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Integer flow and Petersen minor

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ABSTRACT

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Tutte [45] conjectured that every bridgeless Petersen-minor free graph admits a nowhere-zero 4-flow. Let $(P_{10})_{\bar{\mu}}$ be the graph obtained from the Petersen graph by contracting μ edges from a perfect matching. In chapter 1 we prove that every bridgeless $(P_{10})_{\bar{3}}$-minor free graph admits a nowhere-zero 4-flow.

Walton and Welsh [48] proved that if a coloopless regular matroid M does not have a minor in $\{M(K_{3,3}), M^*(K_5)\}$, then M admits a nowhere zero 4-flow. Lai *et al* [27] proved that if M does not have a minor in $\{M(K_5), M^*(K_5)\}$, then M admits a nowhere zero 4-flow. We prove in chapter 2 that if a coloopless regular matroid M does not have a minor in $\{M((P_{10})_{\bar{3}}), M^*(K_5)\}$, then M admits a nowhere zero 4-flow. This result implies Walton and Welsh [48] and Lai *et al* [27].

The odd-edge-connectivity of a graph G, denoted by $\lambda_o(G)$, is the size of the smallest odd edge-cut of G. In chapter 3, some methods are developed to deal with small even edge-cuts and therefore, extending some earlier results from edge-connectivity to odd-edge-connectivity. One of the main results in chapter 3 solves an open problem that every odd-$(2k+1)$-edge-connected graph has k edge-disjoint parity subgraphs. Another main theorem in the chapter generalizes an earlier result by Galluccio and Goddyn *(Combinatorica 2002)* that the flow index of every odd-7-edge-connected graph is strictly less than 4. It is also proved in this paper if $\lambda_o(G) \geq 4\lceil \log_2 |V(G)| \rceil$, then G admits a nowhere-zero 3-flow which is a partial result to the weak 3-flow conjecture by Jaeger and improves an earlier result by Lai and Zhang[24].

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And finally, I would like to thank the Department of Mathematics and Eberly College of Arts and Sciences at West Virginia University for providing me with an excellent study environment and support during my study as a graduate student.

DEDICATION

To

my father *Zuyou* *, my mother* *Yonglian*

and

my wife *Ju*

Contents

Chapter 1

Introduction

1.1 Nowhere-zero 4-flows of graphs

The concept of integer flow was introduced by Tutte as a generalization of map coloring problem. For terms that are not defined here, readers can refer to textbooks [5] or [**?**] for graphs, [54] for flows, and [30] or [50] for matroids.

A nontrivial 2-regular connected graph will be called a **circuit**, and a disjoint union of circuits a **cycle**. Thus the empty set $\emptyset$ is the only independent cycle.

Let $G = (V, E)$ be a graph with vertex set V and edge set E and let D be an orientation of G. For a vertex $v \in V(G)$, let $E^+(v)$ (or $E^-(v)$) be the set of all arcs of $D(G)$ with their tails (or heads, respectively) at the vertex v. G is said to **admit a nowhere-zero** k**-flow** if there exists an ordered pair (D, f), where $f : E(G) \to \{\pm 1, \pm 2, \cdots \pm (k-1)\}$ such that

$$\sum_{e \in E^+(v)} f(e) = \sum_{e \in E^-(v)} f(e)$$

for every vertex $v \in V(G)$.

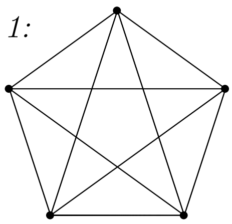
Let G and H be two graphs. If G contains a subgraph which is contractible to H, then H is a **minor** of G and we say G contains an H-mimor.

The following conjecture is one of the major open problems in graph theory.

Conjecture 1.1.1 (Tutte [45]) *Every bridgeless graph without a Petersen minor admits a nowhere-zero 4-flow.*

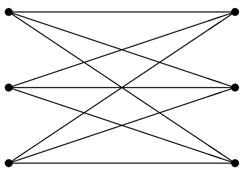
For planar graphs, admitting a nowhere-zero 4-flow is equivalent to having a face 4-coloring. Hence, by the 4-Color Theorem [1, 2, 3, 31], Conjecture 2.1.1 has been verified for all planar graphs. Furthermore, it was also announced that Conjecture 2.1.1 was verified for all cubic graphs [32, 33]. By Kuratowski Theorem, a graph is planar if and only if it contains neither K_5-minor nor $K_{3,3}$-minor. By applying the 4-Color Theorem, Conjecture 2.1.1 was further verified for $K_{3,3}$-minor free graphs [48], K_5-minor free graphs [27], and P_{10}^--minor free graphs [39]. Each of these families contains the family of all planar graphs and may not be necessarily cubic. Graphs K_5, $K_{3,3}$, P_{10} and P_{10}^- are illustrated in the following figure.

Figure 1:



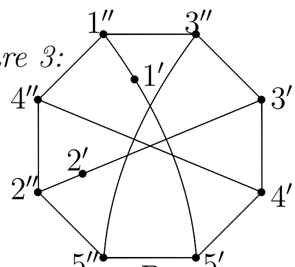
K_5

Figure 2:



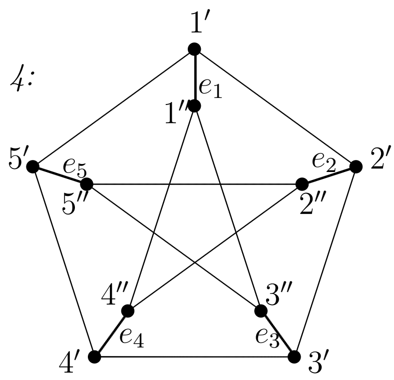
$K_{3,3}$

Figure 3:



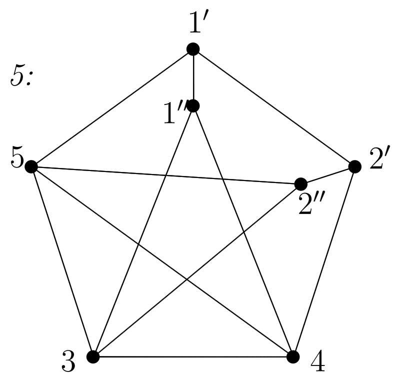
P_{10}^-

Figure 4:



Petersen graph P_{10}

Figure 5:



$(P_{10})_{\bar{3}} = P_{10}/\{e_3, e_4, e_5\}$

Let P_{10} be the Petersen graph with the exterior pentagon $1'2'3'4'5'1'$, interior pentagon $1''3''5''2''4''1''$ and a perfect matching $M = \{e_i = i'i'' : i = 1, 2, 3, 4, 5\}$. Let $(P_{10})_{\bar{\mu}}$ be the graph obtained from P_{10} by contracting F, where $F \subseteq M$ and $|F| = \mu$.

Remark. It is not hard to see that if M and M' are two perfect matchings of P_{10}, $F \subseteq M$, $F' \subseteq M'$ and $|F| = |F'|$, then $P_{10}/F \cong P_{10}/F'$. Hence $(P_{10})_{\bar{\mu}}$ is well defined.

The following is the main theorem of this section and will be proved in chapter 2.

Theorem 1.1.2 *Let G be a bridgeless graph. If G does not have a $(P_{10})_{\bar{3}}$-minor, then G admits a nowhere-zero 4-flow.*

1.2 Nowhere-zero 4-flows of matroids

We shall assume familiarity with graph theory and matroid theory. For terms that are not defined in this note, see Bondy and Murty [5] for graphs, and Oxley [30] or Welsh [50] for matroids.

In this article, $\mathbf{Z}, \mathbf{Z}^+$ and $\mathbf{Z}_n$ denote the additive group of the integers, the set of all positive integers, and the cyclic group of order n, respectively, and $\mathcal{R}$ denotes the family of all regular matroids. As in [30], the set of all circuits of a matroid M is denoted by $\mathcal{C}(M)$. We further denote the set of all cycles of a matroid M by $\mathcal{C}_0(M)$. Note that as we allow empty unions, the empty set is also a cycle (in both graphs and matroids). For matroids $N_1, N_2, \cdots, N_k$, let $EX(N_1, N_2, \cdots, N_k)$ denote the collection of matroids such that a matroid $M \in EX(N_1, N_2, \cdots, N_k)$ if and only if M does not have a minor isomorphic to any one in $\{N_1, N_2, \cdots, N_k\}$. The Fano matroid F_7 is the vector matroid over GF(2) of the following matrix A:

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

The definition of flow has a natural extension to regular matroids. Let M be a regular matroid and D_M be its incidence matrix of circuits against elements. An **orientation**

$(w(D_M), w(D_{M^*}))$ is an assignment of $+, -$ signs to the "1" entries of D_M and D_{M^*}, respectively, so that the resulting matrices $w(D_M)$ and $w(D_{M^*})$ satisfy

$$w(D_M)w(D_{M^*})^T = 0.$$

Let A be an abelian group. For an element $a \in A$, and for integers $+1, -1, 0$, we adopt the convention to write $(+1) \cdot a = a$, $(-1) \cdot a = -a$ and $0 \cdot a = 0$. Let $F^*(M, A) = \{f : E(M) \mapsto A \setminus \{0\}\}$ denote the set of all functions from $E(M)$ into $A \setminus \{0\}$. A map $f \in F^*(M, A)$ can be viewed as an $|E(M)|$-dimensional column vector. For a regular matroid M with an orientation $(w(D_M), w(D_{M^*}))$, a map $f \in F^*(M, A)$ satisfying

$$w(D_{M^*}) \cdot f = 0$$

is a **nowhere zero A-flow** (A-NZF for short) of M. When $A = \mathbf{Z}$, a $\mathbf{Z}$-NZF f of M is called a **nowhere zero k-flow** (k-NZF for short) of M if $\forall e \in E(M)$, $0 < |f(e)| < k$.

The matroid version of Conjecture 2.1.1 is as follows

Conjecture 1.2.1 *If M is a coloopless regular matroid such that $M \in EX(M(P_{10}), M^*(K_5))$, then M admits a 4-NZF.*

Applying the Four-Color theorem, and the duality between colorings and nowhere zero flows, a result by Walton and Welsh implies the following.

Theorem 1.2.2 *(Walton and Welsh [48]) If $M \in EX(M(K_{3,3}), M^*(K_5)) \cap \mathcal{R}$ is a coloopless matroid, then M admits a 4-NZF.*

Proving a conjecture of Jensen and Toft [22], Lai, Li and Poon applied the Four-Color Theorem to prove the following Theorem 3.1.7, which is an approach to Conjecture 1.2.1.

Theorem 1.2.3 *(Lai, Li and Poon, [27]) If $M \in EX(M(K_5), M^*(K_5)) \cap \mathcal{R}$ is a coloopless matroid, then M admits a 4-NZF.*

The main objective of this chapter is to prove the following theorem, which generalizes Theorem 3.1.7, and is also an approach to Conjecture 1.2.1.

Theorem 1.2.4 *If M is a coloopless matroid such that $M \in EX(M((P_{10})_{\bar{3}}), M^*(K_5)) \cap \mathcal{R}$, then M admits a 4-NZF.*

1.3 Odd-edge-connectivity of graphs

It is evident that odd-edge-connectivity (see Definition 4.2.1) plays a more important role than edge-connectivity in the study of some flow and cycle cover related problems. In this article, some earlier results in those areas are extended from λ-edge-connected graphs to odd-λ-edge-connected graphs.

For graphs with large odd-edge-connectivity, small edge-cuts (of even size) may still exist. However, there are not many results or methods developed yet to deal with small even edge-cuts. For some integer flow problems and cycle cover problems, it is pointed out in [36, 21] that 2-edge-cut does not exit in any smallest counterexample (to some well-known flow conjectures and cycle cover conjectures). The 3-flow conjecture by Tutte [**?**] was originally proposed for odd-5-edge-connected graphs. By excluding 4-edge-cut, Kochol [23], with a sophisticated linear algebra approach, proved that 3-flow conjecture is equivalent for 5-edge-connected graphs.

