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Cycles and Bases of Graphs and Matroids

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ABSTRACT

Cycles and Bases of Graphs and Matroids

Ping Li

The objective of this dissertation is to investigate the properties of cycles and bases in matroids and in graphs. In [62], Tutte defined the circuit graph of a matroid and proved that a matroid is connected if and only if its circuit graph is connected. Motivated by Tutte's result, we introduce the 2nd order circuit graph of a matroid, and prove that for any connected matroid M other than $U_{1,1}$, the second order circuit graph of M has diameter at most 2 if and only if M does not have a restricted minor isomorphic to $U_{2,6}$.

Another research conducted in this dissertation is related to the eulerian subgraph problem in graph theory. A graph G is eulerian if G is connected without vertices of odd degrees, and G is supereulerian if G has a spanning eulerian subgraph. In [3], Boesch, Suffey and Tindel raised a problem to determine when a graph is supereulerian, and they remarked that such a problem would be a difficult one. In [55], Pulleyblank confirmed the remark by showing that the problem to determine if a graph is supereulerian, even within planar graphs, is NP-complete. Catlin in [8] introduced a reduction method based on the theory of collapsible graphs to search for spanning eulerian subgraphs in a given graph G. In this dissertation, we introduce the supereulerian width of a graph G, which generalizes the concept of supereulerian graphs, and extends the supereulerian problem to the supereulerian width problem in graphs. Further, we also generalize the concept of collapsible graphs to s-collapsible graphs and develop the reduction method based on the theory of s-collapsible graphs. Our studies extend the collapsible graph theory of Catlin. These are applied to show for any integer $n > 2$, the complete graph K_n is $(n-3)$-collapsible, and so the supereulerian width of K_n is $n-2$. We also prove a best possible degree condition for a simple graph to have supereulerian width at least 3.

The number of edge-disjoint spanning trees plays an important role in the design of networks, as it is considered as a measure of the strength of the network. As disjoint spanning trees are disjoint bases in graphic matroids, it is important to study the properties related to the number of disjoint bases in matroids. In this dissertation, we develop a decomposition theory based on the density function of a matroid, and prove a decomposition theorem that partitions the ground set of a matroid M into subsets based on their densities. As applications of the decomposition theorem, we investigate problems related to the properties of disjoint bases in a matroid. We showed that for a given integer $k > 0$, any matroid M can be embedded into a matroid M' with the same rank (that is, $r(M) = r(M'))$ such that M' has k disjoint bases. Further we determine the minimum value of $|E(M')| - |E(M)|$ in terms of invariants of M. For a matroid M with at least k disjoint bases, we characterize the set of elements in M such that removing any one of them would still result in a matroid with at least k disjoint bases.

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DEDICATION

To

my father Xuegen Li , my mother Yuling Li

Contents

Chapter 1

Preliminaries

1.1 Notation and Terminology

We consider finite graphs with possible multiple edges and loops, and follow the notation of Bondy and Murty [4] for graphs, and Oxley [58] or Welsh [64] for matroids, except otherwise defined. Thus for a connected graph G, $\omega(G)$ denotes the number of components of G. For a matroid M, we use r_M (or r, when the matroid M is understood from the context) denotes the rank function of M, and $E(M)$, $\mathcal{C}(M)$ and $\mathcal{B}(M)$ denote the ground set of M, and the collections of the circuits, and the bases of M, respectively. Furthermore, if M is a matroid with $E = E(M)$, and if $X \subset E$, then $M - X$ is the restricted matroid of M obtained by deleting the elements in X from M, and M/X is the matroid obtained by contracting elements in X from M. As in [58] and [64], we use $M - e$ for $M - \{e\}$ and M/e for $M/\{e\}$.

The **spanning tree packing number** of a connected graph G, denoted by $\tau(G)$, is the maximum number of edge-disjoint spanning trees in G. A survey on spanning tree packing number can be found in [59]. By definition, $\tau(K_1) = \infty$. For a matroid M, we similarly define $\tau(M)$ to be the maximum number of disjoint bases of M. Note that by definition, if M is a matroid with $r(M) = 0$, then for any integer $k > 0$, $\tau(M) \geq k$.

Let M be a matroid with rank function r. For any subset $X \subseteq E(M)$ with $r(X) > 0$, the **density** of X is

$$d_M(X) = \frac{|X|}{r_M(X)}.$$

When the matroid M is understood from the context, we often omit the subscript M. We also use $d(M)$ for $d(E(M))$. Following the terminology in [11], the **strength** $\eta(M)$ and the **fractional arboricity** $\gamma(M)$ of M are respectively defined as

$$\eta(M) = \min\{d(M/X) : r(X) < r(M)\}, \text{ and } \gamma(M) = \max\{d(X) : r(X) > 0\}.$$

For an integer $k > 0$ and a matroid M with $\tau(M) \geq k$, we define $E_k(M) = \{e \in E(M) : \tau(M - e) \geq k\}$. Likewise, for a connected graph G with $\tau(G) \geq k$, $E_k(G) = \{e \in E(G) : \tau(G - e) \geq k\}$.

Let M be a matroid and $k \in \mathbf{N}$. If there is a matroid M' with $\tau(M') \geq k$ such that M' has a restriction isomorphic to M (we then view M as a restriction of M'), then M' is a $(\tau \geq k)$**-extension of** M. We shall show that any matroid has a $(\tau \geq k)$-extension. We then define $F(M, k)$ to be the minimum integer $l > 0$ such that M has a $(\tau \geq k)$-extension M' with $|E(M')| - |E(M)| = l$.

For a graph G, $\delta(G)$, $\Delta(G)$, $\kappa(G)$ and $\kappa'(G)$ represents the minimum degree, the maximum degree, the connectivity and the edge connectivity of a graph G, respectively. As in [4], $G[X]$ denotes the subgraph induced by an edge subset $X \subseteq E(G)$. When no confusion arises, we shall often adopt the convention that for an edge subset $X \subseteq E(G)$, X denotes the edge subset as well as the subgraph $G[X]$ of G. For subgraphs H_1, H_2 of G, $H_1 \cup H_2$ and $H_1 \cap H_2$ denote the union and intersection of H_1 and H_2, respectively. For vertices $u, v \in V(G)$, a trail with end vertices being u and v will be referred as a (u, v)-trial. We use $O(G)$ to denote the set of all odd degree vertices in G. A graph G is **Eulerian** if $O(G) = \emptyset$ and G is connected, and is **supereulerian** if G has a spanning Eulerian subgraph.

Let G be a graph, and $s > 0$ be an integer. For any distinct $u, v \in V(G)$, an $(s; u, v)$**-trail-system** of G is a subgraph H consisting of s edge-disjoint (u, v)-trails. A graph is **supereulerian with width** s if $\forall u, v \in V(G)$ with $u \neq v$, G has a spanning $(s; u, v)$-trail-system. The **supereulerian width** $\mu'(G)$ of a graph G is the largest integer s such that G is supereulerian with width k for any integer k with $1 \le k \le s$. Note that if for some vertices u and v, G does not have a spanning (u, v)-trial, then $\mu'(G) = 0$.

A graph G is **s-collapsible** if for any subset $R \subseteq V(G)$ with $|R| \equiv 0 \pmod 2$, G has a spanning subgraph Γ_R such that
(i) both $O(\Gamma_R) = R$ and $\kappa'(\Gamma_R) \ge s - 1$, and
(ii) $G - E(\Gamma_R)$ is connected.

Let $\mathcal{C}_s$ denote the collection of s-collapsible graphs.

Let M be a matroid on a set E. The **corank** $r^*(M)$ of a matroid M is the rank of M^*, the dual of M. For a subset $S \subseteq E$, we abbreviate the expression $r^*(M|S)$ as r^*S, and the **dimension** dS of S is defined to be the number $r^*S - 1$. Following Tutte [62], a subset S of E is called a **flat** of M if it is a union of circuits of M. The null subset of E is considered as a null union of circuits, and therefore a flat. Note that our definition of a flat here is different from that in [58].

For any subset S of E there is an associated flat $\langle S \rangle$, defined as the union of all the circuits of M contained in S. Thus $\langle S \rangle$ is the union of the circuits of $M|S$. Note that $d\langle Z \rangle = dZ = r^*Z - 1$. A flat S is **on** a flat T if either $S \subseteq T$ or $T \subseteq S$. A flat of dimension k is called a **k-flat**. The 1-flats and 2-flats of M are the **lines** and the **planes** of M, respectively. A flat S of M is called **connected** if $M|S$ is a connected matroid.

Let M be a matroid, and let $k > 0$ be an integer. The **kth order circuit graph** $C_k(M)$ of M has vertex set $V(C_k(M)) = \mathcal{C}(M)$, the set of all circuits of M. Two vertices $C, C' \in \mathcal{C}(M)$ are adjacent in $C_k(M)$ if and only if $|C \cap C'| \ge k$. For notational convenience, for a circuit $C \in \mathcal{C}(M)$, we shall use C to denote both a vertex in $C_k(M)$ and a circuit (also as a subset of $E(M)$) of M.

1.2 Main Results

In the coming several chapters, we will present the following main results.

(1) Let M be a connected simple matroid with more than one circuit. Then M does not have a restriction isomorphic to $U_{2,6}$ if and only if $diam(C_2(M)) \leq 2$.

(2) Let M be a connected simple matroid with more than one circuit, but M is not a line, then $C_2(M)$ is 2-connected.

(3) Let M be a matroid and $k > 0$ be an integer. Each of the following holds.
(i) Suppose that $\tau(M) \geq k$. Then $E_k(M) = E(M)$ if and only if $\eta(M) > k$.
(ii) In general, $E_k(M)$ equals to the maximal subset $X \subseteq E(M)$ such that $\eta(M|X) > k$.

(4) Let M be a matroid with $r(M) > 0$. Then each of the following holds.
(i) There exist an integer $m > 0$, and an m-tuple $(l_1, l_2, ..., l_m)$ of positive rational numbers such that

$$\eta(M) = l_1 < l_2 < ... < l_m = \gamma(M),$$

and a sequence of subsets

$$J_m \subset ... \subset J_2 \subset J_1 = E(M);$$

such that for each i with $1 \leq i \leq m$, $M|J_i$ is an η-maximal restriction of M with $\eta(M|J_i) = l_i$.
(ii) The integer m and the sequences in (i) are uniquely determined by M.
(iii) For every i with $1 \leq i \leq m$, J_i is a closed set in M.

(5) For $k \in \mathbf{N}$, let M be a matroid with $\tau(M) \leq k$ and let $i(k)$ denote the smallest i_j in (4) such that $i(k) \geq k$. Then
(i) $F(M,k) = k(r(M) - r(J_{i(k)})) - |E(M) - J_{i(k)}|$.
(ii) $F(M,k) = \max_{X \subseteq E(M)}\{kr(M/X) - |M/X|\}$.

(6) Let $s \geq 1$ be an integer. Then $\mathcal{C}_s$ satisfies the following.
(C1) $K_1 \in \mathcal{C}_s$.
(C2) If $G \in \mathcal{C}_s$ and if $e \in E(G)$, then $G/e \in \mathcal{C}_s$.
(C3) If H is a subgraph of G and if $H, G/H \in \mathcal{C}_s$, then $G \in \mathcal{C}_s$.

(7) Let $s \geq 1$ be an integer. If a graph $G \in \mathcal{C}_s$, then $\mu'(G) \geq s + 1$.

(8) Let $s \geq 1$ be an integer. If $F(G, s+1) \leq 1$, then $G \in \mathcal{C}_s$ if and only if $\kappa'(G) \geq s+1$.

(9) Let n, p, s be positive integers such that $p \geq 2$. Suppose that G is a simple graph on n vertices. If

$$\delta(G) \geq \frac{n}{p} - 1,$$

then when n is sufficiently large (say $n > p(1 + (1 + 2(s+3) + 2(p+1)(s+1)))$), the $\mathcal{C}_s$-reduction of G has at most p vertices.

(10) Let n, p, s be positive integers such that $p \geq 2$. Suppose that G is a simple graph on n vertices. If G is triangle free, and if

$$\delta(G) \geq \frac{n}{2p},$$

then when n is sufficiently large (say $n > 2p(1 + (1 + 2(s+3) + 2(p+1)(s+1)))$), the $\mathcal{C}_s$-reduction of G has at most p vertices.

(11) Let n, p, s be positive integers such that $p \geq 2$. Suppose that G is a simple graph on n vertices. If

$$\delta(G) \geq \frac{n}{p} - 1,$$

then when n is sufficiently large (say $n > p(1 + (1 + 2(s+3) + 2(p+1)(s+1)))$), the $\mathcal{C}_s$-reduction of G has at most p vertices.

Chapter 2

Diameter of Second Order Circuit Graph of Matroids

2.1 Introduction

Matroids and graphs considered in this paper are finite. For undefined notations and terminology, see [4] for graphs and [58] for matroids. Let M be a matroid on a set E. The *corank* $r^*(M)$ of a matroid M is the rank of M^*, the dual of M. For a subset $S \subseteq E$, we abbreviate the expression $r^*(M|S)$ as r^*S, and the *dimension* dS of S is defined to be the number $r^*S - 1$. Following Tutte [62], a subset S of E is called a *flat* of M if it is a union of circuits of M. The null subset of E is considered as a null union of circuits, and therefore a flat. Note that our definition of a flat here is different from that in [58].

For any subset S of E there is an associated flat $\langle S\rangle$, defined as the union of all the circuits of M contained in S. Thus $\langle S\rangle$ is the union of the circuits of $M|S$. Note that $d\langle Z\rangle = dZ = r^*Z - 1$. A flat S is *on* a flat T if either $S \subseteq T$ or $T \subseteq S$. A flat of dimension k is called a *k-flat*. The 1-flats and 2-flats of M are the *lines* and the *planes* of M, respectively. A flat S of M is called *connected* if $M|S$ is a connected matroid.

There have been many studies on the properties of graphs arising from matroids. In

[62], Tutte defined the *circuit graph* of a matroid M, denoted by $C(M)$, whose vertices are the circuits of M, where the two vertices in $C(M)$ are adjacent if and only if they are distinct circuits of the same connected line. Tutte [62] showed that a matroid M is connected if and only if $C(M)$ is a connected graph. In [48] and [49], Maurer defined the base graph of a matroid. The vertices are the bases of M and two vertices are adjacent if and only if the symmetric difference of these two bases is of cardinality 2. He also discussed the graphical properties of the base graph of a matroid. Alspach and Liu [1] studied the properties of paths and circuits in base graphs of matroids. The connectivity of the base graph of matroids is investigated by Liu [44] and [45]. The graphical properties of the matroid base graphs have also been investigated by many other researchers, see [23], [28], [39], [46], among others.

Recent studies by Li and Liu ([40], [41] and [42]) initiate the investigation of graphical properties of matroid circuits graphs. Let M be a matroid, and let $k > 0$ be an integer. The *kth order circuit graph* $C_k(M)$ of M has vertex set $V(C_k(M)) = \mathcal{C}(M)$, the set of all circuits of M. Two vertices $C, C' \in \mathcal{C}(M)$ are adjacent in $C_k(M)$ if and only if $|C \cap C'| \geq k$. For notational convenience, for a circuit $C \in \mathcal{C}(M)$, we shall use C to denote both a vertex in $C_k(M)$ and a circuit (also as a subset of $E(M)$) of M.

In their studies ([40], [41] and [42]), Li and Liu proved that $C_1(M)$ possesses quite good graphical connectivity properties. The purpose of this chapter is to investigate the graphical properties possessed by $C_2(M)$, a spanning subgraph of $C_1(M)$, which represents a relatively loose interrelationship among circuits in the matroid. We have proved in this chapter that for a connected simple matroid M, the diameter of $C_2(M)$ is at most 2 if and only if M does not have a restriction isomorphic to $U_{2,6}$. Moreover, if a connected simple matroid M is not a line, then $C_2(M)$ is 2-connected.

In Section 2, we shall review some former results and develop certain useful lemmas what will be needed in this paper. The last section will be devoted to the proofs of the main results.

2.2 Useful Results on Circuits and Flats

In this section, we summarize some of the useful former results, and developed a few lemmas for our use. A matroid M is trivial if it has no circuits. In the following all matroids will be nontrivial.

Theorem 2.2.1 *(Tutte [62]) Let M be a matroid.*
(i)(Theorem 4.21 [62]). Let L be a line of M and $a \in L$, then $\langle L - \{a\} \rangle$ is the only circuit on L which does not include a.
(ii)(Theorem 4.28 [62]). Let L be a disconnected line on a connected d-flat S of M, where $dS > 1$. Then there exists a connected plane P of M such that $L \subset P \subseteq S$.
(iii)(Theorem 4.281 [62]). Let L be a disconnected line on a plane P of M. Let X and Y be its two circuits, and let Z be any other circuit on P. Then $X \cup Z$ and $Y \cup Z$ are connected lines, the only lines of M which are on both Z and P.
(iv)(Theorem 4.36 [62]). A matroid M without coloops is connected if and only if its circuit graph $C(M)$ is a connected graph.

Also, throughout the rest of this section, M denotes a simple nontrivial matroid, and C_1 and C_2 will denote two distinct circuits of M.

Lemma 2.2.2 *If C_1 and C_2 are different circuits of a matroid M such that $C_1 \cap C_2 = \{e\}$, then:*
(i) If C_1 and C_2 are on a line, then $C_1 \triangle C_2$ is a circuit of M and $|(C_1 \triangle C_2) \cap C_i| \geq 2$.
(ii) If C_1 and C_2 are not on a line, then there are circuits C_3 and C_4 of M such that $e \in C_3 \cap C_4$, C_1, C_2, C_3, C_4 are pairwise different and $|C_i \cap C_j| \geq 2$, for every $i \in \{1, 2\}$ and $j \in \{3, 4\}$.

Proof. First, we establish (i). Observe that $C_1 - C_2$ and $C_2 - C_1$ are serious classes of $M|(C_1 \cup C_2)$. But $M|(C_1 \cup C_2)$ is connected and so for each element $e' \in C_1 \triangle C_2$ there is a circuit C of M such that $e' \in C \subseteq C_1 \triangle C_2 = (C_1 - C_2) \cup (C_2 - C_1)$. But by Theorem

2.2.1 (i), C is the only circuit on $\langle C_1 \triangle C_2 \rangle$. Therefore $C_1 \triangle C_2 = (C_1 \cup C_2) - e \subseteq C$. Then $C_1 \triangle C_2$ is a circuit of M.

Now, we show (ii). There is a connected line L of M such that $C_1 \subseteq L \subseteq C_1 \cup C_2$. Choose a circuit C_3 of M such that $e \in C_3 \subseteq L$ and $C_3 \neq C_1$. There is a connected line L' of M such that $C_2 \subseteq L' \subseteq C_1 \cup C_2$ and $\langle L' \cap C_3 \rangle = \emptyset$. We can choose a circuit C_4 of M such that $e \in C_4 \subseteq L'$ and $C_4 \neq C_2$. □

Lemma 2.2.3 *If C_1 and C_2 are on a disconnected line of a connected matroid M, then there is a circuit C_0 such that $C_1 \cup C_0$ and $C_2 \cup C_0$ are connected lines with $|C_1 \cap C_0| \geq 2$ and $|C_2 \cap C_0| \geq 2$.*

Proof. By assumption, $M|(C_1 \cup C_2)$ is a disconnected line. Since M is connected, by Theorem 2.2.1(ii), there exists a connected plane P of M such that $C_1 \cup C_2 \subset P$. By Theorem 2.2.1(iii), let C be any other circuit on P, then $C_1 \cup C$ and $C_2 \cup C$ are connected lines. Assume that there is not a circuit C_0 on P such that $|C_1 \cap C_0| \geq 2$ and $|C_2 \cap C_0| \geq 2$. By Lemma 2.2.2, we know that there is a circuit C_3 with $|C_1 \cap C_3| \geq 2$. Then $|C_2 \cap C_3| = 1$. By Lemma 2.2.2, $C_2 \triangle C_3$ is a circuit and $|(C_2 \triangle C_3) \cap C_2| \geq 2$. But also $|(C_2 \triangle C_3) \cap C_1| \geq 2$, a contradiction. □

Lemma 2.2.4 *Let C_1 be a circuit of M. If $e \notin C_1$, then there is a circuit C_2 containing e and C_1 and C_2 are on a connected line.*

Proof. Since M is connected, there is a circuit C' containing e and $|C_1 \cap C'| \neq 0$. Let C_2 be such a circuit and $|C_1 \cup C_2| - r(C_1 \cup C_2)$ is minimal. Since M is binary, $C_1 \triangle C_2$ is a disjoint union of circuits of M. If $C_1 \triangle C_2$ is a circuit, then C_1 and C_2 are on a connected line. If $C_1 \triangle C_2$ is not a circuit, then there is $C_3 \subset C_1 \triangle C_2$ containing e. So we have $|C_1 \cup C_3| - r(C_1 \cup C_3) < |C_1 \cup C_2| - r(C_1 \cup C_2)$ which is a contradiction. So C_1 and C_2 are on a connected line. □

Lemma 2.2.5 *If $|C_1 \cap C_2| = 0$, and $r(C_1) < r(C_1 \cup C_2) < |C_1 \cup C_2| - 2$, then there is a circuit C in $M|(C_1 \cup C_2)$ such that C and C_1 are on a connected line with $|C_1 \cap C| \geq 2$, $|C_2 \cap C| \geq 2$.*

Proof. There is $e \in C_2$ such that C_1 does not span e because, by hypothesis, $r(C_1) < r(C_1 \cup C_2)$. By Lemma 2.2.4, there is a connected line L such that $C_1 \cup e \subseteq L \subseteq C_1 \cup C_2$. If C is a circuit of M such that $e \in C \subseteq L$ and $C \neq C_1$, then $L - C_1 \subseteq C$ and so $|C \cap C_2| \geq 2$. Since $|C_1| \geq 3$, it is possible to choose C such that $|C \cap C_1| \geq 2$. □

2.3 Main Results

In this section, we shall prove our main results.

