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AN EFFICIENT METHOD FOR SOLVING SINGULARLY PERTURBED TWO POINTS FRACTIONAL BOUNDARY-VALUE PROBLEMS

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College of Science

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An Efficient Method for Solving Singularly Perturbed Two Points Fractional Boundary-Value Problems

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This thesis is submitted in partial fulfillment of the requirements for the degree of Master of Science in Mathematics

Under the Supervision of Professor Muhammed Ibrahem Syam

May 2015

Declaration of Original Work

I, Suha Atallah Abdelhadi, the undersigned, a graduate student at the United Arab Emirates University (UAEU), and the author of this PhD dissertation, entitled "*An Efficient Method for Solving Singularly Perturbed Two Points Fractional Boundary-Value Problems*", hereby, solemnly declare that this thesis is an original research work that has been done and prepared by me under the supervision of Professor Muhammed I. Syam, in the College of Science at the UAEU. This work has not been previously formed as the basis for the award of any academic degree, diploma or a similar title at this or any other university. The materials borrowed from other sources and included in my thesis have been properly cited and acknowledged.

Student's Signature___________________________ Date _______________

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Abstract

In this thesis, we present numerical method for approximating the solutions of singularly perturbed two points boundary value problems in both cases: ordinary derivatives and fractional derivatives. We use the Caputo derivation for the fractional case. The method starts with solving the reduced problem then the boundary layer correction problem. A series method; namely, the Adomian decomposition method is used to solve the boundary layer correction problem, and then the series solution is approximated by the $[m, m]$ Pade' approximation of order. Numerical and theoretical results are presented to show the efficiency of the method. Singularly perturbed problems arise frequently in many real-life applications and they are among the hardest numerical approximation problems. Fractional Calculus has been in the minds of mathematicians for 300 years and still contains many mesteries. In recent decades, fractional calculus has been the object of ever increasing interest, due to its applications in different areas of science and engineering.

Keywords: Fractional Calculus, Caputo fractional derivative, Adomian decomposition Method, Pade' approximation, and Reduced layer correction Method.

Title and Abstract (in Arabic)

طريقة فعالة لحل المعادلات التفاضلية الكسرية المحيطية المعتلة

الملخص

في هذه الأطروحة، قمنا بعرض طريقة عددية لتقريب حلول المعادلات التفاضلية المحيطية المعتلة في الحالتين العادية والكسرية. تم استخدام مشتقة كوبوتو للحالة الكسرية. تتكون النظرية من جزئين و هما حل معادلة من الدرجة الأولى ثم جزء التصحيح عند حد الفترة المعطاة ثم نستخدم طريقة حل المتتاليات و المسمى بنظرية أدوميان كومبوزشن و بعدها نستخدم نظرية بادي للتقريب. تم عرض بعض النتائج العددية والنظري للتأكد من فعالية الطريقة. يبرز هذا النوع من المسائل باستمرار في تطبيقات حياتية و يتواجد في مسائل التقريب العددية الصعبة. كان علم التفاضل و التكامل الكسري في عقول علماء الرياضيات من قبل ثلاثمائة عام و ما زال يحتوي على غموض إلا أنه ينال اهتماما متزايدا في القرن الحالي نظرا لتطبيقاته في مجالات العلوم و الهندسة المختلفة.

الكلمات الرئيسة: التفاضل والتكامل الكسري، مشتقة كبوتو الكسرية، نظرية أدوميان ديكومبوسيشن، التقريب باستخدام بادي، ونظرية ريديوس لاير كوريكشن.

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A special thanks to my family. Words can’t express how grateful I am to my father, my mother, all my brothers and their wives, my sister and her husband for all of the sacrifices that you’ve made on my behalf. Your prayer for me was what sustained me thus far. I would also like to thank all of my friends who supported me in writing and incented me to strive towards my goal.

Dedication

This work is dedicated to my father Atallah Abdelhadi, my mother Sara Abdelhadi, my advisor, Professor Muhammed I. Syam, my co-advisor, Dr. Qasem M. Al-Mdallal, my brothers and sister, and to all my ralatives and friends.

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Chapter 1: Introduction

In this thesis, we present a numerical approach for solving a class of fractional singularly perturbed two points boundary value problems of the form

$$-\varepsilon D^{\alpha}y(x) + u(x,y)y'(x) + v(x,y)y(x) = 0, \;\; x\epsilon I := [0,1], \;\; 1 < \alpha \leq 2, \qquad (1.1)$$

subject to

$$y(0) = \beta_1, \; y(1) = \beta_2 \qquad (1.2)$$

where $\varepsilon > 0$ is a small positive parameter, β_1, β_2 are given constants, $u(x,y), v(x,y)$ are sufficiently smooth functions such that $u\big(x,y(x)\big) \neq 0$ for all $x\epsilon I$. In Equation (1.1), D^{α} denotes the Caputo fractional derivative.

The proposed numerical technique consists of two steps. In the first set, we get the reduced problem by setting $\varepsilon = 0$. In this case, problem (1.1) becomes a first order ordinary differential equation. We assume that the solution of the reduced problem satisfies the second boundary condition. This solution behaves like the solution of problem (1.1) - (1.2) on most of the interval (0,1] except for small interval around $x = 0$. To overcome this problem, we apply the second step which is the boundary layer correction by stretching the coordinate x by measure of a scaling parameter w. Then, we rewrite problem (1.1) in terms of w. By setting $\varepsilon = 0$, we get a second order fractional differential equation. We will redesign the boundary conditions to get the second solution. The general solution of problem (1.1) – (1.2) will be a combination of the solutions of the two steps.

We organize this thesis as follows. In chapter one, we present the main definitions and concepts which we will use herein. Caputo derivative, Adomain decomposition approach, and Pade' approximation. In chapter two, we study problem (1.1)-(1.2) when $\alpha = 2$. We discuss the cases when it is linear and nonlinear. Similar study is given in chapter three for $1 < \alpha < 2$. Numerical and theoretical results will be presented in chapters two and three.

1.1 The Gamma Function

In this section, we present the definition of the gamma function which will be used in fractional derivatives.

One of the important functions is the gamma function which is defined by

$$\Gamma(z) = \int_{0}^{+\infty} e^{-t} t^{z-1} dt.$$

The following are some basic properties of the gamma function:

1) $\Gamma(n) = (n-1)!\ , n = 1,2, \ldots$
2) $\Gamma(n+1) = n\Gamma(n),\ n > 0$

To explain the definition of the Gamma function, we compute $\Gamma(1/2)$ using the definition

$$\Gamma(1/2) = \int_{0}^{+\infty} e^{-t} t^{\frac{1}{2}-1} dt$$

$$= \int_{0}^{+\infty} \frac{e^{-t}}{\sqrt{t}} dt.$$

Making the substitution $t = x^2$, we get

$$\Gamma(1/2) = \int_{0}^{+\infty} e^{-t} t^{\frac{1}{2}-1} dt$$

$$= 2\int_{0}^{+\infty} \frac{e^{-t}}{\sqrt{t}} dt$$

then

$$[\,\Gamma(1/2)]2 \;=\; [\,2\,\int_0^{+\infty} e^{-x^2}dx]\,[2\,\int_0^{+\infty} e^{-y^2}dy].$$

$$= 4\int_0^{+\infty}\int_0^{+\infty} e^{-(x^2+y^2)}dxdy$$

Let $x = \gamma\ cos\ \theta\ and\ y = r\sin\theta$. Then

$$[\,\Gamma(1/2)]^2 = 4\int_0^{\pi/2}\int_0^{+\infty} re^{-r^2}drd\theta$$

$$= 4(\int_0^{\pi/2} d\theta)(\int_0^{+\infty} re^{-r^2}dr)$$

$$= 4\left(\frac{\pi}{2}\right)\left(\frac{1}{2}\right) = \pi.$$

Thus, $\Gamma(1/2) = \sqrt{\pi}$.

1.2 Introduction to Fractional Calculus

Fractional calculus is three centuries old as the conventional calculus, but not very popular amongst science and/or engineering communities. The beauty of this subject translates the reality of nature better! Therefore to make this subject available as popular subject and engineering community, adds another dimension to understand or describe basic nature in better way. Perhaps fractional calculus is what nature understands and to talk with nature in this language is therefore efficient. For past three centuries this subject was with mathematicians and only in last few years, this is pulled to several (applied) fields of engineering and science and economics. However recent attempt is on to have definition of fractional derivative as local operator specifically to fractal science theory. Next decade will see several applications based on this three hundred years (old) new subject, which can be thought of as superset of fractional differintegral calculus, the conventional integer order calculus being a part of it. Differintegration is operator doing differentiation and sometimes integrations in a general sense. Also the applications and discussions are limited to fixed fractional order differintegrals and the variable order of differintegration is kept as future research subject. Perhaps the Fractional Calculus will be the calculus of 21^{st} century.

Fractional order systems, or systems containing fractional derivatives and integrals, have been studied by many in engineering and science area. During the period 1922-1990, many reliable discussions devoted specifically to the subject. It should be noted that there are growing number of physical systems whose behavior can be compactly described using fractional calculus system theory. Of specific interest to electrical engineers are long electrical lines, electrochemical process, dielectric polarization, colored noise, viscoelestic materials, Chaos and electromagnetism fractional poles. For more details, see [11].

There are several definitions for the fractional derivative. In this thesis, we focus only on one of them, namely, the Caputo fractional operator Podlubny [10].

First, we define the Rieman- Liouville fractional integral operator.

Definition 1.2.1. The Riemann-Liouville fractional integral operator I_a^α of order $\alpha > 0$ on the usual Lebsgue space $L_1[a,b]$ is given by

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau, \qquad (1.2.1)$$

where $\Gamma(\mathrm{z}) = \int_0^\infty \mathrm{t}^{\mathrm{z}-1} \mathrm{e}^{-\mathrm{t}} \mathrm{dt}$ is the Euler Gamma function.

For any $f \in L_1[a,b], \alpha, \beta \geq 0,\ and\ \gamma > -1,$ the following properties hold:

1) I_a^α exists for any $t \in [a,b]$,
2) $I_a^\alpha I_a^\beta = I_a^{\alpha+\beta}$,
3) $I_a^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}$.