In this chapter, we are to develop some general approaches to deal with small even cuts.

1.3.1 Parity Subgraphs

Definition 1.3.1 *Let H be a subgraph of a bridgeless graph $G = (V, E)$. H is a **parity subgraph** of G if for every vertex $v \in V(G)$, $d_G(v) \equiv d_H(v) (mod\ 2)$.*

It was proved by Tutte and Nash-Williams [43, 29] that every $2k$-edge-connected graph contains at least k edge-disjoint spanning trees, and proved by Itai and Rodeh [18] that every spanning tree of a graph G contains a parity subgraph. The combination of these two theorems yields the following result as a direct corollary.

Theorem 1.3.2 *Every $2k$-edge-connected graph G contains at least k edge-disjoint parity subgraphs of G.*

It is well-known that the search of parity subgraphs plays a central role in the proofs of some important theorems in integer flow areas. For example, the 4-flow theorem is proved by Jaeger [19] with following approach: The 4-edge-connectivity guarantees the existence of two edge-disjoint parity subgraphs (by Theorem 4.1.2) and therefore a 2-cycle cover of G. The 8-flow theorem was proved by Jaeger [19] with following similar approach: The 3-edge-connectivity guarantees the existence of three edge-disjoint parity subgraphs in $2G$ and therefore, a 3-cycle cover of G.

Theorem 4.1.2 is to be generalized in this article by relaxing the edge-connectivity to odd-edge-connectivity and therefore, solves an open problem proposed in [51, 12, 54].

We generalize the theorem and get the following

Theorem 1.3.3 *Every odd-$(2k+1)$-edge-connected graph G contains at least k edge-disjoint parity subgraphs of G.*

1.3.2 Flow Index

Circular flow, introduced in [16] as a real line extension of integer flow problem. The following is one of the many equivalent definitions for circular flows and the corresponding flow indices.

Definition 1.3.4 *Let $\mathcal{D}$ be the set of all orientations of G, and (A, B) be any edge-cut of G. The **flow index** of G is defined by*

$$\phi(G) = \min_{D \in \mathcal{D}} \left\{ \max_{(A,B)} \frac{|[A, B]_D|}{|[B, A]_D|} \right\} + 1.$$

Theorem 1.3.5 (*Galluccio and Goddyn,* [*15*]) *let* G *be a 6-edge-connected graph, then the flow index* $\phi(G) < 4$.

We generalize the theorem above and get:

Theorem 1.3.6 *Let* G *be an odd-7-edge-connected graph, then* $\phi(G) < 4$.

1.3.3 Nowhere-zero 3-flows

The following is an approach of Jaeger's weak 3-flow conjecture [19]. Lai and Zhang proved the following theorem

Theorem 1.3.7 ([*24*]) *Every* $4\lceil \log_2 n \rceil$*-edge-connected multigraph with* n *vertices admits a nowhere-zero 3-flow.*

As a generalization, we prove

Theorem 1.3.8 *Let* G *be a multigraph with* n *vertices. If its odd-edge-connectivity is more than* $4\lceil \log_2 n \rceil$, *then* G *admits a nowhere-zero* 3*-flow.*

Chapter 2

4-NZF in almost P_{10} minor free graphs

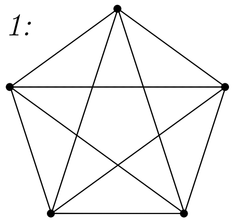
2.1 Introduction

The concept of integer flow was introduced by Tutte as a generalization of map coloring problem. The following conjecture is one of the major open problems in graph theory.

Conjecture 2.1.1 (Tutte [45]) *Every bridgeless graph without a Petersen minor admits a nowhere-zero 4-flow.*

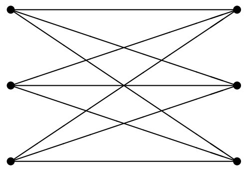
For planar graphs, admitting a nowhere-zero 4-flow is equivalent to having a face 4-coloring. Hence, by the 4-Color Theorem [**?**, 2, 3, **?**], Conjecture 2.1.1 has been verified for all planar graphs. Furthermore, it was also announced that Conjecture 2.1.1 was verified for all cubic graphs [32, 33]. By Kuratowski Theorem, a graph is planar if and only if it contains neither K_5-minor nor $K_{3,3}$-minor. By applying the 4-Color Theorem, Conjecture 2.1.1 was further verified for $K_{3,3}$-minor free graphs [48], K_5-minor free graphs [27], and P_{10}^--minor free graphs [39]. Each of these families contains the family of all planar graphs and may not be necessarily cubic. Graphs K_5, $K_{3,3}$, P_{10} and P_{10}^- are illustrated in the following figure.

Figure 1:



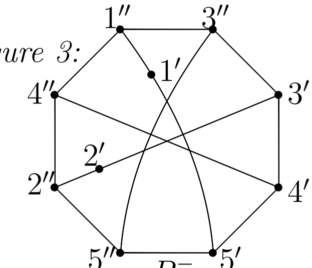
K_5

Figure 2:



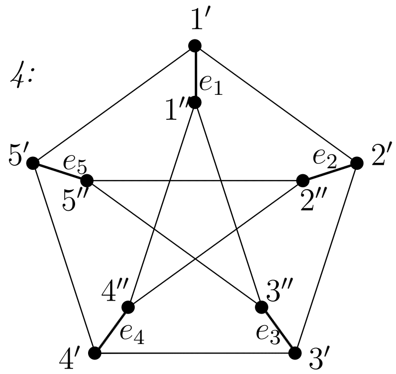
$K_{3,3}$

Figure 3:



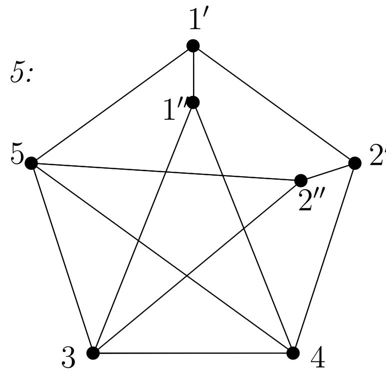
P_{10}^{-}

Figure 4:



Petersen graph P_{10}

Figure 5:



$(P_{10})_{\bar{3}} = P_{10}/\{e_3, e_4, e_5\}$

Let P_{10} be the Petersen graph with the exterior pentagon $1'2'3'4'5'1'$, interior pentagon $1''3''5''2''4''1''$ and a perfect matching $M = \{e_i = i'i'' : i = 1, 2, 3, 4, 5\}$. Let $(P_{10})_{\bar{\mu}}$ be the graph obtained from P_{10} by contracting F, where $F \subseteq M$ and $|F| = \mu$.

Remark. It is not hard to see that if M and M' are two perfect matchings of P_{10}, $F \subseteq M$, $F' \subseteq M'$ and $|F| = |F'|$, then $P_{10}/F \cong P_{10}/F'$. Hence $(P_{10})_{\bar{\mu}}$ is well defined.

The following is our main theorem.

Theorem 2.1.2 *Let G be a bridgeless graph. If G does not have a $(P_{10})_{\bar{3}}$-minor, then G admits a nowhere-zero 4-flow.*

2.2 Notations and terminologies

For terms that are not defined here, readers can refer to textbooks [5], [**?**], and [54] (for flows).

Let $G = (V, E)$ be a graph with vertex set V and edge set E and let D be an orientation of G. For a vertex $v \in V(G)$, let $E^+(v)$ (or $E^-(v)$) be the set of all arcs of $D(G)$ with their tails (or heads, respectively) at the vertex v. G is said to **admit a nowhere-zero k-flow** if there exists an ordered pair (D, f), where $f : E(G) \to \{\pm 1, \pm 2, \cdots \pm (k-1)\}$ such that

$$\sum_{e \in E^+(v)} f(e) = \sum_{e \in E^-(v)} f(e)$$

for every vertex $v \in V(G)$.

A graph G is a 4-**flow snark** if it is bridgeless and does not admit a nowhere-zero 4-flow. Let G and H be two graphs. If G contains a subgraph which is contractible to H, then H is a **minor** of G and we say G contains an H-mimor. A 4-flow snark G is **minor-prime** if every proper minor of G is not a 4-flow snark. With the definitions above, Conecjture 2.1.1 can be restated as follows.

Conjecture 2.2.1 *The Petersen graph is the only minor-prime* 4*-flow snark.*

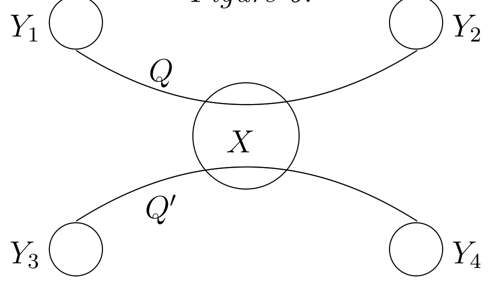
Let H be a minor of a connected graph G. Then there is an onto mapping $f : V(G) \mapsto V(H)$ such that $f^{-1}(v)$ induces a connected subgraph $G[f^{-1}(v)]$ of G for every $v \in V(H)$ and H can be obtained from a spanning subgraph of G by contracting the edges of $G[f^{-1}(v)]$ for all $v \in V(H)$. Here f is called a **minor-mapping** and $f^{-1}(v)$ is called a v-**domain** of f.

A k-**separator** of a graph G is an ordered triple $(H_1, H_2; T)$ such that $H_1 \cup H_2 = G$ and $V(H_1 \cap H_2) = T$, where T is a vertex subset of G and $|T| = k$. Sometimes we say T is a k-separator if there is no confusion. A k-separator $(H_1, H_2; T)$ of G is **trivial** if one of H_1 and H_2, say H_1, is acyclic. G is **quasi** k-**connected** if G is 3-connected and every t-separator of G with $t \leq k$ is trivial.

Let x be a vertex of G. The vertex x **separates** G into q parts $H_1, \cdots, H_q$ if $G = H_1 \cup \cdots \cup H_q$ and $V(H_i \cap H_j) = \{x\}$ for every pair of $i \neq j$.

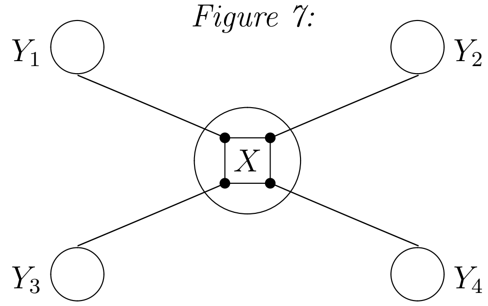
Let X be a connected subgraph of G and Y_1, Y_2, Y_3, Y_4 be four disjoint connected subgraphs of $G - V(X)$ and $X \cap N(Y_i) \neq \emptyset$ for $i = 1, 2, 3, 4$ where $N(Y_i)$ denotes the set of neighbors of Y_i. Let $J = \{Y_1, Y_2, Y_3, Y_4\}$. For each 2×2-partition $P = \{\{a, b\}, \{c, d\}\}$ of $\{1, 2, 3, 4\}$, X is P**-splittable** if X contains two disjoint paths Q and Q' such that Q joins $X \cap N(Y_a)$ and $X \cap N(Y_b)$, Q' joins $X \cap N(Y_c)$ and $X \cap N(Y_d)$, $i \in \{a, b, c, d\}$. An example of a $\{\{1, 2\}, \{3, 4\}\}$-splittable subgraph is illustrated in Figure 6. X is k**-splittable** with respect to J if there are k distinct 2×2 partitions $P_1, \cdots, P_k$ of $\{1, 2, 3, 4\}$ such that X is P_i-splittable for each $i = 1, \cdots, k$. (Remark: $k \leq 3$) An example of a 2-splittable subgraph is illustrated in Figure 7.