Theorem 2.3.1 . *Let M be a connected simple matroid with more than one circuit. The following statements are equivalent:*
(i) M does not have a restriction isomorphic to $U_{2,6}$.
(ii) $diam(C_2(M)) \leq 2$.

Proof. Assume that (i) holds. Let C_1 and C_2 be two circuits of M. We shall show that in $C_2(M)$, either C_1 and C_2 are adjacent, or there is a C_3 which is adjacent to both C_1 and C_2. Since if $|C_1 \cap C_2| \geq 2$, then C_1 and C_2 are adjacent in $C_2(M)$, we assume that $|C_1 \cap C_2| \leq 1$.

Case 1. $C_1 \cap C_2 = \{e\}$. By Lemma 2.2.2, we know (ii) holds.

Case 2. $|C_1 \cap C_2| = 0$. In this case, we have $r(C_1 \cup C_2) \leq |C_1 \cup C_2| - 2$.

Subcase 2.1. If $r(C_1 \cup C_2) = |C_1 \cup C_2| - 2$, then $M|(C_1 \cup C_2)$ is a disconnected line. By Lemma 2.2.3, we can find a circuit C_0 such that $C_1 \cup C_0$ and $C_2 \cup C_0$ are connected lines with $|C_1 \cap C_0| \geq 2$ and $|C_2 \cap C_0| \geq 2$.

Subcase 2.2. If $r(C_1 \cup C_2) < |C_1 \cup C_2| - 2$, then $M|(C_1 \cup C_2)$ is connected. If $r(C_1 \cup C_2) > r(C_1)$ or $r(C_1 \cup C_2) > r(C_2)$, without loss of generality, we assume that $r(C_1 \cup C_2) > r(C_1)$. By Lemma 2.2.5, there is a circuit C_0 in $M|(C_1 \cup C_2)$ such that C_0 and C_1 are on a connected line and $|C_1 \cap C_0| \geq 2$, $|C_2 \cap C_0| \geq 2$.

Subcase 2.3. If $r(C_1 \cup C_2) < |C_1 \cup C_2| - 2$ and $r(C_1 \cup C_2) = r(C_1) = r(C_2)$, assume that in $C_2(M)$, C_1 and C_2 do not have a common adjacent vertex. Since M is a simple matroid, then $|C_2| \geq 3$. Hence there are elements $f_1, f_2, f_3 \in C_2$ and $f_i \neq f_j$ for any $i \neq j$ $(i, j = 1, 2, 3)$.

Subcase 2.3.1. $\{f_1, f_2, f_3\} = C_2$. Then $r(C_1 \cup C_2) = r(C_1) = r(C_2) = 2$. Therefore $M|(C_1 \cup C_2)$ is isomorphic to $U_{2,6}$ which is a contradiction.

Subcase 2.3.2. $\{f_1, f_2, f_3\} \subset C_2$. Therefore in $M|(C_1 \cup \{f_i, f_j\})$, any circuit containing f_i and f_j has length 3 $(i \neq j, i, j = 1, 2, 3)$. Hence we can find a circuit $C_3 = \{f_1, f_2, h\}$ and $C_4 = \{f_2, f_3, g\}$ such that $h, g \in C_1$. Since $r(\{f_1, f_2, f_3\}) = 3$, by Lemma 2.2.2, $\{f_1, h, f_3, g\}$ is a circuit of M. Then we can get a vertex adjacent to both C_1 and C_2 which is also a contradiction.

Conversely, if M has a restriction X isomorphic to $U_{2,6}$, let C_1 and C_2 be two different circuits of X and $C_1 \cap C_2 = \emptyset$, then the distance between C_1 and C_2 in $C_2(M)$ is 3. If not, then we have a circuit C' of M with $|C' \cap C_1| \geq 2$ and $|C' \cap C_2| \geq 2$. Let $Y = (C' \cap C_1) \cup (C' \cap C_2)$. Then $|Y| \geq 4$ and $Y \subseteq X$. Since $M|X \cong U_{2,6}$, and since $|Y| \geq 4$, Y must properly contain a circuit of M, contrary to the circuit axioms.

This completes the proof of the theorem. □

Theorem 2.3.2 . *Let M be a connected simple matroid with more than one circuit, but M is not a line, then $C_2(M)$ is 2-connected.*

Proof. We argue by contradiction. Assume that $C_2(M)$ has a cut vertex C_0. Let C_1 and C_2 be circuits of M such that they are in two different components of $C_2(M) - C_0$ and $|C_1 \cap C_2| = 1$. By Lemma 2.2.2(ii), we know that C_1 and C_2 are on a connected line. By Lemma 2.2.2(i), $C_0 = C_1 \triangle C_2$. But M is not a line, then there is another circuit C_3 in different components of $C_2(M) - C_0$ with C_i and $|C_3 \cap C_i| = 1$ $(i = 1 or 2)$. Assume that $i = 1$. By Lemma 2.2.2, we know C_1 and C_3 are both adjacent to $C_1 \triangle C_3$ which is also a circuit of M. We get a contradiction.

We prove the theorem. □

Theorem 2.3.3 . *If M is a connected matroid with girth at least $2k-1$, then $C_k(M)$ is connected.*

Proof. Let CD be an edge of $C(M)$. By definition, C and D are different circuits of a connected line of M. If $|C \cap D| \geq k$, then CD is an edge of $C_k(M)$. If $|C \cap D| < k$, then $C \triangle D \subseteq C'$, for some circuit C' of M. But $|C \cap C'| \geq |C - D| \geq (2k-1) - (k-1) = k$ and so CC' is an edge of $C_k(M)$. Similarly, DC' is an edge of $C_k(M)$. Therefore $C_k(M)$ is connected because $C(M)$ is connected. □

Chapter 3

Removable Elements in Matroids

3.1 Introduction

The number of edge-disjoint spanning trees in a network, when modeled as a graph, often represents certain strength of the network [18]. The well-known spanning tree packing theorem of Nash-Williams [52] and Tutte [61] characterizes graphs with k edge-disjoint spanning trees, for any integer $k > 0$. For any graph G, the problem of determining which edges should be added to G so that the resulting graph has k edge-disjoint spanning trees has been studied, see Haas [21] and Liu et al [43], among others. However, it has not been fully studied that for an integer $k > 0$, if a graph G has k edge-disjoint spanning trees, what kind of edge $e \in E(G)$ has the property that $G - e$ also has k-edge-disjoint spanning trees. The research of this chapter is motivated by this problem. In fact, we will consider the problem that, if a matroid M has k disjoint bases, what kind of element $e \in E(M)$ has the property that $M - e$ also has k disjoint bases.

We consider finite graphs with possible multiple edges and loops, and follow the notation of Bondy and Murty [4] for graphs, and Oxley [58] or Welsh [64] for matroids, except otherwise defined. Thus for a connected graph G, $\omega(G)$ denotes the number of components of G. For a matroid M, we use ρ_M (or ρ, when the matroid M is understood from the context) denotes the rank function of M, and $E(M)$, $\mathcal{C}(M)$ and $\mathcal{B}(M)$ denote the ground set of M, and the collections of the circuits, and the bases of M, respectively. Furthermore, if M is a matroid with $E = E(M)$, and if $X \subset E$, then $M - X$ is the restricted matroid of M obtained by deleting the elements in X from M, and M/X is the matroid obtained by contracting elements in X from M. As in [58] and [64], we use $M - e$ for $M - \{e\}$ and M/e for $M/\{e\}$.

The **spanning tree packing number** of a connected graph G, denoted by $\tau(G)$, is the maximum number of edge-disjoint spanning trees in G. A survey on spanning tree packing number can be found in [59]. By definition, $\tau(K_1) = \infty$. For a matroid M, we similarly define $\tau(M)$ to be the maximum number of disjoint bases of M. Note that by definition, if M is a matroid with $\rho(M) = 0$, then for any integer $k > 0$, $\tau(M) \geq k$. The following theorems are well known.

Theorem 3.1.1 *(Nash-Williams [52] and Tutte [61]) Let G be a connected graph with $E(G) \neq \emptyset$, and let $k > 0$ be an integer. Then $\tau(G) \geq k$ if and only if for any $X \subseteq E(G)$, $|E(G - X)| \geq k(\omega(G - X) - 1)$.*

Theorem 3.1.2 *(Edmonds [19]) Let M be a matroid with $\rho(M) > 0$. Then $\tau(M) \geq k$ if and only if $\forall X \subseteq E(M)$, $|E(M) - X| \geq k(\rho(M) - r(X))$.*

Let M be a matroid with rank function r. For any subset $X \subseteq E(M)$ with $r(X) > 0$, the **density** of X is

$$d_M(X) = \frac{|X|}{r_M(X)}.$$

When the matroid M is understood from the context, we often omit the subscript M. We also use $d(M)$ for $d(E(M))$. Following the terminology in [11], the **strength** $\eta(M)$ and the **fractional arboricity** $\gamma(M)$ of M are respectively defined as

$$\eta(M) = \min\{d(M/X) : r(X) < r(M)\}, \text{ and } \gamma(M) = \max\{d(X) : r(X) > 0\}.$$

Thus Theorem 3.1.2 above indicates that

$$\tau(M) = \lfloor \eta(M) \rfloor. \tag{3.1}$$

For an integer $k > 0$ and a matroid M with $\tau(M) \geq k$, we define $E_k(M) = \{e \in E(M) : \tau(M - e) \geq k\}$. Likewise, for a connected graph G with $\tau(G) \geq k$, $E_k(G) = \{e \in E(G) : \tau(G - e) \geq k\}$. Using Theorem 3.1.1, Gusfield proved that high edge-connectivity of a graph would imply high spanning tree packing number.

Theorem 3.1.3 *(Gusfield [20]) Let $k > 0$ be an integer, and let $\kappa'(G)$ denote the edge-connectivity of a graph G. If $\kappa'(G) \geq 2k$, then $\tau(G) \geq k$.*

The next result strengthens Gusfield's theorem, and indicates a sufficient condition for a graph G to satisfy $E_k(G) = E(G)$.

Theorem 3.1.4 *(Theorem 1.1 of [13]) Let $k > 0$ be an integer, and let $\kappa'(G)$ denote the edge-connectivity of a graph G. Then $\kappa'(G) \geq 2k$ if and only if $\forall X \subseteq E(G)$ with $|X| \leq k$, $\tau(G - X) \geq k$. In particular, if $\kappa'(G) \geq 2k$, then $E_k(G) = E(G)$.*

A natural question is to characterize all graphs G with the property $E_k(G) = E(G)$. More generally, for any graph G with $\tau(G) \geq k$, we are to determine the edge subset $E_k(G)$. These questions can be presented in terms of matroids in a natural way. The main purpose of this chapter is to characterize $E_k(M)$, for any matroid with $\tau(M) \geq k$. The next theorem is our main result.

Theorem 3.1.5 *Let M be a matroid and $k > 0$ be an integer. Each of the following holds.*
(i) Suppose that $\tau(M) \geq k$. Then $E_k(M) = E(M)$ if and only if $\eta(M) > k$.
(ii) In general, $E_k(M)$ equals to the maximal subset $X \subseteq E(M)$ such that $\eta(M|X) > k$.

For a connected graph G with $M(G)$ denoting its cycle matroid, let $\eta(G) = \eta(M(G))$ and $\gamma(G) = \gamma(M(G))$. Then Theorem 3.1.5, when applied to cycle matroids, yields the corresponding theorem for graphs.

Corollary 3.1.6 *Let G be a connected graph and $k > 0$ be an integer. Each of the following holds.*
(i) If $\tau(G) \geq k$, $E_k(G) = E(G)$ if and only if $\eta(G) > k$.
(ii) In general, $E_k(G)$ equals to the maximal subset $X \subseteq E(G)$ such that every component of $\eta(G[X]) > k$.

In the next section, we shall discuss properties of the strength and the fractional arboricity of a matroid M, which will be useful in the proofs of our main results. We will prove a decomposition theorem in Section 3, which will be applied in the characterizations of $E_k(M)$ and $E_k(G)$ in Section 4. In the last section, we shall develop polynomial algorithms to locate the sets $E_k(M)$ and $E_k(G)$.

3.2 Strength and Fractional Arboricity of a Matroid

Both parameters $\eta(M)$ and $\gamma(M)$, and the problems related to uniformly dense graphs and matroids (defined below) have been studied by many, see [11, 9, 10, 14, 17, 26, 25, 27, 27, 54, 60], among others. From the definitions of $d(M), \eta(M)$ and $\gamma(M)$, we immediately have, for any matroid M with $r(M) > 0$,

$$\eta(M) \leq d(M) \leq \gamma(M). \tag{3.2}$$

As in [11], a matroid M satisfying $\eta(M) = \gamma(M)$ is called a **uniformly dense matroid**. Both $\eta(M)$ and $\gamma(M)$ can also be described by their behavior in some parallel extension of the matroid. For an integer $t > 0$, let M_t denote matroid obtained from M by replacing each element $e \in E(M)$ by a parallel class of t elements. (See Page 252 of [33]). This matroid M_t is usually referred as **the t-parallel extension** of M. For $X \subseteq E(M)$, we use X_t to denote both the matroid $(M|X)_t$ and the set $E((M|X)_t)$.

Theorem 3.2.1 *(Theorem 4 of [11], and Lemma 1 of [33]) Let M be a matroid and let $s \geq t > 0$ be integers. Then*
(i) $\eta(M) \geq \frac{s}{t}$ if and only if $\eta(M_t) \geq s$.

(ii) $\gamma(G) \leq \frac{s}{t}$ *if and only if* $\gamma(M_t) \leq s$.
(iii) $t\eta(M) = \eta(M_t)$.
(iv) $t\gamma(M) = \gamma(M_t)$.

Theorem 3.2.2 *(Theorem 6 of [11]) Let* M *be a matroid. The following are equivalent.*
(i) $\eta(M) = d(M)$.
(ii) $\gamma(M) = d(M)$.
(iii) $\eta(M) = \gamma(M)$.
(iv) $\eta(M) = \frac{s}{t}$, *for some integers* $s \geq t > 0$, *and* M_t, *the* t*-parallel extension of* M, *is a disjoint union of* s *bases of* M.
(v) $\gamma(M) = \frac{s}{t}$, *for some integers* $s \geq t > 0$, *and* M_t, *the* t*-parallel extension of* M, *is a disjoint union of* s *bases of* M.

For each integer $k > 0$, define

$$\mathcal{T}_k = \{M : \tau(M) \geq k\}.$$

Proposition 3.2.3 *The matroid family* $\mathcal{T}_k$ *satisfies the following properties.*
(C1) If $r(M) = 0$, *then* $M \in \mathcal{T}_k$.
(C2) If $M \in \mathcal{T}_k$ *and if* $e \in E(M)$, *then* $M/e \in \mathcal{T}_k$.
(C3) Let $X \subseteq E(M)$ *and let* $N = M|X$. *If* $M/X \in \mathcal{T}_k$ *and if* $N \in \mathcal{T}_k$, *then* $M \in \mathcal{T}_k$.

Proof: Recall that the bases of the contraction M/X has the following form (see, for example, Corollary (3.1.9) of by [58]).

$$\mathcal{B}(M/X) = \{B' \subseteq E - X : B' \cup B_X \in \mathcal{B}(M)\}, \text{ where } B_X \in \mathcal{B}(M|X). \quad (3.3)$$

Since when $r(M) = 0$, $\eta(M) = \infty$, (C1) follows from the definition of η immediately.

If e is a loop of M, then e is not in any basis of M and so by (3.3), $M/e = M - e$. Thus $\tau(M/e) = \tau(M - e) = \tau(M) \geq k$. Therefore $M/e \in \mathcal{T}_k$.

Suppose e is not a loop. Let $B_1, \ldots, B_k$ be disjoint bases of M. We assume that $\forall i \in \{1, 2, \cdots, k\}$, if $e \notin B_i$, then $C_i = C_M(e, B_i)$ is the unique circuit of $B_i \cup e$. Since e

is not a loop, $\exists e_i \in C_i - e$. Define $B'_i = B_i \cup e - e_i$, if $e \notin B_i$; and $B'_i = B_i$, if $e \in B_i$. It follows that $B'_1, B'_2, \ldots, B'_k$ are bases of M such that for any $i \neq j$, $B_i \cap B_j = e$. Note that if $X = \{e\}$, then $B_X = \{e\} \in \mathcal{B}(M|X)$. It follows by (3.3) that $B'_i - e$ is a basis of M/e, and all $\{B'_i - e\}$ are disjoint. Hence $M/e \in \mathcal{T}_k$. This proves (C2).

Let $B''_1, B''_2, \ldots, B''_k$ be disjoint bases of N and $B'_1, B'_2, \ldots, B'_k$ be disjoint bases of M/N. By (3.3), $B'_1 \cup B''_1, B'_2 \cup B''_2, \ldots, B'_k \cup B''_k$ are disjoint bases of M, and so $M \in \mathcal{T}_k$. □

Lemma 3.2.4 *Let M be a matroid with $r(M) > 0$, and let $l \geq 1$ be fractional number. Each of the following holds.*
(i) (Lemma 10 of [11]) If $X \subset E(M)$ and if $\eta(M|X) \geq \eta(M)$, then $\eta(M/X) = \eta(M)$.
(ii) (Theorem 17 of [11]) If $X \subset E(M)$ and if $d(X) = \gamma(M)$, then $\eta(M|X) = \gamma(M|X) = d(X) = \gamma(M)$.
(iii) A matroid M is uniformly dense if and only if $\forall X \subseteq E(M)$, $d(X) \leq \eta(M)$.
(iv) A matroid M is uniformly dense if and only if for any restriction N of M, $\eta(N) \leq \eta(M)$.
(v) If $d(M) \geq l$, then there exists a subset $X \subseteq E(M)$ with $r(X) > 0$ such that $\eta(M|X) \geq l$.

Proof: (iii). If $\forall X \subseteq E(M)$, $d(X) \leq \eta(M)$, then in particular, $d(M) \leq \eta(M)$. It follows by (3.2) that $d(M) = \eta(M)$, and so by Theorem 3.2.2, M is uniformly dense. Conversely, suppose that there exists an $X \subseteq E(M)$ with $d(X) > \eta(M)$. Then by (3.2), $\gamma(M) \geq d(X) > \eta(M)$, contrary to the assumption that M is uniformly dense.

(iv). By (iii) of this lemma, if M is uniformly dense, then for any restriction N, $\eta(N) \leq d(E(N)) \leq \eta(M)$. On the other hand, if M is not uniformly dense, then $\gamma(M) > \eta(M)$. By the definition of $\gamma(M)$, there exists an $X \subset E(M)$ such that $d(X) = \gamma(M)$. It follows by (ii) of this lemma that $\eta(M|X) = d(X) = \gamma(M) > \eta(M)$, contrary to the assumption. Hence M must be uniformly dense.

(v). By (3.2), $\gamma(M) \geq d(M) \geq l$. By definition of $\gamma(M)$, there exists a subset $X \subseteq E(M)$ with $r(X) > 0$, such that $d(X) = \gamma(M)$. Let $N = M|X$. By (ii) of this lemma, $\eta(N) = \gamma(N) = d(N) = \gamma(M) \geq d(M) \geq l$. □

For each rational number $l > 1$, define

$$\mathcal{S}_l = \{M : \eta(M) \geq l\}. \tag{3.4}$$

Corollary 3.2.5 *Let $p > q > 0$ be integers. The matroid family $\mathcal{S}_l$ satisfies the following properties.*
(C1) If $r(M) = 0$, then $M \in \mathcal{S}_l$.
(C2) If $M \in \mathcal{S}_l$ and if $e \in E(M)$, then $M/e \in \mathcal{S}_l$.
(C3) Let $X \subseteq E(M)$ and let $N = M|X$. If $M/X \in \mathcal{S}_l$ and if $N \in \mathcal{S}_l$, then $M \in \mathcal{S}_l$.

Proof: As (C1) follows from the definition of η and (C2) follows from Lemma 3.2.4(i), it suffices to prove (C3) only. Since $l = \frac{p}{q}$, and since both $\eta(M/X) \geq \frac{p}{q}$ and $\eta(M|X) \geq \frac{p}{q}$, it follows by Theorem 3.2.1 that $M_q/(X_q) = (M/X)_q \in \mathcal{T}_p$ and $M_q|X_p = (M|X)_q \in \mathcal{T}_p$. By Proposition 3.2.3(C3), $M_q \in \mathcal{T}_p$, and so by Theorem 3.2.1, $M \in \mathcal{S}_l = \mathcal{S}_{\frac{p}{q}} = \{M : \tau(M_q) \geq p\}$. This verifies (C3). □

Lemma 3.2.6 *Let M be a matroid with $\tau(M) \geq k$. Suppose that $X \subseteq E(M)$ satisfies $\eta(M|X) \geq k$. Then $E_k(M|X) \subseteq E_k(M)$.*

Proof: Let $N = M|X$. It is trivial if $E_k(N) = \emptyset$. Assume $E_k(N) \neq \emptyset$. Let $e \in E_k(N)$. Then $\tau(N-e) \geq k$. By definition of contraction, $(M-e)/(N-e) = M/N$. Since $M \in \mathcal{T}_k$, by Proposition 3.2.3(C2), $M/N \in \mathcal{T}_k$. Since $N - e \in \mathcal{T}_k$ and $(M - e)/(N - e) \in \mathcal{T}_k$, by Proposition 3.2.3(C3), $M - e \in \mathcal{T}_k$. Therefore $e \in E_k(M)$. □

Lemma 3.2.7 *Let M be a matroid, and N be a restriction of M. If $M/N, N \in \mathcal{T}_k$, and if both $E_k(N) = E(N)$ and $E_k(M/N) = E_k(M/N)$, Then $E_k(M) = E(M)$.*

Proof: Let $e \in E(M)$. There are two cases to be considered.

