GENERALIZING IS NECESSARY OR EVEN UNAVOIDABLE

Michael F. Otte, Tânia M. Mendonça, and Luiz de Barros

The problems of geometry and mechanics have driven forward the generalization of the concepts of number and function. This shows how application and generalization together prevent that mathematics becomes a mere formalism. Thoughts are signs and signs have meaning within a certain context. Meaning is a function of a term: This function produces a pattern. Algebra or modern axiomatic come to mind, as examples. However, strictly formalistic mathematics did not pay sufficient attention to the fact that modern axiomatic theories require a complementary element, in terms of intended applications or models, not to end up in a merely formal game.

Keywords: Complementarity; Genetic epistemology; Mathematical cognition

La generalización es necesaria o incluso inevitable

Los problemas de geometría y mecánica han motivado la generalización de los conceptos de número y función. Esto muestra cómo la aplicación y la generalización previenen que las matemáticas sean un mero formalismo. Los pensamientos son signos y los signos tienen un significado dentro de un cierto contexto. El significado es una función de un término: esta función produce un patrón. El álgebra o la moderna axiomática vienen a la mente como ejemplos. Sin embargo, las matemáticas estrictamente formales no prestaron suficiente atención al hecho de que las teorías axiomáticas modernas requieren un elemento complementario, en términos de aplicaciones intencionadas o modelos, para no terminar en un juego meramente formal.

Términos clave: Cognición matemática; Complementariedad; Epistemología genética

"The most notorious ill-fortune must, in the end, yield to the untiring courage of philosophy—as the most stubborn city to the ceaseless vigilance of an enemy."
Edgar Allan Poe

Otte, M. F., Mendonça, T. M., Gonzaga, L., & de Barros, L. (2015). Generalizing is necessary or even unavoidable. *PNA, 9*(3), 143-164.

Generalization and Human Subjectivity

We must generalize, if we want to secure the future of mathematical and scientific culture. Without generalization mathematics would become a kind of chess, as mathematicians since Peirce (1839-1914) and Poincaré (1854-1912), had repeatedly stressed. A characteristic of mathematical thought is, says Peirce "that it can have no success where it cannot generalize". And in a similar vein Poincaré states that "there is no science but the science of the general".

Now the chess game, in particular, has nowadays become the domain of the computer, of the machine. Nobody knows this better than the current world chess champion Magnus Carlsen. His most important teacher and coach in lonely Norway had been the computer. Characterized by its electronic training partner, Carlsen plays such an unorthodox style that many grandmasters, arrested in the tradition, has to capitulate unexpectedly. Because, as Carlsen says, no man can win against the computer today, the game of chess has become psychological warfare. And no one understands it better to wear down his opponents through a tough delaying strategy than Magnus Carlsen. As soon as the complexity of a situation grows beyond sensible efforts, it is more profitable to concentrate on the psychology of your opponent. As his biographer Somen Agdestein says: "While Kasparov was concerned about deep ideas, Magnus is only interested in what works to beat that particular opponent at that particular day" (Agdestein, 2013, preface). Similarly, in the formal teaching situation the student will focus on what the teacher wants to hear from him and will not bother with ideas or universal insights.

School mathematics is essentially algebra. Algebra in turn is a language or technology that works without thinking, as Leibniz was the first to realize. In addition, he thought that the algebraic calculus assumes meaning from a pre-established harmony between the order of signs and the order of things. Nowadays, in face of a highly varied and dynamical world, the notion of pre-established harmony makes no sense anymore and meaningfulness depends on generalization and new applications. The broader view on our lives provides meaning.

Because knowledge is a process of mutual adaptation of subject and object, a process, which changes both parts involved, we find the very same complementarities in the determination of the subject and object of knowledge. As Rorty says critically and with a derogatory undertone, "The distinction of the mental and the physical is parasitic on the universal-particular distinction, rather than conversely" (1979, p. 31). That is our interest in the universal contributes to our being human subjects.

So we must create new concepts and ideas, or ideal objects. To generalize means just this, introduce new ideal objects. What is in question here is not just to frame new formal definitions and some syntactical changes, but to overcome some objective constraints. The belief that generalization can be studied merely syntactically leads to paradox (Goodman, 1965). And to the complementarity of ideal and existent objects, or of meanings and things, corresponds a complementarity between mind and brain, or between an epistemological and an ontological subject (Klein, 2014).

Our empirical contacts with reality seem limited to the observation of so-called facts, that is, are limited to what we can observe locally. Philosophy and science are concerned, in contrast, with laws of nature and as these laws establish relations between universals, rather than particular existents, we have here a case of generalization as defined. Now, to generalize and extend factual knowledge requires an act of faith in the continuity of nature (Lovejoy, 1964). The so-called principle of continuity has been of fundamental importance to all the exact sciences, including mathematics. It works, because objective reality itself is not without logic, particularly in the field of mechanics and geometry. This is present to everybody, who operates with his/her hands and his/her body. Therefore mechanics and geometry, the first positive sciences are so confidently based on the principle of continuity. And at a certain level the human mind or consciousness is nothing but a reflection of this logic.

At the same time the mind is the most active, unpredictable and unstable thing under heaven, that you can think of, hence Plato's "invention of reason" (Chatelet, 1992). We humans are besieged and harassed by hopes, dreams, expectations and ideas. The influence, which these things have on the mind of man cannot be overestimated. We live more in a world of signs and possibilities, than in a universe of determinate things. This means that the knowledge process is to be described as the semiotic process of interpretation and is, therefore, a kind of interactive process between objects and ideas. This is what we mean by the notion of "complementarity" (Otte, 2003a).

