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INTRODUCTION: THE PHENOMENOLOGICAL, EPISTEMOLOGICAL, AND SEMIOTIC COMPONENTS OF GENERALIZATION

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In the first part of this article, I argue that generalization involves three related components: phenomenological, epistemological, and semiotic. I also argue that the concept of generalization conveyed by theories of knowing (e.g., rationalist and empiricist) depends on the manner in which these theories understand the above three components and their interrelations. I elaborate my argument in reference to a cultural-historical dialectical concept of generalization. In the second part of the article, I provide an overview of the articles contained in this special issue and discuss their contributions to educational research.

Keywords: Epistemology; Generalization; Phenomenology; Rationalism; Semiotics

Introducción: Los componentes fenomenológico, epistemológico y semiótico de la generalización

En la primera parte de este artículo, sostengo que la generalización incluye tres componentes entrelazados: un componente fenomenológico, un componente epistemológico y un componente semiótico. También sostengo que el concepto de generalización que presentan las teorías del conocimiento (por ejemplo, teorías racionalistas y empiristas) depende de la manera en que esas teorías conciben los tres componentes anteriores y sus interrelaciones. Mi argumento es elaborado a partir de un concepto cultural histórico dialéctico de generalización. En la segunda parte del artículo, hago un resumen de los artículos contenidos en este número especial y discuto sus contribuciones a la investigación en educación.

Términos clave: Epistemología; Fenomenología; Generalización; Racionalismo; Semiótica

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> "Generalization is essential because it is this process that distinguishes mathematical creativity from mechanizable or algorithmic behavior."
> (Michael Otte, 2003, p. 187)

GENERALIZATION

Theories of knowing often resort to the concept of generalization in the accounts of how we come to know about things in the world. And, as we all know, generalization is also a key concept in education, particularly in teaching and learning mathematics. Yet, a definition of generalization is not easy to come by.

In its etymology, generalization is formed of two simpler terms: generalize and ation, and means a general inference. The etymology points to the idea that generalization operates within a logical or epistemological realm within which the alluded inference is recognized as right or true. The etymology also reveals the idea that there must be some *ground* from where the inference occurs: We always infer (i.e., we always "bring about") something from something else. Last but not least, that which we bring about is of a general nature.

From these brief remarks we understand why rationalist and empiricist epistemologies come up with different concepts of generalization. For instance, the ground of the inference is not the same nor does it play the same role in both epistemological traditions. Nor do these epistemological traditions understand inference in the same way. Kant, for instance, in his famous 1770 inaugural dissertation, "On the Form and Principles of the Sensible and Intelligible World", (Kant, 1770/1894) urged his contemporaries to distinguish and to keep apart the knowledge that results from the sensible and tangible world (the world of *phenomena*) and the knowledge that results from pure reason (the world of *noumena*, i.e., the conceptual world of things in themselves). The ground of these two worlds is not the same. Or so was Kant's view in the inaugural dissertation.

Although we may or may not agree with the empiricist or rational traditions, these traditions may be useful to interrogate the ideas of generalization that, implicitly or explicitly, we adopt and convey in our educational practices. As I have suggested elsewhere (Radford, 2013a), it might be worthwhile considering that a generalization involves at least three interrelated components.

First, there is a phenomenological component that has to do with the choice of the sensible determinations—the manner in which intuition, attention, and intention interact in order to deal with the particular objects that constitute the base or the ground of the generalization.

Second, there is an epistemological-ontological component, which allows the knower to extrapolate or to generalize something from an array of particular objects, so that something is inferred and asserted about another object.

Third, there is a semiotic component that involves semiotic means (such as oral and written language, but also gestures and signs like diagrams and formulas) through which things are inferred and asserted about the generalized object.

