REFUTATIONS AND THE LOGIC OF PRACTICE

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When arguments are refuted in mathematics classrooms, the ways in which they are refuted can reveal something about the logic of practice evolving in the classroom, as well as about the epistemology that guides the teachers' teaching. We provide four examples that illustrate refutations related to the logic of practice, in which sufficiency and relevance are grounds for refutation, as opposed to falsehood.

Keywords: Epistemology; Logic of practice; Proofs; Refutation; Relevance; Sufficiency; Teaching

Refutaciones y la Lógica de la Práctica

Cuando se refutan argumentos en el aula, las maneras en que se refutan pueden revelar algo acerca del desarrollo de la lógica de la práctica en el aula, así como de la epistemología que guía la enseñanza. Presentamos cuatro ejemplos que ilustran refutaciones relacionadas con la lógica de la práctica, en los que la suficiencia y pertinencia, y no la falsedad, son los motivos de refutación.

Términos clave: Enseñanza; Epistemología; Lógica de la práctica; Pertinencia; Pruebas; Refutación; Suficiencia

At first glance, refutations may seem to have little to do with teaching proof. Proofs are concerned with showing what conclusions follow from a set of premises, whereas refutations only tell us what conclusions do not follow. There are, of course, special cases, like proof by contradiction and contraposition, in which one seeks to refute one statement in order to prove its negation. However, we are not concerned with these cases here. Instead we are interested in the kind of refutations that appear in the proving process through which proofs evolve in mathematics classrooms, but which are not evident in the finished proof.

We are interested in these refutations in relation to what Toulmin (1958) calls the logic of practice, which underlies proving processes in classrooms. That is, the logic upon which arguments are based in actuality, rather than the logic upon which one might like them to be based. As mathematics classrooms are contexts for learning, arguments in them are based on a logic in transition from

the everyday logic the students bring to the class to a mathematical logic accepted by the teacher. When arguments are refuted, the ways in which they are refuted can reveal something about the logic of practice as well as the teachers' purpose in engaging in argument in the first place and what epistemology guides their teaching.

BACKGROUND

As Balacheff (2002) noted, the field of mathematics education includes approaches based on a number of distinct epistemologies. The role seen for refutations depends on epistemological factors. For example, for those whose focus is on the logical correctness of formal texts called "proofs", refutations do not play a role except perhaps in the special cases of proofs by contradiction and contraposition. Others' epistemology is based to some extent on Lakatos' (1976) view of mathematics, in which mathematics does not proceed by a process of proving theorems conclusively and then moving on, but rather through a cycle of proofs and refutations, with proofs being always provisional and refutations providing the mechanism for the improvement of theorems and their proofs. For those with this epistemology, proof is inextricably linked to refutations, and approaches to teaching proof from this perspective include an exploration of refutations as an essential element (Balacheff, 1988, 1991; Sekiguchi 1991). Another epistemology for which proofs are essential is that founded on the concept of cognitive unity in which argumentation processes—which may include refutations—provide the basis for proof development (Boero, Garuti, Lemut, & Mariotti, 1996). Studying the role of refutations in classroom proving processes is important if one takes on an epistemology that gives an important role to refutations, e.g., one based on Lakatos (1976) for cognitive unity, but also for descriptive and comparative work looking at current teachers' practices, as a way to reveal the implicit epistemologies guiding teaching. It is such an interest in teaching practices that inspires our work. In classrooms we observe a proving process through which teacher and students produce a proof, and which can include refutations in important ways. In this paper we describe a number of examples of refutations embedded in proving processes, their roles in those processes, and what these roles suggest about the teaching practices and implicit epistemologies underlying them.

One of Toulmin's (1958) aims was to describe the layout of arguments in a way that is independent of the field in which they occur. In this paper we diagram arguments using a method derived from Toulmin's basic layout for an argument (see Figure 1). In this layout an argument is considered to consist of data, which lead to a conclusion, through the support of a warrant.
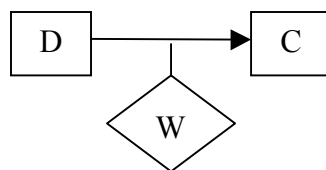
Figure 1. Toulmin's basic layout
(D: Data, W: Warrant, C: Conclusion)

