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Integer flows and Modulo Orientations

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ABSTRACT

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Tutte's 3-flow conjecture(1970's) states that every 4-edge-connected graph admits a nowhere-zero 3-flow. A graph G admits a nowhere-zero 3-flow if and only if G has an orientation such that the out-degree equals the in-degree modulo 3 for every vertex. In the 1980ies Jaeger suggested some related conjectures. The generalized conjecture to modulo k-orientations, called circular flow conjecture, says that, for every odd natural number k, every (2k-2)-edge-connected graph has an orientation such that the out-degree equals the in-degree modulo k for every vertex. And the weaker conjecture he made, known as the weak 3-flow conjecture where he suggests that the constant 4 is replaced by any larger constant.

The weak version of the circular flow conjecture and the weak 3-flow conjecture are verified by Thomassen (JCTB 2012) recently. He proved that, for every odd natural number k, every $(2k^2 + k)$-edge-connected graph has an orientation such that the out-degree equals the in-degree modulo k for every vertex and for $k = 3$ the edge-connectivity 8 suffices. Those proofs are refined in this paper to give the same conclusions for $9k$-edge-connected graphs and for 6-edge-connected graphs when $k = 3$ respectively.

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To

Dai-Qun
Zhi-Yue
and
Zhi-Yi

Contents

Chapter 1

Introduction

1.1 Notation and Terminology

We use [6] for terminology and notations not defined here. Graphs in this dissertation are finite and may have multiple edges but no loops. Let G be a graph. We use $V(G)$ and $E(G)$ to denote the set of vertices and the set of edges of G, respectively. Two vertices $u,\ v$ are adjacent if $uv \in E(G)$.

For a graph G and for $v \in V(G)$, **the neighborhood** $N_G(v)$ denotes the set of all vertices adjacent to v in G. The cardinality of $N_G(v)$ is called the **degree** of v in G, and is denoted by $d_G(v)$ or $d(v)$. For a vertex subset A of G.

A **edge cut** of G is a subset F of $E(G)$ such that $G-F$ is disconnected. A k**-edge cut** is a edge cut of k elements. If G has at least one pair of distinct nonadjacent vertices, the **edge-connectivity** $\kappa(G)$ of G is the minimum k for which G has a k-edge cut. G is said to be k**-edge-connected** if $\kappa(G) \geq k$. For a vertex subset or an edge subset X of G, $G[X]$ denotes the subgraph of G **induced** by X. If $A \subseteq V(G)$, we let $G - A = G[V(G) - A]$. When $A = \{v\}$, we use $G - v$ for $G - \{v\}$.

Let $X \subseteq E(G)$. The **contraction** G/X is the graph obtained from G by identifying the two ends of each edge in X and then deleting the resulting loops. If H is a subgraph of G, we write G/H for $G/E(H)$. Note that even if G is a simple graph, contracting some edges of G may result in a graph with multiple edges.

1.2 Integer Flows

Integer flow was originally introduced by Tutte [57, 58] as a generalization of map coloring problems. The following are some definition about basic integer flow concepts.

Definition 1.2.1 *Let G be a graph, D be an orientation of G and $f : E(G) \longrightarrow \mathbb{Z}$ be a map. For a vertex $v \in V(G)$, let $E^+(v)$(or $E^-(v)$) be the set of all arcs of $D(G)$ with their tails (or heads, respectively) at the vertex v.*

Definition 1.2.2 *An integer flow of a graph G is an ordered pair (D, f) such that $\sum_{e\in E^+(v)} f(e) = \sum_{e\in E^-(v)} f(e)$ every vertex $v \in V(G)$.*

Definition 1.2.3 *A k-flow is an integer flow (D, f) such that $|f(e)| < k$ for each $e \in E(G)$. A k-flow is nowhere-zero if $f(e) \neq 0$ for each $e \in E(G)$.*

The following are the most famous conjectures in the theory of integer flows proposed by Tutte.

Conjecture 1.2.4 (3-flow conjecture, Tutte) *Every 4-edge-connected graph admits a nowhere-zero 3-flow.*

Conjecture 1.2.5 (4-flow conjecture, [59]) *Every bridgeless graph containing no subdivision of the Petersen graph admits a nowhere-zero 4-flow.*

Conjecture 1.2.6 (5-flow conjecture, [58]) *Every bridgeless graph admits a nowhere-zero 5-flow.*

A weak version of Conjecture 1.2.4 was proposed by Jaeger.

Conjecture 1.2.7 (Jaeger [26]) *There is an integer h such that every h-edge-connected graph admits a nowhere-zero 3-flow.*

Conjecture 1.2.7 is recently verified by Thomassen.

Theorem 1.2.8 (Thomassen [56]) *Every 8-edge-connected graph admits a nowhere-zero 3-flow.*

This theorem is further improved in the dissertation as follows.

Theorem 1.2.9 *Every 6-edge-connected graph admits a nowhere-zero 3-flow.*

Note that it was proved by Kochol[32] that it suffices to prove the 3-flow conjecture for 5-edge-connected graphs. So our result is just one step to the 3-flow conjecture (Conjecture 1.2.4).

1.3 Modulo Orientations

Definition 1.3.1 *Let G be a graph and k be an odd integer, $k \geq 3$. An orientation D of G is a* modulo k-orientation *if*

$$d^+(x) - d^-(x) \equiv 0 \pmod{k}$$

for every vertex $x \in V(G)$.

A graph admits a nowhere-zero 3-flow if and only if it has a modulo 3-orientation(see [27], [28] or [63]). Jaeger(1984) generalized the 3-flow conjecture to the following one which he called the circular flow conjecture.

Conjecture 1.3.2 (Jaeger [27])*For every odd natural number k, every $(2k-2)$-edge-connected graph has a modulo k-orientation.*

He also suggested the weaker version of circular flow conjecture that the connectivity $(2k-2)$ is replaced by any large integer function of k. This weaker conjecture has also been proved by Thomassen in the same paper[56].

Theorem 1.3.3 (Thomassen [56]) *Every* $(2k^2 + k)$*-edge-connected graph has a modulo* k*-orientation, where* k *is an odd integer* ≥ 3.

The quadratic bound is reduced to linear one in this dissertation.

Theorem 1.3.4 *Every* $9k$*-edge-connected graph has a modulo* k*-orientation, where* k *is an odd integer* ≥ 3.

Chapter 2

Nowhere-zero 3-flows for 6-edge-connected graphs

2.1 Introduction

2.1.1 3-Flow Conjecture and Weak 3-flow conjecture

One major open problem in the integer flow theory is the following conjecture, which is the dual version of Grötzsch's 3-coloring theorem for planar graphs (see [18], [19], [1], [55]).

Conjecture 2.1.1 (Tutte) *Every graph with no* 1*-edge-cut and no* 3*-edge-cut admits a nowhere-zero* 3*-flow.*

This open problem first appeared in the literatures in the 1970'ies, such as, [51], and [6] (Open Problem 48). It has been recognized as one of major open problems in graph theory, and has appeared in many standard textbooks and reference books, such as, [8] (Open Problem 97), [61] (Conjecture 7.3.28), [11] (p. 157), [30] (Section 13.3), [63] (Conjecture 1.1.8).

The 3-flow conjecture (Conjecture 2.1.1) by Tutte was originally proposed for graphs with no 1-edge-cut and no 3-edge-cut. It was pointed out in [26], [28], [47] that a 2-edge-cut does not exist in any smallest counterexample to some well-known flow conjectures (including this conjecture). Kochol [32] further proved that it suffices to prove this conjecture for 5-edge-connected graphs.

A weak version of Conjecture 2.1.1 was proposed by Jaeger.

Conjecture 2.1.2 (Jaeger [26]) *There is an integer h such that every h-edge-connected graph admits a nowhere-zero 3-flow.*

The followings are some early partial results on Conjecture 2.1.2.

Theorem 2.1.3 (Lai and Zhang [35]) *Every $4\lceil \log_2 n_o \rceil$-edge-connected graph with at most n_o odd-degree vertices admits a nowhere-zero 3-flow.*

Theorem 2.1.4 (Alon, Linial and Meshulam [2], see also [3].) *Every $2\lceil \log_2 n \rceil$-edge-connected graph with n vertices admits a nowhere-zero 3-flow.*

Conjecture 2.1.2 is recently verified by Thomassen.

Theorem 2.1.5 (Thomassen [56]) *Every 8-edge-connected graph admits a nowhere-zero 3-flow.*

This theorem is further improved in this paper as follows.

