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A POSITIVITY CRITERION FOR THE WAVE EQUATION AND GLOBAL EXISTENCE OF LARGE SOLUTIONS

MARIUS BECEANU AND AVY SOFFER

ABSTRACT. In dimensions one to three, the fundamental solution to the free wave equation is positive. Therefore, there exists a simple positivity criterion for solutions. We use this to obtain large global solutions to two well-studied energy-supercritical semilinear wave equations, as well as some new results in the subcritical and critical cases.

1. INTRODUCTION

In this paper, we present large data global existence results for two energy-supercritical equations on \mathbb{R}^{3+1} . Both results are based on a simple positivity criterion for solutions to the free wave equation on \mathbb{R}^{3+1} .

Similar results hold in dimensions one and two, but for simplicity we focus on the three-dimensional case. For the same reason, we confine our study to smooth initial data and, for the most part, to classical solutions.

1.1. Quadratic nonlinearity. The first equation we study is the quadratic semilinear wave equation on \mathbb{R}^{3+1} satisfying the null condition

$$u_{tt} - \Delta u = u_t^2 - |\nabla u|^2, \quad u(0) = u_0, \quad u_t(0) = u_1. \quad (1.1)$$

Equation (1.1) is $\dot{H}^{3/2} \times \dot{H}^{1/2}$ -critical, i.e. L^∞ -critical. The nonlinearity is quadratic in $\nabla_{t,x} u$. Two is the Strauss exponent, meaning that there exist quadratic nonlinearities in $\nabla_{t,x} u$ (e.g. u_t^2) that lead to blow-up for arbitrarily small initial data, but all higher order nonlinearities produce global solutions for sufficiently small initial data.

However, equation (1.1) has a better than expected behavior, because it satisfies the null condition (see [Kla2] or [Chr]; in fact, this equation is the canonical example of an equation that satisfies the null condition). Therefore, small initial data lead to global solutions.

For large data, global solutions to wave equations satisfying the null condition have been constructed by e.g. [WaYu], [Yan], [MPY], [LOY]. The paper [Li] proves global well-posedness for arbitrary large initial data for a supercritical wave equation, using its specific structure (as we also do below, but for another equation and using a different structure).

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The most important fact about equation (1.1) is that by the substitution $v = e^{-u}$ it reduces to a linear wave equation on \mathbb{R}^{3+1} , that is $v_{tt} - \Delta v = 0$. This transformation was suggested by Nirenberg and was first used in [Kla1].

Consequently, this equation has an infinite number of conserved quantities, i.e. $\|(e^{-u} - 1, e^{-u}u_t)\|_{\dot{H}^k \times \dot{H}^{k-1}}$. However, these quantities are not coercive, so cannot be used to always prove the global existence of solutions.

For example, the conserved energy has the form

$$E[u](t) = \int_{\mathbb{R}^3 \times \{t\}} e^{-2u} (u_t^2 + |\nabla u|^2) dx.$$

If $u \rightarrow +\infty$ then energy no longer controls the $\dot{H}^1 \times L^2$ norm of the solution. The same applies to the higher order conserved quantities.

Conserved quantities do not preclude the finite time blow-up of solutions to (1.1). In fact, there can be ODE-type blow-up for sufficiently large initial data.

Although relatively simple to study, due to the presence of infinitely many conserved quantities, equations (1.1) and (1.6) can serve as a model for more complicated semilinear equations, telling us what to expect in those cases.

In this paper we state sufficient (and in some cases necessary) conditions for the existence of global solutions to (1.1). These are classical solutions (i.e. they satisfy the equation pointwise). We need not assume finite energy (though the energy is always locally finite), but there is also a result about finite energy solutions, Proposition 1.2.

The first result refers to radially symmetric solutions.

Proposition 1.1. *Consider smooth radial initial data (u_0, u_1) such that*

$$(u_0)_r + |u_1| < \frac{1}{r}. \quad (1.2)$$

Then the equation (1.1)

$$u_{tt} - \Delta u = u_t^2 - (\nabla u)^2, \quad u(0) = u_0, \quad u_t(0) = u_1$$

admits a global smooth solution u on \mathbb{R}^{3+1} such that for every $(r, t) \in \mathbb{R}^{3+1}$

$$u_r(r, t) + |u_t(r, t)| < \frac{1}{r}.$$

Furthermore, if $u_0 \in L^\infty$, then $r(u_0)_r, ru_1 \in L^\infty$ and $(u_0)_r + |u_1| \leq \frac{1-\epsilon}{r}$ for some $\epsilon > 0$ if and only if $u \in L_{t,x}^\infty$. In this case,

$$\inf u_0 - \ln(1 + \|r(u_0)_r\|_{L^\infty} + \|ru_1\|_{L^\infty}) \leq u \leq \sup u_0 + \ln(1/\epsilon). \quad (1.3)$$

Also, let $c_0 := e^{\sup u_0 - \inf u_0} \epsilon^{-1} (1 + \|r(u_0)_r\|_{L^\infty} + \|ru_1\|_{L^\infty})$. Then $ru_r, ru_t \in L_{t,x}^\infty$ and

$$u_r(r, t) + |u_t(r, t)| \leq \frac{1}{r} \left(1 - \frac{1}{c_0}\right), \quad |u_r(r, t)| + |u_t(r, t)| \leq \frac{c_0 - 1}{r}. \quad (1.4)$$

Our conditions are true for all small initial data (in a suitable sense) and also allow for a wide class of large initial data, e.g. by taking $(u_0)_r$ to be large and negative.

One can prove along the same lines that the solutions depend continuously on the initial data. Moreover, under reasonable assumptions the solutions have finite energy and disperse:

Proposition 1.2. *Consider equation (1.1) with smooth radial initial data (u_0, u_1) . If $u_0 \in L^\infty$, $(u_0, u_1) \in \dot{H}^1 \times L^2$, and $(u_0)_r + |u_1| \leq \frac{1-\epsilon}{r}$ for some $\epsilon > 0$, then $u \in L_t^\infty \dot{H}_x^1 \cap L_t^2 L_x^\infty$, $u_t \in L_t^\infty L_x^2$, and*

$$\begin{aligned} \sup_t \|(u(t), u_t(t))\|_{\dot{H}^1 \times L^2} &\leq e^{\sup u_0 - \inf u_0} \epsilon^{-1} \|(u_0, u_1)\|_{\dot{H}^1 \times L^2}, \\ \|u\|_{L_t^2 L_x^\infty} &\lesssim \max(1, e^{\sup u_0} \epsilon^{-1}) e^{-\inf u_0} \|(u_0, u_1)\|_{\dot{H}^1 \times L^2}. \end{aligned} \quad (1.5)$$

One can also obtain control of higher norms.

Condition (1.2) is optimal, meaning that it is necessary for global existence of solutions:

Proposition 1.3. *Consider smooth radial initial data (u_0, u_1) such that for some $r_0 > 0$ $(u_0)_r(r_0) + |u_1(r_0)| \geq \frac{1}{r_0}$. Then the corresponding solution to equation (1.1) on \mathbb{R}^{3+1} blows up in finite time, at a time t_0 with $|t_0| \leq r_0$.*

More precisely, there exist t_0 with $|t_0| \leq r_0$ and $x_0 \in \mathbb{R}^3$ such that for t close to t_0

$$\|u(t)\|_{L_x^\infty(|x-x_0| \leq 1)} \geq C + |\ln |t - t_0||, \quad \|u(t)\|_{H_x^{3/2}(|x-x_0| \leq 1)} \geq C |\ln |t - t_0||^{1/2}.$$

We prove blow-up in the (critical for the equation) L_{loc}^∞ and $H_{loc}^{3/2}$ senses, meaning that the solution becomes unbounded near a point. This also means it cannot be continued as a classical solution.

Nevertheless, the proof suggests that at least in some cases it may be possible to continue the solution past the blow-up point. This and more aspects of blow-up will be explored elsewhere.

The same ideas used in the study of equation (1.1) apply to the more general semilinear equation (also with a nonlinearity that satisfies the null condition)

$$u_{tt} - \Delta u = f(u)(u_t^2 - |\nabla u|^2), \quad u(0) = u_0, \quad u_t(0) = u_1. \quad (1.6)$$

For simplicity we assume that $f(u)$ is a smooth function, e.g. $f(u) = 1$, $f(u) = -u$, $f(u) = \sin u$, or $f(u) = -\arctan u$.

We introduce the auxiliary function

$$F(t) = \int_0^t e^{-\int_0^s f(\sigma) d\sigma} ds. \quad (1.7)$$

Note that F is strictly increasing and therefore injective.

The subsequent discussion has to take into account whether $F(\pm\infty)$ is finite or infinite. For example, if $f(u) = 1$ then $F(-\infty) = -\infty$, but $F(+\infty)$ is finite; if $f(u) = -1$ then the situation is reversed; if $f(u) = u$ then $F(\pm\infty)$

are finite; if $f(u) = -u$ or $f(u) = -\arctan u$ then $F(\pm\infty) = \pm\infty$. These four are all the possible cases.

