2001

Static and dynamic inventory models under inflation, time value of money and permissible delay in payment

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Static and Dynamic Inventory Models Under Inflation, Time Value of Money and Permissible Delay in Payment

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Thesis submitted to the
College of Engineering and Mineral Resources
at West Virginia University
in partial fulfillment of the requirements
for the degree of

Master of Science
in
Industrial Engineering

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2001

Keywords: Inventory Control, Nonlinear Optimization, Permissible Delay in Payment, Inflation, Time Value of Money

Abstract

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In this research a number of mathematical models were developed for static and dynamic deterministic single-item inventory systems. Economic factors such as inflation, time value of money and permissible delay in payment were considered in developing the models. Nonlinear optimization techniques were used to obtain the optimal policies for the systems.

First, a static single-item inventory model was considered in which shortages are allowed and a delay is permitted in payment. In this case, suppliers allow the customers to settle their accounts after a fixed delay period during which no interest is charged.

An extension of the model was then considered in which all cost components of the model are subject to inflation and discounting, with constant rates over the planning horizon. The mathematical model of the system was developed and a nonlinear optimization technique, Hooke and Jeeves search method, was used to obtain the optimal policies for the system.

A dynamic deterministic single-item inventory model was also considered in which the demand was assumed to be a linear function of time. Suppliers allow for a delay in payment and the cost components are subject to inflation and discounting with constant rates and continuous compounding. The Golden search technique was used to obtain the optimum length of replenishment cycle such that the total cost is minimized.

Computer applications using Visual Basic and *Mathematica* were developed and several numerical examples were solved.

Dedication

I would like to dedicate this thesis to my parents. Without their support and encouragement throughout my entire academic career I would not be where I am today.

Acknowledgment

As I look back over the course of my graduate study, I am truly thankful to many people who have each helped me during my time at WVU.

I wish to thank my advisor, Dr. Wafik Iskander, for his advice, encouragement and patience. I am grateful for the enormous time he spent reviewing every single page of the thesis and providing constructive revision.

I also want to thank Dr. Alan McKendall and Dr. Ralph Plummer for serving as members of my committee and for their help and support.

I would also like to extend my appreciation to Dr. Majid Jaraiedi for his support during my graduate study. I always enjoyed his guidance and advice.

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Chapter One

Introduction

In today's highly competitive market, production and financial decisions are completely related to each other and must be made simultaneously. Everyday, new products are introduced to the market with different life cycles and particular demand patterns. This, together with sensitive customers' expectations, has forced manufacturers, suppliers and business firms to focus on their supply chains. In a classic supply chain, raw materials are ordered by the manufacturer and shipped to temporary warehouses before further process. The final products are then produced and shipped to intermediate warehouses in order to be shipped to customers. To minimize the cost of the system and maintain a high service level, an effective supply chain strategy must take into account the interrelation between financial and production/inventory decisions at different levels of the supply chain.

The effect of inflation and time value of money on the determination of the total cost of inventory systems cannot be ignored. Inflation has become an embedded aspect of the economy, all over the world. The inventories of raw materials and products are capital investments of production plants and must compete from financial point of view, with the other assets for the firm's limited funds. The effect of inflation and time value of money should be considered in developing the proper mathematical representations of production and inventory costs of the supply chain.

There are other financial issues, involved with managing the flow of inventory over the supply chain, which affect the optimal policy of production and procurement for the system. For example in developing the mathematical models for different inventory systems, it is always assumed that the supplier is paid as soon as the items are received by the system. In practice, however, this is not always true. The suppliers often offer their customers a fixed period of delay in payment. This grace period allows the customer to settle the account for payment of the amount owed to the supplier without charged interest. However, a relatively high interest rate is charged if the payment is not settled by the end of the grace period.

From the customer standpoint, this grace period for settling the account can be considered as a loan from the supplier without paying interest. On the other hand, receiving this credit from the supplier will stimulate the demand, which is one of the major goals of the supplier.

In the next sections the methods developed for the analysis of inventory systems are presented for several single item inventory models.

1.1. Inventory Systems

In this section, general methods used for the analysis of inventory problems are reviewed. Inventory can be defined as the accumulation of a commodity that will be used to satisfy some future demand for that commodity. The commodity could be raw material, purchased parts, semi-finished products, finished products in manufacturing, spare parts in maintenance operations, purchased products in retailing, or purchased supplies in service operations.

A schematic way to describe an inventory system is considered in this section. Consider the system shown in Figure1.1 where the inventory level of an item, is affected by an input process and an output process. Let $P(t)$ be the rate at which material is added to inventory at time t and $W(t)$ be the rate at which material is withdrawn from inventory. Usually, it is assumed that the output is being withdrawn to satisfy a demand, with rate $D(t)$. It is assumed that $D(t)$ is not a controllable variable. The output rate will equal the demand rate unless the inventory is depleted and in this case it is said to be in an Out-of-stock or stock-out condition.

The input process is partially under control in that it could be decided when and how much to order from the sources of supply. Because of variable time delays in the supplier filling the orders, the actual input rate, $P(t)$, may be different from the desired one.

The state of the inventory system may be described by variables such as the following:

$I(t)$ = the on-hand inventory level at time t

$B(t)$ = the backorder level at time t

$O(t)$ = the on-order position at time t

$N(t)$ = the net inventory at time t

$X(t)$ = the inventory position at time t

The on-hand inventory is the quantity of material in stock at a given time. When the inventory is out of stock, that is $I(t) = 0$, any demand that occurs is considered to be a shortage. Some of this demand may be (backordered), i.e. it is accumulated and is to be satisfied as soon as possible.

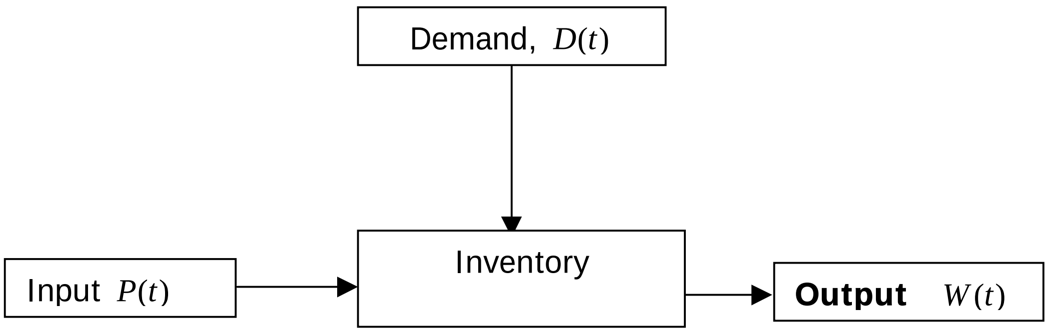


Figure 1.1: An inventory system

The net inventory is defined as the on-hand inventory level minus the backorder level

$$N(t) = I(t) - B(t) \tag{1.1}$$

The inventory position of the system is defined as the net inventory plus the on-order quantity, $O(t)$.

$$X(t) = N(t) + O(t) = I(t) - B(t) + O(t) \tag{1.2}$$

With the assumption that all shortages are backordered, expressing the state of the system in terms of the input, output and demand rate, one should have

$$N(t) = N(0) + \int_0^t [P(u) - D(u)]du \tag{1.3}$$

it should also be noted that

$$I(t) = \max[0, N(t)] \tag{1.4}$$

$$B(t) = \max[0, -N(t)] \tag{1.5}$$

and

$$W(t) = \begin{cases} D(t), & if \quad N(t) > 0 \\ Min\{D(t), P(t)\} & if \quad N(t) = 0 \\ P(t) & if \quad N(t) < 0 \end{cases} \tag{1.6}$$

When to order and how much to order are the basic decisions in an inventory system, and the answers to these questions determines the inventory policy of the system.

To define and solve inventory problems, the following issues must be considered:

Definition of the controllable decision variables in the system, which identify the ordering policy. For example, the desired policy might be to order Q units when the inventory position drops to r units. The decision variables are Q and r. In another situation the desired policy might be to order Q units and have b as the maximum backorder level permitted.

The effectiveness of the system must be measured based on an appropriate measure, which is normally the relevant costs and revenues of the inventory system.

A mathematical model must be constructed to express the measure of effectiveness properly. The value of the effectiveness measure usually varies with different alternatives.

1.2 Measures of Effectiveness

Certain revenues and costs are involved in inventory systems. The analysis performed to determine a good policy involves identification of the relevant economic factors and construction of a mathematical model to show how they are related to the decision variables in the inventory policy. The revenue and cost parameters then must be estimated from accounting or other sources. For a given product, the revenue in a period of time will be a function of the inventory provided and the demand realized.

The major cost components include procurement, inventory holding, shortage, and system operating costs. Procurement costs, both in purchasing and production situations, consist of a component that is independent of the procurement lot size and a component that varies with the lot size. The former is the fixed cost per lot, sometimes called the “ordering cost” in purchasing or the “setup cost” in production. To purchase a lot requires processing of purchase orders, receiving reports, accounting records, and so on, as well as fixed expenses in physically transporting, receiving and storing the lot. If the lot is to be produced instead of purchased by the firm, then the setup cost includes the paperwork processing costs and costs associated with preparing machines, equipment, and workers to produce this particular product and changing the setup after the lot has been produced. These costs will be incurred regardless of the size of the lot. The variable costs per lot depend on the price schedule used by the supplier or on the variable production costs in manufacturing.

In modeling the procurement costs, it is often assumed that the cost of a lot is given by $A + f(Q)$, where A is the fixed cost per lot and $f(Q)$ is the total variable cost. In case it is appropriate to assume $f(Q)$ to be a linear function of Q, one may write, $f(Q) = PQ$ where P is the constant unit cost.

Inventory holding costs result from out-of-pocket losses such as inventory taxes, insurance, damage, deterioration, handling, and storage space requirements. In addition, there are opportunity losses associated with the funds tied up in inventory. Inventories are equivalent to sums of money that are unavailable for investment in other opportunities open to the firm.

A common method of modeling inventory holding costs is to assume that they are proportional to the average inventory. If $I(t)$ is the inventory at time t, the average inventory over a period $(0,T)$ is defined as

$$\bar{I} = \frac{1}{T}\int_0^T I(t)dt \qquad (1.7)$$

As it is shown in Figure 1.2, $\bar{I}$ is the area under the inventory curve divided by T. If h is the cost to carry a unit of inventory for one unit of time, the average inventory holding cost per unit time over the interval $(0,T)$ is equal to $h\bar{I}$ and the total inventory carrying cost over $(0,T)$ is $Th\bar{I}$.

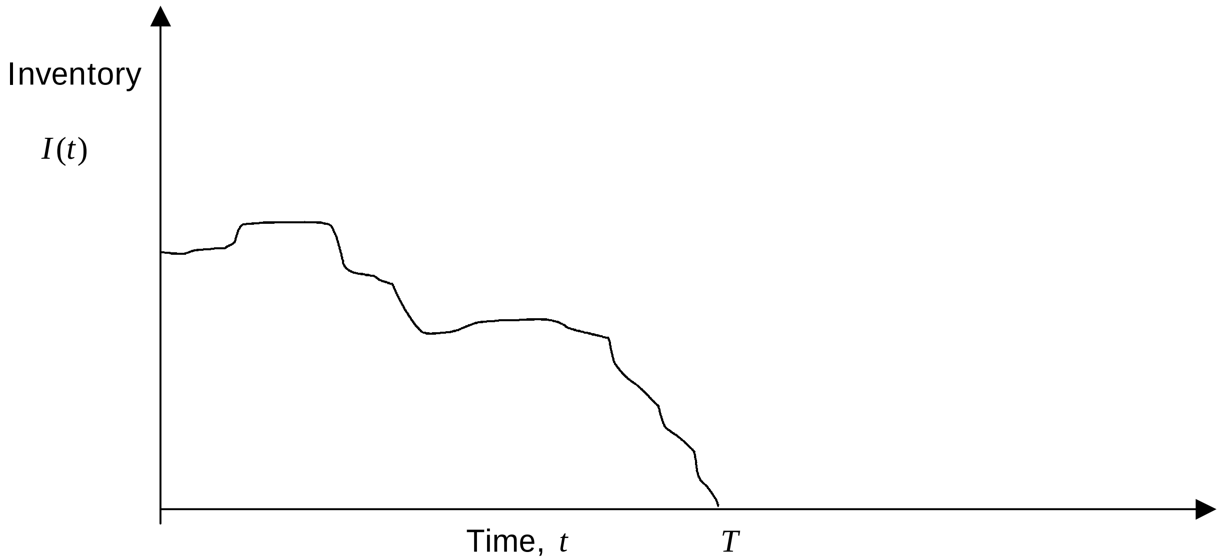


Figure 1.2: Average Inventory

To determine the amount of inventory carrying cost per unit per year, it can be assumed that h consists of a cost proportional to the dollar value of a unit of inventory plus a cost that is independent of the dollar value, to incorporate costs such as those associated with storage and handling. Hence one can write $h = FP + w$, where F is the cost of carrying $1 of inventory for one unit of time, P is the dollar value of a unit, and w is the cost per unit of inventory per unit time. The factor F is called the inventory carrying cost rate. In many inventory models w is omitted, and the average inventory holding cost per unit time becomes, $h = FP\bar{I}$. If a demand for an out-of-stock item occurs, two types of loses must be considered, the backorder and lost sale costs. If the demands are backordered there will be some added costs such as costs of expediting, special handling and shipping of the backordered items, information-processing costs and loss of goodwill because of customer dissatisfaction. Depending on the situation, some other costs may occur. For instance, if the item is to be used in a production operation, the operation may have to shut down with loss of production while the inventory is in a backorder condition. Other parts of the backorder costs may be the cost of informing the customer about the out-of-stock situation and cost of routing the information to find when the item will be available to ship.

When a customer cancels an order, there would be a loss of revenue since the sale is not completed. And in some cases customers may take their business somewhere else, and the system loses these customer for good, which may have a significant impact on future planning and forecasting. Backorder and lost sales costs are very difficult to measure, as they include many different components.

A good way to model the shortage cost is to consider two cost components. A constant loss, π, associated with each unit demanded when the inventory is out of stock. This would be more appropriate for the lost sale case. If the item is backordered it is usually assumed that the loss is proportional to the time required to fill the backorder. Thus, a cost, $\hat{\pi}$, is defined as the cost of carrying a backorder of one unit for one unit of time. The total shortage loss over a period $(0,T)$, can hence be defined as

$$\pi b + \hat{\pi} T \overline{B} \tag{1.8}$$

Where b is the total number of shortages and $\overline{B}$ is the average backorder position during $(0,T)$. The average backorder position is equal to:

$$\overline{B} = \frac{1}{T}\int_0^T B(t)dt \tag{1.9}$$

Where $B(t)$ is the backorder position at time t.

1.3. A review of the Inventory Systems Analysis

In this section, some basic Inventory Models are reviewed and a common approach for analyzing inventory systems is revisited.

In the first step of the problem analysis, the structure of the system must be understood. The objective, constraints, variables and parameters of the system need to be determined. A mathematical model is then constructed to measure the effectiveness of specific choices of the decision variables. These decision variables may include the ordering lot size, amount of backorder, and reorder point. After constructing the objective function, the constraints of the system need to be mathematically expressed as functions of the decision variables. To complete the analysis one should determine the values of the decision variables, which optimize the objective function subject to the constraints. In the next section, some deterministic single item models with static and dynamic demands are revisited.

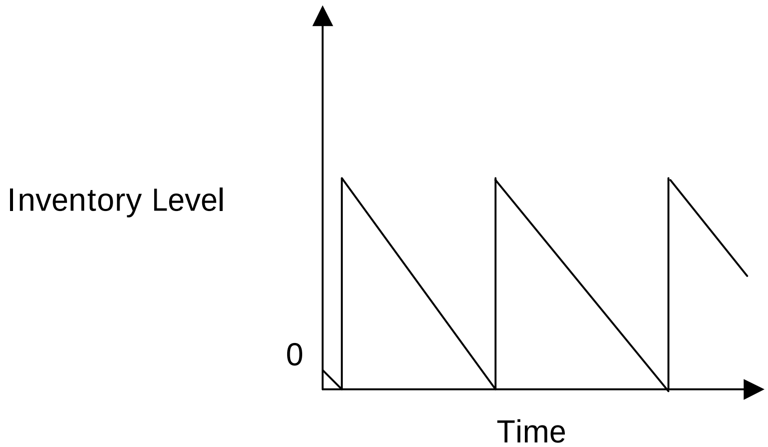
1.4. Deterministic Single Item Models with Static Demand

If the demand rate for a commodity is known with certainty and is constant, the system is called deterministic with static demand. It is assumed that all shortages are backlogged and are satisfied when the orders are received. It is also assumed that the procurement lead time is constant and equal to τ , and the entire order is delivered as a single package.

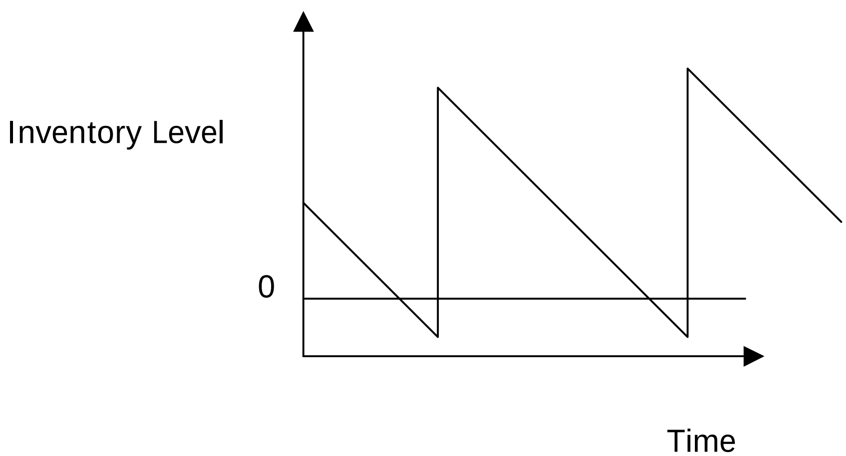
The main objective of this system is to determine when to release an order and how much should be ordered. Since the demand rate is constant, the policy will have equal size lots. The order is placed when the inventory level drops to a certain level, called the reorder point.

As mentioned earlier, the input (production) rate, amount of backorder permitted and the reorder point are the controllable variables in the system. Based on this assumption, four different models for the single item deterministic problem can be considered. These models are shown in Figures. 1.3 and 1.4.

When the backorder cost is considered to be very high or literally infinity, no backlogging is allowed.

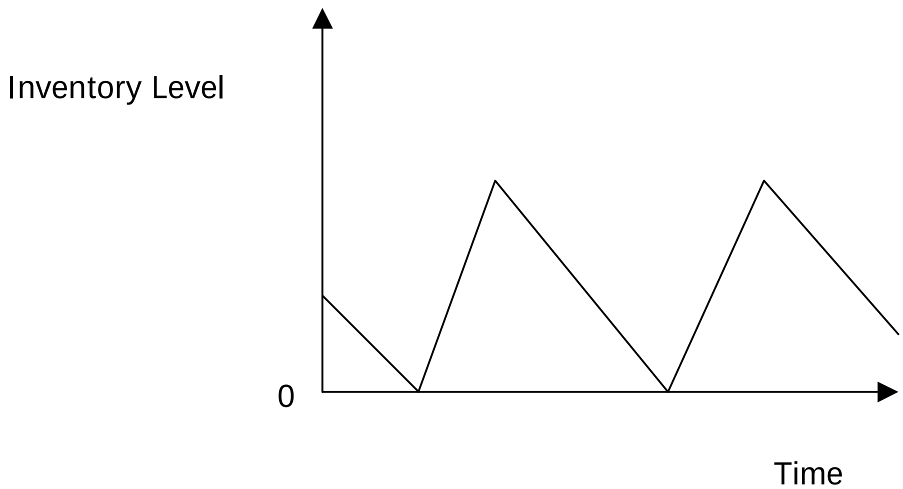


a) Model I. Infinite Input rate, backlogging not allowed.

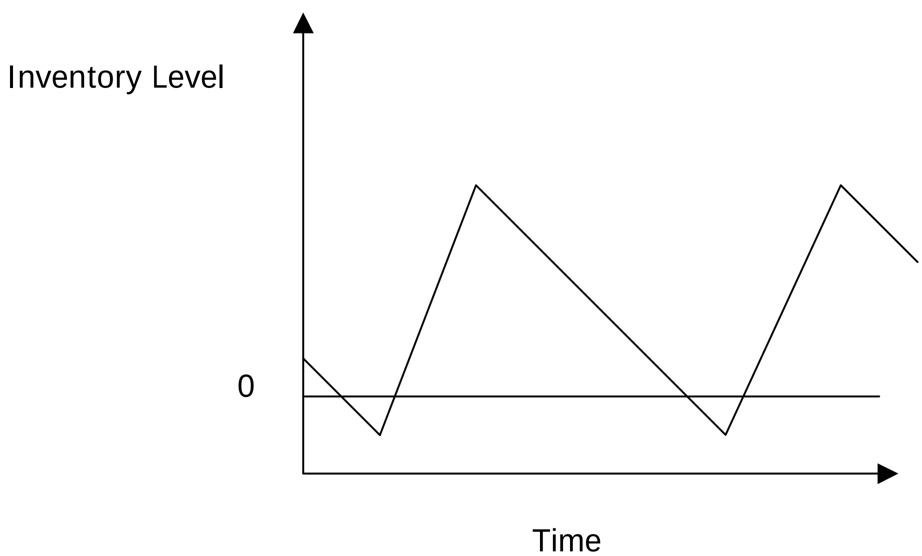


b) Model II. Infinite Input rate, backlogging allowed

Figure 1.3: Inventory behavior for a single item deterministic model with infinite input rate.



a) Model III. Finite Input rate, backlogging not allowed



b) Model IV. Finite Input rate , Backlogging Allowed.

Figure 1.4: Inventory behavior for a single item deterministic model with finite input rate.

The mathematical model for case (II), Infinite Input rate and backlogging allowed is presented. The mathematical model for case (I) can be obtained as a special case of this model. Cases (III) and (IV) will not be considered in this research. The following assumptions are made in order to construct the model:

a) The demand rate of each item is known and constant.

b) The unit cost of each item is constant.

c) The replenishment lead-time is constant.

d) Shortages are permitted.

The following notation will be used:

A = fixed ordering cost associated with a replenishment.

p =unit cost of production (or purchase)

h = inventory carrying cost per unit per year, $h = Fp$, where F is the annual inventory carrying rate.

π = shortage cost per unit short

$\hat{\pi}$ = shortage cost per unit year of shortage

Q = the order quantity

I_{max} = maximum on-hand inventory level

b = maximum backorder level permitted

T = time interval between replenishments

K = average annual cost

In this model it is assumed that a lot size of Q units is received at one particular time. The lot size is fixed and the demand is constant, so the cycle length is constant and equal to $T = Q/D$. The decision variables in this model are Q and b. The Q units satisfy the shortage and also build up the inventory at the warehouse up to point $I_{\max}$. (Figure 1.5.)