Theorem 2.1.6 *Every 6-edge-connected graph admits a nowhere-zero 3-flow.*

There is a long list of publications related to Conjecture 2.1.1 and the stronger Conjecture 2.1.8 below, such as, [1], [2], [3], [4], [10], [12], [13], [14], [18], [19], [22], [23], [24], [25], [26], [28], [29], [31], [32], [33], [34], [35], [36], [37], [38], [40], [41], [43], [44], [47], [49], [51], [52], [53], [54], [55], [62], [65], [66], etc.. Note that many results of those papers are for graphs with some special properties (instead of edge-connectivity), such as, local density, local structure, random structure, symmetrical structure, embedding property, degree, odd-cuts distribution. Many of them remain the best known results for the graph families they concern, and are not corollaries of Theorem 2.1.6

2.1.2 Group connectivity

Group connectivity was introduced in [29] as a generalization of integer flow, and an inductive approach for flow problems.

Definition 2.1.7 *Let* Γ *be an abelian group with "0″ as the additive identity (zero). Let* G *be a graph and* $\beta : V(G) \mapsto \Gamma$. *The mapping* β *is* zero-sum *if* $\sum_{v\in V(G)} \beta(v) = 0$. *The graph* G *is* Γ-connected *if, for every zero-sum mapping* β*, there is an orientation* D_β *and a weight* f_β *of* $E(G)$ *such that*

$$\sum_{e\in E^+_{D_\beta}(v)} f_\beta(e) \; - \; \sum_{e\in E^-_{D_\beta}(v)} f_\beta(e) \; = \; \beta(v)$$

for every vertex $v \in V(G)$. *And a zero-sum mapping* β *is called a* boundary.

Conjecture 2.1.8 (Jaeger, Linial, Payan and Tarsi [29]) *Every* 5*-edge-connected graph is* Z_3*-connected.*

Note that the 5-edge-connectivity is sharp for Conjecture 2.1.8 since some 4-edge-connected counterexamples were discovered in [29] and [38].

Theorem 2.1.9 (Thomassen [56]) *Every* 8*-edge-connected graph is* Z_3*-connected.*

This theorem is further improved in this dissertation as follows.

Theorem 2.1.10 *Every* 6*-edge-connected graph is* Z_3*-connected.*

2.1.3 Generalized Tutte orientation

Modulo 3-orientation was first introduced by Tutte in the study of orientable cycle double covering [57].

Definition 2.1.11 *An orientation D of a graph G is called a* modulo 3-orientation *or* Tutte orientation *if*

$$d^+_D(v) \equiv d^-_D(v) \pmod 3$$

for every vertex $v \in V(G)$.

It was observed in [57] that a graph G admits a nowhere-zero 3-flow if and only if G has a Tutte orientation. This concept is further generalized in [4] as follows.

Definition 2.1.12 *Let $\beta : V(G) \mapsto Z_3$ such that $\sum_{v\in V(G)} \beta(v) = 0$. An orientation D_β of G is called* a generalized Tutte orientation *with respect to β if*

$$d^+_{D_\beta}(v) - d^-_{D_\beta}(v) \equiv \beta(v) \pmod 3$$

for every vertex $v \in V(G)$.

Generalized Tutte orientations are a special case of group connectivity. Indeed, it is not hard to see that G is Z_3-connected if and only if G has a generalized Tutte orientation for every zero-sum mapping β.

2.1.4 Circular flow

Definition 2.1.13 *Let k, d be two integers such that $0 < d \leq \frac{k}{2}$. An integer flow (D, f) of a graph G is called* a circular $\frac{k}{d}$-flow *if $f : E(G) \mapsto \{\pm d, \pm(d+1), \cdots, \pm(k-d)\} \cup \{0\}$.*

The concept of circular flow, introduced by Goddyn, Tarsi and Zhang in [17], is a generalization of integer flows, and a dual version of circular colorings ([60], [7]). We refer to [67], [68] for surveys.

It is proved in [17] that if a graph G admits a nowhere-zero circular p-flow, then G admits a nowhere-zero circular q-flow for every $q \geq p$.

Definition 2.1.14 *Let G be a bridgeless graph. The* circular flow index *of G, denoted by $\phi(G)$, is the smallest rational number q such that G admits a nowhere-zero circular q-flow.*

It is proved in [17] that the number q in Definition 2.1.14 indeed exists.

The following theorem was proved in [16] as an approach to Conjecture 2.1.1 (and Conjecture 2.1.2).

Theorem 2.1.15 (Galluccio and Goddyn [16], also see [39].) *For every* 6*-edge-connected graph* G*, the circular flow index* $\phi(G) < 4$*.*

Theorem 2.1.6 in the present paper improves Theorem 2.1.15. Specifically, $\phi(G)$ is now a rational number ≤ 3.

2.2 The set function $\tau(A)$

The idea which makes the proof in [56] work is a set function called $t(A)$ with values $0, 1, 2, 3$. In the present paper we use the same function except that we allow it to have negative values. We therefore call it $\tau(A)$. This function has values $-3, -2, -1, 0, 1, 2, 3$, and $t(A) = |\tau(A)|$.

Suppose $\beta : V(G) \mapsto Z_3 = \{0, 1, 2\}$. As in Definition 2.1.7, we call β a boundary of the graph G.

Let x be a vertex of G and let μ be an integer such that

$$|\mu| \leq d(x),$$

$$\mu \equiv \beta(x) \pmod 3, \text{ and, } \mu \equiv d(x) \pmod 2. \tag{2.1}$$

Then $d(x) - |\mu|$ is even and there is a natural way to direct the edges incident with x, which we call $E(x)$, such that $d^+(x) - d^-(x) \equiv \beta(x) \pmod 3$ as follows: First we choose $\frac{d(x)-|\mu|}{2}$ pairs of edges and direct each pair in opposite directions; Then we direct all the remaining $|\mu|$ edges away from x if $\mu \geq 0$ or towards x if $\mu \leq 0$. Such an orientation of $E(x)$ satisfies that $d^+(x) - d^-(x) \equiv \beta(x) \pmod 3$ since $d^+(x) = \frac{d(x)+\mu}{2}$ and $d^-(x) = \frac{d(x)-\mu}{2}$.

We may have multiple choices for μ. For example, if $d(x) = 5$ and $\beta(x) = 1$, then we can have $\mu = 1$ or $\mu = -5$. Following [56], denote $\tau(x)$ be the μ, satisfying Equation (2.1), such that $|\mu|$ is minimum. Notice that if $\beta(x) = 0$ and $d(x)$ is odd, then $|\tau(x)| = 3$ and we can direct $\frac{d(x)-3}{2}$ pairs of edges in opposite directions and the remaining 3 edges either all away from x or all toward x. So $\tau(x) = -3$ or $\tau(x) = 3$ under this conditions. Otherwise we have

$$\tau(x) = \beta(x) \text{ if } d(x) \equiv \beta(x) \pmod 2$$

and

$$\tau(x) = \beta(x) - 3 \text{ if } d(x) \not\equiv \beta(x) \pmod 2 \text{ and } \beta(x) \neq 0.$$

Note that the mapping τ may not be a single valued function since, for the case of $d(x) \equiv 1 \pmod 2$ and $\beta(x) = 0$, $\tau(x)$ has two values: 3 and -3.

The mapping τ is further extended to any nonempty vertex subset A with respect to $\beta(A) \equiv \sum_{x\in A} \beta(x) \pmod 3$ and $d(A) = |[A, V(G)\backslash A]|$, where $[A, V(G)\backslash A]$ is the set of edges between A and $V(G)\backslash A$. The mapping $\tau : \mathcal{P}(V(G)) \mapsto \{0, 1, -1, 2, -2, 3, -3\}$ is defined as follows, for each non-empty $A \subset V(G)$,

$$\tau(A) \equiv \begin{cases} \beta(A) & \pmod 3 \\ d(A) & \pmod 2 \end{cases} \tag{2.2}$$

where $\mathcal{P}(V(G))$ is the power set of $V(G)$ (the collection of all subsets of $V(G)$).

For a graph G' with the boundary β', we use the notations $d'(A)$, $\beta'(A)$ and $\tau'(A)$ for the corresponding values of the vertex subset A of $V(G')$.

Lifting Operation. Let x be a vertex of G. If xy and xz are edges with y and z distinct, then the deletion of the edges xy and xz and the addition of the edge yz is called the *lifting* of xy and xz (see Figure 2.1). Also, if one of xy and xz, say xz, is a directed edge, then we direct yz toward z if xz is toward z or away from z otherwise.