Definition 1.2.2. Suppose that $\alpha > 0,\ t > a,\ \alpha, a, t \in \mathbb{R}$. The fractional operator

$$D^\alpha f(t) \coloneqq \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, & n-1 < \alpha < n \in \mathbb{N}, \\ \frac{d^n}{dt^n} f(t), & \alpha = n \in \mathbb{N}, \end{cases} \qquad (1.2.2)$$

is called the Caputo fractional derivative or Caputo fractional differential operator of order α .

This operator is introduced by the Italian mathematician Caputo in 1967, see Caputo [5].

Now, let's compute $D^{\frac{1}{2}}t$ by using definition (1.2.2).

Example 1.2.1. Let $\alpha = \frac{1}{2}$ and $f(t) = t$. Then, for $n = 1$, applying definition (1.2.2) gives

$$D^{1/2}t = \frac{1}{\Gamma(\frac{1}{2})}\int_0^t \frac{1}{(t-\tau)^{1/2}}\,d\tau.$$

Taking into account the properties of the Gamma function and using the substitution $u = t - \tau,$ the final result for the Caputo fractional derivative of the function $f(t) = t$ is obtained as

$$D^{1/2}t = \frac{1}{\sqrt{\pi}}\int_0^t \frac{1}{(t-\tau)^{\frac{1}{2}}}\,d\,\tau$$

$$= -\frac{1}{\sqrt{\pi}}\int_{\sqrt{t}}^0 \frac{du}{\sqrt{u}}$$

$$= \frac{1}{\sqrt{\pi}}\int_0^{\sqrt{t}} \frac{du}{\sqrt{u}}$$

$$= \frac{2}{\sqrt{\pi}}(\sqrt{t} - 0\,).$$

Thus, it holds

$$D^{1/2}t = \frac{2\sqrt{t}}{\sqrt{\pi}}. \tag{1.2.3}$$

It is worth to mention that the operator $D^n,\ n \in \mathbb{N}$ used in the following sections is the standard integer-order differentiation operator, i.e., $D^n = \frac{d^n}{dt^n}$.

Lemma 1.2.1 *Let* $n-1 < \alpha < n, n \in \mathbb{N}, \alpha \in \mathbb{R}$ and $f(t)$ be such that $D^\alpha f(t)$ exist. Then

$$D^\alpha f(t) = I^{n-\alpha} D^n f(t). \tag{1.2.4}$$

Remark 1.2.1 (Linearity): *Let* $n-1 < \alpha < n, n \in \mathbb{N},\ \alpha, \lambda \in \mathbb{C}$ and the functions $f(t)$ and $g(t)$ be such that both $D^\alpha f(t)$ *and* $D^\alpha g(t)$ exist. The Caputo fractional derivative is a linear operator, i.e.,

$$D^\alpha\big(\lambda f(t) + g(t)\big) = \lambda D^\alpha f(t) + D^\alpha g(t). \tag{1.2.5}$$

Remark 1.2.3 (Non-commutation): *Suppose* that $n-1 < \alpha < n, m, n \in \mathbb{N}, \alpha \in \mathbb{R}$ and the functions $f(t)$ is such that $D^\alpha f(t)$ exists. Then in general

$$D^\alpha D^m f(t) = D^{\alpha+m} f(t) \neq D^m D^\alpha f(t). \tag{1.2.6}$$

Other properties for the Caputo fractional derivative are given below:

1) $I^\alpha D^\alpha f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0^+)\frac{t^k}{k!}$,
2) $D^\alpha I^\alpha f(t) = f(t)$,
3) $D^\alpha c = 0$, where c is constant,
4) $D^\alpha t^\gamma = \begin{cases} 0, & \gamma < \alpha,\ \gamma \in \{0,1,2,\dots\} \\ \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} t^{\gamma-\alpha}, & otherwise \end{cases}$.

For more details, see [8].

1.3 Adomian decomposition Method

The Adomian decomposition method (ADM) is a well–known systematic method for practical solution of linear or nonlinear and deterministic or stochastic operator equations, including ordinary differential equations (ODEs), partial differential equations (PDEs), integral equations, etc. The ADM is a powerful technique, which provides efficient algorithms for analytic approximate solutions and numeric simulations for real-world applications in the applied sciences and engineering. It permits us to solve both nonlinear initial value problems (IVPs) and boundary value problems (BVPs) without unphysical restrictive assumptions such as required by linearization, perturbation, and so forth. The method accurately computes the series solution in a rapidly convergent series with components that are elegantly computed. The accuracy of the analytic approximation solutions obtained can be verified by direct substitution.

The main advantage of the method is that it can be applied directly for all types of differential and integral equations, linear or nonlinear, homogeneous or inhomogeneous, with constant coefficients or with variable coefficients. Another important advantage is that the method is capable of greatly reducing the size of computation work while still maintaining high accuracy of the numerical solution.

Consider the following second order initial value problem of the form

$$L(y) - N(y) = f(t), \; y = y(x) \tag{1.3.1}$$

$$y(0) = c_0, \qquad y'(0) = c_1$$

where L is the linear operator and N is the nonlinear operator.

The nonlinear term, $N(y)$, can be expressed by an infinite series of the Adomian polynomials

$$N(y) = \sum_{n=0}^{\infty} A_n, \tag{1.3.2}$$

where

$$A_n(y_0, y_1, \dots, y_n) = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left(N\left[\sum_{k=0}^{\infty} \lambda^k y_k\right]\right)\Big|_{\lambda=0},\ n \geq 0.$$

Next, write $y(x)$ as

$$y(x) = \sum_{k=0}^{\infty} y_k(x). \tag{1.3.3}$$

From equations (1.3.1)-(1.3.3) one can see that

$$y(x) = L^{-1}\big(N(y)\big) + L^{-1}\big(f(t)\big). \tag{1.3.4}$$

Thus,

$$\sum_{k=0}^{\infty} y_k = \sum_{n=0}^{\infty} L^{-1}(A_n) + L^{-1}\big(f(t)\big). \tag{1.3.5}$$

The iterates are determined by the following recursive way

$$y_0 = L^{-1}\big(f(t)\big) + \varphi(t)$$

$$y_n = L^{-1}(A_{n-1}),\ n = 1,2,\dots \tag{1.3.6}$$

To explain the idea of ADM, we discuss the following example.

Example 1.3.1. Consider the following initial value problem

$$\frac{dy}{dt} = -y^2(t) + 1,\ y(0) = 0.$$

The exact solution is $y(t) = \frac{e^{2t}-1}{e^{2t}+1}$. Following the procedure described above, one can see that

$$L(y) = y',\ N(y) = -y^2, \text{ and } f(x) = 1.$$

Then, $L^{-1}(\,.\,) = \int_0^t .\,dx$. Thus,

$$A_0 = y_0^2$$

$$A_1 = 2y_0y_1$$

$$A_2 = 2y_0y_2 + y_1^2$$

$$A_3 = 2y_0y_3 + 2y_1y_2$$

and so on. By using formula (1.3.6) we get

$$y_0 = t$$

$$y_1 = \frac{1}{3}t^3$$

$$y_2 = \frac{2}{15}t^5$$

$$y_3 = \frac{-17}{315}t^7$$

$$y_4 = \frac{62}{2835}t^9$$

and so on. Thus,

$$y(t) = t - \frac{1}{3}t^3 + \frac{2}{15}t^5 - \frac{17}{315}t^7 + \frac{62}{2835}t^9 + \cdots$$

Graphs of the exact and the approximate solutions are given in Figure (1.3.1).

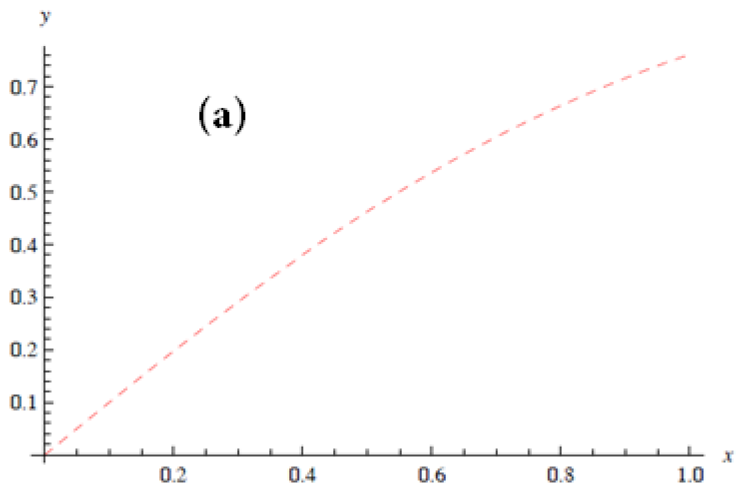

Figure 1.3.1a: The graph of the approximate solution

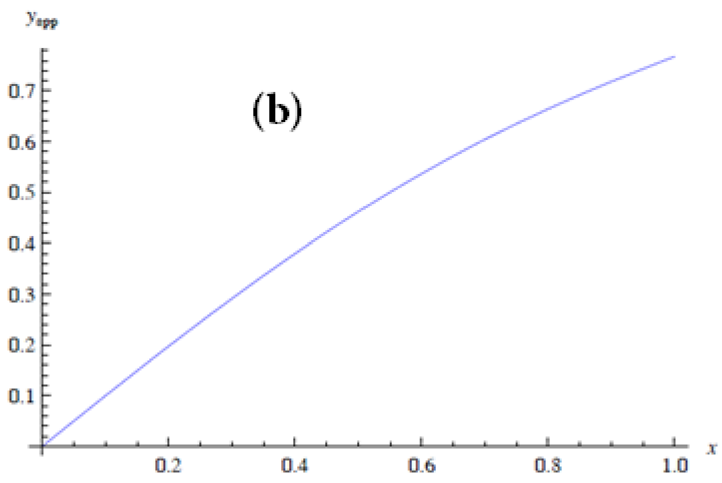

Figure 1.3.1b: The graph of the exact solution

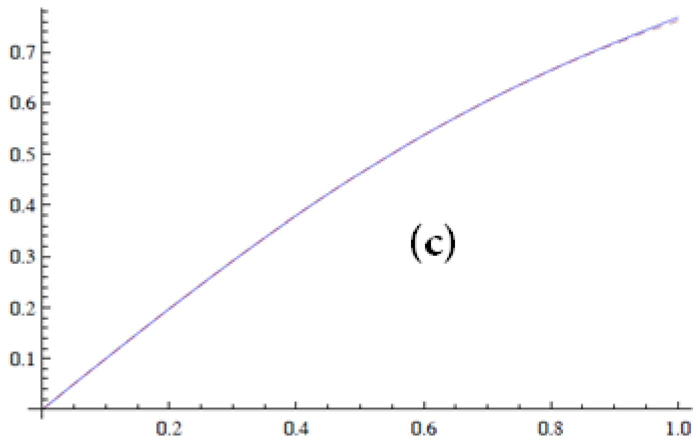

Figure 1.3.1c: The graph of the approximate solution and the exact solution

For more details, see Adomian [1], Syam [15], and Wazwaz [16,17].