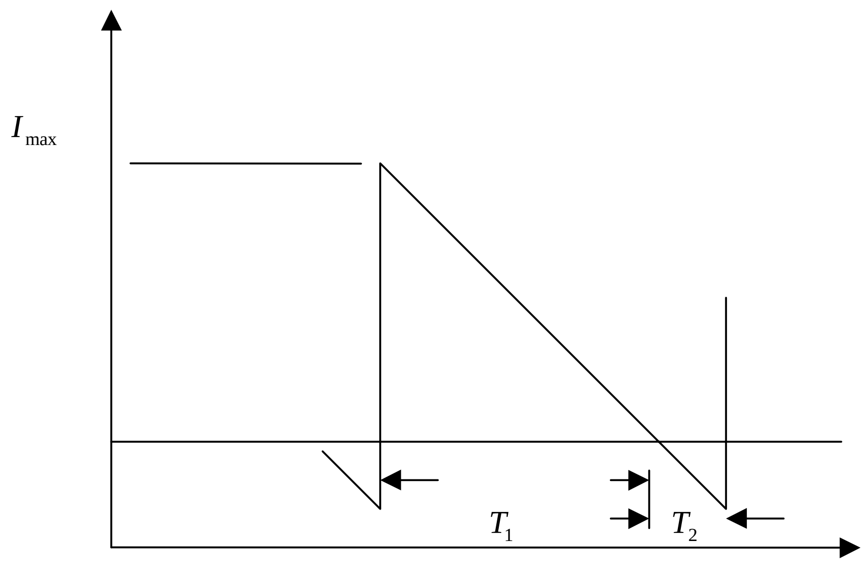


Figure 1.5: a cycle of an inventory model with infinite input rate and allowed Shortage

The best approach to obtain the optimal order quantity and backorder is to construct a mathematical model for the average cost per cycle and then minimize the average cost per year. The average cost per cycle is the sum of procurement cost, inventory cost and shortage cost. These costs are identical for all cycles during the planning horizon.

The average inventory is the area under the inventory graph during which the system is carrying stock divided by T (Figure 1.5). Let T_1 be the portion of cycle time during which the system is carrying inventory and is equal to $T_1 = \frac{Q-b}{D} = \frac{I_{max}}{D}$. The

average inventory level, $\bar{I}$, is equal to: $\bar{I} = \frac{(Q-b)(Q-b)}{2D} \Big/ T = \frac{(Q-b)^2}{2Q}$.

The average backorder position over the cycle, is the area under the backorder triangle divided by T. Let T_2 be the part of cycle time during which a backorder position of b is built up, and is equal to $T_2 = \frac{b}{D}$. The average backorder position over the cycle is: $\overline{B} = \frac{b}{2D}.b \Big/ T = \frac{b^2}{2Q}$.

At this point, the average cost per cycle is developed as the sum of ordering, purchasing, inventory and shortage costs, and is equal to:

$$A + pQ + hT\bar{I} + \hat{\pi}T\overline{B} + \pi b$$

To obtain the average annual cost, the average cost per cycle must be multiplied by the number of the cycles per year, D/Q.

The annual cost is denoted by $K(Q,b)$ which is a function of two decision variables, the order quantity Q, and the shortage position b.

$$K(Q,b) = \frac{AD}{Q} + pD + \frac{Fp(Q-b)^2}{2Q} + \frac{(2\pi Db + \hat{\pi}b^2)}{2Q} \tag{1.10}$$

The optimal value for the order quantity and maximum backorder level can be obtained by solving the following equations:

$$\frac{\partial K(Q,b)}{\partial Q} = 0$$

$$\frac{\partial K(Q,b)}{\partial b} = 0$$

After solving those equations the optimal order quantity and backorder level are obtained as follows:

$$Q^* = \sqrt{\left(\frac{2AD}{Fp} - \frac{(\pi D)^2}{Fp(Fp+\hat{\pi})}\right)} * \sqrt{\frac{Fp+\hat{\pi}}{\hat{\pi}}} \tag{1.11}$$

$$b^* = \frac{(FpQ^* - \pi D)}{Fp+\hat{\pi}} \tag{1.12}$$

If we consider model I in Figure 1.3(a), the cost equation would be:

$$K(Q) = \frac{AD}{Q} + pD + Fp\frac{Q}{2} \tag{1.13}$$

and the optimal solution is obtained by solving the equation

$$\frac{dK(Q,b)}{dQ} = 0$$

which results in

$$Q^* = \sqrt{\frac{2AD}{Fp}} \tag{1.14}$$

This model is the fundamental model in inventory theory known as the Economic Order Quantity (EOQ) model.

1.5. Statement of the Problem and Research Objective

This research is concerned with developing mathematical models for static and dynamic deterministic single item inventory systems, considering economic factors such as inflation, time value of money and permissible delay in payment.

In the first part of the research, a static single item inventory model is considered in which shortages are allowed and a delay in payment is permitted. The appropriate mathematical model presenting the present value of the total annual cost of the system is developed which is a nonlinear function of two decision variables: (1) the length of replenishment cycle, and (2) the length of the period during which the inventory level is positive. In order to minimize the total annual cost of the system, a nonlinear optimization procedure is applied. A computer software is provided to implement the procedure.

In the second part of the research a dynamic deterministic single item inventory model is considered in which the demand is assumed to be a linear function of time. The supplier allows a grace period to pay for the goods acquired. The grace period is a fraction of the replenishment cycle. The mathematical model presenting the present worth of the total cost of the system during the planing horizon will be developed. The objective of the model is to minimize the total cost of the system, by obtaining the optimal number of replenishment cycles during the planning horizon. A numerical example will be provided. In developing the mathematical model related to this problem, a software called *Mathematica* was used which is a product of Wolfram Research

Institute. *Mathematica* is an outstanding tool for mathematical modeling, analysis, and large-scale programming.

Chapter two of this document presents the literature related to the problem. Chapter three gives an introduction to inventory problems that take into account the effects of inflation, discounting, and permissible delay in payment. Chapter four gives a detailed formulation of the static single item models. Optimal solutions and numerical examples are presented as well. Chapter five is concerned with developing the mathematical formulations related to dynamic models. Chapter six presents conclusions and suggestions for future research.

Chapter Two

Literature Review

This chapter reviews the literature available in four main areas:

1. Inventory control models that take into account the effect of inflation and time value of money with constant demand rate (static models).
2. Inventory control models with time-varying demands (dynamic models)
3. Inventory control models which take into account the effect of inflation and time value of money, with time-varying demand rate.
4. Inventory control models with permissible delay in payments

2.1. Static inventory systems and Inflation

One of the pioneering works in this field is by J. A. Buzzacott [1975]. As stated in his paper, the annual inflation rates in most western countries ranged from 8 to 20 percent and it was no longer reasonable to use the classical EOQ model without investigating its modification, when inflation results in cost increasing with time. In his paper a classical EOQ model was investigated, in which he assumed that there is a constant inflation rate of k \$/\$/unit time, i.e. if the cost at time t is $b(t)$, at time $t+\delta$ the cost would be:

$$b(t+\delta) = b(t) + kb(t)\delta$$

$$db(t)/dt = kb(t)$$

As $\delta \to 0$

which results in

$$b(t) = b_0 e^{kt}$$

Where b_0 is the cost at time zero.

The three basic costs of the system, (1) Inventory carrying cost, (2) purchase cost and (3) setup or ordering cost were treated by taking into consideration the effect of inflation, and the optimal order quantity was calculated.

R. B. Misra [1975] considered a general EOQ model and investigated the effect of inflation and time value of money. He considered, different inflation rates for various costs associated with the inventory system. Models that consider separate inflation rates for each cost component of the system are more general and more realistic. Building a mathematical model for this system is fairly straightforward, but the optimization of the model is very difficult. Misra classified the costs in two separate categories. The first category consists of all costs, which follow are inflation rate that is effective inside the company. The second category includes the costs that follow the inflation rate of the general economy. These are called the internal and external inflation rates.

The replenishment cost is increased at the internal inflation rate while the purchasing cost is increased at the external inflation rate. The inventory carrying cost, out-of pocket costs such as costs of insurance, taxes, etc., and the amount of capital tied up in inventory would change with the external inflation rate. The storage cost depends on whether the company owns the warehouse or rents it.

The total cost equation was developed for the system to minimize the present worth of all future costs. The expression for the optimal order quantity was then obtained.

M. J. Chandra and M. L. Bahner [1985] considered more general inventory models. In the EOQ model studied by Buzzacott and Misra, it was assumed that no shortage is permitted and the input (replenishment) rate is close to infinity, but those assumptions were relaxed in the two models studied by Chandra and Bahner. In the first model, shortage was allowed and in the second one a finite replenishment rate was assumed. The approach of Misra [1979] was used to develop the cost expression for each of the two systems, and the optimal values of the decision variables were obtained.

Bhaba R. Sarker and Haixu Pan [1994] studied the effect of inflation and time value of money on the optimal ordering quantities and the maximum allowable shortage in a finite replenishment inventory system. The earlier studies by Misra [1979] and Chandra and Bahner [1985] dealt with either a classical EOQ model or with models in which shortage and finite replenishment are considered separately. Sarker and Pan studied the effect of inflation and time value of money in a model with shortage and finite replenishment rate, using the same approach of Misra. They developed the present value of the total cost incurred during the planning horizon and then obtained the optimal order quantity and maximum allowable shortage using a direct search optimization technique.

T. K. Datta and A. K. Pal [1991], studied the effect of inflation and time value of money on an inventory model with dynamic demand and allowed shortage. The demand was assumed to be linear and time-dependent. In this paper the approach of Misra [1979] was used to evaluate the present value of the total cost during the planning horizon, T. The total cost included cost components for replenishment shortage and purchasing. The number of replenishment cycles in the planning horizon, m, and the fraction of each cycle length during which the inventory is carried for the cycle, k were considered as decision variables.

2.2. Inventory systems with time-varying demand (dynamic models)

W. A Donaldson [1977] analyzed an inventory system in which the demand for an item in the time interval *(t, t+dt)* is given by *f(t)dt.* The instantaneous demand rate at time *t* is thus be represented by the continuous function *f(t).* In his model, no shortages were allowed, inventory at time *t= 0* was assumed to be zero, and replenishment lead-time was assumed to be zero. A planning horizon equal to *H* was used. The objective was to determine the optimal times t_i, $i = 0,1,2,...,n-1$, at which to reorder, and the number of orders, n, so that the total cost over *(0, H)* is minimized and inventory is zero again at time $t_n = H$.

Edward A. Silver [1979] considered the situation of a deterministic demand pattern having a linear trend. His objective was to select the timing and sizes of replenishments so as to keep the total value of replenishment and carrying costs as low as possible. He considered an approximate solution procedure, known as the Silver-

Meal heuristic, which was developed for the general deterministic case with time-varying demand pattern. A special case of a positive linear trend was also considered, which resulted in a very simple decision rule.

R. I. Phelps [1980] considered the classical deterministic inventory model for the case of constant time between replenishments and a linear trend in demand. The optimum policy was derived and shown to apply to both positive and negative trends. This policy was applied on two examples considered in earlier papers by Donaldson [1977] and Silver [1979]. The Phelps method is computationally easier than any of earlier methods and does not require any heuristic adjustment. It has the operationally desirable feature of constant intervals between replenishments.

E. Ritchie [1984] derived a simple optimal policy for the case of linear increasing demand, which is analogous to the EOQ for constant demand. The exact solution for linear increasing demand was published earlier by Donaldson [1977], but that solution does not have the simplicity of the EOQ formula, which had led to development of heuristic methods such as the Silver-Meal heuristic. An exact solution, which has the simplicity of the EOQ formula, was needed which has been the objective of Ritchie's paper.

Maitreyee Deb and K. Chaudhuri [1987] considered the inventory replenishment policy for an item having a deterministic demand pattern with a linear (positive) trend and shortages. They developed a heuristic to determine the decision rule for selecting the times and sizes of replenishments over a finite time-horizon so as to minimize the total cost. The model of Donaldson [1977] was modified by introducing the concept of inventory shortage which makes the problem much more mathematically complex. A heuristic was therefore developed, by following Silver, to determine the decision rule for selecting the time and sizes of replenishments and keeping the carrying and shortage costs as low as possible.

T. M. Murdeshwar [1988] derived an analytical procedure for the above problem. The objective was to obtain the optimal number of replenishment points and the times at which the inventory reduces to zero.

S. K. Goyal [1988] stated that the heuristic method and the total relevant cost model given by Deb and Chaudhuri [1987] for determining the economic replenishment policy for an item having a deterministic demand with a positive linear trend and shortages are incorrect. He pointed out the error, presented a correct heuristic, and developed a model for the replenishment interval, which is consistent with Silver's model [1979] when shortages are not permitted.

Upendra Dave [1989-a] developed a single item order-level lot-size-type inventory model for items with a deterministic time-dependent demand. The model, which allows for shortages, was developed for a fixed finite planning horizon for which the initial and the final inventory levels are zero. The optimal number of replenishments to be made and the corresponding replenishment points during the given horizon were determined in the model.

Upendra Dave [1989-b] also derived a heuristic decision rule for the replenishment of items with a linearly increasing demand rate over a finite-planning horizon during which shortages are allowed. He stated that the exact total cost expression given by Deb and Chaudhuri [1987] is incorrect. He corrected their error and rederived the Silver-Meal heuristic for items with a linearly increasing demand pattern and shortages. The numerical results indicated that the use of a heuristic incurs negligible cost penalties.

2.3. Inventory models with dynamic demand and inflation

T. K. Datta and A. K. Pal [1991], studied the effect of inflation and time value of money for an inventory model with dynamic demand. They assumed that the demand is a linear function of time and allowed for shortages in the model. Their approach is similar to the one used by Misra [1979]. The present value of the total cost during the planning horizon, T, which includes replenishment cost, shortage cost and purchasing cost, was developed. In their cost expression, the number of replenishment cycles in the planning horizon, m, and the fraction of each cycle length during which the inventory is carried for each cycle, k, were considered as the decision variables.

Moncer A. Hariga [1994] developed dynamic programming models for three commonly used replenishment policies with time varying demand and shortages. In the case of linear time dependent demand, a sensitivity analysis was performed. Datta and Pal assumed that the replenishment cycles are equal in length and the inventory carrying times of the cycles are identical, but Hariga did not put any restriction on the replenishment intervals and the length of the inventory carrying time of each cycle. All models in his paper were formulated using a dynamic programming approach. This approach could be applied to any type of demand function and would provide a readymade sensitivity analysis for the length of the planning horizon, which is usually difficult to estimate in practice.

Hariga [1995] also redeveloped the models of Datta and Pal by relaxing the assumption that inventory carrying times during replenishment cycles are equal. His model is applicable to both growing and declining markets, with general continuous time-dependent demand rates.

M. A. Hariga and M. Ben-Daya [1996] dealt with the inventory replenishment problem over a fixed planning horizon for items with linearly time-varying demand and inflationary conditions. They developed models and optimal solution procedures with and without shortages. They did not put any restriction on the length of the replenishment cycles, thus making their proposed methods the first optimal solution procedures for this problem.

J. Ray and K. S. Chaudhuri [1997] developed a finite time-horizon deterministic economic order quantity (EOQ) inventory model with shortages, where the demand rate at any instant depends on the on-hand inventory (stock-level) at that instant. The effect of inflation and time value of money was taken into account. The model deals with the inventory-level-dependent demand rate with shortages and effect of inflation and time value of money.

2.4. Inventory systems with permissible delay in payment

S. K. Goyal [1985] derived mathematical models for obtaining the economic order quantity of an item for which the supplier permits a fixed delay in settling the amount owed. He solved some examples to illustrate his method. As Goyal states, in practice a supplier permits a fixed period for settling the account and usually there is no charge if the outstanding amount is settled within the allowed fixed period. This means that the supplier is actually giving the customer a loan without interest during the grace period.

The reason that a supplier offers this grace period to the customer is that he tries to stimulate the demand for his product. The supplier usually expects that the profit increases due to rising sales volume, which can compensate for the capital loss incurred during the grace period.

During the period before the account has to be settled the customer can sell or use the items and earn revenue. Hence, logically the customer would like to delay the settlement of the account up to the last moment of the permissible period allowed by supplier. As Goyal concludes, the economic replenishment interval and order quantity generally increase, although the annual cost decrease considerably. The saving in cost is a result of the permissible delay in settling the replenishment account and it comes from the ability to delay the payment without paying any interest.

Suresh Chand and James Ward [1987] analyzed the same problem under the assumptions of the classical EOQ model, which are different from Goyal's assumptions, and obtained different results.

Hark Hwang and Seong Whan Shinn [1996] examined the problem of determining the retailer's optimal price and lot-size simultaneously when the supplier permits delay in payment for an order of a product whose demand rate is represented by a constant price elasticity function. They assumed that inventory is depleted not only by customers' demand but also by deterioration. In their model, replenishment is instantaneous, and shortage is not allowed. They developed a mathematical model for the system as well as a solution procedure.

Kun-Jen Chung [1998] studied the problem of the economic order quantity for an item for which the supplier permits a grace period in settling the account. First, he showed that the total annual variable cost function is convex. Then, with convexity, he developed a theorem to determine the economic order quantity. The theorem also revealed that the economic order quantity, with permissible delay in payments, is generally higher than the economic order quantity given by the classical economic order quantity model.

Hung-Chang Liao, Chih-Hung Tsai and Chao-Ton Su [2000] developed an inventory model for initial-stock-dependent consumption rate when a delay in payment is permissible. In their model, shortages are not allowed. The effect of the inflation rate, deterioration rate, initial-stock-consumption rate and delay in payment are discussed.

Bhaba Sarker, A. M. M. Jamal and Shaojun Wang [2000] developed a model to determine an optimal ordering policy for deteriorating items under inflation, permissible delay in payment, and allowable shortage. The purpose of their research was to aid the retailers in economically stocking the inventory under the influence of different decision criteria such as time value of money, inflation rates, purchase price of the product, and deterioration rate.

In the next chapter an introduction is given for the inventory problems that take into account the effects of inflation, time value of money and permissible delay in payment. This will be followed by the development of new models as defined in the research objectives.

Chapter Three

Effects of Inflation, Time Value of Money and Permissible Delay in Payment on Optimal Policy of Inventory Systems

As mentioned earlier, economic factors such as inflation and interest rate may have substantial impact on the economic order quantity and time interval between consecutive orders. Several studies have examined the inflationary effect on the optimal policy of the inventory systems. In this chapter the effects of time value of money and inflation on a classical EOQ model are investigated, then an inventory model will be considered in which the demand is a linear function of time, and the effects of inflation and time value of money on the model are presented.

Special types of financial contracts between the supplier and the customer may have a significant effect on the optimal policy of the inventory system. A situation with a permissible delay in payment in which the supplier allows some grace period before the customer settles the account to pay for the goods bought, has received some attention. The mathematical formulation of the economic order quantity with permissible delay in payment as presented by S. K. Goyal [1985], will also be reviewed in this chapter.

3.1. The effect of inflation and time value of money on the EOQ model

In this section the effect of inflation and time value of money on the classical EOQ model is discussed. The discussion and mathematical formulation are based on the research work of Ram B. Misra [1975] and M. Jaya Chandra and Michael L. Bahner [1985].

As mentioned earlier, several studies investigated the inventory models with inflationary conditions. This chapter reviews the most frequently used inventory model, the general EOQ, under conditions of constant inflation rate, and time discounting.

First the notations and assumptions are presented, then the EOQ model with the effect of time discounting is reviewed, and finally the cost model of a classical EOQ model under inflation and time value of money is developed and the optimal order quantity is obtained.

3.2. Mathematical formulation for EOQ model considering the time value of money

Notations used in the model are as follows:

D = demand rate, unit/year

p = purchase cost, \$/unit

A = replenishment cost, \$/order

h = inventory cost, \$/unit/yr

f = inflation rate \$/\$/yr

r = discount rate, representing the time value of money \$/\$/yr

$R = r - f$

F = annual inventory carrying cost. \$/\$/yr

T = replenishment period, yr

N = number of periods during the planning horizon

P = the present value of the total cost of the inventory system for the first period \$

P_t = the present value of total cost of all cycles over the time horizon \$

The total cost of a classical EOQ model consists of: (1) Ordering cost; (2) purchasing cost and (3) inventory carrying cost. It is assumed that the ordering and purchasing costs are paid at the beginning of each period, and the inventory carrying cost is paid continuously during the period.

The inventory level at time t is $I(t) = D(T-t)$, and hence, the inventory carrying cost at time t, is equal to

`

$$FpI(t) = FpD(T-t)$$

Assuming continuous compounding, the present value of carrying cost at time t, is equal to

$$FpD(T-t)e^{-rt}$$

The present value of the total cost of the inventory system for the first period can be expressed as:

$$P = A + pDT + FpD\int_0^T (T-t)e^{-rt}\,dt \qquad (3.1)$$

The time value of money exists in each cycle of replenishment so one has to consider its effect over the time horizon, NT .

Equation (3.1) represents the total cost at the beginning of the first cycle, the present value of total cost of all cycles over the time horizon is assumed to be P_t . Figure (3.1) shows that if the beginning of the first cycle is set as a reference point of the present value, then P_t is given by:

$$P_t = P(1+\exp(-rT)+\exp(-2rT)+...+\exp[-(N-1)rT]) \tag{3.2}$$

$$P_t = P\left(\frac{1-\exp(-NrT)}{1-\exp(-rT)}\right) \tag{3.3}$$

for the infinite planning horizon, $N \to \infty$, it can be concluded that:

$$P_t = P\left(\frac{1}{1-\exp(-rT)}\right) \tag{3.4}$$

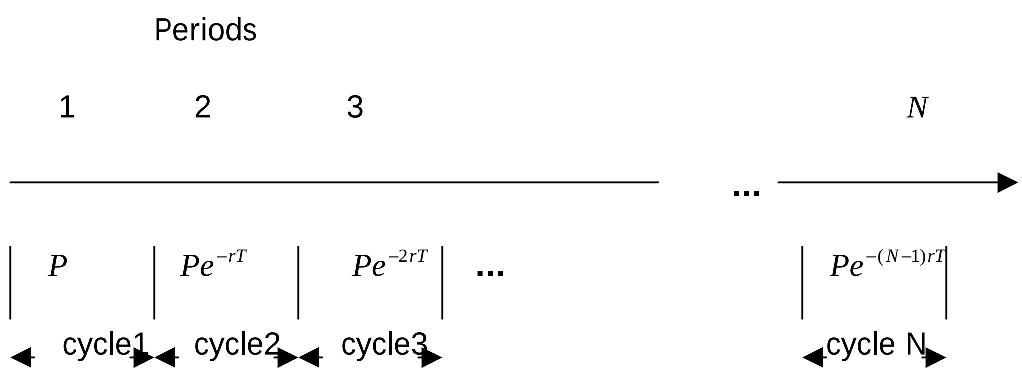


Figure 3.1: Cash Flow Diagram

3.3.The Optimal Solution

It is observed that the present value of the total cost function, P_t , in equation (3.4) is a function of T , the length of a period, and P_t is minimum if the condition $\frac{dP_t}{dT}=0$ holds.

At this point, the present value of the total cost function of the system is expanded and the optimal order quantity is calculated.

Consider the present value of the total cost in one cycle:

$$P = A + pDT + FpD\int_0^T (T-t)e^{-rt}\,dt$$

Upon integration by parts and after some simplifications:

$$P = A + pDT + FpD\left\{\frac{1}{r}\left[T + \frac{1}{r}(e^{-rT}-1)\right]\right\} \tag{3.5}$$

then

$$P_t = \frac{A}{1-e^{-rT}} + \frac{pDT}{1-e^{-rT}} + \frac{FpDT}{r(1-e^{-rT})} - \frac{FpD}{r^2}$$

$$P_t = \frac{Ar + (r+F)pDT}{r(1-e^{-rT})} - \frac{FpD}{r^2} \tag{3.6}$$

After taking the first derivative of P_t with respect to T ,

$$\frac{dP_t}{dT} = \frac{(r+F)pDr(1-e^{-rT}) - r^2e^{-rT}[(r+F)pDT + Ar]}{r^2(1-e^{-rT})^2} = 0 \tag{3.7}$$

it can be concluded that:

$$e^{-rT} = \{(r+F)pDr\} \Big/ \{(r+F)pDr + r^2[(r+F)pDT + Ar]\} \tag{3.8}$$

Hence:

$$e^{rT} = \frac{(r+T)pDr}{(r+F)pDr} + \frac{r^2(r+F)pDT}{(r+F)pDr} + r^2\frac{Ar}{(r+F)pDr} \tag{3.9}$$

Upon simplification:

$$e^{rT} = 1 + rT + \frac{r^2 A}{(r+F)pD} \tag{3.10}$$

On the other hand:

$$e^{rT} = 1 + rT + \frac{(rT)^2}{2!} + \frac{(rT)^3}{3!} + \frac{(rT)^4}{4!} + \ldots \tag{3.11}$$

One can ignore $\frac{(rT)^3}{3!}$ and higher terms if rT is small enough ($rT \approx 0.1$):

$$e^{rT} = 1 + rT + \frac{(rT)^2}{2!} \tag{3.12}$$

By equating the right hand sides of equations (3.10) and (3.12):

$$\frac{r^2 A}{(r+F)pD} = \frac{(rT)^2}{2!} \tag{3.13}$$

The optimal time interval between two replenishments would be:

$$T^* = \sqrt{\frac{2A}{(r+F)pD}} \tag{3.14}$$

and

$$Q^* = \sqrt{\frac{2AD}{(r+F)p}} \tag{3.15}$$

3.4. EOQ under inflation and time value of money

In this section the effect of inflation and time value of money is considered with an EOQ model.