Figure 6:



X is $\{\{1, 2\}, \{3, 4\}\}$-splittable

Figure 7:



X is 2-splittable

2.3 Lemmas

Lemma 2.3.1 (Catlin [9]) *If G is a minor-prime 4-flow snark, then the girth of G is at least five.*

Lemma 2.3.2 (Lai, Li and Poon [27]) *If a bridgeless graph G does not admit a nowhere-zero 4-flow, then G has a K_5-minor.*

Lemma 2.3.3 (Thomas and Thomson, Lemma 4.4 of [39]) *If G is a minor-prime 4-flow* *snark, then G is quasi-4-connected (that is, every k-separator of G is trivial for each $k \leq 3$).*

Obviously, Lemma 2.3.3 generalizes Lemma 2.3.1 and Theorem 3.7.15 of [54].

Proposition 2.3.4 *Let X be a connected subgraph of G and Y_1, Y_2, Y_3, Y_4 be four disjoint connected subgraphs of $G - V(X)$ where $V(X) \cap N(Y_i) \neq \emptyset$ for $i = 1, 2, 3, 4$. Let k be the greatest integer that X is k-splittable with respect to $J = \{Y_1, Y_2, Y_3, Y_4\}$.*

(i) If $k \leq 1$, say, X is $\{\{1,2\},\{3,4\}\}$-splittable or 0-splittable, then X has a 1-separator $(H_1, H_2; \{x\})$ such that $V(X) \cap [N(Y_1) \cup N(Y_2)] \subseteq V(H_1)$ and $V(X) \cap [N(Y_3) \cup N(Y_4)] \subseteq V(H_2)$;

(ii) If $k = 0$, then there exists a cut vertex x of X that separates X into four parts H_1, H_2, H_3, H_4 such that $V(X) \cap N(Y_i) \subseteq V(H_i)$ for each i.

Proof. (i) Let G_1 be the graph induced by $X \cup Y_1 \cup Y_2 \cup Y_3 \cup Y_4$. Let G_2 be the graph obtained from G_1 by contracting each Y_i into a single vertex y_i for $i = 1, 2, 3, 4$, and deleting all edges between y_i and y_j for all $\{i, j\} \subset \{1, 2, 3, 4\}$. Note that G_2 is connected since $V(X) \cap N(Y_i) \neq \emptyset$ for $i = 1, 2, 3, 4$.

Since X is neither {{1,3},{2,4}}-splittable nor {{1,4},{2,3}}-splittable, it is impossible that there is a pair of disjoint paths joining $\{y_1, y_2\}$ and $\{y_3, y_4\}$. By Menger theorem, there is a cut vertex $x \in V(G_2)$ that separates $\{y_1, y_2\}$ and $\{y_3, y_4\}$. It is obvious that $x \in V(X)$. That is, X has a 1-separator $(H_1, H_2; x)$ that $N_{G_2}(y_1) \cup N_{G_2}(y_2) \subseteq V(H_1)$ and $N_{G_2}(y_3) \cup N_{G_2}(y_4) \subseteq V(H_2)$.

(ii) Continue from (i). Assume that there is a path P_1 joining y_1 and y_2 in the graph $G_2 - \{x\}$ (without passing through x). Note that x is a cut vertex that separates $\{y_1, y_2\}$ and $\{y_3, y_4\}$. Thus, this path P_1 is contained in the induced subgraph $G_2[V(H_1 - x) \cup \{y_1, y_2\}]$ and there is another path P_2 joining y_3 and y_4 in the induced subgraph $G_2[H_2 \cup \{y_3, y_4\}$ since H_2 is connected. This contradicts that X is 0-splittable. So every path from y_1 to y_2 must go through x. Symmetrically, every path from y_3 to y_4 must go through x as well. That implies each component of $X - x$ is adjacent to at most one of $\{y_1, y_2, y_3, y_4\}$. ■

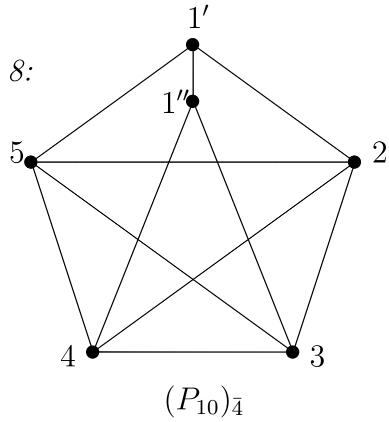
2.4 Proof of the main theorem

Let G be a minor-prime 4-flow snark. By Lemma 2.3.3, G is quasi 4-connected. By Lemma 2.3.2, K_5 is a minor of G. Let $V(K_5) = \{v_1, v_2, v_3, v_4, v_5\}$, and $f : V(G) \mapsto V(K_5)$ be a minor-mapping.

If G does not contain a $(P_{10})_{\bar{4}}$-minor, then v_a-domain $f^{-1}(v_a)$ is at most 0-splittable with respect to $\{f^{-1}(v_{i_j}) : j = 1, 2, 3, 4\}$ for every $\{a, i_1, i_2, i_3, i_4\} = \{1, 2, 3, 4, 5\}$. By Lemma 2.3.4(ii), each $f^{-1}(v_a)$ has a cut vertex v_a^* that separates $N(f^{-1}(v_{i_j}))$ for $j = 1, 2, 3, 4$. Hence $\{v_i^*, v_j^*\}$ is a 2-separator of G. Since $G \neq K_5$, there exist $\{i, j\} \subseteq \{1, 2, 3, 4, 5\}$ such that $\{v_i^*, v_j^*\}$ is a non-trivial 2-separator. This contradicts the fact that G is quasi 4-connected. Hence G contains $(P_{10})_{\bar{4}}$ as a minor.

Let $f : V(G) \to (P_{10})_{\bar{4}}$ be a minor mapping where the vertex set of $(P_{10})_{\bar{4}}$ is $\{v_{1'}, v_{1''}, v_2, v_3, v_4, v_5\}$, the contraction of the edge $v_{1'}v_{1''}$ yields a K_5, $v_{1'}$ is adjacent to v_2 and v_5, and $v_{1''}$ is adjacent to v_3 and v_4. Let $U_i = f^{-1}(v_i)$ for $i \in \{1', 1'', 2, 3, 4, 5\}$ (see the following figure). Denote $U_1 = U_{1'} \cup U_{1''}$ and choose a minor-mapping f such that $|U_1|$ is as small as possible. Now assume that G does not contain a $(P_{10})_{\bar{3}}$-minor.

Figure 8:



$(P_{10})_{\bar{4}}$

Claim 1 $|U_{1'}| = |U_{1''}| = 1$.

Proof. It is clear that $|U_{1'}| \geq 1$ and $|U_{1''}| \geq 1$.

Let $e = u_{1'}u_{1''}$ be an edge between $U_{1'}$ and $U_{1''}$ where $u_{1'} \in U_{1'}$, $u_{1''} \in U_{1''}$. Since each of $U_{1'}$ and $U_{1''}$ is connected, there are spanning trees T_1 and T_2 of $U_{1'}$ and $U_{1''}$, respectively. Let $T = T_1 \cup T_2 \cup \{e\}$. T is a spanning tree of U_1.

Assume there exist $w_2 \in N(U_2) \cap U_{1'}$ and $w_5 \in N(U_5) \cap U_{1'}$ such that $w_2 \neq w_5$. Since T_1 is a spanning tree of $U_{1'}$, there is a unique path P_2 from $u_{1'}$ to w_2 in T_1, and a path P_5 from $u_{1'}$ to w_5 in T_1. Without loss of generality, we may assume P_2 is no shorter than P_5. Since $w_2 \neq w_5$, P_5 does not contain w_2. Let C_2 be the set of vertices of the component of $T_1 \setminus P_5$ that contains w_2. Now we define a new minor mapping f_1 by $f_1^{-1}(v_i) = f^{-1}(v_i)$ for $i = 1'', 3, 4, 5$, $f_1^{-1}(v_{1'}) = f^{-1}(v_{1'}) \setminus C_2$ and $f_1^{-1}(v_2) = f^{-1}(v_2) \cup C_2$. We call this operation **moving w_2 from $U_{1'}$ to U_2.**

$$|f_1^{-1}(v_{1'}) \cup f_1^{-1}(v_{1''})| = |f^{-1}(v_{1'}) \cup f^{-1}(v_{1''})| - |C_2| < |f^{-1}(v_{1'}) \cup f^{-1}(v_{1''})|$$

That contradicts the choice of f. So we have $N(U_2) \cap U_{1'} = N(U_5) \cap U_{1'} = \{u\}$ for some u. Similarly, $N(U_3) \cap U_{1''} = N(U_4) \cap U_{1''} = \{v\}$ for some v.

Since G is quasi 4-connected, if $\{u, v\}$ is a 2-separator, then $|U_1| = 2$. If $|U_1| \geq 3$, then $\{u, v\}$ is not a 2-separator and there exists $w \in U_1 \setminus \{u, v\}$ such that $w \in N(U_i)$ for some $i = 2, 3, 4, 5$. Without loss of generality, we can assume $w \in N(U_2)$.

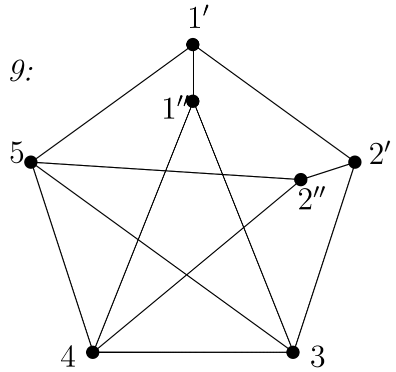
Since $N(U_2) \cap U_{1'} = \{u\}$, $w \notin U_{1'}$. If the path P from u to v in T passes through w, then we can move w from $U_{1''}$ to $U_{1'}$, that contradicts $w \notin U_{1'}$. If P does not pass through w, then we can move w from $U_{1''}$ to U_2, that contradicts the choice of f. ■

From Claim 1, we can let $U_{1'} = \{u_{1'}\}$ and $U_{1''} = \{u_{1''}\}$.

Claim 2 *U_2 is at most 1-splittable with respect to $J = \{1', 3, 4, 5\}$ with a possible partition $\{\{1', 5\}, \{3, 4\}\}$.*

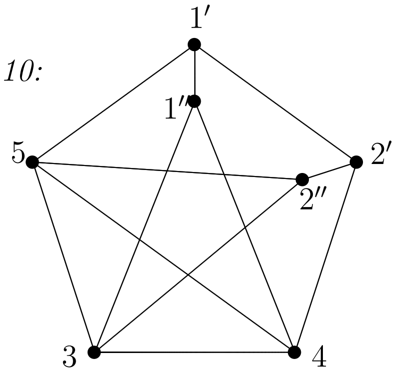
Proof. U_2 is neither $\{\{1', 3\}, \{4, 5\}\}$-splittable nor $\{\{1', 4\}, \{3, 5\}\}$-splittable. Otherwise we can have $(P_{10})_{\bar{3}}$-minors illustrated in Figures 5 and 6, respectively. ■

Figure 9:



U_2 *is* $\{\{1', 3\}, \{4, 5\}\}$*-splittable*

Figure 10:



U_2 *is* $\{\{1', 4\}, \{3, 5\}\}$*-splittable*

By Proposition 2.3.4(i), U_2 has a 1-separator $(A_2, B_2; x_2)$ such that $[N(U_{1'}) \cup N(U_5)] \cap U_2 \subseteq A_2$ and $[N(U_3) \cup N(U_4)] \cap U_2 \subseteq B_2$, as we can see in Figure 11.