Case 1: $e \in E(M) - E(N) = E(M/N)$. Since $E_k(M/N) = E(M/N)$, $\tau(M/N - e) \geq k$. But $(M - e)/N = M/N - e \in \mathcal{T}_k$, and $N \in \mathcal{T}_k$, by Proposition 3.2.3(C3), $M - e \in \mathcal{T}_k$. Hence $e \in E_k(M) \subseteq E(M)$.

Case 2: $e \in E(N)$. Since $E_k(N) = E(N)$, $\tau(N - e) \geq k$. Note that $(M - e)/(N - e) \cong M/N \in \mathcal{T}_k$. By Proposition 3.2.3(C3), $M - e \in \mathcal{T}_k$, and so $e \in E_k(M) \subseteq E(M)$.

As for any $e \in E(M)$, $e \in E_k(M)$, we have $E_k(M) = E(M)$. □

3.3 A Decomposition Theorem

Throughout this section, we assume that M is a matroid with $r(M) > 0$. A subset $X \subseteq E(M)$ is an **η-maximal subset** and $M|X$ is an **η-maximal restriction** if for any subset $Y \subseteq E(M)$, Y properly contains X, we have $\eta(M|Y) < \eta(M|X)$.

Lemma 3.3.1 *If $X \subseteq E(M)$ is an η-maximal subset, then X is a closed set in M.*

Proof: Let $\eta(M|X) = \frac{s}{t}$ for some integers $s \geq t > 0$. It follows by Theorem 3.2.1(i) that $M|X$ has s bases $B_1, B_2, \cdots, B_s$ such that every elements of X lies in at most t of these bases. Suppose that X is not closed. Then there exists an $e \in cl_M(X) - X$, and so $r(X \cup e) = r(X)$. Thus $B_1, B_2, \cdots, B_s$ are also bases of $M|(X \cup e)$, and every element in $X \cup e$ lies in at most t of these bases. By Theorem 3.2.1(i), $\eta(M|(X \cup e)) \geq \frac{s}{t} = \eta(M|X)$, contrary to the assumption that X is an η-maximal subset. □

Lemma 3.3.2 *Let W, $W' \subset E(M)$ be subsets of $E(M)$, and let $l \geq 1$ be an integer. If $\eta(M|W) \geq l$ and $\eta(M|W') \geq l$, then $\eta(M|(W \cup W')) \geq l$.*

Proof: Let $N = M|(W \cup W')$. Since $N/W = (M|W')/(W \cap W')$, it follows by Corollary 3.2.5 (C2) that $\eta(N/W) = \eta((M|W')/(W \cap W')) \geq \eta(M|W') \geq l$. Hence both $N/W \in \mathcal{S}_l$ and $M|W \in \mathcal{S}_l$. It then follows by Corollary 3.2.5 (C3) that $N \in \mathcal{S}_l$. Thus $\eta(N) \geq l$. □

If N_1 and N_2 are two restrictions of M, we denote by $N_1 \cup N_2 = M|(E(N_1) \cup E(N_2))$, the restriction of M to the union of the ground sets of N_1 and N_2. This notation can be extended to any finite union of restrictions.

Lemma 3.3.3 *Let N be a restriction of M. Then M must have an η-maximal restriction L such that both $E(N) \subseteq E(L)$ and $\eta(L) \geq \eta(N)$.*

Proof: Suppose that $\eta(N) = l$ for some rational number $l \geq 1$. Let $\mathcal{F}_N$ be the collection of all restrictions N' of M such that $\eta(N') \geq l$. Define $L = \bigcup_{N' \in \mathcal{F}_N} N'$. As $N \in \mathcal{F}_N$, $E(N) \subseteq E(L)$. By Lemma 3.3.2, $\eta(L) \geq l$. By the definition of L, L must be η-maximal. □

Lemma 3.3.4 *For any restriction N of M, $\eta(N) \leq \gamma(M)$.*

Proof: By (3.2), $\eta(N) \leq d(N) \leq \gamma(M)$, and so it follows from the definition of $\gamma(M)$. □

Theorem 3.3.5 *Let M be a matroid with $r(M) > 0$. Then each of the following holds.*
(i) There exist an integer $m > 0$, and an m-tuple $(l_1, l_2, ..., l_m)$ of positive rational numbers such that

$$\eta(M) = l_1 < l_2 < ... < l_m = \gamma(M), \tag{3.5}$$

and a sequence of subsets

$$J_m \subset ... \subset J_2 \subset J_1 = E(M); \tag{3.6}$$

such that for each i with $1 \leq i \leq m$, $M|J_i$ is an η-maximal restriction of M with $\eta(M|J_i) = l_i$.
(ii) The integer m and the sequences (3.5) and (3.6) are uniquely determined by M.
(iii) For every i with $1 \leq i \leq m$, J_i is a closed set in M.

Proof: Let $\mathcal{R}(M)$ denote the collection of all η-maximal restrictions of M. By Lemma 3.3.3, $\mathcal{R}(M)$ is not empty. Since $E(M)$ is finite,

$$|\mathcal{R}(M)| \text{ is a finite number.} \tag{3.7}$$

Define

$$sp_\eta(M) = \{\eta(N) : \ N \in RHO(M)\}.$$

By (3.7), $|sp_\eta(M)|$ is finite. Since $M \in RHO(M)$, $|sp_\eta(M)| \geq 1$.

Let $m = |sp_\eta(M)|$. Denote

$$sp_\eta(M) = \{l_1, l_2, ..., l_m\}, \text{ such that } l_1 < l_2 < ... < l_m.$$

By Corollary 3.2.5(C3), and by the definition of $\gamma(M)$, we have

$$\eta(M) = l_1, \text{ and } \gamma(M) = l_m. \tag{3.8}$$

For each $j \in \{1, 2, ..., m\}$, let N_j denote the η-maximal restriction of M with $\eta(N_j) = l_j$, and define

$$J_j = E(N_j). \tag{3.9}$$

By the definition of $\mathcal{S}_l$,

$$\mathcal{S}_{l_1} \supset \mathcal{S}_{l_2} \supset ... \supset \mathcal{S}_{l_m}. \tag{3.10}$$

Hence by (3.8), (3.9) and (3.10),

$$E(M) = J_1 \supseteq J_2 \supseteq ... \supseteq J_m. \tag{3.11}$$

Since $RHO(M)$ and $sp_\eta(M)$ are uniquely determined by M, the integer m, the m-tuple $(l_1, l_2, ..., l_m)$ and the sequence (3.6) are all uniquely determined by M.

(iii). This follows from Lemma 3.3.1. □

For a matroid M, the m-tuple $(l_1, l_2, ..., l_m)$ and the sequence in (3.6) will be referred as the **η-spectrum** and the **η-decomposition** of M, respectively.

Corollary 3.3.6 *Let M be a matroid with η-spectrum (3.5) and η-decomposition (3.6) such that $m > 1$. Then each of the following holds.*
(i) M/J_2 is a uniformly dense matroid with $\eta(M/J_2) = \gamma(M/J_2) = \eta(M)$.
(ii) For any integer k with $l_1 \leq k < l_m$, $E(M)$ has a unique subset Z_k such that Z_k is η-maximal and such that $\eta(M|Z_k) > k$.

Proof: (i) Since $m > 1$, $\eta(M|J_2) = l_2 > l_1 = \eta(M)$. It follows by Lemma 3.2.4 that $\eta(M/J_2) = \eta(M)$. To see that M/J_2 is uniformly dense, we argue by contradiction. Suppose that M/J_2 is not uniformly dense, and that $\gamma(M/J_2) > \eta(M/J_2)$. It follows by the definition of γ that there is a subset $J' \subset E(M/X_2)$ such that $d_{M/J_2}(J') = \gamma(M/J_2)$. By Lemma 3.3.3, M/J_2 has an η-maximal subset J'' (containing J') such that $\eta((M/J_2)|J'') = l' > \eta(M) = l_1$. If $l' \geq l_2$, then by Lemma 3.3.2, $\eta(M|(J_2 \cup J')) \geq l_2$, and so J_2 is not η-maximal, contrary to the conclusion of Theorem 3.3.5. Thus we may assume that $l_2 > l' > l_1$. Since J'' is η-maximal in M/J_2, by Lemma 3.2.4((i), $J_2 \cup J''$ is also η-maximal, and so by Theorem 3.3.5, the η-spectrum of M much contain l'. It follows that $(l_1, l_2, ..., l_m)$ cannot be the η-spectrum of M, contrary to the assumption of the corollary. This proves (i).

(ii) Let $j < m$ be the smallest integer such that $l_j > k$, and let $Z_k = J_{l_j}$. Then (ii) of this corollary follows from Theorem 3.3.5. □

The unique subset Z_k stated in Part (ii) of Corollary 3.3.6 will be called the **η-maximal subset at level** k of M.

Corollary 3.3.7 *Let M be a matroid with with η-spectrum (3.5). Then M is uniformly dense if and only if $m = 1$.*

Proof: By definition, M is uniformly dense if and only if $\gamma(M) = \eta(M)$. Since $l_1 = \eta(M)$ and $l_m = \gamma(M)$, it follows that M is uniformly dense if and only if $m = 1$. □

3.4 Characterization of the Removable Elements with Respect to Having k Disjoint Bases

The main purpose of this section is to investigate the behavior of the set $E_k(M)$. We first observe that matroids M with $E_k(M) = \emptyset$ can be characterized in terms of the density of M.

Proposition 3.4.1 *Let $k > 0$ be an integer, and M be a matroid with $\tau(M) \geq k$. Then $E_k(M) = \emptyset$ if and only if $d(M) = k$.*

Proof: Since $\tau(M) \geq k$, M has disjoint spanning bases $B_1, B_2, \cdots, B_k$, and so

$$kr(M) = \sum_{i=1}^{k} |B_i| \leq |E(M)| = d(M)r(M),$$

where equality holds if and only if $k = d(M)$. It follows by Theorem 3.2.2 (iv) (with $s = k$ and $t = 1$) that $k = d(M)$ if and only if $E(M) = \bigcup_{i=1}^{k} B_i$, and so if and only if $E_k(M) = \emptyset$. □

Accordingly, when $\tau(M) \geq k$, $E_k(M) \neq \emptyset$ if and only if $d(M) > k$. We have the following characterization.

Theorem 3.4.2 *Let $k \geq 2$ be an integer. Let M be a graph with $\tau(M) \geq k$. Then each of the following holds.*
(i) $E_k(M) = E(M)$ if and only if $\eta(M) > k$.
(ii) In general, if $\eta(M) = k$ and if $m > 1$, then $E_k(M) = J_2$ equals the η-maximal subset at level k of M.

Proof: Since $\tau(M) \geq k$, it follows by (1) that $\eta(M) \geq k$.

(i). If $\eta(M) = k$, then by Theorem 3.3.5 or by Corollary 3.3.6, there exists an unique subset $J \subset E(M)$ (say, $J = J_2$ in the η-decomposition of M) such that M/J is uniformly dense with $\eta(M/J) = \gamma(M/J) = \eta(M) = k$. It follows by Theorem 3.2.2 that $d(E(M/J)) = k$, and so by Proposition 3.4.1, for any $e \in E(M) - J = E(M/J)$, $\tau((M - e)/J) = \tau(M/J - e) < k$. Thus by $\tau((M - e)|J) = \tau(M|J) \geq k$ and by Proposition 3.2.3(C3), $\tau(M - e) < k$. This proves the necessity of (i).

We shall argue by contradiction to prove the sufficiency. Assume that the sufficiency of (i) fails, and that

$$M \text{ is a counterexample with } r(M) \text{ minimized.} \tag{3.12}$$

Then

$$\eta(M) > k \text{ but } E_k(M) \neq E(M). \tag{3.13}$$

Claim 1: M does not have a restriction N with $r(N) < r(M)$ and $\eta(N) > k$.

Suppose not, and that M has a restriction N with $\eta(N) > k$. As $r(N) < r(M)$, it follows by (3.12) that $E_k(N) = E(N)$. By Lemma 3.2.4, $\eta(M/N) \geq \eta(M) > k$. Since $\eta(N) > k$, $r(N) > 0$, and so $r(M/N) < r(M)$. By (3.12), $E_k(M/N) = E(M/N)$. By (1), both $M/N, N \in \mathcal{T}_k$, and so by Lemma 3.2.7 that $E_k(M) = E(M)$, contrary to (3.13). This proves Claim 1.

The next claim follows from Claim 1 and Lemma 3.2.4 (iv).

Claim 2: M is uniformly dense.

By (3.12) and by (3.13), we may assume that

$$\tau(M) \geq k \text{ and } \eta(M) > k, \text{ but } \exists e \in E(M),\ \tau(M-e) \leq k-1. \tag{3.14}$$

Fix $e \in E(M)$ so that $\tau(M-e) \leq k-1$ as in (3.14). It follows by (3.2) and by $\tau(M-e) \leq k-1$ that $\eta(M-e) < k$. On the other hand, by Claim 2, M is uniformly dense, and so by Theorem 3.2.2,

$$k < \eta(M) = d(M) = \frac{|E(M)|}{r(M)}.$$

This implies $|E(M)| \geq kr(M)+1$. Since M has $k \geq 2$ disjoint bases, e cannot be a coloop of M, and so $r(M-e) = r(M)$. Hence

$$d(E-e) = \frac{|E(M-e)|}{r(M-e)} \geq k.$$

By Lemma 3.2.4(v), $E(M)$ has a subset $X \subseteq E(M)$ with $r(X) > 0$ such that $\eta(M|X) \geq k$. Hence $\tau(M|X) = \lfloor \eta(M|X) \rfloor \geq k$. By Corollary 3.2.5 (C2), $\eta(M/X) \geq \eta(M) > k$. Since $r(X) > 0$, $r(M/X) < r(M)$.

By $e \in E(M/X)$, and by (3.12), $\tau((M-e)/N) = \tau(M/N-e) \geq k$. As $\tau(N) \geq k$, it follows by Proposition 3.2.3(C3) that $\tau(M-e) \geq k$, contrary to (3.14). This proves the sufficiency of (i).

(ii). We assume that $\eta(M) = k$. If $d(M) = k$, then by Proposition 3.4.1, $E_k(M) = \emptyset$. On the other hand, by Theorem 3.2.2, M is uniformly dense and so by Lemma 3.3.7, the η-maximal subset of level k of M is an empty set. Thus if $d(M) = k$, then (ii) holds with $E_k(M) = \emptyset$.

Now assume that $d(M) > k$. By Lemma 3.2.4(v), $\gamma(M) \geq d(M) > k = \eta(M)$, and so M is not uniformly dense. By Lemma 3.3.7, if M has (3.5) as its η-spectrum and sequence (3.6) as its η-decomposition, then $m > 1$. Hence by Lemma 3.3.6(ii), the η-maximal subset of level k of M equals J_2. It follows by Part (i) of this theorem that $E_k(M|J_2) = J_2$. By Lemma 3.2.6,

$$J_2 = E_k(M|J_2) \subseteq E_k(M). \tag{3.15}$$

On the other hand, by Lemma 3.3.6(i), M/J_2 is uniformly dense with $\eta(M/J_2) = \eta(M) = k$, and so by Proposition 3.4.1, $E_k(M/J_2) = \emptyset$. By Theorem 3.3.5(iii), J_2 is closed in M, and so

$$E_k(M) \subseteq E(M) - E(M/J_2) = J_2. \tag{3.16}$$

Combining (3.15) and (3.16), we have $E_k(M) = J_2$, which proves Part (ii) of the theorem. □

Applying Theorem 3.4.2 to cycle matroids of connected graphs, we obtain the corresponding theorem for graphs.

Corollary 3.4.3 *Let $k \geq 2$ be an integer, and G be a connected graph with $\tau(G) \geq k$. Let (3.5) and (3.6) denote the η-spectrum and η-decomposition of $M(G)$, respectively. Then each of the following holds.*
(i) $E_k(G) = E(G)$ if and only if $\eta(G) > k$.
(ii) In general, if $\eta(G) = k$ and if $m > 1$, then $E_k(G) = J_2$ equals the η-maximal subset at level k of $M(G)$.

3.5 Polynomial Algorithms Identifying the Excessive Elements

We remark that there exists a polynomial algorithm which can identify the excessive element subset $E_k(M)$ for any given integer $k > 0$ and any matroid M.

Modifying an algorithm of Kruth (see Page 368 of [64]), Hobbs in [24] obtained an algorithm in $O(|E(M)|^3(r(M)^4)$ time (referred as **Hobbs' Algorithm** below) such that for any matroid M, it computes $\eta(M)$ and $\gamma(M)$, and finds the η-maximal subset J of M such that $\eta(M|J) = \gamma(M)$. By Theorem 3.3.5, this η-maximal subset J of M equals J_m in (3.6).

For any matroid M, Hobbs' Algorithm outputs $i_m = \gamma(M)$ and J_m in (3.6). If $E(M) \neq J_m$ (which means $m > 1$), then by Lemma 3.2.4 (i), we replace M by M/J_m, and run Hobbs' Algorithm to get $\gamma(M) = i_{m-1}$ and the η-maximal subset J' of M/J_m, and so $J_{m-1} = J' \cup J_m$. This process can be repeated m times to generate all subsets $J_1, J_2, \cdots, J_m$ in (3.6). In particular, by Theorem 3.4.2, it also computes $E_k(M)$.

Chapter 4

Reinforcing a matroid to have k disjoint bases

4.1 Introduction

In this chapter, we use $\mathbf{N}$ and $\mathbf{Q}_+$ to denote the set of all natural numbers and the set of all positive fractional numbers, respectively, and consider finite matroids and graphs. Undefined notations and terminology can be found in [58] or [64] for matroids, and [4] for graphs. Thus for a connected graph G, $\omega(G)$ denotes the number of components of G. For a matroid M, r_M (or r, when the matroid M is understood from the context) denotes the rank function of M, and $E(M)$, $\mathcal{I}(M)$, $\mathcal{C}(M)$ and $\mathcal{B}(M)$ denote the ground set of M, and the collections of independent sets, the circuits, and the bases of M, respectively. Furthermore, if M is a matroid with $E = E(M)$, and if $X \subset E$, then $M - X$ is the restricted matroid of M obtained by deleting the elements in X from M, and M/X is the matroid obtained by contracting elements in X from M. As in [58] or [64], we use $M - e$ for $M - \{e\}$ and M/e for $M/\{e\}$.

For a matroid M, let $\tau(M)$ denote the maximum number of disjoint bases of M. For a graph G, define $\tau(G) = \tau(M(G))$, where $M(G)$ denotes the cycle matroid of G. Thus if G is a connected graph, then $\tau(G)$ is the **spanning tree packing number** of G.

Readers are refereed to [59] for a survey on $\tau(G)$. The well-known spanning tree packing theorem of Nash-Williams [52] and Tutte [61] characterizes graphs with k edge-disjoint spanning trees, for any integer $k > 0$. Edmonds [19] proved the corresponding theorem for matroids.

Let $k > 0$ be an integer. For any matroid M with $\tau(M) \geq k$, which element $e \in E(M)$ has the property that $\tau(M - e) \geq k$? Characterizations of all such elements have been found in [36] and [35]. For a graph G, the problem of determining which edges should be added to G so that the resulting graph has k edge-disjoint spanning trees has been studied, see Haas [21] and Liu et al [43], among others. As the arguments in these papers are involved vertices, it is natural to consider the possibility of extending these results to matroids. Since matroids in general do not have a concept corresponding to vertices, one can no longer add an element to a matroid as adding an edge in graphs. Therefore, we need to reformulate the problem so that it would fit the matroid setting while generalizing the graph theory results.

Let M be a matroid and $k \in \mathbf{N}$. If there is a matroid M' with $\tau(M') \geq k$ such that M' has a restriction isomorphic to M (we then view M as a restriction of M'), then M' is a $(\tau \geq k)$**-extension of** M. We shall show that any matroid has a $(\tau \geq k)$-extension. We then define $F(M, k)$ to be the minimum integer $l > 0$ such that M has a $(\tau \geq k)$-extension M' with $|E(M')| - |E(M)| = l$. The main purpose of this chapter is to determine $F(M, k)$ in terms of other invariants of M.

By definition, if M is a matroid with $r(M) = 0$, then $\forall k \in \mathbf{N}$, $\tau(M) \geq k$. Accordingly, for a connected graph G, if $|V(G)| = 1$, then $\tau(G) \geq k$ for any $k \in \mathbf{N}$. For a graph G, then **edge arboricity** of G, denoted by $a_1(G)$, is the minimum number of spanning trees of G whose union equals $E(G)$. For a matroid, we define the similar concept $\gamma_1(M)$, which is the minimum number of bases of M whose union equals $E(M)$. The following theorems are well known.

Theorem 4.1.1 *(Nash-Williams [53]) Let G be a connected graph with $|V(G)| > 1$, and let $k > 0$ be an integer. Then $a_1(G) \leq k$ if and only if $\forall X \subseteq E(G)$, $|X| \leq kr(G[X])$.*

Theorem 4.1.2 *(Edmonds [19]) Let M be a matroid with $r(M) > 0$. Then $\gamma_1(M) \leq k$ if and only if $\forall X \subseteq E(M)$, $|X| \leq kr(X)$.*

Thus Theorem 4.1.2 above indicates that

$$\gamma_1(M) = \lceil \gamma(M) \rceil. \tag{4.1}$$

Our main result of this chapter can now be stated as follows.

Theorem 4.1.3 *For $k \in \mathbf{N}$, let M be a matroid with $\tau(M) \leq k$ and let $i(k)$ denote the smallest i_j in (3.5) such that $i(k) \geq k$. Then*
(i) $F(M,k) = k(r(M) - r(J_{i(k)})) - |E(M) - J_{i(k)}|$.
(ii) $F(M,k) = \max_{X \subseteq E(M)}\{kr(M/X) - |M/X|\}$.

In the next section, we shall present some of the useful properties related to strength and fractional arboricity of a matroid M, and to the decomposition of M. Section 3 will be devoted to the proofs of the main results. In the last section, we shall show some applications of our main results.

4.2 Preliminaries

Both $\eta(M)$ and $\gamma(M)$ have been studied by many, see [11], [25] and [27], among others.