The inflation rate, f, is considered to be constant. To build the total cost expression of the system, three components of the total cost of the first cycle are considered:

(1) Ordering cost, which is paid at the beginning of the period, and is equal to A.

(2) Purchasing cost, which is paid at the beginning of the period, and is equal to pDT

(3) Inventory carrying cost which is paid continuously during the period, and is equal to $FpD(T-t)e^{-rt}e^{ft}$

For convenience, the difference between inflation rate and interest rate is set to be R, $R = r - f$.

The present value of the total cost of the system in the period is equal to

$$P = A + pDT + FpD(\frac{1}{R})\left\{T + \frac{1}{R}(e^{-RT} - 1)\right\} \qquad (3.16)$$

As mentioned before, inflation and time value of money exist all over the planning horizon, hence to obtain the optimal cycle length, the present value of the total inventory cost during the planning horizon should be developed. To do so an infinite planning horizon is considered.

The present value of the total cost of the system during an infinite planning horizon is as follows:

$$P_t = P\sum_{n=0}^{\infty} e^{-nRT} \tag{3.17}$$

$$P_t = P\left(\frac{1}{1-e^{-RT}}\right)$$

To obtain the optimal ordering interval, T , one should solve the following equation:

$$dP_t/dT = 0 \tag{3.18}$$

Following the same procedure as the previous section leads to an optimal order quantity equal to:

$$Q^* = \left(\frac{2AD}{(R+F)p}\right)^{1/2} \tag{3.19}$$

3.5. An inventory model with linear time-dependent demand rate under inflation and time value of money

In this section an inventory model with a linear time dependent demand rate is considered and the effects of inflation and time value of money on the model are discussed.

T.K. Datta and A.K. Pal [1991] developed an inventory model with linear time - dependent demand rate considering the effects of inflation and time value of money. In their model, shortage was allowed. In this section, a simplified version of the Datta and Pal's problem is presented.

The notations used are the same as before. The model is presented under the following assumptions:

(1) The system operates for a prescribed time horizon, H .

(2) Replenishment is instantaneous.

(3) The demand rate, $D(t)$, is a linearly increasing function of time, t, i.e.,

$$D(t) = a + bt, \quad a, b > 0 \text{ and } 0 \leq t \leq H .$$

(4) Shortages are not allowed during the planning horizon.

(5) m replenishments are made during the planning horizon, H, and the length of each replenishment cycle is equal to H/m. m is the decision variable.

The graphical representation of the inventory system is given in Figure 3.2.

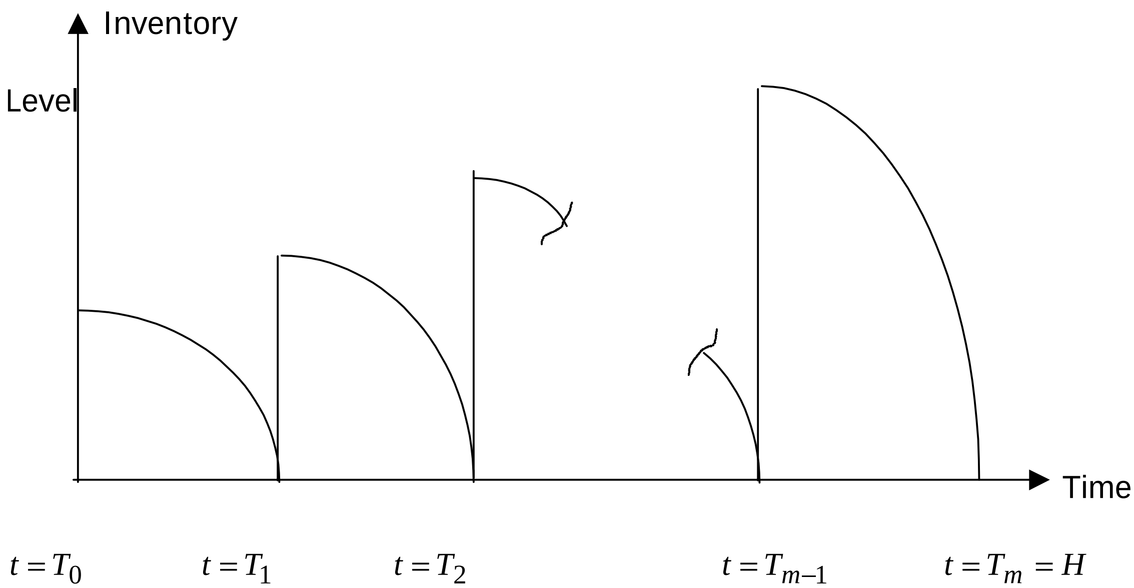


Figure 3.2: Graphical representation of the inventory level

3.5.1. Mathematical formulation of the Inventory system

The replenishments are made at $t = T_0, T_1, T_2, ..., T_{m-1}$. The last replenishment is made at time T_{m-1}, which covers the demand during the last cycle. We must have,

$$T_j = \frac{H * j}{m}, \quad j = 0,1,2,3,..., m.$$

The total inventory cost of the system is comprised of three components: (1) ordering cost, (2) holding cost, and (3) purchasing cost. The methodology used to develop the total inventory cost is the discounted cash flow method, in which the present value of the total cost of the system during the planning horizon is developed first, then the optimal ordering policy of the system is determined. In order to develop the mathematical formulations of the system *Mathematica* software is used.

There are m replenishments during the planning horizon, hence the present value of the total ordering cost during the planning horizon is given by:

$$CR = A\sum_{j=0}^{m-1} \exp(-RT_j) = \frac{A(\exp(-HR)-1)}{\exp(-HR/m)-1} \qquad (3.20)$$

The present worth of the total purchasing cost during the planning horizon is given by:

$$CP = p\sum_{j=0}^{m-1}\int_{T_j}^{T_{j+1}} (a+bt)\exp(-RT_j)dt$$

$$= -\left(\frac{1}{2(\exp(HR/m)-1)m^2}\right) * (\exp(-HR+\frac{HR}{m})H(-bH - b\exp(HR)H + bH\exp(HR/m)$$

$$-bH\exp(HR+\frac{HR}{m}) + 2b\exp(HR)(\exp(-\frac{HR}{m})^m H - 2am + 2a\exp(HR)m + 2a\exp(\frac{HR}{m})m$$

$$-2a\exp(HR+\frac{HR}{m})m-2b\exp(HR)[\exp(-\frac{HR}{m})]^{m}Hm+2b\exp(HR+\frac{HR}{m})[\exp(-\frac{HR}{m})]^{m}Hm)p$$

(3.21)

Present value of the inventory holding cost for m cycles, during the planning horizon is the following:

$$CH=Fp\sum_{j=0}^{m-1}\int_{T_j}^{T_{j+1}}[(t-T_j)(a+bt)\exp(-Rt)]dt$$

=

$$-\frac{1}{m^2R^3}(Fp(bH\left(\frac{\exp(-2HR/m)}{(1-\exp(-HR/m)^2}-(\exp(-HR/m)^{1+m}\left(\frac{\exp(-HR/m)}{(1-\exp(HR/m)^2}+\frac{m\exp(HR/m)}{\exp(HR/m)-1}\right)\right)$$

$$*R(m-m\exp(HR/m)+HR)-\left(\frac{1}{\exp(HR/m)-1}-\frac{(\exp(-HR/m)^{m}}{(\exp(HR/m)-1)}\right)(-2bm^2+2bm^2\exp(HR/m)$$

$$-2bHmR-am^2R+a\exp(HR/m)m^2R-bH^2R^2-aHmR^2)))$$

(3.22)

Hence the present value of the total inventory cost of the system during the entire planning horizon is equal to:

$$TC(m)=CH+CP+CR \qquad (3.23)$$

The above total inventory cost is a function of m, which is a discrete variable. For a given positive integer $m=1,2,3,...$, etc, the value of total inventory cost is obtained and a list of total cost values is prepared. The minimum in the above list would be the optimum total inventory cost and the corresponding m would be the number of equal length replenishment cycles during the planning horizon.

3.5.2. Numerical example

An example is presented here to illustrate the application of the model developed in the previous section. Assume that, an inventory system with a linear increasing trend in demand, has the following properties:

$a = 10$, $b = 1$, $H = 10$, $r = 0.15$, $f = 0.5$, $A = 2$, $F = 0.32$, $p = 1$

The different values of m are substituted in equation (3.23) and the corresponding $TC(m)$ values are obtained. The results obtained are presented in table 3.1 and figure 3.3

Table 3.1. Total cost vs. number of cycles

m	$TC(m)$ in $
1	287.95
2	188.704
3	157.568
4	142.806
5	134.506
6	129.414
7	126.148
8	124.019
9	122.647
10	121.804
11	121.346
12	121.175*
13	121.226
14	121.451

*minimum value.

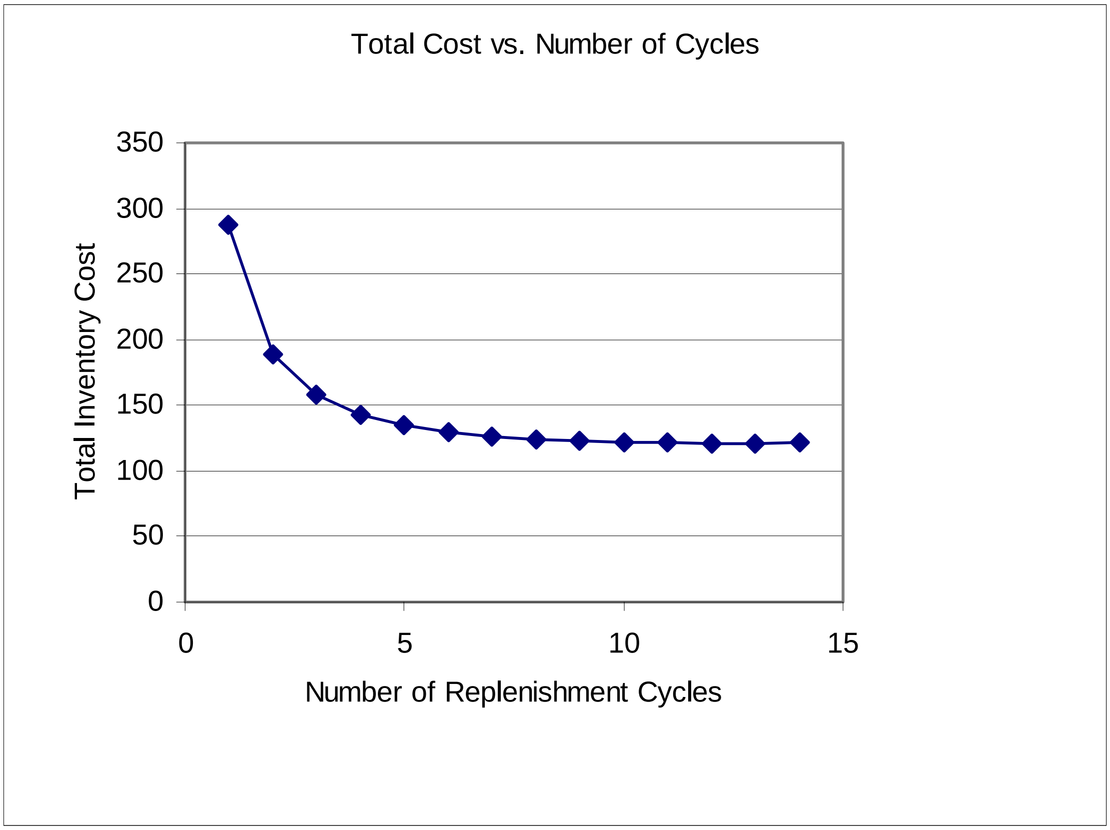


Figure 3.3: Total Inventory Cost and Number of Replenishment Cycles

Table 3.2 compares the optimal number of replenishment cycles and the corresponding total inventory cost for the system used in the above example with values of *R* between -10% and 100%.

The results show that as the inflation rate increases and the difference between inflation rate and discount rate grows, the total cost of inventory system increases significantly, which is expected.

Table 3.2.

Optimal Inventory Cost and Number of Replenishment Cycles Different values of R

$R = f - r$ (%)	No. of Replenishment Cycles	Total Inventory Cost
-10	12	121.175
5	10	252.907
10	10	335.678
30	7	1215.93
50	6	5253.76
80	5	57822.0
100	6	319459

3.6. Economic order quantity under conditions of permissible delay in payment

In deriving the Economic Order Quantity formula, it is assumed that the supplier is paid for the items delivered as soon as the items are received. However, in practice suppliers may allow for a certain fixed period for settling the account and paying the amount owed to them for the items supplied. Typically there is no extra charge if the outstanding amount is settled within the allowed grace period. Beyond this period an interest will be charged.

When a supplier allows for a fixed period for settling the account, he is actually giving his customer a loan without interest during this period. The customer should be able to bring in revenue and also earn interest during the grace period, and logically the customer should delay settling the replenishment account until the end of the permissible grace period.

The mathematical formulation of the inventory model is presented based on the work of S. K. Goyal [1985].

3.6.1. Mathematical Formulation

First the assumptions and additional notations needed for model are introduced.

Notations :

I_c = interest charges per \$ investment in stock per year, \$/\$/yr

I_d = interest that can be earned per \$ in a year, \$/\$/yr

A = Ordering cost for one order, \$

t = permissible delay in settling the account , yr

$Z(t) =$ total annual cost, $

Assumptions :

1. The demand for the item is constant
2. Shortages are not allowed
3. During the grace period, the revenue earned by sales is deposited in an account, which brings in interest, and at the end of the grace period the customer starts paying for the interest charges on the items in stock.
4. Planning horizon is infinite

The annual variable cost of the system has four different components:

(1) Ordering or setup cost which is equal to $\frac{A}{T}$.

(2) Inventory Carrying Cost (excluding interest charges). As it can be seen from Figure (3.4), the average inventory is equal to $\frac{DT}{2}$, so the stock holding cost per year is $DTh/2$.

(3) Cost of interest charges for the items kept in the stock. The sales revenue during the grace period is used to earn interest, but after the end of fixed period the items still in the stock have to be financed at the interest rate I_c. The inventory level at the time of settling the account is equal to $D(T-t)$, and the interest is payable during the time $(T-t)$.

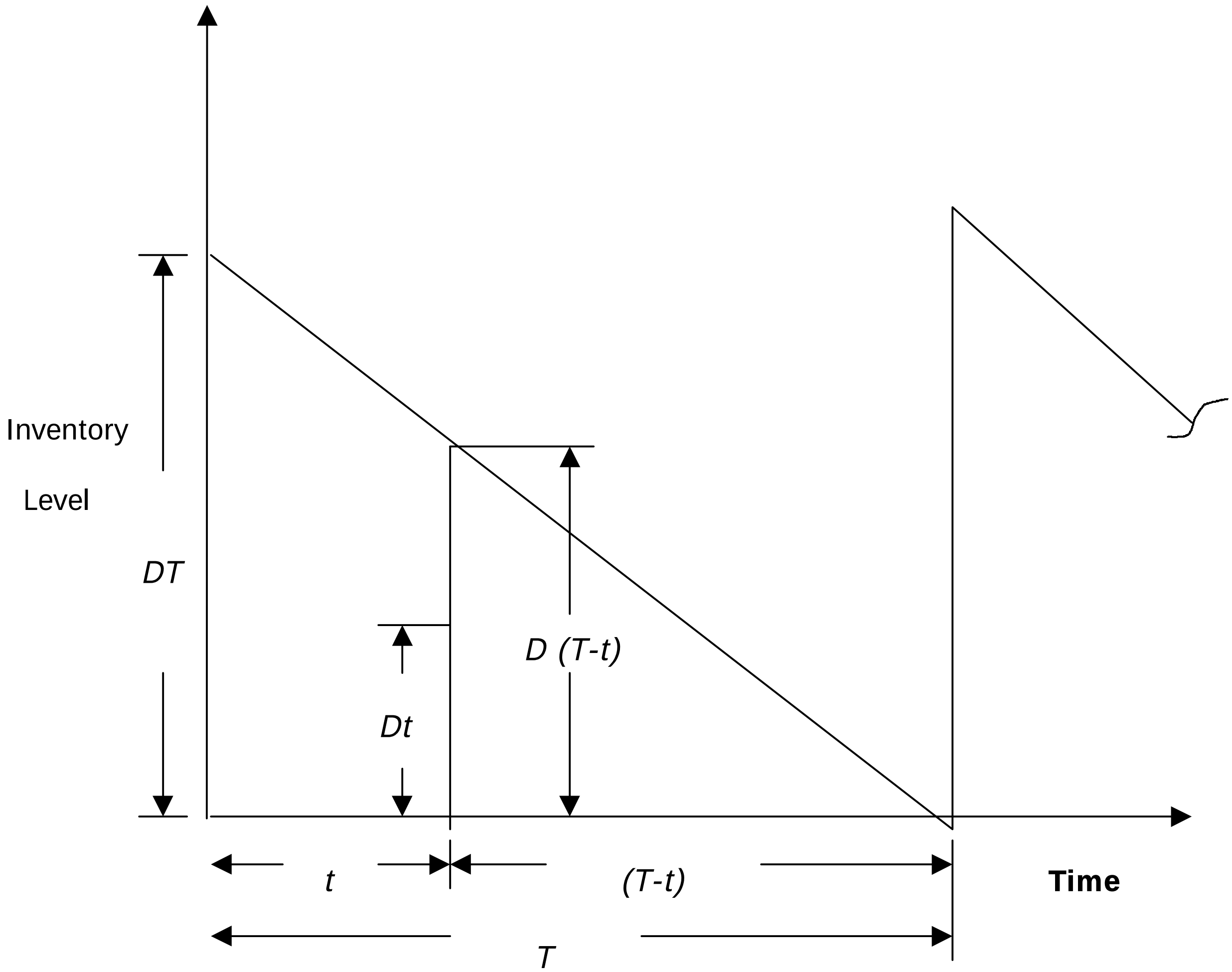


Figure 3.4: the inventory diagram when $T \geq t$

Interest payable in one cycle is equal to:

$$\frac{Dp(T-t)^2 I_c}{2}$$

and the interest payable per year for the inventory system is equal to:

$$\frac{Dp(T-t)^2 I_c}{2T} = \frac{DpTI_c}{2} + \frac{Dpt^2 I_c}{2T} - DptI_c$$

(4) Interest earned during the grace period:

The maximum amount of sale during the grace period is equal to Dtp if $T \geq t$, and if $T < t$, the maximum sale will be equal to DTp .

The interests earned during the grace period for the two cases are as follows:

(a) $T \geq t$, See Figure 3.4.

Interest earned in one cycle is equal to :

$$\frac{Dpt^2 I_d}{2}$$

Interest earned during one year is equal to :

$$\frac{Dpt^2 I_d}{2T}$$

(b) $T < t$, See Figure 3.5.

In this case the interest earned during one cycle is equal to :

$$[\frac{DT^2 p}{2} + DTp(t-T)]I_d = DTpI_d(t-\frac{T}{2})$$

and the interest earned in one year would be equal to :

$$DpI_d\left(t-\frac{T}{2}\right)$$

Obviously the interest earned is subtracted from the total cost in order to obtain the net total cost per year.

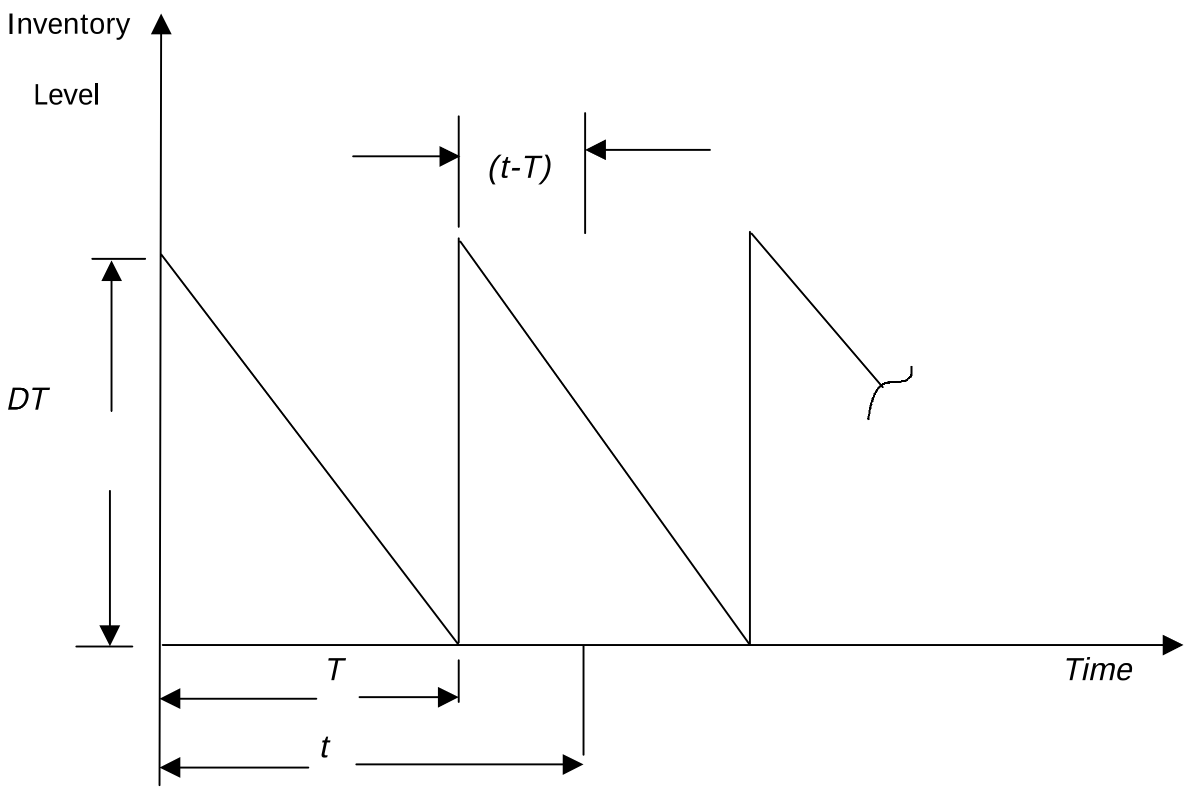


Figure 3.5: The inventory diagram when $T < t$

In the next section the mathematical models to obtain the optimal order quantity for the inventory systems, based on Goyal [1985], are presented.

3.6.2. Economic Order quantity when $T \geq t$

The total variable cost for the system when $T \geq t$, is given as:

$$Z(T) = \frac{A}{T} + \frac{DTh}{2} + \frac{DTpI_c}{2} + \frac{Dpt^2 I_c}{2T} - DptI_c - \frac{Dpt^2 I_d}{2T}$$

After some simplifications, one can obtain the annual cost as :

$$Z(T) = \frac{(2A + Dpt^2(I_c - I_d))}{2T} + \frac{DT}{2}(h + pI_c) - DptI_c. \tag{3.24}$$

To minimize the cost, $\frac{dZ(T)}{dT} = 0$, and obtain $T = T_1^*$

$$T_1^* = \sqrt{\frac{(2A + Dpt^2(I_c - I_d)}{D(h + pI_c)}} \tag{3.25}$$

It should be mentioned that $t > 0$ and $(I_c - I_d) \geq 0$.