All General Phenomena are Essentially of the Character of Signs

A reality and its description are of different logical types: The menu is not the meal. A representation cannot be reduced to the object represented, a concept cannot be reduced to its extension, a set cannot be reduced to its elements, as Russell's paradoxes of set theory have shown clearly. All general phenomena are essentially of the character of signs. From an absolute and static point of view, symbol and object are even incomparably different. Only from a genetic point of view, we may bridge this abyss between the definite and the only vaguely described and incompletely determined. Meaningfulness is, after all, based on selective loss of detail and on generalization.

To generalize, we have said, means to introduce new ideal objects. But, what are ideal objects? And, how can we humans have grasp of, or insight into, the realm of ideas or abstract objects? Platonists, like the inhabitants of "Cantor's paradise" would answer to the first question by conceiving of ideal objects as extensions of concepts. On this account the number symbols, 1, 2, 3, are proper names of certain ideal objects, of "multitudes composed of monads, of unities" as Euclid says. And it is form, which adds coherence to a multitude, makes unity out of it (Klein, 1985, p. 48).

There is a very different view of universals and ideal objects, however, derived from Aristotle' metaphysics, rather than from Plato's theory of forms. It is generality as continuity. It means that a general is something not specified in every respect, like when we speak about a man, or an apple in general. The idea of a general, says Peirce, "involves the idea of possible variations" (Peirce, 1931-1935, p. 102). This difference in the notion of universals reflects a certain opposition between the so-called "context of discovery" and the "context of justification". We justify mathematical statements by formal proofs and these are based on a conception of mathematics established by relations of identity and difference, rather than of continuity.

Peirce describes the concentration on the process of generalization as follows.

> *What I propose to do tonight is, following the lead of those mathematicians who question whether the sum of the three angles of a triangle is exactly equal to two right angles, to call in question the perfect accuracy of the fundamental axiom of logic. This axiom is that real things exist, or in other words, what comes to the same thing, that every intelligible question whatever is susceptible in its own nature of receiving a definite and satisfactory answer, if it be sufficiently investigated by observation and reasoning... Let me be quite understood. As far as all ordinary and practical questions go I insist on this axiom as much as ever.* (Peirce, 1857-1886, pp. 545-546)

If an intelligible question whatever is not susceptible in its own nature of receiving a definite and satisfactory answer, we must generalize, to find that answer. And the answer we find consists of some general relationships between universals, for example, between the angle sum of a general triangle and the possible relations between two straight lines. In addition, it is obvious from Peirce's statement that theories become realities *sui generis*. Thus, a theory consists of a specific form and a set of intended applications or problems. Universals are, according to this view, conceived of as free objectual variables.

Such variables appear in propositions like: An apple is a fruit, or: Man is good and bad, or: an equilateral triangle is a general triangle, or, if a and b are natural numbers, then $a+b=b+a$. "In a proposition like 'an apple is a fruit' it would be unnatural to interpret 'an apple' as a placeholder, because this presupposes that we have given individual names to all the apples in this world" (Quine, 1990, p. 99). When we affirm that a falling stone has a definite acceleration, we do not refer to a particular stone that fell to the ground five minutes ago.

There are ideas of an apple or a stone or a triangle in general, but they turn out to be ideas of particular triangles, put to a certain use. On such an account, a general triangle is a free variable, and is to be treated as representative of a kind and not as a collection of determinate triangles. The variability is determined by theory, or more specifically, by an axiomatic description of it. For example, the sentence "an equilateral triangle is a general triangle" is true in the context of affine geometry, but is false in Euclidean geometry. So if the task, for instance, is to prove the theorem that the medians of a triangle intersect in one point, the triangle on which the proof is to be based can be assumed equilateral. The triangle plays a merely instrumental or functional role in the proof. What the proof is really about, is the determination of the center of gravity of three equal masses. And in the usual proofs of the angle sum of a Euclidean triangle, again the triangle is only functional to establishing the relationship between parallelism and the angle sum.

And, when Joule set up his little machinery, he was interested in the law-like relationship between heat and motion, conceiving of them as equivalent representations of the universal idea of energy. This idea, in turn, is functional to establishing the relationship between the phenomena. Ideal objects, like energy, or the famous general triangle are continua that can variously and differently be specified if need be.

Euclid, Apollonius or Pappus worked on geometrical configurations, understood as free variables, which led Fermat to find fault in their procedure, because they did not present matters "generally enough" (Klein, 1985, p. 13). Fermat obviously had a different view of generality, in terms of quantification, then the Greek scholars.

> *Under one interpretation the statement (to be proved, my insertion) refers to a definite totality.... and it says something about each one of them. Under the other interpretation no such totality is supposed and the sentence has much more conditional character.* (Mueller, 1969, p. 290)

Diagrammatic reasoning, as employed by Apollonius or Euclid could thus better be described in terms of the notion of "thought experiment", that is, "involving an idealized physical object, which can be represented in a diagram" (Mueller, 1969, p. 291).

Relational Thinking

Number, or more generally arithmetic, was to the Pythagoreans "mainly a science of the visible universe, a cosmology, i.e. a science of the unity and order of this universe" (Klein, 1985, p. 45), to Dedekind it was a means to better distinguish between things. Dedekind was—besides Grassmann and Peano—among the first mathematicians, who answered the question, what numbers are, not by an analysis of their ontological status as single entities, but as elements of a structure, or series, such that asking for properties of numbers, which do not stem from the relations they bear to one another, becomes pointless. This opened the door to an axiomatic presentation of arithmetic, about 2000 years after Euclid's axiomatization of geometry.

Hilbert says that the free variables occurring in axiomatic statements are of this kind of "Aristotelian" generality.

> *For example, the statement that if A is a numerical symbol, then* $A+1=1+A$ *is universally true, is from our finitary perspective incapable of negation. We will see this better if we consider that this statement cannot be interpreted as a conjunction of infinitely many numerical equations by means of "and" but only as a hypothetical judgment which asserts something for the case when a numerical symbol is given.* (Hilbert, 1964, p. 91, our translation)

In addition, Hilbert himself gave, speaking about mathematical generalizations, the following examples of ideal elements.