A matroid M satisfying $\eta(M) = \gamma(M)$ is called a **uniformly dense matroid**. The both $\eta(M)$ and $\gamma(M)$ can also be described by their behavior in some parallel extension of the matroid.

Definition 4.2.1 *Let M be a matroid and let $\phi : E(M) \mapsto \mathbf{N}$ be a function. For each $e \in E(M)$, let $X_e = \{e^1, e^2, \cdots, e^{\phi(e)}\}$ be a set such that $X_e \cap X_{e'} = \emptyset$, $\forall e, e' \in E(M)$ with $e \neq e'$. The* **ϕ-parallel extension** *of M, denoted by M_ϕ, is obtained from M*

*by replacing each element $e \in E(M)$ by a class of $\phi(e)$ parallel elements X_e. Thus $E(M_\phi) = \bigcup_{e \in E(M)} X_e$ such that a subset $Y \subseteq E(M_\phi)$ is independent in M_ϕ if and only if both $\{e \in E(M) : X_e \cap Y \neq \emptyset\}$ is independent in M and $\forall e \in E(M)$, $|X_e \cap Y| \leq 1$. For $t \in \mathbf{N}$, if $\forall e \in E(M)$, $\phi(e) = t$ is a constant function, we write M_t for M_ϕ, and call M_t the t-**parallel extension** of M.*

Let $E' = \{e^1 : e \in E(M)\} \subseteq E(M_\phi)$. Then the bijection $e \leftrightarrow e^1$ between $E(M)$ and E' yields a matroid isomorphism between M and $M_\phi|E'$. Under this bijection, we shall view $M = M_\phi|E'$ as a restriction of M_ϕ.

4.3 Characterization of the Must-Added Elements with Respect to Having k Disjoint Bases

The main purpose of this section is to prove Theorems 4.1.3. We will start with a lemma.

Lemma 4.3.1 *Let M be a matroid and let $k > 0$ be an integer. Each of the following holds.*
(i) $\eta(M) \geq k$ if and only if $F(M, k) = 0$.
(ii) If $\gamma(M) \leq k$, then

$$F(M, k) = kr(M) - |E(M)|.$$

and for some $\phi : E(M) \mapsto \mathbf{N}$, M_ϕ is a matroid that contains M as a restriction such that $\eta(M_\phi) = \gamma(M_\phi) = k$, and such that $|E(M_\phi)| - |E(M)| = F(M, k)$.

Proof: (i) By (4.1), $\eta(M) \geq k$ if and only if $\tau(M) \geq k$. By the definition of $F(M, k)$, $\tau(M) \geq k$ if and only if $F(M, k) = 0$. This proves (i).

(ii) Since $\gamma(M) \leq k$, it follows by (4.1) that M has disjoint bases $B_1, \cdots B_k$ such that $E(M) = \bigcup_{i=1}^k B_i$. Define $\phi(e) = |\{B_i : e \in B_i\}|$. Then $\phi : E(M) \mapsto \mathbf{N}$. Let $L = M_\phi$ be the ϕ-parallel extension of M. Then by Definition 4.2.1, M is contained in L as a restriction. Moreover, both $|E(L)| = \sum_{i=1}^k |B_i| = kr(M)$ and $\tau(L) = k$. It follows by Theorem 3.2.2 that $\eta(L) = \gamma(L) = k$. Hence $F(M,k) = |E(L)| - |E(M)| = kr(M) - |E(M)|$. □

When $k = 2$, the cycle matroid version of Lemma 4.3.1 has been frequently applied in the study of supereulerian graphs, see Theorem 7 of [7] and Lemma 2.3 of [12], among others. (For a literature review on supereulerian graphs, see [6] and [16].)

Proof of Theorem 4.1.3(i): Let M be a matroid with $r(M) > 0$. If $\tau(M) \geq k$, then by (4.1) and by Theorem 3.3.5, $i(k) = i_1$, and so

$$E(M) = J_{i(k)}, \text{ and } F(M,k) = 0.$$

Thus Theorem 4.1.3(i) follows trivially with $\tau(M) \geq k$. Hence we assume that $\tau(M) < k$. By Theorem 3.3.5, we must have $m > 1$. Let $i(k)$ be the smallest i_j in η-spectrum (3.6) of M such that $i_j \geq k$. By Theorem 3.3.5, $\eta(M|J_{i(k)}) \geq k$. Let $M' = M/J_{i(k)}$. By the assumption that $\eta(M) < k$ and by Lemma 3.2.4(i), $\eta(M') = \eta(M)$. By the choice of $i(k)$, $\gamma(M') < k$, and so by Lemma 4.3.1,

$$F(M',k) = kr(M') - |E(M')|, \tag{4.2}$$

and there must be a function $\phi' : E(M') \mapsto \mathbf{N}$ such that $M'_{\phi'}$ satisfies $\eta(M'_{\phi'}) = \gamma(M'_{\phi'}) = k$. Define $\phi : E(M) \mapsto \mathbf{N}$ as follows:

$$\phi(e) = \begin{cases} \phi'(e) & \text{if } e \notin J_{i(k)} \\ 1 & \text{if } e \in J_{i(k)} \end{cases}.$$

Then M_ϕ is a matroid that contains M as a restriction, such that $J_k(M) \subset E(M_\phi)$. By the definition of ϕ, $M_\phi|J_{i(k)} = M|J_{i(k)} \in \mathcal{S}_k$. Since $M_\phi/J_{i(k)} = M'_{\phi'} \in_k$, it follows by Proposition 3.2.5(C3) that $M_\phi \in_k$. Thus by (4.2) and by Lemma 3.3.1,

$$\begin{aligned} F(M,k) &= F(M',k) = kr(M') - |E(M')| \\ &= k(r(M) - r(J_{i(k)})) - |E(M) - J_{i(k)}|, \end{aligned}$$

and so Theorem 4.1.3(i) is established. □

To continue our proof for Theorem 4.1.3, we introduce the following function: for any $X \subseteq E(M)$, define

$$f_k(M,X) = kr(M/X) - |M/X|, \text{ and } F_k(M) = \max_{X \subseteq E(M)} \{f_k(M,X))\}. \tag{4.3}$$

The function $f_k(M, X)$ was introduced by Bruno and Weinberg [5] to investigate the principal partition of matroids. They are closely related to the strength and fractional arboricity of matroids, as to be shown in Lemma 4.3.2 below.

Lemma 4.3.2 *Let M be a matroid with $r(M) > 0$, and let $k > 0$ be an integer. Each of the following holds.*
(i) $F_k(M) = 0$ if and only if $\eta(M) \geq k$.
(ii) $F_k(M) = f_k(M, \emptyset)$ if and only if $\gamma(M) \leq k$.
(iii) Let $i(k)$ denote the smallest i_j in (3.5) such that $i(k) \geq k$, and $J_{i(k)}$ the corresponding set in the η-decomposition (3.6) of M. Then $F_k(M/J_{i(k)}) = F(M, k)$.
(iv) For any $e \in E(M)$, $F_k(M) \geq F_k(M/e)$. In particular, $F_k(M) \geq F(M, k)$.
(v) If $X_0 \subset E(M)$ satisfies $F_k(M) = f_k(M, X_0)$, then $F_k(M) = f_k(M/X_0) = F_k(M/X_0) = f_k(M/X_0, \emptyset)$ and $\gamma(M/X_0) \leq k$.

Proof: (i) By definition (4.3), $F_k(M) = 0$ if and only if $\forall X \subseteq E(M)$, $f_k(M, X) = kr(M/X) - |E(M/X)| \leq 0$. By the definition of $\eta(M)$, $\forall X \subseteq E(M)$, $kr(M/X) - |E(M/X)| \leq 0$ if and only if $\eta(M) \geq k$.

(ii) By the definition of $F_k(M)$, $F_k(M) = f_k(M, \emptyset$ if and only if $\forall X \subseteq E(M)$,

$$k(r(M) - r(X)) - |E - X| \leq kr(M) - |E|;$$

and so if and only if $\forall X \subseteq E(M)$ with $r(X) > 0$, $\frac{|X|}{r(X)} \leq k$. By the definition of $\gamma(M)$, this happens if and only if $\gamma(M) \leq k$.

(iii) By Theorem 3.3.5, $\gamma(M/J_{i(k)}) < k$. By (ii) of this lemma, by Lemma 3.3.1, and by Theorem 4.1.3,

$$\begin{aligned} F_k(M/J_{i(k)}) &= f_k(M/J_{i(k)}, \emptyset) = r(M/J_{i(k)}) - |M/J_{i(k)}| \\ &= r(M) - r(J_{i(k)}) - |E| - |J_{i(k)}| = F(M, k). \end{aligned}$$

(iv) For any $e \in E(M)$, by the definition of $F_k(M)$ in (4.3), $F_k(M) \geq F_k(M/e)$. It follows by (iii) of this lemma that $F_k(M) \geq f_k(M, X) = F(M, k)$.

(v) By (iv), and by the choice of X_0, we have

$$F_k(M) \geq F_k(M/X_0) \geq f_k(M/X_0, \emptyset) = f_k(M, X_0) = F_k(M).$$

Thus we must have both $F_k(M) = f_k(M/X_0)$ and $F_k(M/X_0) = f_k(M/X_0, \emptyset)$. It follows by (ii) that $\gamma(M/X_0) \leq k$. This proves (v). □

Lemma 4.3.3 *Suppose that $X_0 \subseteq E(M)$ satisfies $f_k(M, X_0) = F_k(M)$. Then $\eta(M|X_0) \geq k$.*

Proof: By Lemma 4.3.1(i), it suffices to show that $F_k(M|X_0) = 0$. For any $Y \subseteq X_0$, as

$f_k(M|X_0, Y) = k(r(X_0)-r(Y))-|X_0|+|Y|$, and $f_k(M, X_0) = kr(r(M)-(X_0))-|E(M)|+|X_0|$.

It follows that $f_k(M|X_0, Y) + f_k(M, X_0) = f_k(M, Y) \leq F_k(M) = f_k(M, X_0)$. Thus by definition, $f_k(M|X_0, Y) \leq 0$. This implies that $F_k(M|X_0) = 0$, and so $\eta(M|X_0) \geq k$. □

Proof of Theorem 4.1.3(ii): By Lemma 4.3.2(iv), it suffices to show that $F_k(M) \leq F(M, k)$. We shall argue by induction on $|E(M)|$ to proceed the proof.

Suppose first that $F_k(M) = 0$. Then by Lemma 4.3.2(i), $F_k(M) = 0$ if and only if $\eta(M) \geq k$. By Lemma 4.3.1(i), we have $F(M, k) = 0 = F_k(M)$ in this case. Thus we assume that $F_k(M) > 0$.

By Lemma 4.3.1(i), $F_k(M) > 0$ if and only if $\eta(M) < k$. If $\gamma(M) \leq k$, then by Lemma 4.3.1(ii), and by Lemma 4.3.1(ii),

$$F_k(M) = f_k(M, \emptyset) = kr(M) - |E(M)| = F(M, k).$$

Hence we may assume that Theorem 4.1.3(ii) holds for smaller values of $|E(M)|$, and that

$$\eta(M) < k < \gamma(M). \tag{4.4}$$

By induction, we may assume that M does not have loops. By Theorem 3.3.5, and by (4.4), both $i(k)$, the smallest i_j in (3.5) such that $i_j \geq k$, and $J_{i(k)}$, the corresponding set in (3.6), exist.

Let $X_0 \subset E(M)$ be such that $F_k(M) = f_k(M, X_0)$. By (4.4), $X_0 \neq \emptyset$. Since M is loopless, $r(X_0) > 0$, and so $|E(M/X_0)| < |E(M)|$. By Lemma 4.3.2(v) and by induction, we have

$$F_k(M) = f_k(M/X_0) = F_k(M/X_0) = F(M/X_0, k), \text{ and } \gamma(M/X_0) \leq k. \qquad (4.5)$$

Suppose that $F(M, k) = l$. Then there exists a matroid M' with $M' \in \mathcal{S}_k$, which contains M as a restriction and satisfies $|E(M') - E(M)| = l$. Note that $X_0 \subseteq E(M) \subseteq E(M')$. Let $W = E(M') - E(M)$, and $W_0 = W - cl_{M'}(X_0)$. Then $|W_0| \leq |W|$.

Since $M' \in \mathcal{S}_k$, it follows by Proposition 3.2.5(C2) that $M'/X_0 \in \mathcal{S}_k$. Since M is a restriction of M', M/X_0 is a restriction of M'/X_0. It follows by the definition of $F(M/X_0, k)$ and by (4.5) that

$$F_k(M) = F(M/X_0, k) \leq |E(M'/X_0) - E(M/X_0)| \leq |W_0| \leq |W| = F(M, k).$$

This, together with Lemma 4.3.2(iv), implies Theorem 4.1.3(ii). □

4.4 Applications

Let G be a graph, and $M = M(G)$ is the cycle matroid of G. Let $F(G, k) = F(M(G), k)$, and $f_k(G, X) = f_k(M(G), X)$, for any edge subset $X \subseteq E(G)$. Let $\omega(G)$ denote the number of connected components of G. The next theorem follows immediately from Theorem 4.1.3.

Theorem 4.4.1 *(Theorems 3.4 and 3.10 of [43]) For $k \in \mathbf{N}$, let G be a connected graph with $\tau(M(G)) \leq k$ and let $i(k)$ denote the smallest i_j in (3.5) such that $i(k) \geq k$. Then*
(i) $F(G, k) = k(|V(G)| - |V(G[J_{i(k)}])| + \omega(G[J_{i(k)}]) - 1) - |E(G) - J_{i(k)}|$.
(ii) $F(G, k) = \max_{X \subseteq E(G)}\{f_k(G, X)\}$.

The problem of reinforcing graphs to have k edge-disjoint spanning trees has also been investigated by others. In [21], the following is proved.

Theorem 4.4.2 *(Haas, Theorem 1 of [21]) The following are equivalent for a graph G, and integers $k > 0$ and $l > 0$.*
(i) $E(G)| = k(|V(G)| - 1) - l$ and for subgraphs H of G with at least 2 vertices, $|E(H)| \leq k(|V(H)| - 1)$.
(ii) There exists some l edges which when added to G result in a graph that can be decomposed into k spanning trees.

Proof: Assume that (i) holds. Then by (3.1), $\gamma(M(G)) \leq k$. It follows by the assumption that $E(G)| = k(|V(G)| - 1) - l$ and by Lemma 4.3.1(ii) that $F(G, k) = l$, and so (i) is obtained.

Assume (ii) holds. Since adding l edges to G can result in a graph in $\mathcal{S}_k$, by (3.1) and by (4.1), $\gamma(M(G)) \leq k$. By Lemma 4.3.1(ii),

$$k(|V(G)| - 1) - |E(G)| = F(G, k) = l,$$

and so (i) must hold. □

Chapter 5

Supereulerian Width of Graphs

5.1 Introduction

Graphs in this paper are finite and may have multiple edges but no loops. Terminology and notations not defined here are referred to [4]. In particular, for a graph G, $\delta(G)$, $\Delta(G)$, $\kappa(G)$ and $\kappa'(G)$ represents the minimum degree, the maximum degree, the connectivity and the edge connectivity of a graph G, respectively. For subgraphs H_1, H_2 of G, $H_1 \cup H_2$ and $H_1 \cap H_2$ denote the union and intersection of H_1 and H_2, respectively, as defined in [4]. For vertices $u, v \in V(G)$, a trail with end vertices being u and v will be referred as a (u, v)-trial. We use $O(G)$ to denote the set of all odd degree vertices in G. A graph G is **Eulerian** if $O(G) = \emptyset$ and G is connected, and is **supereulerian** if G has a spanning Eulerian subgraph.

Let G be a graph, and $s > 0$ be an integer. For any distinct $u, v \in V(G)$, an $(s; u, v)$**-trail-system** of G is a subgraph H consisting of s edge-disjoint (u, v)-trails. A graph is **supereulerian with width** s if $\forall u, v \in V(G)$ with $u \neq v$, G has a spanning $(s; u, v)$-trail-system. The **supereulerian width** $\mu'(G)$ of a graph G is the largest integer s such that G is supereulerian with width k for any integer k with $1 \leq k \leq s$. Luo et al in [47] defined graphs with $mu'(G) \geq 2$ as **Eulerian-connected** graphs and investigated, given an integer $r > 0$, the minimum value $\psi(r)$ such that if G is a $\psi(r)$-edge-connected graph, then $\forall X \subseteq E(G)$ with $|X| \leq r$, $\mu'(G - X) \geq 2$. Note that if for some vertices u and v, G does not have a spanning (u, v)-trial, then $\mu'(G) = 0$. The vertex counter-part of $\mu'(G)$, called the spanning connectivity of a graph, has been intensively studied, as can be seen in Chapters 14 and 15 of [29].

Throughout this paper, as in [4], $G[X]$ denotes the subgraph induced by an edge subset $X \subseteq E(G)$. When no confusion arises, we shall often adopt the convention that for an edge subset $X \subseteq E(G)$, X denotes the edge subset as well as the subgraph $G[X]$ of G.

In [3], Boesch et al first raised a problem to determine when a graph is supereulerian. They remarked that such a problem would be a difficult one. In [55], Pulleyblank confirmed the remark by showing that the problem to determine if a graph is supereulerian, even within planar graphs, is NP-complete.

In [8], Catlin introduced collapsible graphs as a tool to study supereulerian graphs. Catlin (Theorem of [8]) and Lai et al (Theorem 2.3(iii) of [37]) showed that if G is collapsible, then $\mu'(G) \geq 2$. Most of the studies on supereulerian graphs with width at most 2 can be found in Catlin's survey [6] and its update [16]. By definition, we have the obvious inequality

$$\mu'(G) \leq \kappa'(G), \text{ for any connected graph } G. \tag{5.1}$$

Knowing when the equality in (5.1) will hold is one of the most natural questions. One purpose of this paper is an effort to investigate graphs G such that for a given integer k, $\mu'(G) \geq k$ if and only if $\kappa'(G) \geq k$. Motivated by Catlin's work in [8], we extend the concept of collapsible graphs to s-collapsible graphs, and use it to develop an associate reduction method using s-collapsible graphs in Section 2. In Section 3, we study the s-collapsibility of complete graphs and some other dense graphs, and verify that for any graph G with at most 6 vertices and not isomorphic to $K_{3,3}$, $\mu'(G) \geq 3$ if and only if $\kappa'(G) \geq 3$. In the last section, we apply the reduction method associate with s-collapsible graphs to study the structure of reduced graphs under a degree condition. These allow us to obtain a best possible degree condition for supereulerian graphs with width at least 3.

5.2 Reductions with s-Collapsible Graphs

Throughout this paper, we adopt the convention that any graph G is 0 edge-connected, and let $s \geq 1$ denote an integer. For sets X and Y, the **symmetric difference** of X and Y is

$$X \Delta Y = (X \cup Y) - (X \cap Y).$$

Definition 5.2.1 *A graph G is s-**collapsible** if for any subset $R \subseteq V(G)$ with $|R| \equiv 0$ (mod 2), G has a spanning subgraph Γ_R such that*
(i) both $O(\Gamma_R) = R$ and $\kappa'(\Gamma_R) \geq s-1$, and
(ii) $G - E(\Gamma_R)$ is connected.

A spanning subgraph Γ_R of G with both properties in Definition 5.2.1 is an (s, R)-**subgraph** of G. Let $\mathcal{C}_s$ denote the collection of s-collapsible graphs. Then $\mathcal{C}_1$ is the collection of all collapsible graphs, defined in [8]. By definition, for $s \geq 1$, any $(s+1, R)$-subgraph of G is also an (s, R)-subgraph of G. This implies that

$$\mathcal{C}_{s+1} \subseteq \mathcal{C}_s, \text{ for any positive integer } s. \tag{5.2}$$

Proposition 5.2.2 *Let G be a graph, and let $s \geq 1$ be an integer. Then the following are equivalent.*
(i) $G \in \mathcal{C}_s$.
(ii) For any $X \subseteq V(G)$ with $|X| \equiv 0$ (mod 2), G has a spanning connected subgraph L_X such that $O(L_X) = X$ and such that $\kappa'(G - E(L_X)) \geq s-1$.

Proof. (i) $\Longrightarrow$ (ii). Given $X \subseteq V(G)$ with $|X| \equiv 0$ (mod 2), let $R = O(G)\Delta X$. Since $G \in \mathcal{C}_s$, G has a spanning subgraph Γ_R such that $O(\Gamma_R) = R$, $\kappa'(\Gamma_R) \geq s-1$, and $G - E(\Gamma_R)$ is connected. Let $L_X = G - E(\Gamma_R)$. Then L_X is a spanning connected subgraph such that $O(L_X) = R\Delta O(G) = X\Delta O(G)\Delta O(G) = X$. Moreover $\kappa'(G - E(L_X)) = \kappa'(\Gamma_R) \geq s-1$.

(ii) $\Longrightarrow$ (i). Given $R \subseteq V(G)$ with $|R| \equiv 0$ (mod 2), let $X = R\Delta O(G)$. By (ii), G has a spanning connected subgraph L_X such that $O(L_X) = X$ and such that $\kappa'(G - E(L)) \geq$

$s-1$. Let $\Gamma_R = G - E(L_X)$. Then both $\kappa'(\Gamma_R) \geq s-1$ and $O(\Gamma_R) = O(G)\Delta X = R$. As $G - E(\Gamma_R) = L_X$ is connected, $G \in \mathcal{C}_s$. □

For a graph G, and for $X \subseteq E(G)$, the **contraction** G/X is obtained from G by identifying the two ends of each edge in X and then by deleting the resulting loops. If H is a subgraph of G, then we write G/H for $G/E(H)$. When H is connected, we use v_H to denote the vertex in G/H onto which H is contracted.