Observation 1. *Let G' be the graph constructed from G by lifting two edge xy and xz. Then, for any boundary β and any β-orientation of G', there is a corresponding β-orientation of G such that xy and xz are directed in opposite directions from x.*

Pre-direction Operation. Let β be a boundary of a graph G. If xy is an undirected

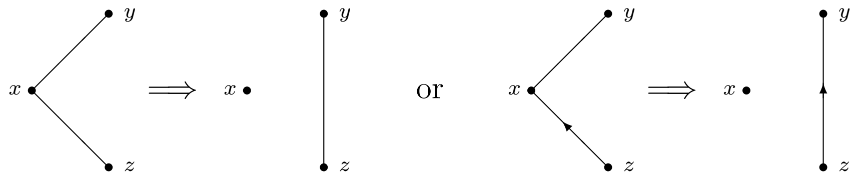

Figure 2.1: *Lifting of xy and xz*

edge or a directed edge from x to y of G, then the removing of xy, decreasing $\beta(x)$ by 1 and increasing $\beta(y)$ by 1 is called the *pre-directing* of xy.

Observation 2. *Let G' be the graph constructed from G by pre-directing edge xy and let β' be the corresponding boundary modified from β. Then, for any β'-orientation of G', the corresponding orientation of G constructed from the one of G' by adding the directed edge xy from x to y is a β-orientation.*

Proposition 2.2.1 *For any vertex subset A of the graph G,*

(1) If $d(A) \geq 6$, then $d(A) \geq 4 + |\tau(A)|$.

(2) If $d(A) > 4 + |\tau(A)|$, then $d(A) \geq 6 + |\tau(A)|$.

(3) If $d(A) < 6 + |\tau(A)|$, then $d(A) \leq 4 + |\tau(A)|$.

Proposition 2.2.1 follows from the fact that $d(A)$ and $|\tau(A)|$ have the same parities (by Equation (2.2)).

Proposition 2.2.2 *Let G be a graph and β be a boundary G. Suppose G' is the resulting graph constructed from G by lifting or pre-directing edges, and β' is the boundary of G' modified from β. Let A be a vertex subset such that $d(A) \geq 6 + |\tau(A)|$. Then*

$$d'(A) \geq 4 + |\tau'(A)|$$

if one of the following is satisfied:

(a) $\beta'(A) = \beta(A)$ and $d'(A) = d(A)$ or $d(A) - 2$,

(b) $\beta'(A) = \beta(A) \pm 1$ and $d'(A) = d(A) - 1$.

Proof. If (a) is satisfied, then, by Equation (2.2), we have that

$$\tau'(A) \equiv \begin{cases} \beta'(A) \equiv \beta(A) \equiv \tau(A) \pmod 3, \\ d'(A) \equiv d(A) \equiv \tau(A) \pmod 2. \end{cases}$$

Hence, $|\tau'(A)| \equiv |\tau(A)| \pmod 6$, and, furthermore, $|\tau'(A)| = |\tau(A)|$ since $|\tau'(A)| \leq 3$ and $|\tau(A)| \leq 3$. So

$$d'(A) \geq d(A) - 2 \geq 6 + |\tau(A)| - 2 = 4 + |\tau'(A)|.$$

If (b) is satisfied, then, by Equation (2.2), we have that

$$|\tau'(A) - \tau(A)| \equiv \begin{cases} |\beta'(A) - \beta(A)| \equiv 1 \pmod 3, \\ |d'(A) - d(A)| \equiv 1 \pmod 2. \end{cases}$$

Hence, $|\tau'(A) - \tau(A)| \equiv 1 \pmod 6$ and, furthermore, $|\tau'(A) - \tau(A)| = 1$ since $|\tau'(A) - \tau(A)| \leq 6$. So $|\tau(A)| \geq |\tau'(A)| - 1$ and, therefore,

$$d'(A) = d(A) - 1 \geq 5 + |\tau(A)| \geq 4 + |\tau'(A)|.$$

■

2.3 Main results

Theorem 2.3.1 below is similar to Theorem 1 in [56]. In the conclusion of Theorem 2.3.1 there is an additional condition on the minimum indegrees and outdegrees which can also easily be added to Theorem 1 in [56] (with 4 replaced by 6). The main modification, however, is the upper bound on the degree of the vertex z_0 incident with the edges with prescribed orientation. In [56], we allow that vertex z_0 to have degree at most 11. In Theorem 2.3.1, the condition on the degree $d(z_0)$ depends on the τ-value.

Theorem 2.3.1 *Let G be a graph, β be a boundary of G, $z_0 \in V(G)$ and D_{z_0} be a pre-orientation of $E(z_0)$. Assume that*

(i) $|V(G)| \geq 3$;

(ii) under the orientation D_{z_0}, the edges incident with z_0 are directed such that

$$d^+(z_0) - d^-(z_0) \equiv \beta(z_0) \pmod 3;$$

(iii) $d(z_0) \leq 4 + |\tau(z_0)|$ *and* $d(A) \geq 4 + |\tau(A)|$ *for each nonempty vertex subset A not containing z_0 such that* $|V(G)\backslash A| > 1$.

Then the pre-orientation D_{z_0} of $E(z_0)$ can be extended to an orientation D of the entire graph such that, for each vertex x distinct from z_0, we have the following conclusions:

(a) $d^+(x) - d^-(x) \equiv \beta(x) \pmod 3$,

(b) $\min\{d^+(x), d^-(x)\} \geq h(x)$ *where* $h(x) = \frac{d(x)-4-|\tau(x)|}{2}$.

Theorem 2.1.10 is a corollary of Theorem 2.3.1. Applying Theorem 2.3.1, Theorem 2.1.10 is proved as follows.

Proof of Theorem 2.1.10. Suppose G is a 6-edge-connected graph.

Let G' be the graph obtained from G by adding an isolated vertex z_0.

For an arbitrary boundary $\beta : V(G) \to Z_3$, define $\beta' : V(G') \to Z_3$ such that $\beta'(z_0) = 0$ and $\beta'(x) = \beta(x)$ if $x \neq z_0$.

We now verify the conditions of Theorem 2.3.1 for G' and β'.

Condition (ii) is obviously satisfied.

As G is 6-edge-connected, $|V(G)| \geq 2$. So $|V(G')| \geq 3$ and Condition (i) holds. Furthermore, $d'(z_0) = 0 \leq 4 + |\tau'(z_0)|$ and, for any nonempty vertex subset A not containing z_0 such that $|V(G')\backslash A| > 1$, we have $d'(A) = |[A, V(G')\backslash A]| = |[A, V(G)\backslash A]| \geq 6$ by the connectivity of G. So by Proposition 2.2.1, $d'(A) \geq 4 + |\tau'(A)|$, proving Condition (iii).

Then, following Theorem 2.3.1, there exists an orientation of G', which is also an orientation of G since $E(G) = E(G')$, such that each vertex $x \in V(G)$ satisfies $d^+(x) - d^-(x) \equiv \beta(x) \pmod 3$. So, the graph G is Z_3-connected. ■

By the definition of group connectivity/generalized Tutte orientation, Theorem 2.1.6 is an immediate corollary of Theorem 2.1.10.

2.4 Proof of Theorem 2.3.1

The proof is by induction. We assume (reductio ad absurdum) that G is a counterexample such that $|E(G)|$ is minimum.

The proof is divided into two parts. The first part follows closely the proof of Theorem 1 in [56]. As we have modified the condition on $d(z_0)$, and added Conclusion (b) and also work with the function $\beta(x)$ instead of the prescribed outdegree $p(x)$ in [56], we include complete proofs in most cases.

The second part contains further reductions, and in this part Conclusion (b) is used in the induction hypothesis.

Part I. Basic reductions following [56]

Claim 1 *If A is a vertex subset not containing z_0 such that $|A| > 1$ and $|V(G)\backslash A| > 1$, then*

$$d(A) \geq 6 + |\tau(A)|.$$

If $d(A) < 6 + |\tau(A)|$, then $d(A) \leq 4 + |\tau(A)|$ by Proposition 2.2.1. And by Condition (iii) we have $d(A) \geq 4 + |\tau(A)|$. So $d(A) = 4 + |\tau(A)|$.

We first contract A and get an orientation of the edges out of $G[A]$ by induction. Then we contract $V(G)\backslash A$ into a single vertex as a new z_0, and again we use induction to extend the orientation to the edges inside of $G[A]$ (see Figure 2.2). Notice that the Conclusion (b) remains true during the inductions. □

Claim 2 *$d(x) = 4 + |\tau(x)|$ and $h(x) = 0$ if $x \neq z_0$.*

If $d(x) \neq 4+|\tau(x)|$, then by Condition (iii) $d(x) > 4+|\tau(x)|$ and by Proposition 2.2.1 we have that $d(x) \geq 6 + |\tau(x)|$ and $h(x) \geq 1$.

Assume that x has been chosen such that $h(x)$ is maximum.

<u>Case 1</u>: x has only one neighbor.

Let y be the neighbor of x and let $A = \{x, y\}$.