1.4 Rational Function Approximation

The class of algebraic polynomials has some distinct advantages for use in approximation:

- There are sufficient number of polynomials to approximate any continuous function on a closed interval to within an arbitrary tolerance.
- Polynomials are easily evaluated at arbitrary values.
- The derivatives and integrals of polynomials exist and are easily determined.

The disadvantage of using polynomials for approximation is their tendency to oscillate. This often causes error bounds in polynomial approximation to significantly exceed the average approximation error, since error bounds are determined by the maximum approximation error. We now consider a method that spread the approximation error more evenly over the approximation interval, this technique involve rational functions.

A rational function r of degree N has the form

$$r(x) = \frac{P(x)}{Q(x)} = \frac{P_0 + P_{1X} + \cdots + P_n X^n}{q_0 + q_1 x + \cdots + q_m x^m}.$$

where $P(x)$ and $Q(x)$ are polynomials whose degrees sum to N. Suppose r is used to approximate a function F on a closed interval I containing zero. For r to be defined at zero requires that $q_0 \neq 0$. In fact, we can assume that $q_0 = 1$, for if this is not the case we simply replace $P(x)$ by $P(x)/\,q_0$. Consequently, there are $N + 1$ parameters $q_1, q_2, \dots, q_m, p_0, p_1, \dots, p_n$ available for the approximation of f by r. The Padé approximation technique, which is the extension of Taylor polynomial approximation to rational functions, choose the $N + 1$ parameters so that

$$f^{(k)}(0) = r^{(k)}(0) \text{ , for each } k = 0, 1, \ldots\ldots, N.$$

when $n = N$ and $m = 0$, the Padé approximation is just the Nth Maclaurin polynomial. Consider the difference

$$f(x) - r(x) = f(x) - \frac{P(x)}{Q(x)} = \frac{f(x)Q(x) - P(x)}{Q(x)} = \frac{f(x)\sum_{i=0}^{m} q_i x^i - \sum_{i=0}^{n} p_i x^i}{Q(x)}$$

and suppose f has the Maclaurin series expansion $f(x) = \sum_{i=0}^{\infty} a_i x^i$. Then

$$f(x) - r\,(x) = \frac{\sum_{i=0}^{\infty} a_i x^i \sum_{i=0}^{m} q_i x^i - \sum_{i=0}^{n} p_i x^i}{Q(x)}.$$

The object is to choose the constants $q_1, q_2, \ldots, q_m$ and $p_0, p_1, \ldots., p_n$ so that

$f^{(k)}(0) - r^{(k)}(0) = 0$, for each $k = 0, 1, \ldots\ldots, N$. Coefficients are determined by setting

$$f(x) = r_N(x) .$$

where

$$c_i = \frac{f^{(i)}(0)}{i!}.$$

Thus,

$$c_0 + c_1 x + c_2 x^2 + \cdots + c_N x^N = \frac{p_0 + p_1 x + \cdots + p_n x^n}{1 + q_1 x + \cdots + q_m x^m}$$

and equating coefficients

$$p_0 = c_0 \quad \text{and} \quad c_{n+1} + q_1 c_n + \cdots + q_m c_{n-m+1} = 0$$

$$p_1 = q_1 c_0 + c_1$$

$$p_2 = q_2 c_0 + q_1 c_1 + c_2$$

$$p_n = q_m c_{n-m} + \cdots + q_1 c_{n-1} + c_n$$

Example 1.4.1. Find the [5,5] Pade' approximation of $\arctan(x)$.

First, we start by the Maclaurin series of arctan:

$$f(x) \approx x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9$$

and

$$x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 = \frac{p_0+p_1x+p_2x^2+p_3x^3+p_4x^4+p_5x^5}{1+q_1x+q_2x^2+q_3x^3+q_4x^4+q_5x^5}.$$

x^0 through x^5	x^6 through x^{10}
$p_0 = 0$	$\frac{1}{5}q_1 - \frac{1}{3}q_3 = 0$
$p_1 = 1$	$-\frac{1}{7}+\frac{1}{5}q_2 - \frac{1}{3}q_4 = 0$
$p_2 = q_1$	$-\frac{1}{7}q_1 + \frac{1}{5}q_3 - \frac{1}{3}q_5 = 0$
$p_3 = -\frac{1}{3} + q_2$	$\frac{1}{9} - \frac{1}{7}q_2 + \frac{1}{5}q_4 = 0$
$p_4 = -\frac{1}{3}q_1 + q_3$	$\frac{1}{9}q_1 - \frac{1}{7}q_3 + \frac{1}{5}q_5 = 0$
$p_5 = \frac{1}{5} - \frac{1}{3}q_2 + q_4$	

Table 1.4.1: Relation between the coefficients of the Pade' approximation

Thus, [5,5] Padé approximation of $\arctan(x)$ is

$$\frac{x+\frac{7}{9}x^3+\frac{64}{945}x^5}{1+\frac{10}{9}x^2+\frac{5}{21}x^4}.$$

Graphs of Pade' approximation and Maclaurin series of arctan are given in figure (1.4.1).

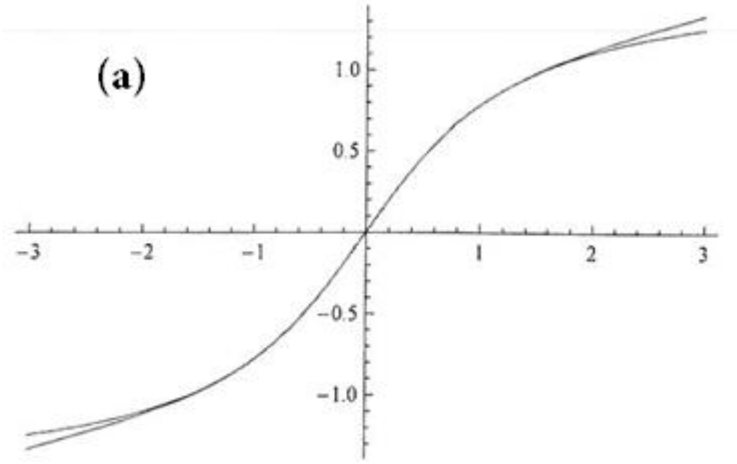

Figure 1.4.1a: The graph of arctan(x) and Pade’ approximation

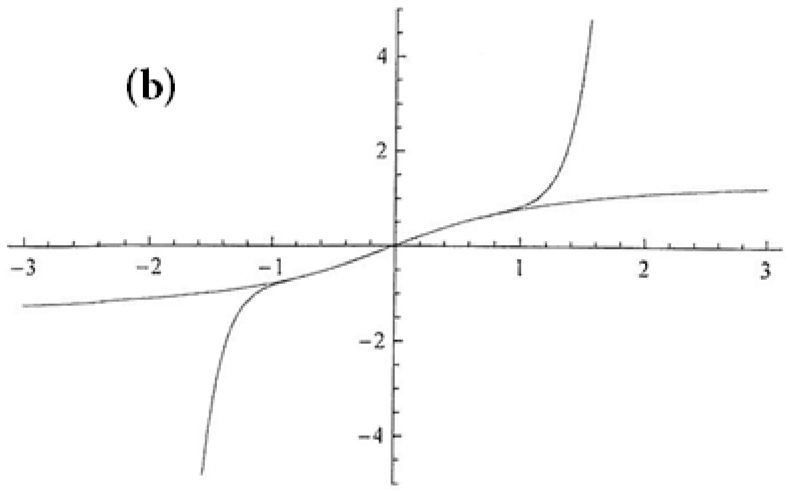

Figure 1.4.1b: The graph of arctan(x) and Maclurian series

We notice that the Pade’ approximation gives better approximation than the Maclaurin series.

Chapter 2: Boundary Layers of Ordinary Boundary Value Problems

Singularly perturbed boundary value problems often arise in applied sciences and engineering, reaction diffusion equations are one good example, Shao [12]. A well-known fact is that the solution of such problems displays sharp boundary or interior layers when the singular perturbation parameter ε is very small. Hence the primary objective in singular perturbation analysis of such problems is to develop asymptotic approximations to the true solution that are uniformly valid with respect to the perturbation parameter, Chandra and Kumar [6]. Numerically, the presence of the perturbation parameters leads to difficulties when classical numerical techniques are used to solve such problems and convergence will not be uniform, see [7]. This is due to the presence of boundary layers in these problems, see for example O'Mally [9].

This suggests having numerical methods where the error in the approximate solution tends to zero independently of the parameter ε ; that is, uniform convergence is desired, see Attili [4].

In this chapter, we present a numerical method for solving a class of nonlinear singularly perturbed two-point boundary value problems with Known boundary layer at one end of the form

$$\varepsilon \frac{d^2y}{dx^2} + u(x,y)\frac{dy}{dx} + v(x,y)y = f(x), \quad x \in (0,1)$$

subject to

$$y(0) = \beta_1, \qquad y(1) = \beta_2$$

Using singular perturbation analysis the method consists of solving two problems; namely, a reduced problem and a boundary layer correction problem. We use Padé approximation to obtain the solution of the latter problem and to satisfy the condition at infinity. Numerical results will be given to illustrate the method.

We will divide this chapter into two sections. In section one, we study the linear case when u and v are functions of x only while in section two we discuss the nonlinear case.

2.1 The Linear Problem

Let us consider the linear two-point boundary value singular perturbation problem of the form

$$L(y) = \varepsilon \frac{d^2y}{dx^2} + u(x)\frac{dy}{dx} + v(x)y = f(x); \quad x \epsilon (0,1) \tag{2.1.1}$$

subject to

$$y(0) = \beta_1, \; y(1) = \beta_2 \tag{2.1.2}$$

where $0 < \varepsilon \ll 1$, α and β are given numbers, $u(x)$ *and* $v(x)$ are sufficiently smooth functions with $u(0) = 0$, $u'(0) \leq 0$, $|u(x)| \geq u_0 > 0$, $|u'(x)| \geq \frac{|u'(0)|}{2}$ and $-v(x) \geq v_0 > 0$ for every $x \in [0,1]$. These conditions imply the existence of a unique solution to (2.1.1)-(2.1.2) exhibiting a boundary layer at one end.