In the radial case we have the following necessary and sufficient result:

Proposition 1.4. *Consider smooth radial (u_0, u_1) . If $F(\pm\infty) = \pm\infty$, then equation (1.6) admits a corresponding smooth solution u on \mathbb{R}^{3+1} . If in addition $u_0 \in L^\infty$, then $r(u_0)_r, ru_1 \in L^\infty$ if and only if $u \in L_{t,x}^\infty$. In this situation, $ru_r, ru_t \in L_{t,x}^\infty$ as well.*

If $F(-\infty) = a \in \mathbb{R}$, suppose that

$$-(u_0)_r + |u_1| < \frac{F \circ u_0 - a}{rF' \circ u_0} = \frac{\int_{-\infty}^{u_0} e^{\int_s^{u_0} f(\sigma) d\sigma} ds}{r}. \quad (1.8)$$

If $F(+\infty) = b \in \mathbb{R}$, suppose that

$$(u_0)_r + |u_1| < \frac{b - F \circ u_0}{rF' \circ u_0} = \frac{\int_{u_0}^{\infty} e^{\int_s^{u_0} f(\sigma) d\sigma} ds}{r}. \quad (1.9)$$

Then equation (1.6) admits a corresponding smooth solution u such that

$$-u_r + |u_t| < \frac{\int_{-\infty}^u e^{\int_s^u f(\sigma) d\sigma} ds}{r} \text{ and/or } u_r + |u_t| < \frac{\int_u^{\infty} e^{\int_s^u f(\sigma) d\sigma} ds}{r}.$$

If $u_0 \in L^\infty$, then $r(u_0)_r, ru_1 \in L^\infty$ and

$$-(u_0)_r + |u_1| \leq \frac{\int_{-\infty}^{u_0} e^{\int_s^{u_0} f(\sigma) d\sigma} ds - \epsilon}{r} \text{ and/or } (u_0)_r + |u_1| \leq \frac{\int_{u_0}^{\infty} e^{\int_s^{u_0} f(\sigma) d\sigma} ds - \epsilon}{r} \quad (1.10)$$

for some $\epsilon > 0$ if and only if $u \in L_{t,x}^\infty$. In this case, one also has that $ru_r, ru_t \in L_{x,t}^\infty$ and a condition similar to (1.10) holds for all times $t \in \mathbb{R}$.

This is a large data result that generalizes Proposition 1.1. All the conclusions can be made quantitative. Under similar conditions we can also obtain finite energy and dispersive solutions and control of higher norms.

Conditions (1.8) and (1.9) are optimal, since their failure leads to finite time blow-up, as in Proposition 1.3. We omit the very similar proof.

In the nonradial case, we obtain the following:

Proposition 1.5. *Consider smooth initial data (u_0, u_1) . If $F(\pm\infty) = \pm\infty$, then equation (1.6)*

$$u_{tt} - \Delta u = f(u)(u_t^2 - |\nabla u|^2), \quad u(0) = u_0, \quad u_t(0) = u_1$$

admits a corresponding smooth solution on \mathbb{R}^{3+1} . If in addition $D^2 u_0, \nabla u_1 \in L^{3/2,1}$, then $u \in L_{t,x}^\infty$.

If $F(-\infty) = a \in \mathbb{R}$, but $F(+\infty) = +\infty$, suppose that $\inf u_0 > -\infty$ and $u_1 \geq |\nabla u_0|$. Then equation (1.6) admits a corresponding smooth solution u on $\mathbb{R}^3 \times [0, \infty)$ with $u \geq \inf u_0$.

Alternatively, suppose that (u_0, u_1) decay at infinity together with their derivatives and

$$-\Delta u_0 + f(u_0)(\nabla u_0)^2 \geq |\nabla u_1 - f(u_0)u_1 \nabla u_0|. \quad (1.11)$$

Then equation (1.6) admits a corresponding smooth solution u on \mathbb{R}^{3+1} with $u \geq 0$. If in addition $D^2u_0, \nabla u_1 \in L^{3/2,1}$, then $u \in L_{t,x}^\infty$.

Similar results apply to the case when $F(-\infty) = -\infty$, but $F(+\infty) = b \in \mathbb{R}$.

In particular, equation (1.1) also falls under the hypotheses of Proposition 1.5.

Here $L^{3/2,1}$ is a Lorentz space; for their definition and properties see [BeLö]. In terms of the more familiar Lebesgue spaces, one has that $L^{3/2-\epsilon} \cap L^{3/2+\epsilon} \subset L^{3/2,1} \subset L^{3/2}$.

One can easily show (using the substitution $v = F(u)$) that the condition (1.11) allows for large initial data. This is obvious for the other condition.

Under similar conditions we can also obtain finite energy and dispersive solutions and control of higher norms.

In the nonradial case there is no expectation that our conditions are optimal. Nevertheless, we can obtain a more general result. A solution to (1.6) can be continued as long as $F(u) > a$ and/or $F(u) < b$ (and indefinitely if $F(\pm\infty) = \pm\infty$). As soon as $F(u) = a$ or $F(u) = b$, one has blow-up in the L_{loc}^∞ and $H_{loc}^{3/2}$ sense, as in Proposition 1.3.

We next prove that all sufficiently nice solutions to equation (1.6) either disperse or blow up in finite time; solitons, infinite time blow-up, or other more complicated types of behavior are excluded. This can be construed as a version of the soliton resolution conjecture for this problem, except there are no solitons. More generally, such a result is called asymptotic completeness.

Proposition 1.6. *For Schwartz-class initial data (u_0, u_1) , equation (1.6)*

$$u_{tt} - \Delta u = f(u)(u_t^2 - |\nabla u|^2), \quad u(0) = u_0, \quad u_t(0) = u_1$$

always admits a smooth solution u that either blows up in finite time (in L_{loc}^∞ and $H_{loc}^{3/2}$, see Proposition 1.3) or is globally defined on \mathbb{R}^{3+1} . In the latter case $u(t)$ is a Schwartz-class function for each $t \in \mathbb{R}$, $\sup_t \|u(t)\|_{H^n} < \infty$ for each n , u and all its derivatives disperse (e.g. $\|D^n u(t)\|_{L^\infty} \lesssim_n |t|^{-1}$), and u scatters (behaves like a solution of the free wave equation as $t \rightarrow \pm\infty$).

In the radial case we have a more precise classification, with necessary and sufficient conditions for global existence (see Proposition 1.4).

The assumption that (u_0, u_1) are of Schwartz class is not needed; we only need control of finitely many seminorms of the initial data. The question of optimal norms is an interesting one, but will be examined elsewhere.

Remark 1.7. The previous results remain mostly valid (with the same proof) if we add to the equation (1.1) a smooth source term $G \leq 0$, i.e.

$$u_{tt} - \Delta u = u_t^2 - |\nabla u|^2 + G.$$

For the more general equation (1.6) one can instead add a magnetic potential term of the form $A_1 u_t + A_2 \cdot \nabla u$.

Remark 1.8. A similar finite time blow-up/dispersion dichotomy is true for the energy-supercritical Schrödinger equation

$$iu_t - \Delta u = (\nabla u)^2, \quad u(0) = u_0.$$

This is also a $\dot{H}^{3/2}$ -critical equation and one can also use a Nirenberg transformation, $e^u = v$, to solve it. Note that the evolution is not unitary.

Finally, although the main topic of this paper is the existence of global solutions, we also briefly look into the question of local existence. We state a result that holds in the general case; undoubtedly, there can be improvements for radial solutions.

Proposition 1.9. *Consider equation (1.6) with smooth initial data (u_0, u_1) and assume that $F(-\infty) = a \in \mathbb{R}$, but $F(+\infty) = +\infty$. If $u_0, \nabla u_0, u_1 \in L^\infty$, then there exists a corresponding smooth solution u on $\mathbb{R}^3 \times (-T, T)$, where*

$$T = \frac{F(\inf u_0) - a}{(\|\nabla u_0\|_{L^\infty} + \|u_1\|_{L^\infty}) \sup F'(u_0)} \quad (1.12)$$

Furthermore, $u \in L_{t,x}^\infty(\mathbb{R}^3 \times [-t, t])$ for any $t < T$.

Similar results hold in the case when $F(+\infty) = b \in \mathbb{R}$.

Conversely, following the argument in the proof, one can easily construct examples of smooth initial data (u_0, u_1) with $u_0, \nabla u_0$, or $u_1 \notin L^\infty$, which lead to blow-up at time 0 (hence the failure of local existence). We omit the construction. For more such results concerning the lack of local existence, see [Lin].

1.2. Monomial nonlinearity. The other equation we study in this paper is the focusing semilinear wave equation on \mathbb{R}^{3+1} with a monomial nonlinearity

$$u_{tt} - \Delta u - |u|^N u = 0, \quad u(0) = u_0, \quad u_t(0) = u_1. \quad (1.13)$$

This equation is $\dot{H}^{1/2}$ -critical for $N = 2$, energy-critical for $N = 4$, and energy-supercritical for $N > 4$. In general, the equation is \dot{H}^{s_c} -critical, where $s_c = 3/2 - 2/N$.

An equivalent formulation is

$$u(t) = \cos(t\sqrt{-\Delta})u_0 + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}u_1 + \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}}|u(s)|^N u(s) ds. \quad (1.14)$$

For equation (1.13) we prove the existence of global solutions for a suitable class of large initial data. The distinguishing feature of our result is that the solutions can be infinite on open sets, so some of them may be said to blow up in finite time in the usual sense. However, they still solve the weak formulation of the equation — (1.14) — on the whole of \mathbb{R}^{3+1} .

As another application, we state a criterion for the global existence, dispersion, and scattering of solutions to equation (1.13) in the energy-critical

case $N = 4$. This criterion extends the well-known one of [KeMe1], see below.

We also obtain a more general criterion for the global existence, dispersion, and scattering that applies to the whole $\dot{H}^{1/2}$ -supercritical ($N > 3$) range.

The most important feature of equation (1.13) that we use is that the nonlinearity preserves sign, meaning that if the solution is positive, then the nonlinearity seen as a source term is also positive. Conceivably, the same methods could also work for other types of nonlinearities.