> *The method of ideal elements is used even in elementary plane geometry. The points and straight lines of the plane originally are real, actually existent objects. One of the axioms that hold for them is the axiom of connection: one and only one straight line passes through two points. It follows from this axiom that two straight lines intersect at most at one point. There is no theorem that two straight lines always intersect at some point, however, for the two straight lines might well be parallel. Still we know that by introducing ideal elements, viz., infinitely long lines and points at infinity, we can make the theorem that two straight lines always intersect at one and only one point come out universally true. These ideal "infinite" elements have the advantage of making the system of connection laws as simple and perspicuous as possible. Moreover, because of the symmetry between a point and a straight line, there results the very fruitful principle of duality for geometry.*
>
> *Another example of the use of ideal elements are the familiar complex-imaginary magnitudes of algebra which serve to simplify theorems about the existence and number of the roots of an equation. Just as infinitely many straight lines, viz., those parallel to each other, are used to define an ideal point in geometry, so certain systems of infinitely many numbers are used to define an ideal number. This application of the principle of ideal elements is the most ingenious of all. If we apply this principle systematically throughout an algebra, we obtain exactly the same simple and familiar laws of division which hold for the familiar whole numbers 1, 2, 3, 4… We are already in the domain of higher arithmetic.* (Hilbert, 1964, p. 85)

What is a Theory?

The twofold character of the general is not least expressed in the history of mathematics by two different interpretations of the continuity principle, two interpretations, over which Cauchy and Poncelet quarreled (Belhoste, 1991, p. 55). Rather than conceiving of this principle in terms of structure and variation or invariance, Cauchy thought of continuity in terms of approximation and limit, as part of a kind of inductive or bottom-up strategy, that is, by a transition from the potential infinite to the actual infinite. The program of rigorization by arithmetization searched to solve the foundational problems in a reductionistic manner, by defining the continuum as a set.

The axiomatic movement, in contrast, as anticipated in the works of Poncelet or Grassmann, tried to employ, so to say, a top-down strategy, solving the foundational problems of mathematics by extending and generalizing its relational structures and its rules of inference. Grassmann's dropping of the commutativity of a general product and his definition of the anti-commutative vector product provides a pertinent example here. Grassmann was well aware of the fact that this loosening of formal constraints could count as genuine generalization only, as soon as there corresponded a semantic field a possible application to the new structure and he found this application in electrodynamics (Otte, 2003b, p. 189).

Now the rigor movement of arithmetization could succeed only, if it generalized the number concept itself. Ideas like motion, law of nature etc. have been represented by mathematicians in terms of the idea of a mathematical function. In classical mathematics from Leibniz to Euler, a mathematical function had been framed, either analytically in terms of formula or rule, or as a continuous geometrical curve. When by the end of the 18th century the importance of the relationship between the notions of function and of continuity, became more clearly perceived and the basic property of an abstract mathematical function became realized in terms of its continuity, the function concept had to be generalized. As Bochner (1974) observes, "the conceptions of function and of continuity have evolved simultaneously" (p. 845).

Since Bolzano or Cauchy mathematicians considered curves as point sets, rather than conceiving of them in analytical terms as formulas, because the number of conceivable functions was much larger this way and the range of possible applications of the function concept increased considerably. But now the number concept had to be generalized and enlarged accordingly, by creating the real and complex numbers to successfully arithmetize the continuum, and makes the approach work. This proved not to be possible in a constructive way, however, but had to be done axiomatically.

Nevertheless, the movement of arithmetization criticized and deplored the fact, that the axiomatic characterization of numbers ignores the question of existence and leads to a situation where every number-symbol becomes infinitely ambiguous. Neither Peano nor Hilbert are capable of defining what the number one is, says Russell. Modern axiomatic theories became, on the one hand, intensional theories, in the sense that the axioms as a set of postulates not only determine the intensions of the theoretical terms, but also descriptively constitute the extensions or referents. In Euclidean geometry, for example, the objects about which the theory speaks seem to be given by intuition, and independently of the theory. In Hilbert's geometry or Peano's arithmetic, the situation is quite different. So, strictly formalistic mathematics did not pay sufficient attention to the fact that modern axiomatic theories require a complementary element in terms of intended applications or models, not to fall prey to the semantic paradoxes. Russell had to introduce his "vicious circle principle" to avoid the paradoxes in the conceptual system.

This principle seems necessary for nominalism or constructivism only

> *because the construction of a thing can certainly not be based on a totality of things to which the thing to be constructed itself belongs. If however it is a question of objects that exist independently of our constructions there is nothing in the least absurd in the existence of totalities which can be described only by reference to this totality.* (Gödel, 1944, p. 136)

Hence results Gödel's intuitive Platonism. And Gödel believes that we have direct, intuitive access to new ideas and are able to frame them into new axioms. The reason for the indispensability of intuition is simply that we cannot reach the objective world by means of language and cannot finitely describe the meaning of any idea, concept or symbol. Gödel says what follows.

> *Obviously… the certainty of mathematics is to be secured not by proving certain properties by a projection onto material systems namely, the manipulation of physical symbols, but rather by cultivating (deepening) knowledge of the abstract concepts themselves… The procedure must thus consist, at least to a large extent, in a clarification of meaning that does not consist in giving definitions. Now in fact, there exists today the beginning of a science which claims to possess a systematic method for such a clarification of meaning, and that is the phenomenology founded by Husserl… Namely, it turns out that in the systematic establishment of the axioms of mathematics, new axioms, which do not follow by formal logic from those previously established, again and again become evident…*
>
> *I would like to point out that this intuitive grasping of ever newer axioms that are logically independent from the earlier ones, which is necessary for the solvability of all problems even within a very limited domain, agrees in principle with the Kantian conception of mathematics… I believe it to be a general feature of many of Kant's assertions that literally understood they are false, but in a broader sense contain deep truths. In particular, the whole phenomenological method… goes back in its central idea to Kant.*
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> (Gödel, 2001, p. 61)

On the other hand, all that we perceive and can think about, appears before our mind's eyes framed by symbols and other types of representations. Hence the importance of diagrams analytical languages and the axiomatic method. In summary, we see that the duality of the conceptual or analytical approach and the perception of the related semantic fields or intended applications can be fruitfully related to each other only from an evolutionary or genetic point of view.

