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SOLVING NONLINEAR BOUNDARY VALUE PROBLEMS USING THE HOMOTOPY ANALYSIS METHOD

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United Arab Emirates University

College of Science

Department of Mathematical Sciences

**SOLVING NONLINEAR BOUNDARY VALUE PROBLEMS USING
THE HOMOTOPY ANALYSIS METHOD**

Ghada Ayed Janem

This thesis is submitted in partial fulfilment of the requirements for the degree of
Master of Science in Mathematics

Under the Supervision of Professor Fathi M Allan

April 2015

Declaration of Original Work

I, Ghada Ayed Janem, the undersigned, a graduate student at the United Arab Emirates University (UAEU), and the author of this thesis entitled “*Solving nonlinear boundary value problems using the Homotopy Analysis Method*”, hereby, solemnly declare that this thesis is an original research work that has been done and prepared by me under the supervision of Professor Fathi M Allan, in the College of Science at UAEU. This work has not been previously formed as the basis for the award of any academic degree, diploma or a similar title at this or any other university. The materials borrowed from other sources and included in my thesis have been properly cited and acknowledged.

Student's Signature _____

Date _____

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Abstract

Analytical solutions of differential equations are very important for all researchers from different discipline. Obtaining such solutions is difficult in most cases, especially if the differential equation is nonlinear. One of the mostly used methods are the series methods, where the solution is represented as an infinite series. Different methods are available to evaluate the terms of this series. These methods include the well-known Taylor series method, the Adomian decomposition method, the Homotopy iteration method, and the Homotopy analysis method.

In this thesis we give a survey of the different series methods available to solve initial and boundary value problems. The methods to be presented are the Taylor series method, the Adomina decomposition method, and the Homotopy analysis method. The main features of each method will be presented and the error analysis will be discussed as well. For the Homotopy analysis method, the error is controlled by introducing the parameter known as \hbar , then the error is controlled by monitoring the value of the solution at a specific point for different values of \hbar . This produces what is known as the \hbar curve. The mathematical foundation of this method is not very well established, and the method will not work at all times. The error for the Taylor series and the Adomian decomposition method is controlled by adding more terms to the series solution which might be costly and difficult to calculate especially if the differential equation is nonlinear.

In this study we will show that the error can be controlled by other means. A modified Taylor series method has been developed and will be discussed. The method is based on controlling the error through different choices of the point of expansion. The mathematical foundation of the method and application of the method to differential equations with singularities and eigenvalue problems will be presented.

Keywords: Adomian Decomposition Method, Homotopy Analysis Method, Modified Taylor series method.

Title and Abstract (in Arabic)

حل المعادلات التفاضلية الحدية و اللاخطية باستخدام طريقة الهوموتوبي التحليلية

الملخص

الحلول التحليلية للمعادلات التفاضلية مهمة جدا لجميع الباحثين من مختلف المجالات العلمية و من الصعب الحصول على مثل هذه الحلول في معظم الحالات، خاصة إذا كانت المعادلة التفاضلية لاخطية. من هذه الطرائق المتبعة طريقة المتسلسلات اللانتهية، حيث يتم تمثيل الحل باعتباره سلسلة لا نهاية لها. ومن هذه الطرائق المعروفة طريقة سلسلة تايلور، وطريقة أدوميان التحليلية، وطريقة هوموتوبي المتكرره والتحليلية.

في هذه الأطروحة سنقدم عرضا لمجموعة من هذه الطرائق بالاختصاص طريقة سلسلة تايلور، وطريقة أدوميان التحليلية وطريقة هوموتوبي التحليلية. وستعرض السمات الرئيسية لكل طريقة وسيتم مناقشة تحليل الخطأ أيضا. بالنسبة لطريقة الهوموتوبي التحليلية، يتم التحكم بالخطأ عن طريق إدخال المعلمة المعروفة باسم \hbar ، ثم يتم التحكم بالخطأ من خلال رصد قيمة الحل عند نقطة محددة باستخدام قيم مختلفة من \hbar ، مما ينتج عنه ما يعرف بمنحنى \hbar . التعليل الرياضي لهذه الطريقة ما زال غير مكتمل، وهذه الطريقة لا تعمل في بعض الحالات. بينما يتم التحكم بالخطأ لسلسلة تايلور وطريقة أدوميان التحليلية بإضافة مزيد من الحدود للسلسلة التي قد تكون مكلفة وصعبة الحساب خاصة إذا كانت المعادلة التفاضلية لاخطية.

في هذه الدراسة سوف نظهر أن الخطأ يمكن السيطرة عليه بوسائل أخرى. وقد تم تطوير طريقة تعديل سلسلة تايلور وسيتم مناقشتها. وتستند هذه الطريقة على السيطرة على الخطأ من خلال خيارات مختلفة للنقطة التي يتم إيجاد السلسلة حولها. وسنعرض التحليل الرياضي لهذه الطريقة وتطبيقها على المعادلات التفاضلية الحدية والمعادلات التفاضلية ذات القيم الذاتية.

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Dedication

To my beloved parents and family

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List of Abbreviations

ADM	Adomian Decomposition Method
BVP	Boundary Value Problem
DE	Differential Equation
Eq	Equation
HAM	Homotopy Analysis Method
IVP	Initial Value Problem
ODE	Ordinary Differential Equation

Chapter 1: Introduction

Many scientific and engineering problems can be modeled by ordinary or partial differential equations which are mostly nonlinear differential equations. It is difficult to obtain closed form solutions for such problems. In most cases only approximate solutions either analytical ones or numerical ones can be obtained. For that reason, the nonlinear equations should be solved using special techniques.

Recently, many researchers have been devoted to new developed methods to construct an analytic solution of non-linear equations. Such methods include the Taylor series method, the Adomian Decomposition method and the Homotopy Analysis method.

Review of the solution of the ordinary differential equations with variable coefficients by the method of power series was discussed in [17, 39, 44]. Detailed analysis of the method and the basic definitions such as the regular and irregular singular points, radius of convergence, with some examples will be introduced in chapter two.

The other widely used method to generate series solution for the differential equations is the Adomian Decomposition method. It was introduced first by Adomian in the early of 1980's. It gives the solution as a rapid convergent infinite series with components that are elegantly computed. Since then, the method has been applied to wide class of equation [6, 7, 8, 9, 40]. A modified version of Adomian method was used to derive the analytic solution of ordinary differential equations in [43].

Error analysis and convergence criterion of the method was investigated by many authors. In [17] Cherruault and Adomian investigate the convergence of the method when applied to a special class of boundary value problems. However, in [18] it was shown that Adomian Decomposition method does not converge in general. In particular, if the method is applied to linear equations.

Recently, many authors tried to modify ADM to accelerate the convergence of the series solution, simultaneously reduces the size of the work. Wazwaz proposed an efficient modification of the standard Adomian Decomposition method that improve convergence and reduce the error [42]. Also Abbasbandy suggested a modification of ADM in [4]. More details about the method and examples will be presented in chapter three.

The third method that will be discussed in this thesis is the Homotopy Analysis method (HAM). This method is based on a continuous variation from an initial trial to the exact solution. It has been developed by the Chinese mathematician S.J. Liao in his PhD thesis in 1992 (see [29] and the reference there in). He derived the method, proved the convergence and developed the error criteria for the method. Recently the method has been employed by many authors to tackle certain class of initial value problems. For example, Liao [26] employed the method to generate an analytical solution for the Blasius equation. He was able to generate 35 terms of the series expansion of the solution and the solution was accurate up to 6 decimal numbers.

In [26], Liao applied the Homotopy analysis method, to study the temperature distributions of a laminar viscous flow over a semi-infinite flat plate. He was able to derive

up to 30 terms of the series solution of the problem. Allan and Syam [15] used the method to solve the non-homogenous Blasius problem that describes the fluid flow over a moving surface. They used it to show that the problem has either no solution or one solution or two solutions. The existence and uniqueness of the solution depends on the flat plate relative velocity.

Allan [11] used the Homotopy Analysis method to derive the Adomian Decomposition method. He proved that the Adomian Decomposition method is a special case of the Homotopy Analysis method. Lately, Allan and Haji [14] employed the same method to study the characteristics of nano-fluid flow over a moving flat plate. Liao and Chang [24] used the Homotopy Analysis method to derive a solution for the classical problem of nonlinear progressive waves in deep water. To further improve the convergence, he applied the Pade expansion. As a result, the calculated phase speed at the 20th order approximation of the solution agrees well with previous perturbation solutions.

Most of the examples presented are initial value problems, very little is known about the application of the method to boundary value problems. Allan and Hajji use the method to derive an analytical solution to multi-level boundary value problems derived from the flow through multi-channels porous media.

Analysis of the method, its convergence and error analysis will be presented and several examples will be presented in chapter four to demonstrate the validity of the work. In chapter five, we will discuss a new method based on a modification of Taylor series approach to solve singular initial value problems and boundary value problems. The

method is based on expanding the solution at different choices of the expansion point. The solution then will be a function of both the independent variable x and the expansion point x_0 . Study of the solution as the expansion point varies will be presented. Mathematical analysis of the method and error control will be discussed and several examples will be presented. Conclusion remarks are presented in Chapter six.

Chapter 2: Series solution about ordinary points

Most differential equations, especially nonlinear ones, do not accept solutions which can be implicitly or explicitly defined in terms of elementary functions. However, the problem of finding solutions is not difficult. In fact, there are available numerical methods, series methods and graphical methods for solving DE's.

For certain differential equations, implicit solutions in terms of elementary functions can be found. However, it is frequently true that such implicit solutions are less useful than numerical or series ones. Implicit solutions are for most differential equations, are difficult to find. The subject of this chapter is the use of power series as a method for deriving solutions to linear and nonlinear ordinary differential equations.

2.1 Introduction to a power series

The basic idea of power series solution is to find the solution of the form

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} a_n (x - x_0)^n \\ &= a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n + \cdots \end{aligned} \quad (2.1)$$

where $a_0, a_1, a_2, \dots, a_n$ are constants to be determined, x_0 is the domain of definition of the differential equation and x is the independent variable. However, if infinitely many values of a_n 's are non-zeros, then the series is infinite and one should consider the convergence of the series. In what follows we will introduce some definitions that will be needed in this chapter.

2.1.1 Definition of a convergence

A power series (2.1) is said to converge at a point x if $\lim_{m \rightarrow \infty} \sum_{n=0}^m a_n(x - x_0)^n = \sum_{n=0}^{\infty} a_n(x - x_0)^n$ exists. It is clear that the series converges for $x = x_0$, it may converge for all x , or it may converge for some values of x and not for others.

2.1.2 Definition of absolute convergence

The power series $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ is said to converge absolutely if $\sum_{n=0}^{\infty} a_n|x - x_0|^n$ converges. One of the most useful tests for the absolute convergence of a power series is the ratio test. It states that for a fixed value of x if $|x - x_0| \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = l$, then the power series converges absolutely at that value of x if $l < 1$, and diverges if $l > 1$. If $l = 1$ the test is inconclusive, i.e, the series may either converge or diverge. However, there is another method to test the absolute convergence that is if a power series converges at $x = x_1$, then it converges absolutely for all x such that $|x - x_0| < |x_1 - x_0|$, and if it diverges at $x = x_1$, then it diverges for all x such that $|x - x_0| > |x_1 - x_0|$.

2.1.3 Definition of radius of convergence

The radius of convergence of a power series is defined to be the positive constant ρ such that the power series converges absolutely for $|x - x_0| < \rho$ and diverges for $|x - x_0| > \rho$. The interval $I = \{x: |x - x_0| < \rho\}$ is called the interval of convergence of the series.

2.2 Power series method for linear differential equation

Consider the second-order linear differential equation,

$$a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = 0 \quad (2.2)$$

Suppose $a_2(x)$ is nonzero for all x . Then we can divide throughout to obtain

$$y''(x) + \frac{a_1(x)}{a_2(x)}y'(x) + \frac{a_0(x)}{a_2(x)}y(x) = 0 \quad (2.3)$$

Suppose that a_1/a_2 and a_0/a_2 are analytic functions at x_0 , then we can derive the power series solution of the differential equation (2.2) as follows.

Let

$$y(x) = \sum_{i=0}^{\infty} a_n(x - x_0)^n \quad (2.4)$$

then

$$y'(x) = \sum_{i=1}^{\infty} na_n(x - x_0)^{n-1} \quad (2.5)$$

and

$$y''(x) = \sum_{i=2}^{\infty} n(n-1)a_n(x - x_0)^{n-2} \quad (2.6)$$

Then replace y, y', y'', \dots in (2.2) one can solve for the coefficients $a_i, i = 1, 2, 3, \dots$ as it will be shown in the next example.

Example [2.1]: Consider the differential equation

$$y'' - xy = 0 \quad (2.7)$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \quad y'(x) = \sum_{n=1}^{\infty} a_n n x^{n-1}, \quad y''(x) = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}$$

Plugging y and y'' into the Eq (2.7) gives

$$\sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} - x \sum_{n=0}^{\infty} a_n x^n = 0$$

By making a shift in the index on the first sum the second sum leads to

$$2a_2 + \sum_{k=1}^{\infty} (k+2)(k+1)a_{k+2} x^k - \sum_{k=1}^{\infty} a_{k-1} x^k = 0$$

Collecting powers of the same power of x leads to

$$2a_2 + \sum_{k=1}^{\infty} [(k+2)(k+1)a_{k+2} - a_{k-1}] x^k = 0$$

If this series is a solution, then all these coefficients must be zero. Thus

$$2a_2 = 0, \text{ or } a_2 = 0$$

and

$$(k+2)(k+1)a_{k+2} - a_{k-1} = 0$$

or

$$a_{k+2} = \frac{a_{k-1}}{(k+2)(k+1)} \tag{2.8}$$

For example, for $k = 1, 2, 3, \dots$

$$a_3 = \frac{1}{6} a_0$$

$$a_4 = \frac{1}{12} a_1$$

$$a_5 = \frac{a_2}{20} = 0$$

$$a_6 = \frac{a_3}{30} = \frac{1}{180} a_0$$

Substituting these coefficients into the original power series for $y(x)$ gives

$$y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$$

which we can break up into the sum of two linearly independent series solutions

$$y_1(x) = a_0 \left[1 + \frac{1}{6}x^3 + \frac{1}{180}x^6 + \dots \right]$$

$$y_2(x) = a_1 \left[x + \frac{1}{12}x^4 + \frac{1}{504}x^7 + \dots \right]$$

and

$$y(x) = y_1(x) + y_2(x)$$

thus

$$y(x) = a_0 \left[1 + \frac{1}{6}x^3 + \frac{1}{180}x^6 + \dots \right] + a_1 \left[x + \frac{1}{12}x^4 + \frac{1}{504}x^7 + \dots \right]$$

The coefficients a_0 and a_1 can be determined using the initial conditions. It is worth mentioning that it is always possible to find the recurrence relation similar to (2.8) in all cases of linear differential equations.