Hence the economic order quantity is equal to:

$$Q(T_1^*) = DT_1^* = \sqrt{\frac{D(2A + Dpt^2(I_c - I_d)}{(h + pI_c)}} \tag{3.26}$$

and the minimum annual cost is equal to:

$$Z(T_1^*) = \sqrt{D(2A + Dpt^2(I_c - I_d))(h + pI_c)} - DptI_c \tag{3.27}$$

As Goyal states, the economic order quantity obtained under the condition of permissible delay in payments is generally higher than the order quantity given by a classical EOQ model.

If $I_d = I_c$, the economic order quantity given by (3.26) is equal to the order quantity obtained by classical EOQ model.

It rarely happens that $I_c - I_d \leq 0$, and in this case the economic order quantity calculated by (3.26) would be lower than the order quantity obtained by classical EOQ model.

3.6.3. Economic Order Quantity when $T < t$

In this case, there wouldn't be any item kept in the stock after the fixed period, t, hence there is no interest charge paid in this case.

The total annual cost can be represented as follows :

$$Z(T) = \frac{A}{T} + \frac{DTh}{2} - DpI_d\left(t - \frac{T}{2}\right)$$

or

$$Z(T) = \frac{A}{T} + \frac{DT}{2}(h + pI_d) - DptI_d \qquad (3.28)$$

To obtain the minimal annual cost, $\frac{dZ(T)}{dT} = 0$

Hence the optimum interval between two successive orders is given by:

$$T_2^* = \sqrt{\frac{2A}{D(h + pI_d)}} \qquad (3.29)$$

and the economic order quantity is given by:

$$Q(T_2^*) = DT_2^* = \sqrt{\frac{2AD}{(h + pI_d)}} \qquad (3.30)$$

Goyal points out that at $T = t$, the total annual variable cost can be obtained by substituting $T = t$, in (3.24) or (3.28):

$$Z(T) = \frac{A}{t} + \frac{Dth}{2} - \frac{DptI_d}{2} \qquad (3.31)$$

Goyal explains the optimal operating policy as follows:

Step 1: Determine T_1^* from (3.25). If $T_1^* \geq t$, obtain $Z(T_1^*)$ from (3.24)

Step 2: Determine T_2^* from (3.29). If $T_2^* < t$, obtain $Z(T_2^*)$ from (3.28)

Step 3: If $T_1^* < t$ and $T_2^* \geq t$, then evaluate $Z(t)$ from (3.31).

Step 4: Compare $Z(T_1^*)$, $Z(T_2^*)$ and $Z(t)$. Select the replenishment interval and the order quantity associated with the least annual cost value evaluated in steps 1, 2 and 3.

Chapter Four

Inventory Models with Allowable Shortages under inflationary conditions and Permissible Delay in Payment

As mentioned earlier, it is usual nowadays to see that customers are allowed a grace period before settling their accounts with the supplier and paying for the goods bought. In chapter three, a classical EOQ model was presented which considers the conditions of permissible delay in payment, based on the work of S. K. Goyal [1985].

In this chapter, a single item inventory model with allowable shortages is considered with permissible delay in payment. Shortages are important, particularly in a model that considers a delay in payment. Shortages can affect the quantity ordered to benefit from the delay in payment. This chapter is concerned with determining an optimal ordering policy for a deterministic single item model with allowable shortages and a permissible delay in payment, with and without considering the effects of inflation and time value of money. In the next section the first model disregarding the discount and inflation rate is presented and the mathematical formulation of the inventory system is developed. The effects of discount and inflation rate will then be considered in the following section.

4.1. A Single Item Inventory Model (EOQ) with Shortages and Permissible Delay in Payment

The assumptions and notations used in the model are presented.

Assumptions of the model:

(1) The demand for the commodity is known with certainty and constant.

(2) Shortages are allowed and are fully backlogged and satisfied when replenishment orders are received.

(3) The revenue earned during the grace period is invested, hence brings in interest.

(4) After the grace period, if there are still items kept in the stock (i.e. unpaid), the customer is charged a penalty (high interest) on the unpaid value.

(5) Planning horizon is infinite.

(6) Six cost component are considered:

(a) Setup cost

(b) Holding cost

(c) Purchasing cost

(d) Shortage cost

(e) Cost of interest charges for the items kept in the stock after the grace period

(f) Interest earned during the grace period

The following notations are used throughout the chapter:

D = demand rate, units/year

p = purchase cost, \$/unit

A = setup cost, \$/order

h = inventory cost, \$/unit/yr

F = annual inventory carrying cost. \$/\$.yr

π = shortage cost per unit short per unit time, \$/unit/yr

T = replenishment period, yr

T_1 = length of period with positive stock of the items, yr

I_c = interest charges per $ value in stock per year, $/$.yr

I_e = interest that can be earned per $ in a year, $/$.yr

f = inflation rate, / yr

r = discount rate representing the time value of money, / yr

$R = r - f$

N = number of replenishment cycles during the planning horizon (when it is finite)

M = permissible delay in settling the account , yr

HC = present value of the inventory carrying cost for the first cycle, $

TC = present value of the total cost for the first cycle, $

PC_1 = present value of the first part of the purchasing cost in the first cycle, $

PC_2 = present value of the second part of the purchasing cost in the first cycle, $

PC = present value of the purchasing cost in the first cycle which is equal to $PC_1 + PC_2$

SC = present value of the shortage cost during the first cycle, $

IC = present value of the cost of interest charges during the first cycle, $

EI = interest earned during the first cycle, $

TCP = present value of the total annual cost, $

In the next section the mathematical presentations of the different cost components of the inventory system are developed. The objective of the proposed model is to determine the length of the inventory replenishment cycle, T, and the length of the period with positive stock of items, T_1, such that the total cost of the system is minimized. There are two distinct cases in this type of inventory system:

Case 1: Payment (end of the grace period) is before the total consumption of inventory ($M \leq T_1$).

Case 2: Payment is after consumption of the inventory, ($T_1 < M$).

4.1.1. Payment before the Total Depletion of Inventory

This is the case where the grace period expires before the total consumption of the inventory (Figure 4.1). The total cost of the system is comprised of the six aforementioned cost components.

(a) Setup cost

Setup cost is paid at the beginning of each replenishment cycle and is equal to A.

(b) Holding cost

The average inventory during the period 0 to T_1 is equal to $DT_1/2$, and the stock holding cost per cycle is equal to

$$HC = \frac{hDT_1{}^2}{2}$$

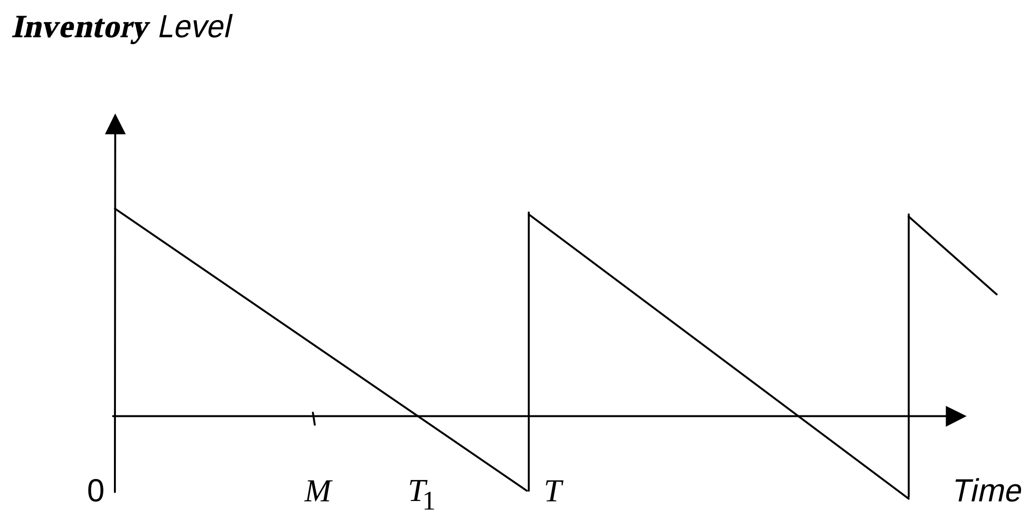


Figure 4.1: The single item inventory system with shortages and permissible delay in payment

(c) Purchasing cost

Total purchasing cost during the first period is given by:

$$PC = pDM + pD(T - T_1) + pD(T_1 - M) = pDT$$

(d) Shortage Cost

The inventory system backlogs the shortages incurred from the end of positive inventory level period until the end of replenishment cycle, $t \in [T_1, T]$.

The shortage cost per cycle is given by,

$$SC = \frac{\pi D(T - T_1)^2}{2}$$

(e) Cost of interest charges for items kept in stock after the grace period

An interest is charged for the goods kept after the grace period, interest rate I_c, the amount of interest charges is equal to:

$$CI = \frac{DpI_c(T_1 - M)^2}{2}.$$

(f) Interest earned during the grace period

The customer earns interest during the grace period and the amount earned is as follows:

$$EI = I_e Dp(T - T_1 + \frac{M}{2})M$$

The total inventory cost of the system during one cycle is given by:

$$TC = A + HC + PC + SC + CI - EI \qquad (4.1)$$

To obtain the average annual cost, TCP, the total cost per cycle should be multiplied by the number of replenishment cycles per year, $1/T$. Hence the total annual cost is given by

$$TCP = \frac{A}{T} + \frac{hDT_1^2}{2T} + pD + \frac{\pi D(T - T_1)^2}{2T} + \frac{DpI_c(T_1 - M)^2}{2T} - \frac{I_e Dp(T - T_1 + \frac{M}{2})M}{T} \qquad (4.2)$$

The total annual cost of the inventory system is a nonlinear function of two independent variables: (1) the length of replenishment cycle, T , and (2) the length of the period with positive stock of the items, T_1.

The optimal policy of the inventory system may be obtained by solving the following equations:

$$\frac{\partial TCP}{\partial T} = 0$$

$$\frac{\partial TCP}{\partial T_1} = 0$$

It is not easy to obtain closed forms for T and T_1 from the above equations. Instead the values may be obtained by trial and error. As an alternative, a non-derivative nonlinear optimization technique known as Hooke and Jeeves method will be used to obtain the optimal policy of the system using equation (4.2). Details of the technique are given in Appendix A. A numerical example is provided in section 4.3.

4.1.2. Payment after the Total Depletion of Inventory

This section considers the case in which the grace period expires after the complete depletion of the inventory (Figure 4.2.). As a result, the interest charges for inventory kept after the grace period are reduced to zero, because the supplier is paid in full at the time, M . The interest earned per cycle is the interest earned during the positive inventory period plus the interest earned during the time period (T_1, M) .

The total cost of the system includes (a) setup cost, (b) holding cost, (c) shortage cost, (d) purchasing cost and (e) interest earned during the grace period. The first four cost components are the same as the cost components of the previous case. The interest earned during the grace period is determined as follows:

$$EI = I_e\left(\frac{DpT_1^{\ 2}}{2} + DpT_1(M - T_1) + DpM(T - T_1)\right)$$

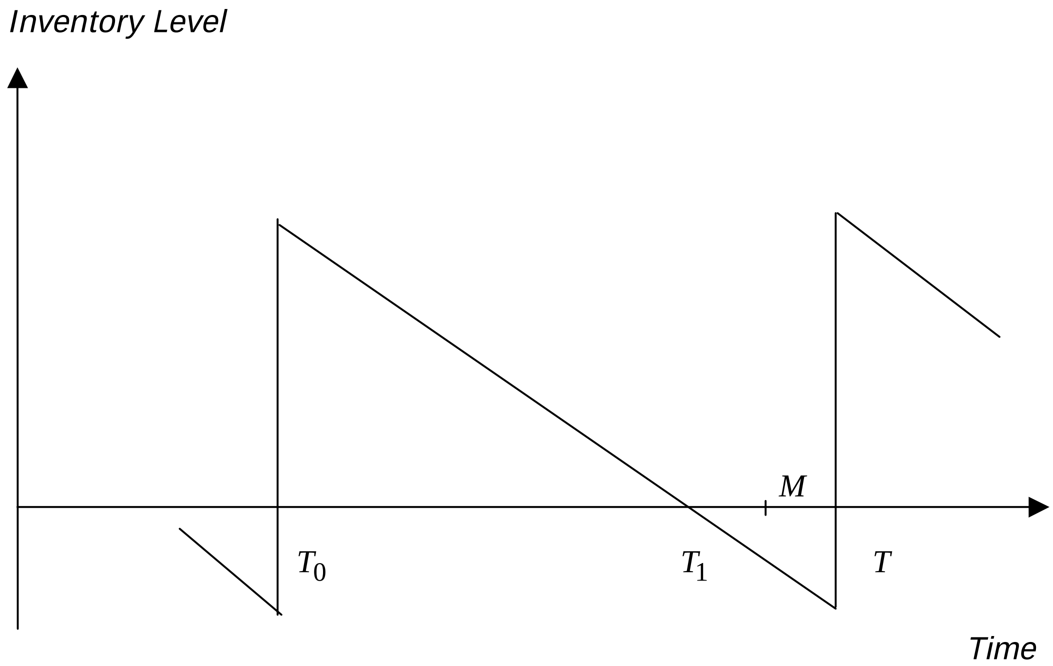


Figure 4.2: The single item inventory system with shortages and permissible delay in payment, when the grace period expires after depletion of the inventory.

The annual total cost of the system is obtained by multiplying the total cost of the system per cycle by the number of replenishment cycles per year and is given by:

$$TCP = \frac{A}{T} + \frac{hDT_1^{\,2}}{2T} + \frac{pDT}{T} + \frac{\pi D(T-T_1)^2}{2T} -$$

$$I_e\left(\frac{DpT_1^{\,2}}{2} + DpT_1(M-T_1) + DpM(T-T_1)\right)\Big/T \qquad (4.3)$$

As mentioned in the previous section, the optimal ordering policy of the system may be obtained by solving the following equations:

$$\frac{\partial TCP}{\partial T} = 0$$

$$\frac{\partial TCP}{\partial T_1} = 0$$

Since no closed form equation could be obtained for the optimal values of T and T_1 using the above equations, one may have to solve these equations by trial and error. Instead a derivative free method, Hooke and Jeeves Algorithm, is used to obtain the optimal policy of the system.

In the next section the effects of inflation and time value of money are considered on the single item deterministic system with shortages and permissible delay in payment.

4.2. Effects of Inflation and Time Value of Money on a Single Item Inventory Model (EOQ) with Shortages and Permissible Delay in Payment

In this section the effects of inflation and time value of money on the inventory models discussed in the previous sections are investigated. It is assumed that all cost components of the inventory system are subject to a constant inflation rate and both inflation and discounting are subject to continuous compounding.

The present value of the total cost of the inventory system is developed first, then the optimal lengths of the replenishment cycle and the period with positive stock are obtained. A finite planning horizon is considered in this case.

Again, two distinct scenarios can be recognized: (1) the case in which the grace period expires before the total consumption of inventory ($M \le T_1$), and (2) the case in which the payment is due after the depletion of the inventory, ($T_1 \le M$).

The appropriate mathematical models are developed for both cases and presented below.

4.2.1. Grace Period Expires before the Total Depletion of Inventory

In this case, the total cost of the inventory system is comprised of six components: setup cost, purchasing cost, inventory carrying cost, shortage cost, interest payable and interest earned during the grace period. Present valuess of the individual costs at the beginning of each cycle are presented below (Figure 4.1).

(a) Setup Cost

Setup cost is paid at the beginning of the cycle and is equal to A.

(b) Holding Cost

The holding cost is continuously paid for the amount of inventory kept during the period in which the stock level is positive. The amount of inventory in hand at time t is :

$$I(t) = D(T_1 - t) \qquad 0 \le t \le T_1$$

The holding cost in the first cycle is given by,

$$HC = Fp\int_0^{T_1} I(t)e^{-Rt}dt \tag{4.4}$$

$$= FpD\int_0^{T_1}(T_1 - t)e^{-Rt}dt$$

$$= FpD\frac{e^{-RT_1} + RT_1 - 1}{R^2}$$

(c) *Purchasing Cost*

The purchasing cost is comprised of two components, the first one is paid at the end of the grace period, M , and covers the items that satisfy the demand during the current period up to M , and the amount of backlog from the preceding period. The second component is paid continuously after the first payment until the end of period with positive inventory, T_1.

The expression for the present value of the first component of the payment is given by:

$$PC_1 = pDMe^{-RM} + pD(T - T_1)e^{-RM} \tag{4.5}$$

The present worth of the second component of the purchasing cost, which is paid continuously after the expiration of grace period, is given by:

$$PC_2 = pD\int_M^{T_1} e^{-Rt}dt \tag{4.6}$$

$$= \frac{pD}{R}(e^{-RM} - e^{-RT_1})$$

The present worth of purchasing cost during the first cycle is given by:

$$PC = PC_1 + PC_2 \tag{4.7}$$

(d) Shortage Cost

The shortage cost occurs if a demand for the commodity takes place when it is out of stock. The amount of shortage at time t is equal to:

$$B(t) = D(t - T_1) \qquad\qquad T_1 \le t \le T$$

The mathematical expression for the present value of the shortage cost during the first cycle is given by:

$$SC = \pi \int_{T_1}^{T} D(t - T_1) e^{-Rt} dt \tag{4.8}$$

$$= \pi D\left(\frac{e^{-RT_1} + e^{-RT}(R(-T + T_1) - 1)}{R^2} \right)$$

(e) Cost of Interest Charges

The expression for the present worth of the interest charges for the value of goods paid after the grace period is as follows:

$$IC = pI_c D \int_{M}^{T_1} (t - M) e^{-Rt} dt \tag{4.9}$$

$$= pI_c D\left(\frac{e^{-RM} + e^{-RT_1}(MR - RT_1 - 1)}{R^2} \right)$$

(f) Interest Earned During the Grace Period

The customer pays no interest during the grace period. Instead he accumulates revenues on the sale or use of the products, and earn interest on the revenue. The present value of the interest earned during the first replenishment cycle is given by:

$$EI = I_e pD \int_{0}^{M} \int_{t}^{M} e^{-R\tau} d\tau dt + I_e pD(T - T_1) \int_{0}^{M} e^{-Rt} dt \tag{4.10}$$

$$= DI_e pM\left(\frac{1-e^{-RM}-R^2Me^{-RM}}{R^2}\right) + I_eDp\left(\frac{1}{R}-\frac{e^{-RM}}{R}\right)(T-T_1)$$

The present value of the total inventory cost during the first cycle is the summation of aforementioned costs:

$$TC_1 = A + HC + PC + SC + IC - EI \qquad (4.11)$$

The effects of inflation and time value of money exist in each cycle of replenishment. In order to develop an appropriate model for the system, a finite time horizon H is considered. The number of replenishment cycles during this time horizon would be $N = \frac{H}{T}$. Equation (4.11) gives the total cost at the beginning of each cycle. If TCP is considered as the present value of the total cost of the inventory system over the planning horizon, then :

$$TCP = TC_1\left(1+\exp(-RT)+\exp(-2RT)+...+\exp(-(N-1)RT)\right) \qquad (4.12)$$

$$= TC_1\left(\frac{1-\exp(-NRT)}{1-\exp(-RT)}\right)$$

$$=\Bigg(A + FpD\frac{e^{-RT_1}-RT_1+1}{R^2} + pDMe^{-RM} + pD(T-T_1)e^{-RM} + \frac{pD}{R}(e^{-RM}-e^{-RT_1})$$

$$+ pD\left(\frac{e^{-RT_1}+e^{-RT}(R(T-T_1)-1)}{R^2}\right) + pI_cD\left(\frac{e^{-RM}+e^{-RT_1}(MR-RT_1-1)}{R^2}\right)$$

$$- DI_e pM\left(\frac{1-e^{-RM}-R^2Me^{-RM}}{R^2}\right) - I_eDp\left(1-e^{-RM}\right)\frac{(T-T_1)}{R}\Bigg)\left(\frac{1-\exp(-NRT)}{1-\exp(-RT)}\right)$$

Two different cases can be considered, (i) $r > f$ (i.e. $R = r - f > 0$), and (ii) $r < f$ (i.e. $R = r - f < 0$). Expressions of the total cost of the system for both aforementioned cases are presented below.

(i) $r > f$ ($R = r - f > 0$)

If $r > f$, when $H \to \infty$, $N \to \infty$ and the total cost given by (4.12) will converge to:

$$TCP = TC_1\left(\frac{1}{1-\exp(-RT)}\right) \tag{4.13}$$

(ii) $r < f$ ($R = r - f < 0$)

In this case, when $N \to \infty$, the total cost given by (4.12) will not be convergent. Instead, a finite horizon would be considered. If one takes $H = 1$ year, then $N = \frac{1}{T}$ and the present worth of the total cost would be:

$$TCP = TC_1\left(\frac{1-\exp(-R)}{1-\exp(-RT)}\right) \tag{4.14}$$

The values of T and T_1, which minimize TCP, may be obtained by simultaneously solving $\partial TCP(T,T_1)/\partial T = 0$, and $\partial TCP(T,T_1)/\partial T_1 = 0$. Since $TCP(T,T_1)$ involves a complicated exponential function, it is not easy to calculate the partial derivations of the total cost function and obtain closed form expressions for the optimal values of T, T_1 from the partial derivatives of the total cost function. Hence the direct search method of Hooke and Jeeves is used again to obtain the optimal ordering policy and minimum annual cost of the system. A numerical example is provided in section 4.4 to illustrate the procedure used and the general behavior of the system with respect to different parameters of the system.

In the next section the case is considered in which the grace period expires after the complete depletion of the inventory, $T_1 < M$.

4.2.2. Grace Period Expires after the Total Depletion of Inventory

The amount of order in each period can be chosen such that the inventory in hand is consumed before the fixed grace period expires. In this section, the mathematical expressions, for the cost components of the system are developed. The inventory system in this situation includes five cost components, (a) setup cost, (b) holding cost (c) purchasing cost, (d) shortage cost, and (e) interest earned during the grace period. In this case, the customer pays for the entire order at the end of the grace period, and no interest is charged for late payment. Setup, holding, and shortage costs are the same as in last case, but purchasing cost and interest earned during the grace period will change in the new situation. The mathematical expressions for these cost components are developed below.

(i) *Purchasing Cost*

The total amount of purchasing cost in each replenishment cycle is paid at the end of the grace period. Hence the present worth of the purchasing cost in the first cycle is given by:

$$PC_1 = pDTe^{-RM} \tag{4.15}$$

(ii) Interest Earned During the Grace Period

$$EI_1 = I_e pD \int_0^{T_1} \int_t^M e^{-R\tau} d\tau dt + I_e pD(T - T_1) \int_0^M e^{-Rt} dt \tag{4.16}$$

$$= I_e pD \left(\frac{1 - e^{-RT_1} - RT_1 e^{-RM}}{R^2} \right) T_1 + I_e Dp \left(\frac{1}{R} - \frac{e^{-RM}}{R} \right) (T - T_1)$$

The present value of the total inventory cost during the first cycle is the summation of (1) Setup cost, (2) Inventory holding cost, (3) Purchasing cost, (4) shortage cost, and (5) Interest earned during the grace period.

$$TC = A + HC + PC + SC - EI \tag{4.17}$$

For a planning horizon of H with a number of cycles $N = \frac{H}{T}$, the present value of the total cost of the system is given by:

$$TCP = TC \left(1 + \exp(-RT) + \exp(-2RT) + ... + \exp(-(N-1)RT)\right) = TC \frac{1 - e^{-NRT}}{1 - e^{-RT}}$$

Similar to the previous section, two cases are considered, (i) $R > 0$ for which $TCP = TC\left(\frac{1}{1 - e^{-RT}} \right)$, when $H \to \infty$ and (ii) $R < 0$ for which $TCP = TC\left(\frac{1 - e^{-R}}{1 - e^{-RT}} \right)$, when $H = 1$.

As in the previous models, in order to obtain the optimal ordering policy of the system, the following equations should be simultaneously solved.