If we set $\varepsilon = 0$, we obtain the reduced problem

$$u(x){y_1}' + v(x)y_1 = f(x); \; y_1(1) = \beta_2, \quad x \epsilon (0,1) \tag{2.1.3}$$

To solve (2.1.3) and since it is first order, we need to impose one of the boundary conditions. Thus, we take $y(1) = \beta_2$ and drop the one at $x = 0$. The resulting problem is easily solvable since it is linear.

Over most of the interval, this solution behaves like the solution of (2.1.1)-(2.1.2) but at the other end around $x = 0$, there is a region in which the solution varies greatly from the solution of (2.1.1)-(2.1.2). To satisfy the other condition, we will use the substitution $x = \varepsilon t$, the stretching transformation which means

$$dx = \varepsilon dt,\ \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{\epsilon}\frac{dy}{dt} \text{ and } \frac{d^2y}{d^2x} = \frac{1}{\epsilon^2}\frac{d^2y}{dt^2}.$$

This transforms (2.1.1) into

$$\frac{1}{\epsilon}\frac{d^2y}{dt^2} + u(x)\frac{1}{\epsilon}\frac{dy}{dt} + v(x)y = f(x)$$

or

$$\frac{d^2y}{dt^2} + u(\varepsilon t)\frac{dy}{dt} + \varepsilon v\,(\varepsilon t)y = f(\varepsilon t). \tag{2.1.4}$$

Taking $\varepsilon = 0$ leads to

$$\frac{d^2y}{dt^2} + u(0)\frac{dy}{dt} = 0$$

or

$$y_2'' + u(0)y_2' = 0,$$

which is called the boundary layer correction problem. It compensates for the fact that the solution to (2.1.3) does not satisfy the boundary condition at $x = 0$ and this solution satisfies $\lim_{t\to\infty} y_2(t) = 0$. leading to the boundary condition to be imposed at $t = 0$; that is, $y_2(0) = \varepsilon - y_1(0)$. The boundary layer correction problem becomes

$$y_2'' + u(0)y_2' = 0 \tag{2.1.5}$$

subject to

$$y_2(0) = \alpha - y_1(0), \quad \lim_{t\to\infty} y_2(t) = 0. \tag{2.1.6}$$

Notice that this problem is independent of ε and at the same time it is not easy to apply the limit condition. For that reason we will employ the shooting method to solve (2.1.5)-(2.1.6). In this case, assume $y_2'(0) = \gamma$ and hence (2.1.5)-(2.1.6) will be transformed to an initial value problem of the form

$$y_2'' + u(0)y_2' = 0 \tag{2.1.7}$$

subject to

$$y_2(0) = \alpha - y_1(0), \; y_2'(0) = \gamma. \tag{2.1.8}$$

The value of γ will be adjusted iteratively until the limit condition is satisfied up to some tolerance. As a result the solution to the original problem (2.1.1)-(2.1.2) will be a combination of the reduced and the boundary layer correction problems; that is,

$$y(x) = y_1(x) + y_2\left(\frac{x}{\varepsilon}\right). \tag{2.1.9}$$

Based on the above discussion, we will have the following result:

Theorem 2.1. The solution to the system (2.1.1)-(2.1.2) is given as

$$y(x) = y_1(x) + y_2\left(\frac{x}{\varepsilon}\right) + o(\varepsilon).$$

Using the transformation $x = \varepsilon t$, where y_1 and y_2 are respectively the solutions to

$$u(x)y_1' + v(x)y_1 = f(x);\;\; y_1(1) = \beta_2, \quad x \in (0,1)$$

and

$$y_2'' + u(0)y_2' = 0$$

subject to

$$\mathrm{y}_2(0) = \alpha - \mathrm{y}_1(0), \qquad \mathrm{y}_2'(0) = -\mathrm{u}\,(0)\big(\alpha - \mathrm{y}_1(0)\big).$$

The proof follows from the discussion proceeded the theorem. For the condition $y_2'(0) = -\mathrm{u}(0)(\alpha - y_1(0))$ and since the problem is linear, then using reduction of order, the solution to (2.1.5)-(2.1.6) is given by

$$y_2(t) = \frac{-\gamma}{u(0)}e^{-u(0)t} + \alpha - y_1(0) + \frac{\gamma}{u(0)}.$$

Taking the limit $\lim_{t\to\infty} y_2(t) = 0$ leads to

$$\alpha - y_1(0) + \frac{\gamma}{u(0)} = 0 \qquad (2.1.6)$$

or

$$\gamma = -u(0)\big(\alpha - y_1(0)\big).$$

Numerically, the [2,2] Pade’ polynomial has the form

$$\bar{y}_2 = \frac{P_2(t)}{Q_2(t)} = \frac{p_0 + p_1 t + p_2 t^2}{1 + q_1 t + q_2 t^2}$$

with

$p_0 = \alpha - y_1(0),\ p_1 = \left(\alpha - y_1\ (0)\right)\frac{u(0)}{2} + \ \gamma,\ P_2 = \ \frac{u(0)^2}{12}\left(\alpha - y_1(0)\right) +$

$\frac{u(0)}{2}\gamma,\ q_1 = \ \frac{u(0)}{2}$ and $q_2 = \ \frac{u(0)^2}{12}$. Differentiating $\overline{y}_2^{\,2,2}$ with respect to t, we obtain

$$\bar{y}_2\ '\ (t) = \frac{[1 + q_1 t + q_2 t^2](p_1 + 2p_2 t) - \ [p_0 + p_1 t + p_2 t^2](q_1 + 2q_2 t)}{[1 + q_1 t + q_2 t^2]^2}$$

Evaluating at $t = 0$, leads to

$$\bar{y}_2\ '_{(0)} = \frac{p_1 - p_0 q_1}{1} = \left(\alpha - y_1(0)\right)\frac{u(0)}{2} + \gamma - \left(\alpha - y_1(0)\right)\frac{u(0)}{2} = \gamma$$

which agrees with $y_2'(0) = \gamma$.

Taking the limit of $\bar{y}\ _2^{\,2,2}$ as t approaches infinity implies

$$\lim_{t\to\infty} \bar{y}_2 \quad = \frac{p_2}{q_2} \approx \gamma.$$

2.2 The Nonlinear Problem

Consider the class of nonlinear singular perturbation problems of the form

$$\varepsilon\frac{d^2y}{dx^2} + u(x,y)\frac{dy}{dx} + v\ (x,y)y = f(x); \quad x \in (0,1) \tag{2.2.1}$$

subject to

$$y(0) = \ \beta_1,\ y(1) = \ \beta_2 \tag{2.2.2}$$

where $0\ <\ \varepsilon \ll 1$, α and β are given numbers, $u\ (x,y)$ and $v\ (x,y)$ are sufficiently smooth functions with $u(x,y(x))\ \geq\ M\ >\ 0$ for every $x\ \in\ [\,0,1]$.

Once again if we set $\varepsilon = 0$, we obtain the reduced problem

$$u\ (x\ ,y_1)y_1{}' + v(x,y_1)y_1 = f(x).$$

As before, we impose the condition at $x = 1$ since the boundary layer is in the neighborhood of $x = 0$ leading to the reduced problem of the form

$$u\,(x\,, y_1)y_1' + v(x, y_1)y_1 = 0;\; y_1(1) = \beta_2,\; x\,\epsilon\,(0,1). \tag{2.2.3}$$

The solution to this problem satisfies (2.2.1)-(2.2.2) on most of the interval (0,1) and a way from $x = 0$. If this problem is separable then it can be integrated easily and if not, any numerical method for initial value problem will be used to approximate the solution such that Taylor or Runge-Kutta methods.

Close to the boundary layer and to satisfy the other condition, we use as before the substitution $x = \varepsilon t$, the stretching transformation which transforms (2.2.1) into

$$\frac{d^2y}{dt^2} + u(\varepsilon t, y)\frac{dy}{dt} + \varepsilon v\,(\varepsilon t\,, y)y = \varepsilon f(\varepsilon t)$$

or

$$y_2'' + u(\varepsilon t\,, y_2)y_2' + \varepsilon v(\varepsilon t\,, y_2)y_2 = \varepsilon f(\varepsilon t).$$

Taking $\varepsilon = 0$ leads to

$$y_2'' + u(0, y_1(0) + y_2){y_2}' = 0 \tag{2.2.4}$$

subject to

$$y_2(0) = \alpha\, - y_1(0),\; \lim_{t\to\infty} y_2(t) = 0. \tag{2.2.5}$$

Using the shooting method, this problem can be replaced by the initial value problem of the form

$$y_2''\; +\; u(0, y_1(0)\; +\; y_2){y_2}' =\; 0 \tag{2.2.6}$$

subject to

$$y_2(0) = \alpha\, - y_1(0),\; {y_2}'(0) =\; \gamma. \tag{2.2.7}$$

The value of γ will be adjusted iteratively until the limit condition is satisfied up to some tolerance. As a result the solution to the original problem (2.2.1)-(2.2.2) will be a combination of the reduced and the boundary layer correction problems; that is,

$$y(x) = y_1(x) + y_2\left(\frac{x}{\varepsilon}\right).$$

To obtain γ by applying the limit condition $\lim_{t\to\infty} y_2(t) = 0$ and to solve (2.2.6)-(2.2.7), we approximate $y_2(t)$ by a series solution of the form $y_2(t) = \sum_{n=0}^{\infty} a_n t^n$ then substitute in (2.2.6)-(2.2.7). The resulting coefficients a_n depend on γ. We then use Padé approximation to approximate y_2(t) of the form $y_2(t) = \frac{P_n(t)}{Q_n(t)}$ and then apply the limit condition on the resulting Padé approximation to obtain γ.

2.3 Numerical Results

For numerical testing, we will consider the following example.

Example 2.3.1: Consider the second-order equation

$$\varepsilon y'' + y'(x) = 1 + 2x\,, 0 \le x \ \le 1; \quad y(0) = 0,\ y(1) = 1 \tag{2.3.1}$$

where the exact solution is $y(x) = x(x + 1 - 2\varepsilon\ ) + \frac{(2\varepsilon-1)(1-e^{\frac{-x}{\varepsilon}})}{1-e^{-1/\varepsilon}}$.