Example [2.2]: Consider the following IVP, [10]

$$e^x y''(x) + xy(x) = 0 \tag{2.9}$$

with the initial conditions

$$y(0) = A, y'(0) = B$$

We can introduce the solution of the Taylor series by an infinite series given

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \quad (2.10)$$

Substituting Eq (2.10) into both sides of Eq (2.9) gives

$$e^x \left(\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} \right) = - \sum_{n=0}^{\infty} a_n x^{n+1}$$

or, equivalently

$$\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \right) (2a_2 + 6a_3x + 12a_4x^2 + \dots) \\ + x(a_0 + a_1x + a_2x^2 + a_3x^3 + \dots) = 0$$

Thus

$$2a_2 + 6a_3x + 12a_4x^2 + \dots + 2a_2x + 6a_3x^2 + 12a_4x^3 + \dots + a_2x^2 + 3a_3x^3 + 6a_4x^4 \\ + \dots + a_0x + a_1x^2 + a_2x^3 + a_3x^4 = 0$$

The coefficients $a_n, n \geq 0$, are determined by equating coefficients of like powers of x through recurrence relation. It is clear that the recurrence relation is difficult to derive.

Alternatively, we multiply the series, term by term to find

$$2a_2 + (6a_3 + 2a_2 + a_0)x + (12a_4 + 6a_3 + a_2 + a_1)x^2 + (12a_4 + 3a_3 + a_2)x^3 + \\ \dots = 0$$

Now

$$a_0 = A, \quad a_1 = B, \quad a_2 = 0, \quad a_3 = -\frac{1}{6}a_0, \quad a_4 = \frac{1}{12}(A - B)$$

$$\text{Thus, } y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots$$

$$= A + Bx + \frac{-1}{6}Ax^3 + \frac{1}{12}(A - B)x^4 + \dots$$

$$= y_1(x) + y_2(x)$$

where

$$y_1(x) = \left[1 - \frac{1}{6}x^3 + \frac{1}{12}x^4 + \dots \right] A$$

and

$$y_2(x) = \left[x - \frac{1}{12}x^4 + \dots \right] B$$

2.3 Power series method for nonlinear differential equation

For the nonlinear differential equations the approach is the same as the linear case. However, calculating the coefficients a_n 's of the series solution is a little bit more difficult. In the next examples, we will suggest a new approach to find the coefficients that will reduce the calculation time and make it easier for the use of software such as Mathematica or Matlab.

Example [2.3]: Consider the nonlinear linear differential equation

$$y' + y^2 = 1, \quad y(0) = 2 \tag{2.11}$$

Which has an exact solution given by

$$y(x) = \frac{1 + 3e^{2x}}{-1 + 3e^{2x}}$$

Now let $y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$

then,

$$y'(x) = \sum_{n=1}^{\infty} a_n n x^{n-1}$$

Plugging y and y' into the Eq (2.11) gives

$$(a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \dots) + (a_0 + a_1x + a_2x^2 + a_3x^3 + \dots)(a_0 + a_1x + a_2x^2 + \dots) = 1$$

And after simplification it becomes

$$\begin{aligned} (a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \dots) + (a_0^2 + a_0a_1x + a_0a_2x^2 + a_0a_3x^3 + \dots + a_0a_1x \\ + a_1^2x^2 + a_1a_2x^3 + a_1a_3x^4 + \dots + a_0a_2x^2 + a_1a_2x^3 + a_2^2x^4 + a_2a_3x^5 \\ + \dots + a_0a_3x^3 + a_1a_3x^4 + a_2a_3x^5 + a_3^2x^6 + \dots) = 1 \end{aligned}$$

The terms can be rearranged as

$$\begin{aligned} (a_1 + a_0^2) + (2a_2 + 2a_0a_1)x + (3a_3 + 2a_0a_2 + a_1^2)x^2 + (4a_4 + 2a_0a_3 + 2a_1a_2) \dots \\ = 1 \end{aligned}$$

This equating the coefficient of each power of x with 0 leads to

$$a_1 + a_0^2 = 1, \text{ or} \quad a_1 = 1 - a_0^2$$

$$2a_2 + 2a_0a_1 = 0, \text{ or} \quad a_2 = a_0^3 - a_0$$

Similarly

$$3a_3 + 2a_0a_2 + a_1^2 = 0, \text{ or} \quad a_3 = \frac{1}{3}(-3a_0^4 + 4a_0^2 - 1)$$

Then the solution in a series form will be

$$y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$$

$$y(x) = a_0 + (1 - a_0^2)x + (a_0^3 - a_0)x^2 + \frac{1}{3}(-3a_0^4 + 4a_0^2 - 1)x^3 + \dots$$

The coefficient a_0 can be found using the initial condition. If for example we assume that

$y(0) = 2$, then $a_0 = 2$, $a_1 = -3, \dots$ which leads to the power series solution

$$y(x) = 2 - 3x + 6x^2 - 11x^3 + \dots$$

As mentioned earlier, the calculations of the coefficients a_n will more complicated and difficult especially if the method is applied to a higher order differential equation with sever nonlinear terms. In addition, it will be very difficult to write a computer program to evaluate the coefficients using procedures describes above. This initiates a need to develop a new approach to solve for the coefficients. The new approach will be described in the next example.

New approach for evaluating the coefficients

Consider the nonlinear differential equation described example [2.3] solve the ODE using the power series method, we set

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

and

$$y'(x) = \sum_{n=1}^{\infty} a_n n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots$$

Plugging y and y' into the ODE gives

$$=(a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots) + (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots)(a_0 + a_1 x + a_2 x^2 + \dots) = 1 \quad (2.12)$$

Setting $x = 0$ in Eq (2.12) gives

$$a_1 + a_0^2 = 1, \text{ then } a_1 = 1 - a_0^2$$

then differentiate Eq (2.12) with respect to x leads to

$$(2a_2 + 6a_3 x + 12a_4 x^2 + \dots) + (a_1 + 2a_2 x + 3a_3 x^2 + a_3 x^3 + \dots)(a_0 + a_1 x + a_2 x^2 + \dots) + (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots)(a_1 + 2a_2 x + 3a_3 x^2 + a_3 x^3 + \dots) = 0 \quad (2.13)$$

and set $x = 0$ implies

$$2a_2 + (a_1 \cdot a_0) + (a_0 \cdot a_1) = 0$$

$$2a_2 + 2a_0a_1 = 0, \text{ then } a_2 = -a_0a_1$$

$$a_2 = -a_0(1 - a_0^2)$$

and

$$a_2 = (-a_0 + a_0^3)$$

Differentiate Eq (2.13) with respect to x again and set $x = 0$ implies

$$6a_3 + (2a_2 \cdot a_0) + (a_1 \cdot a_1) + (a_1 \cdot a_1) + (a_0 \cdot 2a_2) = 0$$

or

$$a_3 = \frac{1}{3}(4a_0^2 - 3a_0^4 - 1)$$

Then the solution will be

$$y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$$

$$y(x) = a_0 + (1 - a_0^2)x + (a_0^3 - a_0)x^2 + \frac{1}{3}(-3a_0^4 + 4a_0^2 - 1)x^3 + \dots$$

The above mentioned technique can be programmed easily using Mathematica to solve the nonlinear differential equations in general. The solution of example [2.3] is presented in figure 2.1 which shows the series solution and the exact solution together, while figure 2.2 represents the error over the interval $[0, 0.4]$. Table 2.1 is the list of error values over the same interval for the same example.

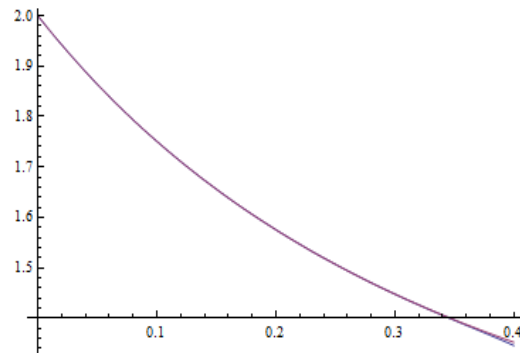


Figure 2.1: The exact and the power series solution for the nonlinear differential equation

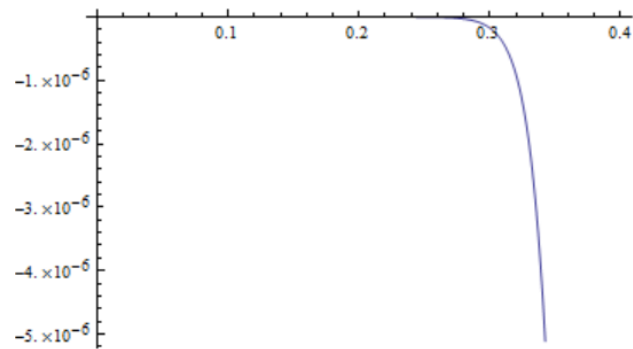


Figure 2.2: The error solution for the nonlinear differential equation

x_i	Exact solution	Error solution
0	2	0
0.1	1.75069	1.78913×10^{-6}
0.2	1.57546	5.21129×10^{-12}
0.3	1.44779	1.74146×10^{-7}
0.4	1.35205	0.000276044
0.5	1.27953	0.0826308

Table 2.1: The exact solution and the error over the interval $[0, 0.5]$

Example [2.4]: Consider the nonlinear DE

$$y' = \frac{y^2}{1-xy}, \quad y(0) = 1 \quad (2.14)$$

Substituting Eq (2.10) into Eq (2.14) yields

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \left(\sum_{n=0}^{\infty} a_n x^{n+1} \right) \left(\sum_{n=1}^{\infty} n a_n x^{n-1} \right) + \left(\sum_{n=0}^{\infty} a_n x^n \right)^2$$

or, equivalently we can write Eq (2.14) as

$$y' - xy y' - y^2 = 0 \quad (2.15)$$

let

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \quad y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Plugging y and y' into the Eq (2.15) gives

$$\begin{aligned}
&=(a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \dots) - x(a_0 + a_1x + a_2x^2 + a_3x^3 + \dots)(a_1 + \\
&2a_2x + 3a_3x^2 + 4a_4x^3 + \dots) - (a_0 + a_1x + a_2x^2 + a_3x^3 + \dots)(a_0 + a_1x + a_2x^2 + \\
&a_3x^3 + \dots) = 0
\end{aligned}$$

Therefore

$$a_0 = 1, \quad a_1 - a_0^2 = 0, \quad a_1 = a_0^2, \quad a_1 = 1, \quad a_2 = \frac{3}{2}, \quad a_3 = \frac{8}{3}$$

Therefore

$$y(x) = 1 + x + \frac{3}{2}x^2 + \frac{8}{3}x^3 + \dots$$

Chapter 3: Adomian Decomposition Method

3.1 Introduction

The Adomian decomposition method has proven to be very efficient in handling many problems in various fields of science and engineering such as physical and biological problems which can be modeled by differential equations that can be easily solved by Adomian's method. For example Al-Khaled and Allan [22] applied ADM to solve Voltera Fredholm equation. Abbasbandy and Darvishi used Adomian decomposition method to solve Burger-equation [4]. Other examples of the use of the Adomian Decomposition method can be found in [12-15].

The method it has been extended by Wazwaz to solve Voltera integral equation, Tan and Abbasbandy applied ADM to solve quadratic Riccati differential equation [40], recently Kamel Al-Khalid and Fathi Allan [13, 21] developed the ideas of Wazwaz study and applied it to high order nonlinear Voltera Fredholm differential equation.

The basic idea of ADM consists of splitting the given equation into two parts, linear part and nonlinear part, inverting the operator of the highest-order derivative contained on both sides in the linear operator, identifying the boundary or the initial conditions and the terms that involving the independent variables alone as initial approximation, decomposition the unknown function into a series whose components are to be determined, decomposing the nonlinear function in terms of special polynomials

called Adomian's polynomials, then finding the successive terms of the series solution by recurrent relation using Adomian polynomials.

The Adomian Decomposition method has many advantages. The main advantage of the method is that it can be applied directly for all types of differential and integral equations linear and nonlinear, homogeneous or inhomogeneous, with constant coefficients or with variable coefficients.

Another advantage is that the method is capable to reduce the size of computational work without reducing the high accuracy of the numerical solution.

Moreover one of the disadvantages of this method is that it gives a good approximation only in the neighborhood of the initial time, as it will be shown later. In addition, Allan [11] proved that the solution given by ADM is just a special case of the Homotopy Analysis method which will be discussed in the next chapter.

3.2 Adomian Decomposition method for nonlinear differential equation

Consider the differential equation

$$F(x, y(x)) = 0 \quad (3.1)$$

which can be split in two components

$$L(y(x)) + N(y(x)) = 0 \quad (3.2)$$

where N and L are the nonlinear and the linear parts of F . The operator L is assumed to be an invertible operator. Solving for $L(y)$ leads to

$$L(y) = -N(y) \quad (3.3)$$

Applying the inverse operator of Eq (3.3) leads to

$$y(x) = -L^{-1}(N(y)) + \vartheta(x) \quad (3.4)$$

where $\vartheta(x)$ is the constant of integration which satisfies the condition $L(\vartheta) = 0$. Now assuming that the solution $y(x)$ can be represented as infinite series of the form

$$y(x) = \sum_{n=0}^{\infty} y_n(x) \quad (3.5)$$

Furthermore, suppose that the nonlinear term $N(y)$ can be written as infinite series in terms of the Adomian polynomials A_n of the form

$$N(y) = \sum_{n=0}^{\infty} A_n \quad (3.6)$$

where the Adomian polynomials A_n of $N(y)$ are evaluated using the following formula:

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} N \left(\sum_{n=0}^{\infty} (\lambda^n y_n) \right) \Big|_{\lambda=0}$$

Then substituting Eqs (3.5) and (3.6) in Eq (3.4) gives

$$\sum_{n=0}^{\infty} y_n(x) = \vartheta(x) - L^{-1} \left(\sum_{n=0}^{\infty} A_n \right) \quad (3.7)$$

Then equating the terms in the linear system of Eq (3.7) gives the recurrent relation

$$y_0(x) = \vartheta(x) \quad y_{n+1} = -L^{-1}(A_n) \quad n \geq 0$$

Then, the solution will be estimated as $y(x) = \sum_{n=0}^N y_n(x)$. In the following sections we will provide some application of the ADM to certain class of DEs.