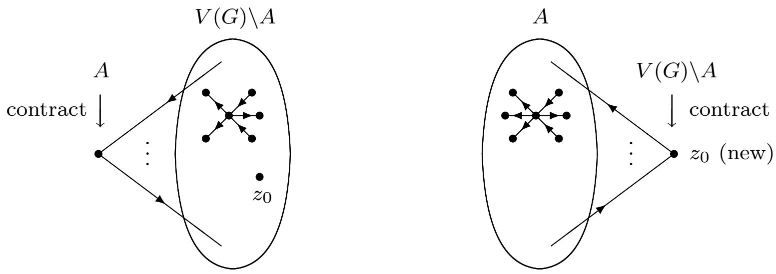

Figure 2.2: *Direct G by inductions*

If $y = z_0$, then $d(V(G)\backslash A) = d(A) = d(z_0) - d(x) \leq (4 + |\tau(z_0)|) - (6 + |\tau(x)|) \leq 1$ which contradicts Condition (iii) for the vertex subset $V(G)\backslash A$.

So $y \neq z_0$. By the maximality of $h(x)$ we have that $h(y) \leq h(x)$ and then $d(A) = d(y) - d(x) = 2(h(y) - h(x)) + |\tau(y)| - |\tau(x)| \leq |\tau(y)| \leq 3$ which contradicts Claim 1 if $|V(G)\backslash A| > 1$. So we must have that $\{x, y\} = V(G) - z_0$ and therefore all undirected edges of G are between x and y and all directed edges are between z_0 and y. We just direct $\frac{d(x)+\tau(x)}{2}$ edges away from x and the other $\frac{d(x)-\tau(x)}{2}$ edges towards x. So Conclusion (a) holds for x. Also it holds for y since $d^+(y) = d^-(x) + d^-(z_0)$, $d^-(y) = d^+(x) + d^+(z_0)$ and $d^+(y) - d^-(y) \equiv -\beta(x) - \beta(z_0) \equiv \beta(y) \pmod 3$.

For Conclusion (b), we have $\min\{d^+(x), d^-(x)\} \geq \frac{d(x)-|\tau(x)|}{2} = h(x) + 2 \geq h(x)$ and $\min\{d^+(y), d^-(y)\} \geq h(y)$ since $d^+(y) \geq d^-(x) \geq h(x) \geq h(y)$ and $d^-(y) \geq d^+(x) \geq h(x) \geq h(y)$. So Conclusion (b) is also satisfied.

This contradicts that G is a counterexample.

Case 2: x has at least two neighbors.

Let y and z be the two neighbors of x. We lift xy and xz to yz, reduce $h(x)$ by 1 and apply induction. We verify Condition (iii) for the resulting graph G' with $\beta' = \beta$.

Obviously Condition (iii) holds for each single vertex.

Let A be a vertex subset not containing z_0 such that $|A| > 1$ and $|V(G')\backslash A| > 1$.

We have that $\beta'(A) = \beta(A)$ and $d'(A) = d(A) - 2$ or $= d(A)$.

By Claim 1 and Proposition 2.2.2-(a), we have $d'(A) \geq 4 + |\tau'(A)|$. So G' satisfies the theorem and then there exists an orientation of G' such that Conclusion (a) and (b) are satisfied for G'. In particular the edge yz gets some direction, say from y to z, and there are at least $h(x)-1$ pairs edges incident with x directed in opposite directions. Now we delete the edge yz and orient xy away from y and xz towards z, then the resulting orientation of G satisfies the theorem which contradicts that G is a counterexample. □

Claim 3 *For any two vertices x, y distinct from z_0, there is at most one edge joining x and y.*

Let F be the set of all parallel edges between x and y ($|F| \geq 2$).

Case 1. $|V(G)| > 3$. Let $G' = G/F$ be the contracted graph and w be the new vertex from the contraction. By Claim 1, G' with the modified boundary $\beta(w) \equiv \beta(x) + \beta(y)$ (mod 3) satisfies Condition (iii) for the new vertex w, and each subset A. By induction, G' has an orientation D' satisfying Conclusions (a) and (b).

Extend the orientation D' to G by orienting each edge of F from y to x (temporarily). Note that

$$[[d^+_G(x) - d^-_G(x)] - \beta(x)] + [[d^+_G(y) - d^-_G(y)] - \beta(y)] \equiv 0 \pmod 3$$

since $\beta(w) \equiv \beta(x) + \beta(y) \pmod 3$. Let

$$[d^+_G(y) - d^-_G(y)] - \beta(y) \equiv \theta \in \{0, 1, 2\} = Z_3.$$

Then reverse the direction(s) of θ edges of F. It is easy to see that the modified orientation satisfies Conclusion (a). Note that there is no need to verify Conclusion (b) because of Claim 2.

Case 2. $|V(G)| = 3$. In this case, all edges of $G - F = E(z_0)$ are pre-oriented. Let D' be the orientation of F from y to x. Then a modification of D' can be obtained by repeating the second paragraph of Case 1. □

Claim 4

$$|V(G)| > 3.$$

Suppose $V(G) = \{z_0, x, y\}$. By Claim 2, we have

$$d(x) = 4 + |\tau(x)| \text{ and } d(y) = 4 + |\tau(y)|.$$

By Claim 3, there is at most one edge joining x and y.

Case 1: There is no edge joining x and y.

Then $d(z_0) = d(x) + d(y) \geq 4 + 4 > 4 + |\tau(z_0)|$ which contradicts Condition (iii).

Case 2: There is exactly one edge joining x and y.

Without loss of generality, let $d(x) \leq d(y)$. Thus, $d(x) \geq 4$ and $2d(x) \leq d(x) + d(y) = d(z_0) + 2 \leq 4 + |\tau(z_0)| + 2 \leq 4 + 3 + 2 = 9$. Therefore,

$$d(x) = 4, \ \tau(x) = 0 \ \text{ and } \beta(x) = 0.$$

Furthermore,

$$d(z_0) = d(x) + d(y) - 2 = d(y) + 2 \text{ and } \beta(y) + \beta(z_0) \equiv -\beta(x) \equiv 0 \pmod 3. \qquad (2.3)$$

So $|\tau(y)| \equiv |\beta(y)| \equiv |\beta(z_0)| \equiv |\tau(z_0)| \pmod 3$ and $|\tau(y)| \equiv |d(y)| \equiv |d(z_0)| \equiv |\tau(z_0)| \pmod 2$.

Therefore, $|\tau(y)| = |\tau(z_0)|$ and, by (2.3), $d(z_0) = d(y) + 2 = 4 + |\tau(y)| + 2 = 6 + |\tau(z_0)|$ which contradicts Condition (iii). $\Box$

Claim 5 *Every vertex x of G distinct from z_0 has at least three neighbors.*

Suppose x has at most two neighbors. By Claim 3, z_0 must be a neighbor of x and there are at least $d(x) - 1$ edges joining x and z_0. Then $d(\{x, z_0\}) \leq d(z_0) + d(x) - 2(d(x) - 1) = d(z_0) - d(x) + 2$.

Let $A = V(G) - x - z_0$. We have $|V(G) \backslash A| = 2 > 1$ and by Claim 4, $|A| > 1$. Then $d(A) = d(\{x, z_0\}) \leq d(z_0) - d(x) + 2 \leq 7 - 4 + 2 = 5 < 6 + |\tau(A)|$, a contradiction to Claim 1. $\Box$

Claim 6

$$\tau(x) \neq 0$$

for every $x \in V(G) - z_0$.

Suppose that $\tau(x) = 0$ for some vertex x other than z_0. By (2.2), $\beta(x) = 0$ and by Claim 2, $d(x) = 4$.

If x has four distinct neighbors, then we can lift the edges incident with x randomly. Otherwise by Claim 3 and Claim 5, x has three neighbors and one of them is z_0 such that there are two edges joining x and z_0. Let y and z be the two neighbors of x distinct from z_0. We can delete the four edges incident with x and add two edges yz_0 and zz_0 to complete the lifting. Let G' be the resulting graph. Define the corresponding boundary β' such that $\beta'(v) = \beta(v)$ if $v \neq x$. Then, by Observation 1, it suffices to verify Condition (iii) for G' and β'.

For any single vertex of G', the condition clearly holds.

Now consider a vertex subset A of G' not containing z_0 such that $|A| > 1$ and $|V(G')\backslash A| > 1$.

If A contains all the neighbors of x, then $d'(A) = d(A) - d(x) = d(A + x)$. By Claim 1, $d(A + x) \geq 6 + |\tau(A + x)|$ since $|V(G)\backslash(A + x)| = |V(G')\backslash A| > 1$. So, $d'(A) \geq 6$ and by Proposition 2.2.1, we have $d'(A) \geq 4 + |\tau'(A)|$.