Lemma 5.2.3 *Suppose that H is a connected subgraph of G, and $R \subseteq V(G)$ with $|R| \equiv 0$ (mod 2). Define*

$$R' = \begin{cases} R - V(H) & \text{if } |R \cap V(H)| \equiv 0 \text{ (mod 2)} \\ (R - V(H)) \cup \{v_H\} & \text{if } |R \cap V(H)| \equiv 1 \text{ (mod 2).} \end{cases}$$

If G/H has an (s, R')-subgraph $\Gamma_{R'}$, and if $H \in \mathcal{C}_s$, then G has an (s, R)-subgraph Γ_R.

Proof. Let $\Gamma_{R'}$ be an (s, R')-subgraph of G/H. Define $R^* = V(H) \cap O(G[E(\Gamma_{R'})])$. Thus R^* consists of vertices in H that are incident with an odd number of edges in $E(\Gamma_{R'})$. By the definition of R', $|R^*| \equiv |R \cap V(H)|$ (mod 2). Define $R'' = R^*\Delta(R \cap V(H))$. Then $|R''| \equiv |R^*| + |R \cap V(H)| \equiv 0$ (mod 2). Since $H \in \mathcal{C}_s$, H has an (s, R'')-subgraph $\Gamma_{R''}$. Define

$$\Gamma_R = G[E(\Gamma_{R'}) \cup E(\Gamma_{R''})].$$

Since $\kappa'(\Gamma_{R'}) \geq s-1$ and $\kappa'(\Gamma_{R''}) \geq s-1$, we conclude that $\kappa'(\Gamma_R) \geq s-1$. By the definition of R' and R'',

$$O(\Gamma_R) = O(G[E(\Gamma_{R'})])\Delta O(\Gamma_{R''} = (R - V(H)) \cup (R \cap V(H)) = R.$$

Moreover, $G - E(\Gamma_R) = G[E(G/H - E(\Gamma_{R'})) \cup E(H - E((\Gamma_{R''}))]$. Since $\Gamma_{R'}$ is an (s, R')-subgraph of G/H, and since $\Gamma_{R''}$ is an (s, R'')-subgraph of H, $\Gamma_{R'}$ contains a spanning tree of G/H and $\Gamma_{R''}$ contains a spanning tree of H. It follows that $G - E(\Gamma_R)$ has a spanning tree of G, and so by definition, Γ_R is an (s, R)-subgraph of G. □

Corollary 5.2.4 *Let $s \geq 1$ be an integer. Then $\mathcal{C}_s$ satisfies the following.*
(C1) $K_1 \in \mathcal{C}_s$.

(C2) If $G \in \mathcal{C}_s$ and if $e \in E(G)$, then $G/e \in \mathcal{C}_s$.
(C3) If H is a subgraph of G and if $H, G/H \in \mathcal{C}_s$, then $G \in \mathcal{C}_s$.

Proof. (C1) and (C2) follow immediately from definitions, and (C3) follows from Lemma 5.2.3. □

Corollary 5.2.5 *Let $s \geq 1$ be an integer. If a graph $G \in \mathcal{C}_s$, then $\mu'(G) \geq s+1$.*

Proof. Let u and v be two distinct vertices of G. Let $X = \emptyset$. Since $G \in \mathcal{C}_s$, by Proposition 5.2.2, G has a spanning connected subgraph L_X with $O(L_X) = \emptyset$ and $\kappa'(G - E(L_X)) \geq s-1$. Since L_X is a spanning eulerian subgraph, L_X can be partitioned into two edge-disjoint (u,v)-trails T_1, T_2. By Menger's theorem, $G - E(L_X)$ has $s-1$ edge-disjoint (u,v)-trials, $T_3, T_4, \cdots, T_{s+1}$. Since $T_1 \cup T_2 = L_X$ is spanning, $\{T_1, T_2, \cdots, T_{s+1}\}$ is spanning $(s+1; u, v)$-trail-system. □

A subgraph H of G is $\mathcal{C}_s$**-maximal** if $H \in \mathcal{C}_s$ and if G has no subgraph in $\mathcal{C}_s$ that properly contains H.

Lemma 5.2.6 *Let G be a graph and let $s > 0$ be an integer. Each of the following holds.*
(i) Let L_1, L_2 be vertex induced subgraphs of G. If $V(L_1) \cap V(L_2) \neq \emptyset$ and if $L_1, L_2 \in \mathcal{C}_s$, then $L_1 \cup L_2 \in \mathcal{C}_s$.
(ii) The graph G has a unique set of $\mathcal{C}_s$-maximal subgraphs $H_1, H_2, \cdots, H_c$, and if $G' = G/(\cup_{i=1}^{c} E(H_i))$, then G' contains no nontrivial subgraph in $\mathcal{C}_s$.

Proof. (i) Let $L = L_1 \cup L_2$, and $L' = L/L_2$. Let v' denote the vertex of L' onto which L_2 is contracted. Since L_1, L_2 are vertex induced subgraphs of G,

$$L' = L/L_2 = (L_1 \cup L_2)/L_2 = L_1/(L_1 \cap L_2),$$

is a contraction of L_1, it follows by Corollary 5.2.4(C2) that $L' \in \mathcal{C}_s$. As $L_2 \in \mathcal{C}_s$ and by Corollary 5.2.4(C3), $L \in \mathcal{C}_s$.

(ii) The existence and the uniqueness of this set $\mathcal{C}_s$-maximal subgraphs $H_1, H_2, \cdots, H_c$ follow from Corollary 5.2.4(C1) and from (i). Let $V(G') = \{u_1, u_2, \cdots, u_c\}$, where u_i is the vertex onto which the subgraph H_i is contracted, $(1 \le i \le c)$. Suppose that G' has a nontrivial subgraph $H' \in \mathcal{C}_s$. We may assume that $V(H') = \{u_1, u_2, \cdots, u_t\}$ with $t \ge 2$. Then by repeat applications of Corollary 5.2.4(C3),

$$H = G[E(H') \cup \left(\bigcup_{i=1}^{t} E(H_i)\right)] \in \mathcal{C}_s,$$

contrary to the assumption that these H_i's are $\mathcal{C}_s$-maximal. □

A graph is **$\mathcal{C}_s$-reduced** if it contains no nontrivial subgraph in $\mathcal{C}_s$. By Lemma 5.2.6, the graph $G' = G/(\cup_{i=1}^{c} E(H_i))$ is $\mathcal{C}_s$-reduced, called the **$\mathcal{C}_s$-reduction** of G.

Corollary 5.2.7 *Let $s \ge 1$ be an integer. Let T be a spanning tree of a graph G. If $\forall e \in E(T)$, e lies in a subgraph $H_e \in \mathcal{C}_s$, then $G \in \mathcal{C}_s$.*

Proof. The hypothesis implies that G has a nontrivial subgraph in $\mathcal{C}_s$. Let H be a subgraph of G such that $H \in \mathcal{C}_s$ with $|V(H)|$ maximized. If $G = H$, then done. Assume that $|V(H)| < |V(G)|$. Since T is a spanning tree, there must be an edge $e \in E(T) - E(H)$ but e is incident with a vertex in H. By assumption, G has a subgraph $H_e \in \mathcal{C}_s$ such that $e \in E(H_e)$. Since $V(H) \cap V(H_e) \ne \emptyset$, by Lemma 5.2.6(i), $H \cup H_e \in \mathcal{C}_s$, contrary to the maximality of H. Hence we must have $G = H$. □

Lemma 5.2.8 *Let $s \ge 1$ be an integer. Suppose that H is a connected subgraph of G. For any $x \in V(G)$, define $x' = x$ if $x \in V(G) - V(H)$ and $x' = v_H$ if $x \in V(H)$. If $H \in \mathcal{C}_s$, then for any $u, v \in V(G)$ with $u \ne v$, the following are equivalent.*
(i) G has a spanning $(s+1; u, v)$-trail-system.
(ii) If $u' \ne v'$, then G/H has a spanning $(s+1; u', v')$-trail-system; and if $u' = v' = v_H$, then G/H is supereulerian.

Proof. (i) $\Longrightarrow$ (ii). Let $T_1, T_2, \cdots T_{s+1}$ be edge-disjoint (u, v)-trials in G such that $\cup_{i=1}^{s+1} T_i$ is spanning in G. For $i \in \{1, 2, \cdots, s+1\}$, let $Z_i = V(G) - V(T_i)$, and define

$$T_i' = (T_i \cup H)/H - Z_i, \text{ for } i \in \{1, 2, \cdots, s+1\},$$

Then in G/H, if $u' \neq v'$, $T'_1, T'_2, \cdots T'_{s+1}$ are edge-disjoint (u', v')-trails. Since $\cup_{i=1}^{s+1} T_i$ is spanning in G, $\{T'_1, T'_2, \cdots T'_{s+1}\}$ is a spanning $(s+1; u', v')$-trail-system of G/H. If $u' = v'$, then since $u \neq v$ in G, we must have $u' = v' = v_H$, and so $T'_1, T'_2, \cdots T'_{s+1}$ are edge-disjoint closed trails in G/H. Since $\cup_{i=1}^{s+1} T_i$ is spanning in G, $\cup_{i=1}^{s+1} T'_i$ is a spanning closed trail in G/H, and so G/H is supereulerian.

(ii) $\Longrightarrow$ (i). Suppose first that $u' = v' = v_H$, and G/H is supereulerian. Let T' denote a spanning closed trial in G/H. Let $X' = O(G[E(T')])$.

Since T' is an Eulerian subgraph of G/H, $X' \subseteq V(H)$ with $|X'| \equiv 0 \pmod 2$. Since $H \in \mathcal{C}_s$, by Proposition 5.2.2, H has a spanning connected subgraph L_X with $O(L_{X'}) = X'$ such that $\kappa'(H - E(L_{X'})) \geq s - 1$. Thus $H - E(L_{X'})$ has $s - 1$ edge-disjoint (u, v)-paths $T_1, T_2, \cdots, T_{s-1}$. Let $\Gamma = T' \cup L_{X'}$ be an edge-induced subgraph of G. Since T' is spanning and connected in G/H, and since $L_{X'}$ is spanning and connected in H, Γ is a spanning connected subgraph of G with $O(\Gamma) = O(T')\Delta O(L_{X'}) = O(T')\Delta O(T') = \emptyset$. Thus Γ is a spanning Eulerian subgraph of G, and so Γ can be partitioned into two edge-disjoint (u, v)-trails T_s and T_{s+1}, such that $T_s \cup T_{s+1} = \Gamma$ is spanning in G. It follows that $\{T_1, T_2, \cdots, T_{s+1}\}$ is a spanning $(s + 1; u, v)$-trail-system.

Therefore we assume that $u' \neq v'$. Let $\{T'_1, T'_2, \cdots T'_{s+1}\}$ be a spanning $(s + 1; u', v')$-trail-system of G/H. Let $L' = \cup_{i=1}^{s+1} T'_i$. Let $G[E(T'_i)]$, $(1 \leq i \leq s + 1)$, and $G[E(L')]$ denote the edge induced subgraphs of G. Let

$$Y_i = O(G[E(T'_i)]) \cap V(H), \;\; 1 \leq i \leq s + 1.$$

Since for each i, T'_i is a (u, v)-trail in G/H,

$$O(G[E(T'_i)]) \subseteq V(H) \cup \{u, v\}, \;\; 1 \leq i \leq s + 1. \tag{5.3}$$

To complete the proof of the lemma, we consider the following cases to show that a spanning $(s + 1; u, v)$-trail-system always exists.

Case 1 $u, v \notin V(H)$.

Then $u' = u$ and $v' = v$. Since $u, v \notin V(H)$, by (5.3), $|Y_i| \equiv 0 \pmod 2$. Without loss of generality, we assume that $Y_i \neq \emptyset$ when $1 \leq i \leq t$, and $Y_i = \emptyset$, for all $i > t$. Since each

T_i' is an (u, v)-trail containing v_H, for each i with $1 \le i \le t$, there must be $u_i, v_i \in Y_i$ such that T_i' contains a (u, u_i)-trail J_i and a (v_i, v)-trail J_i' such that J_i and J_i' are edge-disjoint. Define sets W_i and W as follows:

$$W_i = \begin{cases} \{u_i, v_i\} & \text{if } u_i \neq v_i \\ \emptyset & \text{otherwise} \end{cases} \quad \text{where } 1 \le i \le t, \text{ and } W = \Delta_{i=2}^{t} W_i.$$

Note that if $t = 1$, then $W = \emptyset$, and that it is possible that for $i \neq j$, $u_i = u_j$ or $v_i = v_j$. As each $|W_i| \equiv 0 \pmod 2$, we also have $|W| \equiv 0 \pmod 2$. Define

$$X = (\cup_{i=1}^{t} Y_i) \Delta W.$$

Since both $|\cup_{i=1}^{t} Y_i| \equiv 0 \pmod 2$ and $|W| \equiv 0 \pmod 2$, $|X| \equiv 0 \pmod 2$. Since $H \in \mathcal{C}_s$, and since $X \subseteq V(H)$, by Proposition 5.2.2, H has a spanning connected connected subgraph L_X with $O(L_X) = X$, such that $\kappa'(H - E(L_X)) \ge s - 1$.

Since $\kappa'(H - E(L_X)) \ge s - 1$, relabelling the u_i's and the J_i's if necessary, $H - E(L_X)$ has edge-disjoint (u_i, v_i)-trials J_i'', $(2 \le i \le t)$. Define edge induced subgraphs as follows:

$$T_i = \begin{cases} J_i \cup J_i' \cup J_i'' & \text{if } 2 \le i \le t \\ T_i' & \text{if } t + 1 \le i \le s + 1. \end{cases}$$

Thus for all $2 \le i \le s + 1$, these T_i's are edge-disjoint (u, v)-trials. For $i = 1$, define

$$T_1 = J_1 \cup J_1' \cup L_X \cup \left(L' - \bigcup_{i=2}^{s+2} E(T_i) \right).$$

Since each T_i is a (u, v)-trail, every vertex in $L_X \cup \left(\bigcup_{i=2}^{s+1} T_i - \bigcup_{i=2}^{t} T_i\right) - \{u_1, v_1\}$ has an even degree. By the definition of W, either $u_1 = v_1$ or $u_1, v_1 \in X$, and so T_1 is also a (u, v)-trial, edge-disjoint from $\bigcup_{i=2}^{s+1} T_i$. Since L_X is spanning in H and L' spans G/H, $\cup_{i=1}^{s+1} T_1$ is spanning in G. Thus $\{T_1, T_2, \cdots, T_{s+1}\}$ is a spanning $(s + 1; u, v)$-trail-system.

Case 2 $u \notin V(H)$ and $v \in V(H)$. (The case when $u \in V(H)$ and $v \notin V(H)$ is similar and will be omitted).

Then $u' = u$ and $v' = v_H$. Since $u \notin V(H)$, by (5.3), $\forall i$ with $1 \le i \le s$, $|Y_i| \equiv 1$ (mod 2). Since T_i' is a (u, v_H)-trial in G/H, $\exists u_i \in Y_i$, such that T_i' contains a (u, u_i)-trail J_i in G. Without loss of generality, assume that $u_i \neq v$ if $1 \le i \le t$, and $v = u_j$ if $j > t$.

Let $W_i = Y_i - \{u_i\}$, $1 \le i \le t$; and $W_j = Y_j - \{v\}$, $t < j \le s+1$. Then $|W_i| \equiv 0 \pmod 2$, $1 \le i \le s+1$. Define

$$X = \left(\Delta_{i=1}^{s+1} W_i\right) \Delta\{u_1, v\}.$$

Then $|X| \equiv 0 \pmod 2$. Since $H \in \mathcal{C}_s$, by Proposition 5.2.2, H has a spanning connected subgraph L_X with $O(L_X) = X$, such that $\kappa'(H - E(L_X)) \ge s - 1 \ge t$. Thus $H - E(L_X)$ has edge-disjoint (u_i, v)-paths J'_i, $2 \le i \le t$. Define edge induced subgraphs as follows:

$$T_i = \begin{cases} J_1 \cup L_X \cup \left(L' - \bigcup_{i=2}^{s+1} E(J_i)\right) & \text{if } i = 1 \\ J_i \cup J'_i & \text{if } 2 \le i \le t \\ J_i & \text{if } t < i \le s+1. \end{cases}$$

Note that $O(T_1) = O(J_1) \Delta O(L_X) \Delta O(L' - \bigcup_{i=2}^{t} J_i) = \{u, v\}$. As L_X is connected, $T_1, T_2, T_3, \cdots, T_{s+1}$ are edge-disjoint (u, v)-trails in G. Since L' is spanning in G/H and L_X is spanning in H, $\{T_1, T_2, \cdots, T_{s+1}\}$ is a spanning $(s+1; u, v)$-trail-system of G. □

Corollary 5.2.9 *Let $s \ge 1$ be an integer, G be a graph and let H be a subgraph of G such that $H \in \mathcal{C}_s$. Each of the following holds.*
(i) $G \in \mathcal{C}_s$ if and only if $G/H \in \mathcal{C}_s$.
(ii) $\mu'(G) \ge s+1$ if and only if $\mu'(G/H) \ge s+1$.

Proof. (i) follows from Corollary 5.2.4. Since $\mu'(G/H) \ge s + 1 \ge 2$ implies that G/H is supereulerian, (ii) follows by Lemma 5.2.8. □

Let $s \ge 1$ be an integer. For a graph G, let $\tau(G)$ denote the maximum number of edge-disjoint spanning trees of G. By the well known spanning tree packing theorem of Nash-Williams [52] and Tutte [61], every $2k$-edge-connected graph must have k edge-disjoint spanning trees. (For a direct proof of this fact, see [20], or Theorems 1.1 and 1.3 of [13]). By Corollary 5.2.5 that a relationship between $\mathcal{C}_s$ membership and the value of $\tau(G)$ is observed:

$$\text{if } G \in \mathcal{C}_s, \text{ then } \tau(G) \ge \lfloor \tfrac{s+1}{2} \rfloor. \tag{5.4}$$

Let $F(G, s)$ denote the minimum number of additional edges that must be added to G to result in a graph G' with $\tau(G') \geq s$. The value of $F(G, s)$ has been studied and determined in [43], whose matroidal versions are proved in [36].

Theorem 5.2.10 *(Catlin, [8]) If $F(G, 2) \leq 1$, then $G \in \mathcal{C}_1$ if and only if $\kappa'(G) \geq 2$.*

We extend this Catlin's Theorem to other values of s.

Theorem 5.2.11 *Let $s \geq 1$ be an integer. If $F(G, s+1) \leq 1$, then $G \in \mathcal{C}_s$ if and only if $\kappa'(G) \geq s+1$.*

Proof. Suppose first that $G \in \mathcal{C}_s$. By Corollary 5.2.5, we have $\kappa'(G) \geq \mu'(G) \geq s+1$. Hence we assume that $\kappa'(G) \geq s+1$ to prove that $G \in \mathcal{C}_s$. By Theorem 5.2.10, we may assume that $s > 1$. Let $n = |V(G)|$.

Since $F(G, s+1) \leq 1$, G has spanning trees $T_1, T_2, \cdots, T_s$ and a spanning forest F with $|E(F)| = n-2$. Let F' and F'' denote the two components of F. For each i with $1 \leq i \leq s$, let $H_i = T_i \cup F$. By definition, $F(H_i, 2) = 1$. If $\kappa'(H_i) = 1$, then there must be an edge $e_i \in E(T_i)$ such that if T_i', T_i'' are two components of $T_i - e_i$, then $V(F') = V(T_i')$ and $V(F'') = V(T_i'')$. It follows that if for every i, $\kappa'(H_i) = 1$, then $\{e_1, e_2, \cdots, e_s\}$ is an edge cut of G separating $V(F')$ and $V(F'')$, contrary to the assumption that $\kappa'(G) \geq s+1$. Hence we may assume that $\kappa'(H_1) \geq 2$. By Theorem 5.2.10, $H_1 \in \mathcal{C}_1$. Let $X \subseteq V(G)$ be a subset with $|X| \equiv 0 \pmod 2$. Since $H_1 \in \mathcal{C}_1$, by Proposition 5.2.2, H_1 has a spanning connected subgraph L_X with $O(L_X) = X$. Since $G - E(L_X)$ contains spanning trees $T_2, \cdots, T_s$, we have $\kappa'(G - E(L_X)) \geq s-1$. By Proposition 5.2.2 again, $G \in \mathcal{C}_s$. □

We need a theorem of Nash-Willaims in deriving a corollary of the theorem above. For an explicit proof of this theorem, see Theorem 2.4 of [65].

Theorem 5.2.12 *(Nash-Willaims [53]) Let G be a graph. If*

$$\frac{|E(G)|}{|V(G)|-1} \geq s+1.$$

then G has a nontrivial subgraph L with $\tau(L) \geq s+1$.

Corollary 5.2.13 *Let G be a connected graph, and $s \geq 1$ be an integer.*
(i) If $\tau(G) \geq s+1$, then $G \in \mathcal{C}_s$.
(ii) If G is $\mathcal{C}_s$-reduced, then for any nontrivial subgraph H of G, $\frac{|E(H)|}{|V(H)|-1} < s+1$.
(iii) If $\kappa'(G) \geq s+1$ and G is $\mathcal{C}_s$-reduced, then

$$F(G, s+1) = (s+1)|V(G)| - |E(G)| - (s+1) \geq 2.$$

Proof. (i) If $\tau(G) \geq s+1$, then $F(G, s+1) = 0$ and $\kappa'(G) \geq \tau(G) \geq s+1$. By Theorem 5.2.11, $G \in \mathcal{C}_s$.

(ii) If for some connected subgraph H of G, $\frac{|E(H)|}{|V(H)|-1} \geq s+1$, then by Theorem 5.2.12, H (and so G) has a nontrivial subgraph L with $\tau(L) \geq s+1$. By Theorem 5.2.11, $L \in \mathcal{C}_s$, contrary to the assumption that G is $\mathcal{C}_s$-reduced.

(iii) The formula $F(G, s+1) = (s+1)|V(G)| - |E(G)| - (s+1)$ follows from Lemma 3.1 of [36] (or indirectly, Theorem 3.4 of [43]). The inequity follows from Theorem 5.2.11. □

The following theorem of Chen is useful when dealing with graphs with small order.