3.3 Numerical Results

Example [3.3a]: Consider the nonlinear DE.

$$y'(t) + y^2(t) = -1$$

with initial condition $y(0) = 0$

To solve this problem, the highest degree differential operator (written here as L) is put on the left side, in the following way

$$L(y) = -1 - y^2$$

with $L = \frac{d}{dt}$ and $L^{-1} = \int_0^t (\cdot) dx$

Assume the solution of the form

$$y = y_0 + y_1 + y_2 + y_3 + \dots$$

Replacing in the previous expression, we obtain

$$(y_0 + y_1 + y_2 + y_3 + \dots) = \vartheta(x) + L^{-1}[-1 - (y_0 + y_1 + y_2 + y_3 + \dots)^2]$$

Now we identify y_0 with some explicit expression on the right, and y_i , $i = 1, 2, 3, \dots$

with some expression on the right containing terms of lower order than i . For instance,

$$y_0 = y(0) + L^{-1}(-1) = -t$$

$$y_1 = -L^{-1}(A_0) = -L^{-1}(y_0^2) = -L^{-1}(t^2) = -t^3/3$$

$$y_2 = -L^{-1}(A_1) = -L^{-1}(2y_0y_1) = -2t^5/15$$

$$y_3 = -L^{-1}(y_1^2 + 2y_0y_2) = -17t^7/315$$

In this way any contribution can be explicitly calculated at any order. If we settle for the four first terms, the approximant is the following

$$\begin{aligned}
 y &= y_0 + y_1 + y_2 + y_3 + \dots \\
 &= - \left[t + \frac{1}{3}t^3 + \frac{2}{15}t^5 + \frac{17}{315}t^7 + \dots \right]
 \end{aligned}$$

Example [3.3b]: Consider the nonlinear differential equation

$$y'(t) + y^2(t) = 1, \quad y(0) = 2$$

Solution

The differential equation can be split into the two parts $L(y) = y' - 1$ and $N(y) = -y^2$,

where $L = \frac{d}{dt}$, and $L(y) = -N(y)$

Let

$$y = y_0 + y_1 + y_2 + y_3 + \dots$$

then

$$(y_0 + y_1 + y_2 + y_3 + \dots) = y(0) + L^{-1}[-1 - (y_0 + y_1 + y_2 + y_3 + \dots)^2]$$

$$y_0 = y(0) + L^{-1}(1) = (2 + t)$$

$$y_1 = -L^{-1}(y_0^2) = -L^{-1}(2 + t)^2 = -\left(4t + 2t^2 + \frac{t^3}{3}\right)$$

$$\begin{aligned}
 y_2 &= -L^{-1}(2y_0y_1) = -L^{-1}(2(2 + t)\left(-4t - 2t^2 - \frac{t^3}{3}\right)) \\
 &= -\left(-8t^2 - \frac{16}{3}t^3 - \frac{4}{3}t^4 + \dots\right)
 \end{aligned}$$

$$y_3 = -L^{-1}(y_1^2 + 2y_0y_2) = -\left(\frac{48}{3}t^3 + \frac{142}{12}t^4 + \dots\right)$$

then

$$\begin{aligned}
y &= y_0 + y_1 + y_2 + y_3 + \dots \\
&= [2 - 3t + 6t^2 - 11t^3 + \dots]
\end{aligned}$$

Example [3.3c]: Consider the nonlinear ordinary differential equation [11] given by

$$y' = \frac{y^2}{1-xy}, \quad y(0) = 1 \quad (3.8)$$

Using the ADM, Eq (3.8) can be written as

$$\begin{aligned}
y'(1 - xy) &= y^2 \\
y' &= xy y' + y^2 \\
L(y) &= xy y' + y^2 \quad (3.9)
\end{aligned}$$

Applying L_x^{-1} to both sides of Eq (3.9) and using the initial condition we find

$$y = L^{-1}(-N(y)). \text{ Now, let } L(y) = 0. \text{ Then } L^{-1}(xy y') + L^{-1}(y^2) = 0$$

thus

$$\begin{aligned}
y(x) &= 1 + L^{-1}(xy y') + L^{-1}(y^2) \\
y(x) &= 1 + xL^{-1}\left(\sum_{n=0}^{\infty} A_n\right) + L^{-1}\left(\sum_{n=0}^{\infty} B_n\right)
\end{aligned}$$

The decomposition method expands each of the nonlinear terms yy' and y^2 formally in a power series, given by

$$\begin{aligned}
yy' &= A_0 + A_1 + A_2 + A_3 + \dots = \sum_{n=0}^{\infty} A_n \\
y^2 &= B_0 + B_1 + B_2 + B_3 + \dots = \sum_{n=0}^{\infty} B_n
\end{aligned}$$

where A_n and B_n are so called Adomian polynomials corresponding to the nonlinear terms yy' and y^2 , respectively, and

$$A_0 = y_0 y_0' = 1(0) = 0$$

$$B_0 = y_0^2 = (1)^2 = 1$$

thus

$$y_1 = L_x^{-1}(xA_0) + L_x^{-1}(B_0) = x$$

$$\begin{aligned} A_1 &= \frac{d}{d\lambda} [(y_0 + \lambda y_1)(y_0 + \lambda y_1)'] \\ &= [y_1(y_0 + \lambda y_1)' + (y_0 + \lambda y_1)y_1'] \end{aligned} \quad (3.10)$$

Substitute $\lambda = 0$ in (3.10) implies

$$A_1 = [y_1(y_0)' + (y_0)y_1']$$

thus

$$B_1 = \frac{d}{d\lambda} [(y_0 + \lambda y_1)^2] = 2(y_0 + \lambda y_1)y_1 \quad (3.11)$$

substitute $\lambda = 0$, in (3.11) implies

$$B_1 = 2y_0 y_1$$

$$y_2 = L_x^{-1}(xA_1) + L_x^{-1}(B_1) = \frac{3x^2}{2}$$

$$\begin{aligned} A_2 &= \frac{d^2}{d\lambda^2} [(y_0 + \lambda y_1 + \lambda^2 y_2)(y_0 + \lambda y_1 + \lambda^2 y_2)'] \\ &= (2y_2)(y_0' + \lambda y_1' + \lambda^2 y_2') + (y_1 + 2\lambda y_2)(y_1' + 2\lambda y_2') + (y_1 + 2\lambda y_2) (y_1' + \\ &\quad 2\lambda y_2') + (y_0 + \lambda y_1 + \lambda^2 y_2)(2y_2') \end{aligned} \quad (3.12)$$

substitute $\lambda = 0$, in (3.12) implies

$$A_2 = 2y_2y_0' + 2y_1y_1' + 2y_0y_2'$$

$$B_2 = \frac{d^2}{d\lambda^2} [(y_0 + \lambda y_1 + \lambda^2 y_2)^2]$$

$$B_2 = \frac{d}{d\lambda} 2(y_0 + \lambda y_1 + \lambda^2 y_2)(y_1 + 2\lambda y_2)$$

$$B_2 = 2(y_1 + 2\lambda y_2)(y_1 + 2\lambda y_2) + (y_0 + \lambda y_1 + \lambda^2 y_2)(2y_2) \quad (3.13)$$

Substitute $\lambda = 0$ in (3.13), implies

$$B_2 = 2y_1^2 + 4y_0y_2$$

$$y_3 = L_x^{-1}(xA_2) + L_x^{-1}(B_2) = \frac{8}{3}x^3$$

The solution of Eq (3.8) in a series form is therefore

$$y(x) = 1 + x + \frac{3}{2}x^2 + \frac{8}{3}x^3 + \dots$$

From the above discussion we can verified, the components y_0, y_1, y_2, \dots were determined easily in a recurrent way. Only simple integrals were used in the computational work. It was indicated that the accuracy can be improved by adding more terms.

Chapter 4: Homotopy Analysis Method (HAM)

4.1 Introduction

The Homotopy Analysis method , which was first introduced in 1992 by Liao [24, 26, 28] , is another method used to derive an analytic solution for nonlinear operators. It consists of introducing embedding parameters and embedding operators where the solution is assumed to depend continuously on these parameters. The method has been used by several authors and proved to be very effective in deriving an analytic solution of nonlinear differential equations [1, 2, 25, 29, 30].

4.2 The Homotopy Analysis method for the solution of $y(x) = 0$

To explain HAM, consider a nonlinear equation

$$y(x) = 0 \tag{4.1}$$

First of all we construct such a Homotopy

$$\mu[x, q] = (1 - q)[y(x) - y(x_0)] + qy(x) \tag{4.2}$$

where (x_0) is an initial guess and (q) is called Homotopy parameter, or including parameter.

Now, at $q = 0$ and $q = 1$, one can get

$$\mu[x, 1] = y(x), \text{ and } \mu(x, 0) = y(x) - y(x_0)$$

Thus as q increases from 0 to 1, $\mu[x, q]$ varies continuously from $y(x) - y(x_0)$ to $y(x)$

Now let $\mu(x, q) = 0$

$$(1 - q)[y(x) - y(x_0)] + qy(x) = 0 \quad (4.3)$$

Since the above family of algebraic equations depends on the Homotopy parameter q , then

we can set $x = \phi(q)$,

then (4.3) can be written as

$$(1 - q)[y(\phi(q)) - y(x_0)] + qy(\phi(q)) = 0 \quad (4.4)$$

At $q = 0$ we have

$$y(\phi(0)) - y(x_0) = 0$$

We may take $\phi(0) = x_0$, then the function $\phi(q)$ agrees with the initial approximation at

$q = 0$

More over at $q = 1$, we have $y(\phi(1)) = 0$, then $\phi(1)$ is a solution of the nonlinear algebraic equation.

Therefore, as the Homotopy parameter q increases from 0 to 1, $\phi(q)$ varies from initial guess (x_0) to the solution of $y(x) = 0$.

Equations like (4.4) is called a zeroth-order deformation equation.

Now, expand $\phi(q)$ as a Maclurian series, gives the Homotopy series

$$\phi(q) = x_0 + \sum_{k=1}^{\infty} x_k q^k \quad (4.5)$$

Let $x_k = D_k(\phi)$, Where $D_k(\phi)$ is called the k th-order homotopy derivatives of ϕ

then,

$$x_k = \frac{1}{k!} \left. \frac{\partial^k \phi(q)}{\partial q^k} \right|_{q=0}$$

According to the theory of Taylor series, this power series is unique as one regards q as a small parameter.

If the Homotopy series (4.5) is convergent at $q = 1$, then using the relationship $\emptyset(1) = x$, one has the so called Homotopy series solution

$$x = x_0 + \sum_{k=1}^{\infty} x_k$$

Now, by taking the first order Homotopy derivative on both sides of the zeroth order Homotopy derivative equation (4.4) gives the so called first order deformation equation:

$$x_1 y'(x_0) + y(x_0) = 0, \text{ then } x_1 = \frac{-y(x_0)}{y'(x_0)}$$

After that, if we take the second order Homotopy derivative on both sides of the zeroth order deformation equation (4.4), which means differentiate twice with respect to q , then divide by $2!$ and finally set $q = 0$, we get $x_2 y'(x_0) + \frac{1}{2} x_1^2 y''(x_0) = 0$

$$\text{whose solution is } x_2 = \frac{-x_1^2 y''(x_0)}{2y'(x_0)} = \frac{-y^2(x_0) y''(x_0)}{2[y'(x_0)]^3}$$

Finally, the first order Homotopy series approximation,

$$x \cong x_0 + x_1, \text{ and the second order Homotopy series approximation } x \cong x_0 + x_1 + x_2$$

To control the radius of convergence of the series solution, Liao introduced an auxiliary parameter $\hbar \neq 0$.

The new kind of zeroth order deformation equation becomes like

$$(1 - q)\{y(\emptyset(q)) - y(x_0)\} = q\hbar y(\emptyset(q)) \quad (4.6)$$

Now we have the corresponding first order deformation equation by taking first order Homotopy derivative on both sides of (4.6) and setting $q = 0$ leads to

$$x_1 y'(x_0) - \hbar y(x_0) = 0$$

Therefore, $x_1 = \frac{\hbar y(x_0)}{y'(x_0)}$

By the same way if we take the second order Homotopy derivative on both sides of (4.4) we can have the corresponding second order deformation equation

$$x_2 y'(x_0) - (1 + \hbar)x_1 y'(x_0) + \frac{1}{2} x_1^2 y''(x_0) = 0$$

On the other hand x_1 and x_2 given by (4.4) are special case of x_1 and x_2 given by (4.6) with $\hbar = -1$.

4.3 The Homotopy Analysis method for the solution of the nonlinear differential equation

Consider the differential equation

$$N[y(x)] + L[y(x)] = 0$$

where N is a nonlinear operator, x denotes the independent variables and y is the unknown function.

By the means of HAM

$$(1 - q)L[(\phi(x; q) - y_0(x))] = q\hbar[N[\phi(x; q)] + L[\phi(x; q)]] \quad (4.7)$$

Where $q \in [0, 1]$ is the embedding parameter, $\hbar \neq 0$ is an auxiliary parameter used to control the error, L is an auxiliary linear operator, $\phi(x, q)$ is an unknown function, $y_0(x)$ is an initial guess.

When the embedding parameter $q = 0$ and $q = 1$, equation (4.7) becomes $\phi(x; 0) = y_0(x)$ and $\phi(x; 1) = y(x)$.