These three components are not independent from each other. In fact, let me note that, as the previous etymological analysis suggests, a generalization is always practiced within a certain mode of knowing—what Foucault (1966) used to call an *episteme*—that is, a mode of perceiving things, thinking and talking about them, and distinguishing between true and false. What this means is that our (perceptual, emotional, intellectual, aesthetic, etc.) relationships to things in the world are not direct but mediated. Thus, to take the example of perception, we do not merely see or look at something; we see and look at something in certain ways. The historical epistemologist Marx Wartofsky notes that "Our empirical knowledge of the world is not simply acquired by looking and seeing, but by looking at and seeing as" (Wartofsky, 1968, p. 420). A distinction should hence be made between the object and the object of perception. The latter is the former as transformed in the act of perceiving. This transformation has to do with a cultural frame of concepts that, as we grow, we come to share. Wartofsky goes on to say that

> *what it is we choose to notice, and what escapes our attention even when it stares us in the face, and what we take whatever it is we notice to be, is a function, in large part, of that framework of concepts into which we enter when we are weaned from out mother's breast.* (Wartofsky, 1968, p. 420)

In a similar vein, the prominent art historian Michael Baxandall notes that to see Piero della Francesca's *Annunciation* requires one to visually and conceptually interpret the painting in a certain way, so that what is to be seen is adjusted and in doing so becomes meaningful. In the painting there is a column between the Angel Gabriel and Mary, and both of them seem to be paying attention to the column rather than to each other.[1] Baxandall (1972) says that

> *regarding knowledge of the story, if one did not know about the Annunciation it would be difficult to know quite what was happening in Piero's painting; as a critic once pointed out, if all Christian knowledge were lost, a person could well suppose that both figures, the Angel Gabriel and Mary, were directing some sort of devout attention to the column. This does not mean that Piero was telling his story badly; it means he could depend on the beholder to recognize the Annunciation subject promptly enough for him to accent, vary and adjust it in rather advanced ways.* (p. 36)

Within this conception of perception, our sensorial-perceptual organs are not conceived of as merely part of our phylogenetic evolved biological equipment: "percep-

[1] For a picture of della Francesca's Annunciation see http://www.wga.hu/html_m/p/piero/2/91/91annun1.html

tion itself is a highly evolved and specific mode of human action or praxis… its characterization as only biological or physiological or more generally, in 'natural' contexts, is inadequate" (Wartofsky, 1979, p. 189). Our sensorial-perceptual organs evolve rather culturally and historically.

> *Some of the mental equipment a man orders his visual experience with is variable, and much of this variable equipment is culturally relative, in the sense of being determined by the society which has influenced his experience. Among these variables are categories with which he classifies his visual stimuli, the knowledge he will use to supplement what his immediate vision gives him, and the attitude he will adopt to the kind of artificial object seen.* (Baxandall, 1972, p. 40)

To better grasp the imbrication of the phenomenological dimension in its concomitant epistemological-ontological dimension, it should be pointed out that, in perception, it is not only the potential objects of the phenomenological world that are transformed, but in fact perceptual reality at large. The case of space can be invoked in this regard. As Erwin Panofsky (1991)—another eminent art historian—remarks in his famous study of Renaissance art, perspective drawing involves a transformation of our concept of space from something substantial into something functional. That is, space becomes considered as something homogeneous, of which geometric space becomes the paradigm.

> *Homogeneity of geometric space is that all its elements, the 'points' which are joined in it, are mere determinations of position, possessing no independent content of their own outside of this relation, this position which they occupy in relation to each other. Their reality is exhausted in their reciprocal relation: it is a purely functional and not a substantial reality.* (Panofsky, 1991, p. 30)

The invention of perspective is not a purely aesthetic phenomenon restricted to artists and painters; it is the other way around: It incarnates a social phenomenon where reality and the objects in it started being perceived in new ways.

Let us now turn to the semiotic dimension, which includes not only signs and semiotic systems but also artefacts and the body as sources of meaning-making processes in knowledge production. I have argued above that this component cannot be neglected in the study of generalization. The semiotic dimension, much as the phenomenological one, is imbricated in the epistemological-ontological dimension in that it entails a discrimination between the possible semiotic, artefactual and embodied forms through which knowledge can properly be dealt with. Pre-Pythagorean mathematicians, for instance, asserted properties of odd and even numbers through recourse to pebbles. The Euclidean tradition replaced the pebble semiotic system with letters and segments to represent numbers—as in Euclid's *Elements* (Heath, 1956). The new semiotic system allowed the mathematicians to deal with generality in a different way.

On the one hand, while the pebble sign system rests on an idea of number as a compound of units that are counted, the Euclidean sign system rests on a double meaning of numbers: They result from counting and from measuring.