Phenomenology and Mathematics

Phenomenology is the science of pure possibility and so, according to Gödel, it must precede the science of real fact. Another way to characterize phenomenology consists in pointing out that it is studying that, which is common to all phenomena, while natural science is concerned with natural phenomena, and sociology looks into the nature of social phenomena, etc. Phenomenology, says Peirce (1931-1935), "studies the kinds of elements universally present… to the mind" (p. 186). From this point of view, Peirce concludes that three categories, which he calls, in an abstract manner first, second and third, belong to all phenomena, because they are the ingredients of any sign and everything, which is present to the minds must be of the nature of a representation. Now, a sign, according to Peirce

> *is something which stands to somebody for something in some respect or capacity. It addresses somebody, that is, creates in the mind of that person an equivalent sign, or perhaps a more developed sign. That sign which it creates I call the interpretant of the first sign. The sign stands for something, its object. It stands for that object, not in all respects, but in reference to a sort of idea. "Idea" is here to be understood in a sort of Platonic sense* (Peirce, 1931-1935, Vol. 2, p. 228).

To Peirce, phenomenology is the science of appearances and it is abstract in the sense that its subject matter is general and hypothetical, just as the constructs of mathematics are. However, as mathematics is, according to Peirce "the science which draws necessary conclusions", it appears as an essential instrument to phenomenological analysis and plays a functional role in its considerations. And a phenomenology which does not reckon with "pure mathematics… will be the same pitiful clubfooted affair which Hegel produced" (Peirce, 1931-1935, Vol. 5, p. 40).

What is wrong with Hegel's phenomenology? Phenomenology has to start from what is necessarily to be recognized or taken into account in any effort to generalize. Phenomenology therefore is part of any theory of development and evolution. Hegel's view of development knows only of continuity, however, and excludes pure chance and brutish, unresolved fact. It is essentially a theory of conceptual development. Some Hegelian philosophers seem, Peirce writes what follows.

> *To think that the real subject of a proposition can be denoted by a general term of the proposition; that is, that precisely what it is that you are talking about can be distinguished from other things by giving a general description of it. Kant already showed, in a celebrated passage of his cataclysmic work, that this is not so (see Kant, 1787, B 626, our insertion); and recent studies in formal logic have put it in a clearer light. We now find that, besides general terms, two other kinds of signs are perfectly indispensable in all reasoning.* (Peirce, 1931-1935, Vol. 8, p. 41)

One of these kinds is the *index*, the other the *icon.* It is in the dynamics of mathematical reasoning only, where they interact and come together. Peirce calls icons and indices explicitly non-symbolic thought-signs (Peirce, 1931-1935, Vol. 6, p. 338), and distinguishes them by saying that icons are pictures or diagrams or other images such as have to be used to explain the meanings of words, whereas indices have the role of designating the subject of discourse. One might think, Peirce himself adds

> *that there would be no use for indices in pure mathematics, dealing, as it does, with ideal creations, without regard to whether they are anywhere realized or not. But the imaginary constructions of the mathematician, and even dreams, are so far approximate to reality as to have a certain degree of fixity, in consequence of which they can be recognized and identified as individuals.* (Peirce, 1931-1935, Vol. 2, p. 305)

The indices occurring in pure mathematics refer to entities or objects that belong to a model, rather than to "the real world", that is, they indicate objects in constructed semantic universes. That mathematics, on the one hand, does not make existential claims, only outlining possibilities and in the other hand makes essential use of indices, in order to represent statements of fact, is fundamental for Peirce's conception of mathematics as "diagrammatic reasoning".

Mathematics proceeds by hypothetic-deductive reasoning, and "hypothesis substitutes, for a complicated tangle of predicates attached to one subject, a single conception" (Peirce, 1857-1886, Vol. 3, p. 337), that is, introduces an ideal object. Peirce calls the process of generalization "abduction" and he stresses the importance of what he called theorematic reasoning, in contrast to corollarial reasoning, which relies only on that which is enunciated in the premises. If a proof is possible only by reference to other things not mentioned in the original statement and to be introduced by intuition and abductive reasoning, if we need, for example, auxiliary constructions, which were not mentioned in the premises of the theorem, in order to be able to carry out a geometrical argument, such a proof is theorematic. The mathematician constructs and manipulates or modifies a diagrammatic representation of the premises in order to find out that foreign idea—to use Peirce's expression—which must be added to the set of explicit premises already available. Theorematic reasoning implies generalization, that is, the introduction of new ideal objects. By the abductive inference, says Peirce, "a number of reactions called for by one occasion get united in a general idea which is called out by the same occasion" (Peirce, 1892, p. 552).

> *The abductive suggestion comes to us like a flash. It is an act of insight, although of extremely fallible insight. It is true that the different elements of the hypothesis were in our minds before; but it is the idea of putting together what we had never before dreamed of putting together, which flashes the new suggestion before our contemplation.* (Peirce, 1931-1935, Vol. 5, p. 181)

Is Mathematics Capable of Coping with Change?

The classic thinking of ancient Greece was marked by aporetic situations and paradoxes. Zeno's paradox of the race of Achilles and the tortoise represents a defining expression. Originally intended as a defense of Parmenides, this paradox characterizes the epistemology of Plato too.