Theorem 5.2.14 *(Chen [15]) If G satisfies $\kappa'(G) \geq 3$ and $|V(G)| \leq 11$, then $G \in \mathcal{C}_1$ if and only if G cannot be contracted to the Petersen graph.*

5.3 Complete Graphs and Other Examples

In this section, we shall study the $\mathcal{C}_s$ membership and the μ' values of certain graphs, which will be useful in our arguments in the other sections. We start with a simple example. For an integer $l > 1$, and a graph H, lH denote the graph obtained from H by replacing each edge of H by a set of l parallel edges joining the same pair of vertices. For example, lK_2 is the loopless connected graph with two vertices and l edges. By Corollaries 5.2.5 and 5.2.13 and as $\mu'(G) \leq \kappa'(G)$ for any graph G, we have

Corollary 5.3.1 *Let $l \geq 2, s \geq 1$ be integers. Then $lK_2 \in \mathcal{C}_s$ if and only if $l \geq s+1$.*

We next consider the problem that for a give integer $s \geq 1$, determine the value of n such that $K_n \in \mathcal{C}_s$.

Lemma 5.3.2 *Let $s \geq 2, n \geq 2$ be positive integers.*
(i) If both $s \equiv n \equiv 1$ (mod 2), and if $n^2 < (3+s)n - 3$, then $K_n \notin \mathcal{C}_s$.
(ii) If $s+n \equiv 1$ (mod 2) or if $s \equiv n \equiv 0$ (mod 2), and if $n^2 < (3+s)n-2$, then $K_n \notin \mathcal{C}_s$.

Proof. In the proofs below, for each n satisfying the inequalities, we will choose a particular $R \subseteq V(K_n)$, and argue by contradiction to show that K_n cannot have an (s, R)-subgraph.

(i) Take $R \subset V(G)$ with $|R| = n - 1 \equiv 0$ (mod 2). Since $\kappa'(\Gamma) \geq s - 1$, $s - 1 \equiv 0$ (mod 2) and $O(\Gamma) = R$, $\forall v \in R$, we must have $d_\Gamma(v) \geq s$. It follows that $2|E(\Gamma)| \geq s(n-1)+(s-1)$. As $n^2 < (3 + s)n - 3$,

$$|E(K_n) - E(\Gamma)| = |E(K_n)| - |E(\Gamma)| \leq \frac{n(n-1)}{2} - \frac{s(n-1)+(s-1)}{2} < n - 1,$$

and so $K_n - E(\Gamma)$ cannot be connected, contrary to the assumption that Γ is an (s, R)-subgraph of K_n.

(ii) We first present the proof for the case when $s \equiv 1$ and $n \equiv 0$ (mod 2). Let $R = V(K_n)$. As $s \equiv 1$ (mod 2), $\delta(\Gamma) \geq s$, and so $2|E(\Gamma)| \geq sn$. Since $n^2 < (3 + s)n - 2$,

$$|E(K_n) - E(\Gamma)| = |E(K_n)| - |E(\Gamma)| \leq \frac{n(n-1)}{2} - \frac{sn}{2} < n - 1.$$

and so $K_n - E(\Gamma)$ cannot be connected, contrary to the assumption that Γ is an (s, R)-subgraph of G.

The case when $s \equiv 0$ and $n \equiv 1$ (mod 2) is similar.

What is left is to show that case when $n \equiv s \equiv 0$ (mod 2). Let $R = \emptyset$. As $s \equiv 0$ (mod 2), $\delta(\Gamma) \geq s$, and so $2|E(\Gamma)| \geq sn$. Since $n^2 < (3 + s)n - 2$,

$$|E(K_n) - E(\Gamma)| = |E(K_n)| - |E(\Gamma)| \leq \frac{n(n-1)}{2} - \frac{sn}{2} < n - 1,$$

and so $K_n - E(\Gamma)$ cannot be connected, contrary to the assumption that Γ is an (s, R)-subgraph of G. □

Theorem 5.3.3 *Let $s \geq 2$ and $n \geq 2$ be integers. Then $K_n \in \mathcal{C}_s$ if and only if $n \geq s+3$.*

Proof. Suppose that $G \in \mathcal{C}_s$. By Corollary 5.2.5, $\kappa'(G) \geq \mu'(G) \geq s+1$. Thus if $n \leq s+1$, then $\kappa'(G) \leq s$ and so $K_n \notin \mathcal{C}_s$. By Lemma 5.3.2 with $n = s+2$, $K_{s+2} \notin \mathcal{C}_s$. This completes the proof for necessity.

To prove the sufficiency, we note that if we can prove $K_{s+3} \in \mathcal{C}_s$, then for any $n > s+3$, K_n/K_{s+3} contains a spanning tree isomorphic to $K_{1,n-(s+3)}$ with the contraction image of K_{s+3}being a vertex of degree $n-(s+3)$, such that every edge of this spanning tree lies in a $(s+1)K_2$. By Corollary 5.3.1 and by Corollary 5.2.7, $K_n/K_{s+3} \in \mathcal{C}_s$. Thus by Corollary 5.2.4(C3), $K_n \in \mathcal{C}_s$. Hence it suffices to show that $K_{s+3} \in \mathcal{C}_s$. For any integer $n > 0$, let $V(K_n) = \{v_1, v_2, \cdots, v_n\}$, where the subscript are taken mod n.

Let $R \subseteq V(K_n)$ be a subset with $|R| \equiv 0 \pmod 2$. It suffices to show that for any possible values of $|R|$, K_n always has an (s,R)-subgraph Γ_R.

Case 1 $n = 2k+1$, for some integer $k > 2$.

Let $C_n = v_1v_2...v_nv_1$ be a Hamilton cycle of K_n. As $n \geq 7$ and $s = n-3$, $K_n - E(C_n)$ is an s-edge-connected, s-regular graph. Let $M = \{v_iv_{2k-i} : \text{with } i = 1, 2, \cdots, k-1\} \cup \{v_{k+1}v_{2k}\}$. Then M is a perfect matching of $K_n - E(C_n) - v_1$. Since $n \geq 7$, it is routine to check that $\kappa'(K_n - E(C_n) - M) \geq n-4 = s-1$.

By symmetry and since n is odd, we may assume that $v_1 \notin R$. Again by symmetry, we may assume that if $|R| > 0$, then $|R| = \{v_i, v_{2k-i+3} : i = 2, 3, 4, \cdots, l+1\}$ if $|R| = 2l$ with $l \leq k$.

If $|R| = 0$, then let $\Gamma_R = K_n - E(C_n)$; if $|R| = 2l$ for some $0 < l < k$, then let $\Gamma_R = K_n - E(C_n) - \{v_iv_{2k-i+3} : 2 \leq i \leq l\}$. Then $O(\Gamma_R) = R$ with $\kappa'(\Gamma_R) \geq s-1$, and $G - E(\Gamma_R)$ is connected. Therefore by definition, $K_n \in \mathcal{C}_s$.

Case 2 $n = 2k$, for some integer $k > 4$.

Let $M_1 = \{v_iv_{k+i} : i = 1, 2, \cdots, k\}$, $M_2 = \{v_iv_{k+i+1} : i = 2, 3, \cdots, k-1\} \cup \{v_kv_{k+1}\}$, and $M_3 = \{v_iv_{k+i+2} : i = 2, 3, \cdots, k-2\} \cup \{v_{k-1}v_{k+1}, v_kv_{k+2}\}$. Let $L = G[M_1 \cup M_2 \cup M_3)]$. As $n = 2k \geq 10$, it is routine to verify that $\kappa'(K_n - E(L)) \geq n-4 = s-1$.

By symmetry and since n is even, we may assume that if $|R| > 0$, then $R = \{v_1, v_{k+1}, \cdots, v_l, v_{k+l}\}$ if $0 < |R| = 2l \leq 2k$.

If $|R| = 0$, then let $\Gamma_R = K_n - E(L)$; if $|R| = 2l$ for some $0 < l \leq k$, then let $\Gamma_R = K_n - E(L - \{v_i v_{k+i} : 1 \leq i \leq l\})$. In any case, we have $O(\Gamma_R) = R$ with $\kappa'(\Gamma_R) \geq s - 1$, and $G - E(\Gamma_R)$, containing a Hamilton cycle $v_1 v_{k+2} v_k v_{k+1} v_{k-1} v_{2k} v_{k-2} v_{2k-1} \cdots v_2 v_{k+3} v_1$, is connected. Therefore by definition, $K_n \in \mathcal{C}_s$.

Case 3 $n \in \{4, 5, 6, 8\}$.

Note that when $n = 4$ and $s = 4 - 3 = 1$, $K_4 \in \mathcal{C}_1$ by Corollary 5.2.13. Hence we assume that $n \geq 5$.

For $n = 5$, let $C_5 = v_1 v_3 v_5 v_2 v_4 v_1$. If $|R| = 0$, then let $\Gamma_R = C$; if $R = \{v_3, v_4\}$, then let $\Gamma_R = C_5 + v_3 v_4\}$; if $R = \{v_2, v_3, v_4, v_5\}$, then let $\Gamma_R = C_5 + v_3 v_4 - v_2 v_5$. In any case, $O(\Gamma_R) = R$ and both Γ_R and $G - E(\Gamma_R)$ are connected. By symmetry and by the definition $\mathcal{C}_s$, $K_5 \in \mathcal{C}_2$.

Suppose that $n = 6$, and let $C_6 = v_1 v_2 v_3 v_4 v_5 v_6 v_1$, and $H = C + v_2 v_5$. If $|R| = 0$, then $\Gamma_R = H + \{v_1 v_3, v_4 v_6\}$; if $R = \{v_1, v_3\}$, then $\Gamma_R = H + v_4 v_6$; if $R = \{v_1, v_3, v_4, v_6\}$, then $\Gamma_R = H$; if $R = V(K_6)$, then $\Gamma_R = C_6$. In any case, we have $O(\Gamma_R) = R$ with $\kappa'(\Gamma_R) \geq 2$, and $G - E(\Gamma_R)$ connected. By symmetry and by the definition $\mathcal{C}_s$, $K_5 \in \mathcal{C}_3$.

Suppose that $n = 8$, and let $K_{4,4}$ denote the complete bipartite graph with vertex bipartition $\{v_1, v_3, v_5, v_7\}$ and $\{v_2, v_4, v_6, v_8\}$. Let $M = \{v_1 v_4, v_3 v_6, v_5 v_8, v_7 v_2\}$. Let $L = K_{4,4} - M$. If $|R| = 0$, then let $\Gamma_R = K_8 - E(L)$; if $R = \{v_1, v_2\}$, then let $\Gamma_R = K_8 - E(L - v_1 v_2)$; if $R = \{v_1, v_2, v_3, v_4\}$, then let $\Gamma_R = K_8 - E(L - \{v_1 v_2, v_3 v_4\})$; if $R = \{v_1, v_2, v_3, v_4, v_5, v_6\}$, then let $\Gamma_R = K_8 - E(L - \{v_1 v_2, v_3 v_4, v_5 v_6\})$; and if $R = V(K_8)$, then let $\Gamma_R = K_8 - E(L - \{v_1 v_2, v_3 v_4, v_5 v_6, v_7 v_8\})$. In any case, we have $O(\Gamma_R) = R$ with $\kappa'(\Gamma_R) \geq 4$, and $G - E(\Gamma_R)$ connected. By symmetry and by the definition $\mathcal{C}_s$, $K_8 \in \mathcal{C}_5$. □

Example 5.3.1 *We present some examples G with $\kappa'(G) = \mu'(G) = 3$. Let $C_n = v_1 v_2 \cdots v_n v_1$ denote a cycle on n vertices and let $v_0 \notin \{v_1, v_2, \cdots, v_n\}$ be a vertex. The* **wheel** *on $n + 1$ vertices, denoted by W_n, is obtained from C_n and v_0 by adding n new*

edges v_0v_i, $(1 \le i \le n)$. These new edges v_0v_i, $(1 \le i \le n)$, are referred as the spoke edges of W_n. The graph W_n' is obtained from W_n by contracting a spoke edge. Isomorphically, we can write $W_n' = W_n/\{v_0v_n\}$. The following can be routinely verified.
(i) $\mu'(K_n) = \kappa'(K_n) = n-1$.
(ii) if $G \in \{W_n, W_n'\}$ for $n \ge 3$, then $\mu'(G) = \kappa'(G) = 3$.

It suffices to verify (ii). Since both properties $\mu'(G) \ge 3$ and $\kappa'(G)$ are preserved under taking contractions, and since W_n' is a contraction of W_n, it suffices to show that $\mu'(W_n) \ge 3$. Let $u, v \in V(W_n$ be two distinct vertices. If $\{u,v\} = \{v_i, v_j\}$ for some $0 < i < j \le n$, then $v_iv_0v_j, v_iv_{i+1}\cdots v_j, v_jv_{j+1}\cdots v_nv_1\cdots v_i$ is a spanning $(3; u, v)$-trail-system. If $u = v_0$ and $v = v_1$, then $v_0v_1, v_0v_2v_1, v_0v_3\cdots v_nv_1$ is a spanning $(3; u, v)$-trail-system.

5.4 Smallest Graph G with $\mu'(G) < \kappa'(G) = 3$

The main result of this section will determine the smallest graph G with $\mu'(G) < \kappa'(G) = 3$. For a vertex $v \in V(G)$,

$$E_G(v) = \{e \in E(G) : \ e \text{ is incident with } v \text{ in } G\}.$$

We start with a conditional reduction lemma.

Lemma 5.4.1 *Let G be a graph and let $H = 2K_2$ be a subgraph of G. Denote $V(H) = \{z_1, z_2\}$ and $E(H) = \{e_1, e_2\}$. Suppose that*

$$|E_G(z_i) - E(H)| \le 2, \ \textit{for each } i = 1, 2. \tag{5.5}$$

Let v_H denote the vertex in G/H onto which H is contracted. For each vertex $v \in V(G)$, define $v' = v$ if $v \in V(G) - V(H)$ and $v' = v_H$ if $v \in V(H)$.
(i) For any $u, v \in V(G)$, if $\{u', v'\} - \{v_H\} \ne \emptyset$, and if G/H has a spanning $(3; u'v')$-trail-system, then G has a spanning $(3; u, v)$-trail-system.
(ii) If $\{u, v\} = V(H)$ and if $G - E(H)$ has a spanning (u, v)-trail, then G has a spanning $(3; u, v)$-trail-system.

Proof. (i) Let T_1', T_2', T_3' be a spanning $(3; u', v')$-trail-system in G/H.

Case 1 $v_H \notin \{u', v'\}$. Then v_H has even degrees in each T_i'. By (5.5), at most two of T_1', T_2', T_3' can contain v_H, and so we may assume that $v_H \in V(T_1') - V(T_3')$. Hence we may assume that $z_1 \in V(G[E(T_1')])$.

If $z_1 \notin O(G[E(T_1')])$, then by (5.5), $E_G(z_1) - E(H) \subseteq E(T_1')$. It follows that $(G[E(T_1') \cup E(H)], G[E(T_2')], G[E(T_3')])$ is a spanning $(3; u'v')$-trail-system in G. By symmetry, we assume that By (5.5),

$$z_1 \in O(G[E(T_j')]) \text{ if and only if } z_2 \in O(G[E(T_j')]), \text{ for each } j \in \{1, 2\}. \quad (5.6)$$

Define $T_1 = G[E(T_1') \cup \{e_1\}]$ and $T_3 = G[E(T_3')]$. For T_2, let

$$T_2 = \begin{cases} G[E(T_2') \cup \{e_2\}] & \text{if } z_1 \in O(G[E(T_2')]) \\ G[E(T_2')] & \text{if } z_1 \in O(G[E(T_2')]) \end{cases}.$$

Since $v_H \in V(T_1' - V(T_3'))$, T_1, T_2 and T_3 are (u, v)-trials in G. Since $\cup_{i+1}^3 T_i'$ is a spanning in G/H, and since $V(H) \subseteq V(T_1)$, $\{T_1, T_2, T_3\}$ is a spanning $(3, u, v)$-trail-system of G.

Case 2 $v_H \in \{u', v'\}$. We shall assume that $u' \neq v_H$ and $v' = v_H$. Without loss of generality, we assume that $v = z_1$. By the definition of $(3; u', v')$-trail-system, $v_H \in O(T_j')$, for each $j \in \{1, 2, 3\}$. By (5.5), $|E(T_j') \cap (E_G(z_1) \cup E_G(z_2))| = 1$, and so v is in at most two of the $O(G[E(T_j')])$'s. We then assume that $v \notin O(G[E(T_3')])$. For each $j \in \{1, 2, 3\}$, define

$$T_j = \begin{cases} G[E(T_j')] & \text{if } z_1 \in O(G[E(T_j')]) \\ G[E(T_j') \cup \{e_j\}] & \text{if } z_1 \notin O(G[E(T_j')]). \end{cases}$$

It is routine to verify that $\{T_1, T_2, T_3\}$ is a spanning $(3, u, v)$-trail-system of G.

(ii) Let $T_i = G[\{e_i\}]$, for $i = 1, 2$. If $G - E(H)$ has a spanning (u, v)-trail T_3, then $\{T_1, T_2, T_3\}$ is a spanning $(3, u, v)$-trail-system of G. □

Example 5.4.1 *Let $n > 2$ be an integer, and let $C_n = v_1 v_2 ... v_n v_1$ denote a cycle on n vertices. For $i = 1, 2, ..., n-1$, let e_i denote the edge of C_n with end vertices v_i and v_{i+1}. The graph $2C_n - e$ is obtained from C_n by adding a new edge e_i', parallel to e_i, for each $i = 1, 2, ..., n-1$. It is routine to show that $\mu'(2C_n - e) = 3$.*

Lemma 5.4.2 *Let $G = K_{3,3}$. Then $\mu'(G) = 2$.*

Proof. By Theorem 5.2.10, $K_{3,3} \in \mathcal{C}_1$, and so by Corollary 5.2.5, $\mu'(K_{3,3}) \geq 2$. It suffices to show that for some $u, v \in V(K_{3,3})$, $K_{3,3}$ does not have a spanning $(3; u, v)$-trail-system.

We shall adopt the notation in Figure 1 for $K_{3,3}$. Suppose that $K_{3,3}$ has a spanning $(3; v_1, v_3)$-trail-system $\{P_1, P_2, P_3\}$. Let $e_1 = v_1v_2, e_2 = v_1v_4$, and $e_3 = v_1v_6$; and $f_1 = v_3v_2$, $f_2 = v_3v_4$ and $f_3 = v_3v_6$. Since P_1, P_2, P_3 are edge-disjoint, we must have

$$|\{e_1, e_2, e_3\} \cap E(P_i)| = 1 = |\{f_1, f_2, f_3\} \cap E(P_i)|, \forall i \in \{1, 2, 3\}. \quad (5.7)$$

By (5.7), we may assume that $e_i \in E(P_i)$, $(1 \leq i \leq 3)$. If $f_1 \notin E(P_1)$, then since $K_{3,3}$ is 3-regular, P_1 must use v_2v_5, which will force f_1 lying in no P_i's, contrary to (5.7). Therefore, we must have $f_1 \in E(P_1)$. Similarly, we must have $f_2 \in E(P_2)$. As $K_{3,3} - \{v_2, v_4\}$ has cannot have a spanning (v_1, v_3)-trail. This proves that $K_{3,3}$ does not have a spanning $(3; v_1, v_3)$-trail-system. □.

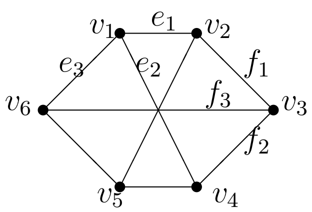

Figure 1. The graph $K_{3,3}$

Theorem 5.4.3 *Let G be a graph on n vertices.*
(i) (Lemma 5 of [8]) If $n \leq 4$, and if $\kappa'(G) \geq 2$, then $\mu'(G) \geq 2$ if and only if $G \neq K_{2,2}$.
(ii) If $n \leq 6$, and if $\kappa'(G) \geq 3$, then $\mu'(G) \geq 3$ if and only if $G \neq K_{3,3}$.

Proof of (ii). We argue by contradiction and assume that

$$G \text{ is a counterexample with } |E(G)| + |V(G)| \text{ minimised.} \quad (5.8)$$

If $1 \leq n \leq 3$, then $\kappa'(G) \geq 3$ implies that $F(G, 3) \leq 1$, and so (ii) follows by Theorem 5.2.11 and Corollary 5.2.5. By the definition of $\mu'(G)$, $\mu'(G) \geq 3$ if and only if every block H of G satisfying $\mu'(H) \geq 3$. Therefore, by (5.8), we assume that

$$\kappa(G) \geq 2, 4 \leq n \leq 6 \text{ and } G \text{ is minimally 3-edge-connected, and } \mathcal{C}_2\text{-reduced.} \quad (5.9)$$

Note that by Theorem 5.2.14, every such graph has a spanning eulerian subgraph. By (5.9) and by $n \leq 6$, we further conclude that

$$\text{every such graph } G \text{ has a Hamilton cycle } C = v_1 v_2 \cdots v_n v_1. \tag{5.10}$$

Let $\tilde{G}$ denote the simplification of G, and let $f(G, C) = |E(\tilde{G})| - n$ denote the number of chords of C in $\tilde{G}$. We choose C so that $f(G, C)$ is minimized. If $f(G, C) = 0$, then G is spanned by a $2C_n - e$, and so by Example 5.4.1, $\mu'(G) \geq 3$, contrary to (5.8). Hence we have

Claim 1 $f(G, C) \geq 1$.

A subgraph $2K_2$ of G satisfying (5.5) and Lemma 5.4.1(ii) in G will be referred as a **contractible** $2K_2$ of G. Claim 2 below follows from (5.8) and Lemma 5.4.1, and from the fact that when $n \geq 5$, that $f(G, C) \leq 1$ forces G to have a contractible $2K_2$.