Now, as q increase from 0 to 1, the solution will be change from the initial guess $y_0(x)$ to the solution $y(x)$.

By expanding $\phi(x; q)$ in Taylor series with respect to q ,

$$\phi(x; q) = y_0(x) + \sum_{m=1}^{\infty} y_m(x)q^m,$$

$$\text{where } y_m(x) = \frac{1}{m!} \frac{\partial^m \phi(x; q)}{\partial q^m} \Big|_{q=0}$$

If the auxiliary parameter \hbar is convergent at $q = 1$, then

$$y(x) = y_0(x) + \sum_{m=1}^{\infty} y_m(x)$$

which must be the original solution of the nonlinear equation.

4.4 Examples of HAM

Example [4.1]: Consider the nonlinear problem $y' - y^2 = 0$, $y(0) = 1$

Let, $L(y) = y'$

then, $y'_0(x) = 0$, $y_0 = A$, $A = 1$, $y_0(x) = 1$

$$(1 - q)(y'(x) - y_0'(x)) = \hbar q(y'(x) - y^2(x))$$

$$y = y_0 + y_1q + y_2q^2 + y_3q^3 + \dots$$

$$\begin{aligned} (1 - q)(y'_0 + y'_1q + y'_2q^2 + y'_3q^3 + \dots - y'_0(x)) \\ = \hbar q(y'_0 + y'_1q + y'_2q^2 + y'_3q^3 + \dots) \\ - (y_0 + y_1q + y_2q^2 + y_3q^3 + \dots)^2 \end{aligned}$$

From the first derivative

$$\begin{aligned} (-1)(y'_1q + y'_2q^2 + y'_3q^3 + \dots) + (1 - q)(y'_1 + 2y'_2q + 3y'_3q^2 + \dots) \\ = \hbar(y'_0 + y'_1q + y'_2q^2 + y'_3q^3 + \dots - (y_0 + y_1q + y_2q^2 + y_3q^3 + \dots)^2) \\ + \hbar q(y'_1 + 2y'_2q + 3y'_3q^2 + \dots \\ - 2((y_0 + y_1q + y_2q^2 + y_3q^3 + \dots) - (y_1 + 2y_2q + 3y_3q^2 + \dots))) \end{aligned}$$

Take $q = 0$, to get

$$y'_1 = \hbar y'_0 - \hbar y_0^2$$

$$y'_1 = -\hbar y_0^2 = -\hbar$$

$$y_1 = -\hbar x + c$$

$$\text{Since } y_1(0) = 0, \quad c = 0$$

$$y_1 = -\hbar x$$

From the second derivative

$$\begin{aligned}
& (-1)(y'_1 + 2y'_2q + 3y'_3q^2 + \dots) + (1 - q)(2y'_2 + 6y'_3q) \\
& \quad + (-1)(y'_1 + 2y'_2q + 3y'_3q^2 + \dots) \\
& = \hbar(y'_1 + 2y'_2q + 3y'_3q^2 + \dots) \\
& \quad - 2(y_0 + y_1q + y_2q^2 + y_3q^3 + \dots)(y_1 + 2y_2q + 3y_3q^2 + \dots) \\
& \quad + \hbar(y'_1 + 2y'_2q + 3y'_3q^2 + \dots) \\
& \quad - 2(y_0 + y_1q + y_2q^2 + y_3q^3 + \dots)(y_1 + 2y_2q + 3y_3q^2 + \dots) \\
& \quad + \hbar q(2y'_2 + 6y'_3q + \dots)
\end{aligned}$$

Take $q = 0$

$$-y'_1 + 2y'_2 - y'_1 = \hbar y'_1 - 2y_0y_1\hbar + \hbar y'_1 - 2y_0y_1\hbar$$

$$2y'_2 = 2y'_1 + 2\hbar y'_1 - 4y_0y_1\hbar$$

$$y'_2 = y'_1 + \hbar y'_1 - 2y_0y_1\hbar$$

$$y_2 = -\hbar x + 1 + \hbar(-\hbar x) - 2(1)(-\hbar x)\hbar$$

$$y_2 = -\hbar x - \hbar^2 x + \hbar^2 x^2$$

Then the third derivative

$$\begin{aligned}
& (-1)(2y'_2 + 6y'_3q + \dots) + (-1)(2y'_2 + 6y'_3q + \dots) + (1 - q)(6y'_3 + \dots) \\
& \quad + (-1)(2y'_2 + 6y'_3q + \dots) \\
& = \hbar((2y'_2 + \dots) \\
& \quad - 2(y_1 + 2y_2q + 3y_3q^3 + \dots)(y_1 + 2y_2q + 3y_3q^2 + \dots) \\
& \quad - 2(y_0 + y_1q + y_2q^2 + y_3q^3 + \dots)(2y_2 + 6y_3q + \dots)) \\
& \quad + \hbar((2y'_2 + 6y'_3q + \dots) \\
& \quad - 2(y_1 + 2y_2q + 3y_3q^2 + \dots)(y_1 + 2y_2q + 3y_3q^2) \\
& \quad - 2(y_0 + y_1q + y_2q^2 + \dots)(2y_2 + 6y_3q + \dots)) \\
& \quad + \hbar((2y'_2 + 6y'_3q + \dots) \\
& \quad - 2(y_1 + 2y_2q + 3y_3q^2 + \dots)(y_1 + 2y_2q + 3y_3q^2) \\
& \quad - 2(y_0 + y_1q + y_2q^2 + \dots)(2y_2 + 6y_3q + \dots)) + \hbar q(6y'_3 + \dots).
\end{aligned}$$

Take $q = 0$

$$y'_3 = y'_2 + \hbar y'_2 - \hbar y_1^2 - 2y_0y_2\hbar$$

$$y'_3 = -\hbar - 2\hbar^2 + 4\hbar^2x - \hbar^3 + 4\hbar^3x - 3\hbar^3x^2$$

$$y_3 = -\hbar x - 2\hbar^2x + 2\hbar^2x^2 - \hbar^3x + 2\hbar^3x^2 - \hbar^3x^3$$

Then, $y(x) = y_0 + y_1(x) + y_2(x) + y_3(x)$

Figure 4.1 shows the \hbar curve which is the variation of the value of $y'(0)$ as the parameter \hbar varies over the interval $[-2, 0]$. It shows a horizontal line over specific values of \hbar .

So choosing any value of \hbar on the interval where the curve is horizontal will guarantee the convergence of the series.

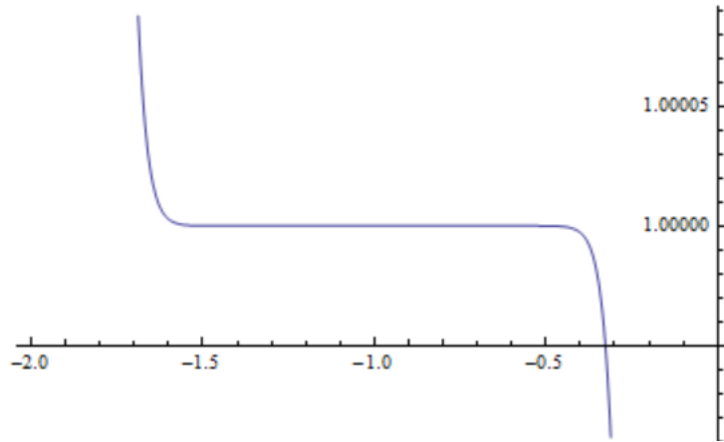


Figure 4.1: The \hbar -curve for $y'(0)$ by using Homotopy Analysis method

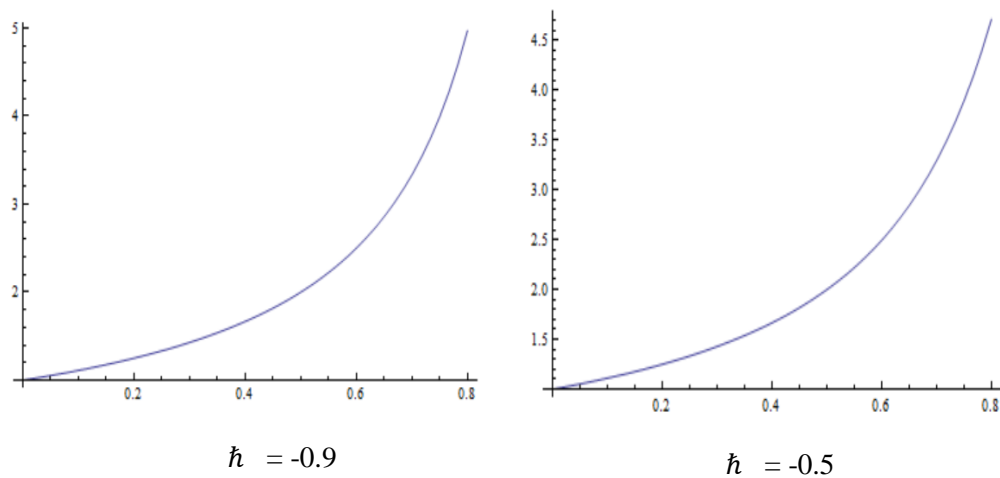


Figure 4.2: The solution for x between $[0, 0.8]$ for $\hbar = -0.9$ and $\hbar = -0.5$

Figure 4.2 shows the series solutions for the two different values of \hbar that resulted from the Homotopy Analysis method on the interval $[0, 0.8]$.

As it is clear, one can find that the solution depends on the parameter \hbar as well. This parameter is used to control the error by watching the value of $y'(0)$ for different values

of \hbar . It is clear that the value $y'(0)$ is stable for all values of \hbar in the interval between $(-1.5, -0.3)$. So picking \hbar in the interval should decrease the error.

Example [4.2]: Consider the nonlinear problem

$$y'' + y^2 = 0, \quad y(0) = 1 \quad y'(0) = 0$$

Using the same procedures described above, one can find the \hbar -curve shown in fig 4.3

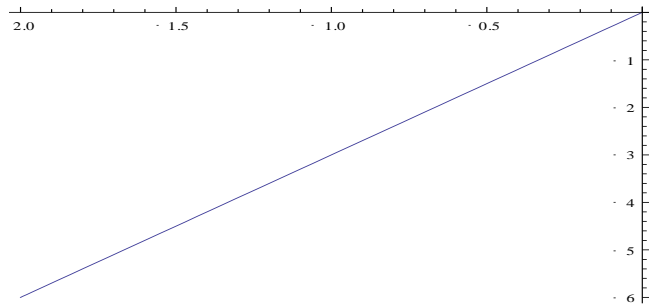


Figure 4.3: The \hbar -curve for $y''(0)$ using the Homotopy Analysis method

Figure 4.3 shows that the horizontal line is not achieved, which means that \hbar cannot be used as an error controller for this problem. This initiates the need for a modification of the method. This will be achieved in the next chapter.

Chapter 5: Modified Taylor series solution method

5.1 Modified Taylor series solution of initial value problems with singularities

5.1.1 Introduction

In mathematics, in general a singularity is a point at a given mathematical object is not defined, or a point where it fails to be well-behaved of an exceptional set in some particular way, such as many problems in various fields give rise to ODE of the form

$$y'' = f(x, y, y'), \quad a < x < b \quad (5.1)$$

For some interval of the real line with initial condition. An initial value problem associated to the second order DE (5.1) is singular if one of the following situations occurs. The interval (a, b) infinite if $f(x, y, y')$ is unbounded at some value of y' or y or $f(x, y, y')$ is unbounded at some $x_0 \in [0, 1]$.

Consider the n^{th} order linear differential equation of the form

$$f_n(x)y^n + f_{n-1}(x)y^{n-1} + \dots + f_1(x)y' + f_0(x)y = 0 \quad (5.2)$$

subject to the initial conditions:

$$y(x_0) = \alpha_0, y'(x_0) = \alpha_1, \dots, y^{n-1}(x_0) = \alpha_{n-1} \quad (5.3)$$

The point x_0 is said to be a singular point if $f_n(x_0) = 0$ and it is said to be a regular point for the differential equation (5.2) if all functions $(x - x_0) \frac{f_{n-1}}{f_n}$, $(x - x_0)^2 \frac{f_{n-2}}{f_n}, \dots, (x - x_0)^{n-1} \frac{f_0}{f_n}$ are analytic at x_0 . Otherwise, the point is said to be an irregular singular point. It is very well known that near the regular singular point

one can find the series expansion of the solution using the well-known method of Frobenius. For the differential equations with irregular singular points, the series expansion of the solution near the irregular singular points is not possible. However, there are special treatments for these kinds of problems such that the exponential transformation method, where the transformation is given by

$$y(x) = \text{Exp}(F(x))Y(x) \quad (5.4)$$

where

$$F(x) = \frac{A_k}{(x - x_0)^k} + \frac{A_{k-1}}{(x - x_0)^{k-1}} + \frac{A_{k-2}}{(x - x_0)^{k-2}} + \dots + \frac{A_1}{(x - x_0)^1}$$

and

$$Y(x) = \sum_{i=0}^{\infty} a_i (x - x_0)^{i+r}$$

is the Frobenius series. Then finding all derivatives of the function defined by (5.4) and plug in the differential equation (5.2) after dividing by the function $f_n(x)$, one can find the unknown coefficients A_k and a_i and the constant r . The difficulty associated with this approach is that the determination of the coefficients is time consuming and too much involved.

Numerical techniques available for the solution of these kinds of problems also face the fact that the initial conditions are given at x_0 and the evaluation of the terms of the Taylor series solution at any step requires dividing by $x - x_0$ which results in overflow and therefore will fail to obtain the solution.

Taylor series solution of ordinary differential equations is very well known practice for many years. Recently it has been proven that Taylor series expansion is very efficient in handling special type of differential equations. Huabsomboon et al [20] used the Taylor-series expansion methods to solve the second kind integral equations. Maleknejad et al [32] obtained numerical solution for the second kind Fredholm integral equations system by using a Taylor-series expansion method. Maleknejad and Aghazadeh [31] employed the Taylor series expansion to solve numerically the Volterra integral equations of the second kind with convolution kernel. Hile Ren et al [38] considered the solution of a class of second kind integral equations using simple Taylor series expansion method.