Otherwise, let y_1, y_2, z_1, z_2 be the neighbors of x such that the resulting edges after the lifting are y_1z_1 and y_2z_2 (see Figure 2.3 and it is possible that $z_1 = z_2 = z_0$), and suppose $y_2 \notin A$. Then $d'(A) = d(A) - 2$ if both y_1 and z_1 are contained in A or $d'(A) = d(A)$ otherwise. By Proposition 2.2.2, we also have $d'(A) \geq 4 + |\tau'(A)|$ since $\beta'(A) = \beta(A)$ and by Claim 1, $d(A) \geq 6 + |\tau(A)|$. $\square$

Claim 7 *There is no edge joining x and y if $x \neq z_0$, $y \neq z_0$ and $\tau(x)\tau(y) < 0$. And there is no directed edge from z_0 to x if $\tau(x) < 0$, or, from x to z_0 if $\tau(x) > 0$.*

(Note that, for the case of $\beta(x) = 0$ and $d(x)$ is odd, the vertex x has multiple τ-values: 3 and -3. That is, the τ-value for such vertex is considered as either positive or negative.)

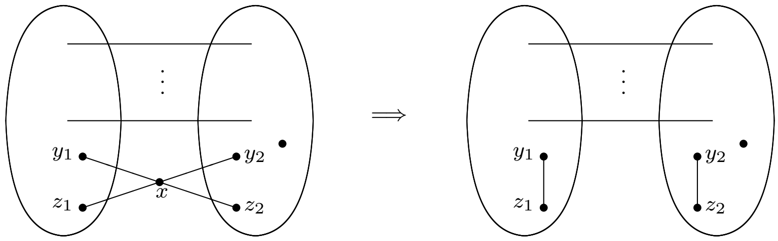

Figure 2.3: *Lift the edges incident with x*

Assume that $e = xy$ is an edge of G such that $\tau(x) > 0, \tau(y) < 0$ and $z_0 \notin \{x, y\}$, or $e = z_0y$ (or $e = xz_0$) such that $\tau(y) < 0$ and z_0y is pre-oriented away from z_0 (or $\tau(x) > 0$ and xz_0 is pre-oriented into z_0, respectively). Let $G' = G - e$ be the resulting graph after pre-directing $e = xy$ (if $z_0 \notin \{x, y\}$) or deleting e (if $z_0 \in \{x, y\}$) and let β' be the modified boundary such that $\beta'(x) = \beta(x) - 1$ and $\beta'(y) = \beta(y) + 1$. Note that, for each $v \in \{x, y\} - \{z_0\}$, both $|\beta(v)|$ and $|\tau(v)|$ are reduced by 1 when they are replaced by $|\beta'(v)|$ and $|\tau'(v)|$, respectively. So, Condition (iii) remains satisfied for v since $d'(v) = 4 + |\tau'(v)|$. If z_0 is an endvertex of e, then both $|\beta(z_0)|$ and $|\tau(z_0)|$ increase by 1, or they both decrease by 1. In either case, Condition (iii) remains satisfied for z_0. Furthermore, Condition (iii) remains satisfied for each non-empty, non-trivial, proper subset A of $V(G') - \{z_0\}$ (by Claim 1 and Proposition 2.2.2).

By induction, G' has an orientation D' satisfying β'. By including the pre-directed edge (or deleted directed edge) e, the extended orientation D' in G satisfies Conclusion (a). Note that there is no need to check the Conclusion (b) for D' because the h-value is zero for every vertex u other than z_0 (by Claim 2). □

Summary.

The following is a summary of those structural results in Part I (following [56]) about the smallest counterexample G.

($\star$) For every non-empty, proper subset A of $V(G) - \{z_0\}$,

$$\begin{cases} d(A) = 4 + \tau(A) & \text{if } |A| = 1 \quad \text{(by Claim 2)}, \\ d(A) \geq 6 + \tau(A) & \text{if } |A| > 1 \quad \text{(by Claim 1)}; \end{cases}$$

($\star$) $G - \{z_0\}$ contains no parallel edges (by Claim 3);

($\star$) $|N(x)| \geq 3$ for every $x \in V(G) - \{z_0\}$ (by Claim 5);

($\star$) $\tau(x) \neq 0$ for every $x \in V(G) - \{z_0\}$ (by Claim 6) and

($\star$) $\tau(x)\tau(y) > 0$ for every edge $xy \in E(G - \{z_0\})$ (by Claim 7).

Part II. Additional reductions not contained in [56].

Let $V^+ = \{x \in V(G) - z_0 : \tau(x) = 1 \text{ or } 2\}$ and $V^- = \{x \in V(G) - z_0 : \tau(x) = -1 \text{ or } -2\}$.

Claim 8 *Either $V(G) - z_0 = V^+$ or $V(G) - z_0 = V^-$.*

First we prove that $|\tau(x)| \neq 3$ for any vertex $x \in V(G) - z_0$. Suppose $|\tau(x)| = 3$ for some x distinct from z_0.

By Claim 5 let y be a neighbor of x distinct from z_0. By Claim 6 we have $\tau(y) > 0$ or $\tau(y) < 0$. We can choose $\tau(x) = -3$ or $\tau(x) = 3$ such that $\tau(x)\tau(y) < 0$ and get a contradiction to Claim 7. So, Claim 6 implies that $\{V^+, V^-\}$ is a partition of $V(G) - z_0$.

Now assume that $V^+ \neq \emptyset$ and $V^- \neq \emptyset$.

Again by Claim 7 there is no edge such that one end in V^+ and another in V^-. Therefore all the edges with one end in V^+ or in V^- must be incident with z_0. And then we have that $d(z_0) = d(V^+) + d(V^-) \geq 4 + |\tau(V^+)| + 4 + |\tau(V^-)| \geq 8 > 4 + |\tau(z_0)|$ which contradicts Condition (iii) for z_0. $\square$

Without loss of generality, suppose $V(G) - z_0 = V^+$. Otherwise if $V(G) - z_0 = V^-$, we can reverse all the directions of the edges incident with z_0 and replace $\beta(x)$ by $3 - \beta(x)$ for each vertex x (including z_0) if $\beta(x) \neq 0$. Then the resulting graph satisfies $V(G) - z_0 = V^+$ and is still a minimum counterexample.

By Claim 8 (and the assumption that $V(G) - z_0 = V^+$), Claim 2 and by (2.2), we have that, for each vertex $x \in V(G) - z_0$,

$$d(x) = 4 + \tau(x) = 4 + \beta(x) \text{ with } \tau(x) = \beta(x) = 1 \text{ or } 2. \tag{2.4}$$

Claim 9 *All edges incident with z_0 are directed away from z_0 and $d(z_0) \leq 5$.*

Let x be a neighbor of z_0. By Claim 7 xz_0 is directed away from z_0 since $\tau(x) > 0$.

So $d^-(z_0) = 0$ and $d(z_0) = d^+(z_0)$. If $d(z_0) = 6$, then $\beta(z_0) = 0$ and $\tau(z_0) = 0$. And if $d(z_0) = 7$, then $\beta(z_0) = 1$ and $\tau(z_0) = 1$. Both cases imply that $d(z_0) = 6 + |\tau(z_0)|$ and, therefore, contradict Condition (iii). So $d(z_0) \leq 5$. □

Claim 12 will deal with some special structure before the final step of the proof. And the following Claims 10 and 11 are preparations for the proof of Claim 12.

Claim 10 *Let A be a vertex subset not containing z_0 such that $|A| > 1$ and $|V(G)\backslash A| > 1$. If $d(A) = 6 + |\tau(A)|$, then $\tau(A) \neq 0, -1, -2$. In particular,*

$$d(A) \geq 7. \tag{2.5}$$

Suppose (reductio ad absurdum) that $d(A) = 6 + |\tau(A)|$ and $\tau(A) = 0, -1$, or -2 for some vertex subset A.

We first contract A into a single vertex, say a, and by induction we get an orientation of all edges out of $G[A]$. By Conclusion (b) $\min\{d^+(a), d^-(a)\} \geq h(a) = 1$. So there exists an edge oriented from A to $V(G)\backslash A$. Let uv be such an edge with $u \in A$ and $v \in V(G)\backslash A$.

Now we remove uv, contract $V(G)\backslash A$ into a single vertex as a new z_0, decrease $\beta(u)$ by 1 and increase $\beta(z_0)$ by 1 (see Figure 2.4). Again we use induction to extend the orientation to the edges in $G[A]$. It suffices to verify Condition (iii) for resulting graph G' and modified boundary β'.