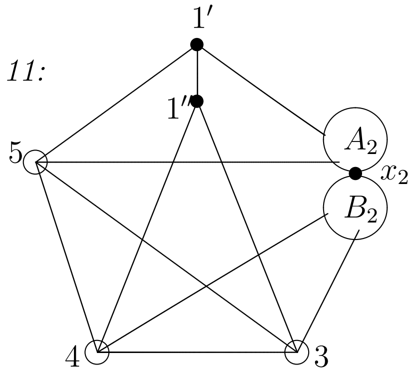
Symmetrically, we have the following conclusions (as shown in Figure 12):

(i) U_5 is at most 1-splittable with respect to $J = \{1', 2, 3, 4\}$ with the only possible (2×2)-partition $\{\{1', 2\}, \{3, 4\}\}$ and it has a 1-separator $(A_5, B_5; x_5)$ such that $[N(U_{1'}) \cup N(U_2)] \cap U_5 \subseteq A_5$, and $[N(U_3) \cup N(U_4)] \cap U_5 \subseteq B_5$.

(ii) U_3 is at most 1-splittable with respect to $J = \{1'', 2, 4, 5\}$ with the only possible (2×2)-partition $\{\{1'', 4\}, \{2, 5\}\}$ and it has a 1-separator $(A_3, B_3; x_3)$ such that $[N(U_{1''}) \cup N(U_4)] \cap U_3 \subseteq A_3$, and $[N(U_2) \cup N(U_5)] \cap U_3 \subseteq B_3$.

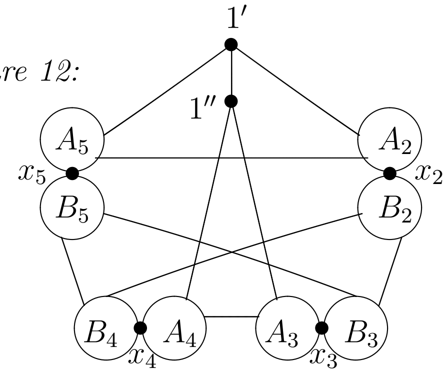
(iii) U_4 is at most 1-splittable with respect to $J = \{1'', 2, 3, 5\}$ with the only possible (2×2)-partition $\{\{1'', 3\}, \{2, 5\}\}$ and it has a 1-separator $(A_4, B_4; x_4)$ such that $[N(U_{1''}) \cup N(U_3)] \cap U_4 \subseteq A_4$, and $[N(U_2) \cup N(U_5)] \cap U_4 \subseteq B_4$.

Figure 11:



U_2 *is at most* 1*-splittable*
w.r.t. $J = \{1', 3, 4, 5\}$

Figure 12:



Each U_i *is at most* 1*-splittable*
for $i = 2, 3, 4, 5$

Claim 3 $\{N(u_{1''}) \cap A_2 - \{x_2\}\} \cup \{N(u_{1''}) \cap A_5 - \{x_5\}\} \neq \emptyset$.

Proof. Otherwise, $T = \{u_{1'}, x_2, x_5\}$ is a non-trivial 3-separator of G that separates G with $A_2 \cup A_5 \cup U_{1'}$ as one part. By Lemma 2.3.3, G is quasi 4-connected, therefore, $A_2 \cup A_5 \cup U_{1'}$ is trivial, but it is not acyclic. ■

Similarly, $\{N(u_{1'}) \cap A_3 - \{x_3\}\} \cup \{N(u_{1'}) \cap A_4 - \{x_4\}\} \neq \emptyset$.

Without loss of generality, we assume that

$$\{N(u_{1'}) \cap A_3 - \{x_3\}\} \neq \emptyset, \quad \{N(u_{1''}) \cap A_2 - \{x_2\}\} \neq \emptyset. \tag{2.1}$$

Claim 4 U_2 *is not* $\{\{1'', 5\}, \{3, 4\}\}$*-splittable.*

Proof. Otherwise G has a $(P_{10})_{\bar{3}}$-minor as in Figure 13 (note that the edge between $U_{1'}$ and U_3 is given by (2.1). ■

Symmetrically, U_5 is not $\{\{1'', 2\}, \{3, 4\}\}$-splittable.

Claim 5 U_2 *is at most 0-splittable with respect to* $J = \{1', 3, 4, 5\}$.

Proof. By way of contradiction, assume U_2 is not 0-splittable with respect to $J = \{1', 3, 4, 5\}$. By Claim 2, U_2 is $\{\{1', 5\}, \{3, 4\}\}$-splittable.

Let $\{P_{1',5}, P_{3,4}\}$ be a pair of vertex disjoint paths in U_2 that P_{ij} joins $N(U_i) \cap U_2$ and $N(U_j) \cap U_2$ for $i, j \in \{1', 3, 4, 5\}$.

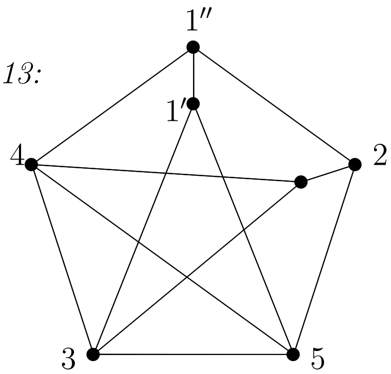
It is obvious that $P_{3,4}$ must contain the cut vertex x_2 for otherwise A_2 contains a path joining $N(u_{1''})$ and $N(U_5)$. This contradicts Claim 4. Therefore, $N(U_{1'}) \cap (A_2 - x_2) \neq \emptyset$, $N(U_5) \cap (A_2 - x_2) \neq \emptyset$ and both of them are contained in the same component of $A_2 - x_2$, called C_2, while $N(U_{1''}) \cap (A_2 - x_2)$ is contained in another component of $A_2 - x_2$.

Symmetrically, $A_5 - x_5$ has a component C_5 that contains $N(U_{1'}) \cap (A_5 - x_5)$ and $N(U_2) \cap (A_5 - x_5)$ and is disjoint with $N(U_{1''})$.

Here we have obtained a 3-separator $(H_1, H_2; T)$ with $T = \{u_{1'}, x_2, x_5\}$ as the cut and $H_1 = C_2 \cup C_5 \cup \{u_{1'}, x_2, x_5\}$. Note that neither H_1 nor H_2 is trivial. This contradicts Lemma 2.3.3. ■

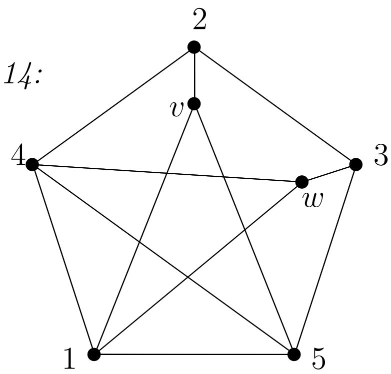
Similarly, U_3 is at most 0-splittable with respect to $J = \{1'', 3, 4, 5\}$.

Figure 13:



when U_2 is $\{\{1'', 5\}, \{3, 4\}\}$-splittable

Figure 14:



Final Step:

By Claim 5 and Proposition 2.3.4(ii), x_2 separates U_2 into four parts $U_2(1')$, $U_2(5)$, $U_2(4)$ and $U_2(3)$ such that $N(U_i) \cap U_2 \subseteq U_2(i)$ for $i \in \{1', 3, 4, 5\}$.

By Claim 3, $N(u_{1''}) \cap A_2 - \{x_2\} \neq \emptyset$. Assume that $N(u_{1''}) \cap A_2 - \{x_2\} \subseteq U_2(1') - x_2$. Then $\{u_{1'}, u_{1''}, x_2\}$ is a 3-separator of G with $U_2(1') \cup U_1$ as a part. Both parts of G separated by $\{u_{1'}, u_{1''}, x_2\}$ contain cycles. This contradicts Lemma 2.3.3. So, there exists a vertex $v \in U_2(5) \cap N(u_{1''}) - \{x_2\}$ since $A_2 = U_2(1') \cup U_2(5)$.

Similarly, there is a vertex $w \in N(u_{1'}) \cap A_3 - \{x_3\}$, which deduces $w \in U_3(4)$. Now we have a $(P_{10})_{\bar{3}}$-minor as in Figure 14.

Chapter 3

4-NZF in regular matroids

3.1 Introduction

We shall assume familiarity with graph theory and matroid theory. For terms that are not defined in this note, see Bondy and Murty [5] for graphs, and Oxley [30] or Welsh [50] for matroids.

In this chapter, $\mathbf{Z}, \mathbf{Z}^+$ and $\mathbf{Z}_n$ denote the additive group of the integers, the set of all positive integers, and the cyclic group of order n, respectively, and $\mathcal{R}$ denotes the family of all regular matroids. As in [30], the set of all circuits of a matroid M is denoted by $\mathcal{C}(M)$. We further denote the set of all cycles of a matroid M by $\mathcal{C}_0(M)$. Note that as we allow empty unions, the empty set is also a cycle (in both graphs and matroids). For matroids $N_1, N_2, \cdots, N_k$, let $EX(N_1, N_2, \cdots, N_k)$ denote the collection of matroids such that a matroid $M \in EX(N_1, N_2, \cdots, N_k)$ if and only if M does not have a minor isomorphic to any one in $\{N_1, N_2, \cdots, N_k\}$. The Fano matroid F_7 is the vector matroid over GF(2) of the following matrix A:

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Flow was initially defined for graphs. For a discussion on flow and flow conjectures, see Jaeger [20] or Zhang [54]. The definition of flow has a natural extension to regular matroids. Let M be a regular matroid and D_M be its incidence matrix of circuits against elements. An **orientation** $(w(D_M), w(D_{M^*}))$ is an assignment of $+, -$ signs to the "1" entries of D_M and D_{M^*}, respectively, so that the resulting matrices $w(D_M)$ and $w(D_{M^*})$ satisfy

$$w(D_M)w(D_{M^*})^T = 0.$$

Let A be an abelian group. For an element $a \in A$, and for integers $+1, -1, 0$, we adopt the convention to write $(+1) \cdot a = a$, $(-1) \cdot a = -a$ and $0 \cdot a = 0$. Let $F^*(M, A) = \{f : E(M) \mapsto A \setminus \{0\}\}$ denote the set of all functions from $E(M)$ into $A \setminus \{0\}$. A map $f \in F^*(M, A)$ can be viewed as an $|E(M)|$-dimensional column vector. For a regular matroid M with an orientation $(w(D_M), w(D_{M^*}))$, a map $f \in F^*(M, A)$ satisfying

$$w(D_{M^*}) \cdot f = 0$$

is a **nowhere zero A-flow** (A-NZF for short) of M. When $A = \mathbf{Z}$, a $\mathbf{Z}$-NZF f of M is called a **nowhere zero k-flow** (k-NZF for short) of M if $\forall e \in E(M)$, $0 < |f(e)| < k$.

For positive integers k and m, an **m-cycle k-cover** of a matroid M is a family of cycles $C_1, C_2, \cdots, C_m$ of M such that every element of $E(M)$ lies in exactly k members of these C_i's. It has been observed that a graph G admits a 4-NZF if and only if G has a 3-cycle 2-cover (for example, see Zhang [54]). The following fact will be needed, a formal proof of it can be found in [27].

Proposition 3.1.1 *(Proposition 1.1 of [27]) For a matroid $M \in \mathcal{R}$, M admits a 4-NZF if and only if M has a 3-cycle 2-cover.*

Let P_{10} be the Petersen graph. Tutte proposed the famous 4-flow conjecture as follows.

Conjecture 3.1.2 *(Tutte [45] and [46], Matthews [28]) Let G be a 2-edge-connected graph. If G does not have a P_{10}-minor, then G admits a 4-NZF.*

The matroid version of the conjecture is as follows

Conjecture 3.1.3 *If M is a coloopless regular matroid such that $M \in EX(M(P_{10}), M^*(K_5))$, then M admits a 4-NZF.*

The Four-Color theorem can be stated in terms of nowhere zero flows as follows.

Theorem 3.1.4 *(Appel and Haken [1], Appel, Haken and Hoch [2], Robertson, Sanders, Seymour and Thomas [31]) Every 2-edge-connected planar graph admits a 4-NZF.*

Recently Robertson et. al. prove Conjecture 3.1.2 for cubic graphs.