Some of our results also hold in the defocusing case, i.e. when the nonlinearity in (1.13) has the plus sign.

Equation (1.13) is well-studied. We only give a brief survey of the known results.

The global existence of solutions for small initial data was proved by, among others, [Pec], [LiSo], and [ShSt]. This equation can have ODE-type blow-up in finite time for $N > 0$, for sufficiently large initial data. Another approach to blow-up is due to [Lev].

For $N = 4$ (the energy-critical case) equation (1.13) also has soliton solutions $u(x, t) = W(x)$, where W solves the semilinear elliptic equation

$$-\Delta W = W^5. \quad (1.15)$$

We distinguish the ground state soliton $Q > 0$ given by the explicit formula

$$Q(x) = \frac{1}{\left(1 + \frac{|x|^2}{3}\right)^{1/2}}. \quad (1.16)$$

In fact, each soliton is part of an infinite family of solitons obtained by rescaling, but this will play no role in our proof.

Recently, [KeMe1] classified all solutions smaller than the ground state soliton for the energy-critical equation. Define the energy of a solution by

$$E[u] = \int_{\mathbb{R}^3 \times \{t\}} \frac{u_t^2 + |\nabla u|^2}{2} - \frac{u^6}{6} dx. \quad (1.17)$$

The result of [KeMe1] states that if $E[u] < E[Q]$ and $\|\nabla u_0\|_{L^2} < \|\nabla Q\|_{L^2}$ then the solution scatters, while if $E[u] < E[Q]$ and $\|\nabla u_0\|_{L^2} > \|\nabla Q\|_{L^2}$ then the solution blows up.

This classification was then extended, in a less precise manner, by [DKM1] to radial solutions of arbitrary size. These results also extend to the energy-supercritical case (where the lack of solitons makes the classification simpler), but only under the assumption that the solution stays bounded in the critical Sobolev norm; see [DMK2], [DoLa], [DuRo].

Another approach to the energy-critical equation (1.13) belongs to [KrSc1], [KST], [KNS1], [KNS2], [Bec], and [KNS3], which studied solutions in a neighborhood of the ground state soliton.

Results for the focusing supercritical equation (1.13) include [KrSc2], [BeSo1], and [LOY]. These papers construct particular classes of large global solutions.

Many more results are known in the defocusing case (plus sign of the non-linearity in (1.13)). For the energy-critical equation, global well-posedness was proved by [Str1] and [Gri]. In the energy-supercritical case, some results — [Tao1], related to an idea from [GSV], [Roy1], [Roy2], and [Str2] — refer to slightly supercritical equations, while others — [KeMe2], [KiVi1], [KiVi2], [Bul1], [Bul2], [Bul3] — are conditional results. The paper [BeSo2] uses a modified supercritical nonlinearity. Also see the recent result [Tao2], which shows blow-up for a defocusing supercritical wave equation.

The following is the statement of our main result in this context. For convenience, we assume that N is an even integer.

Proposition 1.10. *Assume that $N \geq 0$ is an even integer and (u_0, u_1) are smooth and*

- i. either radial and outgoing according to Definition 3.2 and $u_0 \geq 0$;*
- ii. or radial and $(u_0)_r + u_0/r \geq |u_1|$;*
- iii. or not necessarily radial and $u_0 \geq 0$, $u_1 \geq |\nabla u_0|$;*
- iv. or not necessarily radial functions that decay at infinity together with their derivatives, with $-\Delta u_0 \geq |\nabla u_1|$.*

Then there exists a global solution u (on $\mathbb{R}^3 \times [0, \infty)$) in cases i and iii and on \mathbb{R}^{3+1} in cases ii and iv) to equation (1.13), having (u_0, u_1) as initial data. Moreover, u is nonnegative or infinite.

In the first case if $u_0 \geq v_0 \geq 0$, in the second case if

$$(u_0)_r + u_0/r \pm u_1 \geq (v_0)_r + v_0/r \pm v_1 \geq 0,$$

in the third case if

$$u_0 \geq v_0, \quad u_1 - |\nabla u_0| \geq v_1 + |\nabla v_0| \geq v_1 - |\nabla v_0| \geq 0,$$

and in the fourth case if

$$-\Delta u_0 - |\nabla u_1| \geq -\Delta v_0 + |\nabla v_1| \geq -\Delta v_0 - |\nabla v_1| \geq 0,$$

then one has for the corresponding solutions that $u \geq v \geq 0$.

The last conclusion shows that (at least in some cases) solutions depend monotonically on the initial data. One can also compare two solutions of arbitrary sign in the same manner, provided they are both well-defined.

We emphasize again that, in general, these global solutions only exist in a weak sense and some of them may actually blow up in finite time in the usual sense.

Our approach to the study of equation (1.13) seems to be new. One can hope that this unified treatment will lead to a better understanding of the boundary between blow-up and global solutions.

Reverting to the usual notions of solution existence and blow-up, one has the following criterion (whose proof we omit):

Corollary 1.11. *Assume $N \geq 0$ and consider smooth initial data (u_0, u_1) and (v_0, v_1) , such that (u_0, u_1) give rise to a nonnegative, measurable solution u for equation (1.13) which is finite almost everywhere on $\mathbb{R}^3 \times I$. If*

- i. (u_0, u_1) and (v_0, v_1) are radial and outgoing and $u_0 \geq |v_0|$*
- ii. or (u_0, u_1) and (v_0, v_1) are radial and $(u_0)_r + u_0/r - |u_1| \geq |(v_0)_r + v_0/r| + |v_1|$*
- iii. or $u_1 - |\nabla u_0| \geq |v_1| + |\nabla v_0|$*
- iv. or $-\Delta u_0 - |\nabla u_1| \geq |\Delta v_0| + |\nabla v_1|$,*

then (v_0, v_1) also give rise to a solution v on $\mathbb{R}^3 \times I$ to equation (1.13) which is finite almost everywhere, with $|v| \leq u$.

Here we assume that $I \subset [0, \infty)$ in cases i and iii.

If u disperses, then so does v . If a Lebesgue or Strichartz norm of v blows up in finite time, then the same is true for u .

In order to take full advantage of this criterion, we need to know beforehand that a certain positive solution u exists globally or that a solution v blows up. The simplest choice is the ground state soliton Q defined by (1.16):

$$Q(x) = \frac{1}{\left(1 + \frac{|x|^2}{3}\right)^{1/2}}.$$

Note that both $(rQ)_r > 0$ and $-\Delta Q = Q^5 > 0$. A more refined choice for comparison is $(1 \pm \epsilon)Q$.

Corollary 1.12. *Assume $N = 4$ and consider smooth initial data (u_0, u_1) such that either (in the radial case) $|(u_0)_r + u_0/r| + |u_1| \leq Q_r + Q/r$ or (in the general case) $|\Delta u_0| + |\nabla u_1| \leq -\Delta Q = Q^5$. Then (u_0, u_1) give rise to a global solution u of equation (1.13) on \mathbb{R}^{3+1}*

$$u_{tt} - \Delta u - |u|^4 u = 0, \quad u(0) = u_0, \quad u_t(0) = u_1,$$

such that $|u(x, t)| \leq Q(x)$.

If in addition $|(u_0)_r + u_0/r| + |u_1| \leq (1 - \epsilon)(Q_r + Q/r)$ or $|\Delta u_0| + |\nabla u_1| \leq (1 - \epsilon)Q^5$, then u disperses ($\|u\|_{L_{t,x}^8} < \infty$) and scatters.

If on the contrary $(u_0)_r + u_0/r - |u_1| \geq (1 + \epsilon)(Q_r + Q/r)$ or $-\Delta u_0 - |\nabla u_1| \geq (1 + \epsilon)Q^5$, then the solution u blows up in finite time, in the sense that there exist finite $a < 0 < b$ such that $\|u\|_{L_{t,x}^8(\mathbb{R}^3 \times (a, 0))} = \|u\|_{L_{t,x}^8(\mathbb{R}^3 \times (0, b))} = +\infty$.

Remark 1.13. This criterion extends the result of [KeMe1]. Indeed, consider a solution u with initial data $(u_0 = 0, u_1 = Q_r + Q/r)$. Since the first component is zero, the energy of u , defined by (1.17), is given by

$$E[u] = \frac{1}{2} \int_{\mathbb{R}^3} u_1^2 dx = 2\pi \int_0^\infty \left(\frac{1}{(1 + r^2/3)^{3/2} r} \right)^2 r^2 dr = \frac{3\sqrt{3}\pi^2}{8}.$$

At the same time,

$$\frac{1}{2} \|\nabla Q\|_{L^2}^2 = \frac{1}{2} \int_{\mathbb{R}^3} Q_r^2 dx = 2\pi \int_0^\infty \left(\frac{r/3}{(1 + r^2/3)^{3/2}} \right)^2 r^2 dr = \frac{3\sqrt{3}\pi^2}{8}.$$

One can also see directly that

$$\int_{\mathbb{R}^3} (2Q_r Q/r + Q^2/r^2) dx = 4\pi \int_0^\infty 2Q_r Q r + Q^2 dr = 0,$$

so

$$\|\nabla Q\|_{L^2}^2 = \int_{\mathbb{R}^3} Q_r^2 dx = \int_{\mathbb{R}^3} (Q_r + Q/r)^2 dx = \|u_1\|_{L^2}^2.$$

Since $\|\nabla Q\|_{L^2}^2 = \|Q\|_{L^6}^6$, it follows that $E[Q] = \frac{1}{3}\|\nabla Q\|_{L^2}^2$, so $E[u] = \frac{3}{2}E[Q]$.