The series terms can be calculated term by term such as the direct Taylor series expansion [20, 32, 38] or collectively such as the Adomian Decomposition method [22, 42] and the Homotopy Analysis method [12, 19, 23, 40].

For the Initial value problems, it is well known that the Taylor series expansion [24, 31, 41] of the solution will be about the initial point and the error is controlled by adding more terms to guarantee the convergence of the solution over a specific interval. Sometimes adding more terms might be difficult and time consuming which is the case of Adomian Decomposition method [11] and the Homotopy Analysis method [14, 29]. As discussed in the previous chapter, to control the error of the Homotopy Analysis method an additional parameter, \hbar is used and the behavior of the solution is monitored at one of the initial conditions as the parameter \hbar varies. However not much theory is available to

support the use of this additional parameter. This idea initiates a new approach to decrease the error and increase the convergence of the solution.

In this chapter we will discuss a new approach to solve initial value problem using the series expansion method [33, 34]. The method is based on different choices of the expansion point used to expand the solution about. Details properties of the suggested method will be discussed and different linear and nonlinear initial value problems will be presented to demonstrate the efficiency of the method in the following.

Example [5.1]:

Consider the following equation

$$x^2y'' - 2y = 0$$

which has a singular point at $x = 0$.

It can be shown by direct substitution that the following functions are linearly independent solutions for $x \neq 0$

$$y_1(x) = x^2, \quad y_2(x) = x^{-1}$$

Thus, in any interval not containing the origin, the general solution is $y(x) = c_1x^2 + c_2x^{-1}$. Thus, $y = c_1x^2$ is bounded and analytic at the origin.

However, $y = c_2x^{-1}$ does not have a Taylor series expansion about $x = 0$, for that the standard numerical techniques will fail, because the use of the standard Taylor series expansion of the solution near the irregular singular point will not be possible due to the fact that evaluating the series coefficients will require a division by 0. In what follows,

we will develop a new approach to control the error in the Taylor series expansion of analytic functions. The approach will be used to solve differential equations with irregular singularities and eigenvalue problems.

5.1.2 The Taylor series expansion of the function $f(x)$

Consider the Taylor series expansion of the function $f(x)$ about x_0

$$f(x) = T_n(x, x_0) + R_n(x, x_0) \quad (5.5)$$

where

$$T_n(x, x_0) = \sum_0^n a_n (x - x_0)^n$$

and

$$R_n(x, x_0) = \frac{f^{n+1}(\xi)}{(n+1)!} (x - x_0)^{n+1}$$

The variable ξ is between x and x_0 . Let us assume $|f^{n+1}(\xi)| \leq M$ then for fixed x_0 , one can control the error and increase the interval of convergence of the series either by adding more terms which might be costly in most cases, or one can reduce the error by changing the expansion point x_0 .

However, looking at the error term Eq (5.5) one can see that the error is also controlled by the quantity $(x - x_0)^n$. So selecting different x_0 will give different values for the error and the closer the point x_0 to the point x , the less error will be obtained.

The relation between the interval of convergence and the choice of x_0 for a fixed number of terms is given by the following theorem.

Theorem (5.1): Let $f(x)$ be analytic on the interval I and assume that $|f^n(x)| \leq M$ for all $x \in I$ and all $n \in \mathbb{N}$ then given $\varepsilon > 0$ and $n_0 \in \mathbb{N}$, there exists an interval $[a, b] \subseteq I$ such that $|T_{n_0}(x, x_0) - f(x_0)| < \varepsilon$ for all $x_0 \in [a, b]$ and $n \geq n_0$.

Proof

The proof will be given by finding the interval of convergence directly

Consider the error term in the Taylor series expansion of the function $f(x)$, then

$$|T_{n_0}(x, x_0) - f(x_0)| = \left| \frac{f^{n_0+1}(\xi)}{(n+1)!} (x - x_0)^{n_0+1} \right| < \varepsilon$$

and

$$\left| \frac{f^{n_0+1}(\xi)}{(n+1)!} (x - x_0)^{n_0+1} \right| < \left| \frac{M}{(n+1)!} (x - x_0)^{n_0+1} \right| < \varepsilon$$

then

$$|(x - x_0)^{n_0+1}| < \frac{\varepsilon(n_0+1)!}{M}$$

Then solving for x_0 yields

$$-\sqrt[n_0+1]{\frac{\varepsilon(n_0+1)!}{M}} + x < x_0 < \sqrt[n_0+1]{\frac{\varepsilon(n_0+1)!}{M}} + x \quad (5.6)$$

Therefore the required interval is $[a, b] = \left[-\sqrt[n_0+1]{\frac{\varepsilon(n_0+1)!}{M}} + x, \sqrt[n_0+1]{\frac{\varepsilon(n_0+1)!}{M}} + x \right]$

The application of the method to certain examples is presented below.

Example [5.2]:

Consider the Taylor series expansion of the function $f(x) = e^x$ on the interval $[0,2]$. The rate of decrease in the error is proportional to the number of terms. The error of e^x is bounded by,

$$\text{error} = \frac{e^\zeta}{(n+1)!} x^{n+1} \leq \frac{e^2}{(n+1)!} x^{n+1}$$

So it is expected that the error will increase as $x \rightarrow 2$. Now using Eq (5.6) with

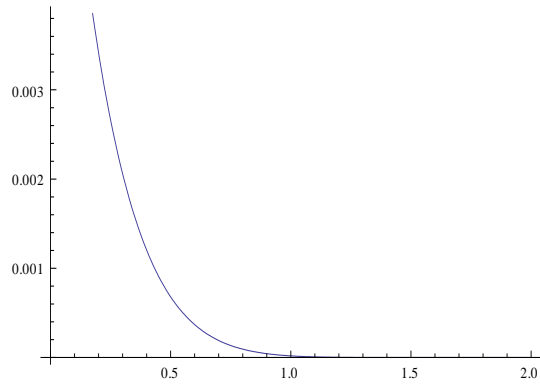
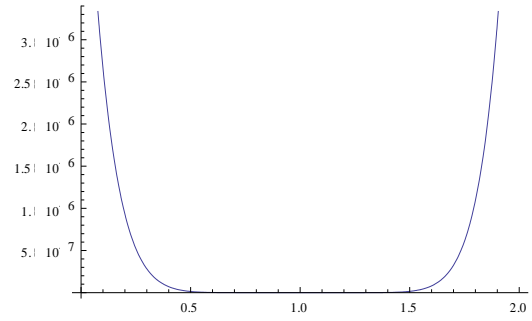
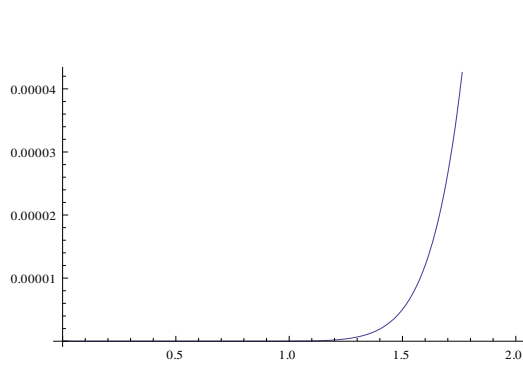
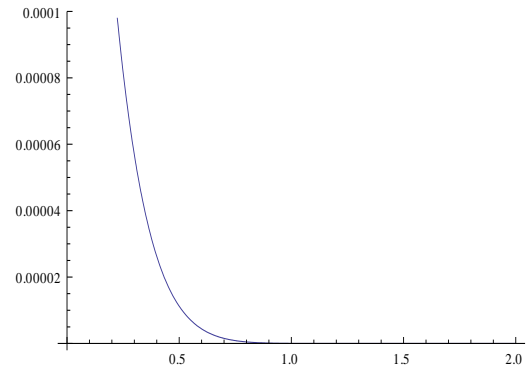
$N \geq n_0 = 8$, $\varepsilon = 0.000001$, and $M = e^2$, the new interval will be

$$x - \sqrt[9]{0.000001 * \frac{9!}{e^2}} < x_0 < x + \sqrt[9]{0.000001 * \frac{9!}{e^2}}$$

One can reduce the error by taking x_0 in the interval $[0,2]$ provided that $N \geq n_0 = 8$

$$- \sqrt[9]{0.000001 * \frac{9!}{e^2}} + x < x_0 < \sqrt[9]{0.000001 * \frac{9!}{e^2}} + x$$

and the lower end is greater than 0 and the upper end less than 2. Figure 5.1 shows the variation of the error as x_0 varies in the interval $[0, 2]$. The error associated with the different choices of x_0 is presented in tables 5.1 through 5.4

(a) $x_0 = 0.5$ (b) $x_0 = 1.0$ (c) $x_0 = 1.5$ (d) $x_0 = 2.0$ Figure 5.1: The variation of the error as x_0 varies in the interval $[0, 2]$

x_0	Error
0	8.44956×10^{-9}
0.2	8.6817×10^{-11}
0.4	4.66294×10^{-15}
0.6	4.21885×10^{-15}
0.8	9.21867×10^{-11}
1.0	9.33863×10^{-9}
1.2	1.97045×10^{-7}
1.4	1.93264×10^{-6}
1.6	0.0000120213
1.8	0.0000552736
2.0	0.000204942

Table 5.1: The error obtained for example [5.2] where $x_0 = 0.5$

x_0	Error
0	6.8046×10^{-6}
0.2	9.30456×10^{-7}
0.4	7.11964×10^{-8}
0.6	1.8879×10^{-9}
0.8	3.75966×10^{-12}
1.0	0.0
1.2	3.91331×10^{-12}
1.4	2.04519×10^{-9}
1.6	8.028×10^{-8}
1.8	1.0921×10^{-6}
2.0	8.31419×10^{-6}

Table 5.2: The error obtained for example [5.2] where $x_0 = 1.0$

x_0	Error
0	0.000412194
0.2	0.000115757
0.4	0.0000262114
0.6	4.38691×10^{-6}
0.8	4.65591×10^{-7}
1.0	2.29683×10^{-8}
1.2	2.35993×10^{-10}
1.4	1.06581×10^{-14}
1.6	1.24345×10^{-14}
1.8	2.50588×10^{-10}
2.0	2.5385×10^{-8}

Table 5.3: The error obtained for example [5.2] where $x_0 = 1.5$

x_0	Error
0	0.0086648
0.2	0.00341533
0.4	0.0012041
0.6	0.000368514
0.8	0.0000937065
1.0	0.0000184968
1.2	2.52924×10^{-6}
1.4	1.93532×10^{-7}
1.6	5.13184×10^{-9}
1.8	1.02212×10^{-11}
2.0	0.0

Table 5.4: The error obtained for example [5.2] where $x_0 = 2.0$

Table 5.1, 5.2, 5.3 and 5.4 show the error associated with the series estimation of the solution of example [5.2]. The error is relatively very small for all $x \in [0,2]$.

5.1.3 Solution of initial value problems with irregular singular points

In what follows we will discuss the use of the Modified Taylor series to solve initial value problems [14, 20]. As mentioned earlier, the use of the standard Taylor series expansion of the solution near the irregular singular point will not be possible due to the fact that evaluating the series coefficients will require a division by 0. In addition the standard numerical techniques will fail as well. This initiates the needs for a new method to solve these types of problems.

Consider the second order initial value problem

$$y'' + p(x)y' + q(x)y = r(x)$$

Subject to the initial conditions

$$y(a) = \alpha, \quad y'(a) = \beta$$

One can assume that the solution is given by the series.

$$y(x) = \sum_{n=0}^N a_n (x - x_0)^n + \mathcal{R}_N$$

with

$$\mathcal{R}_N = \frac{y^{N+1}(\zeta)}{N+1} (x - x_0)^{N+1}$$

With the initial choice

$$y(x_0) = a_0 = \alpha, \quad y'(x_0) = a_1 = \beta$$

Then ignoring the error term and using the series in the original differential equations to evaluate the rest of the coefficients as we will see in the next example.

We present in this section numerical results for three types of initial value problem which illustrate the method of Taylor series expansion.

The first one of these is linear initial value problem, the second one is for initial value problem with regular singular point and the last one about initial value problem with irregular singular point.

Example [5.3]: Consider the linear second order initial value problem

$$y'' + y' - 6y = 0$$

Subject to

$$y(0) = 1, \quad y'(0) = 1$$

With exact solution $y(x) = \frac{1}{5}e^{-3x} + \frac{4}{5}e^{2x}$

Then using the procedure describe above and assuming that $y(x_0) = a_0 = \alpha$ and $y'(x_0) = a_1 = \beta$, one can find that the rest of the coefficients are given by the following:

$$a_2 = -\frac{\alpha}{2}, \quad a_3 = \frac{-\beta}{6}, \quad a_4 = \frac{\alpha}{24}, \dots, \quad a_{2n} = \frac{(-1)^n}{2n!} \alpha, \quad a_{2n+1} = \frac{(-1)^n}{(2n+1)!} \beta$$

Then the solution $y(x)$ will be given in terms of x , x_0 , α and β . To find the values of α and β as a function of x_0 we need to solve the two equations $y(0) = 1$, and $y'(0) = 1$ for both α and β .

Assuming that we will take only 5 terms of the series expansion of the solution and solving for α and β leads to the series solution $T_n(x, x_0)$ depends on both x and x_0 ,

$$y(x, x_0) = T_5(x, x_0).$$

In order to verify numerically the accuracy of the error produced by the modified Taylor series expansion method, the numerical results in figure 5.2, 5.3 and table 5.5, 5.6, 5.7, 5.8 show a good approximation is achieved by choosing different values of x_0 .