On the other hand, the Euclidean sign system allows also representing numbers in an indeterminate, more general way. Indeed, in the historical reconstruction offered by Becker (1936) and Lefèvre (1981), to prove that the sum of two even numbers is an even number, the pre-Pythagorean tradition used two (or a few) particular even numbers (say 8 and 6) displayed in two equal rows each. They showed ostensibly that the sum can also be displayed in two equal rows (in this case, two rows of seven pebbles each). The particular numbers had to be imagined not as the actual chosen numbers, but as any even numbers (see Radford, 2003). In other words, you were required to see in the number 8 or the number 6, not 8 or 6 pebbles but only two equal rows (see Figure 1).

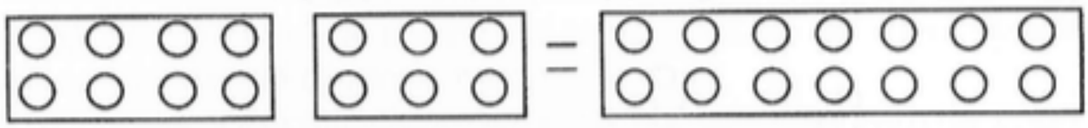

Figure 1. The pre-Pythagorean proof that the sum of even numbers is an even number

In the Euclidean tradition, the proof goes as follows.

> *Proposition 21*
>
> *If as many even numbers as we please be added together, the whole is even.*
>
> *For, let as many even numbers as we please, AB, BC, CD, DE, be added together; I say that the whole AE is even.*
>
> 
>
>
> *For, since each of the numbers AB, BC, CD, DE is even, it has a half part [vii. Def. 6]; so that the whole AE also has a half part. But an even number is that which is divisible into two equal parts; [id.] therefore AE is even. Q. E. D.* (Heath, 1956, p. 413)

The use of letters and diagrams allows Euclid to deal with generality in a more sophisticated way. Through letters he can now designate the even numbers without recourse to particular even numbers. Also, through diagrams, the lengths remain indeterminate. What Euclid does not do, though, is to deal with generality in general. That is, although in natural language the proposition is stated to be true of as "many even numbers as we please", the actual proof is based on any four even numbers. It seems that Euclid's semiotic system does not allow for referring effectively in the proof to as many even numbers as we please. Although the proof could have gone something like: "Let us suppose that we want to add four even numbers AB, BC, CD, and DE", what the proof actually says is: "Let as many even numbers as we please, AB, BC, CD, DE*,* be added together." Many is replaced by four indeterminate even numbers.

It is true that Netz has argued that, in the Euclidean tradition, generalization signifies repeatability (Netz, 1999, p. 246)—in this case, the possibility of redoing the same proof with five, six, seven, etc. any even numbers. Yet, the point that I am trying to make is that semiotic systems are much more than surrogates of things they represent. The semiotic systems to which we resort in dealing with generality are consubstantial with the layer of generality involved. Thus, the layer of generality that can be attained within the pebble semiotic system is not the same as the one that can be attained within the Euclidean semiotic system that includes letters and diagrams and a more elaborated meaning and syntax. Nor is the generality reached with the Euclidean semiotic system equal to the layer of generality that is attained using modern subindex notations and universal quantifiers. Any semiotic system has a limit as to what it can afford to be thought, felt and imagined (Radford, 2014a).

In psychological terms, this point can be rephrased by saying that the form of consciousness or awareness that is reached in terms of mathematical generality varies as the semiotic systems vary. In its simplest expression, this statement amounts to the formidable Vygotskian problem of the relationship between thinking and language (Vygotsky, 1987). Language (in its written or oral form) is not a mere vehicle of thought; thought is imbricated in language. "We pursue our human inquiry through language and in language, and the shape and forms of expression are not simply images of our thought but its structures as well" (Wartofsky, 1977, p. 134). We can reformulate this idea in a more general way: There is a dialectical constitution between consciousness and the semiotic systems we use in our social and cultural experience of the world.

A CLASSROOM EXAMPLE

Let me turn to a classroom example to illustrate the previous ideas. The example comes from a Grade 2 mathematics lesson on pattern generalization. The lesson features the sequence of Figure 2.

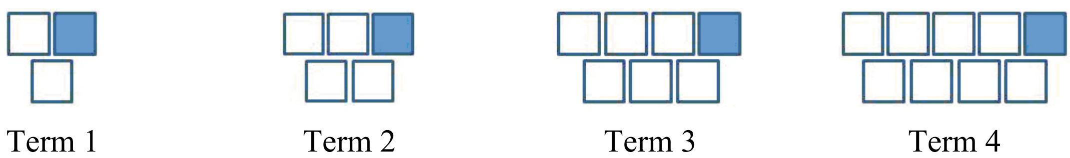

Figure 2. The first four figures of a sequence given to the students in a Grade 2 class

The students were invited to draw Terms 5 and 6 and to anticipate the number of squares in remote terms, such as Terms 10 and 25.