The result of [KeMe1] refers only to solutions u for which $E[u] \leq E[Q]$, so it does not apply to u . Neither does the result of [KNS2] apply, since it refers to solutions u for which $E[u] \leq E[Q] + \epsilon^2$.

On the other hand, Corollary 1.12 shows that u exists globally on \mathbb{R}^{3+1} . If we consider instead the solution u_ϵ with initial data $(u_0 = 0, u_1 = (1 - \epsilon)(Q_r + Q/r))$, then Corollary 1.12 tells us that u_ϵ exists globally and disperses.

It is also possible to construct a small perturbation u_0 of Q such that $0 \leq -\Delta u_0 \leq (1 - \epsilon)(-\Delta Q)$ or $-\Delta u_0 \geq (1 + \epsilon)(-\Delta Q)$ and $E[(u_0, 0)] > E[Q]$. In both cases, the behavior of the solution can be predicted using Corollary 1.12, but not using the results of [KeMe1]. The connection with [KNS2] remains to be studied.

Corollary 1.12 depends essentially on the ground state soliton Q , which is only guaranteed to exist in the energy-critical case $N = 4$, and on the results of [KeMe1].

However, more generally, for $N > 2$ there exists another family of positive stationary solutions to equation (1.13), which can be called singular solitons.

Namely, we look for positive radial solutions of the form $Q_N = C_N |x|^\alpha$ of the semilinear elliptic equation (1.15)

$$-\Delta Q_N = Q_N^{N+1}.$$

Plugging our ansatz in the equation, we obtain

$$-\alpha(\alpha + 1)C_N r^{\alpha-2} = C_N^{N+1} r^{\alpha(N+1)}.$$

Hence $\alpha = -\frac{2}{N}$ and $C_N = (-\alpha(\alpha + 1))^{1/N} = (2(N - 2)/N^2)^{1/N}$, so

$$Q_N(x) = \left(\frac{2(N - 2)}{N^2} \right)^{1/N} |x|^{-2/N}. \quad (1.18)$$

Q_N is scaling-invariant, so by rescaling it we do not obtain anything new, but we can get other solitons by translating it.

Note that Q_N never has finite energy. Q_N logarithmically fails to be in the critical Sobolev space for the equation \dot{H}^{s_c} , where $s_c = 3/2 - 2/N$.

$N > 2$ is the $\dot{H}^{1/2}$ -supercritical case, which includes the energy-critical and energy-supercritical cases. In particular, in the energy-critical case, $Q_4(x) = \frac{1}{\sqrt{2}}|x|^{-1/2}$. Comparing this singular soliton with the usual soliton Q defined by (1.16), we see that $Q_4(x) \leq Q(x)$ for $|x| \in [3 - \sqrt{6}, 3 + \sqrt{6}]$

and $Q_4(x) > Q(x)$ otherwise. Of course, this depends on the scaling of Q , but regardless of scaling we see that neither soliton dominates the other.

Using the singular soliton Q_N we can formulate a criterion for global existence of solutions to (1.13). We omit the proof. Note that $(rQ_N)_r > 0$ and $-\Delta Q_N = Q_N^{N+1} > 0$.

Corollary 1.14. *Assume $N > 2$ and consider smooth initial data (u_0, u_1) , which decay at infinity together with their derivatives, to the equation*

$$u_{tt} - \Delta u \pm |u|^N u = 0, \quad u(0) = u_0, \quad u_t(0) = u_1. \quad (1.19)$$

If $|(u_0)_r + u_0/r| + |u_1| \leq (Q_N)_r + Q_N/r$ (in the radial case) or $|\Delta u_0| + |\nabla u_1| \leq -\Delta Q_N = Q_N^{N+1}$ (in the general case), then the corresponding solution u exists globally on \mathbb{R}^{3+1} and $|u(x, t)| \leq Q_N(x)$.

Note that the initial data need not have finite energy or finite critical Sobolev $\dot{H}^{s_c} \times \dot{H}^{s_c-1}$ norm (where $s_c = 3/2 - 2/N$).

In one sense, this result does not preclude the finite time blow-up of solutions (in particular type II blow-up), since we have no control over the Strichartz norms. Nevertheless, in another sense the solution u exists globally and is finite almost everywhere on \mathbb{R}^{3+1} .

Both conditions are scaling-invariant. Note that the second condition can be expressed in terms of the weighted $|x|^{-2-2/N} \dot{W}^{2,\infty} \times |x|^{-2-2/N} \dot{W}^{1,\infty}$ norm of the initial data, i.e.

$$\||x|^{2+2/N} \Delta u_0\|_{L^\infty} + \||x|^{2+2/N} \nabla u_1\|_{L^\infty} \leq C_N^{N+1} = \left(\frac{2(N-2)}{N^2} \right)^{1+\frac{1}{N}}.$$

The conclusion can also be expressed in terms of a weighted $|x|^{-2/N} L^\infty$ norm, i.e. $\||x|^{2/N} u\|_{L_{t,x}^\infty} \leq C_N = (2(N-2)/N^2)^{1/N}$.

More generally, if $|\Delta u_0| + |\nabla u_1| \leq \alpha(-\Delta Q_N)$ with $\alpha \leq 1$, then $|u| \leq \alpha Q_N$, so the norm of the solution depends linearly on the size of the initial data.

Note that $C_N < 1$ and $C_N \rightarrow 1$ as $N \rightarrow \infty$.

Finally, it is easy to see that the solution is dominated by Q_N regardless of whether the equation is focusing or defocusing.

Remark 1.15. It is easy to see that we actually only need a supersolution to equation (1.15):

$$-\Delta u_0 \geq u_0^{N+1}, \quad u_0 \geq 0. \quad (1.20)$$

Then the solution with initial data $(u_0, 0)$ exists globally on \mathbb{R}^{3+1} , with $|u(x, t)| \leq u_0(x)$, and so does any solution v dominated by u , i.e. with initial data (v_0, v_1) such that $|\Delta v_0| + |\nabla v_1| \leq -\Delta v_0$.

Many results are known about the inequality (1.20). For example, it has no positive solutions for $N \leq 2$, see [BCDN].

Even though the previous result did not say anything about the asymptotic behavior of solutions, under slightly stronger conditions we can prove

that solutions do not blow up in finite time and in the defocusing case actually scatter. For some results it is convenient to assume that we are in the energy-supercritical range $N > 4$.

Theorem 1.16. *Consider $\alpha > 0$, $N > 2$, and smooth initial data (u_0, u_1) , which decay at infinity together with their derivatives, to the equation (1.19)*

$$u_{tt} - \Delta u \pm |u|^N u = 0, \quad u(0) = u_0, \quad u_t(0) = u_1.$$

Assuming that

$$|\Delta u_0(x)| + |\nabla u_1(x)| \leq \frac{C_N^{N+1}}{(\alpha + |x|)^{2+2/N}} = \left(\frac{2(N-2)}{N^2} \right)^{1+\frac{1}{N}} \frac{1}{(\alpha + |x|)^{2+2/N}}, \quad (1.21)$$

then the corresponding solution u exists globally on \mathbb{R}^{3+1} and $\|u\|_{L_{t,x}^\infty} \leq C_N \alpha^{-2/N}$ (in fact $|u(x, t)| \leq C_N(\alpha + |x|)^{-2/N}$).

Assuming that e.g. $(u_0, u_1) \in \dot{H}^{s_c} \times \dot{H}^{s_c-1}$ (where $s_c = 3/2 - 2/N$ is the critical Sobolev exponent), then the appropriate Strichartz norms of u do not blow up in finite time.

In the defocusing case (plus sign in the equation), assuming that $N > 4$ and $(u_0, u_1) \in (\dot{H}^{s_c} \cap \dot{H}^1) \times (\dot{H}^{s_c-1} \cap L^2)$, or in the focusing case (minus sign in the equation), assuming in addition that (u_0, u_1) are radially symmetric, then u disperses ($\|u\|_{L_{t,x}^{2N}} < \infty$) and scatters.

One can also write explicit estimates for all the norms, except in the radially symmetric focusing case.

For the energy-critical equation (1.19) with $N = 4$, the defocusing case is already well understood: all finite energy solutions disperse and scatter. In the focusing case, assuming the initial data are radially symmetric, condition (1.21) and the result of [DMK2] imply that the corresponding solutions disperse and scatter (since the singular soliton Q_4 does not dominate the usual soliton Q).

Condition (1.21) is optimal, since the singular soliton Q_N itself (which fulfills the condition for $\alpha = 0$) provides a counterexample: its $L_{t,x}^{2N}$ norm is infinite on any interval.

The $\dot{H}^{s_c} \times \dot{H}^{s_c-1}$ norm of the initial data can be arbitrarily high. The solutions constructed by Theorem 1.16 are the first examples of large global solutions to a supercritical equation which correspond to generic initial data, are bounded in the critical Sobolev norm, and scatter — i.e. the first “normal” large solutions.

The paper is organized as follows: in the introduction we state the main results, in Section 2 we define some notations, in Section 3 we state and prove the positivity criteria we use in the proof, and in Sections 4 and 5 we prove the results pertaining to equation (1.1), respectively (1.13).

2. NOTATIONS

$A \lesssim B$ means that $|A| \leq C|B|$ for some constant C . We denote various constants, not always the same, by C .

The Laplacian is the operator on \mathbb{R}^3 $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$.