Since $\tau'(A) = \tau(A) - 1 \leq -1$, we have that $\tau'(z_0) = -\tau'(A) = -\tau(A) + 1 = |\tau(A)| + 1$ for the new z_0. Then $d'(z_0) = d'(A) = d(A) - 1 = 6 + |\tau(A)| - 1 = 4 + |\tau'(z_0)|$.

For the vertex u, we have $\tau'(u) = \tau(u) - 1 \geq 0$ since $u \in V^+$. Then $d'(u) = d(u) - 1 = 4 + \tau(u) - 1 = 4 + |\tau'(u)|$.

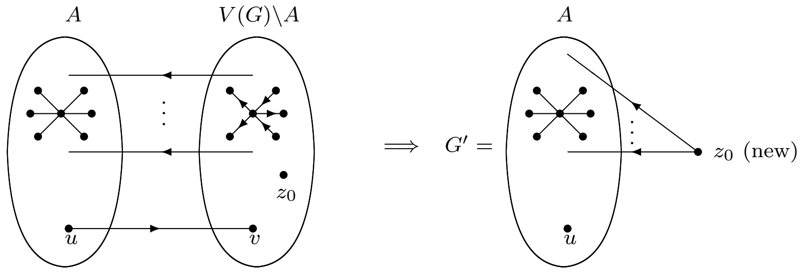

Figure 2.4: *Contract edges out G, delete uv and contract $V(G)\backslash A$*

For any vertex subset B not containing the new z_0 such that $|B| > 1$ and $|V(G')\backslash B| > 1$, if $d'(B) \neq d(B)$, then $d'(B) = d(B) - 1$ and $\beta'(B) = \beta(B) - 1$. By Claim 1 and Proposition 2.2.2, we have $d'(B) \geq 4 + |\tau'(B)|$. □

Claim 11 *Let A be a vertex subset not containing z_0 such that $|A| > 1$, $|V(G)\backslash A| > 1$. If $d(A) = 8$ and $\tau(A) = 0$, then there is an orientation of the contracted graph G/A satisfying the theorem such that there exist two edges directed from A to $V(G)\backslash A$ and their ends in A are distinct.*

By Equation (2.2),

$$\beta(A) = 0. \tag{2.6}$$

We contract A into a single vertex, say a, then by induction we can get an orientation of the contracted graph G/A satisfying the theorem.

Note that $d^+(a) + d^-(a) = 8$ and by Conclusion (a), $d^+(a) \equiv d^-(a) \pmod 3$. By Conclusion (b), $d^+(a) \geq 2$ and $d^-(a) \geq 2$ since $h(a) = \frac{d(A)-4-|\tau(A)|}{2} = 2$. So the only possibility is that $d^+(a) = d^-(a) = 4$. There are four edges directed from A to $V(G)\backslash A$. If their ends in A are not the same one, then this orientation satisfies the claim and we are done.

Now assume that $u \in A$ is the common end of the four edges directed from A to $V(G)\backslash A$. (See Figure 2.5.) Then

$$d(A-u) = |[A-u, V(G)\backslash A]| + |[A-u, \{u\}]| = d^-(a) + (d(u) - d^+(a)) = d(u) \leq 6. \tag{2.7}$$

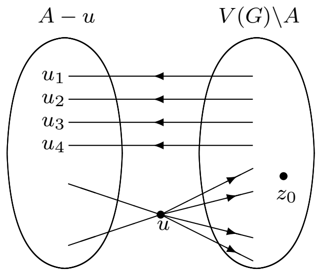

Figure 2.5: *Possibly, $u_i = u$ for some i*

The last inequality in (2.7) follows from Equation (2.4).

By assumption, $|A-u| \neq 0$. Also, $|A-u| \neq 1$. For otherwise, let v be the vertex of A distinct from u. Then, by Claim 3 and Equation (2.4), we have that $\beta(u) \geq 1$, $\beta(v) \geq 1$ and

$$8 = d(A) \geq d(u) + d(v) - 2 = 6 + \beta(u) + \beta(v) \geq 8.$$

Hence, all equalities hold. That is, $\beta(v) = \beta(u) = 1$ which results that $\beta(A) = 2$ and contradicts (2.6).

So $|A-u| > 1$. By Claim 1, we have $d(A-u) \geq 6 + |\tau(A-u)|$. Therefore, by (2.7), $d(A-u) = 6$ and $\tau(A-u) = 0$, a contradiction to Claim 10. □

Claim 12 *If A is a vertex subset not containing z_0 such that $|A| > 1$ and $|V(G)\backslash A| > 1$, then $d(A) \geq 7$. Furthermore, we have $\tau(A) = 1$ if $d(A) = 7$ and $\tau(A) = 2$ if $d(A) = 8$.*

By Claim 1 we have $d(A) \geq 6+|\tau(A)|$. So if $d(A) \leq 8$, then, by (2.2), $d(A) = 6+|\tau(A)|$ with $\tau(A) \in \{0, \pm1, \pm2\}$ or $d(A) = 8 + |\tau(A)|$ with $\tau(A) = 0$.

Then by the last statement of Claim 10 (Inequality (2.5)), $d(A) \geq 7$. Hence, by Claim 10 again, $\tau(A) = 1$ if $d(A) = 7$, and, $\tau(A) = 0$ or 2 if $d(A) = 8$. To prove the claim we only need to prove that $\tau(A) \neq 0$ if $d(A) = 8$.

Assume therefore that

$$d(A) = 8 \text{ and } \tau(A) = 0. \tag{2.8}$$

By Claim 11 we can orient all edges not in $G[A]$ such that each vertex of G/A satisfies the Conclusions (a) and (b) and there exist two edges, say u_1v_1 and u_2v_2, directed from A

to $V(G)\backslash A$ such that $u_1 \neq u_2$, where $u_1, u_2 \in A$. Then we remove u_1v_1 and u_2v_2, contract $V(G)\backslash A$ into a single vertex as a new z_0, decrease $\beta(u_1)$ and $\beta(u_2)$ by 1 and increase $\beta(z_0)$ by 2 for the new z_0 (see Figure 2.6). Again we use induction to direct the edges inside $G[A]$. It suffices to verify Condition (iii) for resulting graph G' and modified boundary β'.

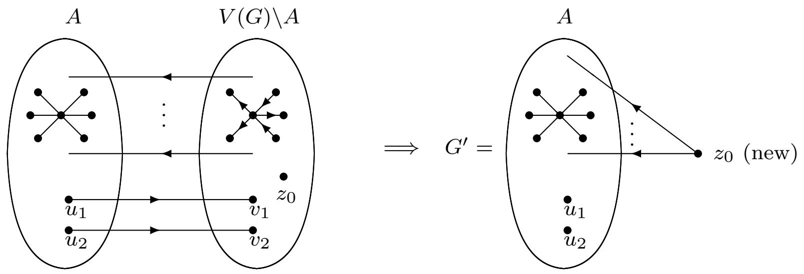

Figure 2.6: *Delete u_1v_1, u_2v_2 and contract $V(G)\backslash A$*

By Equation (2.8), we have $d(A) = 8$ and $\beta(A) = \tau(A) = 0$.

So $d'(z_0) = d(A) - 2 = 6$ and $\beta'(z_0) = \beta(V(G)\backslash A) + 2 = -\beta(A) + 2 = 2$.

Then $\tau'(z_0) = 2$ and $d'(z_0) = 4 + |\tau'(z_0)|$. So, Condition (iii) holds for the new z_0.

For u_i, $i = 1, 2$, by (2.4) we have $d(u_i) = 4 + \tau(u_i)$ with $\tau(u_i) = 1$ or 2. Then $\tau'(u_i) = \tau(u_i) - 1 \geq 0$ and $d'(u_i) = d(u) - 1 = 4 + \tau(u_i) - 1 = 4 + \tau'(u_i) = 4 + |\tau'(u_i)|$.

For a vertex subset B not containing the new z_0 such that $|B| > 1$ and $|V(G')\backslash B| > 1$, we have $d(B) \geq 7$ (implied by Claim 1 and the last statement of Claim 10) since $|V(G)\backslash B| \geq |V(G')\backslash B| > 1$.

So $d'(B) \geq d(B) - 2 \geq 5$ and, by Proposition 2.2.1-(1), we only need to consider the case of $d'(B) = 5$.

Since $d'(B) = 5$, Condition (iii) fails only if $|\tau'(B)| = 3$ and $\beta'(B) = 0$.

Then $d(B) = 7$ (by Inequality (2.5)) and therefore, $u_1, u_2 \in B$. We have $\beta(B) = \beta'(B) + 2 = 2$ and, by (2.2), $\tau(B) = -1$ which contradicts Claim 10. □

Claim 13 *For every vertex $x \in V(G) - z_0$ we have $d(x) = 6$.*

Suppose that there is a vertex x ($\in V(G) - z_0$) with $d(x) \neq 6$.