$\partial TCP(T, T_1) / \partial T = 0$, and $\partial TCP(T, T_1) / \partial T_1 = 0$

However, the total cost function of the system involves complicated exponential functions which makes the task of obtaining closed form solutions for the optimal values of T, T_1 very difficult, if not impossible. Hence the algorithm of Hooke and Jeeves is used to obtain the optimal policy and minimum cost of the system.

4.3. Optimal Solution for Inventory Model (EOQ) with Shortages and Permissible Delay in Payment

The main objective of the models considered in this research is.to minimize the annual cost which is a nonlinear function of the length of the replenishment cycle and the length of the period with positive inventory, T, T_1. Since it is very hard, if not impossible, to derive closed formulae for the decision variables, some numerical analysis method or alternatively a nonlinear optimization technique needed to be used. The direct search method of Hooke and Jeeves was applied to obtain the optimal solutions. The general idea of the search method is to change the value of one variable at a time while keeping the other(s) constant (exploratory search). By using the information obtained at the exploratory search stage, a direction is defined to move towards the minimum point (pattern search). A detailed explanation of the Hooke and Jeeves search algorithm is provided in Appendix A. A computer software was developed to facilitate the implementation of the optimization algorithm. The code is given in Appendix B. The application is implemented using Microsoft Visual Basic 6. As Microsoft premium programming language, Visual Basic 6 is the most popular Object Oriented Programming approach for application development. It utilizes powerful Windows features as the front end. One instance of the front end of the application is presented in figure 4.4.

An example is presented here to illustrate the application of the models developed with the following data:

D = 200 units/yr, A = 10 \$/order, p = 5 \$/unit, F = 0.3 \$/\$/yr, π = 5 \$/unit/yr, I_c = 0.15 \$/\$/yr, I_e = 0.1 \$/\$/yr, M = [0.01, 0.03, 0.05, 0.07, 0.1, 0.12, 0.14] year.

The behavior of the optimal policy of the system with respect to the length of permissible delay in payment is investigated with the help of appropriate graphs and discussions. Relationship between the total annual cost of the system, the length of the replenishment cycle, and length of the period with positive stock and different lengths of grace period are also investigated.

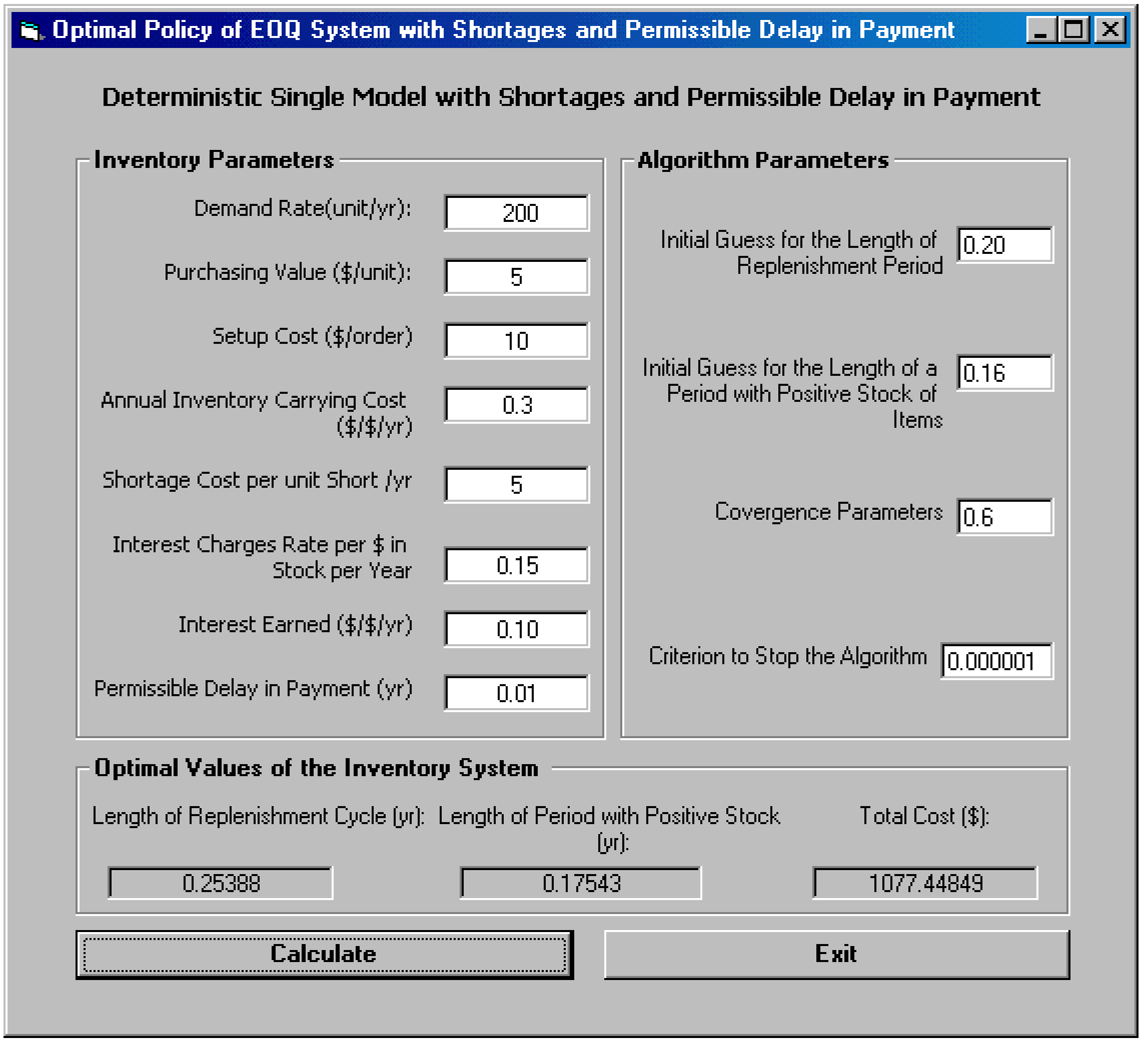


Figure 4.4: Example of Computer Application implementing Hooke and Jeeves Algorithm

The optimal ordering policies obtained for different values of M for the above example are reported in Table 4.1.

Grace Period ,Year	*Replenishment Cycle (T), Year*	*Period With Positive Stock (T1), Year*	*Optimal Order (Q), Units*	*Present value of Total Cost,*
0.01	0.25388	0.17543	50.776	$1077.449
0.03	0.25417	0.17631	50.834	$1074.835
0.05	0.25468	0.17753	50.936	$1072.297
0.07	0.25537	0.17856	51.074	$1069.835
0.10	0.2569	0.18064	51.38	$1066.281
0.12	0.25823	0.18222	51.646	$1064.003
0.14	0.25985	0.18402	51.97	$1061.798

Table 4.1. Optimal Ordering Policies for Different Values of *M.*

As shown in figure 4.5., the length of the replenishment cycle increases as the length of grace period increases.

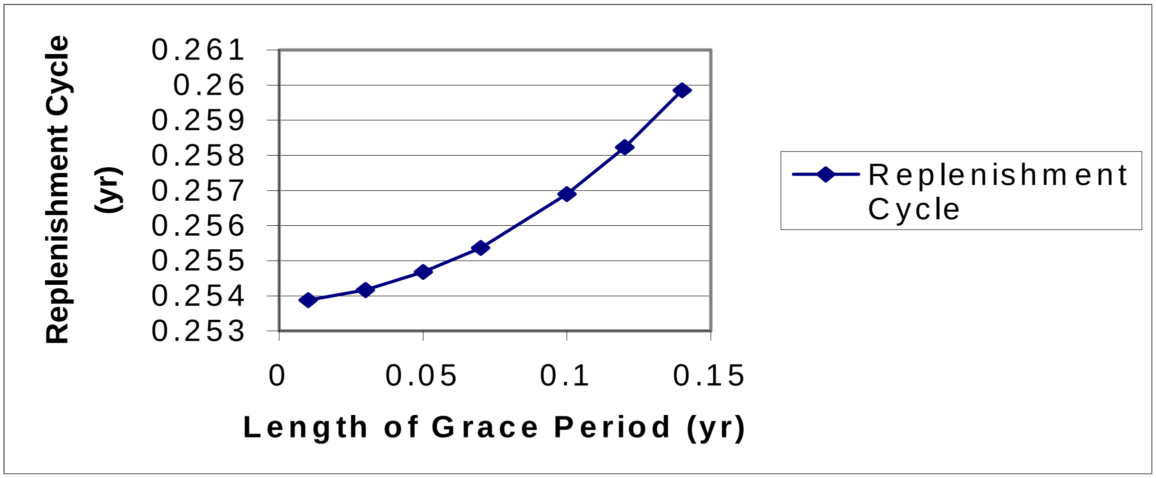


Figure 4.5: Behavior of *T,* with respect to length of grace period.

The length of period in which the inventory level is positive also increases as the length of grace period increases, as shown in figure 4.6.

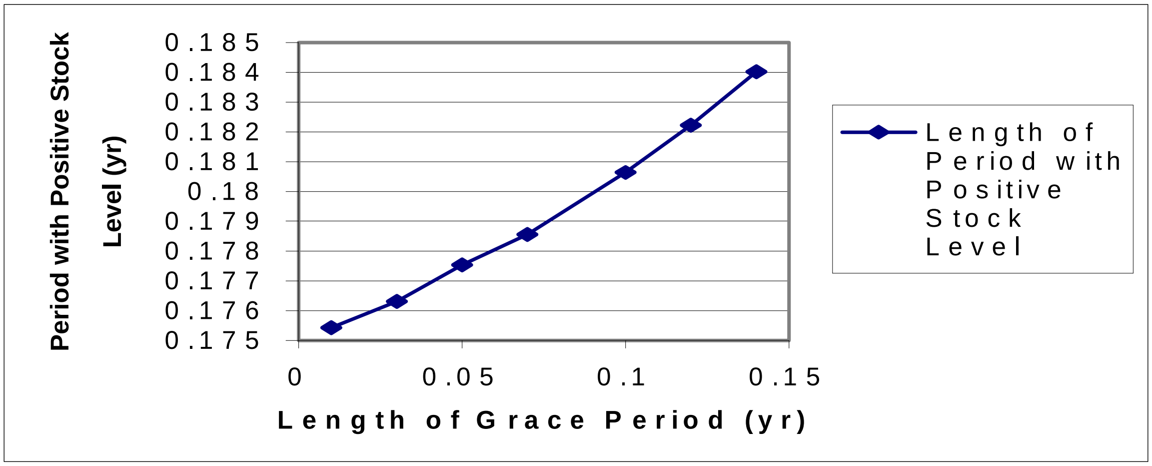


Figure 4.6: Behavior of T_1, with respect to length of grace period.

The total cost of the system decreases as the length of grace period increases, as shown in figure 4.7.

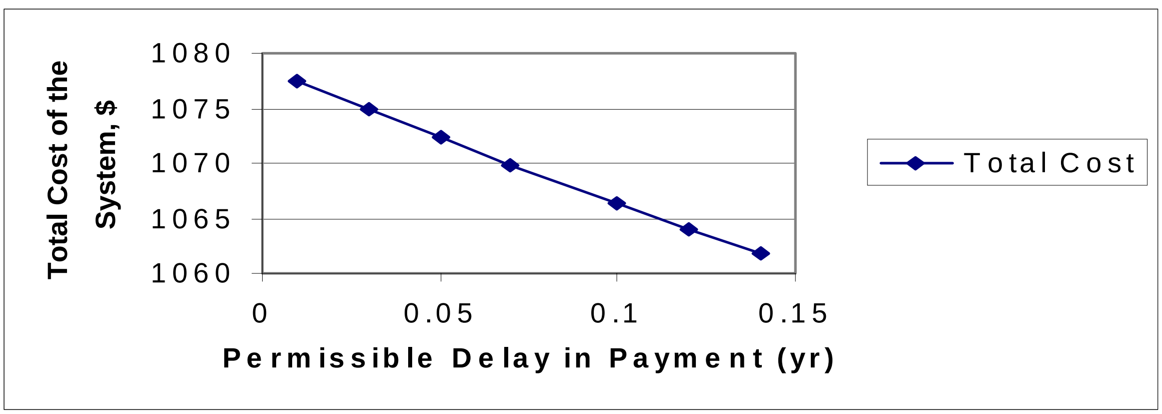


Figure 4.7: Behavior of the total cost of the system with respect to length of the grace period

The total cost function of the system for the above example is represented in figure 4.8.,which shows the behavior of the total cost with respect to two decision variables, Replenishment Cycle, T , and Positive Stock Level Period T_1. The figure shows that the total cost function forms a convex surface.

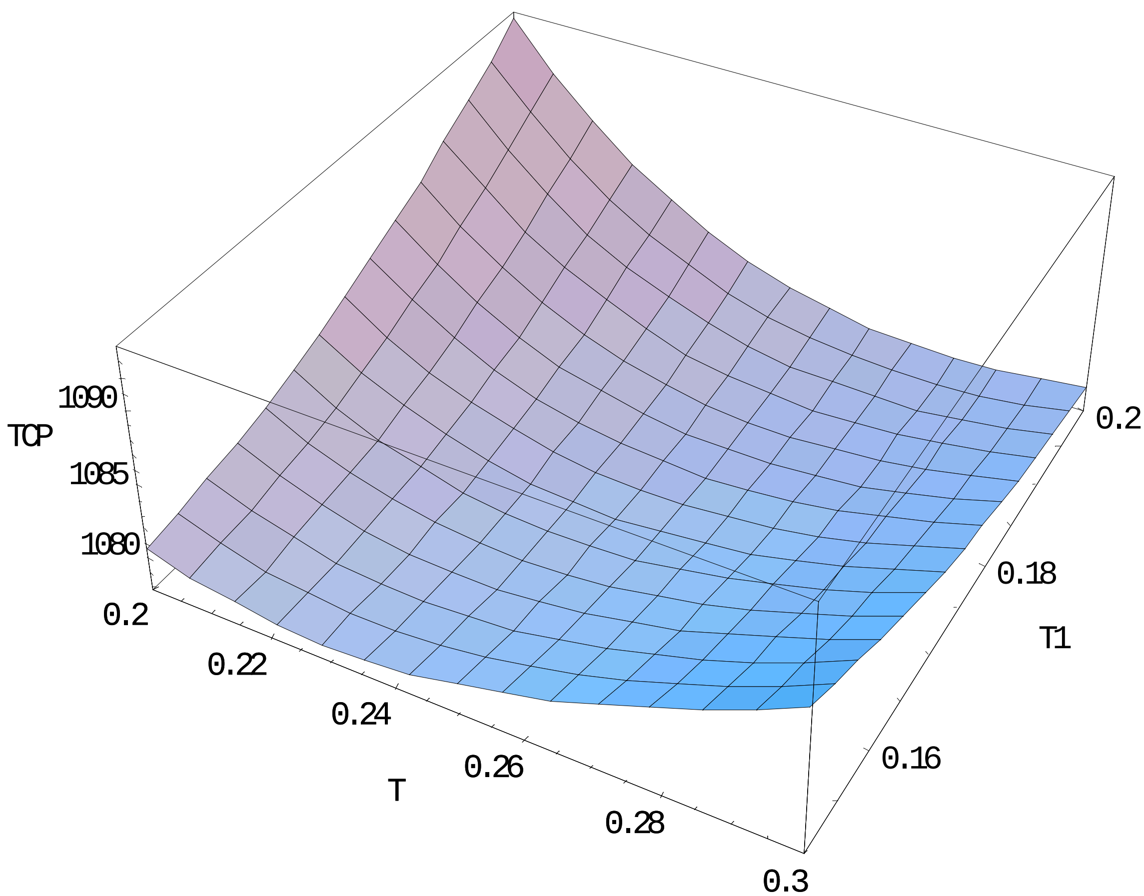


Figure 4.8: Total Cost Function of the system $TCP(T,T_1)$

4.4. Optimal Solution for Inventory Model (EOQ) with Shortages, Inflation, Time Value of Money and Permissible Delay in Payment

In this section a numerical example is provided to illustrate the application of the inventory model developed in section 4.2. The sensitivity of the present value of the total cost of the system to the system parameters is also examined. The same parameter values used in the previous example are considered.

The present value of the total cost of the inventory system is a function of two continuous variable, length of replenishment cycle, T , and length of the period with positive inventory level, T_1. As in the previous example, the direct search procedure of Hooke and Jeeves was applied to obtain the optimal policy of the system. Detailed explanations about the algorithm are given in Appendix A. The algorithm was implemented using Visual basic 6. The code is given in Appendix C. An instance of the application developed is shown in figure 4.9.

The application developed allows the user to change different parameters of the system and obtain the optimal ordering policy and minimum cost.

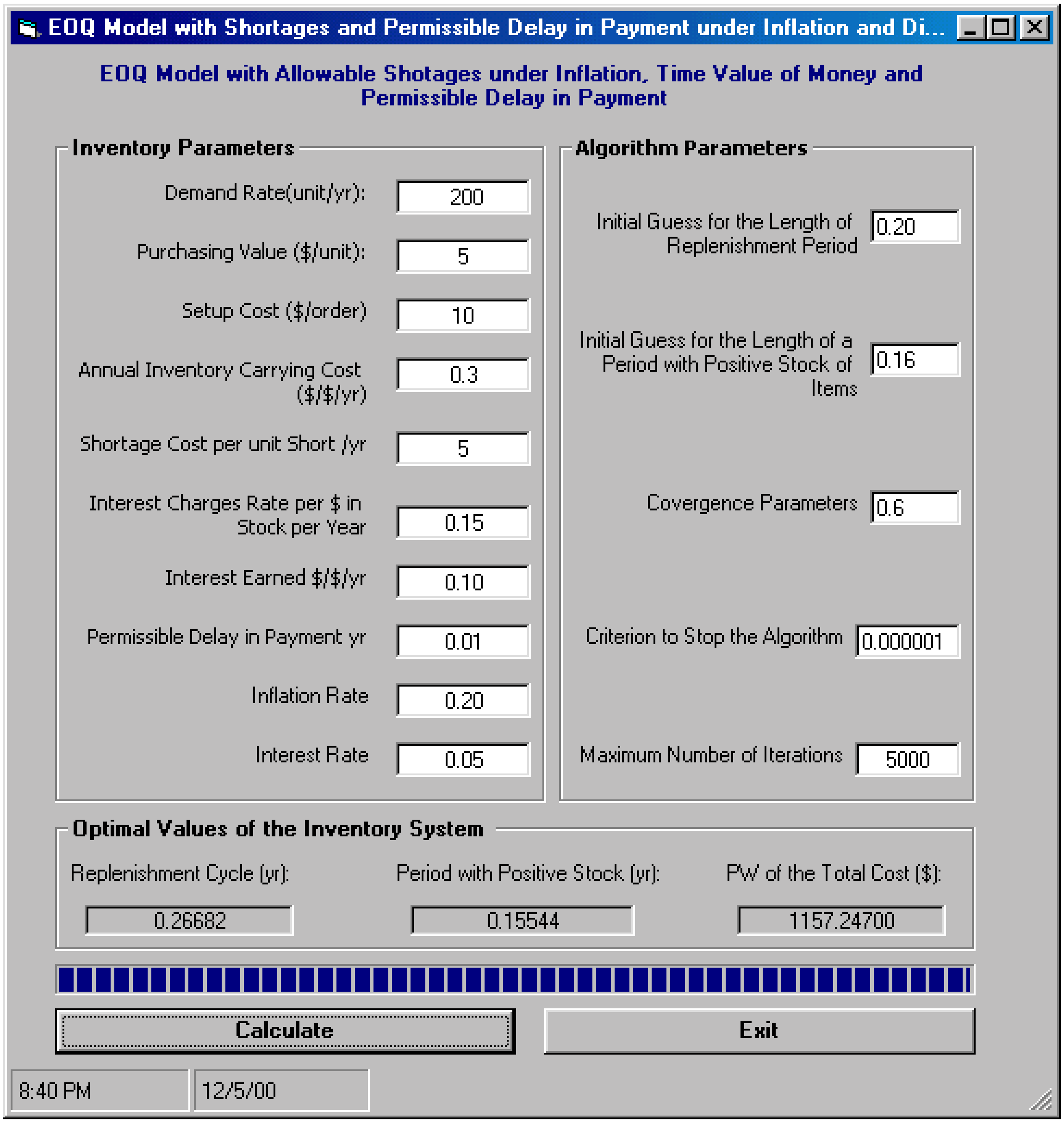


Figure 4.9: Example of the Computer Application developed to implement the Hooke and Jeeves Search Method

The inventory parameters are as follows:

D = 200 units/yr, A = 10 \$/order, p = 5 \$/unit, F = 0.3 \$/\$/yr, π = 5 \$/unit/yr, I_c = 0.15 \$/\$/yr, I_e = 0.1 \$/\$/yr. Different values of M and $R = r - f$, are considered. Several tables are provided which show the length of replenishment cycles, length of period with positive stock, optimal ordering quantities and present values of minimal annual cost for M =[0.01, 0.03, 0.05, 0.07, 0.10, 0.12, 0.15], and R =[0.1, 0.05, - 0.1, - 0.3].

Several figures are also provided to illustrate the relationship between the decision variables and the present value of the total cost of the system for different parameter values. The values of the length of the replenishment cycle, length of period with positive stock, and present value of the total cost of the system are given for different values of grace period at specific values of inflation rate. Additional graphs are provided to illustrate the relationship between the decision variables and the present value of the total cost of the system for different values of inflation rate.

Table 4.2. shows the optimal policy of the system when $R = 0.1$.

Delay in Payment (M),Year	*Replenishment Cycle (T), Year*	*Period With Positive Stock (T1), Year*	*Optimal Order (Q), Units*	*Total Cost, $*
0.01	0.25764	0.20415	50.95200	1025.0850
0.03	0.25624	0.20080	51.34800	1023.1450
0.05	0.25352	0.19617	50.99800	1020.7990
0.07	0.24923	0.19033	50.45200	1018.0360
0.10	0.24013	0.17952	49.36600	1013.0640
0.12	0.23178	0.17034	48.38400	1009.1500
0.15	0.21604	0.15421	46.58600	1002.2510

Table 4.2. Optimal Policy of the System when R= 0.1.

Figure 4.10. shows that the length of replenishment cycle generally decreases as the length of grace period increases.

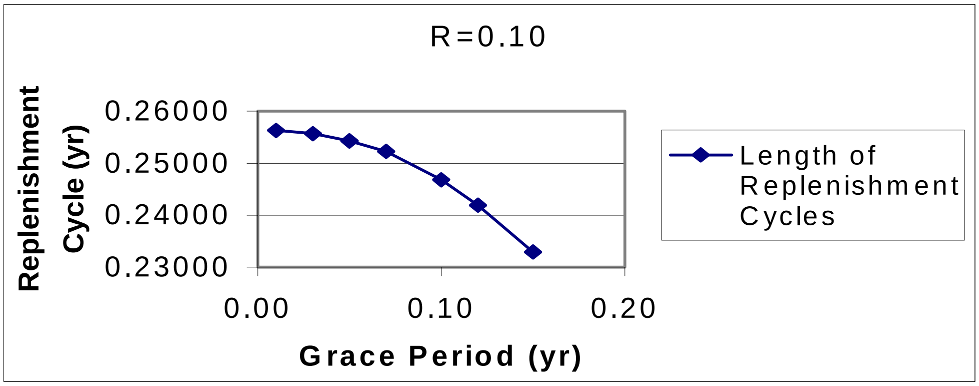


Figure 4.10: Relationship between T and the length of grace period when R= 0.1.

In figure 4.11. the length of the period with positive stock level decreases as the length of grace period increases.