While the trained eye perceives the figures of the sequence in meaningful chunks (e.g., the figures are made up of two rows, the figures are made of diagonals and the dark square, etc.), young students tend to perceive them as a bunch of squares without a specific spatial configuration. The students tend to focus on the activity of counting only. Figure 3 shows one of the students (Carlos) counting the squares on the top row of Term 3. He did the same for all the given terms.

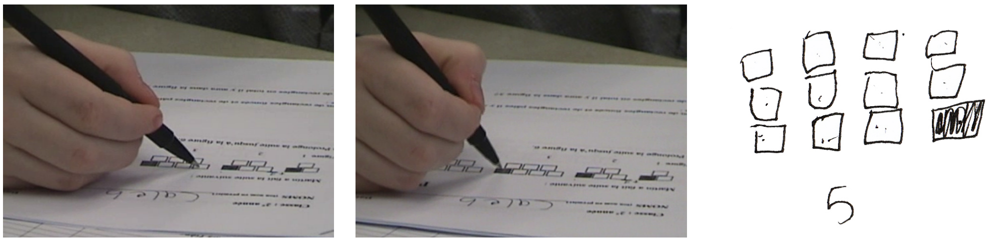

Figure 3. Carlos counting squares

In the first two photos of the Figure 3, while counting aloud, Carlos sequentially points to the squares in the top row of Figure 3. The third photo shows Carlos's term 5. When it was time to draw Term 5, Carlos produced a term with three rows and four columns, putting the dark square at the end. Neither the amount of squares, nor their spatial disposition matched the expected mathematical answer (for a detailed analysis, see Radford, 2012). As Wartofsky reminds us in a previous citation, what we choose to notice and what we choose not to notice are related to the framework of concepts with which we tackle the problem at hand. This framework of concepts defines the *seeing as* through which the terms of the sequence are transformed into objects of perception. This is why perception is not, as Kant (1787/2003) suggested in the *Critique of Pure Reason*, a two-step process: First, starting with a sensuous experience, raw sense data is produced. Second, the raw sense data thus produced is submitted to the faculty of reason or understanding to be processed and endowed with conceptual meaning. Perception is rather sensuous and conceptual through and through:

> *Perceiving is… not an incipient form of human action; it is human action in one of its modes, complexly and subtly involved in all the other modes of more direct productive praxis, or in the motor-activity by which human beings act in the world, and sustain their existence.* (Wartofsky, 1979, p. 210)

The signs and objects that the eye perceive in the world are not part of a mere empirical process but of a process of perceptual semiosis, that is "a process relying on a use of signs dialectically entangled with the way that concrete objects become perceived by the students", as they engage in practical sensuous activity (Radford, Bardini & Sabena, 2006, pp. 685-686). We should not forget, though, that the crucial question here is not a question of the special features of the child's embodied perception, attention, or intention as such; that is, "as some kind of ability of his consciousness". It is rather a question of the characteristics of the child's activity (Leont'ev, 1978, p. 156).

Let me come back to the epistemological-ontological component of generalization. Carlos's framework of concepts has still to be transformed in order to see the terms of the sequence not as a bunch of squares to be counted but as spatially arranged in manners that make the counting process more effective. These forms of mathematical *seeing as* have a long history that goes back to the first human settlements. They were refined and redefined in the course of centuries of practical-intellectual activity. Carlos has still to encounter them. This encounter is not something that will happen all of a sudden. It will take years. It is a process that we have called objectification (Radford, 2008, 2014b), and which will require the use of a variety of sign systems in various sensuous practical-intellectual activities (for details, see Radford, 2014c).