Then, by (2.4), $d(x) = 5$ with $\beta(x) = \tau(x) = 1$.

We now pre-direct the edges incident with x all towards x (if z_0 is a neighbor of x, then, by Claim 9, the edges between them are already directed away from z_0) and then modify β for each neighbor of x accordingly.

If the reduced graph $G' = G - x$ and the modified boundary β' satisfy the condition of the theorem, then, by induction, there exists an orientation described in the theorem for G' and β'. The corresponding orientation by adding back the pre-directed edges toward x satisfies the boundary β for G since $d^+(x) - d^-(x) = 0 - 5 \equiv 1 = \beta(x) \pmod 3$. Hence, it suffices to verify the conditions of the theorem for G' and β'.

For each single vertex of G' we only need verify it for z_0 and each neighbor of x.

By Claim 9 we have that $d(z_0) \leq 5$.

So $d'(z_0) \leq d(z_0) \leq 5 < 6 + |\tau'(z_0)|$ and by Proposition 2.2.1 $d'(z_0) \leq 4 + |\tau'(z_0)|$.

Let y be a neighbor of x distinct from z_0. By (2.4), $\tau'(y) = \tau(y) - 1 \geq 0$ and $d'(y) = d(y) - 1 = 4 + \tau(y) - 1 = 4 + \tau'(y) = 4 + |\tau'(y)|$.

Now let A be a nonempty vertex subset not containing z_0 such that $|A| > 1$ and $|V(G')\backslash A| > 1$. Then $|V(G)\backslash(A + x)| = |V(G')\backslash A| > 1$.

By Claim 12,

$$d(A) \geq 7 \text{ and } d(A + x) \geq 7. \tag{2.9}$$

Hence,

$$5 = d(x) = (d(A) - d'(A)) + (d(A + x) - d'(A)) \geq 14 - 2d'(A)$$

and we have $d'(A) \geq 5$.

By Proposition 2.2.1-(1), we only need to consider the case of $d'(A) = 5$.

Let s be the number of neighbors of x contained in A.

Then, by (2.9),

$$7 \leq d(A) = d'(A) + s = 5 + s \text{ and } 7 \leq d(A + x) = d'(A) + (5 - s) = 10 - s.$$

Thus, $2 \leq s \leq 3$ and $7 \leq d(A) = 6 + (s - 1) \leq 8$.

By Claim 12, we have that $d(A) = 7$ and $\tau(A) = 1$ if $s = 2$, or $d(A) = 8$ and $\tau(A) = 2$ if $s = 3$. Hence, $\tau(A) = s - 1 > 0$.

By (2.2), $\tau'(A) \equiv \beta'(A) \equiv \beta(A) - s \equiv \tau(A) - s = -1 \pmod 3$ and $\tau'(A) \equiv d'(A) = d(A) - s \equiv \tau(A) - s = -1 \pmod 2$.

We have that $\tau'(A) = -1 \pmod 6$ and $|\tau'(A)| = 1$ since $|\tau'(A)| \leq 3$.

Therefore $d'(A) = 4 + |\tau'(A)|$. This verifies the conditions of the theorem and completes the proof of the claim. □

The final Step.

Now we are going to prove that G is not a counterexample as the final step.

By Claim 13, for each vertex $x \in V(G) - z_0$ we have $d(x) = 6$. Furthermore,

$$\beta(x) = \tau(x) = 2. \tag{2.10}$$

By Claim 5 we can lift two undirected edges incident with x, say xu and xv to uv where $u, v \neq z_0$, pre-direct the other edges all towards x and modify the β values of the neighbors accordingly. As before, we only need to verify Condition (iii) for the resulting graph $G' = G - x$ and the modified boundary β' for the purpose of induction.

For single vertices of G', the proofs are similar to those of Claim 13.

Now let A be a vertex subset not containing x and z_0 such that $|A| > 1$ and $|V(G')\backslash A| > 1$. Then $|V(G)\backslash(A + x)| = |V(G')\backslash A| > 1$.

By Claim 12, $d(A) \geq 7$ and $d(A + x) \geq 7$. So $d(A) \geq 8$ and $d(A + x) \geq 8$ since each vertex of G has an even degree and each edge cut of G is of even size. Thus,

$$6 = d(x) = (d(A) - d'(A)) + (d(A + x) - d'(A)) \geq 8 + 8 - 2d'(A). \tag{2.11}$$

And, therefore, $d'(A) \geq 5$.

Assume that $d'(A) = 5$. Then equalities of (2.11) hold and hence, $d(A) = d(A+x) = 8$. By Claim 12, $\tau(A) = \tau(A + x) = 2$ and, furthermore, by (2.2), $\beta(A) = \beta(A + x) = 2$. Then $\beta(x) = 0$ which contradicts that $\beta(x) = 2$ (Equation (2.10)).

So $d'(A) \geq 6$ and by Proposition 2.2.1-(1), we have $d'(A) \geq 4 + |\tau'(A)|$.

By induction, there exists an orientation of G' such that every vertex distinct from x satisfying the boundary condition β' (Conclusion (a)) and the lifted edges xu and xv receive opposite directions and the other four pre-directed edges are all towards x. (Note that, by Claim 2, Conclusion (b) is satisfied automatically.)

So $d^+(x) - d^-(x) = 1 - 5 \equiv 2 = \beta(x) \pmod 3$ and this orientation satisfies the theorem for G and β which implies that G is not a counterexample. This completes the proof. ■

2.5 Remarks

Thomassen proved in [56] that a graph is Z_3-connected (that is, it admits all generalized Tutte orientations) provided $d(A) \geq 6 + |\tau(A)|$ for every non-empty, proper vertex subset of G. In the present paper it is shown that 6 can be lowered to 4. The additive constant 4 may be replaced by 3. For, if it is satisfied for 3 it is automatically satisfied for 4 as well, for parity reasons (see Section 2.2). But, it cannot be lowered to 2. The first example (See figure 2.7), which is 4-regular and 4-connected, was given by Jaeger, Linial, Payan and Tarsi [29] (also see [63] page 232). And an infinite family of 4-regular 4-edge-connected planar graphs was recently given by Lai [38](See figure 2.8 and figure 2.9, the graph in figure 2.8 contains $3k$ blocks in figure 2.9). For each of those examples ([29], [38]), the boundary β is a constant 1 for every vertex. So, it is a challenge to modify the connectivity condition introduced in [56] to obtain group-connectivity information about graphs of odd edge-connectivity, in particular 5-edge-connected graphs.

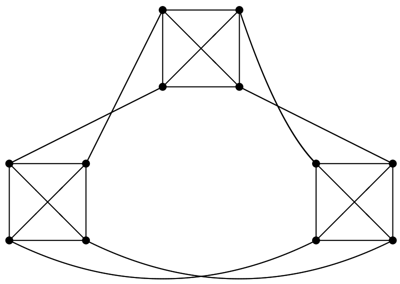

Figure 2.7: *A* 4*-regular,* 4*-edge-connected graph having no* β*-orientations with* $\beta = 1$

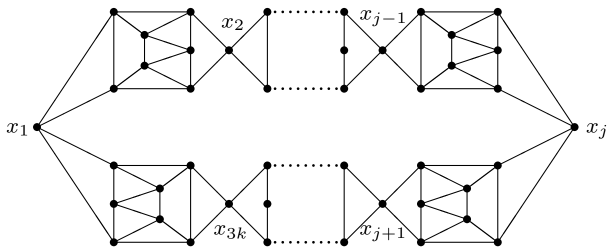

Figure 2.8: 4*-regular,* 4*-edge-connected planar graphs having no* β*-orientations with* $\beta = 1$

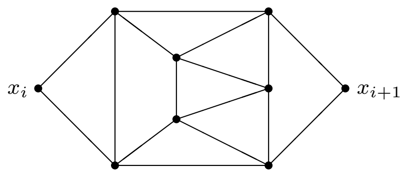

Figure 2.9: *A block of the graph in figure 2.8 where* $x_{3k+1} = x_1$

Chapter 3

Modulo k-orientations in $9k$-edge-connected graphs

3.1 Introduction

Problem 3.1.1 *Let G be a graph and let k be an odd integer, $k \geq 3$. Decide if G has an orientation D (called a modulo k-orientation) such that, for every vertex $v \in V(G)$,*

$$d_D^+(v) \equiv d_D^-(v) \pmod{k}.$$

Note that Problem 3.1.1 is trivial if the integer k is even (as a graph has such an orientation if and only if it is eulerian).