Theorem 3.1.5 *(Robertson, Sanders, Seymour and Thomas, [33]) Every 2-edge-connected cubic graph without a minor isomorphic to the Petersen graph admits a 4-NZF.*

Applying the Four-Color theorem, and the duality between colorings and nowhere zero flows, a result by Walton and Welsh implies the following.

Theorem 3.1.6 *(Walton and Welsh [48]) If $M \in EX(M(K_{3,3}), M^*(K_5)) \cap \mathcal{R}$ is a coloopless matroid, then M admits a 4-NZF.*

Proving a conjecture of Jensen and Toft [22], Lai, Li and Poon applied the Four-Color Theorem to prove the following Theorem 3.1.7, which is an approach to Conjecture 3.1.3.

Theorem 3.1.7 *(Lai, Li and Poon, [27]) If $M \in EX(M(K_5), M^*(K_5)) \cap \mathcal{R}$ is a coloopless matroid, then M admits a 4-NZF.*

The main objective of this chapter is to prove the following theorem, which generalizes Theorem 3.1.7, and is also an approach to Conjecture 3.1.3.

Theorem 3.1.8 *If M is a coloopless matroid such that $M \in EX(M((P_{10})_{\bar{3}}), M^*(K_5)) \cap \mathcal{R}$, then M admits a 4-NZF.*

The definition of flow has no natural extension to binary matroids whereas cycle cover is defined for general matroids. In view of Proposition 3.1.1 and the excluded-minor characterization of regular matroids, Theorem 3.1.8 is equivalent to saying that if a coloopless binary matroid $M \in EX(F_7, F_7^*, M((P_{10})_{\bar{3}}), M^*(K_5))$, then M has a 3-cycle 2-cover. In Section 3 we will show that this result can be extended in the following form.

Corollary 3.1.9 *Let M be a coloopless binary matroid. If $M \in EX(F_7^*, M((P_{10})_{\bar{3}}), M^*(K_5))$, M has a 3-cycle 2-cover.*

As the matroid F_7^* does not have a 3-cycle 2-cover (to be shown in Section 3), Corollary 3.1.9 does not hold if F_7^* is not excluded.

In Section 2, we extract a decomposition theorem for regular matroids without $M(K_5)$ or $M^*(K_5)$ minors from the well known decomposition theorems of Seymour [35] and Wagner [47]. In Section 3, this theorem will be employed to prove Theorem 3.1.8 and Corollary 3.1.9.

3.2 Decomposition of Regular Matroids in $EX(M(K_5), M^*(K_5))$

In this paper, we use $\triangle$ to denote both a set operator and a matroid operator. Given two sets X and Y, the symmetric difference of X and Y is defined as

$$X \triangle Y = (X \cup Y) - (X \cap Y).$$

Definition 3.2.1 *Suppose that M_1, M_2 are binary matroids on E_1 and E_2, respectively. We follow Seymour [35] and define a new binary matroid $M_1 \triangle M_2$ to be the matroid with ground set equal to $E_1 \triangle E_2$ and with its set of cycles equal to*

$$\{C_1 \triangle C_2 \subseteq E_1 \triangle E_2 : C_i \text{ is a cycle of } M_i,\ i = 1, 2\}. \tag{3.1}$$

Definition 3.2.2 *Three special cases of this operation are introduced by Seymour ([35] and [37]) as follows.*

(i) If $E_1 \cap E_2 = \emptyset$ and $|E_1|, |E_2| < |E_1 \triangle E_2|$, $M_1 \triangle M_2$ is a **1-sum** *of M_1 and M_2.*

(ii) If $|E_1 \cap E_2| = 1$ and $E_1 \cap E_2 = \{z\}$, say, and z is not a loop or coloop of M_1 or M_2, and $|E_1|, |E_2| < |E_1 \triangle E_2|$, $M_1 \triangle M_2$ is a **2-sum** *of M_1 and M_2.*

(iii) If $|E_1 \cap E_2| = 3$ and $E_1 \cap E_2 = Z$, say, and Z is a circuit of M_1 and M_2, and Z includes no cocircuit of either M_1 or M_2, and $|E_1|, |E_2| < |E_1 \triangle E_2|$, $M_1 \triangle M_2$ is a **3-sum** *of M_1 and M_2.*

For $i = 1, 2, 3$, an i-sum of M_1, M_2 is denoted as $M_1 \oplus_i M_2$. The 1-sum $M_1 \oplus_1 M_2$ is also written as $M_1 \oplus M_2$. Let R_{10} denote the vector matroid of the following matrix over $GF(2)$:

$$R_{10} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

It is known that R_{10}^* is isomorphic to R_{10}. Based on the notion of matroid sums, Seymour proved the following decomposition theorem for regular matroids.

Theorem 3.2.3 *(Seymour [35]) Let M be a regular matroid. One of the following must hold.*

(i) M is graphic.

(ii) M is cographic.

(iii) $M \cong R_{10}$.

(iv) For some $i \in \{1, 2, 3\}$, $M = M_1 \oplus_i M_2$ is the i-sum of two matroids M_1 and M_2, each of which is isomorphic to a proper minor of M.

If a matroid M is isomorphic to the cycle matroid of a planar graph, then M is called a **planar matroid**. Thus a matroid M is planar if and only if M^* is planar. Let H_8

denote the graph depicted in the figure below.

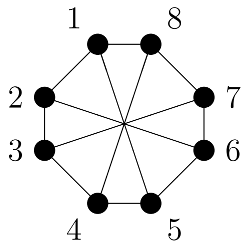


Figure 1: The graph H_8

Wagner's original statement of his decomposition theorem is in pure graph theory terms. A matroid al version is given as follows (see Seymour [35] and [37]).

Theorem 3.2.4 *(Wagner [47]) Let M be a graphic matroid that does not contain a minor isomorphic to $M(K_5)$. One of the following must hold.*

(i) M is a planar matroid.

(ii) $M \cong M(H_8)$.

(iii) $M \cong M(K_{3,3})$.

(iv) For some $i \in \{1, 2, 3\}$, $M = M_1 \oplus_i M_2$ is the i-sum of two matroids M_1 and M_2, such that both M_1 and M_2 are proper minors of M.

Proposition 3.2.5 *(Propositions 4.2.11, 8.3.1 and 12.4.16 of [30]) Each of the following holds:*

(i) The matroid M is not 2-connected, if and only if for some proper non-empty subset T of $E(M)$, $M = (M|T) \oplus (M|(E \setminus T))$. Note that $M|T$ and $M|(E \setminus T)$ are both proper minors of M.

(ii) The matroid M is 2-connected but not 3-connected if and only if $M = M_1 \oplus_2 M_2$ for some matroids M_1 and M_2, each of which is isomorphic to a proper minor of M.

(iii) If M is a 3-connected binary matroid and a 3-sum of M_1 and M_2, then M_1 and M_2 are isomorphic to proper minors of M.

3.3 The Proofs of Theorem 3.1.8 and Corollary 3.1.9

In view of Proposition 3.1.1, we will prove Theorem 3.1.8 by showing that M has a 3-cycle 2-cover given the assumption of the theorem. We first establish some lemmas.

Proposition 3.3.1 *Each of the following holds.*

(i) Each of $M(H_8)$, $M^*(H_8)$, $M(K_{3,3})$, $M^*(K_{3,3})$, R_{10}, F_7 has a 3-cycle 2-cover.

(ii) F_7^* cannot have a 3-cycle 2-cover.

These results follow from known facts about tangential 2-block. See for example the discussion on Tutte's tangential 2-block conjecture in [7]. The results can also be verified directly in a straightforward way.

Lemma 3.3.2 *Suppose that M, M_1, M_2 are binary matroids and that each of M_1 and M_2 has a 3-cycle 2-cover. Then each of the following holds.*

(i) If $M = M_1 \oplus M_2$ is a 1-sum of M_1 and M_2, then M also has a 3-cycle 2-cover.

(ii) If $M = M_1 \oplus_2 M_2$ is a 2-sum of M_1 and M_2, then M also has a 3-cycle 2-cover.

Proof. (i) Suppose that $M = M_1 \oplus M_2$. For $k = 1, 2$, we assume that M_k has a 3-cycle 2-cover, denoted as $C_{k,1}, C_{k,2}, C_{k,3}$. It follows that $\{C_{1,i} \cup C_{2,i} : i = 1, 2, 3\}$ is a 3-cycle 2-cover of M.

(ii) Now assume that $M = M_1 \oplus_2 M_2$. Denote $E(M_1) \cap E(M_2) = \{e\}$. For each $k \in \{1, 2\}$, assume that M_k has a 3-cycle 2-cover, denoted as C_{k1}, C_{k2}, C_{k3}. Note that by the definition of a 2-cover, e appears exactly twice in each set of 3 cycles. Without loss of generality, we may assume that $e \in C_{ki}$, $k, i \in \{1, 2\}$. Now it is easy to verify that $\{C_{11} \triangle C_{22}, C_{12} \triangle C_{23}, C_{13} \triangle C_{21}\}$ is a 3-cycle 2-cover of M. ■

Definition 3.3.3 *Suppose that M_1, M_2 are binary matroids and that each of $Z = \{e_1, e_2, e_3\} = E(M_1) \cap E(M_2)$ is a circuit in both M_1 and M_2. Let $L = M(K_4)$ with $E(L) = \{e_1, e_2, e_3, f_1, f_2, f_3\}$ such that $Z = \{e_1, e_2, e_3\}$ is a circuit of L and $Z' = \{f_1, f_2, f_3\}$ is a cocircuit of L, and such that $\{e_j, f_j\}$ is a perfect matching of K_4, for each $j \in \{1, 2, 3\}$. Define $N_i = M_i \oplus_3 L$, for $i \in \{1, 2\}$.*

With the same notation in Definition 3.3.3, we observe that for each $i \in \{1, 2\}$, if $Z = E(M_i) \cap E(L)$, then $E(N_i) \cap E(L) = Z'$. Moreover, for each $i \in \{1, 2\}$,

$$M_i \oplus_3 L = N_i \text{ and } N_i \Delta L = M_i. \tag{3.2}$$

By Definition 3.3.3, if M_1 and M_2 are coloopless, then N_1 and N_2 are also coloopless. The following is known (need reference).

Lemma 3.3.4 *Let N be a connected binary matroid with $r(N) \geq 4$, and let $Z = \{e_1, e_2, e_3\}$ be a 3-circuit of N. Then for some disjoint subsets $T_1, T_2 \subseteq E(N) - Z$, $(N - T_1)/T_2 \cong K_4$, where Z is a 3-circuit of $(N - T_1)/T_2$.*

Lemma 3.3.5 $M_1 \oplus_3 M_2 = N_1 \Delta N_2$.

Proof. We shall show that both sides have the same set of cycles. By Definition 2.1, for any $C \in \mathcal{C}(M_1 \oplus_3 M_2)$, $C = C_1 \Delta C_2$ with $C_1 \in \mathcal{C}_0(M_1)$, $C_2 \in \mathcal{C}_0(M_2)$ and $C_1 \cap Z = C_2 \cap Z = W$. If $W = \emptyset$, then $C \in \mathcal{C}_0(N_1 \Delta N_2)$, by (1) in Definition 2.1. Similarly, if $W = Z$, then $C_1 \Delta C_2 = (C_1 \Delta Z) \Delta (C_2 \Delta Z) \in \mathcal{C}_0(N_1 \Delta N_2)$. Thus we assume that $2 \geq |W| \geq 1$.

If $|W| = 1$, then without loss of generality, we assume that $W = \{e_1\}$. Thus $C' = \{e_1, f_2, f_3\}$ is a circuit of $L = M(K_4)$, and so by (1), $C_1' = C_1 \Delta C' \in \mathcal{C}_0(M_1 \oplus_3 L) = \mathcal{C}_0(N_1)$, Similarly, $C_2' = C_2 \Delta C' \in \mathcal{C}_0(N_2)$. It follows by (1) that $C_1' \Delta C_2' \in \mathcal{C}_0(N_1 \Delta N_2)$. Since

$$C_1' \Delta C_2' = C_1 \Delta C' \Delta C_2 \Delta C' = C_1 \Delta C_2,$$

it follows that $C_1 \Delta C_2 \in \mathcal{C}_0(N_1 \Delta N_2)$.