The world is, and has ever been, violent, messy and in permanent change. In contrast: Humans are searching for clarity, stability, orientation and self-identity. Platonism was a first reaction to these contrasting situations. Platonism teaches that diversity, change and motion are not real, are only appearances. This static view causes problems of knowledge and explanation. The Theaetetus is one of Plato's dialogues concerning the nature of knowledge. Therein Plato claims that to explain is to analyze complexes into their elements, i.e., those parts which cannot be further analyzed. The primary elements are "unaccountable and unknowable, but perceivable" while the complexes are “knowable and expressible” and so can be objects of “true judgment” (202b).

However, Socrates exposes some difficulties by examining letters. He takes the first two letters of his name, S and O to wonder if the syllable “So” is knowable, while the individual letters are not (203b-d). Theaetetus finds the idea strange, so Socrates deduces that in order to know the syllable, the letters must be known first (203e). They cannot be known, however, because they are not reducible to more simple elements. They can only, as it seems, be explained by themselves. P means just P! They can only be given as forms!

So Socrates returns to talking about elements and complexes to propose that they are in the same class (205d). “The number of the army is the same as the army.” (204d) However, when we make a drawing of a swan, for example, in the end we must not just draw feathers, legs, wings, and head, but must draw the swan! The whole discussion ends in disaster when Socrates finally says

> *but how utterly foolish, when we are asking what is knowledge, that the reply should only be, right opinion with knowledge of difference or of anything! And so, Theaetetus, knowledge is neither sensation, nor true opinion, nor yet definition and explanation, accompanying and added to true opinion.* (Socrates, 210b)

As any representation is nothing but a particular perspective or a certain point of view on something and thus is relative, these ideas or forms, that is, the objects of true knowledge, must be directly known. To know means to intuitively grasp the essence or idea of something and to intuit it as a form. So Plato’s problem, stated in semiotic terms, consists in the inevitable difference between sign and object, between knowledge and the object known, between form and meaning, or between the ideas and their shadows in the everyday world, a world that can only imperfectly approximate an unchanging and ultimate reality. The difference between the sign and the thing represented by it, thus causes some paradoxes of knowledge and explanation that have to be approached from a genetic and evolutionary perspective. The essence of something becomes distributed to all possible representations, past present and future. This totality obviously cannot be conceived of as a well-defined set, but establishes a kind of continuity.

In contrast to these situations, it had been affirmed many times in history that mathematics is essentially a science of the identical and the different, or about equality and difference. Thus, it is essentially about number. Therefore mathematics should be arithmetized. All mathematics, says Russell, for example, “including analytical geometry, may be considered as consisting wholly of propositions about natural numbers” (Russell, 1967, p. 4). Moreover, if the world should be mathematized, it must be finished, static and discrete. Oswald Spengler, in his *The Decline of the West* (1918), expressed these desires.

> *In number, as the sign of the completed and limited, is contained the nature of all reality, that has become recognized and limited at the same time, as Pythagoras, or who it was otherwise, realized with innermost certainty as a result of a great, quite religious intuition.* (p. 110)

One of the most important discoveries of the Pythagorean School is without doubt that of the incommensurability between the side and the diagonal of the square. And this implied the failure of arithmetization and was a rejection of the Pythagorean view that the realm of number provides a model of the entire natural world. Hence the importance of the continuum, and the continuity principle.

Now, the continuum of space is nothing but the realm of change and of possible relations, rather than a set of distinct points. Already the solution to Zeno`s paradoxes, proposed by Aristotle, involves distinguishing between, what he termed a 'continuous' line and a line divided into parts. And he solved the paradox, by saying that "if time is continuous, magnitude is continuous also" and putting the continua of space and time alongside each other. Aristotle is most often regarded as the great representative of a logic and a mathematics, which rests on the assumption of the possibility of clear divisions and rigorous classification.

> *But this is only half the story about Aristotle; and it is questionable whether it is the more important half. For it is equally true that he first suggested the limitations and dangers of classification, and the non-conformity of nature to those sharp divisions which are so indispensable for language [...].* (Lovejoy, 1964, p. 58)

Aristotle thereby became responsible for the introduction of the principle of continuity into natural history. "And the very terms and illustrations used by a hundred later writers down to Locke and Leibniz and beyond, show that they were but repeating Aristotle's expressions of this idea." (p. 58)

It also seems that in Greek mathematics occurred two different kinds of proof. During the first phase of Greek mathematics there a proof consisted in showing or making visible the truth of a statement. This was the epagogic method. "This first phase was followed by an apagogic or deductive phase. During this phase visual evidence was rejected and Greek mathematics became a deductive system" (Koetsier, 1991, p. 180f). Apagogic proof primarily verifies and epagogic proof generalizes. Epagoge is usually translated as induction. But it is perhaps not quite, what we think of as induction, but is rather taking one individual as prototypical for the whole kind. Aristotle writes with respect to epagoge the following ideas.

> *The consideration of similarity is useful both for inductive arguments and for hypothetical reasoning [...] It is useful for hypothetical reasoning, because it is an accepted opinion that whatever holds good of one or several similars, holds good also for the rest.* (Aristotle, 108b 7)

Epagogic proof depends on some law of the uniformity of nature or some continuity principle.

Platonic Ideas and Human Subjectivity

Plato's philosophy originated, as we know, in the scandal of Socrates' conviction and death. How could the people of Athens not have perceived that Socrates was a good man, how could such a seemingly obvious truth become so distorted? Plato held the Sophists responsible, so, he fought against them and against their motto, "Man is the measure of all things" (Protagoras). Which man, one might ask. Was Socrates not coaxing "the mind down from self-assertion—subjective assertion and private definition—and lead it back, through the community, home" (Cavell, 2008, p. 43).