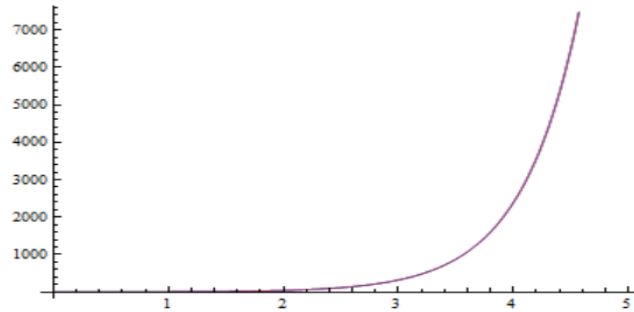
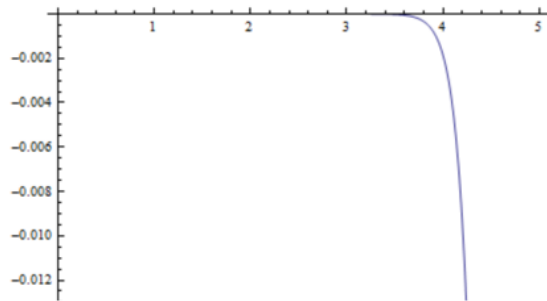
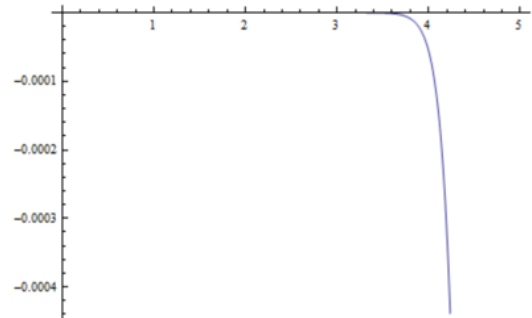
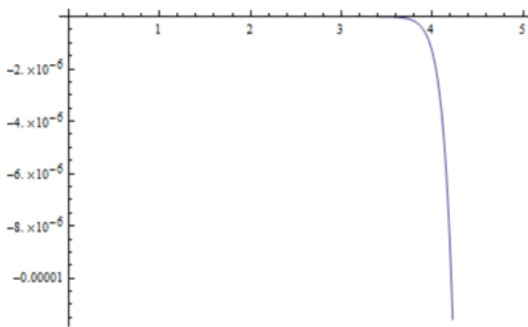
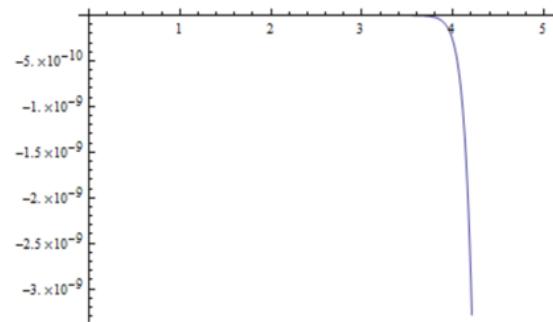


Figure 5.2: The series and the exact solution for the differential equation of example [5.3]

(a) $x_0 = 0.3$ (b) $x_0 = 0.6$ (c) $x_0 = 0.9$ (d) $x_0 = 1.5$ Figure 5.3: The variation of the error as x_0 varies in the interval $[0, 5]$

x_0	Error
0	0
0.2	2.22045×10^{-16}
0.4	4.44089×10^{-16}
0.6	0
0.8	4.44089×10^{-16}
1.0	1.77636×10^{-15}
1.2	1.77636×10^{-15}
1.4	3.55271×10^{-15}
1.6	7.10543×10^{-15}
1.8	0
2.0	6.39488×10^{-14}

Table 5.5: The error obtained for example [5.3] where $x_0 = 0.3$

x_0	Error
0	3.33067×10^{-16}
0.2	4.44089×10^{-16}
0.4	8.88178×10^{-16}
0.6	8.88178×10^{-15}
0.8	8.88178×10^{-16}
1.0	8.88178×10^{-16}
1.2	1.77636×10^{-15}
1.4	1.77636×10^{-15}
1.6	1.06581×10^{-14}
1.8	0
2.0	2.13163×10^{-14}

Table 5.6: The error obtained for example [5.3] where $x_0 = 0.6$

x_0	Error
0	6.99441×10^{-15}
0.2	4.21885×10^{-15}
0.4	1.9984×10^{-15}
0.6	1.77636×10^{-15}
0.8	4.44089×10^{-16}
1.0	0
1.2	0
1.4	0
1.6	0
1.8	3.55271×10^{-15}
2.0	1.42109×10^{-14}

Table 5.7: The error obtained for example [5.3] where $x_0 = 0.9$

x_0	Error
0	5.24025×10^{-14}
0.2	4.52971×10^{-14}
0.4	3.28626×10^{-14}
0.6	1.11022×10^{-14}
0.8	8.43769×10^{-15}
1.0	4.44089×10^{-15}
1.2	3.55271×10^{-15}
1.4	3.55271×10^{-15}
1.6	7.10543×10^{-15}
1.8	7.10543×10^{-15}
2.0	1.42109×10^{-14}

Table 5.8: The error obtained for example [5.3] where $x_0 = 1.5$

Table 5.5, 5.6, 5.7 and 5.8 show the error associated with the series estimation of the solution of example [5.3]. The error is relatively very small for all $x \in [0, 2]$.

Example [5.4]: Consider the second order initial value problem with regular singular point

$$x^2 y'' - 4xy' + 6y = 0, \quad y(1) = 1 \quad y'(1) = 0$$

With the exact solution given by $y(x) = 3x^2 - 2x^3$

Now, following the procedure described before, generate the series solution and compare the series solution with the exact solution as shown in figure 5.4

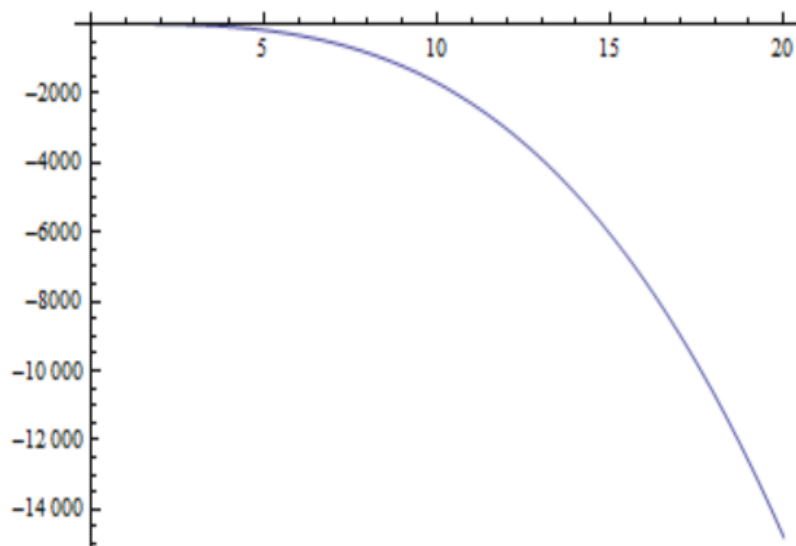
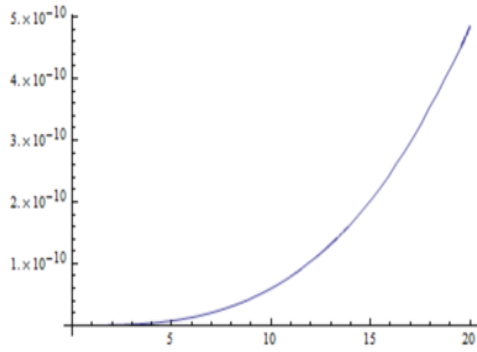
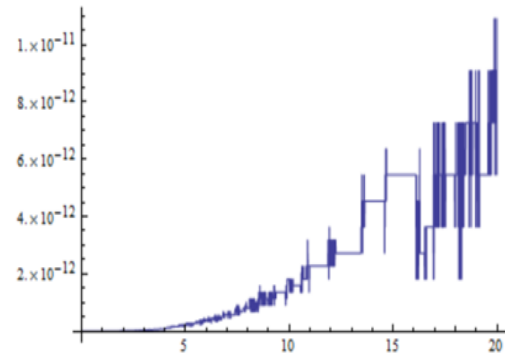


Figure 5.4: The series and the exact solution for the differential equation of example [5.4]

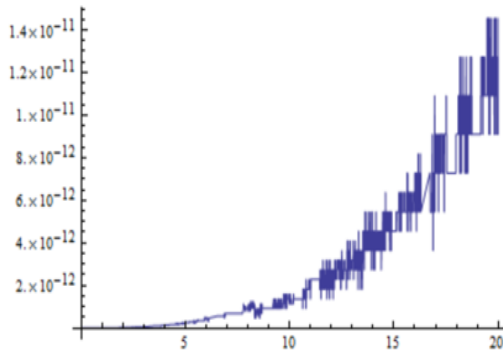
The error between the exact solution and the estimated series solution for different value of x_0 presented in figure 5.5



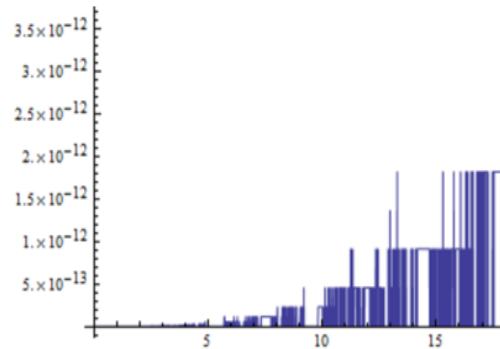
(a) $x_0 = 0.01$



(b) $x_0 = 0.1$



(c) $x_0 = 0.3$



(d) $x_0 = 1.0$

Figure 5.5: The variation of the error as x_0 varies in the interval $[0, 20]$

x_0	Error
0	5.42101×10^{-20}
0.2	4.57967×10^{-16}
0.4	3.77476×10^{-15}
0.6	1.31006×10^{-14}
0.8	3.06422×10^{-14}
1.0	6.06182×10^{-14}
1.2	1.03917×10^{-13}
1.4	1.64313×10^{-13}
1.6	2.46914×10^{-13}
1.8	3.53495×10^{-13}
2.0	4.81393×10^{-13}

Table 5.9: The error obtained for example [5.4] where $x_0 = 0.01$

x_0	Error
0	6.93889×10^{-18}
0.2	1.38778×10^{-17}
0.4	1.11022×10^{-16}
0.6	3.33067×10^{-16}
0.8	8.88178×10^{-16}
1.0	1.77636×10^{-15}
1.2	2.66454×10^{-15}
1.4	3.55271×10^{-15}
1.6	6.21725×10^{-15}
1.8	8.88178×10^{-15}
2.0	1.06581×10^{-14}

Table 5.10: The error obtained for example [5.4] where $x_0 = 0.1$

x_0	Error
0	5.55112×10^{-17}
0.2	1.38778×10^{-17}
0.4	0
0.6	2.22045×10^{-16}
0.8	6.66134×10^{-16}
1.0	1.33227×10^{-15}
1.2	2.66454×10^{-15}
1.4	3.9968×10^{-15}
1.6	6.21725×10^{-15}
1.8	7.99361×10^{-15}
2.0	1.24345×10^{-14}

Table 5.11: The error obtained for example [5.4] where $x_0 = 0.3$

x_0	Error
0	0
0.2	1.52656×10^{-16}
0.4	1.66533×10^{-16}
0.6	0
0.8	0
1.0	0
1.2	1.11022×10^{-16}
1.4	1.11022×10^{-16}
1.6	0
1.8	2.22045×10^{-16}
2.0	0

Table 5.12: The error obtained for example [5.4] where $x_0 = 1.0$

Table 5.9, 5.10, 5.11 and 5.12 show the error associated with the series estimation of the solution of example [5.4]. The error is relatively very small for all $x \in [0,2]$.

Example [5.5]: Consider the second order initial value problem with irregular singular point

$$x^3 y'' + y' - y = x^3 e^x, \quad y(0) = 1 \quad y'(0) = 1$$

Where the exact solution is given by $y(x) = e^x$

Now, following the procedure described before, one can generate the series solution and compare the series solution with the exact solution as shown in figure 5.6

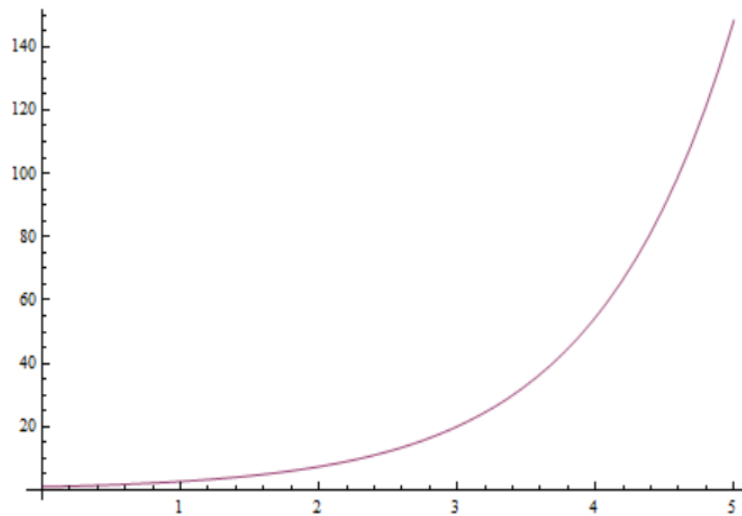


Figure 5.6: The series and the exact solution of the differential equation of example [5.5]

The error between the exact solution and the estimated series solution for different value of x_0 is presented in figure 5.7