We denote by L^p the Lebesgue spaces, by \dot{H}^s homogenous and by H^s inhomogenous Sobolev spaces, and by $L^{p,q}$ Lorentz spaces.

\dot{H}^s and H^s are Hilbert spaces and so is $\dot{H}^1 \times L^2$, under the norm

$$\|(u_0, u_1)\|_{\dot{H}^1 \times L^2} = (\|u_0\|_{\dot{H}^1}^2 + \|u_1\|_{L^2}^2)^{1/2}.$$

For a radially symmetric function $u(x)$, we let $u(r) := u(x)$ for $|x| = r$.

We define the mixed-norm Strichartz spaces on $\mathbb{R}^3 \times [0, \infty)$

$$L_t^p L_x^q := \left\{ f \mid \|f\|_{L_t^p L_x^q} := \left(\int_0^\infty \|f(x, t)\|_{L_x^q}^p dt \right)^{1/p} < \infty \right\},$$

with the standard modification for $p = \infty$, and likewise for the reversed mixed-norm spaces $L_x^q L_t^p$. We use a similar definition for $L_t^p \dot{W}_x^{s,p}$. Also, for $I \subset [0, \infty)$, let $\|f\|_{L_t^p L_x^q(\mathbb{R}^3 \times I)} := \|\chi_I(t)f\|_{L_t^p L_x^q}$, where χ_I is the characteristic function of I .

The global Kato space is defined as follows:

$$\mathcal{K} = \{u : \|u\|_{\mathcal{K}} := \sup_y \int_{\mathbb{R}^3} \frac{|u(x)|}{|x - y|} < \infty\}.$$

3. POSITIVITY CRITERIA FOR SOLUTIONS TO THE WAVE EQUATION

The positivity of the fundamental solution to the free wave equation in dimensions one to three has been known for a long time. In the context of semilinear wave equations, it was used by [Joh] to prove finite time blow-up below the Strauss exponent for small initial data.

We use positivity in a different manner, by deriving sufficient — and in some cases also necessary — positivity criteria for solutions to the wave equation (inspired by the techniques used in [BeSo1]). These are the basis of our main results.

Although not needed in this paper, we first look at the simplest case, that of the one-dimensional equation.

Lemma 3.1. *Consider smooth (u_0, u_1) and suppose that there exists an antiderivative $\partial_x^{-1}u_1$ of u_1 such that $u_0 \geq |\partial_x^{-1}u_1|$. Then the corresponding solution u of the one-dimensional free wave equation*

$$u_{tt} - u_{xx} = 0, \quad u(0) = u_0, \quad u_t(0) = u_1$$

is nonnegative on \mathbb{R}^{1+1} .

Note that, in order to apply this to compact support solutions, it is necessary that $\int_{\mathbb{R}} u_1 dx = 0$.

From the proof it follows immediately that, when u_0 and $\partial_x^{-1}u_1$ have compact support, the condition that $u_0 \geq |\partial_x^{-1}u_1|$ is also necessary for positivity.

Proof of Lemma 3.1. By the well-known d'Alembert formula, $u(x, t) = u_+(r - t) + u_+(r + t)$, where

$$u_{\pm}(x) = \frac{1}{2}(u_0(x) \mp \partial_x^{-1} u_1(x))$$

and $\partial_x^{-1} u_1$ denotes any antiderivative of u_1 . If u_{\pm} are nonnegative, then so is u . Hence it suffices that $u_0(x) \mp \partial_x^{-1} u_1(x) \geq 0$, which is our hypothesis. \square

Skipping the two-dimensional case for now, we next state a positivity criterion for radially symmetric solutions to the free wave equation on \mathbb{R}^{3+1} .

The proof is based on the following construction introduced in [BeSo1]: for radial $u(r) = u(|x|)$, define

$$T(u)(r) = (ru(r))', \quad u(r) = \frac{1}{r} \int_0^r T(u)(s) ds. \quad (3.1)$$

Then, u solves the free wave equation on \mathbb{R}^{3+1} if and only if $U = T(u)$ solves the free wave equation on $[0, \infty) \times \mathbb{R}$

$$U_{tt} - U_{rr} = 0, \quad U(0) = U_0 = T(u_0), \quad U_t(0) = U_1 = T(u_1), \quad (3.2)$$

with Neumann boundary conditions $U_r(0, t) = 0$. Equation (3.2) has solutions of the form (for $r \geq 0$)

$$U(r, t) = \chi_{r \geq t} U_+(r - t) + \chi_{r \leq t} U_-(t - r) + \chi_{r+t \geq 0} U_-(r + t) + \chi_{r+t \leq 0} U_+(-r - t),$$

where by d'Alembert's formula

$$U_{\pm}(r) = \frac{1}{2}(U_0(r) \mp \partial_r^{-1} U_1(r)). \quad (3.3)$$

If we take $(U(t_0), U_t(t_0))$ as initial data in equation (3.2), we obtain a time-translated solution \tilde{U} with

$$\tilde{U}_{\pm}(r) = \frac{1}{2}(U(r, t_0) \mp \partial_r^{-1} U_t(r, t_0)).$$

This decomposition is related to the original one by

$$\tilde{U}_-(r) = \chi_{r \geq 0} U_-(r + t_0), \quad \tilde{U}_+(r) = \chi_{r \geq t_0} U_+(r - t_0) + \chi_{0 \leq r \leq t_0} U_-(t_0 - r) \quad (3.4)$$

for $t_0 \geq 0$ and similar for $t_0 \leq 0$.

In [BeSo1], we also used this construction to define incoming and outgoing radial initial data.

Definition 3.2. We call a radially symmetric pair (u_0, u_1) *outgoing* if

$$(u_0)_r + \frac{u_0}{r} = u_1.$$

In this case, we showed that for $r \geq t \geq 0$

$$u(r, t) = \frac{r - t}{r} u_0(r - t) \quad (3.5)$$

and $u(r, t) \equiv 0$ for $0 \leq r \leq t$. It is easy to see that, for outgoing initial data, if $u_0 \geq 0$ then $u \geq 0$.

The three-dimensional criterion involves one more derivative than the one-dimensional one.

Lemma 3.3. *Suppose that (u_0, u_1) are smooth and radial. Then $(ru_0(r))_r > r|u_1|$ if and only if the solution u to the free wave equation*

$$u_{tt} - \Delta u = 0, \quad u(0) = u_0, \quad u_t(0) = u_1$$

is positive on \mathbb{R}^{3+1} . Moreover, in this case for any $t \in \mathbb{R}$ $(ru(r, t))_r > r|u_t(t, r)|$.

Conversely, if there exists $r_0 \geq 0$ such that $(ru_0)_r(r_0) \leq r_0|u_1(r_0)|$, then $u \leq 0$ somewhere in \mathbb{R}^{3+1} (more precisely, $u(0, t = r_0) \leq 0$ or $u(0, t = -r_0) \leq 0$).

In addition, $u \in L_{t,x}^\infty$ if and only if $(ru_0(r))_r, ru_1 \in L^\infty$ and in this case

$$\|(ru(r, t))_r\|_{L_{t,r}^\infty} + \|ru_t(t, r)\|_{L_{t,r}^\infty} \leq 2\|u\|_{L_{t,x}^\infty} \leq 2(\|(ru_0(r))_r\|_{L^\infty} + \|ru_1\|_{L^\infty}).$$

Our positivity condition implies that $(ru_0(r))' > 0$. Note that u_0 will decay no faster than $1/r$ and (of course) will be positive. In particular, this means that u_0 can have finite energy, but cannot have finite L^2 norm.

Remark 3.4. If we allow equality as well, then $(ru_0(r))_r \geq r|u_1|$ is equivalent to $u \geq 0$ on \mathbb{R}^{3+1} (and to $(ru(r, t))_r \geq r|u_t(t, r)|$ for any $t \in \mathbb{R}$).

Proof of Lemma 3.3. With T defined by (3.1), let $T(u) = U$. Then, by virtue of (3.1), in order to prove that $u > 0$ it suffices to prove that $U > 0$ and in turn this follows once we show that $U_\pm > 0$. However, by (3.3),

$$U_\pm(r) = \frac{1}{2}(U_0(r) \mp \partial_r^{-1}U_1(r)) = \frac{1}{2}((ru_0(r))_r \mp ru_1(r)) > 0. \quad (3.6)$$

By formula (3.4), this implies that $\tilde{U}_\pm > 0$, so, by a computation analogous to (3.6), we get that $(ru(r, t))_r \mp ru_t(t, r) > 0$.

For the converse statement, if $(ru_0)_r(r_0) \leq r_0|u_1(r_0)|$, then either $U_+(r_0) \leq 0$ or $U_-(r_0) \leq 0$. Both cases imply that $U(0, t) \leq 0$ for some t ($t = -r_0$ in the first case, $t = r_0$ in the second case). But by (3.1) $u(0, t) = U(0, t)$, so u is indeed nonpositive somewhere in \mathbb{R}^{3+1} .

The same reasoning applies to the $L_{t,x}^\infty$ norm: following (3.1) and (3.6),

$$\|u\|_{L_{t,x}^\infty} \leq \|U\|_{L_{t,r}^\infty} \leq \|U_-\|_{L^\infty} + \|U_+\|_{L^\infty} \leq \|(ru_0(r))_r\|_{L^\infty} + \|ru_1\|_{L^\infty}.$$

Conversely, again following (3.6),

$$\|(ru_0(r))_r\|_{L^\infty} + \|ru_1\|_{L^\infty} \leq 2 \max(\|U_-\|_{L^\infty}, \|U_+\|_{L^\infty}) \leq 2\|U(0, t)\|_{L_t^\infty} \leq 2\|u\|_{L_{t,x}^\infty}.$$

(Again we used the fact that $u(0, t) = U(0, t)$). Same is true for any other time $t \in \mathbb{R}$, so

$$\|(ru(r, t))_r\|_{L_{t,r}^\infty} + \|ru_t(t, r)\|_{L_{t,r}^\infty} \leq 2\|u\|_{L_{t,x}^\infty}.$$

□

In the proof we also use the following more refined condition for the boundedness of solutions.