Toulmin does not consider refutations within this structure because he is considering arguments as they are, once the assertion is established, not the process of their coming to be. However, Toulmin's first chapter deals extensively with refutations in order to explore how arguments in different fields are based on different criteria. There he gives examples of arguments in which an assertion is made which is true in one field but which can be refuted in another field. Looking at Toulmin's basic layout, three ways in which an argument can involve a refutation immediately suggest themselves. The data of the argument can be refuted, leaving the conclusion in doubt. The warrant of the argument can be refuted, again leaving the conclusion in doubt. Or the conclusion itself can be refuted, implying that either the data or the warrant is invalid, but not saying which. In the language of Lakatos (1976) the first two are local counterexamples, while the latter type is a global counterexample. Sekiguchi (1991) provides examples in a classroom context of several types of refutations within this framework. However, as we noted above, we are more interested in refutations where the focus of the refutation is not the data, conclusion or warrant, but rather the logic underlying the argument.

REFUTATION IN CLASSROOM ARGUMENTS

In our research we have examined classroom arguments at upper elementary and junior high school in Canada, Germany and France. In these contexts refutations sometimes occur, but in different forms and with different functions. Here we provide four examples along with discussion of the insight each gives us into the logic of practice and the teacher's epistemology.

Refutation of a Conclusion Implied by a Question
The conclusion that is refuted may not always be stated directly. In classrooms a common exchange is for the teacher to ask a question with the intent of pointing out an error. For example, in this exchange grade five students have been trying to develop a formula for how many squares there are in a n by n grid—DAR is a guest teacher. They have been working with a concrete model in which three pyramids made of linking cubes are joined to make a roughly box shaped solid made up of $n\frac{1}{2}$, n by $(n+1)$ layers. Here they are considering a 10 by 10 grid

for which the solid has 10.5, 10 by 11 layers. They have multiplied these three numbers to find the total number of linking cubes used: 1155. In Reid (2002), Zack (2002), Zack and Reid (2003), and Zack and Reid (2004) can be found more background and details.

DAR: Right. So, 1155 is what you get if you multiply those three numbers. Is that [1155] how many squares there are in a 10 by 10 [grid]?

Several voices: No.

Here the question "Is that [1155] how many squares there are in a 10 by 10 [grid]?" implies the conclusion "1155 is … how many squares there are in a 10 by 10 [grid]" which the answer "No" refutes. This answer requires no further support as the students and DAR are all aware that there are 385 squares in a 10 by 10 grid. Figure 2 represents this situation. The jagged arrow in the layout indicates a refutation.
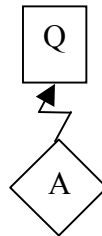

Figure 2. A refutation
(A: Answer, Q: Question)

In terms of the final structure of the argument the statement "1155 is … how many squares there are in a 10 by 10 [grid]" plays no role, as it is false. Even its negation "1155 is not … how many squares there are in a 10 by 10 [grid]" is not important to the final argument. However, in the proving process it is an important statement, as the students have arrived at a point where they might expect 1155 to be the answer—as DAR has guided them to this result ostensibly to find a formula that works—but at the same time they know from counting previously the correct answer is 385. This tension offers a motivation for further exploration of why the product of the three numbers in question is not the expected answer.

Refutation of the Sufficiency of a Warrant, While Accepting the Data and the Conclusion
In the previous example, no warrant was offered to justify the connection between the data and the conclusion. In this example, a warrant is offered, but it is not the warrant that is refuted, but the sufficiency of it to establish the connection between the data and the conclusion. The example comes from a grade nine class which is trying to explain why if two diagonals of a quadrilateral meet at their midpoints and are perpendicular, then the quadrilateral must be a rhombus. In

this and the following examples, the codification of each part of the students' argument is indicated between brackets.

Kaylee: Umm, I said cause if they meet, if they meet at the midpoint, they meet at the midpoint and they're ninety degree angles [D1]

then umm, then, it would have to be—the [...]—then the sides, they have to be umm, like outsides have to be equal lengths. [C1]

T: Why?

Kaylee: Be, ummm, because they meet at, they all meet at ninety degree angles and at the midpoint [D1]

but the segments are different lengths, then [D2]

it can't be a square [C2]

because squares they have to be the same length. [W2]

T: OK, but ...

Kaylee: So it has to be a rhombus. [C1]

T: Could it be a rectangle? Could it be a parallelogram?

S: If it—If none.