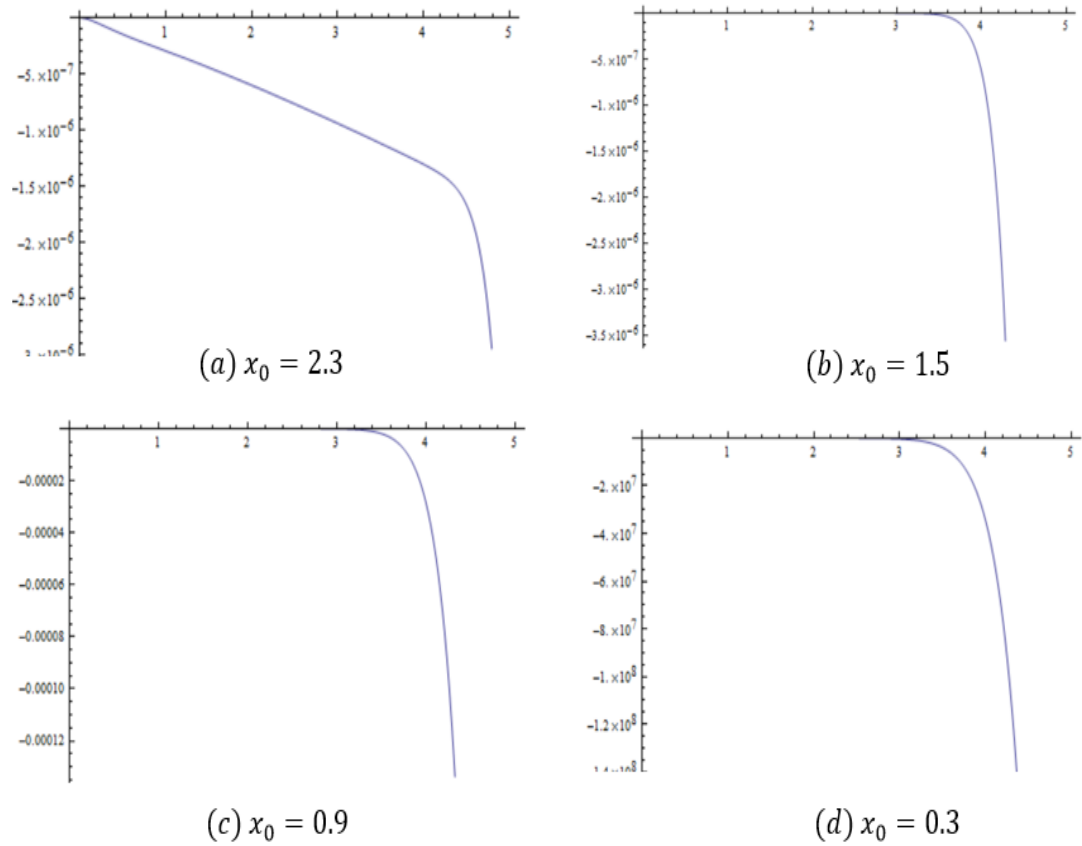


Figure 5.7: The variation of the error as x_0 varies in the interval $[0, 5]$

x_0	Error
0	4.55191×10^{-15}
0.2	4.50019×10^{-8}
0.4	1.14729×10^{-7}
0.6	1.79071×10^{-7}
0.8	2.38157×10^{-7}
1.0	2.95589×10^{-7}
1.2	3.53531×10^{-7}
1.4	4.12855×10^{-7}
1.6	4.73799×10^{-7}
1.8	5.36348×10^{-7}
2.0	6.00402×10^{-7}

Table 5.13: The error obtained for example [5.5] where $x_0 = 2.3$

x_0	Error
0	4.44089×10^{-15}
0.2	2.75953×10^{-11}
0.4	5.43123×10^{-11}
0.6	7.30915×10^{-11}
0.8	9.00529×10^{-11}
1.0	1.07327×10^{-10}
1.2	1.25344×10^{-10}
1.4	1.44127×10^{-10}
1.6	1.63615×10^{-10}
1.8	1.83734×10^{-10}
2.0	2.04412×10^{-10}

Table 5.14: The error obtained for example [5.5] where $x_0 = 1.5$

x_0	Error
0	2.72449×10^{-13}
0.2	8.65974×10^{-15}
0.4	5.10703×10^{-15}
0.6	7.32747×10^{-15}
0.8	8.88178×10^{-15}
1.0	1.02141×10^{-14}
1.2	1.24345×10^{-14}
1.4	1.42109×10^{-14}
1.6	1.86517×10^{-14}
1.8	1.6076×10^{-13}
2.0	3.25162×10^{-12}

Table 5.1: The error obtained for example [5.5] where $x_0 = 0.9$

x_0	Error
0	7.32641×10^{-10}
0.2	8.88178×10^{-16}
0.4	6.66134×10^{-16}
0.6	1.91836×10^{-9}
0.8	3.66299×10^{-9}
1.0	0.000545917
1.2	0.0231008
1.4	0.461156
1.6	5.5859
1.8	47.3746
2.0	307.592

Table 5.16: The error obtained for example [5.5] where $x_0 = 0.3$

Table 5.13, 5.14, 5.15 and 5.16 show the error associated with the series estimation of the solution of example [5.5]. The error is relatively very small for all $x \in [0,2]$.

5.2 Modified Taylor series solution of boundary value problems

5.2.1 Introduction

A boundary value problem consists of a differential equation on a given interval and an obvious condition that a solution must be known at one or various points. The information about a solution to the differential equation may be mostly specified at more than one point. Quite often there are two points, which match physically to the boundaries of several regions, so that it is a two-point boundary value problem [16]. A simple and general form for a two-point boundary value problem is

$$y'' = f(x, y, y'), \quad a < x < b \quad (5.7)$$

Subject to

$$y(a) = \alpha_1 \quad y(b) = \alpha_2$$

Where α_1 and α_2 are known constants and the known endpoints a and b may be infinite or finite. For the linear case of this BVP, (5.7) takes the simpler form

$$y''(x) + p(x)y'(x) + q(x)y(x) = g(x), \quad a < x < b \quad (5.8)$$

Several numerical techniques are available to solve boundary value problems. These methods include the shooting method and the finite difference method. Very little is known in the literature about finding an analytical solution for boundary value problems.

Series method is one way used to estimate the analytical solution for these kinds of problems. In this section we will focus on the series methods of solving those linear and

nonlinear boundary value problems whose exact solutions are too difficult to obtain analytically. The methods to be discussed are the Homotopy Analysis method and Modified Taylor series method.

5.2.2 Solution of boundary value problem using Homotopy Analysis method

In this section we will focus on a special kind of BVP which is the class of eigenvalue problems. When the Homotopy Analysis method is used, we can get an auxiliary linear operator, so we can approximate a nonlinear problem effectively [26, 27]. Details about the use of the method have been already discussed in the previous chapter. The application of the method to the eigenvalue problem will be present in the following two examples.

Example [5.6]: Consider the linear boundary eigenvalue problem

$$y'' + \lambda y = 0 \tag{5.8}$$

with the boundary conditions:

$$y'(0) = 0, \quad y(1) = 0$$

Solution

If we solve this BVP analytically we will get, for $\lambda > 0$, the exact solution:

$$y(x) = A \cos(\sqrt{\lambda}x) + B \sin(\sqrt{\lambda}x)$$

$$y'(x) = -A \sin(\sqrt{\lambda}x) + B \sqrt{\lambda} \cos(\sqrt{\lambda}x)$$

Then using the boundary condition at $x = 0$ leads to

$$y'(0) = \beta \sqrt{\lambda}$$

And the boundary condition at $x = 1$ leads to

$$y(1) = A \cos(\sqrt{\lambda}) = 0$$

$$\text{If } A \neq 0, \quad \cos(\sqrt{\lambda}) = 0 \text{ or, } \lambda_n = \left(\pi\left(\frac{1}{2} + n\right)\right)^2$$

where, the eigenvalues are: $\lambda_n = \left(\frac{2n+1}{2}\right)^2(\pi)^2$, for $n = 0, 1, 2, \dots$

and the corresponding eigenvectors are $y_n(x) = \cos\left(\frac{2n+1}{2}\pi x\right)$, $n = 0, 1, 2, \dots$

Abbasbandy et al [5] employed the Homotopy Analysis method and assumed that the linear and nonlinear operators are given by

$$L[\phi(x, q)] = \phi''(x, q), \quad \text{and}$$

$$N[\phi(x, q)] = \phi''(x, q) + \lambda\phi(x, q)$$

Subject to the boundary condition and the solution expression $L[\phi(x, q)]$, we will choose the initial approximation as $y_0(x) = 1$, and we have the zero- order equation with the initial condition $\phi'(0, q) = 0$, and

$$R_k(y(x, q)) = y''(x, q) + \lambda y(x, q)$$

$$y_k(x) = y_{k-1}(x) + \hbar \int_0^x \int_0^x R_k(y_{k-1}(x)) dx dx$$

Now, the first three terms of the Homotopy Analysis method series solution are

$$y_1(x) = \frac{1}{2} \lambda \hbar x^2$$

$$y_2(x) = \left(\frac{\lambda \hbar^2}{2} + \frac{\lambda \hbar}{2}\right) x^2 + \frac{1}{24} \lambda^2 \hbar^2 x^4$$

$$y_3(x) = \left(\frac{\lambda \hbar^3}{2} + \lambda \hbar^2 + \frac{\lambda \hbar}{2} \right) x^2 + \left(\frac{\lambda^2 \hbar^3}{12} + \frac{\lambda^2 \hbar^2}{12} \right) x^4 + \frac{1}{720} \lambda^3 \hbar^3 x^6$$

According to the k th order approximate solution, the Homotopy Analysis method series solution will be

$$v_k(x) = \sum_{i=0}^k y_i(x)$$

Which depends on the auxiliary parameter \hbar , and the eigenvalue λ . When we solve $v_k(1) \cong 0$, we get a relation between λ and h . The plot of λ as a function of h gives us different plateaus, which matches to an eigenvalues of the DE.

In the first figure, λ has been plotted according to the equation $v_k(1) = 0$ in the λ range [2, 125] and \hbar range [-2, 0] for $k = 25$, also four plateaus of λ can be identified. In the second figure, λ has been plotted according to the equation $v_k(1) = 0$ in the λ range [125, 300] and \hbar range [-1, 0] for $k = 25$, also three plateaus of λ can be identified.

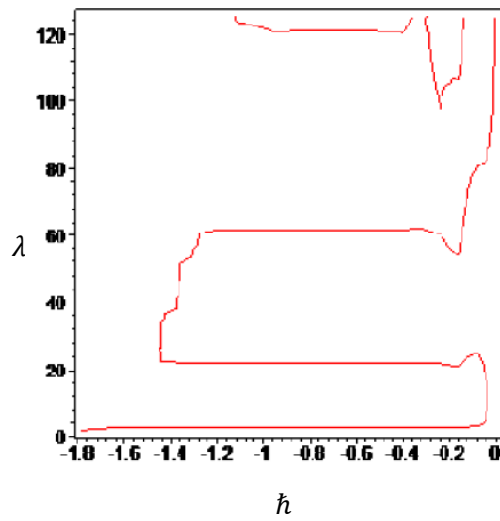


Figure 5.8: The \hbar – curve of solution for λ range $[2, 125]$ and $k = 25$

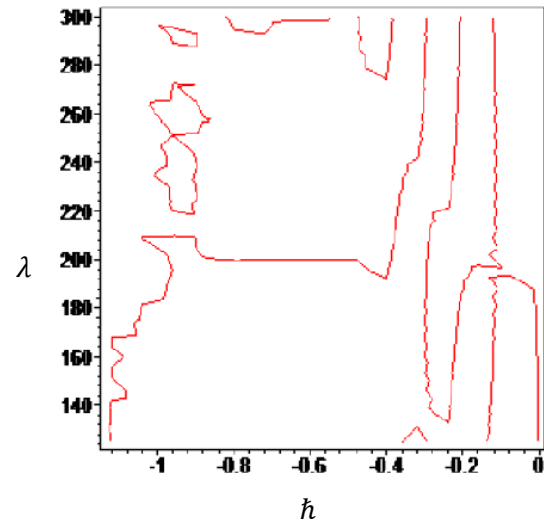


Figure 5.9: The \hbar – curve of solution for λ range $[125, 300]$ and $k = 25$

In the plot of λ as a function of \hbar , it can be seen that some plateaus occur, which matches to an eigenvalue of the DE.

Table 5.17 shows the first 6 approximated eigenvalues with the exact eigenvalues and the absolute error of the original equation which has been approximated by Abbasbandy et al [5].

n	Approximated λ_n	Exact λ_n	Absolute error λ_n
1	2.46740110	2.467401100272	2.7×10^{-10}
2	22.20660990	22.206609902451	2.4×10^{-9}
3	61.68502750	61.685027506808	6.8×10^{-9}
4	120.90266801	120.902653913345	1.5×10^{-5}
5	199.85633513	199.859489122060	3.1×10^{-3}
6	298.31364965	298.555533132953	2.4×10^{-1}

Table 5.17: The eigenvalue and absolute error using HAM

The approximated eigenfunction corresponding to the third eigenvalue λ_3 , and the corresponding error which is used with $\hbar = -0.9$ is shown below in figures 5.10 and 5.11

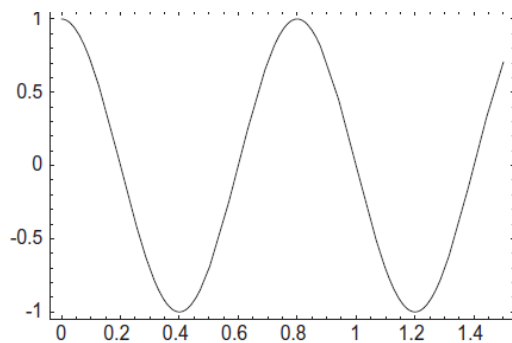


Figure 5.10: The eigenfunction corresponding to λ_3

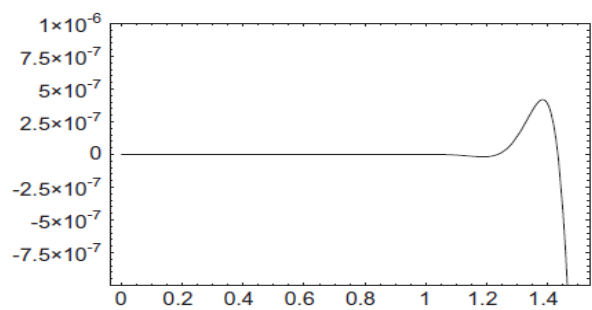


Figure 5.11: The error of eigenfunction corresponding to λ_3

Example [5.7]: Consider the singular eigenvalue problem

$$y''(x) + \left(\lambda + \frac{1}{x}\right)y(x) = 0 \quad (5.10)$$

with the boundary condition

$$y(0) = 0 \quad y(1) = 0$$

Solution

By following the same procedure described in example [5.7], we can find the first two terms of the Homotopy series solution as

$$y_1(x) = \frac{\hbar x^2}{2} + \frac{1}{6} \lambda \hbar x^3$$

$$y_2(x) = \left(\frac{\hbar^2}{2} + \frac{\hbar}{2}\right)x^2 + \left(\frac{\lambda \hbar^2}{6} + \frac{\hbar^2}{12} + \frac{\lambda \hbar}{6}\right)x^3 + \frac{1}{18} \lambda \hbar^2 x^4 + \frac{1}{120} \lambda^2 \hbar^2 x^5$$

According to the k th order approximate solution, the Homotopy Analysis method series solution will be $v_k(x) = \sum_{i=0}^k y_i(x)$.