In his dialogue Phaedo, Plato explains that true knowledge is intuitive recollection, because we have to have a scale and take a prospect to find something beautiful or ugly or to find some things equal or different and therefore we must possess some non-empirical knowledge (e.g. the form or idea of equality or beauty) at birth, implying that the soul existed before birth, to carry that knowledge. In other words, there are two selves to the person, a mind or soul and a corporal existence, which, by means of its senses and activities, establishes relationships between the ideas and the things of the world. The belief in universal and objective truths thus encompasses the belief in a universal and immortal soul of Man. Eternal ideas correspond to eternal souls.

Since Rorty, for example, rejects universals, like mathematical objects or natural laws, he reduces the human subject to a kind of machine, rather seeing the individual in its concrete expression as a reflection of general contexts (be those socio-cultural or natural). Society thereby becomes scattered, turning it into a mere set of isolated individuals, from which then a not explained solidarity is demanded, in order not to turn society into hell. How can Rorty try to appeal to "common humanity", in view of his denial of any universal essence or universal idea? On what empathy and human solidarity could be based? Rorty himself admits that, "Attempts to unite a striving for perfection with a sense of community require us to acknowledge a common human nature" (Rorty, 1989, p. XIII). And further: "Our insistence on contingency, and our consequent opposition to ideas like essence, nature and 'foundation' makes it impossible for us to retain the notion that some actions and attitudes are naturally 'inhuman'." (p. 189)

Peirce's so called "Pragmatic Maxim" reflects the problem of the complementarity of the collective vs. individual in the constitution of the human subject. The Maxim certainly favors scientific operationalism and the interests of the individual subject, being directed against Plato's or Descartes' intuitionism and their appeals to a priori self-evidence. The original 1878 statement of the Maxim runs as follows: "Consider what effects, that might conceivably have practical bearings, we conceive the object of our conception to have. Then, our conception of these effects is the whole of our conception of the object." (Peirce, 1966, p. 124)

Peirce comments on this about 25 years later, in 1902, by a contribution to Baldwin's *Dictionary of Philosophy and Psychology*. The Pragmatic Maxim, there he says that

> *might easily be misapplied,... The doctrine appears to assume that the end of man is action—If it be admitted, on the contrary, that action wants an end, and that end must be something of a general description, then the spirit of the maxim itself, which is that we must look to the upshot of our concepts in order rightly to apprehend them, would direct us towards something different from practical facts, namely, to general ideas, as the true interpreters of our thought.* (Peirce, 1931-1935, Vol. 5, p. 3)

Peirce wants by this comment to justify the creation of such entities as the imaginary numbers or the "doctrine of incommensurables" which represent a contradiction between the definite determinations of the everyday world, on the one hand, and the world of theory and development, on the other hand, between direct action and general intelligibility and reasonableness. To the complementarity of ideas and objects corresponds, therefore another one in the constitution of the human subject. Universal ideas require universal human spirits and souls.

Philosophers distinguish between the ontological self and the epistemological self. We cannot here completely reproduce the discussion about the two selves (Klein, 2014), which are complementary to each other, as are ideas and objects. We shall however try to provide some hints of the importance of this distinction, discussing some examples. The ontological self is a particular existent, while the epistemological self is like a general symbol.

It has been Kant, who had introduced such a distinction into modern philosophy. Kant had emphasized the indispensability of a "transcendental subject", besides the empirical subject, because, to know, he argued, means to know that one knows, that is, knowledge requires reflective self-consciousness or meta-knowledge. To understand the reality of knowledge one has to understand the reality of understanding. And here we meet with the transcendental subject, "which is cognized only by means of the thoughts that are its predicates, and of which, apart from these, we cannot form the least conception" (Kant, 1787, B 404). The continuity of experience or, as Kant puts it, the unity of the "I think", is an important indication of the objectiveness of knowledge. Kant writes: "The 'I think' must accompany all my representations, for otherwise something would be represented in me which could not be thought; that representation which can be given previously to all thought is called intuition." (B 131)

The reflection combines a present question, the actual information or fact with the knowledge already stored in memory. This connection cannot simply consist, as Plato believes, in a reduction of the new to the old—otherwise there would be nothing new—but it must also represent the old in the light of the new and explain it so. The new is as much the starting point, as the goal of knowledge. In science, to understand a concept means to develop a theory, and vice versa, the theory as a whole is logically founded, if it can be understood as an original idea, which has been developed and unfolded and has thereby rendered meaningful. The most far-reaching unfolding of the idea as a theory itself unfolds the original concept, although it is founded on the latter. Hence, these ideas are the goal of theory development. These ideas are, however, at the same time its beginning and its base. This means they have to be intuitively impressive and must motivate activity. Fundamental concepts or basic ideas therefore are self-referential, that is, they themselves organize the process of their own deployment and articulation. The continuity of mind, which Kant wanted to secure by reflection of his "I think", is established in Peirce's transformation of Kant's philosophy by the endless sign process and therefore, there

> *is but one law of mind, namely, that ideas tend to spread continuously and to affect certain others which stand to them in a peculiar relation of affectability. In this spreading they lose intensity, and especially the power of affecting others, but gain generality and become welded with other ideas. We are accustomed to speak of ideas as reproduced, as passed from mind to mind, as similar or dissimilar to one another, and, in short, as if they were substantial things ; nor can any reasonable objection be raised to such expressions. But taking the word "idea" in the sense of an event in an individual consciousness, it is clear that an idea once past is gone forever, and any supposed recurrence of it is another idea.* (Peirce, 1892, p. 534)

Peirce, differently from Kant, considers the continuum of ideas as something objective, as a continuous interpretation, that is, as a sign process, rather than as an event in the subject's mind. The semiotic theory of Peirce

> *Is an attempt to explain the cognitive process of acquiring scientific knowledge as a pattern of communicative activity in which the dialogic partners are, indifferently, members of a community or sequential states of a single person's mind.* (Parmentier, 1994, p. 3)

Is There More to Mathematics than the Capacity to Reason Formally?