T: Cause there are other ones [R1]

like the rectangle one met at the midpoint. It didn't meet at a ninety degree angle though. [B1a]

And then the rhombus we covered. The one that made a kite met at a ninety degree angle, it didn't meet at the midpoint. [B1b]

You're on the start, but I'm not sure that you've clinched it, I'm not sure you've got that final part, but you've got—you're three quarters of the way there my dear.

Kaylee's warrant is a correct statement. Figures with perpendicular bisecting diagonals are not generally squares, as squares have the additional characteristic that their diagonals are the same length. However, the teacher's objection is not to the truth of Kaylee's warrant but to its sufficiency. As the teacher notes, there are other quadrilaterals that have not been considered and excluded. Although she excludes rectangles and kites from consideration at the same time, she uses them to back up her refutation, her point is made: Other quadrilaterals, other than squares and rhombuses, exist, and so excluding squares is not sufficient to guarantee the shape must be a rhombus. Here the refutation is directed at the warrant, but does not refute it—as it is correct. Instead it suggests that the warrant is insufficient in the logic the teacher expects mathematical arguments to follow (see Figure 3). By offering an argument of her own refuting the sufficiency of

Kaylee's warrant, the teacher provides that students with a hint as to the logic she would accept as mathematical.
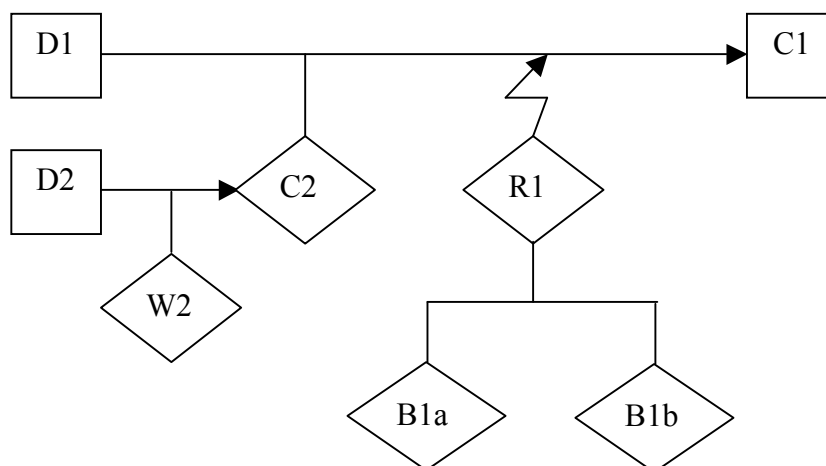

Figure 3. Refutation of sufficiency
(Di: Data, Ci: Conclusion, Wi: Warrant, Ri: Refutation,
B1a and B1b are backings supporting the warrant)

Refutation of the Relevance of Data Offered in Support of a Conclusion
In the previous case the refutation addressed the sufficiency of the warrant, but it is also possible to refute the relevance of the data offered. This example also comes from the grade five class looking for a formula for how many squares there are in an *n* by *n* grid. The students have suggested that by dividing 1155 by three, they can get the correct answer of 385.

DAR: Could we have somebody … suggest a reason why we might want to divide by three—Mona? [C1]

Mona: Because there's three numbers. [D1]

DAR: Because there's three numbers. That's a good reason.

Mona: I guess.

DAR: OK. It's not a great reason [R1]

         but it's a good reason.

         [Calls on another student] [R1]

Here the teacher's refutation is an implicit one. His qualified support, "not a great reason but it's a good reason", and shift of attention away from Mona's response communicates to the class that there is something wrong with what she has said, without specifying exactly what. Neither the data nor the conclusion is refuted—as the class knows them both to be true statements—, and there is no suggestion that the lack of a warrant in Mona's argument is the issue—as would be suggested by the teacher asking "Why would we divide by three when there

are three numbers?". Instead the focus is on the relationship between the facts, not on the facts themselves. It is the unspoken logic of the argument that is refuted. Mona has made a link between two statements, but not in a way that wins acknowledgment from the teacher (see Figure 4). Note that here the teacher's refutation is based only on his authority (Sekiguchi, 1991) and unlike the teacher's refutation in the previous case it does not offer any guidance for what might be an acceptable link. Instead he has the students guess until they come up with something acceptable.
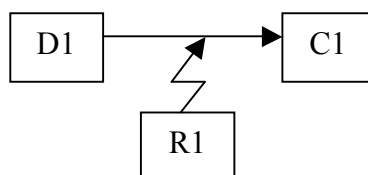

Figure 4. Refutation of relevance
(Di: Data, Ri: Refutation, Ci: Conclusion)

A Complex Refutation

This example follows immediately after the previous one. The students have been working with a concrete model in which three pyramids are joined to make a roughly box shaped solid made up of $n\frac{1}{2}$, n by $n+1$ layers.

