = -0.694147, y_2'(0) = \theta.$$

Using the RKM, we get

$$y_2(s) \approx -0.694147 + \frac{2\theta s}{3} - \frac{2s^2}{\sqrt{\pi}} + \frac{8\theta s^{5/2}}{5} - \frac{32\theta s^3}{9\sqrt{\pi}} + \frac{16\theta s^{7/2}}{7} - \frac{64\theta s^4}{15\sqrt{\pi}}.$$

Using the Pade' approximation of order [2,2], we have $\theta = -0.694147$. In figures 3.2-3.4, we plot the approximate solution for $\epsilon = 0.0001, 0.00001, 0.000001$ respectively.

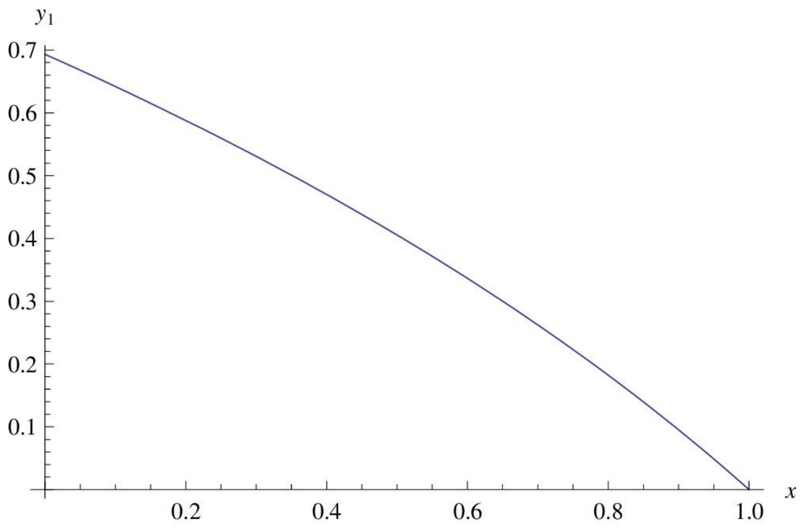

Figure 3.1: The approximate of Example 3.4.1 solution y_1

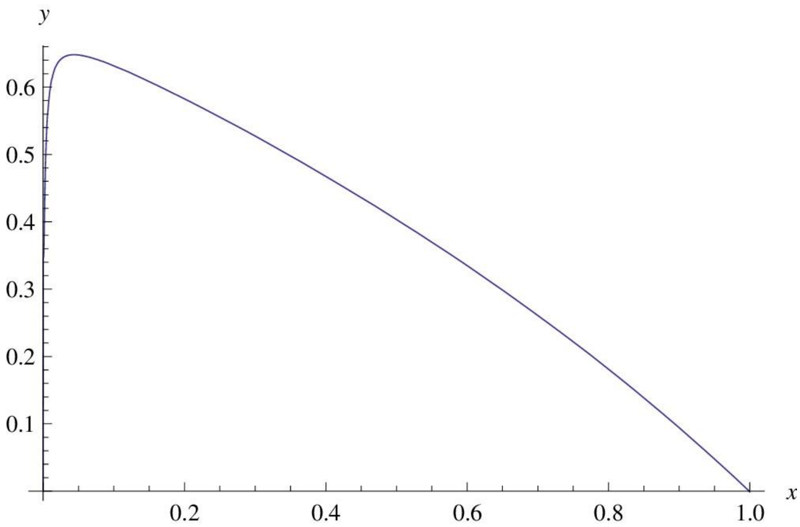

Figure 3.2: The approximate solution y of Example 3.4.1 for $\epsilon = 0.0001$

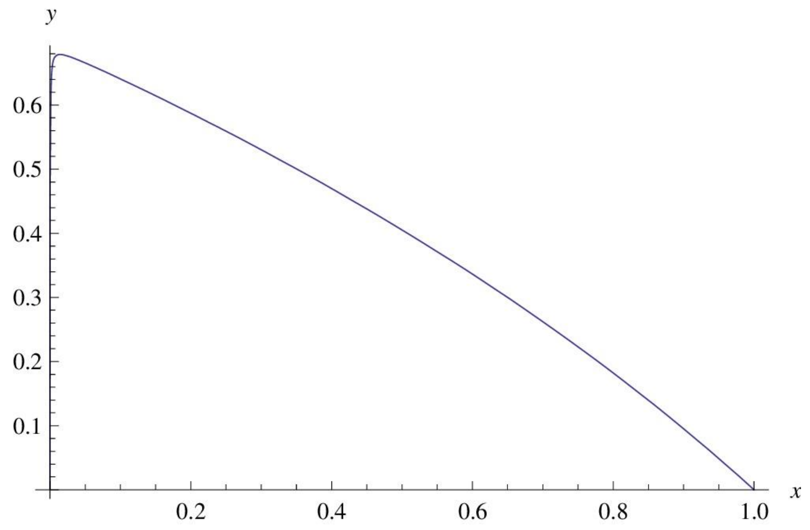

Figure 3.3: The approximate solution y of Example 3.4.1 for $\epsilon = 0.00001$

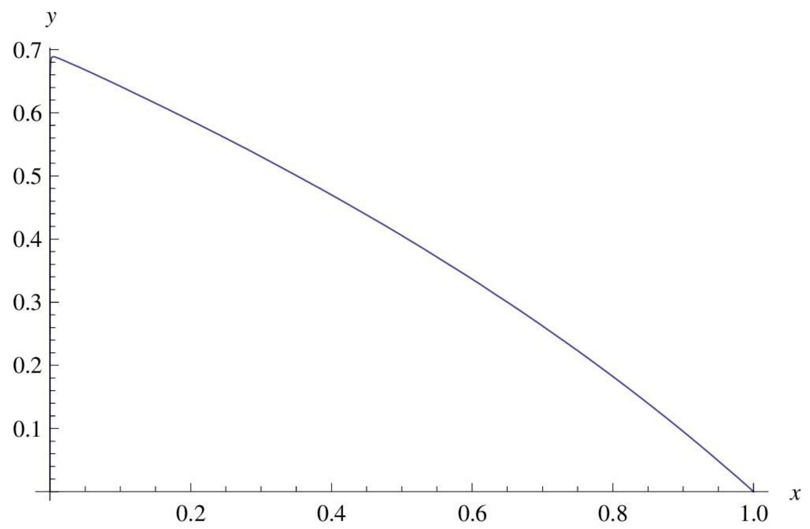

Figure 3.4: The approximate solution y of Example 3.4.1 for $\epsilon = 0.000001$

Example 3.4.2: Consider the following problem

$$-\epsilon D^{\frac{3}{2}}y(x) - yy' - \int_0^x (x-t)y^2(t)dt = f(x), 0 \leq x \leq 1, 0 < \epsilon \ll 1, \qquad (3.4.4)$$

Subject to

$$y(0) = -1, y(1) = 6. \qquad (3.4.5)$$

where

$$f(x) = -5 - x - \frac{25}{3}x^3 - \frac{5}{6}x^4 - \frac{x^5}{30}.$$