Now suppose $|W| = 2$. Then without loss of generality, we assume that $W = \{e_1, e_2\}$. Thus $C'' = \{e_1, e_2, f_2, f_3\}$ is a circuit of $L = M(K_4)$, and so by (1), $C_1'' = C_1 \Delta C'' \in \mathcal{C}_0(M_1 \oplus_3 L) = \mathcal{C}_0(N_1)$, Similarly, $C_2'' = C_2 \Delta C'' \in \mathcal{C}_0(N_2)$. It follows by (1) that $C_1'' \Delta C_2'' \in \mathcal{C}_0(N_1 \Delta N_2)$. Since

$$C_1'' \Delta C_2'' = C_1 \Delta C' \Delta C_2 \Delta C' = C_1 \Delta C_2,$$

it follows that $C_1 \Delta C_2 \in \mathcal{C}_0(N_1 \Delta N_2)$. This proves that $\mathcal{C}_0(M_1 \oplus_3 M_2) \subseteq \mathcal{C}_0(N_1 \Delta N_2)$.

Conversely, pick an arbitrary $D = D_1 \Delta D_2 \in \mathcal{C}_0(N_1 \Delta N_2)$, with $D_i \in \mathcal{C}_0(N_i)$, $i \in \{1, 2\}$. Then $D_1 \cap Z' = D_2 \cap Z' = W'$. Since D_i is a circuit and Z' is a cocircuit, and since N_i is binary, $|W'| \in \{0, 2\}$. If $W' = \emptyset$, then for each i, $D_i \in \mathcal{C}_0(M_i - Z)$ and so by (1), $D_i \in \mathcal{C}_0(M_1 \oplus_3 M_2)$. As $D_1 \Delta D_2$ is a disjoint union of D_1 and D_2, $D_1 \Delta D_2 \in \mathcal{C}_0(M_1 \oplus_3 M_2)$. Thus we assume that $|W'| = 2$. Without loss of generality, we may assume that $W' = \{f_1, f_2\}$. Let $D' = \{f_1, f_2, e_3\}$. Then for $i \in \{1, 2\}$, D' is a circuit in L such that $D'\mathcal{Z}' = \mathcal{W}' = \mathcal{D}_\rangle \mathcal{Z}'$. It follows by (2) that $C_i = D' \Delta D_i$ is a cycle of M_i. Moreover, as

$$C_1 \Delta C_2 = (D' \Delta D_1) \Delta (D' \Delta D_2) = D_1 \Delta D_2,$$

we conclude that $D_1 \Delta D_2 \in \mathcal{C}_0(M_1 \oplus_3 M_2)$. This proves that $\mathcal{C}_0(N_1 \Delta N_2) \subseteq \mathcal{C}_0(M_1 \oplus_3 M_2)$, and so it completes the proof for this lemma. ■

Lemma 3.3.6 *Let $M = M_1 \oplus_3 M_2$ be a 3-connected matroid. With the same notation in Definition 3.3.3, for each $i \in \{1, 2\}$, N_i is a minor of M.*

Proof. By symmetry, it suffices to show that N_1 is a minor of M. Since $M = M_1 \oplus_3 M_2$, and since M is 3-connected, M_2 is also 3-connected (need some modifications, use Seymour [35]). By Lemma 3.4, M_2 has a minor $L \cong M(K_4)$ such that $Z = E(M_1) \cap E(M_2)$ is a 3-circuit of L. It follows that $N_1 = M_1 \oplus_3 L$ is a minor of $M = M_1 \oplus_3 M_2$. ■

Lemma 3.3.7 *Let $M = M_1 \oplus_3 M_2$ be a 3-connected matroid. With the same notation in Definition 3.3.3, if each of N_1 and N_2 has a 3-cycle 2-cover, then M also has a 3-cycle 2-cover.*

Proof. For any $C \in \mathcal{C}(N_1)$, $C = C_1 \Delta C_0$ and $C_1 \cap Z = C_0 \cap Z$ where $C_1 \in \mathcal{C}(M_1)$ and $C_0 \in \mathcal{C}(L)$. Therefore, $C \cap Z' = C_0 \cap Z'$. Since Z' is a cocircuit of L, $C_0 \cap Z' \equiv 0(\text{mod } 2)$, and so $C \cap Z' \equiv 0(\text{mod } 2)$. This implies Z' is a cocircuit of N_1. Similarly, Z' is a cocircuit of N_2. $Z' \in \mathcal{C}(N_1^*) \cap \mathcal{C}(N_2^*)$.

Let $\{C_j^i : j = 1, 2, 3\}$ be a 3-cycle 2-cover of N_i, for $i = 1, 2$. Since $Z' \in \mathcal{C}(N_i^*)$, $|C_i^j \cap Z'| = 2$. Without loss of generality, we can assume that $f_i \notin C_i^j$, then $\{C_i^1 \Delta C_i^2 : i = 1, 2, 3\}$ is a 3-cycle 2-cover of M. ■

proof of Theorem 3.1.8: By way of contradiction, assume Theorem 3.1.8 does not hold. Then there is a matroid $M \in EX(M((P_{10})_{\bar{3}}), M^*(K_5)) \cap \mathcal{R}$ which does not admit a 4-NZF, and $|E(M)|$ is minimum.

Claim 6 *M is 3-connected.*

Otherwise, by proposition 3.2.5, for $i \in \{1, 2\}$, $M = M_1 \oplus_i M_2$ for some proper minors M_1 and M_2. By the minimality of M, both M_1 and M_2 have 3-cycle 2-covers. By Lemma 3.3.2, M has a 3-cycle 2-cover, which contradicts to the choice of M. □

By Claim 6, M can't be 1- or 2-sums. M is not graphic. Otherwise since M is $(P_{10})_{\bar{3}}$-minor free, by theorem 2.1.2, M admits a 4-NZF, a contradiction. $M \ncong R_{10}$ by Proposition 3.3.1. If M is cographic, then $M \in EX(M(K_5))$, and by Theorem 3.1.7, M has a 3-cycle 2-cover.

By Lemma 3.2.3, M is graphic, cographic, R_{10} or $M = M_1 \oplus_i M_2$ for $i = 1, 2, 3$. Therefore, M has to be a 3-sum. Suppose $M = M_1 \oplus_3 M_2$. Follow definition 3.3.3 and Lemma 3.3.6, we get N_i is a minor of M for $i = 1, 2$ and so $N_i \in EX(M((P_{10})_{\bar{3}}), M^*(K_5))$. By the minimality of M, both N_1 and N_2 have 3-cycle 2-covers. By Lemma 3.3.7, M has a 3-cycle 2-cover. A contradiction.

For binary matroids without F_7^* minor, Seymour has established the following decomposition theorem.

Theorem 3.3.8 *(Seymour [37]) Every binary matroid without F_7^* minor may be obtained by means of proper 1-sums or 2-sums from regular matroids and copies of F_7.*

Proof of Corollary 3.1.9: This follows from Proposition 3.3.1, Lemma 3.3.2, Theorem 3.3.8 and Theorem 3.1.8. □

Chapter 4

Odd edge connectivity

4.1 Introduction

It is evident that odd-edge-connectivity (see Definition 4.2.1) plays a more important role than edge-connectivity in the study of some flow and cycle cover related problems. In this paper, some earlier results in those areas are extended from λ-edge-connected graphs to odd-λ-edge-connected graphs.

For graphs with large odd-edge-connectivity, small edge-cuts (of even size) may still exist. However, there are not many results or methods developed yet to deal with small even edge-cuts. For some integer flow problems and cycle cover problems, it is pointed out in [36, 21] that 2-edge-cut does not exit in any smallest counterexample (to some well-known flow conjectures and cycle cover conjectures). The 3-flow conjecture by Tutte [**?**] was originally proposed for odd-5-edge-connected graphs. By excluding 4-edge-cut, Kochol [23], with a sophisticated linear algebra approach, proved that 3-flow conjecture is equivalent for 5-edge-connected graphs.

In this chapter, we are to develop some general approaches to deal with small even cuts. For small even degree vertices, the vertex splitting method is to be applied and the odd-edge-connectivity is to be preserved. For non-trivial small even cuts, contractions of internal highly connected k-tree blocks (as contractible configuration) are to be applied. Note that the determination and verification of contractible configurations (see Definition 4.2.8) differ significantly from each other if the corresponding graph theory properties are different. In this paper, k-tree blocks (see Definition 4.2.3) is to be verified as contractible configurations for some graph properties. The contractibility is either to be verified independently for each problem, or to be proved by applying some existing results (such as, a recent theorem by Barát and Thomassen [6]).

4.1.1 Parity Subgraphs

Definition 4.1.1 *Let H be a subgraph of a bridgeless graph $G = (V, E)$. H is a **parity subgraph** of G if for every vertex $v \in V(G)$, $d_G(v) \equiv d_H(v) (mod\ 2)$.*

It was proved by Tutte and Nash-Williams [43, 29] that every $2k$-edge-connected graph contains at least k edge-disjoint spanning trees, and proved by Itai and Rodeh [18] that every spanning tree of a graph G contains a parity subgraph. The combination of these two theorems yields the following result as a direct corollary.

Theorem 4.1.2 *Every $2k$-edge-connected graph G contains at least k edge-disjoint parity subgraphs of G.*

It is well-known that the search of parity subgraphs plays a central role in the proofs of some important theorems in integer flow areas. For example, the 4-flow theorem is proved by Jaeger [19] with following approach: The 4-edge-connectivity guarantees the existence of two edge-disjoint parity subgraphs (by Theorem 4.1.2) and therefore a 2-cycle cover of G. The 8-flow theorem was proved by Jaeger [19] with following similar approach: The 3-edge-connectivity guarantees the existence of three edge-disjoint parity subgraphs in $2G$ and therefore, a 3-cycle cover of G.

Theorem 4.1.2 is to be generalized in this paper by relaxing the edge-connectivity to odd-edge-connectivity and therefore, solves an open problem proposed in [51, 12, 54].

We generalize the theorem and get the following

Theorem 4.1.3 *Every odd-$(2k+1)$-edge-connected graph G contains at least k edge-disjoint parity subgraphs of G.*

4.1.2 Flow Index

Integer flow was originally introduced by Tutte [40, 41] as a generalization of map coloring problems.

Definition 4.1.4 *Let $G = (V, E)$ be a graph. An ordered pair (D, f) is called an* ***integer flow*** *of G if D is an orientation of $E(G)$ and $f : E(G) \to Z$, the set of integers, such that the total in-flow equals the total out-flow at every vertex. An integer flow (D, f) is a* ***k-flow*** *if $|f(e)| \leq k-1$ for every edge e of G. It is* ***nowhere-zero*** *if $f(e) \neq 0$ for every edge e of G.*

Circular flow, introduced in [16] as a real line extension of integer flow problem. The following is one of the many equivalent definitions for circular flows and the corresponding flow indices.

Definition 4.1.5 *Let $\mathcal{D}$ be the set of all orientations of G, and (A, B) be any edge-cut of G. The* ***flow index*** *of G is defined by*

$$\phi(G) = \min_{D \in \mathcal{D}} \left\{ \max_{(A,B)} \frac{|[A, B]_D|}{|[B, A]_D|} \right\} + 1.$$

Theorem 4.1.6 (*Galluccio and Goddyn, [15]*) *let G be a 6-edge-connected graph, then the flow index $\phi(G) < 4$.*

We generalize the theorem above and get:

Theorem 4.1.7 *Let G be an odd-7-edge-connected graph, then $\phi(G) < 4$.*

4.1.3 Nowhere-zero 3-flows

The following is an approach of Jaeger's weak 3-flow conjecture [19]. Lai and Zhang proved the following theorem

Theorem 4.1.8 ([*24*]) *Every* $4\lceil \log_2 n \rceil$*-edge-connected multigraph with* n *vertices admits a nowhere-zero 3-flow.*

As a generalization, we prove

Theorem 4.1.9 *Let* G *be a multigraph with* n *vertices. If its odd-edge-connectivity is more than* $4\lceil \log_2 n \rceil$*, then* G *admits a nowhere-zero* 3*-flow.*

4.2 Notations and Lemmas

Note: For notations not defined here, see [**?**] or [14].