Elaine:	Because there's three of those triangle thingies in there. [D1]
Maya:	But then why wouldn't you divide it by three and a half [C2] because there's a half? [D2]
DAR:	OK. How many triangles did we put together to make this thing? [D1]
Several voices:	Three. [D1]
Maya:	And then there's the half.
DAR:	… So then we put together three of them and suddenly [D1] we had three times too many, [D3] so that would be a good reason to divide by three if you've got three times too many of something. [W1]

Here Elaine is trying to answer the question why might we divide by three? The data she uses to justify this refers to the three pyramids—"triangle thingies"—in the box. But she does not offer a warrant to support the connection of this data to the conclusion. Maya's refutation consists of a parallel argument, which also makes reference to elements visible in the box like the three pyramids and the half layer. Her refutation is again on the level of the logic of the argument. By

making an argument on the basis of a coincidence or analogy—one half is just as much a property of the box as three is—that leads to a false conclusion, she refutes Elaine's use of similar reasons. DAR then supports Elaine and in so doing implicitly refutes Maya's refutation. He provides a warrant for Elaine's original argument, in the process supplying a linking piece of data that shifts the logic of the argument from analogy—three thingies, so divide by three—to deduction—three times too many, so divide by three.

Maya's refutation offers a challenge not only to Elaine's argument but also to the teacher's practice (see Figure 5). He refuted Mona's argument by simply asserting his authority—C1 is the conclusion from Figure 4. If Maya had not refuted Elaine's argument, he might have used his authority again to endorse it and moved on. In order to refute Maya's refutation, he had to recast Elaine's argument into a more complete, and less refutable form, including reference to a new piece of data (D3) and a warrant (W1) to support the drawing of the conclusion from it. This made the kind of logic he considered acceptable much more explicit.
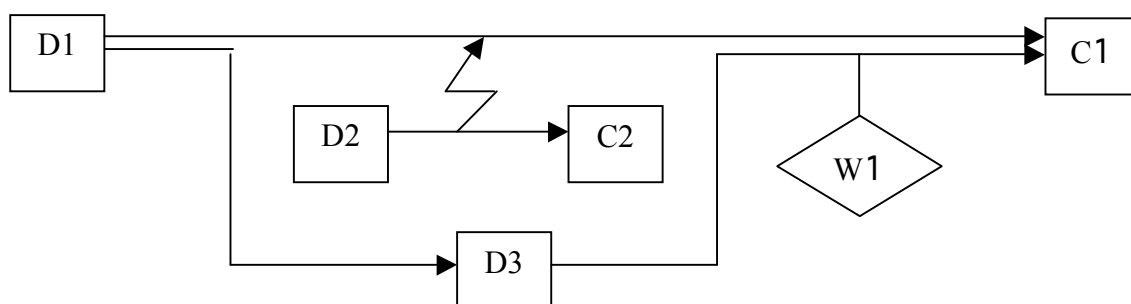

Figure 5. A complex refutation
(Di: Data, Ci: Conclusion, Wi: Warrant)

CONCLUSION

These four examples illustrate some of the insights an examination of refutation in proving processes can provide, both into the nature and evolution of the logic of practice operating and into teaching practices and epistemologies related to proof.

By drawing attention to the insufficiency of a warrant or data, as in the second and third examples, or forcing the teacher to be more explicit about his implicit criteria for acceptable arguments, as in the fourth example, refutations provide hints as to what is the teacher's accepted logic. These hints are of value to students learning to shift from everyday arguments to mathematical arguments, as well as to us as researchers interested in this process.

We can also get insight into teaching practice from refutations. The first example, of a teacher using a refutation to provide motivation for further exploration, suggests an epistemology compatible with a Lakatosian view of mathemat-

ics as improving though confronting conclusions with counterexamples. The third and fourth examples reveal a teacher relying on authority as a means of refutation, suggesting an approach to teaching proving that relies on examples and non-examples as much as or more than direct modelling.

We believe that such research can provide insight into actual practice of teaching proof, which is necessary to any program of reform, as well as any comparison of approaches.

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