Claim 2 There will be no contractible $2K_2$ of G, and when $n \geq 5$, $F(G, C) \geq 2$.

Claim 3 Theorem 5.4.3(ii) holds if $4 \leq n \leq 5$.

By Claim 2, G cannot have a contractible $2K_2$. Therefore, if $n = 4$, G must be either $L(4, 1, 1)$ or K_4 as depicted in Figure 2. By inspection, $\mu'(L(4, 1, 1)) = \mu'(K_4) = 3$.

Assume $n = 5$. By Claim 2, $f(G, C) \geq 2$. To avoid a contractible $2K_2$ in G, when $f(G, C) = 2$, G must be $L(5, 2, 1)$ (see Figure 2). When $f(G, C) \geq 3$, we may assume, by (5.9), that $G = L(5, 3, 1)$ (see Figure 2). Direct verification shows that $\mu'(G) \geq 3$ for $G \in \{L(5, 2, 1), L(5, 3, 1)\}$ (see Appendix for details).

Claim 4 If $e \notin E(K_{3,3})$ is an edge whose ends are in $V(K_{3,3})$, and if $G = K_{3,3} + e$, then $\mu'(G) \geq 3$.

We again use the notation of Figure 1 for $K_{3,3}$. By symmetry, we may assume that $e = v_1 v_i$. By Claim 2, G does not have a contractible $2K_2$, and so $i \notin \{2, 4, 6\}$. Therefore, we may assume that $e = v_1 v_3$. Then it is routine to show that $\mu'(G) \geq 3$. (See Table 6 in Appendix for details).

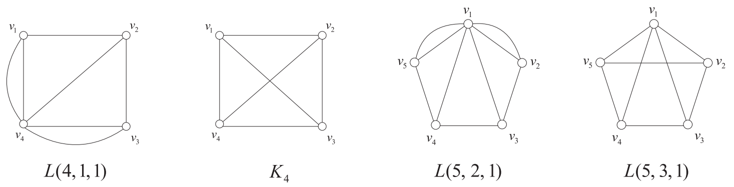

Figure 2 Graphs in Claim 3

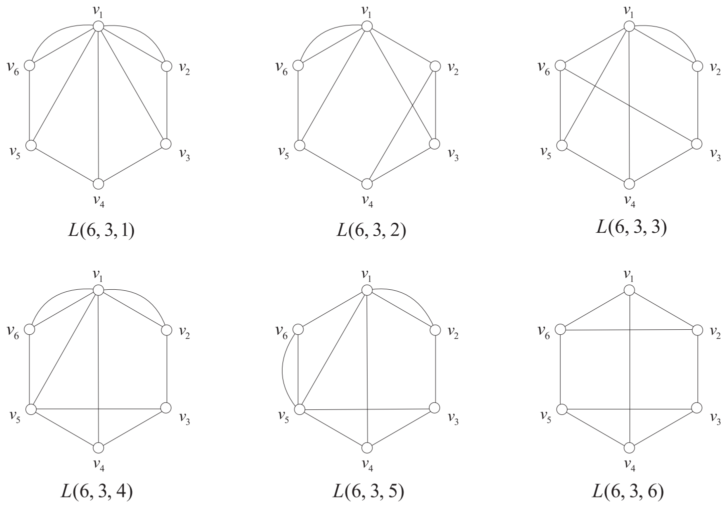

Figure 3 $\tilde{G}$ has 6 vertices with 3 chords of C

We are now ready to prove Theorem 5.4.3(ii). By Claims 3 and 4, we assume that $n = 6$ and G is not spanned by a $K_{3,3}$. Note that when $n = 6$, if $f(G, C) \le 2$, then by (5.9), G must have a contractible $2K_2$. Hence $f(G, C) \ge 3$. Let $d = \Delta(\tilde{G})$.

Suppose that $f(G, C) = 3$. If $d = 5$, then as $\tilde{G}$ is simple and by (5.9), $G \cong L(6, 3, 1)$ (depicted in Figure 3). If $d = 3$, then $G \in \{K_{3,3}, L(6, 3, 6)\}$ (depicted in Figure 3). Assume that $d = 4$ and v_1 has degree 4 in $\tilde{G}$. If v_1 is adjacent to v_2, v_3, v_5, v_6, then to avoid a contractible $2K_2$, either v_4v_2 or $v_4v_6 \in E(G)$. Hence by symmetry, we assume that $G = L(6, 3, 2)$ (depicted in Figure 3). Therefore by symmetry, we may assume that v_1 is adjacent to v_2, v_4, v_5, v_6. To avoid a contractible $2K_2$, v_3 must have degree 3. Hence $G \in \{L(6, 3, 3), L(6, 3, 4), L(6, 3, 5)\}$ (depicted in Figure 3). In any of these cases,

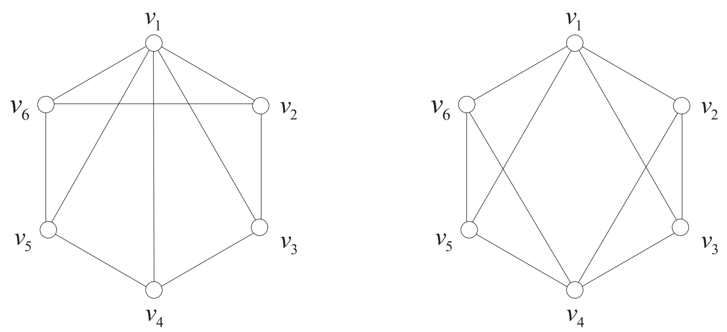

Figure 4 $\tilde{G}$ **has at least 4 chords of** C

$\mu'(G) \geq 3$. (See Appendix for details).

Now suppose that $F(G, C) \geq 4$, $n = 6$, and that

$$G \text{ is not spanned by a } K_{3,3} \text{ or any } L(6, 3, i). \tag{5.11}$$

If $\tilde{G}$ has a vertex v of degree 2, then at least 4 edges in $E(\tilde{G}) - E(C)$ will be joining the vertices of $V(C) - \{v\}$, and so G must have at least one edge e, such that $\kappa'(G - e) \geq 3$. Thus G is not minimally 3-edge-connected, contrary to (5.9). Hence we assume that $\delta(\tilde{G}) \geq 3$. Since $F(G, C) \geq 4$, $d \geq 4$.

Case 1 $d = 5$. We assume that v_1 is adjacent to all other 5 vertices of $\tilde{G}$. Since $\delta(\tilde{G}) \geq 3$, and by (5.9), we must have $G = L(6, 4, 1)$ (depicted in Figure 4). It is routine to show that $\mu'(L(6, 4, 1)) = 3$. (See Appendix for details).

Case 2 $d = 4$, We assume that v_1 is a vertex of degree 4 in $\tilde{G}$.

If v_1 is adjacent to all but v_4. Since $\delta(\tilde{G}) \geq 3$, we may assume, by symmetry, that $v_2v_4 \in E(\tilde{G})$. If $v_6v_2 \in E(\tilde{G})$, then $\kappa'(G - v_1v_2) \geq 3$, contrary to (5.9). Hence $v_4v_6 \in E(\tilde{G})$ and so $G = L(6, 4, 2)$ (depicted in Figure 4). It is routine to show that $\mu'(L(6, 4, 2)) = 3$. (See Appendix for details).

Thus by symmetry, we may assume that v_1 is adjacent to all but v_3. Then either v_3v_5 or $v_3v_6 \in E(\tilde{G})$, and either v_2v_5 or $v_2v_6 \in E(\tilde{G})$. But any of such combination will either violate (5.11) or violate (5.9).

This completes the proof of the theorem. □

5.5 Degree Condition for Supereulerian Graphs with Larger Width

Settling three open problems of Bauer in [2], Catlin and Lai proved the following.

Theorem 5.5.1 *Let G be a 2-edge-connected simple graph G on n vertices.*
(i) (Catlin, Theorem 9 of [8]) If $\delta(G) > \frac{n}{5} - 1$, then when n is sufficiently large, G is supereulerian.
(ii) (Lai, Theorem 5 of [32]) If G is bipartite, or G is triangle free, and if $\delta(G) > \frac{n}{10}$, then when n is sufficiently large, G is supereulerian.

Both bounds in Theorem 5.5.1 are best possible in the sense that there exist an infinite family of non-supereulerian 2-edge-connected graphs G on n vertices with $\delta(G) = \frac{n}{5} - 1$ (for Theorem 5.5.1(i)) and an infinite family of non-supereulerian bipartite graphs on n vertices with $\delta(G) = \frac{n}{10}$ (for Theorem 5.5.1(ii)). The main purpose of this section is to extend the theorem above, with a more general argument than the proofs in both [8] and [32]. We start with some preparation before presenting our main arguments. If G is a graph and G' is the $\mathcal{C}_s$-reduction of G, then for any vertex $u \in V(G')$, G has a maximal $\mathcal{C}_s$-subgraph H_u such that u the the vertex onto which H_u is contracted. The subgraph H_u is called the **preimage** of u in G. It is possible that H_u consists of a single vertex, in which case u is a **trivial vertex** of the contraction.

Lemma 5.5.2 *Let n, p, c be positive integers, and $f(n, p)$ be a function of n and p such that*

$$\text{for every fixed } p > 0, \text{ both } \frac{\partial f}{\partial n} > 0 \text{ and } \lim_{n\to\infty} f(n,p) = \infty.$$

Suppose that G is a simple graph on n vertices. If one of the following holds:
(i) $\delta(G) \geq f(n,p) - 1$,
(ii) G is triangle free and $\delta(G) \geq \frac{f(n,p)}{2}$,
then when n is sufficiently large, any vertex v in the $\mathcal{C}_s$-reduction of G whose degree is at most c must be the contraction image of a connected subgraph N_v with $|V(N_v)| \geq f(n,p)$.

Proof. Let G' be the $\mathcal{C}_s$-reduction of G. Define $W = \{u \in V(G') : d_{G'}(u) \leq c\}$ and pick $v \in W$. Let $N_G(v)$ denote the vertices of G adjacent to v in G. Then $V(N_v)$ contains all vertices in $N_G(v)$ except at most c vertices in $V(G) - V(N_v)$. Hence $|V(N_v)| \geq d_G(v) - c$.

By assumption, we can choose n so large that $f(n,p) > 2(c+1)$. (If for Part (i) only, we can choose n large so that $f(n,p) > c + 1$.) Then $|V(N_v)| \geq d_G(v) - c > 0$. Since $V(N_v)$ has at most c vertices that are adjacent to vertices not in N_v, $\exists z \in V(N_v)$, such that z is adjacent only to vertices in N_v. As $V(N_v)$ must contain all vertices adjacent to z, if (i) holds, then $|V(N_v)| \geq d_G(z) + 1 \geq f(n,p)$.

Suppose that G is triangle free and $\delta(G) \geq \frac{f(n,p)}{2}$. Find a vertex $z \in V(N_v)$ such that z is adjacent only to vertices in N_v as above. Since $|V(N_v - z)| \geq |N_G(z)| > c$, $N_G(z)$ has a vertex z' such that $N_G(z')$ is not adjacent to any vertex in $V(G) - V(N_v)$. Since G is triangle free, $N_G(z) \cap N_G(z') = \emptyset$. Thus $|V(N_v)| \geq |N_G(z)| + |N_G(z')| \geq 2\delta(G) \geq f(n,p)$. Hence in any case,

$$\forall v \in W, |V(N_v)| \geq f(n,p). \tag{5.12}$$

This completes the proof of the lemma. □

Theorem 5.5.3 *Let n, p, s be positive integers such that $p \geq 2$. Suppose that G is a simple graph on n vertices.*
(i) If

$$\delta(G) \geq \frac{n}{p} - 1, \tag{5.13}$$

then when n is sufficiently large (say $n > p(1 + (1 + 2(s+3) + 2(p+1)(s+1)))$), the $\mathcal{C}_s$-reduction of G has at most p vertices.
(ii) If G is triangle free, and if

$$\delta(G) \geq \frac{n}{2p}, \tag{5.14}$$

then when n is sufficiently large (say $n > 2p(1 + (1 + 2(s + 3) + 2(p + 1)(s + 1)))$), the $\mathcal{C}_s$-reduction of G has at most p vertices.

Proof. As the argument to prove both conclusions are similar, we shall prove them simultaneously. For given $p > 0$ and $s > 0$, choose an integer $c = 1+2(s+3)+2(p+1)(s+1)$.

Let G' be the $\mathcal{C}_s$-reduction of G, and assume that $n' = |V(G')| > 1$. Define

$$W = \{v \in V(G') : d_{G'}(v) \leq c\}.$$

Pick any $v \in W$ and any $z \in V(N_v)$. By Lemma 5.5.2 with $f(n,p) = \frac{n}{p}$, (5.12) must hold. By Corollary 5.2.13, we have

$$|E(G')| \leq (s + 1)n' - (s + 3). \tag{5.15}$$

It follows by combining (5.12) and (5.15) that,

$$cn' - cp \leq c|V(G') - W| \leq 2|E(G')| \leq 2(s + 1)n' - (2s + 3). \tag{5.16}$$

As $c > 2(s + 3) + 2(p + 1)(s + 1)$, (5.16) implies

$$n' \leq \frac{2(s + 3) + cp}{c - 2(s + 1)} < p + 1.$$

Hence $n' \leq p$, and so the theorem follows. □

The theorem above can be applied to study the supereulerian width of some dense graphs, as shown in Corollary 5.5.4 below. When $s = 1$ and $p = 5$, Corollary 5.5.4 gives the same results stated in Theorem 5.5.1.

Corollary 5.5.4 *Let n, s be positive integers such that $1 \leq s \leq 2$. Suppose that G is a simple graph on n vertices with $\kappa'(G) \geq s + 1$. Let $p(s) = 2s + 3$.*
(i) If

$$\delta(G) > \frac{n}{p(s)}, \tag{5.17}$$

then when n is sufficiently large, $\mu'(G) \geq s+1$ if and only if the $\mathcal{C}_s$-reduction of G is not a $K_{s+1,s+1}$.

(ii) If G is triangle free, and if

$$\delta(G) > \frac{n}{2p(s)}, \tag{5.18}$$

then when n is sufficiently large, $\mu'(G) \geq s+1$ if and only if the $\mathcal{C}_s$-reduction of G is not a $K_{s+1,s+1}$.

Proof. Let $p = p(s)$. Let G' denote the $\mathcal{C}_s$-reduction of G. By Corollary 5.2.5, we may assume that $|V(G')| > 1$. By Theorem 5.4.3, the conclusions hold if $|V(G')| \leq p-1$. Hence we assume that $|V(G')| \geq p$. By Theorem 5.5.3, when n is sufficiently large, G' has at most p vertices, and so we must have $|V(G')| = p$. Apply Lemma 5.5.2 with $c = p$ and $f(n,p) = \frac{n+1}{p}$. Thus when n is sufficiently large, by Lemma 5.5.2, every vertex in G' has a nontrivial preimage with at least $\lceil f(n,p) \rceil$ vertices. It follows that

$$n \geq \sum_{v \in V(G')} |V(N_v)| \geq pf(n,p) \geq n+1.$$

This contradiction established the corollary. □

Appendix: Checking the supereulerian width of certain graphs in the proof of Theorem 5.4.3

In the tables below, notations in Figures 2, 3 and 4 will be used. For each of these graphs, and for the given vertices u and v, a spanning $(3; u, v)$-trail-system in the given graph is presented, and the missing cases can be obtained by symmetry.

Graphs	$\{u, v\}$	Spanning $(3; u, v)$-trail-systems
$L(5, 2, 1)$	$\{v_1, v_2\}$	$G[\{v_1v_2\}]$, $G[\{v_1v_2\}]$, $G[\{v_1v_5, v_5v_4, v_4v_3, v_3v_2\}]$
	$\{v_1, v_3\}$	$G[\{v_1v_2, v_2v_3\}]$, $G[\{v_1v_3\}]$, $G[\{v_1v_5, v_5v_4, v_4v_3\}]$
	$\{v_2, v_3\}$	$G[\{v_2v_1, v_1v_3\}]$, $G[\{v_2v_3\}]$, $G[\{v_2v_1, v_1v_5, v_5v_4, v_4v_3\}]$
	$\{v_2, v_4\}$	$G[\{v_2v_3, v_3v_4\}]$, $G[\{v_2v_1, v_1v_4\}]$, $G[\{v_2v_1, v_1v_5, v_5v_4\}]$
	$\{v_2, v_5\}$	$G[\{v_2v_1, v_1v_5\}]$, $G[\{v_2v_1, v_1v_5\}]$, $G[\{v_2v_3, v_3v_4, v_4v_5\}]$
	$\{v_3, v_4\}$	$G[\{v_3v_4\}]$, $G[\{v_3v_1, v_1v_4\}]$, $G[\{v_3v_2, v_2v_1, v_1v_5, v_5v_4\}]$
$L(5, 3, 1)$	$\{v_1, v_2\}$	$G[\{v_1v_2\}]$, $G[\{v_1v_3, v_3v_2\}]$, $G[\{v_1v_4, v_4v_5, v_5v_2\}]$
	$\{v_1, v_3\}$	$G[\{v_1v_3\}]$, $G[\{v_1v_2, v_2v_3\}]$, $G[\{v_1v_5, v_5v_4, v_4v_3\}]$
	$\{v_2, v_3\}$	$G[\{v_2v_1, v_1v_3\}]$, $G[\{v_2v_3\}]$, $G[\{v_2v_5, v_5v_4, v_4v_3\}]$
	$\{v_2, v_4\}$	$G[\{v_2v_3, v_3v_4\}]$, $G[\{v_2v_5, v_5v_4\}]$, $G[\{v_2v_1, v_1v_4\}]$
	$\{v_2, v_5\}$	$G[\{v_2v_5\}]$, $G[\{v_2v_1, v_1v_5\}]$, $G[\{v_2v_3, v_3v_4, v_4v_5\}]$
	$\{v_3, v_4\}$	$G[\{v_3v_4\}]$, $G[\{v_3v_1, v_1v_4\}]$, $G[\{v_3v_2, v_2v_5, v_5v_4\}]$
	$\{v_3, v_5\}$	$G[\{v_3v_4, v_4v_5\}]$, $G[\{v_3v_1, v_1v_5\}]$, $G[\{v_3v_2, v_2v_5\}]$

Table 1. Spanning $(3; u, v)$-trail-systems when $n = 5$

Graphs	$\{u, v\}$	Spanning $(3; u, v)$-trail-systems
$L(6,3,1)$	$\{v_1, v_2\}$	$G[\{v_1v_2\}]$, $G[\{v_1v_2\}]$, $G[\{v_1v_6, v_6v_5, v_5v_4, v_4v_3, v_3v_2\}]$
	$\{v_1, v_3\}$	$G[\{v_1v_3\}]$, $G[\{v_1v_2, v_2v_3\}]$, $G[\{v_1v_6, v_6v_5, v_5v_4, v_4v_3\}]$
	$\{v_1, v_4\}$	$G[\{v_1v_4\}]$, $G[\{v_1v_2, v_2v_3, v_3v_4\}]$, $G[\{v_1v_6, v_6v_5, v_5v_4\}]$
	$\{v_2, v_3\}$	$G[\{v_2v_3\}]$, $G[\{v_2v_1, v_1v_3\}]$, $G[\{v_2v_1, v_1v_6, v_6v_5, v_5v_4, v_4v_3\}]$
	$\{v_2, v_4\}$	$G[\{v_2v_3, v_3v_4\}]$, $G[\{v_2v_1, v_1v_4\}]$, $G[\{v_2v_1, v_1v_6, v_6v_5, v_5v_4\}]$
	$\{v_2, v_5\}$	$G[\{v_2v_1, v_1v_5\}]$, $G[\{v_2v_1, v_1v_6, v_6v_5\}]$, $G[\{v_2v_3, v_3v_4, v_4v_5\}]$
	$\{v_2, v_6\}$	$G[\{v_2v_1, v_1v_6\}]$, $G[\{v_2v_1, v_1v_6\}]$, $G[\{v_2v_3, v_3v_4, v_4v_5, v_5v_6\}]$
	$\{v_3, v_4\}$	$G[\{v_3v_4\}]$, $G[\{v_3v_1, v_1v_4\}]$, $G[\{v_3v_2, v_2v_1, v_1v_6, v_6v_5, v_5v_4\}]$
$L(6,3,6)$	$\{v_1, v_2\}$	$G[\{v_1v_2\}]$, $G[\{v_1v_6, v_6v_2\}]$, $G[\{v_1v_4, v_4v_5, v_5v_3, v_3v_2\}]$
	$\{v_1, v_3\}$	$G[\{v_1v_2, v_2v_3\}]$, $G[\{v_1v_4, v_4v_3\}]$, $G[\{v_1v_6, v_6v_5, v_5v_3\}]$
	$\{v_1, v_4\}$	$G[\{v_1v_4\}]$, $G[\{v_1v_2, v_2v_3, v_3v_4\}]$, $G[\{v_1v_6, v_6v_5, v_5v_4\}]$
	$\{v_2, v_3\}$	$G[\{v_2v_3\}]$, $G[\{v_2v_1, v_1v_4, v_4v_3\}]$, $G[\{v_2v_6, v_6v_5, v_5v_3\}]$
	$\{v_2, v_4\}$	$G[\{v_2v_3, v_3v_4\}]$, $G[\{v_2v_1, v_1v_4\}]$, $G[\{v_2v_6, v_6v_5, v_5v_4\}]$
	$\{v_2, v_5\}$	$G[\{v_2v_6, v_6v_5\}]$, $G[\{v_2v_1, v_1v_4, v_4v_5\}]$, $G[\{v_2v_3, v_3v_5\}]$
	$\{v_2, v_6\}$	$G[\{v_2v_6\}]$, $G[\{v_2v_1, v_1v_6\}]$, $G[\{v_2v_3, v_3v_4, v_4v_5, v_5v_6\}]$
	$\{v_3, v_4\}$	$G[\{v_3v_4\}]$, $G[\{v_3v_5, v_5v_4\}]$, $G[\{v_3v_2, v_2v_6, v_6v_1, v_1v_4\}]$

Table 2. Spanning $(3; u, v)$-trail-systems when $n = 6$ and $F(G, C) = 3$: $L(6,3,1)$ and $L(6,3,6)$.