A circuit is a connected 2-regular subgraph, while a cycle is the union of edge-disjoint circuits.

Let G be a an undirected graph. Let $X \subseteq V(G)$, the set of all edges between X and $Y = V(G) - X$, denoted by (X, Y), is an edge-cut of G. If G is a directed graph under an orientation, then the set of arcs from X to Y is denoted by $[X, Y]_D$.

Definition 4.2.1 *A graph* G *is said to be* ***odd-***$(2k+1)$***-edge-connected*** *provided the size of every odd edge cut is at least* $2k+1$*. The* ***odd-edge-connectivity*** *of* G*, denoted by* $\lambda_o(G)$*, is the size of the smallest odd edge-cut of* G*.*

Tutte proposed the 3-flow conjecture [**?**] that every odd-5-connected graph admits a nowhere-zero 3-flow. Later Jaeger([19]) weakened the conjecture and proposed the weak

3-flow conjecture that there is an integer k such that every k-edge-connected graph admits a nowhere-zero 3-flow. Also, Jaeger proved the 4-flow theorem, which is the best approach to the 3-flow conjecture so far.

Definition 4.2.2 *Let D be an orientation of a graph G, let Γ be an Abelian group and let $f : D \to \Gamma$ ba a map. The **boundary** of f is the map $\partial f : V(G) \to \Gamma$ where $\partial f(v) = \sum_{e \in E^+(v)} f(e) - \sum_{e \in E^-(v)} f(e)$ for each vertex $v \in V(G)$. G is said to be Γ-**connected** if for every $b : V(G) \to \Gamma$ with $\sum_{v \in V(G)} b(v) = 0$, there exists a nowhere-zero map $f : D \to \Gamma$ with boundary $\partial f = b$.*

Definition 4.2.3 *Let H be a subgraph of a graph $G = (V, E)$. H is said to be a k-**tree block** provided H is a maximal subgraph with k edge-disjoint spanning trees.*

The following is the key lemma of this paper and will be used in the proof of theorems 4.1.3, 4.1.7 and 4.1.9.

Lemma 4.2.4 *[14, 52] Let $\{T_1, T_2, \cdots, T_k\}$ be a set of edge-disjoint spanning forests of a graph G of maximum total size. If there is an edge $e \in E(G) - \cup_{i=1}^{k} E(T_i)$, then there is a k-tree block H of G containing e.*

By counting the numbers of edges needed for k edge-disjoint spanning trees, we can easily get the following corllary.

Corollary 4.2.5 *If $\delta(G) \geq 2k$, then there is a non-trivial k-tree block of G.*

A polynomial algorithm was obtained in [52] for the detection of all k-tree-blocks of a graphs.

Lemma 4.2.6 *[52] Let G be a graph without k edge-disjoint spanning trees, and let H_1, H_2, $\cdots$, H_t be all k-tree blocks of G. Then*
(i). H_i and H_j are disjoint if $i \neq j$;
(ii). $G' = G/\{H_1, H_2, \cdots, H_t\}$ does not have k edge-disjoint spanning trees.

Lemma 4.2.7 *[53] Let $G = (V, E)$ be a graph with odd edge connectivity λ_o. Assume that there is a vertex $v \in V(G)$ with degree d such that $d(v) \notin \{2, \lambda_o\}$. Arbitrarily label the edges incident with v as $\{e_1, e_2, \cdots, e_d\}$, then there is an integer $i \in \{1, 2, \cdots, d\}$ such that the odd edge connectivity of the new graph G' obtained from G by splitting e_i and e_{i+1} (mod b) away from v remains λ_o.*

Definition 4.2.8 *A graph H is a* ***contractible configuration for property*** *$\mathcal{P}$ if for any graph G containing H as a subgraph, G has property $\mathcal{P}$ if and only if G/H has the property.*

4.3 Proof of Theorem 4.1.3

The following lemma is straightforward.

Lemma 4.3.1 *A k-tree block is a contractible configuration for having k edge-disjoint spanning trees.*

Here, we can go further to generalize Lemma 4.3.1 for the packing problem of parity subgraphs.

Lemma 4.3.2 *A k-tree block is a contractible configuration for having k edge-disjoint parity subgraphs.*

The following definition and lemma are needed in the proof of Lemma 4.3.2.

Definition 4.3.3 *Let $\vec{T} : V(G) \to Z_2^k$ be a zero-sum mapping. A* ***$\vec{T}$-subgraph packing*** *is a set of edge-disjoint subgraphs $\{P_1, P_2, \cdots, P_k\}$ such that $d_{P_i}(v) \equiv T_i(v)$ (mod 2) where $T_i(v)$ is the i^{th} component of the vector $\vec{T}(v)$.*

Lemma 4.3.4 *If G contains k edge-disjoint spanning trees $T_1, \cdots, T_k$, then for any zero-sum mapping $\vec{T} : V(G) \to Z_2^k$, G has a $\vec{T}$-subgraph packing $\{P_1, \cdots, P_k\}$ such that $P_i \subseteq T_i$.*

Proof. Let $T_1, \cdots, T_k$ be k edge-disjoint spanning trees. For every i, let

$$S_i = \{v \in V(G) \mid \vec{T}_i(v) \equiv 1\}(\mod 2)$$

Note that $|S_i|$ is even. Partition S_i into pairs. For each pair of vertices, there is a path in T_i. Let P_i be the symmetric difference of these paths, then $\{P_1, \cdots, P_k\}$ is a $\vec{T}$-subgraph packing that $P_i \subseteq T_i$. ■

Proof of Lemma 4.3.2. Suppose a subgraph H of G is a k-tree block. If G has k edge-disjoint parity subgraphs P_1, P_2, $\cdots$, P_k, then P_1/H, P_2/H, $\cdots$, P_k/H are k edge-disjoint parity subgraphs of G/H.

Suppose G/H has k edge-disjoint parity subgraphs P'_1, P'_2, $\cdots$, P'_k. For each $i \in \{1, 2, \cdots, k\}$, let $S_i = \{v \in V(G) : d_{G \setminus E(P'_i)}(v)$ is odd $\}$ and $T_i(v) = 1$ if $v \in S_i$ and 0 otherwise. By Lemma 4.3.4, there is a $\vec{T}$-subgraph packing $P''_1, \cdots, P''_k$ of H. Now $P'_1 \cup P''_1$, $P'_2 \cup P''_2$, $\cdots$, $P'_k \cup P''_k$ are k edge-disjoint parity subgraphs of G. ■

Proof of Theorem 4.1.3:

By way of contradiction, we assume G is the minimum counter example with respect to the cardinality of edges, and then the cardinality of vertices.

If G itself is a k-tree-block, then, by an observation in [18] (Itai and Rodeh), each spanning tree contains a parity subgraph. So, the minimum counterexample G is not a k-tree-block.

We claim that

$$\delta(G) \geq 2k \tag{4.1}$$

Otherwise, there is a vertex v with even degree at most $2k - 1$. By Lemma 4.2.7, we can split a pair of edges away from v, and the resulting graph G' remains odd-$(2k + 1)$-edge-connected. By the minimality of G, G' has k edge-disjoint parity subgraphs P_1, P_2, $\cdots$, P_k. They are also parity subgraphs of G, which contradicts the choice of G.

By the inequality (4.1) and Corollary 4.2.5, there is a nontrivial k-tree block H of G. Since G/H has less edges and satisfies the conditions of the theorem, G/H has k

edge-disjoint parity subgraphs P'_1, P'_2, $\cdots$, P'_k. By Lemma 4.3.2, G has k edge-disjoint parity subgraphs. This is a contradiction and completes the proof of the theorem. ■

Corollary 4.3.5 *Every odd-$(2k+1)$-connected graph G has a $(2\lfloor \frac{k}{2} \rfloor + 1)$ parity subgraph decomposition.*

Proof. Let P_1, P_2, $\cdots$, P_k be edge-disjoint parity subgraphs and $P = G \setminus \cup_{i=1}^{k} E(P_i)$. If k is even, then P is also a parity subgraph of G, thus we have $k+1$ edge-disjoint parity subgraphs. ■

4.4 Proof of Theorem 4.1.7

The following lemma describes the relation between integer-valued flows and orientations.

Lemma 4.4.1 [*17*]) *Let G be a bridgeless graph and D be an orientation of G and a, b be two positive integers $(a < b)$. The following statements are equivalent.*

(1)

$$\frac{a}{b} \leq \frac{|[A, B]_D|}{|[B, A]_D|} \leq \frac{b}{a}$$

for every edge-cut (A, B) of G;

(2) G admits a nowhere-zero integer flow (D, f_1) such that $a \leq f_1(e) \leq b$ for every $e \in E(G)$.

Before the proof of Theorem 4.1.7, we are to obtain some structural characterizations about graphs G with $\phi(G) = 4$.

Definition 4.4.2 *For an edge-cut $Q = \{e_1, e_2, \cdots, e_q\} = (X, Y)$ and a nowhere-zero 4-flow (D, f) of G. A **balanced partition** of Q is a partition $Q = P_1 \cup \cdots \cup P_t$ such* *that*

$$\sum_{e\in[X,Y]_D\cap P_i} f(e) = \sum_{e\in[Y,X]_D\cap P_i} f(e), \qquad \forall P_i.$$

If a balanced partition of Q has the most number of parts among all balanced partitions, it is called a ***finest balanced partition****. The cut Q is* ***bad with respect to*** *(D,f) if every part P_i of a finest balanced partition is of size 4.*

Lemma 4.4.3 *For a positive 4-flow (D,f), an edge-cut (X,Y) is bad with respect to (D,f) if and only if*

$$\frac{|[X,Y]_D|}{|[Y,X]_D|} = 3 \text{ or } \frac{1}{3}.$$

Proof. The proof is straightforward since (D,f) is a positive flow and $[X,Y]_D \cap P_i \neq \emptyset$ and $[Y,X]_D \cap P_i \neq \emptyset$ for every part P_i of a finest balanced partition of (X,Y). ■

Lemma 4.4.4 *Let G be a graph admitting a nowhere-zero 4-flow. Then $\phi(G) = 4$ if and only if, for every nowhere-zero 4-flow (D,f) of G, there is a bad cut with respect to (D,f).*

Proof. I. Note that every nowhere-zero 4-flow can be converted to a positive 4-flow by changing signs and reversing orientations of some edges. Hence, we are to prove the lemma for positive flows since the operation described above does not affect finest balanced partition of any edge-cut.

II. "$\Rightarrow$": Let (D_1, f_1) be a nowhere-zero 4-flow, and (D,f) be the corresponding positive 4-flow. By Lemma 4.4.1,

$$\frac{|[B,A]_D|}{|[A,B]_D|} \leq 3$$

for every edge-cut (A,B).

If $\phi(G) = 4$, then $\phi(G) \not< 4$. By Definition 4.1.5, there must be an edge cut $Q = (X,Y)$ with

$$\frac{|[Y,X]_D|}{|[X,Y]_D|} \geq 3.$$

Therefore, the equality must hold for $Q = (X, Y)$. Since (D, f) is a positive 4-flow of G, by Lemma 4.4.3, (X, Y) is a bad cut with respect to (D, f).