Corollary 3.5. *Suppose u is a smooth radial solution to the free wave equation on \mathbb{R}^{3+1} :*

$$u_{tt} - \Delta u = 0, \quad u(0) = u_0, \quad u_t(0) = u_1.$$

Then $a \leq u \leq b$ on \mathbb{R}^{3+1} if and only if

$$(ru_0(r))_r - a \geq r|u_1|, \quad b - (ru_0(r))_r \geq r|u_1|,$$

in which case it is also true that for any $t \in \mathbb{R}$

$$(ru(r, t))_r - a \geq |u_t(r, t)|, \quad b - (ru(r, t))_r \geq |u_t(r, t)|.$$

These two conditions taken together imply that $a \leq (ru(r, t))_r \leq b$ and that $|u_t(r, t)| \leq \frac{b-a}{2r}$.

Corollary 3.5 is natural, in the sense that its hypotheses are true for all radially symmetric Schwartz class (u_0, u_1) and some finite $a, b \in \mathbb{R}$. However, it is not necessary to assume any decay at infinity.

Proof of Corollary 3.5. Apply Lemma 3.3 to $u - a$ and to $b - u$. \square

We next state two positivity and boundedness criteria that holds for non-radial solutions. These criteria are less sharp than Lemma 3.3, in the sense that they are sufficient, but not necessary. For one criterion we need to assume that the initial data (u_0, u_1) decay at infinity together with their derivatives. In addition, this criterion requires two derivatives instead of one.

In the statement of the boundedness criterion we also use the global Kato space, defined as follows:

$$\mathcal{K} = \{u : \|u\|_{\mathcal{K}} := \sup_y \int_{\mathbb{R}^3} \frac{|u(x)|}{|x-y|} < \infty\}.$$

Due to the pairing between $L^{3/2,1}$ and $L^{3,\infty}$ and to the fact that $\frac{1}{|x|} \in L^{3,\infty}$, it follows that $L^{3/2,1} \subset \mathcal{K}$.

Also note that, due to the boundedness of the Riesz transforms, $\Delta u \in L^{3/2,1}$ is equivalent to $D^2 u \in L^{3/2,1}$.

Lemma 3.6. *Consider a smooth solution u to the free wave equation on \mathbb{R}^{3+1} with initial data (u_0, u_1) . If $u_0 > 0$ and $u_1 \geq |\nabla u_0|$, then $u > 0$ on $\mathbb{R}^3 \times [0, \infty)$.*

Alternatively, assume that (u_0, u_1) decay at infinity together with their derivatives. If $-\Delta u_0 > |\nabla u_1|$, then $u > 0$ on \mathbb{R}^{3+1} . Also, if $\Delta u_0, \nabla u_1 \in \mathcal{K}$, then $u \in L_{t,x}^\infty$ and

$$\|u\|_{L_{t,x}^\infty} \leq \frac{1}{4\pi} (\|\Delta u_0\|_{\mathcal{K}} + \|\nabla u_1\|_{\mathcal{K}}). \quad (3.7)$$

The condition $-\Delta u_0 > |\nabla u_1| \geq 0$ automatically implies that $u_0 > 0$, since we can write

$$u_0(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{-\Delta u_0(y)}{|x-y|} dy. \quad (3.8)$$

For the same reason, if $-\Delta u_0 \in \mathcal{K}$, then $u_0 \in L^\infty$.

It is easy to generate pairs of initial data that satisfy our hypotheses by, for example, first choosing u_1 , then choosing $-\Delta u_0 > |\nabla u_1|$, and then retrieving u_0 by means of the formula (3.8).

If we assume instead that $u_0 \geq 0$ or that $-\Delta u_0 \geq |\nabla u_1|$ (we allow for equality), then we get that $u \geq 0$.

A sharper way of bounding u is to say that it is dominated by the positive solution with initial data $((-\Delta)^{-1}(|\Delta u_0| + |\nabla u_1|), 0)$. Also note that a solution u with initial data $(u_0, 0)$ and $-\Delta u_0 \geq 0$ has $u_t \leq 0$ for $t \geq 0$ and $u_t \geq 0$ for $t \leq 0$ (so it reaches its maximum at time 0).

By the method of descent (i.e. writing a solution of the two-dimensional wave equation as a solution of the three-dimensional wave equation that is constant in one variable) the same criteria also hold in the two-dimensional case.

A sharper positivity criterion can be written in terms of the Radon transform of the initial data, but it is nonlocal and harder to use in the proof.

Proof of Lemma 3.6. This follows immediately from the usual solution formula for the free wave equation in three dimensions. With no loss of generality take $x = 0$; then for $t > 0$

$$u(0, t) = \frac{1}{4\pi t^2} \int_{|y|=t} u_0(y) dy + \frac{1}{4\pi t} \int_{|y|=t} (u_0)_r(y) + u_1(y) dy. \quad (3.9)$$

Note that $|(u_0)_r| \leq |\nabla u_0|$.

A similar formula ensures the validity of the second criterion: for $t \geq 0$

$$u(0, t) = \frac{1}{4\pi} \int_{|y| \geq t} \frac{-\Delta u_0(y)}{|y|} dy + \frac{1}{4\pi t} \int_{|y|=t} u_1(y) dy \quad (3.10)$$

and

$$\frac{1}{4\pi t} \int_{|y|=t} u_1(y) dy = -\frac{t}{4\pi} \int_t^\infty \int_{S^2} (u_1)_r(r\omega) d\omega dr = -\frac{1}{4\pi} \int_{|y| \geq t} \frac{t(u_1)_r(y)}{|y|^2} dy. \quad (3.11)$$

Then note that $|(u_1)_r| \leq |\nabla u_1|$ and $\frac{t}{|y|^2} \leq \frac{1}{|y|}$.

The boundedness estimate (3.7) follows for example from formulas (3.10) and (3.11). Also note that an even more general statement was already proved in [BeGo]. \square

Finally, we also state a boundedness criterion that only holds locally in time, but requires fewer conditions.

Lemma 3.7. *Consider the free wave equation on \mathbb{R}^{3+1} with smooth initial data u_0 and u_1 , such that $u_0, \nabla u_0, u_1 \in L^\infty$. Then the corresponding solution u satisfies the bounds*

$$\begin{aligned} \sup_{\mathbb{R}^3 \times [-T, T]} u &\leq \sup u_0 + T(\|u_1\|_{L^\infty} + \|\nabla u_0\|_{L^\infty}) \\ \inf_{\mathbb{R}^3 \times [-T, T]} u &\geq \inf u_0 - T(\|u_1\|_{L^\infty} + \|\nabla u_0\|_{L^\infty}). \end{aligned}$$

Proof of Lemma 3.7. This immediately follows from (3.9). \square

4. LARGE SOLUTIONS TO EQUATION 1.1

Proof of Proposition 1.1. Consider a smooth solution u of equation (1.1). By setting $v = e^{-u}$, we obtain the free wave equation on \mathbb{R}^{3+1}

$$v_{tt} - \Delta v = 0, \quad v(0) = v_0 = e^{-u_0}, \quad v_t(0) = v_1 = -e^{-u_0} u_1$$

for v , where the initial data are also smooth.

The free wave equation always has a smooth solution v for smooth initial data. Then, the transformation can be reversed as long as $v > 0$, by setting $u = \ln v$. By Lemma 3.3, v is guaranteed to be positive as long as $(rv_0(r))_r > r|v_1|$. Expressing this in terms of u , we obtain exactly condition (1.2).

By the same lemma, in this case $(rv(r, t))_r > r|v_t(r, t)|$ or in other words $u_r(r, t) + |u_t(r, t)| < \frac{1}{r}$.

Next, $u \in L_{t,x}^\infty$ is equivalent to $v \in L_{t,x}^\infty$ and $\inf v > 0$. Applying Lemma 3.5, we see that it is necessary and sufficient that

$$\sup(rv_0(r))_r + r|v_1| < \infty, \quad \inf(rv_0(r))_r - r|v_1| > 0.$$

But

$$(rv_0(r))_r + r|v_1| = e^{-u_0}(1 - r(u_0)_r + r|u_1|) \leq e^{-\inf u_0}(1 + \|r(u_0)_r\|_{L^\infty} + \|ru_1\|_{L^\infty})$$

and

$$(rv_0(r))_r - r|v_1| = e^{-u_0}r\left(\frac{1}{r} - (u_0)_r - |u_1|\right) \geq e^{-\sup u_0}\epsilon.$$

Thus v is always between these two bounds: $A \leq v \leq B$, where $A = e^{-\sup u_0}\epsilon$ and $B = e^{-\inf u_0}(1 + \|r(u_0)_r\|_{L^\infty} + \|ru_1\|_{L^\infty})$. Taking the logarithm, we obtain exactly the inequality (1.3).