There is, however, something terribly misleading in my account. The way I talk about Carlos seems to presuppose that Carlos is already there, waiting for the encounter with more mathematically sophisticated ways of seeing as. This is the inadequacy of the formulation of the problem of *subject* and *object* in traditional epistemology. The subject appears as already given, and invariable. The subject appears as remembering knowledge (Plato) or as finding it within itself (Leibniz, Descartes and the whole array of rationalists) or discovering it through its sense (Hume and the empiricists) or as constructing it (Kant, Piaget and the constructivists). In all these accounts the problem is posed as if the knower is already there, fully constituted or constituted through its own deeds. What is inadequate in this way of posing the problem of the relationship between subject and object is that it misses the fact that there is a dialectical relationship between knowing and becoming (Radford, 2013b). What I am suggesting is that Carlos is knowing, but not because he is already there—a subject ready to know. In fact, Carlos is not there yet (and never will be!).

He is knowing because he is becoming. Carlos is perpetually becoming. He is continuously being shaped in his endless encounter with cultural and historical forms of perceiving, sensing, feeling, thinking, etc.

And reciprocally: Carlos is becoming because he is knowing, not purely knowing, but knowing-with-others, suffering, hoping, and dreaming with others—the teacher included—in practical-sensuous activity.

This Special Issue

Let me now turn to this Special Issue. The idea of the Special Issue arose during my visit to the Universidad de Granada in October 2014, when this university was celebrating the retirement of one of the founders of its mathematics education group—professor Encarnación Castro. The idea was to gather a number of papers by researchers whose work deals with generalization. I shall comment briefly on each one of the papers.

The first article, "Generalizing is Necessary or Even Unavoidable" (Otte, Mendonça, & de Barros, 2015), is of a theoretical nature and provides us with a very rich panorama of generalization in mathematics. They distinguish between the Platonic and the Aristotelian concepts of generalization and the impact that these traditions have had in the conceptualization of mathematics. The authors argue in favour of a genetic or evolutionary perspective on knowledge and position themselves against the traditional epistemological static relationship of subject and object. They claim that "knowledge is a process of mutual adaptation of subject and object, a process, which changes both parts involved" (p. 142). In their account, generalization is linked to our possibilities of giving a *form* to our ideas. To generalize means to have an idea and to apply it, to give it a form. "Theories and works of art are constructed forms, they are realities in their own right." (p. 158) Drawing on Peirce's phenomenology they argue that semiotic icons play a crucial role in generalization, as icons are an essential in creative thinking: Icons are the only type of signs that bring something new to the mind and they are thus essential to generalization (Otte, 2003).

The second article, "The Distributed Nature of Pattern Generalization" (Rivera, 2015), focuses on the students' ability to generalize patterns. Rivera highlights the interrelation between pattern generalization and mathematical structure, and explores the cognitive and noncognitive factors that influence pattern generalization. In his analysis, he resorts to the sequence shown in Figure 2 (see above) and a variant of it—the same sequence without a dark square. He proposes a conceptual framework of factors influencing pattern generalization, which includes: (a) natures and sources of generalization (something related to what I termed the ground of the generalization at the beginning of this Introduction); (b) types of structures, which include additive, multiplicative, and iterative thinking, and that are important in the construction of functional-based generalizations; (c) attention or awareness; (d) representations; and (e) context, which will favour or encumber the formulation of a general formula (in arithmetic or verbal procedural terms).

The third article, "*Generalización de Patrones y Formas de Pensamiento Algebraico Temprano* (Generalization of Patterns and Forms of Early Algebraic Thinking)" (Vergel, 2015), deals with the way in which young students tackle tasks about pattern generalization. Drawing on the theory of objectification, Vergel pays close attention to the students' semiotic resources, such as gestures, language, and rhythm, to make sense of the sequences to generalize. Particular attention is paid to the manner in which the variable becomes progressively an object of consciousness, and the semiotic resources that allow it. Vergel explores different kinds of sources of generalization (in Rivera [2015]'s sense): He resorts to the well-studied figural sequences (i.e., sequences whose terms are given as figures, as in Figure 2 above), but moves to less investigated sequences, such as those supported by a tabular representation (numbers, associated by their position in the sequence). His results suggest something very interesting: That the lack of spatial clues (that are present in the figural sequences), leads the students to focus on numerical relations out of which, through abduction (in Peirce's sense), the students come up with a formula. The formula, however, does not necessarily bear the features that characterize an algebraic generalization: Variables in the formula are rather the product of an abduction, which, regardless of its sophistication, is not transformed into hypothesis from where to deduce the formula. The result is that the generalized object results from an abductive (and sometimes guessing) act, rather than an analytic one. The generalization is arithmetic, not algebraic. This result invites us to better understand the distinction between arithmetic and algebraic generalizations.