The plot of $v_k(x) \cong 1$, gives λ as a function of h . The contour plot gives us different plateaus, which matches to an eigenvalue of the DE.

In the figure 5.13, λ has been plotted according to the equation $v_k(1) = 1$ in the λ range [7, 8] and h range [-2, 0] for $k = 25$.

In the figure 5.14, λ has been plotted according to the equation $v_k(1) = 1$ in the λ range [36, 36.5] and h range [-1.4, -0.4] for $k = 25$.

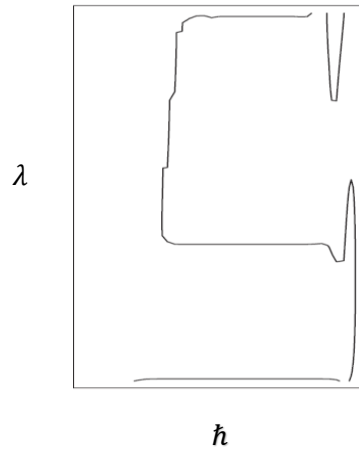


Figure 5.12: \hbar -curve according to $v_k(x) = 1$ and $\lambda \in [0, 36]$

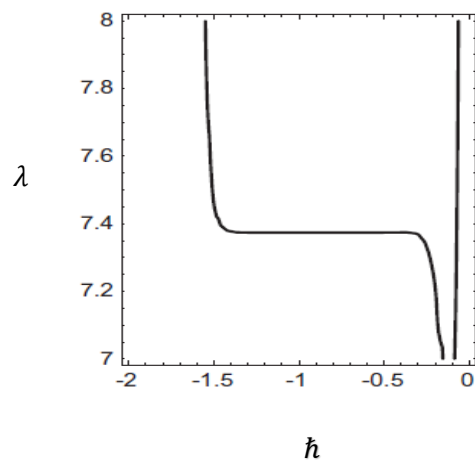


Figure 5.13: \hbar -curve according to $v_k(x) = 1$ and $\lambda \in [7, 8]$

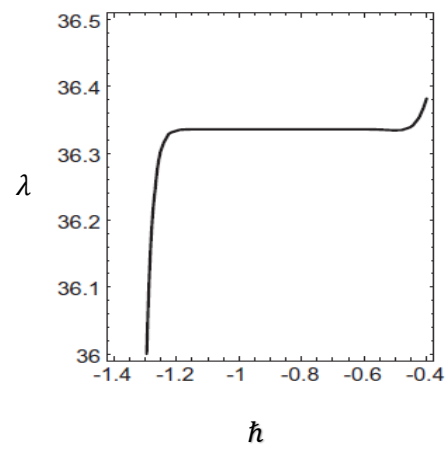


Figure 5.14: \hbar -curve according to $v_k(x) = 1$ and $\lambda \in [36, 36.5]$

n	Approximated λ_n
1	7.37398501
2	36.33601851
3	85.29251075
4	154.10192997
5	352.47100980
6	568.16321224

Table 5.18: The first 6 eigenvalues obtained for example [5.7] of the DE

The approximated eigenfunction corresponding to the second eigenvalue λ_2 , and the corresponding error which is used with $\hbar = -0.8$ is shown below in figures 5.15 and 5.16

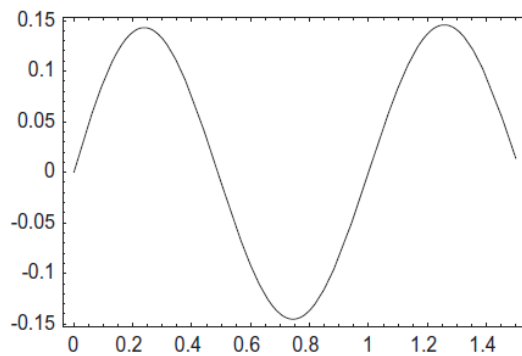


Figure 5.15: The eigenfunction corresponding to λ_2

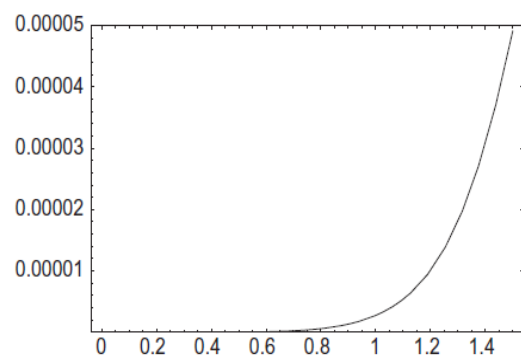


Figure 5.16: The error of eigenfunction corresponding to λ_2

5.2.3 Solution of eigenvalue problem using the Modified Taylor series method

In this section, we will solve the boundary value problems by using the Modified Taylor method. The suggested method is more applicable and accurate than the built in methods used in other software packages.

We will solve several examples for nonlinear BVPs and compared results to those obtained using Mathematica program.

Let us consider the nonlinear Boundary value problem,

$$y'' = f(x, y, y'), \quad a < x < b$$

subject to the conditions

$$y(a) = A, \quad y(b) = B$$

Let's assume one type of differential equations, then following some techniques in the example, after that we can assume the solution by the series

$$y(x) = \sum_{n=0}^N a_n (x - x_0)^n + \mathcal{R}_N$$

with

$$\mathcal{R}_N = \frac{y^{N+1}(\zeta)}{N+1} (x - x_0)^{N+1}$$

With the initial choice

$$y(x_0) = a_0 = \alpha, \quad y'(x_0) = a_1 = \beta$$

Then ignoring the error term and using the series in the original differential equations to evaluate the rest of the coefficients leads to the following values for the first few terms.

So, we can write $a_2, a_3, a_4, \dots, a_n$ in terms of α and β . To find the values of α and β as a function of x_0 we need to solve

$$y(a) = A, \quad y(b) = B \quad \text{for } \alpha, \beta$$

At the end the solution will be in terms of x_0 , after that we will sketch $y(x)$ as a function of x_0 to find the minimum error.

Example [5.8]: Consider the previous nonlinear boundary value problem

$$y'' + \lambda y = 0 \quad y'(0) = 0 \quad y(1) = 0$$

With exact eigenvalues and its corresponding eigenfunctions given by:

$$\lambda_n = \left(\pi \left(\frac{1}{2} + n \right) \right)^2 \quad \text{are the eigenvalues,}$$

$$\text{and } y_n(x) = \text{Cos} \left(\left(\frac{\pi}{2} + n\pi \right) x \right) \quad \text{are the corresponding eigenfunctions}$$

Solution

Applying the same techniques used in the first section for initial value problems but with arbitrary initial conditions and solving for α the first condition at $x = 0$, then solving for the second boundary condition $x = 1$ for β , will give a relation between the expanding point x_0 and the eigenvalue λ . Plotting the contour of this relation will result in the following figures.

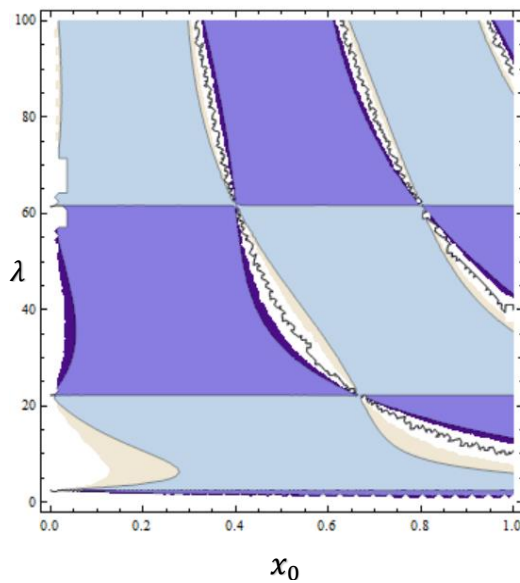


Figure 5.17: (x_0, λ) curve for
 $\lambda \in [0, 100]$ and $x_0 \in [0, 1]$

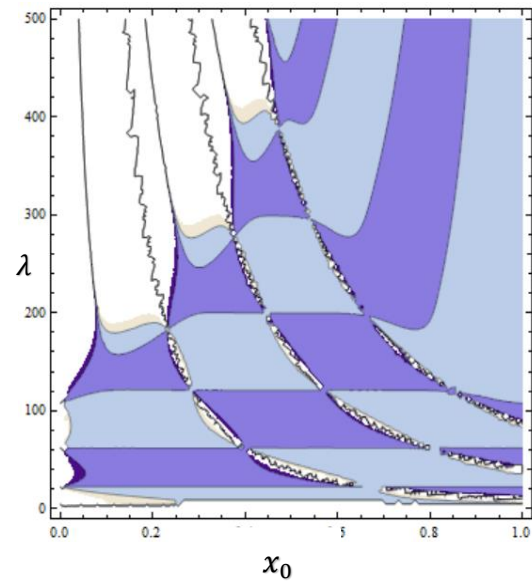


Figure 5.18: (x_0, λ) curve for
 $\lambda \in [0, 500]$ and $x_0 \in [0, 1]$

In the first figure 5.17, λ has been plotted by using Modified Taylor method in the λ range $[0, 100]$ and x_0 range $[0, 1]$ for $k = 35$, also three plateaus of λ can be identified.

In the second figure 5.18, λ has been plotted by using Modified Taylor method in the λ range $[0, 500]$ and x_0 range $[0, 1]$ for $k = 35$, also six plateaus of λ can be identified.

For small values of λ , x_0 can be any number between $[0, 1]$, but for larger values of λ , λ has to be around the middle of the interval.

In the plot of λ as a function of x_0 , it can be seen that some plateaus occur, which matches to an eigenvalue of the Modified Taylor problem.

This table shows the first 6 approximated eigenvalues with the exact eigenvalues and the absolute error of the original equation which has been approximated by Abbasbandy et al [5].

n	Approximated λ_n by Homotopy method	Approximated λ_n by Modified method	Exact λ_n
1	2.46740110	2.4674	2.467401100272
2	22.20660990	22.2066	22.206609902451
3	61.68502750	61.685	61.685027506808
4	120.90266801	120.903	120.902653913345
5	199.85633513	199.863	199.859489122060
6	298.31364965	298.472	298.555533132953

Table 5.19: The eigenvalues using HAM, Modified Taylor method and the exact error

The approximated eigenfunction corresponding to the third eigenvalue λ_3 , and the corresponding error which is used with $x_0 = -0.9$ is shown below as

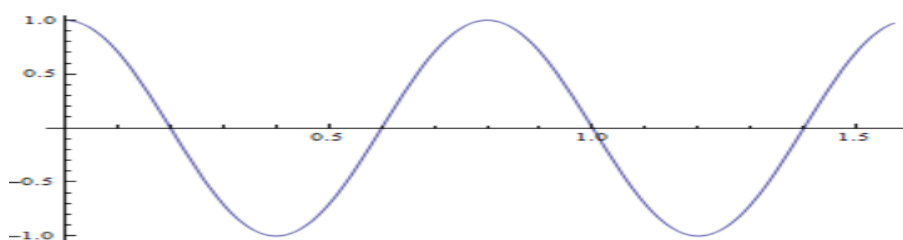


Figure 5.19: The approximated eigenfunction corresponding to λ_3

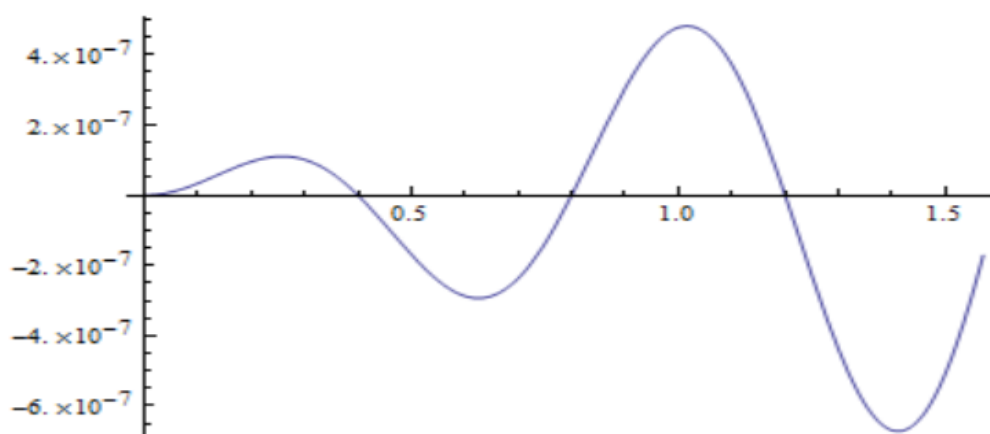


Figure 5.20: The error of eigenfunction corresponding to λ_3

Also, the Modified Taylor series method has been used to obtain numerically the solutions of eigenvalues of the boundary value problems of the DE. All computational work was carried by using Mathematical program. The numerical results show that our method is suited for the boundary value problem.

Chapter 6: Conclusion

In this thesis, we have discussed the several series methods available to solve initial and boundary value problems. The methods are the Taylor series method where we have modified it to control the error by choosing different expansion point. The method was applied to differential equations with singular points and to eigenvalue problems. We have shown that this modification will help in reducing the error and recover all the eigenvalues of the eigenvalue problems. It was shown that the method is reliable to solve all kind of boundary value problems as well.

In addition, we have discussed the Adomian Decomposition method and how it can be used to generate the solution of the initial value problems with the error associated with this method.

The Homotopy Analysis method was also discussed. The control of this method through the use of what is known as the \hbar curve. It was shown how this curve can be used to monitor the error at one point in the domain of definition of the solution. However, it was shown that this method is not all the times reliable and the error could be reduced using other methods such as the Modified Taylors method described above.

Different linear and nonlinear examples of ordinary initial and boundary value problems have been presented.

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