By setting $\varepsilon = 0$, we get the reduced form

$$y_1'(x) = 1 + 2x, \qquad y_1(1) = 1,$$

which has the solution

$$y_1(x) = \ x^2 + x - 1.$$

The boundary layer correction problem is

$$y_2'' + y_2' = 0,\ y_2(0) = 1, {y'}_2(0) = \gamma. \tag{2.3.2}$$

Then,

$$y'_2 + y_2 = c_1$$

which implies that

$$y_2 = c_2 e^{-x} + c_1.$$

Using the initial condition, the solution will be

$$y_2(x) = -\gamma\, e^{-x} + 1 + \gamma$$

with $t = \frac{x}{\varepsilon}$, and $\lim_{x\to\infty} y_2(t) = 0$, implies that $\gamma = -1$.

Hence the solution of (2.3.1) is

$$y(x) \cong y_1(x) + y_2\left(\frac{x}{\varepsilon}\right)$$

$$= x^2 + x - 1 + e^{-\frac{x}{\varepsilon}}. \tag{2.3.3}$$

For $\varepsilon = 10^{-3}$, Table (2.3.1) represents the absolute error. Graphs of the approximate and the exact solutions are given in Figure (2.3.1).

x_i	$y(x_i)$	Approximate y_i	Absolute Error $\lvert y(x_i) - y_i \rvert$
0	0	0	0
0.1	-0.8882	-0.89	0.0018
0.2	-0.7584	-0.76	0.0016
0.3	-0.6086	-0.61	0.0014
0.4	-0.4388	-0.44	0.0012
0.5	-0.249	-0.25	0.001
0.6	-0.0392	-0.04	0.0008
0.7	0.1906	0.19	0.0006
0.8	0.4404	0.44	0.0004
0.9	0.7102	0.71	0.0002
1	1	1	0

Table 2.3.1: The absolute error between the exact and approximate solutions.

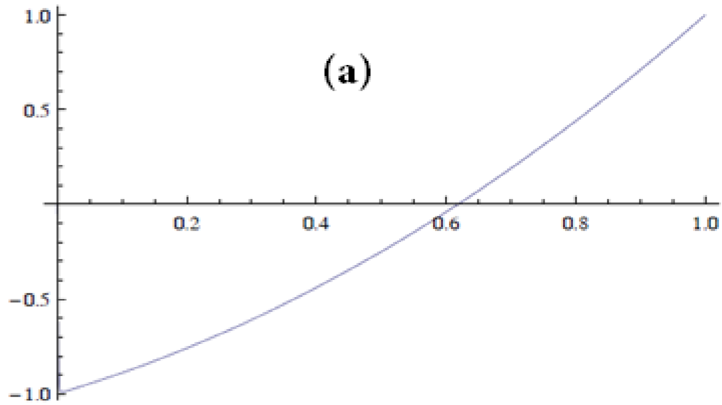

Figure 2.3.1a: The graph of the exact solution at $\varepsilon = 10^{-5}$

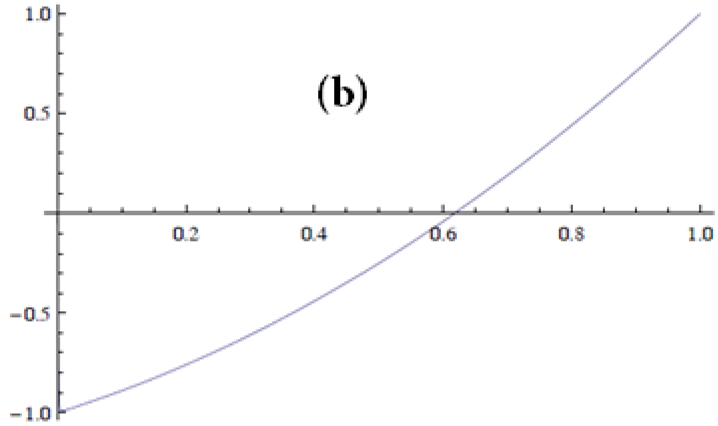

Figure 2.3.1b: The graph of the approximate solution at $\varepsilon = 10^{-5}$

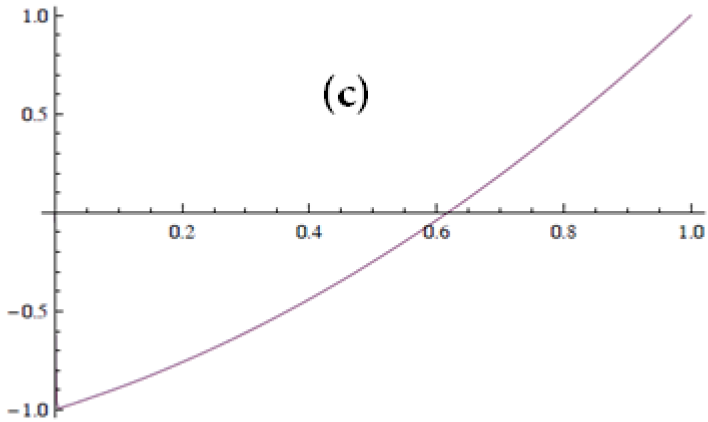

Figure 2.3.1c: The graph of the exact and the approximate solutions at $\varepsilon = 10^{-5}$

Chapter 3: Boundary Layers of Fractional Boundary Value Problems

In this chapter, we discuss a numerical solution of a class of non-linear fractional singularly perturbed two points boundary-value problem. The method of solution consists of solving reduced problem and boundary layer correction problem. A series method is used to solve the boundary layer correction problem, and then the series solution is approximated by the Pade' approximation of order $[m, m]$. Some theoretical results are established and proved. Three numerical examples are discussed to illustrate the efficiency of the present scheme, see [14,13,2,3].

3.1 Reduced and boundary layer correction method

In this section, we consider a class of fractional singularly perturbed two points boundary-value problems with Dirichlet boundary conditions of the form

$$-\varepsilon D^{\alpha} y(x) + u(x,y)y'(x) + v(x,y)y(x) = 0, \;\; x\epsilon I := [0,1], \;\; 1 < \alpha \leq 2, \quad (3.1.1)$$

subject to

$$y(0) = \beta_1, \;\; y(1) = \beta_2 \qquad (3.1.2)$$

where $\varepsilon > 0$ is a small positive parameter, $\beta_1, \; \beta_2$ are given constants, $u(x,y), v(x,y)$ are sufficiently smooth functions such that $u\big(x, y(x)\big) \neq 0$ for all $x\epsilon I$, and $y \,\epsilon\, L_1[a,b] := \left\{z : \; [a,b] \to \mathbb{R}, \left|\int_a^b z(t)dt < \infty\right.\right\}$. Here, D^{α} denotes the Caputo fractional derivative.

The numerical solution of the present problem is based on dividing the main problem (3.1.1) and (3.1.2) into two equivalent problems; namely, a reduced problem and a boundary layer correction problem. The reduced problem is first order initial value problem (IVP) which can be solved analytically. However, the boundary layer correction problem is a nonlinear regular IVP of order α which is solved using a series method; namely Adomain decomposition method (ADM) followed by the Pade' approximation method.

This approach consists of two steps. In the first step, we obtain the following reduced problem of (3.1.1) by setting $\varepsilon = 0$

$$u(x, y_1)y_1'(x) + v(x, y_1)y_1(x) = 0, \quad x \in I. \tag{3.1.3}$$

The solution of the first order differential Eq. (3.1.3) can not satisfy both boundary conditions in (3.1.2). For this reason, we will force the solution of problem (3.1.3) to satisfy the following condition

$$y_1(1) = \beta_2. \tag{3.1.4}$$

In practice, the solution of (3.1.3) and (3.1.4) behaves like the solution of (3.1.1) and (3.1.2) on most of the interval (0,1] except for small interval around $x = 0$ in which the solutions y_1 and y do not match. This problem is handled in the second step by introducing the boundary layer correction by stretching the coordinate x by means of a scaling parameter, say, w. We thus define

$$x = \sqrt[\gamma]{\varepsilon} w, \text{ where } \gamma = \alpha - 1. \tag{3.1.5}$$

Obviously, the first and second derivatives of y with respect to w should be given by

$$\frac{dy}{dx} = \frac{1}{\sqrt[\gamma]{\varepsilon}} \frac{dy}{dw} \text{ and } \frac{d^2y}{dx^2} = \frac{1}{\sqrt[\gamma]{\varepsilon^2}} \frac{d^2y}{dw^2}.$$

However, the effect of the transformation (3.1.5) on Caputo fractional derivative D^α is presented in the following lemma.

Lemma 3.1.1. *Let* $\varepsilon > 0$ and $x = \sqrt[\gamma]{\varepsilon}w$. Then $D^{\alpha}y(x) = \varepsilon^{\frac{-\alpha}{\gamma}}D^{\alpha}y(w)$, for $\alpha \in (1,2]$.

Proof. Let $x = \sqrt[\gamma]{\varepsilon}w$, then

$$D^{\alpha}y(x) = \frac{1}{\Gamma(2-\alpha)}\int_0^x (x-t)^{1-\alpha}y''(t)dt = \frac{1}{\Gamma(2-\alpha)}\int_0^{\sqrt[\gamma]{\varepsilon}w}\left(\sqrt[\gamma]{\varepsilon}w - t\right)^{1-\alpha}y''(t)dt$$

$$= \frac{\varepsilon^{\frac{1-\alpha}{\gamma}}}{\Gamma(2-\alpha)}\int_0^{\sqrt[\gamma]{\varepsilon}w}\left(w - \frac{t}{\sqrt[\gamma]{\varepsilon}}\right)^{1-\alpha} y''(t)dt \qquad (3.1.6)$$

Let $r = \frac{t}{\sqrt[\gamma]{\varepsilon}}$, then

$$dt = \sqrt[\gamma]{\varepsilon}dr \text{ and } \frac{d^2y}{dt^2} = \frac{1}{\sqrt[\gamma]{\varepsilon^2}}\frac{d^2y}{dr^2}.$$

Thus, Eq. (3.1.6) transforms to

$$D^{\alpha}y(x) = \frac{\varepsilon^{\frac{1-\alpha}{\gamma}}}{\Gamma(2-\alpha)}\int_0^w (w-r)^{1-\alpha}\frac{1}{\sqrt[\gamma]{\varepsilon^2}}\frac{d^2y}{dr^2}\sqrt[\gamma]{\varepsilon}dr = \varepsilon^{\frac{-\alpha}{\gamma}}D^{\alpha}y(w),$$

as desired.