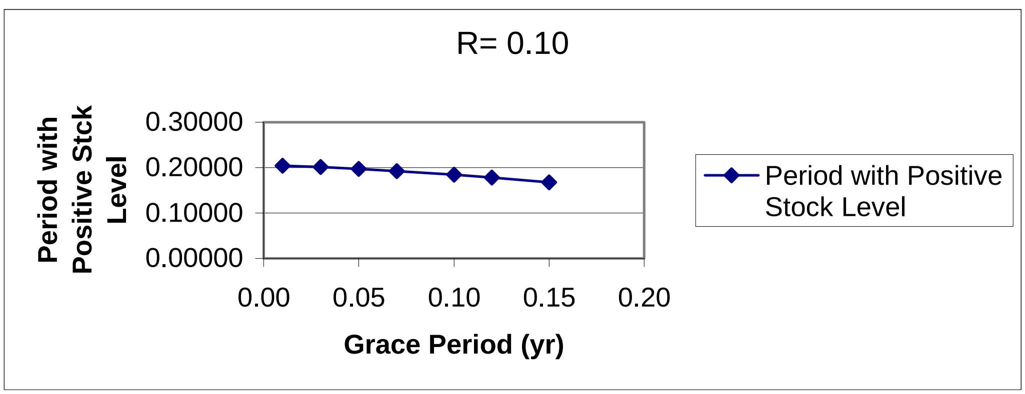


Figure 4.11: Relationship between T_1 and the length of the grace period when R= 0.10

Figure 4.12. shows that the present value of the total cost of the inventory system decreases as the length of the grace period increases.

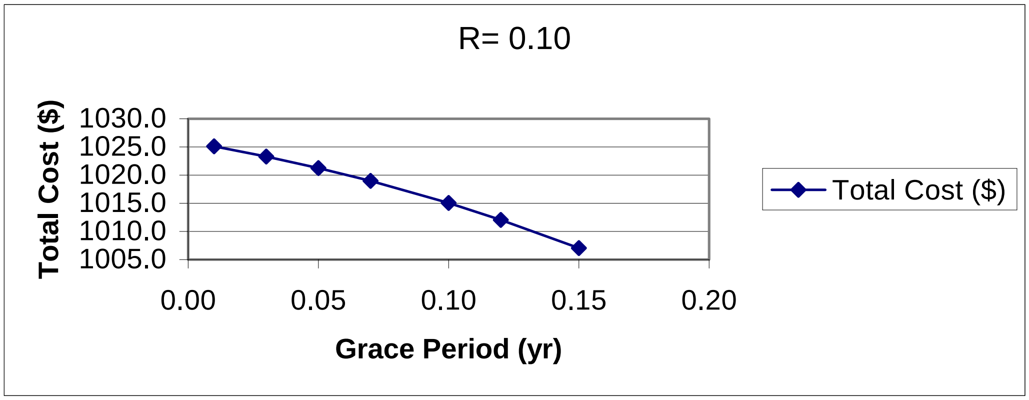


Figure 4.12: Relationship between the total Cost of the System and the Length of the Grace Period.

Figure 4.13. illustrates the shape of the total cost function of the system when M= 0.01 and R = 0.1, which shows that the function makes a convex surface.

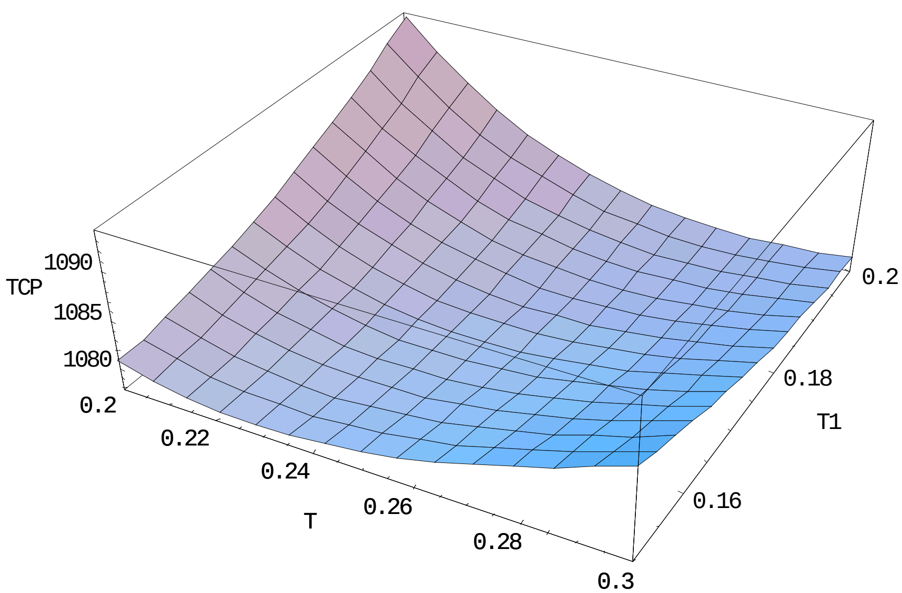


Figure 4.13: Total Cost Function, M = 0.01, R = 0.10

The analysis presented in Table 4.2 and Figures 4.10 through 4.12 is repeated below in Tables 4.3 through 4.5 and Figures 4.14 through 4.20 with the values of R changed from 0.10 to 0.05, -0.10 and -0.30.

Delay in Payment (M),Year	*Replenishment Cycle (T), Year*	*Period With Positive Stock (T1), Year*	*Optimal Order (Q), Units*	*Total Cost, $*
0.01	0.25303	0.18672	50.818	1051.2580
0.03	0.25370	0.18651	50.798	1048.8110
0.05	0.25168	0.18434	50.66	1046.1300
0.07	0.24929	0.18168	50.52	1043.2080
0.1	0.24479	0.17724	50.178	1038.3630
0.12	0.24032	0.17319	49.922	1034.8100
0.15	0.23182	0.16601	49.408	1028.9590

Table 4.3: Optimal Ordering Policy of the System, When R = 0.05.

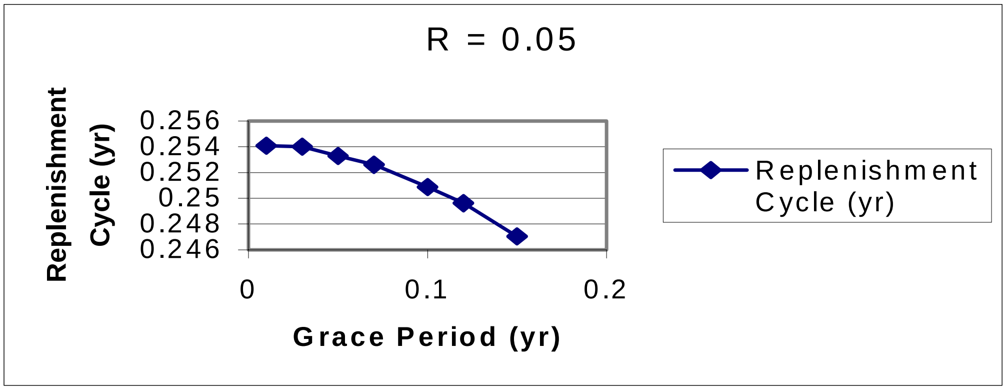


Figure 4.14: Relationship between *T* and the length of the grace period M, when R = 0.05.

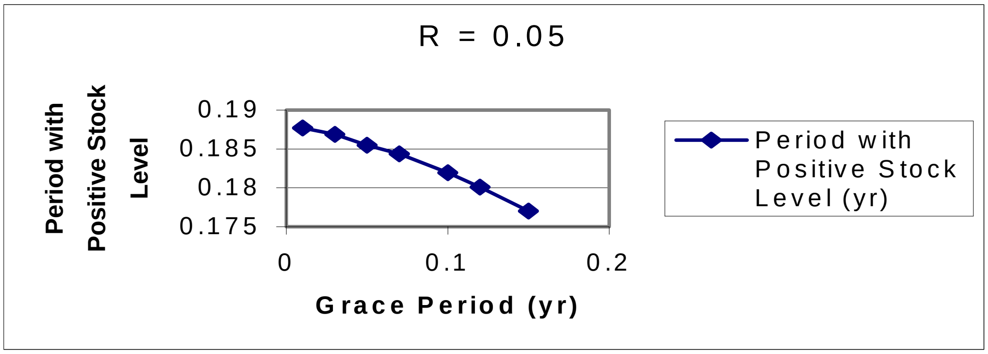


Figure 4.15: Relationship between T_1 and the length of the grace period M When R= 0.05.

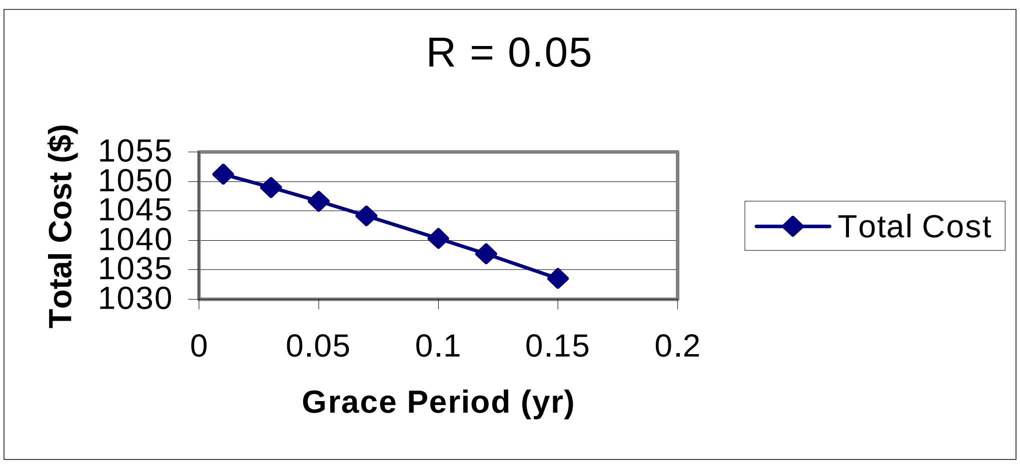


Figure 4.16: relationship between the total cost of the system and *M* When R= 0.05.

Delay in Payment (M),Year	*Replenishment Cycle (T), Year*	*Period With Positive Stock (T1), Year*	*Optimal Order (Q), Units*	*Total Cost, $*
0.01	0.26054	0.1600	52.078	1130.3290
0.03	0.26131	0.16351	52.248	1127.3240
0.05	0.26206	0.16688	52.838	1124.4960
0.07	0.26425	0.17136	53.454	1121.8440
0.10	0.26755	0.17791	54.816	1118.1870
0.12	0.27023	0.18267	55.902	1115.9570
0.15	0.27643	0.19103	57.904	1112.9110

Table 4.4. Optimal Policy of the System when R = - 0.10.

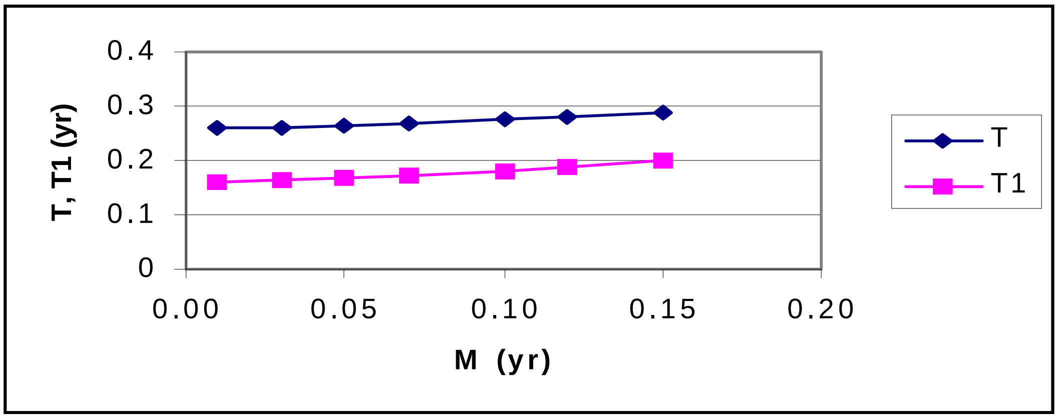


Figure 4.17: Relationship between T, and T1 the length of the grace period, when R = - 0.10.

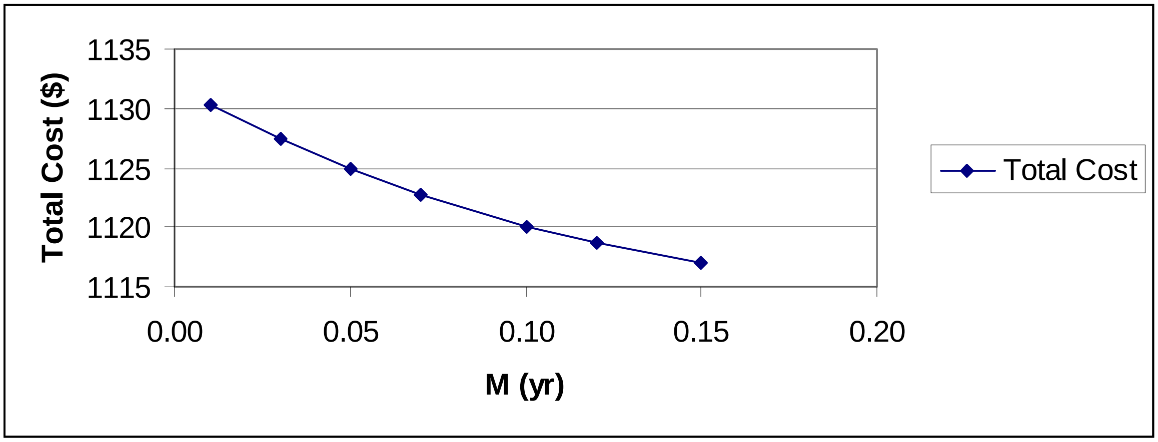


Figure 4.18: Relationship between the total cost of the system and M,

when R= - 0.10.

Delay in Payment (M),Year	*Replenishment Cycle (T), Year*	*Period With Positive Stock (T1), Year*	*Optimal Order (Q), Units*	*Total Cost, $*
0.01	0.29864	0.15146	59.728	1240.1790
0.03	0.30150	0.15903	60.422	1237.8530
0.05	0.30537	0.16746	61.424	1236.0410
0.07	0.31197	0.17684	63.018	1234.60718
0.10	0.32543	0.19295	66.366	1233.6480
0.12	0.33610	0.20447	69.208	1233.4290
0.15	0.35495	0.22352	73.79	1233.8390

Table 4.5. The optimal policy of the system when R = - 0.30 and M varies

from 0.01 to 0.15.

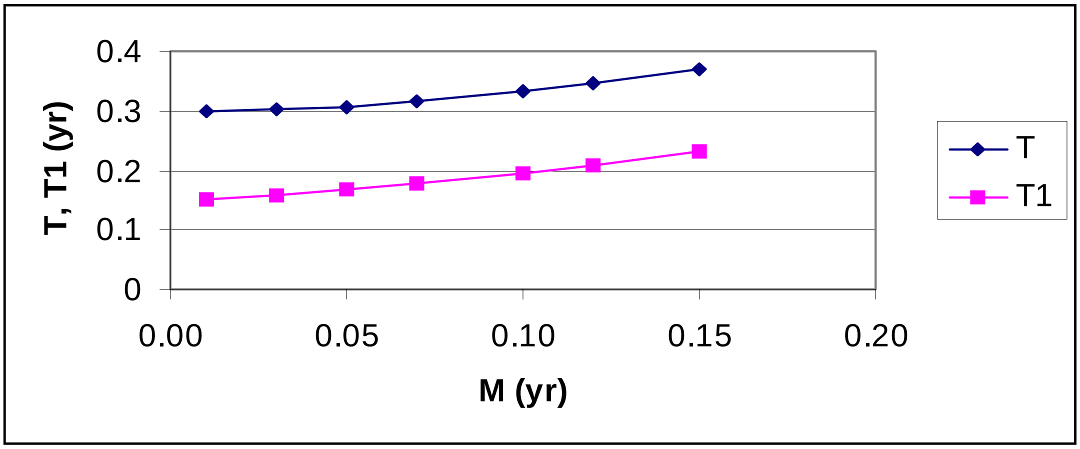


Figure 4.19: Relationship between T, T1 and M, when R= - 0.30

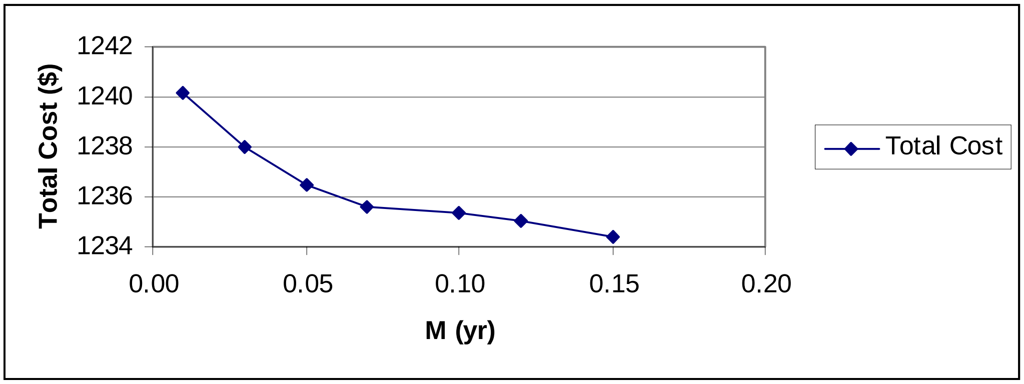


Figure 4.20: Relationship between the total cost and M When R = - 0.30.

Table 4.6 summerizes the optimal solutions and shows the effects of inflation and discounting on the system for M=[0.01, 0.07, 0.15]. then effects are also illustrated in Figures 4.21 through 4.26, which show the relationship between T, T1, and the total cost, and - R for different values of the grace period M.

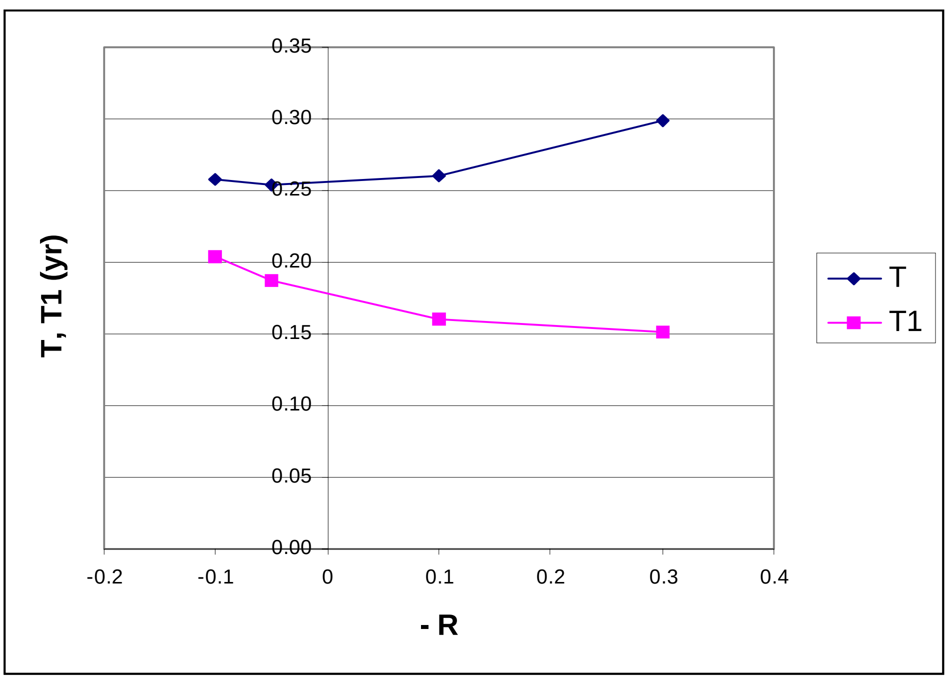


Figure 4.21: Relationship between T, T1 and - R when M= 0.01.

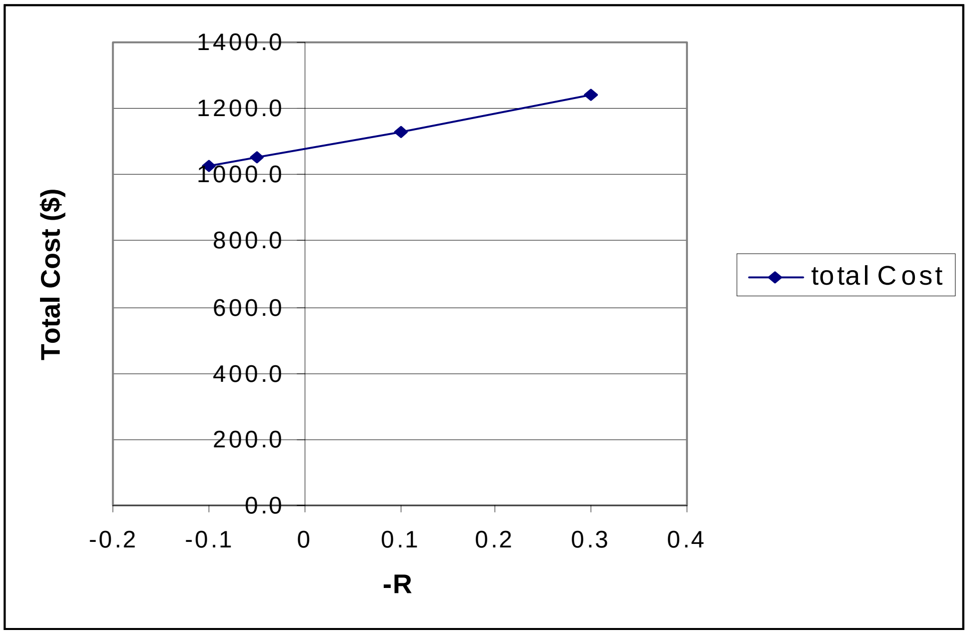


Figure 4.22: Relationship between the total cost and - R when M = 0.01.

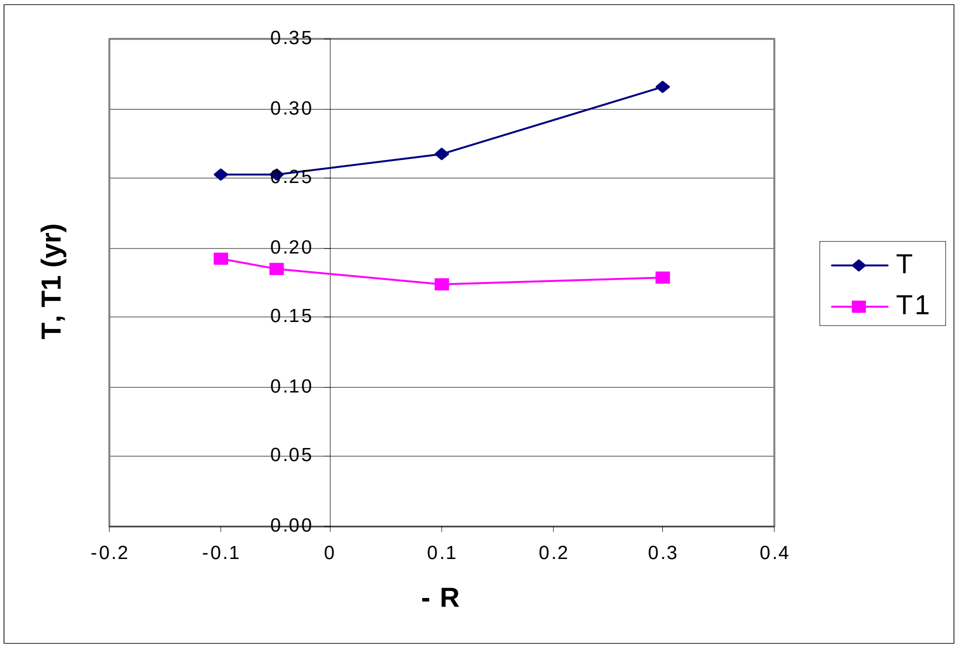


Figure 4.23: Relationship between T, T1 and -R When M=0.07.