When $\epsilon = 0$,

$$-yy' - \int_0^x (x-t)y^2(t)dt = f(x), y(1) = 6. \quad (3.4.6)$$

We discretized the interval [0,1] by $x_i = ih, h = \frac{1}{n}, n \in N$. Let $y_k \approx y(x_k)$ for $k = 0 : n$. Using the backward finite difference method to approximate $y'(x_k)$ and the trapezoidal quadrature to approximate the integral $\int_0^{x_k}(x-t)y^2(t)dt$, we get

$$-y_k \frac{y_k - y_{k-1}}{h} - \frac{h}{2}\sum_{j=0}^{k-1}\left((x_k - x_{j+1})y_{j+1}^2 + (x_k - x_j)y_j^2\right) = f(x_k), y_n = 6.$$

Using Mathematics, one can see that the solution of the above system for $n = 12$ is giving in Figure 3.5. Using the change of variable $x = \epsilon^2 s$, we get

$$-D^{\frac{3}{2}}y_2(s) - (y_2(s) + 5)\frac{dy_2}{ds} = 0$$

subject to

$$y_2(0) = y_0 - y_1(0) = -6, y_2'(0) = \theta.$$

Using the RKM, we get

$$y_2(s) \approx -6 + \theta s - \frac{4\theta}{3\sqrt{\pi}}s^{\frac{3}{2}} + \frac{\theta}{2}s^2 - \frac{8\theta^2}{15\sqrt{\pi}}s^{\frac{5}{2}} - \frac{7\theta^2}{12}s^3 + \frac{7\theta^3}{48}s^4.$$

Using the Pade' approximation of order [2,2], we have $\theta = 0.0927388622769557$. In figures 3.5-3.8, we plot the approximate solution for $\epsilon = 0.001, 0.0001, and\ 0.00001$, respectively.