Consequently, Eq. (3.1.1) transforms to

$$-D^{\alpha}y + \varepsilon^{\frac{\alpha-\gamma-1}{\gamma}}u\left(\sqrt[\gamma]{\varepsilon}w, y\right)\frac{dy}{dw} + \varepsilon^{\frac{\alpha-\gamma}{\gamma}}v\left(\sqrt[\gamma]{\varepsilon}w, y\right)y = 0. \qquad (3.1.7)$$

Since $\gamma = \alpha - 1$, we have

$$\frac{\alpha-\gamma-1}{\gamma} = 0 \text{ and } \frac{\alpha-\gamma}{\gamma} = \frac{1}{\alpha-1} > 0.$$

Consequently, Eq. (3.1.7) will be written as

$$-D^{\alpha}y+u\left(\sqrt[\alpha-1]{\varepsilon}w,y\right)\frac{dy}{dw}+\varepsilon^{\frac{1}{\alpha-1}}v\left(\sqrt[\alpha-1]{\varepsilon}w,y\right)y=0. \tag{3.1.8}$$

Since the solution of the reduced problem (3.1.3) and (3.1.4) does not satisfy the boundary condition at $x=0$, the solution of the boundary layer correction problem should approaches zero as w approaches infinity. Thus, the boundary layer correction problem, when $\varepsilon=0$, should has the form

$$-D^{\alpha}y_2+u(0,y_1(0)+y_2)y_2'=0, \tag{3.1.9}$$

with

$$y_2(0)=\ \beta_1-y_1(0),\ \lim_{w\to\infty}y_2(t)=0. \tag{3.1.10}$$

It should be noted that first condition in (3.1.10) is also required to ensure that the solution of (3.1.1) and (3.1.2) satisfies the condition at $x=0$. Note that, we implement the linear shooting method to transform problem (3.1.9) and (3.1.10) to the following fractional initial value problem

$$-D^{\alpha}y_2+u(0,y_1(0)+y_2){y'}_2=0 \tag{3.1.11}$$

$$y_2(0)=\ \beta_1-y_1(0),\ y_2'(0)=\theta, \tag{3.1.12}$$

where θ will be determined later using the condition at infinity. Since finding the exact solution for problem (3.1.11) and (3.1.12) is a difficult task, we implement the well-known Adomian decomposition method (ADM), for details about this method see [9-13]. To derive the algorithm which serves to solve Eqs. (3.1.11) and (3.1.12), we rewrite Eq. (3.1.11) in the form

$$Ly_2(w)=N, \tag{3.1.13}$$

where $L = D^{\alpha}$ and $N = N(w, y_2, y'_2)$ is a linear operator contains all other terms. As a result of lemma (3.1.1), we may apply I_0^{α} on Eq. (3.1.13) to have

$$y_2(w) = y_2(0) + \ y_2'(0)w - I_0^{\propto} N(w, y_2, y_2') \qquad (3.1.14)$$

Assuming the solution $y_2(w)$ is represented by an infinite series of the form

$$y_2(w) = \ \sum_{n=0}^{\infty} \bar{y}_n(w), \qquad (3.1.15)$$

and the term *N* by an infinite series of polynomials

$$N = \ \sum_{n=0}^{\infty} A_n \, , \qquad (3.1.16)$$

where A_n are the Adomian polynomials given by

$$A_n = \frac{1}{n!} \frac{d^n}{d\mu^n} \left[N \left(\sum_{i=0}^{\infty} \mu^i \bar{y}_i \right) \right]_{\mu=0} , \qquad (3.1.17)$$

then Eq. (3.1.14) has the form

$$\sum_{n=0}^{\infty} \bar{y}_n(w) = \ y_2(0) + y_2'(0)w - \sum_{n=0}^{\infty} I_0^{\propto} A_n(w). \qquad (3.1.18)$$

Thus, Eq. (3.1.18) introduces the following recursive relations

$$\bar{y}_0(w) = \ y_2(0) + y_2'(0)w,$$

$$\bar{y}_{n+1}(w) = \ - I_0^{\propto} A_n(w), \quad n \geq 0. \qquad (3.1.19)$$

The series solution of $y_2(w)$ follows directly and the accuracy of the solution definitely depends on the number of the calculated terms. In the following calculations, the number of terms in the Adomian series (3.1.15) did not exceed 10 terms.

We finally apply the $[m, m]$ Pade' approximation to approximate $y_2(w)$ and then apply the limit condition to obtain the value of θ; the approximation solution of (3.1.1) and (3.1.2) is

$$y(x) = y_1(x) + y_2\left(\frac{x}{\varepsilon^{1/(\alpha-1)}}\right).$$

See [12].

3.2 Numerical Results

In this section, we consider two examples to demonstrate the performance and efficiency of the method.

Example 3.2.1. Consider the linear fractional problem

$$\varepsilon D^{\frac{3}{2}} y(x) - y'(x) = 1 + 2x, \quad 0 < x < 1 \tag{3.2.1}$$

Subject to

$$y(0) = 0 \ \ , \ \ y(1) = 1. \tag{3.2.2}$$

we set $\varepsilon = 0$ to obtain the following reduced problem

$$-y_1'(x) = 1 + 2x, \qquad y_1(1) = 1. \tag{3.2.3}$$

Obviously, y_1 should have the following explicit from

$$y_1(x) = x + x^2 - 1$$

However, the boundary layer correction problem is

$$\varepsilon D^{\frac{3}{2}} y_2(w) + y_2'(w) = 0, \tag{3.2.4}$$

subject to

$$y_2(0) = 1, \qquad y_2'(0) = \theta. \tag{3.2.5}$$

Applying (ADM), we obtain

$$y_2(w) = 1 + \theta w - \frac{4\theta}{3\sqrt{\pi}} w^{\frac{3}{2}} + \frac{\theta}{2} w^2 - \frac{8\theta}{15\sqrt{\pi}} w^{\frac{5}{2}} + \frac{\theta}{6} w^3 - \frac{16\theta}{105\sqrt{\pi}} w^{\frac{7}{2}} + \frac{\theta}{24} w^4 - \frac{32\theta}{954\sqrt{\pi}} w^{\frac{9}{2}} + \frac{\theta}{120} w^5 - \frac{64\theta}{10395\sqrt{\pi}} w^{\frac{11}{2}} + \frac{\theta}{720} w^6 - \frac{128\theta}{135135\sqrt{\pi}} w^{\frac{13}{2}} + \frac{\theta}{5040} w^7 - \frac{256\,\theta}{2027025\sqrt{\pi}} w^{\frac{15}{2}} + \frac{\theta}{40320} w^8 - \frac{512\,\theta}{34459425\sqrt{\pi}} w^{\frac{17}{2}} \tag{3.2.6}$$

Approximating (3.2.6) using the Pade' approximation of order $[3,3]$, we have

$$y_2(w) \approx \bar{y}_2(w) = \frac{p(w,\theta)}{q(w,\theta)},$$

and then, solving the equation

$$\lim_{w\to\infty} \frac{p(w,\theta)}{q(w,\theta)} = 0,$$

we obtain $\theta = 0.000174322$. The graphs of the y_1, y_2 and y at $\varepsilon = 10^{-5}$ are displayed in Figure (3.2.1).

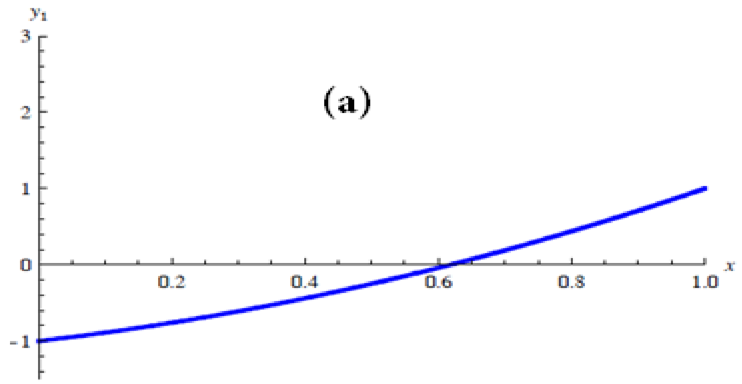

Figure 3.2.1a: The graph of the solution $y_1$at $\varepsilon = 10^{-5}$

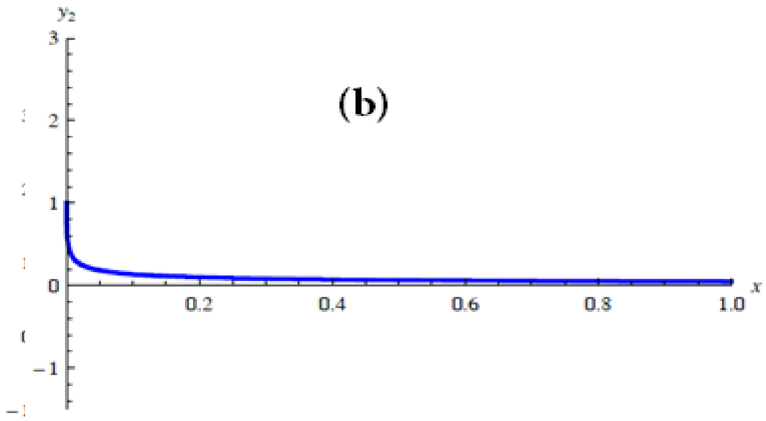

Figure 3.2.1b: The graph of the solution $y_2$at $\varepsilon = 10^{-5}$

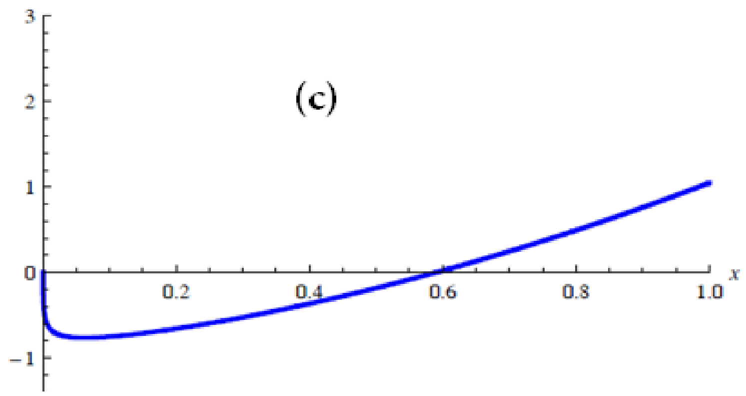

Figure 3.2.1c: The graph of the solution y at $\varepsilon = 10^{-5}$

Example 3.2.2. Consider the nonlinear singular fractional problem

$$-\varepsilon D^{\frac{3}{2}}y(x) - y(x)y'(x) + e^{y(x)}y(x) = 0, \quad x \epsilon I, \tag{3.2.2.1}$$

subject to

$$y(0) = 1 \ , \ y(1) = 4. \tag{3.2.2.2}$$

Following the above discussion, we set $\varepsilon = 0$ to obtain the following reduced problem

$$-y(x)y_1'(x) + e^{y(x)}y_1(x) = 0, \ y_1(1) = 4. \tag{3.2.2.3}$$

It can be easily verified that the solution of (3.2.2.3) is $y_1(x) = -\log(-x + e^{-4} + 1)$. However, using the stretching transformation $x = \sqrt[\gamma]{\varepsilon}w$ with $\gamma = 1/2$, we have the following boundary layer correction problem

$$-D^{\frac{3}{2}}y_2(w) - \big(y_1(0) + y_2(w)\big)y_2'(w) = 0, \tag{3.2.2.4}$$

subject to

$$y_2(0) = 1 - y_1(0) = 1 + \log(1 + e^{-4}), \ y_2'(0) = \theta. \tag{3.2.2.5}$$