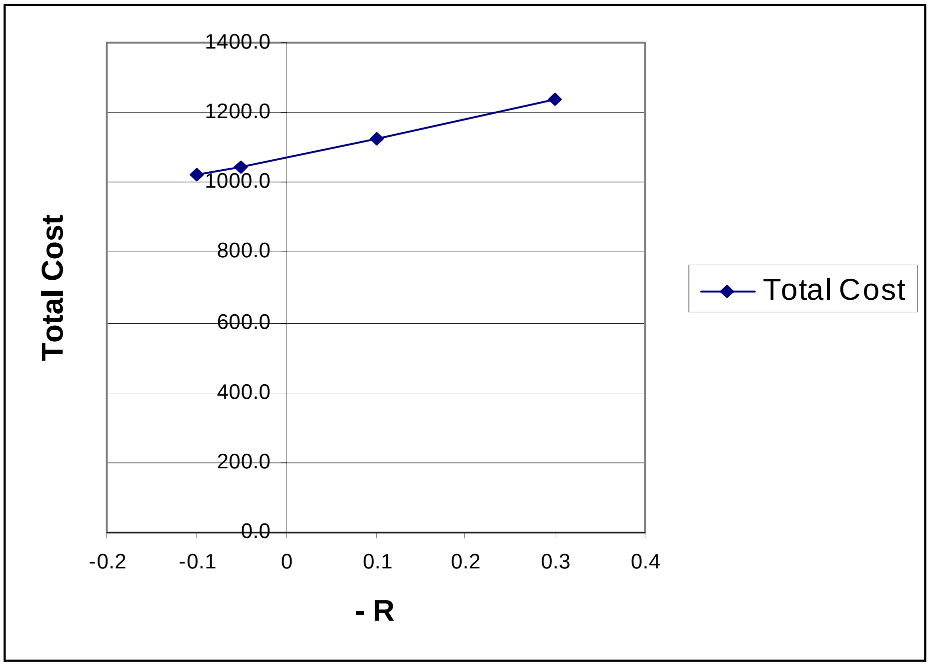


Figure 4.24: Relationship between the total cost and - R when M= 0.07

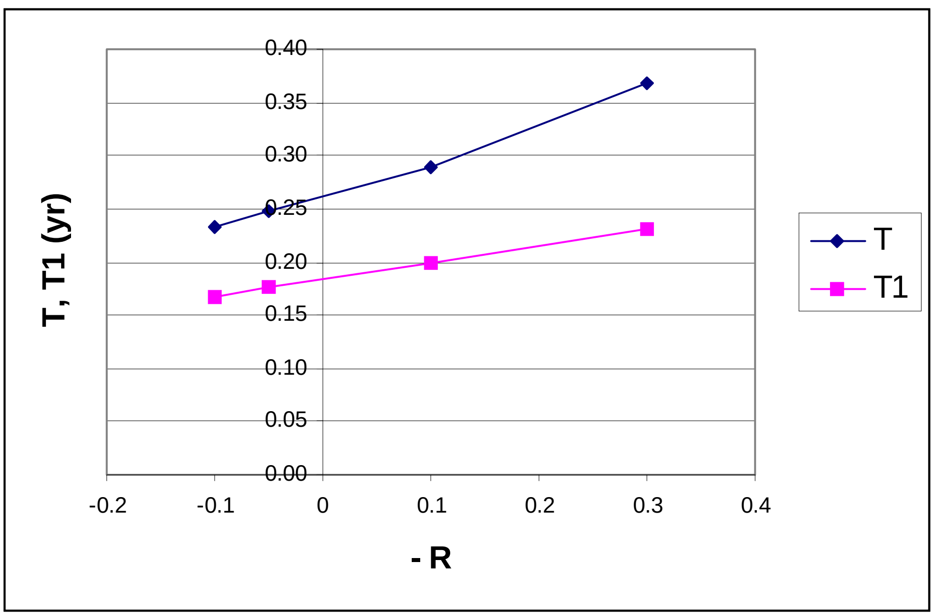


Figure 4.25: relationship between T, T1 and - R when M=0.15.

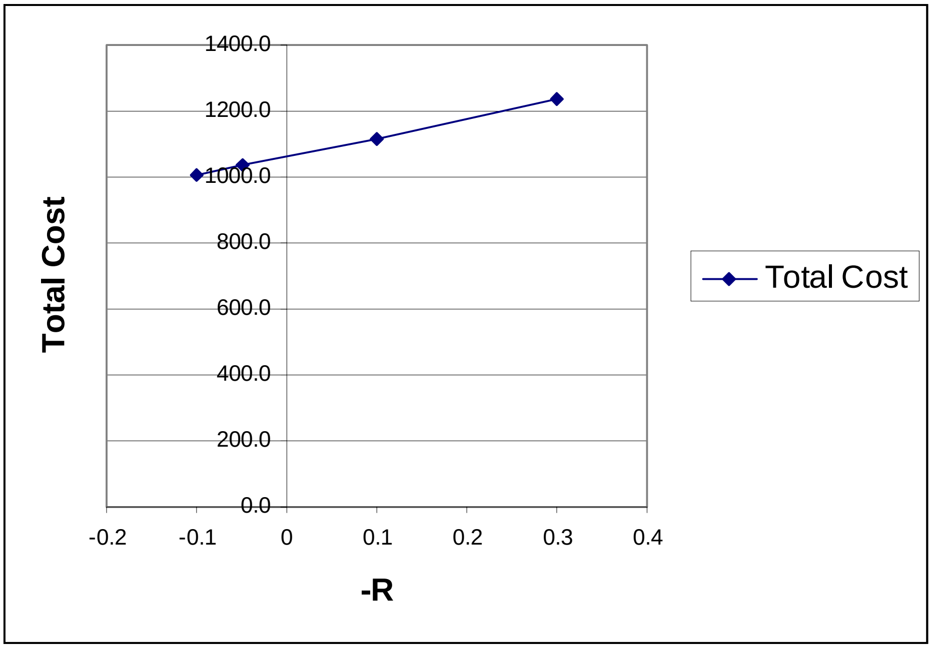


Figure 4.26: Relationship between the total cost of the system and - R, when M=0.15

From the information provided in Tables 4.2 through 4.5 and Figures 4.21 through 4.26, it can be observed that the total cost of the system increases when the inflation rate increases. The length of the replenishment cycle generally tends to increase when the inflation rate increases. The length of the period with positive stock level has a tendency to increase when the length of grace period increases.

Chapter Five

Dynamic Single Item Inventory Models with Inflationary Conditions and Permissible Delay in Payment.

In this chapter inventory models are developed for products with time-varying demand rate. In these models, customers are allowed a period of time to pay back for the goods bought without paying interest and all costs are subject to a uniform inflation rate and discounting. A case is considered in which the grace period granted by the supplier, is a fraction of the replenishment cycle. A mathematical representation of the model will be developed and the optimal policy will be presented for the system.

As stated in previous chapters, it is common these days to see that customers are allowed a grace period to settle the account with the supplier and pay for the goods bought within that period. Customers pay no interest during that period and can postpone the payment till the end of the grace period, but after that period, if the customer has not paid for the goods delivered, an interest will be charged.

Granting a delay period in payment to the customer can be considered as a demand stimulating activity performed by the supplier to encourage the customer to buy more. Hence an appropriate pattern should be considered which properly presents the demand during the planning horizon. In the model presented in this chapter the demand rate is considered as a linear function of time.

As mentioned earlier, inflation is a fundamental feature of today's economy all over the world and large-scale inflation rates are not uncommon in many countries. On the other hand, inventory represents a capital investment of a firm and must compete with other assets for the firm's limited funds. Therefore the effects of inflation and time value of money are explicitly considered in analyzing the inventory system in this chapter.

The length of replenishment cycles are considered to be equal and the purpose of the model is to determine the length of the replenishment period such that the total inventory cost is minimized.

The remainder of this chapter is organized as follows: In the next section the assumptions underlying the model are presented and for more convenience, the notations used throughout the research are reproduced and new terms are added. In the third section the model is developed. The fourth section provides numerical examples to illustrate the application of the model developed.

5.1. Assumptions and notations

The following notations are used throughout the chapter.

h = unit inventory-carrying cost per unit per year, \$/unit.yr

I_c = interest charges per \$ investment in stock per year, \$/\$.yr

I_d = interest that can be earned per \$ in a year, \$/\$.yr

p = unit purchasing price, \$

A = Ordering cost for one order, \$

M = permissible delay in settling the account as a fraction of the replenishment cycle

T = time interval between two consecutive orders, yr

H = length of planning horizon, yr.

r = interest rate, /yr.

f = inflation rate, /yr.

$R = r - f$

CR = total setup (ordering) cost during the planning horizon

CH_j = present worth of holding cost during the *j-th* cycle

CH = present worth of total holding cost during the planning horizon

CP_{j1} = present worth of purchasing for the j-th cycle, for goods demanded during the grace period

CP_{j2} = present worth of the purchasing cost for the goods demanded after the grace period for the j-th cycle

CP = present worth of the purchasing cost during the planning horizon.

CI_j = present value of interest charges during the j-th period

CI = present value of the total interest charges for the items kept after the grace period during the planning horizon

CE_j = present value of the interest earned during the j-th cycle

CE = present value of the total interest earned during the planning horizon

TC = present worth of the total cost of the inventory system

Assumptions

The mathematical model of the inventory replenishment problem is based on the following assumptions:

- All the cost components of the inventory system are subject to an inflation rate, which is constant over the planning horizon.
- Inflation and discounting are subject to continuous compounding.
- The replenishment rate is infinite.
- Shortages are not permitted.
- Lead time is zero
- The system operates for a prescribed planning horizon, H.
- The demand rate is a linear function of the time, i.e.,

$$D(t) = a + bt, \quad a, b > 0 \quad \text{and} \quad 0 \le t \le H.$$

- m replenishments are made during the entire time horizon, H, and the length of each replenishment cycle is equal to H/m.
- A constant fraction of each replenishment cycle, M, is considered to be the grace period granted by the supplier.
- Five cost components will be considered:

  (a) Ordering cost

  (b) Holding cost

  (c) Purchasing cost

  (d) Cost of interest charges for the items kept in stock after the grace period

  (e) Interest earned during the grace period.

- Ordering cost is paid at the beginning of each replenishment period and is subject to the constant inflation rate effective during the planning horizon.
- The holding cost is paid through each replenishment cycle and is affected by inflation rate.
- Two components will be considered for purchasing cost
  - (a) The first part is instantaneously paid at the end of each grace period, M_j, during the replenishment cycles, and covers the cost of goods demanded during the grace period.
  - (b) The second part is continuously paid after the grace period until the beginning of next replenishment cycle, for the goods demanded after the grace period up to the end of the replenishment cycle.

  Both components are subject to constant inflation rate and discounting during the planning horizon.
- An interest is charged for items kept in the stock after the grace period.
- An interest is earned during the grace period.

5.2. Mathematical Formulation of the Proposed Model

In this section the mathematical formulation of the inventory model is presented. The objective of the proposed model is to determine the optimal length of the replenishment cycles during the planning horizon, H, which minimizes the present worth of the total inventory cost.

In this model, there are m equal length replenishment cycles during the planning horizon. Where m is unknown and needs to be determined. Replenishments occur at the beginning of each cycle, $t = T_0, T_1, T_2, ..., T_{m-1}$, and $T_j = \frac{H * j}{m}$, $j = 0,1,2,3,...,m$. As illustrated in figure 5.1.,the grace period for the jth cycle is a constant fraction of that cycle and starts at the beginning of the cycle, $t = T_{j-1}$, and continues up to time $t = M_j$. M represents the constant fraction of each cycle length during which the customer can settle the account and pay for the goods bought, but after that an interest will be charged. Hence $M_j = \frac{H(M + j - 1)}{m}$, $j = 1,2,3,...,m$.

The approach used in developing a model for this problem starts by determining the expressions for the present value of the various costs involved in a cycle.

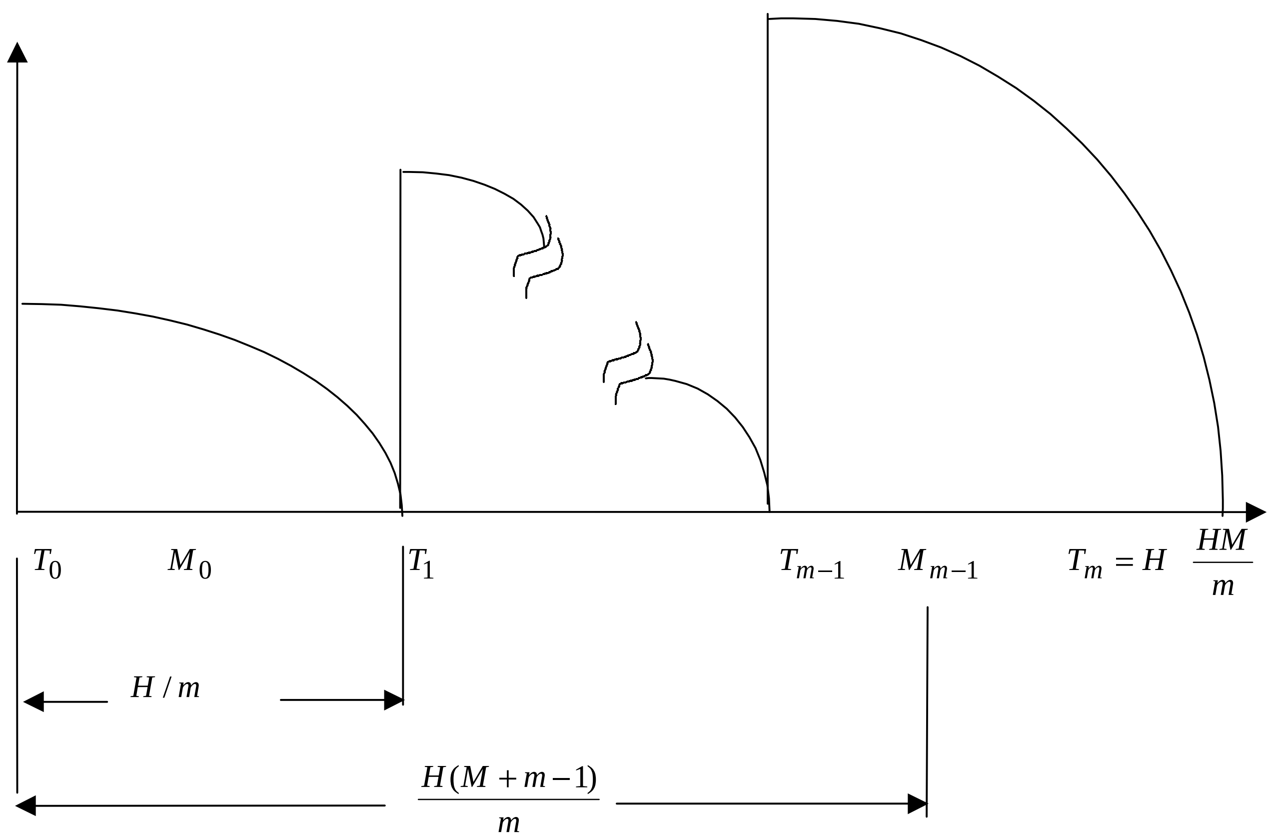


Figure 5.1, Inventory level as a function of time for the proposed model

5.2.1. Ordering cost

Customers pay the ordering cost at the beginning of each period and the cost is subject to a constant inflation rate. The present value of the total replenishment cost incurred during the entire time horizon, H, is given by (the following mathematical equations were developed using a software called *Mathematica*):

$$CR = A\sum_{j=0}^{m-1} \exp(-RT_j) = \frac{A\exp\left(-HR+\frac{HR}{m}\right)(\exp(HR-1)}{-1+\exp\left(\frac{HR}{m}\right)}, \qquad (5.1)$$

5.2.2. Holding Cost

Shortage is not allowed, therefore an inventory is kept during the entire cycle. The present worth of the holding cost during the *j-th* cycle ($j = 1,2,3,...,m$) is as follows:

$$CH_j = Fp \int_{H(j-1)/m}^{Hj/m} (t - H(j-1)/m)(a+bt)\exp(-Rt)dt \qquad (5.2)$$

$$CH_j = FpD\left(\frac{\exp(\frac{-R(j-1)H}{m})(amR+b(2m+H(j-1)R}{mR^3}\right)$$

$$-FpD\left(\frac{\exp(\frac{-HjR}{m})(amR(m+HR)+b(2m^2+H(1+j)mR+H^2jR^2}{m^2R^3}\right)$$

Hence the present worth of the total holding cost during the entire time horizon H , is the following:

$$CH = \sum_{j=1}^{m} CH_j \qquad (5.3)$$

=

$$DFp(\frac{1}{mR^3}(\exp(\frac{HR}{m})(bm(\frac{\exp(-\frac{HR}{m})}{(1-\exp(-\frac{HR}{m}))^2} - (\exp(-\frac{HR}{m})^{1+m}(\frac{\exp(-\frac{HR}{m})}{(1-\exp(-\frac{HR}{m}))^2} + \frac{\exp(\frac{HR}{m})}{\exp(\frac{HR}{m})-1}$$

$$+\frac{\exp(\frac{HR}{m})m}{\exp(\frac{HR}{m})-1}))R + (-1+\frac{1-(\exp(\frac{-HR}{m}))^{1+m}}{1-\exp(\frac{-HR}{m})})(2mb - bHR + amR))) - \frac{1}{m^2R^3}(bH(\frac{\exp(\frac{-HR}{m})}{(1-\exp(\frac{HR}{m}))^2}$$

$$-(\exp(-\frac{HR}{m})^{1+m}(\frac{\exp(-\frac{HR}{m})}{(1-\exp(-\frac{HR}{m}))^2} + \frac{\exp(\frac{HR}{m})(1+m)}{\exp(\frac{HR}{m})-1}))R(m+HR) + (-1+\frac{1-\exp(-HR-\frac{HR}{m})}{1-\exp(-\frac{HR}{m})})$$

$$m(2bm + bHR + amR + aHR^2)))$$

5.2.3. Purchasing Cost

The purchasing cost is comprised of two components. In each cycle the customer pays for the goods demanded during the grace period at the end of the grace period, M_j . The following is the present worth of the purchasing cost for the *j-th* cycle ($j = 1,2,3,...,m$), for the goods demanded during the grace period:

$$CP_{j1} = p\int_{T_{j-1}}^{M_j}(a+bt)\exp(-RM_j)dt\,, \qquad (5.4)$$

$$= p\frac{b\exp(-\frac{H(j+M-1)R}{m})H^2 jM}{m^2} + p\frac{b\exp(-\frac{H(j+M-1)R}{m})H^2 M^2}{2m^2}$$

$$+ p\frac{a\exp(-\frac{H(j+M-1)R}{m})HM}{m} - \frac{b\exp(\frac{-H(j+M-1)}{m})H^2 M}{m^2},$$

$$j = 1,2,3,\ldots,m$$

The customer starts paying for the second portion of the purchasing cost after the grace period and continues till the beginning of the next period for the goods demanded during that time, $t \in [M_j, T_j]$.

Following is the present value of the purchasing cost for the goods demanded after the grace period for the *j-th* cycle:

$$CP_{j2} = p\int_{M_j}^{T_j}(a+bt)\exp(-Rt)dt, \tag{5.5}$$

$$= p\left(-\frac{\exp(-\frac{H(1+j)R}{m})(amR + b(m+HjR))}{mR^2}\right)$$

$$+ p\left(\frac{\exp(-\frac{H(j+M-1)R}{m})(amR + b(m+H(j+M-1)R))}{mR^2}\right), \qquad j = 1,2,3,\ldots,m$$

Hence the present value of the purchasing cost during the planning horizon is:

$$CP = \sum_{j=1}^{m}(CP_{j1} + CP_{j2}) = \tag{5.6}$$

$$\frac{1}{2m^2R^2}(p(2amR((1+\frac{\exp(\frac{HR}{m}-\frac{HMR}{m})(1-(\exp(-\frac{HR}{m})^m)}{\exp(\frac{HR}{m})-1}-\frac{1-\exp(-HR-\frac{HR}{m})}{1-\exp(-\frac{HR}{m})})m$$

$$+(\frac{\exp(-\frac{HMR}{m})(1-\exp(-\frac{HR}{m}))^m}{\exp(\frac{HR}{m})-1})HmR)+b(2H(\frac{\exp(-\frac{HR}{m}-\frac{HRM}{m})}{(1-\exp(-\frac{HR}{m}))^2}-(\exp(-\frac{HR}{m})^{1+m}$$

$$(\frac{\exp(-\frac{HR}{m}-\frac{HMR}{m})}{(1-\exp(-\frac{HR}{m}))^2}+\frac{\exp(\frac{HR}{m}-\frac{HMR}{m})(1+m)}{\exp(\frac{HR}{m})-1}))R((\exp(\frac{HR}{m})-\exp(\frac{HRM}{m}))m+HMR)+$$

$$(\frac{\exp(-\frac{HR}{m})(1-\exp(-\frac{HR}{m}))^m}{\exp(\frac{HR}{m})-1})(2\exp(\frac{HR}{m})(m^2-HmR+HmMR)-2\exp(\frac{HRM}{m})m^2-2H^2MR^2$$

$$+H^2M^2R^2)))).$$

5.2.4. Cost of interest charges for the items kept in stock after the grace period

As mentioned, if the customer does not pay the supplier by the end of the grace period, he will owe interest to the supplier. The items still in stock have to be financed at the interest rate I_c. The present value of interest charges during the *jth* period is given by:

$$CI_j = pI_c \int_{M_j}^{T_j} (a+bt)(t-M_j)\exp(-Rt)dt \tag{5.7}$$

$$= pI_c \left(\frac{\exp(\frac{-H(j+M-1)R}{m})(amR + b(2m + H(j+M-1)R))}{mR^3} \right)$$

$$- pI_c \left(\frac{\exp(\frac{-HjR}{m})(amR(m-H(M-1)R) + b(2m^2 + H(1+j-M)mR - H^2 j(M-1)R^2))}{m^2R^3} \right)$$

$$j = 1,2,3,\ldots, m-1$$

And the present value of the total interest charges for the items kept after the grace period is given by:

$$CI = \sum_{j=1}^{m} CI_j \tag{5.8}$$

$$= I_c\, p(\frac{1}{mR^3}((\exp(-HR/m)^{m-1}(bH(\frac{\exp(-HR/m)}{(1-\exp(-HR/m)^2} - (\exp(-HR/m)^{1+m}(\frac{\exp(-HR/m)}{(1-\exp(-HR/m))^2} +$$

$$\frac{\exp(HR/m)(1+m)}{-1+\exp(HR/m)})(2bm - bHR + bHMR + amR))) - \frac{1}{m^2R^3}(-bH(\frac{\exp(-HR/m)}{(1-\exp(-HR/m)^2} - (\exp(\frac{-HR}{m})^{1+m}$$

$$(\frac{\exp(-HR/m)}{(1-\exp(-HR/m))^2} + \frac{\exp(HR/m)(1+m)}{(-1+\exp(HR/m))}))R(-m-HR+HMR) + (-1+\frac{1-\exp(-HR-HR/m)}{1-\exp(-HR/m)})$$

$$* m(2bm + bHR - bHRM + aHR^2 - aHMR^2))).$$

5.2.5. Interest earned during the grace period

The customer earns money during the grace period. When the supplier allows the customer to pay for the goods bought after a fixed period of time, he is in fact giving him a loan without interest during that period. The customer can enjoy this privilege and continue to accumulate profit and earn interest during the credit period with rate I_e.