In this situation, Lemma 3.5 also implies that

$$A \leq (rv(r, t))_r - r|v_t(r, t)| \leq (rv(r, t))_r + r|v_t(r, t)| \leq B,$$

hence

$$2r|v_t(r, t)| \leq B - A.$$

Taking into account the fact that $A \leq v \leq B$, these statements imply

$$u_r(r, t) + |u_t(r, t)| \leq \frac{1}{r}\left(1 - \frac{A}{B}\right), \quad -u_r(r, t) + |u_t(r, t)| \leq \frac{1}{r}\left(\frac{B}{A} - 1\right),$$

respectively

$$|u_t(r, t)| \leq \frac{1}{2r}\left(\frac{B}{A} - 1\right).$$

Thus we retrieve the conclusions (1.4). \square

Proof of Proposition 1.2. Concerning energy and dispersion, assuming that $u_0 \in L^\infty$ and that $(u_0)_r + |u_1| \leq \frac{1-\epsilon}{r}$, then we have already proved above that $v \geq e^{-\sup u_0}\epsilon$, i.e. $u \leq \sup u_0 + \ln(1/\epsilon)$. Since $(u_0, u_1) \in \dot{H}^1 \times L^2$,

$$\|(v_0 - 1, v_1)\|_{\dot{H}^1 \times L^2} \leq e^{-\inf u_0}\|(u_0, u_1)\|_{\dot{H}^1 \times L^2}.$$

We need to subtract 1 because $\lim_{x \rightarrow \infty} v_0 = e^0 = 1$. As a radial solution of the free wave equation, $v - 1$ conserves energy,

$$\|(v(t) - 1, v_t(t))\|_{\dot{H}^1 \times L^2} = \|(v_0 - 1, v_1)\|_{\dot{H}^1 \times L^2},$$

and also satisfies the endpoint Strichartz estimate [KlMa]:

$$\|v - 1\|_{L_t^2 L_x^\infty} \lesssim \|(v_0 - 1, v_1)\|_{\dot{H}^1 \times L^2}.$$

In order to convert back to u , note that

$$\|(u(t), u_t(t))\|_{\dot{H}^1 \times L^2} \leq e^{\sup u} \|(v(t) - 1, v_t(t))\|_{\dot{H}^1 \times L^2}$$

and $|u| \leq |v - 1| \max(1, e^{\sup u})$. Putting together all these estimates we obtain exactly (1.5). \square

Proof of Proposition 1.3. Let $v = e^{-u}$. Then, as stated above, v satisfies the free wave equation

$$v_{tt} - \Delta v = 0, \quad v(0) = v_0 = e^{-u_0}, \quad v_t(0) = v_1 = -e^{-u_0} u_1.$$

Our hypothesis $(u_0)_r(r_0) + |u_1(r_0)| \geq \frac{1}{r_0}$ implies that $(rv_0)_r(r_0) \leq r_0 |v_1(r_0)|$. Then, by Lemma 3.3, it follows that $v(0, t = r_0) \leq 0$ or $v(0, t = -r_0) \leq 0$.

Suppose $v(0, t = r_0) \leq 0$ and let $t_0 = \inf\{t \geq 0 : \exists r, |r| \leq r_0 - t \text{ and } v(r, t) = 0\}$. By continuity and compactness, $v(r_1, t_0) = 0$ for some r_1 . Then clearly $t_0 \leq r_0$ and $t_0 > 0$ (since at time 0 $v_0 = e^{u_0} \neq 0$). By our definition, on the light cone with $\{(r, t) : t \geq 0, |r - r_1| \leq t_0 - t\}$ v is positive.

This means that we can take $u = \ln v$ and retrieve a smooth solution u of the original equation (1.1) on this cone. At the same time, since $\lim_{(r,t) \rightarrow (r_1, t_0)} v(r, t) = v(r_1, t_0) = 0$, it follows that $\lim_{(r,t) \rightarrow (r_1, t_0)} u(r, t) = -\infty$, which implies the L_{loc}^∞ blow-up.

Consider $x_0 \in \mathbb{R}^3$ with $|x_0| = r_1$, so that $v(x_0, t_0) = 0$. It follows that, for $|x - x_0|, |t - t_0| \leq 1$, $|v(x, t)| \lesssim |x - x_0| + |t - t_0|$.

Therefore, under the same conditions and on the domain of u ,

$$u(x, t) \leq C + \ln(|x - x_0| + |t - t_0|). \quad (4.1)$$

Setting $x = x_0$, we get that $\|u(t)\|_{L_x^\infty(|x-x_0| \leq 1)} \geq C + |\ln |t - t_0||$.

Concerning the $H^{3/2}$ norm, we use the Trudinger-Moser inequality, see ([Tru]) and ([Mos]): for any sufficiently regular bounded domain $\Omega \subset \mathbb{R}^3$ there exists C such that

$$\int_{\Omega} \exp \left(\left(\frac{|u(x)|}{C \|u\|_{H^{3/2}}} \right)^2 \right) - 1 \, dx \leq 1.$$

By making the coordinate change $x - x_0 = |t - t_0|(y - x_0)$, we obtain that $\|u(t)\|_{H^{3/2}(|x-x_0| \leq 1)} \geq C |\ln |t - t_0||^{1/2}$. \square

Proof of Proposition 1.4. Here we make the substitution $v = F(u)$, where $F''/F' = -f$, so we can take $F(u) = \int_0^u e^{-\int_0^s f(\sigma) d\sigma} ds$, see (1.7). Then v solves the free wave equation

$$v_{tt} - \Delta v = 0, \quad v(0) = v_0 = F(u_0), \quad v_t(0) = v_1 = F'(u_0)u_1.$$

If $F(\pm\infty) = \pm\infty$ we can always invert this transformation, so we obtain global smooth solutions for any smooth initial data. On the other hand, if $F(-\infty) = a \in \mathbb{R}$ and/or $F(+\infty) = b \in \mathbb{R}$, then, in order to invert by taking $u = F^{-1}(v)$, we need to impose the condition $v > a$ and/or $v < b$. By Corollary 3.5, it is sufficient that

$$rF'(u_0)(u_0)_r + F(u_0) - a > rF'(u_0)|u_1|,$$

which is equivalent to condition (1.8).

In order to obtain $L_{t,x}^\infty$ solutions, we must ask that $v \in L_{t,x}^\infty$ and, depending on each case, $\inf v > a$ and/or $\sup v < b$.

Suppose that $u_0 \in L^\infty$. By Lemma 3.3, in order for $v \in L_{t,x}^\infty$ we must assume that $r(u_0)_r, ru_1 \in L^\infty$.

By Corollary 3.5, a necessary and sufficient condition for e.g. $\inf v - a > 0$ is that

$$\inf rF'(u_0)(u_0)_r + F(u_0) - a - rF'(u_0)|u_1| > 0.$$

Taking into account the fact that we are assuming $u_0 \in L^\infty$, so $F'(u_0)$ is bounded from above and below, this reduces to the stated condition (1.10). \square

Finally, the nonradial case is not so different from the radial case.

Proof of Proposition 1.5. The proof is almost identical to that of Proposition 1.4. We use the same substitution $v = F(u)$, with F given by (1.7). Then v must be a solution of the free wave equation

$$v_{tt} - \Delta v = 0, \quad v(0) = v_0 = F(u_0), \quad v_t(0) = v_1 = F'(u_0)u_1,$$

which is guaranteed to exist.

This transformation is always invertible when $F(\pm\infty) = \pm\infty$, but in the other cases we need to check whether $v > F(-\infty) = a$ and/or $v < F(+\infty) = b$. This is done using the positivity criteria of Lemma 3.6.

If $F(-\infty) = a \in \mathbb{R}$, then our conditions will in fact imply that $v \geq \inf v_0$ or that $v \geq 0$. Stated in terms of v , the conditions are $\inf v_0 > -\infty$ and $v_1 \geq |\nabla v_0|$ or (v_0, v_1) Schwartz functions and $-\Delta v_0 \geq |\nabla v_1|$. Note that

$$\begin{aligned} \nabla v_0 &= F'(u_0)\nabla u_0, \\ \Delta v_0 &= F'(u_0)\Delta u_0 + F''(u_0)(\nabla u_0)^2, \\ \nabla v_1 &= F'(u_0)\nabla u_1 + F''(u_0)u_1\nabla u_0, \end{aligned}$$

and that by definition $F''/F' = -f$ and $F' > 0$. We obtain exactly condition (1.11).

If $D^2u_0 \in L^{3/2,1} \subset \mathcal{K}$, then $u_0 \in L^\infty$, so $F'(u_0)$ and $F''(u_0) \in L^\infty$. Assuming that $D^2u_0, \nabla u_1 \in L^{3/2,1}$, it also follows that $\nabla u_0, u_1 \in L^{3,1}$.

Consequently $\Delta v_0, \nabla v_1 \in L^{3/2,1}$, so by Lemma 3.6 v is bounded. Under our previous conditions, this also implies that u is bounded. \square

Proof of Proposition 1.6. Let $v = F(u)$, where F is given by (1.7). Note that by our definition $F(0) = 0$. We obtain the free wave equation on \mathbb{R}^{3+1} for v :

$$v_{tt} - \Delta v = 0, \quad v(0) = v_0 = F(u_0), \quad v_t(0) = v_1 = F'(u_0)u_1.$$

It is easy to see that, since u_0 and u_1 are of Schwartz class, then so are $v_0 - F(0) = v_0$ and v_1 . Therefore the solution v is globally defined on \mathbb{R}^{3+1} , $v(t)$ is of Schwartz class for each $t \in \mathbb{R}$, and v disperses.

In particular, by the usual decay estimates, $\lim_{t \rightarrow \infty} \|v(x, t)\|_{L_x^\infty} = 0$. Since v is continuous and for each fixed t $\lim_{x \rightarrow \infty} v(x, t) = 0$, it follows that $\lim_{(x,t) \rightarrow \infty} v(x, t) = 0$.