In the fourth article, “Forms of Generalization in Students Experiencing Mathematical Learning Difficulties” (Santi, & Baccaglini-Frank, 2015) challenge from the outset the traditional view that a special need student “is as a student who requires interventions to restore a currently expected functioning behaviour” and introduce a new paradigm “to frame special needs students’ learning of mathematics.” Although in official documents there is an increasing trend to incorporate special students in regular classrooms, “in reality”, the authors remark, “these students are not always included and they do not actually play a key role (or any role, at that!) in the teaching-learning process” (p. 217). They question the dichotomy between normal and disabled students, often cast in terms of the difficulties that the latter encounter in learning. They remark that “all students have to face difficulties, learning obstacles and failure throughout their education, therefore it is not clear how to identify a clear boundary between normal and disabled students” (p. 218). The paradigm that they introduce goes beyond the usual human being of education and psychology, namely a human being reduced to a cognitive problem-solver mind. Instead, they propose the following.

> *Educational activity should aim at fostering a mode of existence in mathematics, i.e. being and becoming with others to make sense of the world also through mathematics. The aim of education should be to allow all students to make sense of the world in spite of their particular conditions.* (p. 220)

They draw on the theory of objectification and use an application for the iPad and iPhone, *Mak-Trace*, and investigate the verbal and embodied dimensions involved in generalization. They put into evidence interesting forms of what, in the theory of objectification, are called factual and contextual generalizations. The article certainly opens new avenues for rethinking special education.

The fifth paper, by D’Eredità and Ferro (2015), deals with a different topic. As the title of the paper indicates, it deals with “Generalization in Chess Thinking”. I am indeed very glad that D’Eredità and del Ferro, two high-level chess players involved in chess education with the Italian Chess Federation, accepted the challenge to write a paper on generalization. They show how abduction (one of the elements of the structure of generalization suggested in the context of mathematics pattern generalization [Radford, 2013a]) is ubiquitous in chess generalization. Patterns in chess have a perceptual distinctiveness as compared to the forms of perceptual semiosis we have identified in mathematical figural sequences. As D’Eredità puts it,

> *the reasoning of chess players is characterized by a strong visuo-spatial component, as confirmed by neuroscience outputs. This component acts both in a classical, hypothetic-deductive way and in an immediate (based on previous knowledge) way, mainly in a non-verbal modality.* (D’Eredità, 2012, p. 130)

D’Eredità and Ferro capture this perceptual distinctiveness of chess thinking through the theoretical construct of configural concepts (Ferro, 2012). Although chess pieces have a conceptual content (e.g., their position on the chessboard at the beginning of the game, their rule-based movements), they become part of more general concepts —configural concepts—that include other pieces and contextual-based potential actions. More precisely that

> *a configural concept is made up of chess objects and their conceptual relationships. Its meaning comes from the hierarchical linkage of the conceptual relationships between the involved chess objects and from its position in the whole theoretical structure of the pieces in the chessboard.* (Ferro, 2012, p. 15)

As we can see, a configural concept is already a generalization where perception remains omnipresent, although perhaps in a specific refined and sophisticated way, highlighting relationships that remain "embodied in figural patterns, some of which become more salient than others" (D'Eredità & Ferro, 2015).

In his studies with children, Ferro has shown the embodied genetic nature of configural concepts. Referring to one of the students, he says that

> *in lower level of awareness the student used gestures to point to squares, to keep in hand pieces or to tap them over the chessboard. These gestures (in particular the pointing gestures) were modified (or simply contracted) into eyes' actions. When he achieved highest level of awareness he moved his eyes and his head to individuate the squares on the chessboard without using gestures… [The] eyes' motions were not "alone," they were coordinated with the language that the student improved in "calling" the columns or the squares.* (Ferro, 2013, pp. 104-105)

More research is still needed to better understand generalization in chess and its possible relationships to mathematics generalization. D'Eredita and Ferro's work points to new possible connections that we can only hope will be explored in the near future.

Although there always will be new things and aspects to investigate about generalization, in part as a result of the cultural and historical changes in our sensitivities to gauge, explore, pose, and understand problems, I hope that the articles included in this Special Issue will make a contribution to the problem of generalization. I would like to thank Maria C. Cañadas, from the PNA editorial team, for her help in producing this Special Issue. I would also like to thank the reviewers of the papers for helping the authors improve their ideas.

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