The present value of the interest earned during the *jth* cycle ($j = 1,2,3,...,m$) is given by:

$$CE_j = pI_e \int_{T_{j-1}}^{M_j} \int_{t}^{M_j} (a+bt)\exp(-R\tau)\,d\tau dt \tag{5.9}$$

$$= pI_e\left(\frac{-1}{2m^2R^3}(\exp(\frac{-H(-1+j+M)R}{m})(2amR(m+H(-1+j+M)R+b(2m^2+2H(-1+j+M)mR\right.$$

$$+H^2(-1+j+M)^2R^2)))+\frac{1}{2m^2R^3}(b\exp(-\frac{H(-1+j+M)}{m})H^2(-1+j)^2R^2+2a\exp(\frac{-H(-1+j+M)R}{m})$$

$$\left.*H(-1+j)mR^2+2\exp(\frac{-H(-1+j)R}{m})m(amR+b(m+H(-1+j)R)))\right),$$

$$j = 1,2,3,...,m$$

Therefore the present value of the total interest earned during the planning horizon is given by:

$$CE = \sum_{j=1}^{m} CE_j \tag{5.10}$$

$$= I_e p(\frac{1}{mR^3}((\exp(-\frac{HR}{m})^{(M-1)}(bH(\frac{\exp(-\frac{HR}{m})}{(1-\exp(-HR/m))^2} - (\exp(-HR/m))^{1+m}(\frac{\exp(-HR/m)}{(1-\exp(-HR/m))^2}$$

$$+ \frac{\exp(HR/m)(1+m)}{\exp(HR/m)-1}))R + (-1 + \frac{1-(\exp(-HR/m)^{1+m}}{1-\exp(-HR/m)})(2bm - bHR + bHMR + amR))) - \frac{1}{m^2R^3}$$

$$*(\exp(HR/m)(-bH(\frac{\exp(-HR/m)}{(1-\exp(-HR/m)^2} - (\exp(-HR/m)^{1+m}(\frac{\exp(-HR/m)}{(1-\exp(-HR/m)^2} + \frac{\exp(\frac{HR}{m})(1+m)}{\exp(\frac{HR}{m})-1}))$$

$$*R(-m + HMR) + (-1 + \frac{1-(\exp(-HR/m)^{1+m}}{1-\exp(-HR/m)})(2bm^2 - bHmR - bHMmR + am^2R + bH^2MR^2$$

$$-aHMmR^2))))$$

5.2.6. The total cost function

The total cost of the inventory system is comprised of the five aforementioned components, and is given by:

$$TC = CR + CH + CP + CI + CE \ , \tag{5.11}$$

5.3. Optimal Solution and Numerical Examples

The present value of the total cost of the system is a nonlinear function of one variable, *m*. The decision variable is not continuous and hence the optimal value can not be found by taking the derivative and equating it to zero. To obtain the optimal number of replenishment cycles during the planning horizon, a unidimensional optimization technique (Golden search) was used. To use the unidimentional search method (Golden search), one needs to specify an interval in which the optimum value of decision variable, m, lies. The interval is split into two segments according to what is termed the "golden section", in which the ratio of the whole interval to the larger segment is the same as the ratio of larger segment to the smaller one. The algorithm was implemented on a problem with the following parameters, $a = 10$, $b = 1$, $H = 10$, $R = \{0.10, -0.10, -0.20, -0.40\}$, $A = 2$, $F = 0.32$, $p = 1$, $I_e = 0.04$, $I_c = 0.15$, $M = \{0.1, 0.20\}$.

For more details about the golden search refer to Appendix D. The algorithm was implemented on an interval for *m* between 1 and 20.The results obtained are reported in Table 5.1.

Grace Period	R = r - f	Total Cost ($)	Number of Cycles
M = 0.2	+0.1	120.705	12
	-0.1	359.709	14
	-0.2	694.8	14
	-0.4	3077.65	15
M = 0.1	+0.1	121.813	12
	-0.1	362.038	14
	-0.2	698.835	14
	-0.4	3090.59	15

Table 5.1. Optimal number of replenishment cycles Total cost of the system for different value of *R*, with respect to *M.*

Figure 5.2. illustrates the relationship between the total cost of the system and the inflation rate. The figure shows the cost increases significantly with the increase in inflation rate. The total cost of the system also decreases when the length of the grace period increases. The number of replenishment cycles increases with the increase in inflation rate.

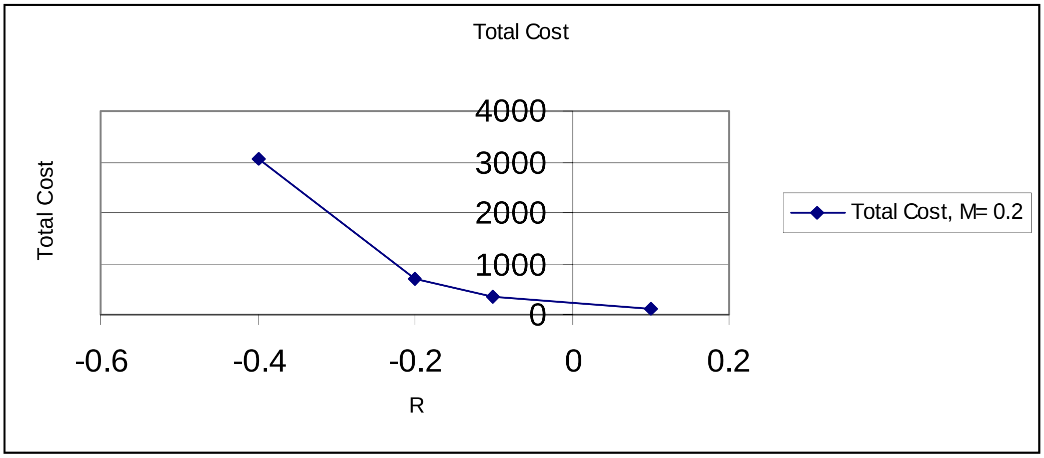


Figure.5.2: relationship between the total cost of the system and the inflation rate, when M = 0.2.

Chapter Six

Conclusions and Future Research

In this research a number of static and dynamic inventory models were developed in which the effects of economic factors such as inflation and time value of money were taken into account. Allowing for a delay in payment is a common practice among suppliers that lets customers pay for the goods bought within a certain period of time. The effects of permissible delay in payment in the inventory models were studied. First a static single item model was considered in which the shortages were allowed and the effect of a permissible delay in payment on the model was investigated. Next, the same model was augmented by considering the effects of inflation and time value of money. Appropriate mathematical models were developed and a search method was used to obtain the optimal policies of the inventory systems. The objective was to find the optimal length of replenishment cycle and the optimal length of the period during which the inventory level is positive.

A single item deterministic model was also considered in which the demand is a linear function of time. Inflation, time value of money and permissible delay in payment were considered in the development of the mathematical model representing the system over a finite planning horizon. The main objective of the problem was to determine the optimal number of replenishment cycles over the planning horizon.

Several extensions can be made to this research. In the first problem in which an EOQ model was considered with shortages and permissible delay in payment, it was assumed that the replenishment rate was infinite and goods were delivered instantaneously as the order was released. In many practical cases, a finite replenishment system takes place in which raw material is processed into products and added to the inventory at a finite rate. The problem of a single item inventory system with finite input rate, no shortages and permissible delay in payment needs to be investigated as an extension of the model developed in this research. One needs to extend the above problems while considering the effects of inflation and time value of money.

In developing the dynamic inventory model with permissible delay in payment, it was assumed that the length of the grace period was a fraction of replenishment cycle. One may consider a case in which the length of the grace period is fixed and does not depend on the length of replenishment cycle. Also in this research shortages were not allowed in the dynamic model. As an extension to the model one may consider a case in which shortages are allowed. Also in developing the dynamic model, replenishment cycles were restricted to be equal in length. One may want to relax this restriction and allow for replenishment cycles with different lengths.

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APPENDICES

Appendix A. Hooke and Jeeves, Unconstrained Minimization Procedure

Using derivatives in solving unconstrained nonlinear programming problems leads to quicker solutions compared to direct search methods, but problems may arise implementing such methods. In problems with a large number of variables, it may be difficult if not impossible to derive close formulae for the variables.

In this section the algorithm proposed by Hooke and Jeeves for solving unconstrained nonlinear problems is presented. The algorithm is comprised of two phases, first an “Exploratory Search” is performed around a base point to find the best direction to move, and second a “Pattern Search” is used to minimize the function.

Assume that the function $f(X)$ needs to be minimized. Elements of X are the decision variables. In order to implement the algorithm, the initial values of the decision variables, elements of X , must be provided as well as the initial incremental changes ΔX . At the first step, the objective function, $f(X)$, is evaluated at the base point provided by the user, then each variable is changed while keeping all the others unchanged. To be specific $x_1^{(0)}$ is changed by the amount of $+\Delta x_1^{(0)}$**,** so that $x_1^{(1)} = x_1^{(0)} + \Delta x_1^{(0)}$. The objective function $f(X)$ is evaluated at the new point; if there is an improvement in the objective function, $x_1^{(1)} = x_1^{(0)} + \Delta x_1^{(0)}$ is considered as the new value of x_1. If there is no improvement in the objective function by increasing the value of x_1, $\Delta x_1^{(0)}$ is subtracted from $x_1^{(0)}$ and again the objective function is evaluated at the new point, $x_1^{(1)} = x_1^{(0)} - \Delta x_1^{(0)}$.If the value of objective function is not improved by either $x_1^{(1)} = x_1^{(0)} \pm \Delta x_1^{(0)}$,the value of x_1 is left unchanged. Then x_2 is changed by the amount of $+\Delta x_2^{(0)}$ and so on till all the decision variables have been changed and the effects of their changes on the objective function have been investigated. After making one or two exploratory searches a pattern search is made. Those variable changes which improved the objective function, form a vector which shows a direction suitable to move along in order to decrease the value of $f(X)$. A series of movements are made along this vector as long as the objective function improves. The extent of the steps in the pattern search for each variable depends on the number of successful steps previously made in each coordinate during the exploratory search in previous cycles.

$f(X)$ if is not improved after the pattern search, a new exploratory search is made in order to find a new direction to move. If the exploratory search does not give a new successful direction, the amount of ΔX is reduced until a new direction can be defined or each Δx_i becomes less than some predefined factor in order to stop the search. In order to stop the algorithm there are three tests that need to be satisfied. The first one compares the change in the objective function with a prescribed small number, after each exploratory search and pattern search. If the objective function does not change by a value that exceeds the specified number, the exploratory search and pattern search fail. The second test is performed in the absence of the aforementioned failure to determine if the objective function increases (failure) or decreases (success) to ensure that the value of the objective function is always improving. The third test compares the amount of Δx_i, after an exploratory search failure, with some prescribed numbers. If the amount of change in each variable is less than the specified number, the test can be terminated.

Appendix B. Computer program used to minimize the total cost function of the EOQ model with shortages and permissible delay in payment when inflation is not considered:

```
Option Explicit
Private Iter_Occured As Long
Private Iter As Integer
Private optimal_point(2) As Variant
Private OptTime() As Variant
Private startpt(0 To 1) As Variant
Private delta(2) As Variant
Private prevbest As Single
Private start_point() As Variant
Private iteration_max As Integer
Private demand As Single
Private purchase As Single


Private Sub cmdCalculate_Click()
startpt(0) = Val(txtReplenishment.Text)
startpt(1) = Val(txtPositive.Text)
Dim stp_shrk As Variant
Dim eps As Variant
Dim ttcost As Single
Dim ii As Integer, No_of_Iterations As Integer
iteration_max = 5000
stp_shrk = Val(txtRho.Text)
eps = Val(txtEpsilon.Text)
```

```
No_of_Iterations = Hooke_Jeeves(startpt(), stp_shrk, eps, iteration_max)
Dim tcost As Single
tcost = Total_Cost(optimal_point())
ttcost = tcost + demand * purchase
lblTotalCost.Caption = Format(ttcost, "##00.00000")
lblOptimalRep.Caption = Format(optimal_point(0), "##0.00000")
lblOptimalPositive.Caption = Format(optimal_point(1), "##0.00000")
End Sub
Public Static Function Total_Cost(OptTime() As Variant) As Single
Dim Iter_Occured As Long
Iter_Occured = Iter_Occured + 1
Dim setup As Single
Dim F As Single
Dim shortCost As Single
Dim intCharges As Single
Dim intEarned As Single
Dim delay As Single
demand = Val(txtDemand.Text)
purchase = Val(txtPurchase.Text)
setup = Val(txtSetup.Text)
F = Val(txtCarry.Text)
shortCost = Val(txtShortage.Text)
intCharges = Val(txtCharges.Text)
intEarned = Val(txtEarned.Text)
delay = Val(txtDelay.Text)
Dim Rep_Cycle As Single, Inv_Hold As Single
Rep_Cycle = OptTime(0)
Inv_Hold = OptTime(1)
Total_Cost = (2 * setup + demand * purchase * F * Inv_Hold * Inv_Hold + shortCost *
demand * (Rep_Cycle - Inv_Hold) * (Rep_Cycle - Inv_Hold) + demand * purchase *
```

```
intCharges * (Inv_Hold - delay) * (Inv_Hold - delay) - 2 * demand * purchase *
intEarned * (Rep_Cycle - Inv_Hold + (delay / 2)) * delay) / (2 * Rep_Cycle)
End Function
Private Sub cmdExit_Click()
End
End Sub

Public Function exploratory_search(delta() As Variant, basepoint() As Variant, prevbest
As Single) As Single
Dim newpoint(2) As Variant
Dim mincost As Single
Dim tmpcost As Single
Dim i As Integer
mincost = prevbest
For i = 0 To 1
   newpoint(i) = basepoint(i)
Next i
For i = 0 To 1
   newpoint(i) = basepoint(i) + delta(i)
   tmpcost = Total_Cost(newpoint())
   If tmpcost < mincost Then
      mincost = tmpcost
   Else
      delta(i) = 0 - delta(i)
      newpoint(i) = basepoint(i) + delta(i)
      tmpcost = Total_Cost(newpoint())

      If tmpcost < mincost Then
         mincost = tmpcost
      Else
```

```
            newpoint(i) = basepoint(i)
        End If
    End If
Next i
For i = 0 To 1
    basepoint(i) = newpoint(i)
Next i

exploratory_search = mincost
End Function



Public Function Hooke_Jeeves(start_point() As Variant, stp_shrk As Variant, epsilon As Variant, itermax As Integer) As Integer
Dim del(2) As Variant
Dim New_Cost As Single
Dim Prev_Cost As Single
Dim steplength As Variant
Dim Temp_Point As Single
Dim xbefore(2) As Variant
Dim newx(2) As Variant
Dim i As Integer
Dim j As Integer
Dim flag As Integer
Dim iters As Integer
For i = 0 To 1
  newx(i) = start_point(i)
  xbefore(i) = newx(i)
    del(i) = Abs(start_point(i) * stp_shrk)
    If del(i) = 0 Then
```

```
        del(i) = stp_shrk
    End If
Next i
steplength = stp_shrk
iters = 0
Prev_Cost = Total_Cost(newx())
New_Cost = Prev_Cost
Do While iters < itermax And steplength > epsilon
    iters = iters + 1
        For j = 0 To 1
        newx(j) = xbefore(j)
    Next j
    New_Cost = exploratory_search(del(), newx(), Prev_Cost)
    flag = 1
    Do While New_Cost < Prev_Cost And flag = 1

        For i = 0 To 1
            If newx(i) <= xbefore(i) Then
                del(i) = 0 - Abs(del(i))
            Else
                del(i) = Abs(del(i))
            End If
            Temp_Point = xbefore(i)
            xbefore(i) = newx(i)
            newx(i) = newx(i) + newx(i) - Temp_Point
        Next i
        Prev_Cost = New_Cost
        New_Cost = exploratory_search(del(), newx(), Prev_Cost)
        If New_Cost >= Prev_Cost Then
            Exit Do
```

```
        End If
        flag = 0
        For i = 0 To 1
            flag = 1
            If Abs(newx(i) - xbefore(i)) > 0.5 * Abs(del(i)) Then
                Exit For
            Else
                flag = 0
            End If
        Next i
        Loop
        If steplength >= epsilon And New_Cost >= Prev_Cost Then
            steplength = steplength * stp_shrk
            For i = 0 To 1
                del(i) = del(i) * stp_shrk
            Next i
        End If
    Loop
    For i = 0 To 1
        optimal_point(i) = xbefore(i)
    Next
Hooke_Jeeves = iters
End Function
```

Appendix C. Computer program used to minimize the total cost function of the EOQ model with shortages and permissible delay in payment when inflation is considered:

```
Option Explicit
Private Iter_Occured As Integer
Private optimal_point(2) As Variant
Private OptTime() As Variant
Private startpt(0 To 1) As Variant
Private delta(2) As Variant
Private prevbest As Single
Private start_point() As Variant
Private iteration_max As Integer


Private Sub cmdCalculate_Click()
startpt(0) = Val(txtReplenishment.Text)
startpt(1) = Val(txtPositive.Text)
Dim stp_shrk As Variant
Dim eps As Variant
Dim ii As Integer, No_of_Iterations As Integer
iteration_max = Val(txtNoIterations.Text)
stp_shrk = Val(txtRho.Text)
eps = Val(txtEpsilon.Text)
No_of_Iterations = Hooke_Jeeves(startpt(), stp_shrk, eps, iteration_max)
Dim NI As Integer
NI = No_of_Iterations * 10
prg1.Max = NI
```

```
prg1.Min = 0
For ii = 0 To NI
   prg1.Value = ii
Next ii
Dim tcost As Single
tcost = Total_Cost(optimal_point())
lblTotalCost.Caption = Format(tcost, "##00.00000")
lblOptimalRep.Caption = Format(optimal_point(0), "##0.00000")
lblOptimalPositive.Caption = Format(optimal_point(1), "##0.00000")
End Sub
Public Static Function Total_Cost(OptTime() As Variant) As Double
Dim D As Single
Dim p As Single
Dim A As Single
Dim F As Single
Dim Pi As Single
Dim Ic As Single
Dim Ie As Single
Dim M As Single
Dim Inflation As Variant
Dim Interest As Variant
Dim R As Variant
D = Val(txtDemand.Text)
p = Val(txtPurchase.Text)
A = Val(txtSetup.Text)
F = Val(txtCarry.Text)
Pi = Val(txtShortage.Text)
Ic = Val(txtCharges.Text)
Ie = Val(txtEarned.Text)
M = Val(txtDelay.Text)
```

```
Inflation = Val(txtInflation.Text)
Interest = Val(txtInterest.Text)
R = Inflation - Interest
Dim T As Variant
Dim T1 As Variant
T = OptTime(0)
T1 = OptTime(1)
Total_Cost = (A + F * p * D * ((Exp(R * T1) - T1 * R - 1) / R ^ 2) + ((p * D * (T - T1
+ M) * Exp(R * M)) + (p * D * (T1 - M) * (Exp(R * T1)))) + (Pi * D * (Exp(R * T1) +
Exp(R * T) * (R * (T - T1) - 1)) / R ^ 2) + (D * p * Ic * (Exp(M * R) + Exp(R * T1) *
(R * T1 - M * R - 1)) / R ^ 2) - (Ie * p * D * ((Exp(M * R) - M * R - 1) / R ^ 2)) - (Ie *
p * D * ((Exp(M * R) - 1) / R) * (T - T1))) * ((1 - Exp(R)) / (1 - Exp(R * T)))
End Function
Private Sub cmdExit_Click()
End
End Sub

Public Function exploratory_search(delta() As Variant, basepoint() As Variant, prevbest
As Single) As Single
Dim newpoint(2) As Variant
Dim mincost As Single
Dim tmpcost As Single
Dim i As Integer
mincost = prevbest
For i = 0 To 1
   newpoint(i) = basepoint(i)
Next i
For i = 0 To 1
   newpoint(i) = basepoint(i) + delta(i)
   tmpcost = Total_Cost(newpoint())
```

```
    If tmpcost < mincost Then
        mincost = tmpcost
    Else
        delta(i) = 0 - delta(i)
        newpoint(i) = basepoint(i) + delta(i)
        tmpcost = Total_Cost(newpoint())

        If tmpcost < mincost Then
            mincost = tmpcost
        Else
            newpoint(i) = basepoint(i)
        End If
  End If
Next i
For i = 0 To 1
    basepoint(i) = newpoint(i)
Next i

exploratory_search = mincost
End Function



Public Function Hooke_Jeeves(start_point() As Variant, rho As Variant, epsilon As
Variant, itermax As Integer) As Integer
Dim del(2) As Variant
Dim New_Cost As Single
Dim Prev_Cost As Single
Dim steplength As Variant
Dim Temp_Point As Single
Dim xbefore(2) As Variant
```

```
Dim newx(2) As Variant
Dim i As Integer
Dim j As Integer
Dim flag As Integer
Dim iters As Integer
For i = 0 To 1
  newx(i) = start_point(i)
  xbefore(i) = newx(i)
    del(i) = Abs(start_point(i) * rho)
    If del(i) = 0 Then
        del(i) = rho
    End If
Next i
steplength = rho
iters = 0
Prev_Cost = Total_Cost(newx())
New_Cost = Prev_Cost
Do While iters < itermax And steplength > epsilon
    iters = iters + 1
    For j = 0 To 1
        newx(j) = xbefore(j)
    Next j
    New_Cost = exploratory_search(del(), newx(), Prev_Cost)
    flag = 1
    Do While New_Cost < Prev_Cost And flag = 1

        For i = 0 To 1
            If newx(i) <= xbefore(i) Then
                del(i) = 0 - Abs(del(i))
```

```
        Else
            del(i) = Abs(del(i))
        End If
        Temp_Point = xbefore(i)
        xbefore(i) = newx(i)
        newx(i) = newx(i) + newx(i) - Temp_Point
    Next i
    Prev_Cost = New_Cost
    New_Cost = exploratory_search(del(), newx(), Prev_Cost)
    If New_Cost >= Prev_Cost Then
        Exit Do
    End If
    flag = 0
    For i = 0 To 1
        flag = 1
        If Abs(newx(i) - xbefore(i)) > 0.5 * Abs(del(i)) Then
            Exit For
        Else
            flag = 0
        End If
    Next i
    Loop
    If steplength >= epsilon And New_Cost >= Prev_Cost Then
        steplength = steplength * rho
        For i = 0 To 1
            del(i) = del(i) * rho
        Next i
    End If
Loop
For i = 0 To 1
```

```
        optimal_point(i) = xbefore(i)
    Next
Hooke_Jeeves = iters
End Function
```

Appendix D. Golden Search Method

In this section details about a unidimensional optimization technique called Golden Search Method is presented. To use the Golden search method, one needs to specify an interval in which the optimum value of decision variable, x, lies. The interval is split into two segments according to "golden section", in which the ratio of the whole interval to the larger segment is the same as the ratio of larger segment to the smaller one. The two ratios employed are: $F_1 = \frac{3-\sqrt{5}}{2} \approx 0.38$ and $F_2 = \frac{\sqrt{5}-1}{2} \approx 0.62$. Note that $F_1 = (F_2)^2$ and $F_1 + F_2 = 1$. Let the three x values designated for the interval in which the optimum of x lies as $x_3^{(0)}$ (the last point), $x_2^{(0)}, x_1^{(0)}$, where $f(x_3^{(0)}) \geq f(x_2^{(0)})$, and let the interval $\Delta^{(k)} = x_3^{(k)} - x_1^{(k)}$. For the k-th stage the (k+1)-th interval is computed as follows. Determine $y_1^{(k)} = x_1^{(k)} + F_1\Delta^{(k)}$ and $y_2^{(k)} = x_1^{(k)} + F_2\Delta^{(k)} = x_3^{(k)} - F_1\Delta^{(k)}$. If $f(y_1^{(k)}) > f(y_2^{(k)})$ then $\Delta^{(k+1)} = (y_2^{(k)} - x_1^{(k)})$ and $x_1^{(k+1)} = x_1^{(k)}$, $x_3^{(k+1)} = y_2^{(k)}$. If $f(y_1^{(k)}) < f(y_2^{(k)})$ then $\Delta^{(k+1)} = (x_3^{(k)} - y_1^{(k)})$ and $x_1^{(k+1)} = y_1^{(k)}$, $x_3^{(k+1)} = x_3^{(k)}$. If $f(y_1^{(k)}) = f(y_2^{(k)})$ then $\Delta^{(k+1)} = (y_2^{(k)} - x_1^{(k)}) = (x_3^{(k)} - y_1^{(k)})$ and $x_1^{(k+1)} = x_1^{(k)}$, $x_3^{(k+1)} = y_2^{(k)}$ or $x_1^{(k+1)} = y_1^{(k)}$, $x_3^{(k+1)} = x_3^{(k)}$.