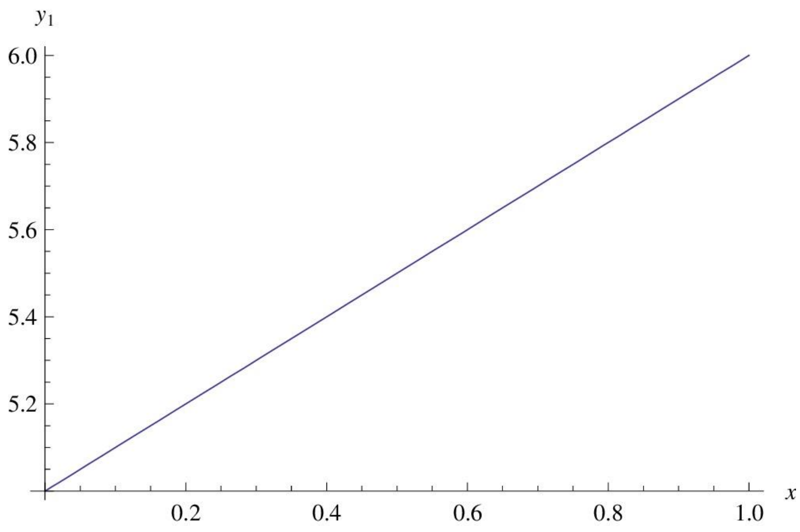

Figure 3.5: The approximate solution of Example 3.4.2 for y_1

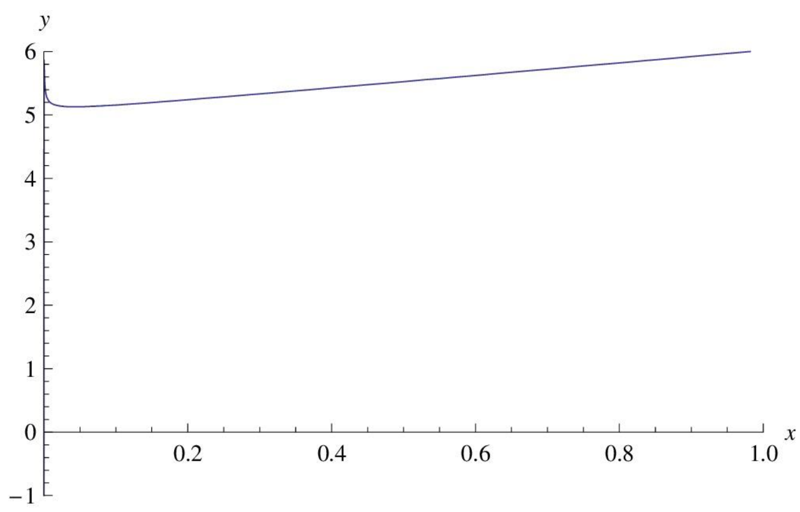

Figure 3.6: Approximate solution of Example 3.4.2 for for $\epsilon = 0.001$

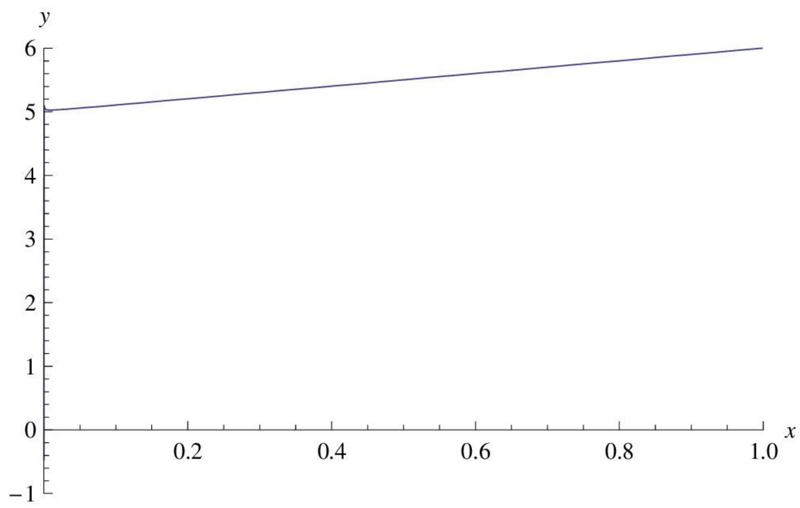

Figure 3.7: The approximate solution y of Example 3.4.2 for for $\epsilon = 0.0001$