Applying (ADM) to solve (3.2.2.4) and (3.2.2.5), we obtain

$$\begin{aligned} y_2(w) = {} & 1 + \log(1 + e^{-4}) + \theta w - \frac{4\theta}{\sqrt[3]{\pi}}w^{\frac{3}{2}} + \frac{\theta}{2}w^2 - \frac{8\theta(\theta+1)}{\sqrt[15]{\pi}}w^{\frac{5}{2}} + \frac{1}{12}\theta(7\theta+2)w^3 \\ & - \frac{8\theta\big(32\theta + \pi(39\theta+6)\big)}{315\pi^{\frac{3}{2}}}w^{\frac{7}{2}} + \frac{\theta\big(128\theta + 3\pi(\theta(28\theta+115)+8)\big)}{576\pi}w^4 \\ & - \frac{4\theta(128\theta(16\theta+21) + 15\pi(\theta(140\theta+171)+8))}{14175\pi^{3/2}}w^{9/2} \end{aligned}$$

$$+\frac{\theta(1792\theta(11\theta+6)15\pi(\theta(2483\theta+1290)+32))}{57600\pi}w^5 -$$

$$\frac{1}{3274425\pi^{\frac{5}{2}}}\left(2\theta\left(13107200\theta^2+768\pi(\theta(224\theta+3417)+838)\theta+315\pi^2(\theta(\theta(560\theta+5903)+1770)+32)\right)\right)w^{\frac{11}{2}}+\frac{1}{232243200\pi^2}(\theta\left(419430400\theta^2+384\pi(13\theta(7168\theta+32157)+53632)\theta+315\pi^2(\theta(\theta(200564\theta+591001)+98968)+1024)\right))w^6.$$

(3.2.2.6)

Approximating (3.2.2.6) using the Pade' approximation of order [3,3], we have

$$y_2(w) \approx \bar{y}_2(w) = \frac{p(w,\theta)}{q(w,\theta)},$$

where

$$
\begin{aligned}
p(w,\theta) = 2.60765\ \times\ 10^{12}& \\
&- \{1.85615\ \times\ 10^{23}(\theta - 0.101132)(\theta\ + 0.0395075) \\
&\times\ ((\theta - 0.132056)\theta + 0.00519482)((\theta - 0.0243776)\theta + 0.000361298) \\
&\times (\theta(\theta + 0.0333639) + (0.00344611)\}w^{\frac{1}{2}} \\
&- \{8.67481\ \times\ 10^{22}(\theta - 0.0940307) \\
&\times (\theta + 0.0439606)(\theta + 1.72007)((\theta - 0.131771)\theta + 0.0051919) \\
&\times ((\theta - 0.022062)\theta + 0.000336622)\ (\theta(\theta + 0.030252) + 0.00338713)\}w \\
&- \{1.74199\ \times\ 10^{23}(\theta - 0.0884453)(\theta + 0.0502881)(\theta + 0.313528) \\
&\times ((\theta - 0.131294)\theta + 0.00521401)((\theta - 0.019656)\theta + 0.000307185) \\
&\times (\theta(\theta + 0.0241632) + 0.00370855)\}w^{\frac{3}{2}} \\
&- \{1.16034\ \times\ 10^{23}\ (\theta - 0.0821132) \\
&\times ((\theta - 0.132305)\theta + 0.00528421)((\theta - 0.0168859)\theta + 0.000274977) \\
&\times (\theta(\theta + 0.00154127) + 0.00420052)(\theta(\theta + 0.140806) + 0.00512562)\}w^2 \\
&- \{3.00284 \\
&\times\ 10^{22}(\theta - 0.0802418)(\theta + 0.862162)((\theta - 0.129726)\theta \\
&+ 0.00525109)((\theta - 0.0201212)\theta + 0.00377281)((\theta - 0.0129097)\theta \\
&+ 0.000242214)(\theta(\theta + 0.0958752) + 0.00259807)\}w^{\frac{5}{2}} \\
&- \{1.8105\ \times\ 10^{22} \\
&\times (\theta - 0.0836548)(\theta - 0.000147157)((\theta - 0.125415)\theta + 0.00526373) \\
&\times ((\theta - 0.0600121)\theta + 0.00526894)((\theta - 0.00566949)\theta + 0.000515556) \\
&\times (\theta(\theta + 0.135897) + 0.00504432)\}w^3
\end{aligned}
$$

and

$$
\begin{aligned}
q(w,\theta) = 2.56117 \times 10^{12} \\
&- \left\{1.82306 \times 10^{23} (\theta - 0.101123)(\theta + 0.0395075)\right. \\
&\times \big((\theta - 0.132056)\theta + 0.00519482\big)\big((\theta - 0.0243776)\theta \\
&\left. + 0.000361298\big) \times (\theta(\theta + 0.0333639) + 0.00344611)\right\} w^{\frac{1}{2}} \\
&- \left\{1.4844 \times 10^{23}(\theta - 0.0934038)\right. \\
&\times (\theta + 0.0437815)\big((\theta - 0.131562)\theta \\
&+ 0.00518463\big)\big((\theta - 0.0222593)\theta + 0.000334289\big)(\theta(\theta + 0.0303015) \\
&\left. + 0.00340829)\right\} w \\
&- \left\{5.6131\right. \\
&\times 10^{22} (\theta - 0.0866065)(\theta + 0.0494551)(\theta \\
&+ 0.981364)\big((\theta - 0.13145)\theta + 0.00520405\big) \\
&\times \big((\theta - 0.0197525)\theta + 0.000307021\big)(\theta(\theta + 0.0254835) \\
&\left. + 0.00375923)\right\} w^{\frac{3}{2}} \\
&- \left\{6.02667 \times 10^{22} (\theta - 0.0806517)(\theta + 0.0598394)(\theta + 0.143598)\right. \\
&\times \big((\theta - 0.132129)\theta + 0.00527472\big)\big((\theta - 0.0170627)\theta \\
&\left. + 0.000276725\big) \times (\theta(\theta + 0.00265845) + 0.00480268)\right\} w^2 \\
&- \left\{1.67279 \times 10^{22}(\theta - 0.0724199)\right. \\
&\times \big((\theta - 0.133461)\theta + 0.00533471\big)\big((\theta - 0.0350406)\theta + 0.00433639\big) \\
&\times \big((\theta - 0.0145521)\theta + 0.000245492\big)(\theta(\theta + 0.113692) \\
&\left. + 0.00369427)\right\} w^{\frac{5}{2}} \\
&- \left\{4.23912\right. \\
&\times 10^{21} (\theta - 0.0815741)(\theta + 0.00096346)\big((\theta - 0.123372)\theta \\
&+ 0.00534966\big)\big((\theta - 0.0738228)\theta + 0.00565972\big)\big((\theta - 0.0152932)\theta \\
&\left. + 0.000244182\big)(\theta(\theta + 0.161018) + 0.00673569)\right\} w^3.
\end{aligned}
$$

Solving the equation

$$\lim_{w\to\infty} \frac{p(w,\theta)}{q(w,\theta)} = 0,$$

we obtain $\theta = 1.47157 \times 10^{-3}$. The graph of the approximate solutions y_1, y_2 and y at $\varepsilon = 10^{-4}$ are displayed in Figure (3.2.2) - (1). This figure is an evident proof to our claim that the solutions y_1 and y match on most interval (0,1] except for small interval around $x = 0$. The graph of the approximate solution y for several values of ε are displayed in Figure (3.3.2) - (2). Obviously, the singularity of the solution at $x = 0$ is accurately captured by the present technique.