III. "$\Leftarrow$": Prove by contradiction. Assume that $\phi(G) < 4$. Then, by Definition 4.1.5,

$$\frac{1}{3} < \frac{|[X,Y]_{D_2}|}{|[Y,X]_{D_2}|} < 3 \tag{4.2}$$

for every edge-cut (X, Y) of G under some orientation D_2. Furthermore, by Lemma 4.4.1, there is a positive 4-flow (D_2, f_2) that agrees with the orientation D_2. Because of (4.2), G has no bad-cut with respect to (D_2, f_2) (by Lemma 4.4.3). This is a contradiction and completes the proof. ■

By Definition 4.4.2, we notice that if an edge cut $Q = (X, Y)$ is not bad with respect to a *nowhere-zero* 4-flow (D, f), then some part P_i of a finest balanced partition of (X, Y) is of size less than 4. Therefore, either there is an edge $e \in P_i$ with $f(e) = 2$ (if $|P_i| = 3$) or there are a pair of edges $e_1, e_2 \in P_i$ with $f(e_1) = f(e_2)$ (if $|P_i| = 2$). Some of these observations are to be use very frequently in later studies.

Lemma 4.4.5 *A cut Q is not bad with respect to a nowhere-zero 4-flow (D, f) if one of the following holds:*
(1). $Q \cap E_{f=\pm 2} \neq \emptyset$;
(2). $|Q| \not\equiv 0 \ (mod\ 4)$

The following lemma is an immediate corollary of Lemma 4.4.4 and is to used in the proof of the main theorem whenever a vertex splitting occurs.

Lemma 4.4.6 *If G' is a graph obtained from G by splitting a pair of incident edges, then $\phi(G') < 4$ implies that $\phi(G) < 4$.*

Proof of Theorem 4.1.7:

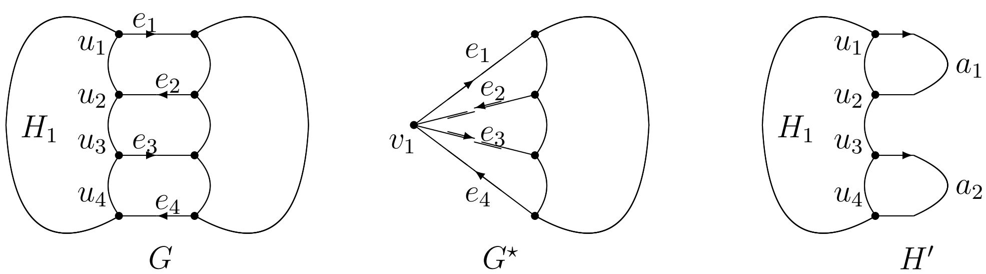
I. By way of contradiction, let G be an odd-7-edge-connected graph such that $\phi(G) \geq 4$ with least number of edges and vertices. By Corollary 4.3.5, G has three edge-disjoint parity subgraphs. Therefore, by a theorem of Jaeger [19], it admits a nowhere-zero 4-flow (D, f). That is, $\phi(G) = 4$.

II. We claim that the graph G itself is not a 3-tree-block. For otherwise, let T_1, T_2, T_3 be three edge-disjoint spanning trees of G. Let C_{e,T_j} be the circuit contained in $T_j + e$ for each edge $e \notin T_j$. Let $C_{1,2} = \triangle_{e \notin T_3} C_{e,T_3}$ and $C_3 = \triangle_{e \in T_3} C_{e,T_2}$. It is easy to see that $T_1 \cup T_2 \subseteq C_{1,2}$ and $T_3 \subseteq C_3$ and $T_1 \cap C_3 = \emptyset$.

For an arbitrary orientation D, let $(D, f_{1,2})$ be a 2-flow of G with support $C_{1,2}$ and (D, f_3) be a 2-flow of G with support C_3. Here, $(D, f = 2f_{1,2} + f_3)$ is a nowhere-zero 4-flow with $T_1 \subseteq E_{f=\pm 2}$. By Lemma 4.4.5, $\phi(G) < 4$. This contradicts that G is a conterexample.

III. We claim that $\delta(G) \geq 7$. Assume that there is a vertex v with degree at most 6. By Lemma 4.2.7, we can split a pair of edges away from v to get a smaller odd-7-edge-connected graph G'. By the minimality of G, $\phi(G') < 4$. By Lemma 4.4.6, $\phi(G) < 4$ as well.

IV. By II, III and by Corollary 4.2.5, G contains some non-trivial k-tree blocks. Let $\{H_1, H_2, \cdots, H_a\}$ be the collection of all k-tree blocks of G. By Lemma 4.2.6, $G^\star = G/\{H_1, \cdots, H_a\}$ does not contain k edge-disjoint spanning trees. By Corollary 4.2.5, $\delta(G^\star) = 2$ or 4.



V. By IV, let v_1 be a vertex in $G^\star$ with degree either 2 or 4. Let H_1 be the nontrivial k-tree block of G corresponding to v_1 (the small degree vertex v_1 is created by contraction of H_1 since $\delta(G) \geq 6$). Since the smaller graph $G^{\star\star} = G/H_1$ remains odd-7-edge-connected, its flow index is less than 4. Let (D_0, f_0) be a positive 4-flow of G/H_1 with no bad cuts. It is easy to see that the size of each part of a finest balanced partition of $E(v_1)$ is 2 since the edge-cut $E(v_1)$ is of size 2 or 4. Without loss of generality, suppose e_1, $e_3 \in E^+(v_1)$, e_2, $e_4 \in E^-(v_1)$ under the orientation D_0. And let u_i be the endvertex of e_i in H_1, for every $i = 1, 2, 3, 4$. Also, suppose $f_0(e_1) = f_0(e_2) = w_1$, $f_0(e_3) = f_0(e_4) = w_2$ with $w_1 \geq w_2 \geq 0$. (For technical reasons, if $|(V(H_1), V(G) - V(H_1)| = 2$, then $w_2 = 0$ since e_3, e_4 do not exist.)

Let H' be the graph obtained from H_1 by adding two arcs a_1 and a_2 that a_1 joins u_2 to u_1 and a_2 joins u_4 to u_3. Let

$$A_1 = \{a_i \big| w_i \text{ is odd, } i = 1, 2\} \text{ and } A_2 = \{a_i \big| w_i \geq 2, i = 1, 2\}.$$

VI. This is the final step of the proof. In this part, we are to extend the flow (D_0, f_0) of $G^{\star\star} = G/H_1$ to the entire graph G by finding a 4-flow (D', f') of H' that agrees with a_1 and a_2 in both orientations and weights. Certainly, the 4-flow (D', f') of H' should not have any bad cut in H'.

Since H_1 is a 3-tree block, it has three spanning trees T^1, T^2 and T^3. For every edge $e \in (E(H) - E(T^1) - E(T^3)) \cup A_1 = B$, there is a circuit C_{e,T^3} in $T^3 \cup e$. Let C^2 be the symmetric difference of these circuits, that is,

$$C^2 = \Delta_{e \in B} C_{e,T^3}$$

Here $E(C^2) \supseteq E(T^2) \cup A_1$ and $E(C^2) \cap E(T^1) = \emptyset$.

Let $M = (T^3 - C^2) \cup A_2$. For every edge $e \in M \cup T^1$, there is a fundamental circuit $C_{e,T^2} \subseteq T^2 \cup e$. Let C^1 be the symmetric difference of these circuits

$$C^1 = \Delta_{e \in T^1 \cup M} C_{e,T^2}$$

Here $E(C^1) \supseteq E(T^1) \cup A_2$, and $C^1 \cup C^2 = E(H')$.

Since C^i $(i = 1, 2)$ contains a spanning tree T^i of H, $C^i \setminus \{a_1, a_2\}$ is connected and, therefore, C^i has a circuit decomposition that a_1 and a_2 are in different circuits if both $a_1, a_2 \in C^i$. Therefore, we are able to extend the orientation of $\{a_1, a_2\}$ to all edges of

$E(C^i)$ so that each member of the circuit decomposition of C^i is oriented independently as a directed circuit (and agrees with a_i if it contains a_i). Let D^i be this eulerian orientation of the cycle C^i. We further notice that D^1 and D^2 may disagree with each other in the intersection of C^1 and C^2 (however, D^1 and D^2 agree with each other on a_1 and a_2 if any of them is contained in the intersection of C^1 and C^2).

Let (D^i, f^i) be a non-negative 2-flow of H' with support C^i for each $i = 1, 2$.

Let D^3 be the orientation of H' obtained from D^1 and D^2 that preserves the orientation on C^1 and $C^2 \setminus C^1$. Note that the orientation of a_1, a_2 remain the same since each D^i agrees with each a_j if it is in C^i. Let (D^3, f^2) be the 2-flow obtained from (D^2, f^2) by reversing orientations and changing signs for negative edges in the intersection of C^1 and C^2.

Then $(D^3, f^3 = 2f^1 + f^2)$ is a positive 4-flow and the weight of every edge in T^1 is 2 since $C^2 \cap T^1 = \emptyset$.

Now, the positive 4-flow (D_0, f_0) of $G^{\star\star} = G/H_1$ can be extended to a positive 4-flow (D, f) of the entire graph G that agrees with both (D_0, f_0) in G' and (D^3, f^3) in H_1.

We only need to show G does not have any bad cut with respect to (D, f) (by Lemma 4.4.4). For any cut Q of G, if $Q \cap E(H_1) = \emptyset$, then it is an edge cut of $G^{\star\star}$, so it is not bad. If $Q \cap E(H) \neq \emptyset$, then $Q \cap H_1$ is an edge-cut of H_1. Note that T^1 is a spanning tree of H_1 and $T^1 \subseteq E_{f=2}$. Hence, there is an edge $e \in Q \cap T^1 \subseteq E_{f=2}$. By Lemma 4.4.5, the flow index of G is less than 4.That contradicts the choice of G. ■

4.5 Proof of Theorem 4.1.9

Lemma 4.5.1 [*13*, *25*] *Let H be a Z_3-connected subgraph of a graph G. Then H is a contractible configuration for having a nowhere-zero 3-flow.*

The following two theorems was proved by Barát and Thomassen, and we will use the second one in our proof. Note these two theorems are also generalizations of Theorem 4.1.8.

Theorem 4.5.2 [6] *Every* $4\lceil \log_2 n \rceil$*-edge-connected multigraph with* n *vertices is* Z_3*-connected.*

Theorem 4.5.3 [6] *Let* G *be a multigraph with* n *vertices. If* G *has* $2\lceil \log_2 n \rceil$ *edge-disjoint spanning trees, then* G *is* Z_3*-connected.*

Proof of Theorem 4.1.9:

By way of contradiction, suppose G is the minimum counterexample with respect to order and size. Let $\lambda_o(G) = 2k + 1$.

Claim: $\delta(G) > 2k$**.**

Otherwise, suppose $d_G(v) \leq 2k$. By Lemma 4.2.7, we can split 2 edges away from G to get G' where G' is still odd-$(2k+1)$-connected. By the minimality of G, G' admits a nowhere-zero 3-flow, so does G.

By the Claim above and Corollary 4.2.5, there is a nontrivial k-tree block H of G. Note that G/H is still odd-$(2k+1)$-connected, so it admits a nowhere-zero 3-flow.

Since H has k edge-disjoint spanning trees where $k > 2\lceil \log_2 n \rceil \geq 2\lceil \log_2 |V(H)| \rceil$, by Theorem 4.5.3, H is Z_3 connected. By Lemma 4.5.1, G admits a nowhere-zero 3-flow. A contradiction.

4.6 Remarks

Note that both Theorems 4.1.7 and 4.1.9 were proved by verifying that k-tree-blocks are contractible configurations for these two problems. But we proved Theorem 4.1.7 without proving a similar lemma that every 3-tree-block is a contractible configuration for $\phi < 4$ problem. We took the advantage of a small value $\delta(G')$ and paid attention only on (at most) two extra edges of a 3-tree-block H_1 instead of all possible zero-sum boundary. This approach does simplify the proof of Theorem 4.1.7. However, the problem that every 3-tree-block is a contractible configuration for $\phi < 4$ flows remains a very interesting open problem.

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