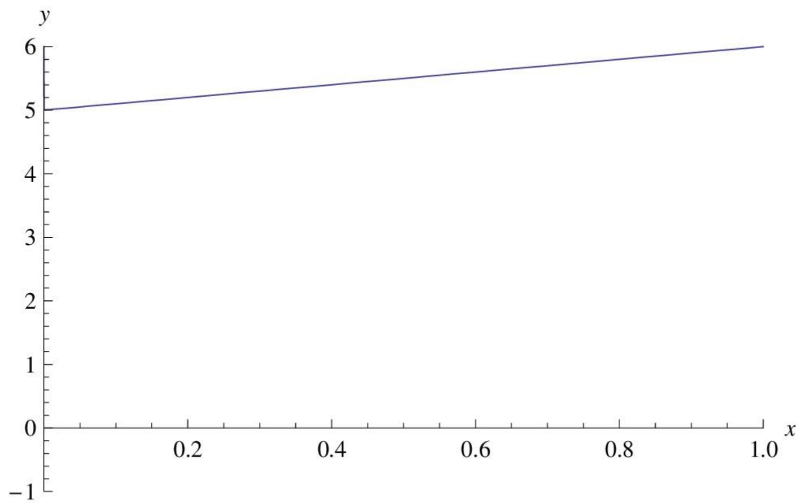

Figure 3.8: The approximate solution y of Example 3.4.2 for for $\epsilon = 0.00001$

Chapter 4: Conclusion

In this thesis we study two classes of fractional nonlinear Volterra integro-differential type of singularly perturbed problems which are the first order and the second order. The first order class has the form

$$\epsilon D^{\alpha}y + u(x,y) + \int_0^x K(x,t)v(t,y)dt = f(x), x \in (0,1), 0 < \alpha \leq 1,$$

subject to

$$y(0) = y_0$$

while the second order class has the form

$$\epsilon D^{\alpha}y + u(x,y)y' + \int_0^x K(x,t)v(t,y)dt = f(x), x \in (0,1), 01 < \alpha \leq 2,$$

subject to

$$y(0) = y_0, y(1) = y_1$$

where $\epsilon > 0$ is a small positive parameter, y_0 is constant, and $K(x,t)$ and $f(x), u(x,t), v(x,t)$ are smooth functions.

In chapter one, we study the classes of first order and second order fractional nonlinear Volterra integro-differential type of singularly perturbed problems. We present some preliminaries which we used in this thesis such as definition of Caputo derivative and its properties. In addition, we present the main definitions of the nonlinear Volterra integro-differential type and the singularly perturbed problems.

In chapter two, we present some theoretical results such as the maximum principle, stability of the numerical scheme, and the uniqueness of the proposed problem. We derive the necessary kernel to be able to implement the reproducing kernel method. Also, we derive the reproducing kernel method for the proposed problem. Two numerical examples are presented to show the efficiency of the numerical scheme.

In Chapter three, we study the classes of second order fractional nonlinear Volterra integro-differential type of singularly perturbed problems. We present some theoretical results such as the maximum principle, stability of the numerical scheme, and the uniqueness of the second order problem. We derive the necessary kernel to be able to implement the reproducing kernel method. Also, we derive the reproducing kernel method for the proposed problem. Two numerical examples are presented to show the efficiency of the numerical scheme.

Theoretical and numerical results show that the reproducing kernel method is working very efficiently especially when ϵ very small. We believe that this technique will work very efficiently for the higher order problem. However, we leave it for the future work.

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List of Publications

Syam, M. I., & Abu Omar, M. (2018). A Numerical Method for Solving a Class of Nonlinear Second Order Fractional Volterra Integro-Differntial Type of Singularly Perturbed Problems. Mathematics, 6(4), 48.

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