The distinction between calculation and proof is useful in order illustrate the issue insofar as there is a difference between following the course of an argument on the one side, and understanding it, on the other.

Suppose I have found a proof for some mathematical theorem, which after having checked out the argument of the proof step by step is now intuitively completely clear to me.

> *Suppose that a great authority announces that there is something wrong with the argument. In that case my experience upon checking over the argument may be quite different from what it was before this announcement was made.*

> *Just as before, I find that the argument appears to be correct; only this time I do not accept it as being correct.* (Stolzenberg, 1984, p. 263)

This remains true even if I cannot find fault with my argument. The distinction between being correct and merely appearing to be correct is exactly the same as that between seeing something and merely following a rule or a chain of arguments, or between arguing logically and formally or having some concrete application in mind.

As Lewis Carroll had shown in his little piece on Achilles and the Tortoise (Carroll, 1895), logic can never force on us the acceptance of anything. And nobody can be forced to perceive something he does not want to see. Perceptual judgments, which seem the basis of all our knowledge, are of the form: "A perceives B as a case of C". This implies that genuine proofs must—at least in the more complicated cases—involve generalization and thus employ what Peirce has been called abductive inferences.

Nevertheless, the difference between an ideal vs. a real state of things, or between the mind and the world, seems only relative. This relativity may, however, lead to the desire that objective reality should be completely intelligible, even though not everything in the world can have a meaning. For example, the well-known Gestalt psychologist Max Wertheimer (1880-1943) comments on the presentation and solution of Zeno's paradoxes by means of a geometric series, resp. the convergence of that series, which is accomplished by multiplying the series by *a* and subtracting that is current in present days mathematics. Wertheimer says that

> *it is correctly derived, proved, and elegant in its brevity. A way to get real insight into the matter, sensibly to derive the formula is not nearly so easy; it involves difficult steps and many more. While compelled to agree to the correctness of the above proceeding, there are many who feel dissatisfied, tricked. The multiplication of* $S = (1 + a + a^2 + a^3 + a^4 + ...)$ *by a together with the subtraction of one series from the other, gives the result; it does not give understanding of how the continuing series approaches this 'value in its growth. Real understanding proceeds by considering what happens in the growth of the series and derives the law of this growth, leading to the limit. Many do not bother really to understand. They are satisfied to have the result.* (Wertheimer, 1945, p. 90)

As an appendix to his book Wertheimer presents an alternative answer. The essential characteristic of it consists in his relying on the meaning of some relevant concepts (fraction etc.). "If I want to understand", he says, "I must realize from the beginning what the first term 1/a means as a part of its whole." (p. 218)

Wertheimer's solution is foundationalist, insofar as it reduces a problem concerning one concept A (series) to the meaning of another one B (fraction). Whereas a complementarist approach would stress the symmetrical aspects of conceivable relations between A and B. We can for instance, interpreting a periodical decimal fraction

(B) as a geometric series (A), directly prove that these decimal fractions represent rational numbers.

A representation, like a mathematical formula, may appear to be an impoverishment of the wealth and force of inner experience and intuition. On the other hand, no sign can be exhausted by individual experience and intuition. Any symbolization implies, in fact a generalization, as a symbol has meaning, which a thing or a feeling has not, and meanings are socially shared and are thus universals, after all (Durkheim, 1995, p. 433). Meanings or concepts serve the knowledge process and the development of discourse.

Zeno's paradoxes seem to present another defeat of the Pythagorean dream about the mathematical nature of the world. A logic of relations—taking relations as external and prior to the relata—is required to deal with these paradoxes. External relations lead to structural holism, as in modern axiomatics. Leibniz is considered to have been first recognizing the relational character of mathematical cognition, but he considers internal relations only, as exemplified by the relation between a father and his child.

If we accept that Achilles must first reach all the points that the tortoise has already reached, what we are actually saying is, that Achilles can reach only these points, that these are that determine his position. We quasi encapsulate Achilles' movement within that of the tortoise; we chain it to the latter. Yet who or what prevents Achilles from running two or three stadiums? We have to symmetrize our perspective, by adopting a relational point of view. Precisely speaking, the task is as follows: Achilles runs ten times as fast as the tortoise, though the tortoise has a one-stadium start. For each of the stages, $x(x>0)$, covered by Achilles, the tortoise has crawled the distance $f(x)=\frac{1}{10}x+1$ stadium.

The relative movement of Achilles and the tortoise is a linear function, as both movements are uniform: $f(x)=ax+b$ (i.e., when Achilles reaches x, the tortoise is at $f(x)$. The problem: At what point does Achilles really catch up with the tortoise? is now: What is the fixed point of $f(x)$? We seemingly have solved the problem by taking a relational point of view, that means, by adopting a "world view" which provides objects and relations between objects with an equal ontological status. This essentially makes up for what has been called a transition from thinking in terms concrete objects to relational thinking, or thinking in terms of ideal objects.

The example of Zeno is about constant movement. If we want to treat the more general case of accelerated motion we must generalize the mathematical concepts of function and number. We have also mentioned this in section What is a theory? So to mathematize means to generalize.