Graphs	$\{u,v\}$	Spanning $(3;u,v)$-trail-systems
$L(6,3,2)$	$\{v_1,v_2\}$	$G[\{v_1v_2\}]$, $G[\{v_1v_3,v_3v_2\}]$, $G[\{v_1v_6,v_6v_5,v_5v_4,v_4v_2\}]$
	$\{v_1,v_3\}$	$G[\{v_1v_3\}]$, $G[\{v_1v_2,v_2v_3\}]$, $G[\{v_1v_6,v_6v_5,v_5v_4,v_4v_3\}]$
	$\{v_1,v_4\}$	$G[\{v_1v_2,v_2v_4\}]$, $G[\{v_1v_3,v_3v_4\}]$, $G[\{v_1v_6,v_6v_5,v_5v_4\}]$
	$\{v_1,v_5\}$	$G[\{v_1v_5\}]$, $G[\{v_1v_6,v_6v_5\}]$, $G[\{v_1v_2,v_2v_3,v_3v_4,v_4v_5\}]$
	$\{v_1,v_6\}$	$G[\{v_1v_6\}]$, $G[\{v_1v_6\}]$, $G[\{v_1v_2,v_2v_3,v_3v_4,v_4v_5,v_5v_6\}]$
	$\{v_2,v_3\}$	$G[\{v_2v_3\}]$, $G[\{v_2v_4,v_4v_3\}]$, $G[\{v_2v_1,v_1v_5,v_5v_6,v_6v_1,v_1v_3\}]$
	$\{v_2,v_4\}$	$G[\{v_2v_4\}]$, $G[\{v_2v_3,v_3v_4\}]$, $G[\{v_2v_1,v_1v_6,v_6v_5,v_5v_4\}]$
	$\{v_2,v_5\}$	$G[\{v_2v_4,v_4v_5\}]$, $G[\{v_2v_1,v_1v_5\}]$, $G[\{v_2v_3,v_3v_1,v_1v_6,v_6v_5\}]$
	$\{v_2,v_6\}$	$G[\{v_2v_1,v_1v_6\}]$, $G[\{v_2v_3,v_3v_1,v_1v_6\}]$, $G[\{v_2v_4,v_4v_5,v_5v_6\}]$
	$\{v_3,v_4\}$	$G[\{v_3v_4\}]$, $G[\{v_3v_2,v_2v_4\}]$, $G[\{v_3v_1,v_1v_6,v_6v_5,v_5v_4\}]$
	$\{v_3,v_5\}$	$G[\{v_3v_1,v_1v_5\}]$, $G[\{v_3v_4,v_4v_5\}]$, $G[\{v_3v_2,v_2v_1,v_1v_6,v_6v_5\}]$
	$\{v_3,v_6\}$	$G[\{v_3v_1,v_1v_6\}]$, $G[\{v_3v_2,v_2v_1,v_1v_6\}]$, $G[\{v_3v_4,v_4v_5,v_5v_6\}]$
	$\{v_4,v_5\}$	$G[\{v_4v_5\}]$, $G[\{v_4v_3,v_3v_1,v_1v_5\}]$, $G[\{v_4v_2,v_2v_1,v_1v_6,v_6v_5\}]$
	$\{v_4,v_6\}$	$G[\{v_4v_5,v_5v_6\}]$, $G[\{v_4v_2,v_2v_1,v_1v_6\}]$, $G[\{v_4v_3,v_3v_1,v_1v_6\}]$
	$\{v_5,v_6\}$	$G[\{v_5v_6\}]$, $G[\{v_5v_1,v_1v_6\}]$, $G[\{v_5v_4,v_4v_3,v_3v_2,v_2v_1,v_1v_6\}]$
$L(6,3,3)$	$\{v_1,v_2\}$	$G[\{v_1v_2\}]$, $G[\{v_1v_2\}]$, $G[\{v_1v_6,v_6v_5,v_5v_4,v_4v_3,v_3v_2\}]$
	$\{v_1,v_3\}$	$G[\{v_1v_2,v_2v_3\}]$, $G[\{v_1v_4,v_4v_3\}]$, $G[\{v_1v_5,v_5v_6,v_6v_3\}]$
	$\{v_1,v_4\}$	$G[\{v_1v_4\}]$, $G[\{v_1v_2,v_2v_3,v_3v_4\}]$, $G[\{v_1v_6,v_6v_5,v_5v_4\}]$
	$\{v_1,v_5\}$	$G[\{v_1v_5\}]$, $G[\{v_1v_6,v_6v_5\}]$, $G[\{v_1v_2,v_2v_3,v_3v_4,v_4v_5\}]$
	$\{v_1,v_6\}$	$G[\{v_1v_6\}]$, $G[\{v_1v_4,v_4v_3,v_3v_6\}]$, $G[\{v_1v_2,v_2v_1,v_1v_5,v_5v_6\}]$
	$\{v_2,v_3\}$	$G[\{v_2v_3\}]$, $G[\{v_2v_1,v_1v_4,v_4v_3\}]$, $G[\{v_2v_1,v_1v_5,v_5v_6,v_6v_3\}]$
	$\{v_2,v_4\}$	$G[\{v_2v_3,v_3v_4\}]$, $G[\{v_2v_1,v_1v_4\}]$, $G[\{v_2v_1,v_1v_6,v_6v_5,v_5v_4\}]$
	$\{v_2,v_5\}$	$G[\{v_2v_1,v_1v_5\}]$, $G[\{v_2v_1,v_1v_6,v_6v_5\}]$, $G[\{v_2v_3,v_3v_4,v_4v_5\}]$
	$\{v_2,v_6\}$	$G[\{v_2v_1,v_1v_6\}]$, $G[\{v_2v_3,v_3v_6\}]$, $G[\{v_2v_1,v_1v_4,v_4v_5,v_5v_6\}]$
	$\{v_3,v_4\}$	$G[\{v_3v_4\}]$, $G[\{v_3v_2,v_2v_1,v_1v_4\}]$, $G[\{v_3v_6,v_6v_5,v_5v_4\}]$
	$\{v_3,v_5\}$	$G[\{v_3v_4,v_4v_5\}]$, $G[\{v_3v_6,v_6v_5\}]$, $G[\{v_3v_2,v_2v_1,v_1v_5\}]$
	$\{v_3,v_6\}$	$G[\{v_3v_6\}]$, $G[\{v_3v_4,v_4v_5,v_5v_6\}]$, $G[\{v_3v_2,v_2v_1,v_1v_6\}]$
	$\{v_4,v_5\}$	$G[\{v_4v_5\}]$, $G[\{v_4v_1,v_1v_5\}]$, $G[\{v_4v_3,v_3v_2,v_2v_1,v_1v_6,v_6v_5\}]$
	$\{v_4,v_6\}$	$G[\{v_4v_5,v_5v_6\}]$, $G[\{v_4v_3,v_3v_6\}]$, $G[\{v_4v_1,v_1v_2,v_2v_1,v_1v_6\}]$
	$\{v_5,v_6\}$	$G[\{v_5v_6\}]$, $G[\{v_5v_1,v_1v_6\}]$, $G[\{v_5v_4,v_4v_1,v_1v_2,v_2v_3,v_3v_6\}]$

Table 3. Spanning $(3;u,v)$-trail-systems when $n=6$ and $F(G,C)=3$: $L(6,3,2)$ and

$L(6,3,3)$.

The examining the spanning $(3;u,v)$-trail-systems for graphs $L(6,3,4)$ and $L(6,3,5)$ below, the edge v_1v_5 is not used in both cases.

Graphs	$\{u,v\}$	Spanning $(3;u,v)$-trail-systems
$L(6,3,4)$	$\{v_1,v_2\}$	$G[\{v_1v_2\}]$, $G[\{v_1v_2\}]$, $G[\{v_1v_6, v_6v_5, v_5v_4, v_4v_3, v_3v_2\}]$
	$\{v_1,v_3\}$	$G[\{v_1v_4, v_4v_3\}]$, $G[\{v_1v_2, v_2v_3\}]$, $G[\{v_1v_6, v_6v_5, v_5v_4, v_4v_3\}]$
	$\{v_1,v_4\}$	$G[\{v_1v_4\}]$, $G[\{v_1v_2, v_2v_3, v_3v_4\}]$, $G[\{v_1v_6, v_6v_5, v_5v_4\}]$
	$\{v_2,v_3\}$	$G[\{v_2v_3\}]$, $G[\{v_2v_1, v_1v_4, v_4v_3\}]$, $G[\{v_2v_1, v_1v_6, v_6v_5, v_5v_3\}]$
	$\{v_2,v_4\}$	$G[\{v_2v_3, v_3v_4\}]$, $G[\{v_2v_1, v_1v_4\}]$, $G[\{v_2v_1, v_1v_6, v_6v_5, v_5v_4\}]$
	$\{v_2,v_5\}$	$G[\{v_2v_1, v_1v_4, v_4v_5\}]$, $G[\{v_2v_1, v_1v_6, v_6v_5\}]$, $G[\{v_2v_3, v_3v_5\}]$
	$\{v_2,v_6\}$	$G[\{v_2v_1, v_1v_6\}]$, $G[\{v_2v_1, v_1v_6\}]$, $G[\{v_2v_3, v_3v_4, v_4v_5, v_5v_6\}]$
	$\{v_3,v_4\}$	$G[\{v_3v_4\}]$, $G[\{v_3v_2, v_2v_1, v_1v_4\}]$, $G[\{v_3v_5, v_5v_1, v_1v_6, v_6v_5, v_4v_4\}]$
	$\{v_3,v_5\}$	$G[\{v_3v_5\}]$, $G[\{v_3v_4, v_4v_5\}]$, $G[\{v_3v_2, v_2v_1, v_1v_6, v_6v_5\}]$
	$\{v_3,v_6\}$	$G[\{v_3v_2, v_2v_1, v_1v_6\}]$, $G[\{v_3v_4, v_4v_1, v_1v_6\}]$, $G[\{v_3v_5, v_5v_6\}]$
$L(6,3,5)$	$\{v_1,v_2\}$	$G[\{v_1v_2\}]$, $G[\{v_1v_3, v_3v_2\}]$, $G[\{v_1v_6, v_6v_5, v_5v_4, v_4v_3, v_3v_2\}]$
	$\{v_1,v_3\}$	$G[\{v_1v_4, v_4v_3\}]$, $G[\{v_1v_2, v_2v_3\}]$, $G[\{v_1v_6, v_6v_5, v_5v_3\}]$
	$\{v_1,v_4\}$	$G[\{v_1v_4\}]$, $G[\{v_1v_2, v_2v_3, v_3v_4\}]$, $G[\{v_1v_6, v_6v_5, v_5v_4\}]$
	$\{v_1,v_5\}$	$G[\{v_1v_4, v_4v_5\}]$, $G[\{v_1v_6, v_6v_5\}]$, $G[\{v_1v_2, v_2v_3, v_3v_5\}]$
	$\{v_1,v_6\}$	$G[\{v_1v_6\}]$, $G[\{v_1v_4, v_4v_5, v_5v_6\}]$, $G[\{v_1v_2, v_2v_3, v_3v_5, v_5v_6\}]$
	$\{v_2,v_3\}$	$G[\{v_2v_3\}]$, $G[\{v_2v_1, v_1v_4, v_4v_3\}]$, $G[\{v_2v_1, v_1v_6, v_6v_5, v_5v_3\}]$
	$\{v_2,v_4\}$	$G[\{v_2v_1, v_1v_4\}]$, $G[\{v_2v_3, v_3v_4\}]$, $G[\{v_2v_1, v_1v_6, v_6v_5, v_5v_4\}]$
	$\{v_2,v_5\}$	$G[\{v_2v_3, v_3v_5\}]$, $G[\{v_2v_1, v_1v_6, v_6v_5\}]$, $G[\{v_2v_1, v_1v_4, v_4v_5\}]$
	$\{v_2,v_6\}$	$G[\{v_2v_1, v_1v_6\}]$, $G[\{v_2v_3, v_3v_5, v_5v_6\}]$, $G[\{v_2v_1, v_1v_4, v_4v_5, v_5v_6\}]$
	$\{v_3,v_4\}$	$G[\{v_3v_4\}]$, $G[\{v_3v_5, v_5v_1, v_1v_4\}]$, $G[\{v_3v_2, v_2v_1, v_1v_6, v_6v_5, v_5v_4\}]$
	$\{v_3,v_5\}$	$G[\{v_3v_5\}]$, $G[\{v_3v_4, v_4v_5\}]$, $G[\{v_3v_2, v_2v_1, v_1v_6, v_6v_5\}]$
	$\{v_3,v_6\}$	$G[\{v_3v_5, v_5v_6\}]$, $G[\{v_3v_2, v_2v_1, v_1v_6\}]$, $G[\{v_3v_4, v_4v_5, v_5v_6\}]$
	$\{v_4,v_5\}$	$G[\{v_4v_5\}]$, $G[\{v_4v_3, v_3v_5\}]$, $G[\{v_4v_1, v_1v_2, v_2v_1, v_1v_6, v_6v_5\}]$
	$\{v_4,v_6\}$	$G[\{v_4v_5, v_5v_6\}]$, $G[\{v_4v_1, v_1v_2, v_2v_1, v_1v_6\}]$, $G[\{v_4v_3, v_3v_5, v_5v_6\}]$
	$\{v_5,v_6\}$	$G[\{v_5v_6\}]$, $G[\{v_5v_6\}]$, $G[\{v_5v_4, v_4v_3, v_3v_2, v_2v_1, v_1v_6\}]$

Table 4. Spanning $(3;u,v)$-trail-systems when $n=6$ and $F(G,C)=3$: $L(6,3,4)$ and

$L(6,3,5)$.

Graphs	$\{u,v\}$	Spanning $(3;u,v)$-trail-systems
$L(6,4,1)$	$\{v_1,v_2\}$	$G[\{v_1v_2\}]$, $G[\{v_1v_3,v_3v_2\}]$, $G[\{v_1v_4,v_4v_5,v_5v_6,v_6v_2\}]$
	$\{v_1,v_3\}$	$G[\{v_1v_3\}]$, $G[\{v_1v_2,v_2v_3\}]$, $G[\{v_1v_6,v_6v_5,v_5v_4,v_4v_3\}]$
	$\{v_1,v_4\}$	$G[\{v_1v_4\}]$, $G[\{v_1v_2,v_2v_3,v_3v_4\}]$, $G[\{v_1v_6,v_6v_5,v_5v_4\}]$
	$\{v_2,v_3\}$	$G[\{v_2v_3\}]$, $G[\{v_2v_1,v_1vv_3\}]$, $G[\{v_2v_6,v_6v_5,v_5v_4,v_4v_3\}]$
	$\{v_2,v_4\}$	$G[\{v_2v_3,v_3v_4\}]$, $G[\{v_2v_1,v_1v_4\}]$, $G[\{v_2v_6,v_6v_5,v_5v_4\}]$
	$\{v_2,v_5\}$	$G[\{v_2v_6,v_6v_5\}]$, $G[\{v_2v_1,v_1v_5\}]$, $G[\{v_2v_3,v_3v_4,v_4v_5\}]$
	$\{v_2,v_6\}$	$G[\{v_2v_6\}]$, $G[\{v_2v_1,v_1v_6\}]$, $G[\{v_2v_3,v_3v_4,v_4v_5,v_5v_6\}]$
	$\{v_3,v_4\}$	$G[\{v_3v_4\}]$, $G[\{v_3v_1,v_1v_5,v_5v_4\}]$, $G[\{v_3v_6,v_6v_1,v_1v_4\}]$
$L(6,4,2)$	$\{v_1,v_2\}$	$G[\{v_1v_2\}]$, $G[\{v_1v_3,v_3v_2\}]$, $G[\{v_1v_6,v_6v_5,v_5v_4,v_4v_2\}]$
	$\{v_1,v_3\}$	$G[\{v_1v_3\}]$, $G[\{v_1v_2,v_2v_3\}]$, $G[\{v_1v_6,v_6v_5,v_5v_4,v_4v_3\}]$
	$\{v_1,v_4\}$	$G[\{v_1v_2,v_2v_4\}]$, $G[\{v_1v_3,v_3v_4\}]$, $G[\{v_1v_6,v_6v_5,v_5v_4\}]$
	$\{v_2,v_3\}$	$G[\{v_2v_3\}]$, $G[\{v_2v_1,v_1v_3\}]$, $G[\{v_2v_4,v_4v_6,v_6v_5,v_5v_4,v_4v_3\}]$
	$\{v_2,v_4\}$	$G[\{v_2v_4\}]$, $G[\{v_2v_3,v_3v_4\}]$, $G[\{v_2v_1,v_1v_6,v_6v_5,v_5v_4\}]$
	$\{v_2,v_5\}$	$G[\{v_2v_4,v_4v_5\}]$, $G[\{v_2v_1,v_1v_6,v_6v_5\}]$, $G[\{v_2v_3,v_3v_1,v_1v_5\}]$
	$\{v_2,v_6\}$	$G[\{v_2v_4,v_4v_6\}]$, $G[\{v_2v_1,v_1v_6\}]$, $G[\{v_2v_3,v_3v_4,v_4v_5,v_5v_6\}]$
	$\{v_3,v_4\}$	$G[\{v_3v_4\}]$, $G[\{v_3v_2,v_2v_4\}]$, $G[\{v_3v_1,v_1v_6,v_6v_5,v_5v_4\}]$

Table 5. Spanning $(3;u,v)$-trail-systems when $n=6$ and $F(G,C)\geq 4$.

The following Table 6 verifies that $\mu'(K_{3,3}+v_1v_3)=3$ (with the notation in Figure 1). Missing cases can be obtained by symmetry.

Graphs	$\{u, v\}$	Spanning $(3; u, v)$-trail-systems
$K_{3,3} + v_1v_3$	$\{v_2, v_3\}$	$G[\{v_2v_3\}]$, $G[\{v_2v_1, v_1v_6, v_6v_3\}]$, $G[\{v_2v_5, v_5v_4, v_4v_1, v_1v_3\}]$
	$\{v_2, v_4\}$	$G[\{v_2v_1, v_1v_3, v_3v_4\}]$, $G[\{v_2v_5, v_5v_4\}]$, $G[\{v_2v_3, v_3v_6, v_6v_1, v_1v_4\}]$
	$\{v_2, v_5\}$	$G[\{v_2v_5\}]$, $G[\{v_2v_3, v_3v_4, v_4v_5\}]$, $G[\{v_2v_1, v_1v_6, v_6v_5\}]$
	$\{v_3, v_4\}$	$G[\{v_3v_4\}]$, $G[\{v_3v_2, v_2v_1, v_1v_4\}]$, $G[\{v_3v_6, v_6v_5, v_5v_4\}]$
	$\{v_3, v_5\}$	$G[\{v_3v_4, v_4v_5\}]$, $G[\{v_3v_6, v_6v_5\}]$, $G[\{v_3v_1, v_1v_2, v_2v_5\}]$
	$\{v_3, v_6\}$	$G[\{v_3v_6\}]$, $G[\{v_3v_4, v_4v_5, v_5v_6\}]$, $G[\{v_3v_2, v_2v_1, v_1v_6\}]$
	$\{v_1, v_3\}$	$G[\{v_1v_3\}]$, $G[\{v_1v_2, v_2v_3\}]$, $G[\{v_1v_6, v_6v_5, v_5v_4, v_4v_3\}]$
	$\{v_4, v_5\}$	$G[\{v_4v_5\}]$, $G[\{v_4v_3, v_3v_6, v_6v_5\}]$, $G[\{v_4v_1, v_1v_2, v_2v_5\}]$

Table 6. Spanning $(3; u, v)$-trail-systems of $K_{3,3} + v_1v_3$

Proof of claim (5.9) If G has an edge e such that $\kappa'(G - e) \geq 3$, then by (5.8) either $\mu'(G - e) \geq 3$, whence $\mu'(G) \geq \mu'(G - e) \geq 3$; or $G - e = K_{3,3}$, whence it is routine to verify that $\mu'(K_{3,3} + e) \geq 3$ (See Claim 4 within the proof of Theorem 5.4.3).

Now suppose that G is not $\mathcal{C}_2$-reduced, and so G has a nontrivial subgraph $H \in \mathcal{C}_2$. Then by (5.8), $\mu'(G/H) \geq 3$, and so by Corollary 5.2.9, $\mu'(G) \geq 3$ also.

Hence we may assume that (5.9) must hold. □

Proof of claim (5.10) Let C denote a spanning eulerian subgrah of G such that

$$\Delta(C) + |E(C)| \text{ is minimized.} \tag{5.19}$$

By (5.9) and by Corollary 5.3.1, no edge in G is parallel to 2 other edges. Let $m(C)$ denote the the number of pairs of multiple edges in C. If $m(C) \geq 2$, then since $n \leq 6$ and since C is eulerian, we must have $n = 6$ and $m(C) = 3$. It follows by $\kappa(G) \geq 2$ in (5.9) that G must have an edge e not in C joining two two vertices not adjacent in C, and so $C \cup e$ has a cycle C_e containing e, But then $C \Delta C_e$ is a spanning eulerian subgraph of G violating he choice of C stated in (5.19). Hence we must have $m(C) \leq 1$.

When $m(C) \leq 1$, since $5 \leq n \leq 6$, C must be an edge-disjoint union of two cycles C_k and C_l, where $(k, l) \in \{(2, 3), (2, 4), (3, 3), (3, 4)\}$, such that $V(C_k) \cap V(C_l) = \{v\}$ for some

v. By (5.9), $\kappa(G) \geq 2$, and so G must have an edge $e \notin E(C)$ such that $G[C \cup e]$ contains a 3-cycle C_e that contains v. Therefore $C \Delta C_e$ violates (5.19), and so the contradiction establishes (5.10). $\square$

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