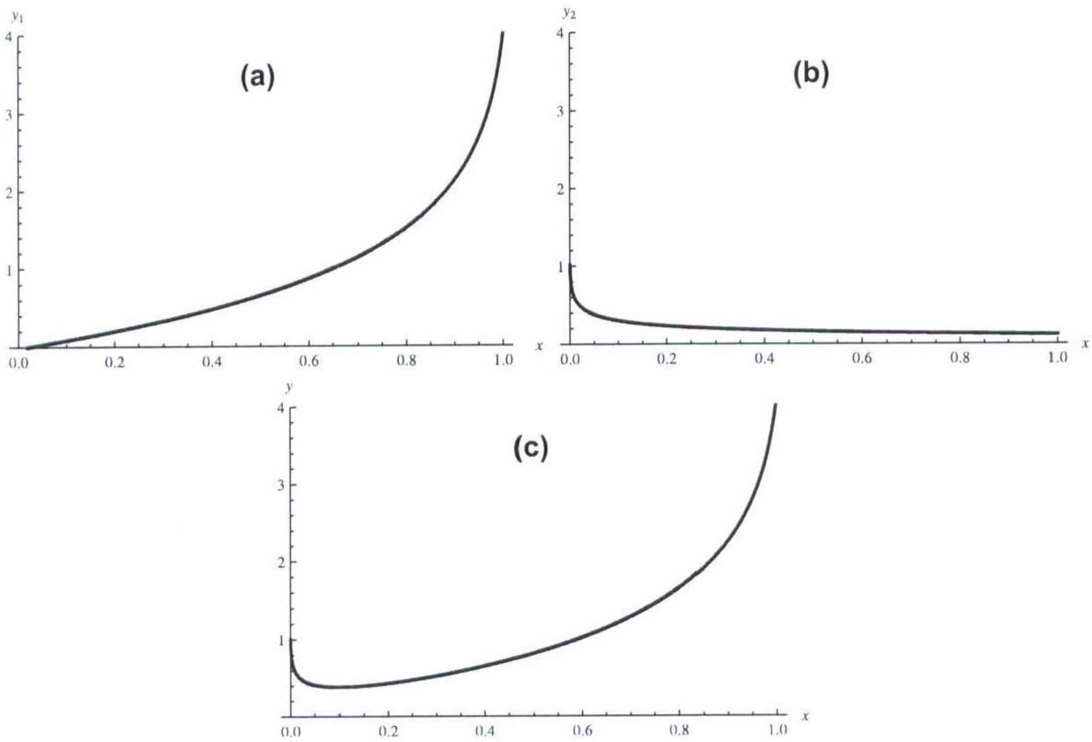

Figure 3.2.2 – 1: Graphs of y_1, y_2 and y at $\varepsilon = 10^{-4}$

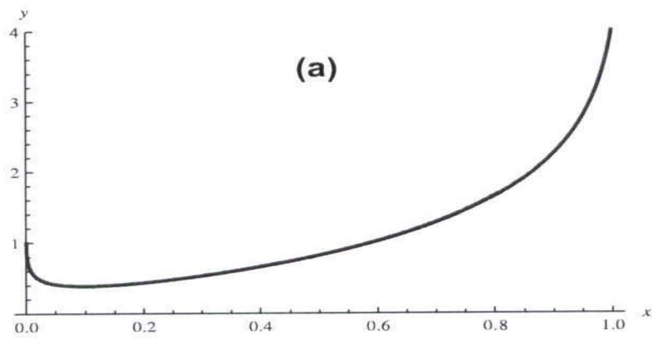

Figure 3.2.2-2a: The graph of the solution y at $\varepsilon = 10^{-4}$

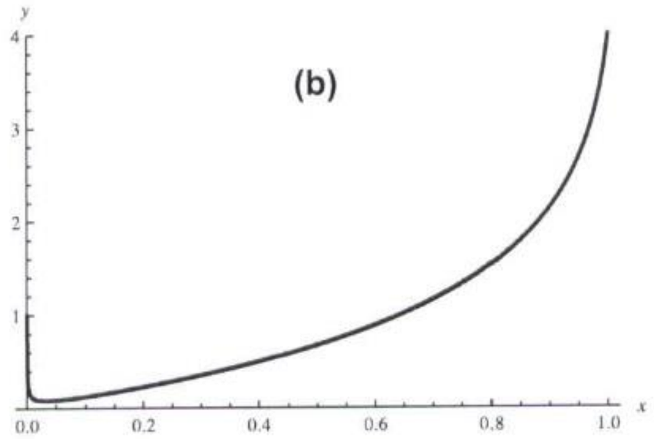

Figure 3.2.2-2b: The graph of the solution y at $\varepsilon = 10^{-5}$

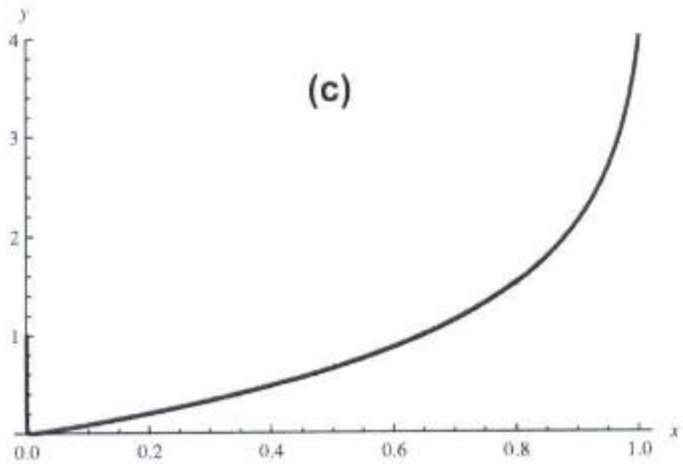

Figure 3.2.2-2c: The graph of the solution y at $\varepsilon = 10^{-6}$

Example 3.2.3: Consider the nonlinear fractional problem

$$-\varepsilon\, Dy^{\frac{3}{2}}(x) - y'(x) + \sqrt{y(x)}\, y(x) = 0, \qquad x\epsilon\, I, \tag{3.2.3.1}$$

subject to

$$y(0) = 0, \quad y(1) = 1. \tag{3.2.3.2}$$

we set $\varepsilon = 0$ to obtain the following reduced problem

$$-\, y_1'(x) + \sqrt{y_1} y_1(x) = 0, \qquad y_1(1) = 1. \tag{3.2.3.3}$$

It can be easily verified that the solution of (3.2.3.3) is $y_1(x) = \frac{4}{(-3+x)^2}$, however using the stretching transformation $x = \sqrt[\gamma]{\varepsilon}\, w$ with $\gamma = ½$, we have the following boundary layer correction problem

$$D^{\frac{3}{2}} y_2(w) + y_2' = 0. \tag{3.2.3.4}$$

subject to

$$y_2(0) = 0 - y_1(0) = -\frac{4}{9},\; y_2'(0) = \theta. \tag{3.2.3.5}$$

Applying (ADM), we obtain

$$\begin{aligned} y_2(w) = -4 + \theta w + \frac{4\theta}{3\sqrt{\pi}} w^{\frac{3}{2}} + \frac{\theta}{2} w^2 + \frac{8\theta}{15\sqrt{\pi}} w^{\frac{5}{2}} + \frac{\theta}{6} w^3 + \frac{16\theta}{105\sqrt{\pi}} w^{\frac{7}{2}} + \frac{\theta}{24} w^4 \\ + \frac{32\theta}{945\sqrt{\pi}} w^{\frac{9}{2}} + \frac{\theta}{120} w^5 + \frac{\theta}{720} w^6 \frac{64\theta}{10395\sqrt{\pi}} w^{\frac{11}{2}} + \frac{128\theta}{135135\sqrt{\pi}} w^{\frac{13}{2}} \\ + \frac{\theta}{5040} w^7 \frac{265\theta}{2027025\sqrt{\pi}} w^{\frac{15}{2}} + \frac{\theta}{40320} w^8 + \frac{512\theta}{34459425\sqrt{\pi}} w^{\frac{17}{2}} \end{aligned}$$

Following the same steps as in the previous example, we obtain $\theta = 6.97287 \times 10^{-4}$

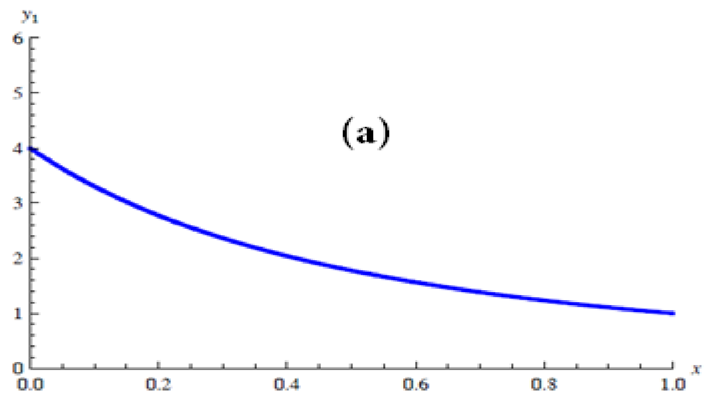

Figure 3.2.3-a The graph of the solution y_1 at $\varepsilon = 10^{-8}$

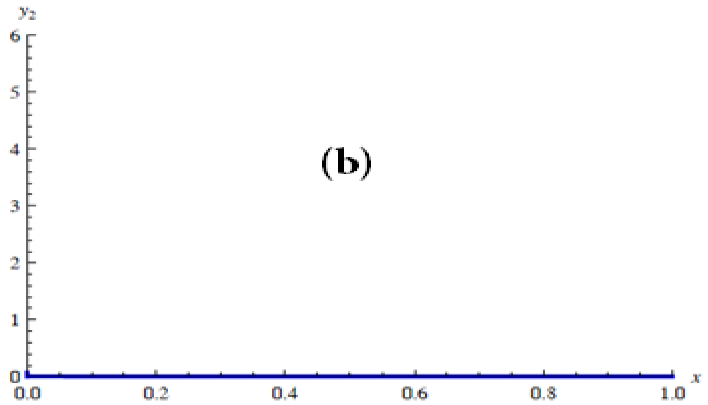

Figure 3.2.3-b The graph of the solution y_2 at $\varepsilon = 10^{-8}$

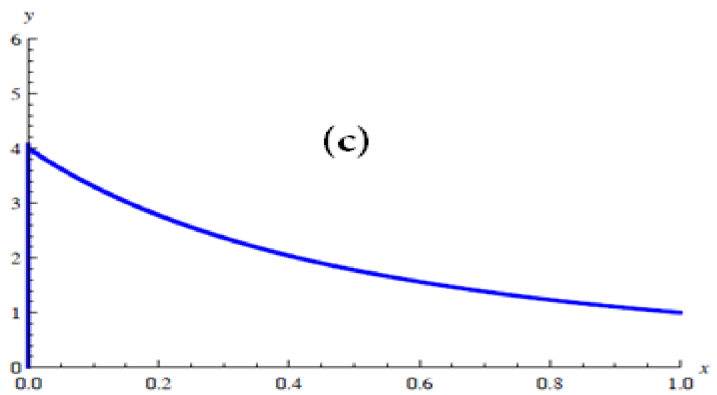

Figure 3.2.3-c The graph of the solution of y at $\varepsilon = 10^{-8}$

Conclusion

In recent years, the fractional differential equations have received more and more attention in many physical applications, phenomena in electromagnetics, acoustics, viscoelasticity, electrochemistry, etc. In this thesis, we have introduced an algorithm for approximating solutions of a class of non- linear singularly perturbed two points boundary-value problems of fractional order $\alpha, 1 < \alpha \leq 2$. The method of solution is based on reduced layer correction method which divides the singular problem into first order IVP and fractional IVP of order α. The fractional IVP is solved using the Adomain decomposition method and Pade' approximation method. Three examples are discussed to illustrate the efficiency of the present scheme. The Mathematica software system has been used for all numerical computations in this thesis.

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