Without loss of generality, assume $F(-\infty) = a \in \mathbb{R}$ and $F(+\infty) = b \in \mathbb{R}$; then $a < F(0) = 0$ and $b > 0$. From the above it follows that there exists some $R > 0$ such that if $|(x, t)| > R$ then $v(x, t) \in (a/2, b/2)$. Since the set $\{(x, t) : |(x, t)| \leq R\}$ is compact, v reaches its maximum and minimum on this set, i.e. $m = \min_{|(x,t)| \leq R} v \leq v \leq M = \max_{|(x,t)| \leq R} v$.

There are two cases. If $m \geq a$ or $M \leq b$, then the solution u blows up in finite time; see the proof of Proposition 1.3 for more details. Otherwise, one has $a < \inf v < \sup v < b$.

In the latter case, one can invert the transformation by taking $u = F^{-1}(v)$ and obtain a global smooth solution u on \mathbb{R}^{3+1} to equation (1.6). In addition, $u = F^{-1}(v)$ is bounded and more generally $(F^{-1})^{(n)}$ is bounded on the domain of v .

We obtain that $|u| \lesssim |v|$ and $|D^n u| \lesssim \sum_{k_1 + \dots + k_n = n} |D^{k_1} v| \dots |D^{k_n} v|$ (with constants that may depend on the solution). Therefore $u(t)$ is a Schwartz function for each t , its Sobolev H^n norms are uniformly bounded, and u and all its derivatives disperse (because v has these properties and dominates u).

Concerning scattering, note that $u = (F^{-1})'(0)v + O(v^2)$. The first term is a solution of the free wave equation, while the second term goes to zero in any H^n Sobolev norm, since $v(t)$ is uniformly bounded in H^n and $\lim_{t \rightarrow \infty} \|D^n v(t)\|_{L^\infty} = 0$ for any $n \geq 0$. \square

Proof of Proposition 1.9. The proof uses the same ideas as that of Proposition 1.5. We make the transformation $v = F(u)$, where F is given by (1.7). Then v is a solution of the free wave equation on \mathbb{R}^{3+1}

$$v_{tt} - \Delta v = 0, \quad v(0) = v_0 = F(u_0), \quad v_t(0) = v_1 = F'(u_0)u_1.$$

Since u_0 and u_1 are smooth functions, so are v_0 and v_1 , leading to a smooth solution v on \mathbb{R}^{3+1} .

In order to reverse the transformation, we need to ensure that $v > a$. Our hypotheses and Lemma 3.7 guarantee this on $\mathbb{R}^3 \times (-T, T)$, where T is defined by (1.12). Also, for any $t < T$, we see that $v \in L_{t,x}^\infty(\mathbb{R}^3 \times [-t, t])$

and $\inf v > a$ on $\mathbb{R}^3 \times [-t, t]$, which implies that $u \in L_{t,x}^\infty(\mathbb{R}^3 \times [-t, t])$ as well. \square

5. LARGE SOLUTIONS TO EQUATION (1.13)

Proof of Proposition 1.10. Consider the following sequence:

$$\begin{aligned} u^0(t) &= \cos(t\sqrt{-\Delta})u_0 + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}u_1, \\ u^{n+1}(t) &= \cos(t\sqrt{-\Delta})u_0 + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}u_1 + \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} |u^n(s)|^N u^n(s) ds. \end{aligned}$$

The first term in the sequence is nonnegative due to our hypothesis, by (3.5) or Lemma 3.3. Inductively we see that all u^n are smooth and nonnegative, hence all the integrals are well-defined.

Furthermore, one proves by induction that the sequence $(u^n)_n$ is monotonically increasing, due to the positivity of the kernel

$$\frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}(x, y) = \frac{1}{4\pi t} \delta_{|x-y|=t}.$$

Since the sequence $(u^n)_n$ is monotonically increasing, it must have a limit (which can be nonnegative or $+\infty$) pointwise in \mathbb{R}^{3+1} . Let $u := \lim_{n \rightarrow \infty} u^n$; clearly,

$$\lim_{n \rightarrow \infty} |u^n|^N u^n = |u|^N u,$$

with the usual convention that $(+\infty)^{N+1} = +\infty$.

By the monotone convergence theorem it follows that

$$u(t) = \cos(t\sqrt{-\Delta})u_0 + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}u_1 + \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} |u(s)|^N u(s) ds,$$

i.e. u is a solution to (1.13) with initial data (u_0, u_1) .

In order to compare two solutions, note that if $u_0 \geq v_0 \geq 0$ in case i, $(ru_0(r))_r \pm ru_1 \geq (rv_0(r))_r \pm rv_1 \geq 0$ in case ii, etc., then $u^0 \geq v^0 \geq 0$ and by induction $u^n \geq v^n \geq 0$ for every n , so $u \geq v \geq 0$. \square

Proof of Corollary 1.12. We use comparison with the solutions with initial data $((1 \pm \epsilon)Q, 0)$.

A straightforward computation shows that

$$\frac{d}{d\epsilon} E[(1 \pm \epsilon)Q] \big|_{\epsilon=0} = 0, \quad \frac{d^2}{d\epsilon^2} E[(1 \pm \epsilon)Q] \big|_{\epsilon=0} < 0,$$

so $E[(1 \pm \epsilon)Q] < E[Q]$ for small $\epsilon > 0$. Using the criterion of [KeMe1], it follows that the solution with initial data $((1 - \epsilon)Q, 0)$ disperses, while the solution with initial data $((1 + \epsilon)Q, 0)$ blows up in finite time.

As an aside, note that our criteria guarantee that any solution with initial data $(CQ, 0)$, $C > 0$, is positive.

It is well-known (see [KeMe1]) that finite $L_{t,x}^8$ norm implies that the solution must exist globally on \mathbb{R}^{3+1} . \square

Proof of Theorem 1.16. Condition (1.21) implies that the solution u is dominated, in the sense of Corollary 1.11, by $Q_N(x-x_0)$ for any x_0 with $|x_0| = \alpha$. It immediately follows that $|u(x, t)| \leq C_N(\alpha + |x|)^{-2/N}$.

In order to prove that the solution u does not blow up, the easiest thing to do is to assume that the initial data have compact support. Due to the finite speed of propagation, u will have compact support at any fixed time $t \in \mathbb{R}$. Since it is bounded, all its Lebesgue norms will be bounded, uniformly on compact time intervals. Therefore the Strichartz norms of u will also be finite.

In the defocusing case, if the initial data have finite energy, then the Morawetz inequality guarantees that $\|u\|_{L_x^\infty L_t^2}^2 \lesssim E[u] < \infty$. On the other hand, we already know that $\|u\|_{L_{t,x}^\infty} \lesssim \alpha^{-2/N}$ and

$$\|u\|_{L_x^{3N/2,\infty} L_t^\infty} \leq \|Q_N\|_{L^{3N/2,\infty}} \lesssim 1.$$

Interpolating between the three bounds we obtain that $\|u\|_{L_{t,x}^{2N}} < \infty$, so u disperses and scatters.

A more general argument to prove that the solution does not blow up in finite time is the following: for $s_c \leq 3/2$, $\dot{H}^{s_c} \cap L^\infty$ is a Banach algebra and

$$\|fg\|_{\dot{H}^{s_c}} \lesssim \|f\|_{\dot{H}^{s_c}} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{\dot{H}^{s_c}}.$$

Then at least for even N

$$\||u(t)|^N u(t)\|_{\dot{H}^{s_c}} \lesssim \|u(t)\|_{\dot{H}^{s_c}} \|u\|_{L_{t,x}^\infty}^N.$$

In fact, similar estimates hold for every $0 < N \leq 3/2$. Since

$$\left\| \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}} f \right\|_{\dot{H}^{s_c}} \leq t \|f\|_{\dot{H}^{s_c}},$$

it follows that for $t \geq 0$

$$\|(u(t), u_t(t))\|_{\dot{H}^{s_c} \times \dot{H}^{s_c-1}} \lesssim \|(u_0, u_1)\|_{\dot{H}^{s_c} \times \dot{H}^{s_c-1}} + \int_0^t (t-s) \|(u(s), u_t(s))\|_{\dot{H}^{s_c} \times \dot{H}^{s_c-1}} \|u\|_{L_{t,x}^\infty}^N ds.$$

By Gronwall's inequality $\|(u(t), u_t(t))\|_{\dot{H}^{s_c}} \lesssim \|(u_0, u_1)\|_{\dot{H}^{s_c} \times \dot{H}^{s_c-1}} \exp(Ct^2 \|u\|_{L_{t,x}^\infty}^N)$.

Using this inequality for time $t \leq 1$ and iterating, we get that

$$\|(u(t), u_t(t))\|_{\dot{H}^{s_c}} \lesssim \|(u_0, u_1)\|_{\dot{H}^{s_c} \times \dot{H}^{s_c-1}} \exp(Ct \|u\|_{L_{t,x}^\infty}^N).$$

Thus $\|u(t)\|_{\dot{H}^{s_c}}$ is bounded on compact time intervals. This immediately implies that the nonlinearity $\||u(t)|^N u(t)\|_{\dot{H}^{s_c}}$ is also bounded on compact intervals and the same for the Strichartz norms.

Following the result of [DuRo], for radial solutions, in either the focusing or the defocusing case, the local boundedness of the critical Sobolev norm implies that the solution disperses and scatters (without any explicit bounds, however, on the size of the global Strichartz norms). \square

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