THEORIES ARE REALITIES *SUI GENERIS*

One must not only have an idea, but has to find a concrete form of the same, to render the idea effective and vice versa. In 1928 Max Dehn (1878-1952), one of Hilbert's most prominent students, gave a talk at the University of Frankfurt on the "Mentality of the Mathematician". He said that

> *Descartes himself believed that, through an illumination, he had discovered a new science…. But this was hardly the case. His great contribution was not the discovery of a completely new idea—that of the unity of algebra and geometry. It is even incorrect to say that he realized the existing idea in a manner more daring than his predecessors…. The historical significance of his contribution lies, above all, in formulation…. The origin of ideas is often unclear, the roots reaching far back into time cannot be unravelled. But the form is always the property of one person, that which is truly individual, which happens but once.* (Dehn, 1983, p. 18)

Theories and works of art are constructed forms, they are realities in their own right. A work of art is just a work of art; a theory is just a theory. It must be grasped as a form sui generis, as a hopefully more intelligible world in itself, rather than as a pale and passive reflection of the empirically given world, before we can inquire into its possible meanings or applications. In a diagram, like in a theory or a work of art, synthesis of representation is realized in the construction and transformation of the representation, that is, in the process of generalization. To generalize in this way one must see something and to see means to construct a representation and to show it in a new way.

The comparison with artistic or musical activity is particularly illuminating, because it shows the importance of the means of representation. In artistic drawing what we achieve is a line, and the line does all the work, and if it fails to do so, no philosophical commentary will rescue or repair a bad work of art. In literature or philosophy, it is the word or the sentence, in mathematics the formula or the diagram, which carry the entire weight, etc. Mastery, Paul Valéry (1998), says that

> *presupposes that ne has the habit of thinking and combining directly from the means of imagining a work only within the limits of the means at hand, and never approaching a work from a topic or an imagined effect that is not linked to the means.* (p. 40)

And the great music director David Barenboim said that

> *Music does not work with spiritual means, although it aims at the spiritual possibilities of man. Music works with sound. Sound is not spiritual, it is a very simple physical means. Sound is related to silence. You have to know as a musician, how to start a note, how long to keep it, how is the transition to the next note, etc.... I have studied composition with Nadia Boulanger in Paris. And she said the sentence: "You have to fill the structure of music with emotions and then analyze the emotions"* (interview in: Der Spiegel, 12/2014, our translation).

Therefore, if somebody, like Wertheimer, tries to grasp the meaning of anything, he has first of all to appreciate how it is represented. This may be termed "poetic imagination" and it is as important to the aesthetic world view as it is essential to the logic of science and mathematics. The essence of something is the essence of a representation of that thing and the essence of a representation is nothing but another representation. Therefore, we might say that the essential meaning or essence or idea is but a continuum of possible representations. There is never a literal application of a theory nor a direct reference of a painting to reality. Having interpreted a theory or an artistic product might change transitorily or permanently our way of seeing the world and of our acting within it. Thus texts and theories are means and objects at the same time. Means and objects are fully differentiable by their respective moments on individual cognitive activity, but they play a completely symmetric part in the development of cognition.

Peirce writes that

> *the work of the poet or novelist is not so utterly different from that of the scientific man. The artist introduces a fiction; but it is not an arbitrary one; it exhibits affinities to which the mind accords a certain approval in pronouncing them beautiful, which if it is not exactly the same as saying that the synthesis is true, is something of the same general kind. The geometer draws a diagram, which if not exactly a fiction, is at least a creation, and by means of observation of that diagram he is able to synthesize and show relations between elements which before seemed to have no necessary connection* (Peirce, 1931-1935, Vol. 1, p. 383).

One might even claim that the "new" algebra of the 16th/17th centuries has been brought about to a large degree by the opportunities offered by the writing system and, even more, by the printing press (Eisenstein, 2005). Oral language is an analog system; written language speaks to the eye. It allows a meta-perspective and is thus are capable of stimulating reflection. And this is an important point.

> *Human beings are the only animals to create visual art. At some moment in the narrative evolution human societies began to draw and paint things and it is safe to say that the act of picture making is only possible because we have the faculty to reflective self-consciousness, that is, we are able to represent ourselves to ourselves and muse about our own beings by becoming objects in our own eyes. This ability is distinct from what has been called pre-reflective self-consciousness, that immersion in everyday experience that we do not have to reflect upon to perceive…* (Hustvedt, 2012, p. 342)

To be able to reflect on the form of some thought, a new representation and a change of perspective is required. The proliferation of texts, made possible by the printing press, extended not only the knowledge but the whole knowledge process became more objective, dynamic and profound. One could compare identical prints, detect contradictions, add supplements, notice and record correlations. Different readers could talk about a precisely quotable argument. Errors could be eliminated and enhancements of knowledge became possible. Copernicus and Tycho Brahe are two centrally important examples of the new possibilities and the new style.

> *Tycho Brahe did have at his disposal, as few had before him, two separate sets of computations based on two different theories, compiled several centuries apart which he could compare with each other. The study of records was no less important for Tycho than it had been for the astronomers of the past. ... His observatory unlike theirs, included a library well stocked with printed materials as well as assistants trained in the new art of printing and engraving. For he took care to install printing presses and a paper mill on the island of Hveen.* (Eisenstein, 2005, p. 244)

CONCLUSION

The opposition between foundationalism and formalism has always dominated the epistemological discussions of mathematics, as well as, of mathematics education. The controversy about the foundations of mathematics between the movement of arithmetization and the axiomatic movement, or Wertheimer's approach to Zeno's paradoxes of motion, are examples that have been discussed in the paper.

As the symbol is categorically different from the symbolized, as theory is to be distinguished from reality and not one of these two is to be subordinated or to be derived from the other, an evolutionary perspective on knowledge is imperative, because otherwise not one of the fundamental questions in the epistemology of mathematics can fruitfully be discussed. So the processes of generalization and application of mathematics are essential to understand what mathematics is and how it works in the context of cultural history or individual cognitive development.

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Michael F. Otte
University of Bielefeld
Universidade Bandeirantes de São Paulo
michaelontra@aol.com

Tânia M. Mendonça
Universidade Bandeirantes de São Paulo
taniammcampos@hotmail.com

Luiz de Barros
Universidade Bandeirantes de São Paulo
lgxbarros@hotmail.com