The general problem for all odd integers k (Problem 3.1.1) was introduced by Jaeger ([27], [28]) as a circular flow problem. It has been further generalized in [4], [38] and [42].

Problem 3.1.2 *Let G be a graph, let k be an integer, $k \geq 3$, and let $\beta : V(G) \mapsto Z_k$. Decide if G has an orientation D such that, for every vertex $v \in V(G)$,*

$$d_D^+(v) - d_D^-(v) \equiv \beta(v) \pmod{k}.$$

It is easy to see that the mapping β must satisfy the following necessary conditions:
(C1) $\sum_{v\in V(G)} \beta(v) \equiv 0 \pmod{k}$ *and*
(C2) *for every vertex $v \in V(G)$, $d(v) - \beta(v)$ is even if k is even.*

Definition 3.1.3 *Let G be a graph and let k be an integer, $k \geq 3$.*

(i) A mapping $\beta : V(G) \mapsto Z_k$ is called a <u>*Z_k-boundary*</u> *of G if it satisfying the necessary conditions (C1) and (C2).*

(ii) Let β be a Z_k-boundary of G. An orientation D is called a <u>*β-orientation*</u> *of G if for every vertex $v \in V(G)$, $d_D^+(v) - d_D^-(v) \equiv \beta(v) \pmod{k}$.*

Due to their close relation with flow theory (see [27], [28] or [63]), some important conjectures have been proposed for Problems 3.1.1 and 3.1.2.

Conjecture 3.1.4 *Let G be a graph and k (≥ 3) be an odd integer.*

(i) (Jaeger [27], also see [28], [63]) *If G is $(2k-2)$-edge-connected, then G has a modulo k-orientation;*

(ii) (Galluccio, Goddyn, Hell [15] and Seymour [48], see also [64] p. 149) *There exists an integer $f(k)$ such that every $f(k)$-edge-connected graph G has a modulo k-orientation.*

Conjecture 3.1.5 *Let G be a graph, k (≥ 3) be an odd integer and β be a Z_k-boundary of G.*

(i) (Lai [38], see also [42]) *If G is $(2k-1)$-edge-connected, then G has a β-orientation;*

(ii) (Lai [38], see also [42]) *There exists an integer $g(k)$ such that if G is $g(k)$-edge-connected, then G has a β-orientation.*

Conjecture 3.1.4-(ii) and Conjecture 3.1.5-(ii) have been proved recently by Thomassen [56] (which also includes the case where k is even).

Theorem 3.1.6 (Thomassen [56]) *Let G be a graph and let β be a Z_k-boundary of G, where $k \geq 3$ is an integer. Then G has a β-orientation if one of the following is satisfied.*

(i) k is even and G is $\frac{k^2+k}{2}$-edge-connected;

(ii) k is odd and G is $(2k^2+k)$-edge-connected.

In the dissertation the quadratic bound is reduced to the following linear bounds.

Theorem 3.1.7 *Let G be a graph and let β be a Z_k-boundary of G, where $k \geq 3$ is an integer. Then G has a β-orientation if one of the following is satisfied.*

(i) k is even and G is $(5k)$-edge-connected;

(ii) k is odd and G is $(10k)$-edge-connected.

And the edge-connectivity is further reduced to $9k$ for a graph having a modulo k-orientation.

Theorem 3.1.8 *Every $9k$-edge-connected graph has a modulo k-orientation, where k is an odd integer ≥ 3.*

Problem 3.1.1 has been extensively studied for various families of graphs in [4], [5], [9], [15], [20], [21], [27], [28], [38], [42], [45], [46], [50], [57], [64] and [67]. Many of them remain the best known results for the graph families they concern.

3.2 Preliminaries

The definition of a Z_k-boundary $\beta : V(G) \mapsto Z_k$ is extended to $\beta : \mathcal{P}(V(G)) \mapsto Z_k$ as follows, where $\mathcal{P}(V(G))$ is the power set of $V(G)$.

Let A be a vertex subset of $V(G)$. Define

$$\beta(A) \equiv \sum_{x \in A} \beta(x) \pmod{k}.$$

Also, let

$$d(A) = |[A, V(G) \backslash A]|,$$

where $[A, V(G)\backslash A]$ is the set of edges between A and $V(G)\backslash A$.

For a graph G' with the boundary β', we use the notations $d'(A)$ and $\beta'(A)$ for the corresponding values of the vertex subset A of $V(G')$.

Let $E(x, y)$ (respectively $E(x, U)$) be the set of all edges between vertex x and vertex y (respectively vertex subset U) and denote $e(x, y) = |E(x, y)|$ (respectively $e(x, U) = |E(x, U)|$).

Proposition 3.2.1 *Let $k > 0$ be an even integer, and G be a graph with a Z_k-boundary β. If G' is the resulting graph constructed from G after contracting, lifting and/or pre-directing operations, then the resulting mapping β', modified from β, satisfies the necessary conditions (C1) and (C2), that is, β' is a Z_k-boundary.*

Proof. By induction we only need consider the cases that G' constructed from G after a single operation. We are to prove it for the contracting operation since the other two cases are trivial. It suffices to verity that $d(A) - \beta(A)$ is even for any vertex set $A \subset V(G)$.

Suppose A is an vertex subset of $V(G)$. Let m be the number of edges with both ends in A. By definitions of $d(A)$ and $\beta(A)$, we have that $d(A) = \sum_{x\in A} d(x) - 2m$ and $\beta(A) \equiv \sum_{x\in A} \beta(x) \pmod 2$ since $\beta(A) \equiv \sum_{x\in A} \beta(x) \pmod k$ and k is even. Then

$$d(A) \equiv \sum_{x\in A} d(x) \equiv \sum_{x\in A} \beta(x) \equiv \beta(A) \pmod 2.$$

■

3.3 Main results

All theorems in this chapter are corollaries of the following technical result, which is a refinement of Theorem 2 in [56]. The additional new idea is the specification of a vertex set V_0 of size at most 1 satisfying condition (iii) below.

Theorem 3.3.1 *Let k be an even integer, $k \geq 4$, and let G be a graph with a Z_k-boundary $\beta : V(G) \mapsto \{0, \cdots, k-1\}$. Let $z_0 \in V(G)$, let D_{z_0} be a pre-orientation of $E(z_0)$ and let $V_0 \subseteq V(G) - z_0$ such that $|V_0| \leq 1$. Assume that*

(i) $|V(G)| \geq 3$;

(ii) $d(z_0) < 7k$, and the edges incident with z_0 are directed such that

$$d^+(z_0) - d^-(z_0) \equiv \beta(z_0) \pmod{k};$$

(iii)If $V_0 \neq \emptyset$, then $e(z_0, V_0) \leq d(V_0) - k$ and

$$d(V_0) \geq \begin{cases} 3k & \text{if } \beta(V_0) = 0 \\ 4k - \beta(V_0) & \text{if } \beta(V_0) > 0 \end{cases}.$$

(iv) For each non-empty vertex subset A not containing z_0 such that $|V(G)\backslash A| > 1$ and $A \neq V_0$, we have that

$$d(A) \geq 4k + \beta(A).$$

Then the pre-orientation D_{z_0} of $E(z_0)$ can be extended to an orientation D of G such that, for each vertex x, we have

$$d^+(x) - d^-(x) \equiv \beta(x) \pmod{k}.$$

Let m be a non-negative integer. We say a graph G with a Z_k-boundary β is $(m+\beta)$-*edge-connected* if $d(A) \geq m + \beta(A)$ for every vertex subset $A \subset V(G)$.

Corollary 3.3.2 *Let G be a graph and let β be a Z_k-boundary of G, where $k \geq 3$ is an integer. Then G has a β-orientation if one of the following is satisfied.*

(i) k is even and G is $(4k + \beta)$-edge-connected;

(ii) k is odd and G is $(9k + \beta)$-edge-connected.

Proof.

Let G' be the graph constructed from G by adding an isolated vertex z_0 and let $V_0 = \emptyset$. Define $\beta' : V(G') \rightarrow Z_k$ such that $\beta'(z_0) = 0$ and $\beta'(x) = \beta(x)$ if $x \neq z_0$. If (i) is satisfied, then G' and β' satisfy the conditions of Theorem 3.3.1. There exists an orientation of G', which is a β-orientation of G since $E(G) = E(G')$ and for each vertex $x \in V(G)$, $d^+(x) - d^-(x) \equiv \beta(x) \pmod{k}$.

Now suppose k is odd and G is $(9k + \beta)$-edge-connected.

Define $\beta' : V(G) \mapsto Z_{2k}$ from β as follows: for each vertex $x \in V(G)$,

$$\beta'(x) = \begin{cases} \beta(x) & \text{if } d(x) - \beta(x) \text{ is even} \\ \beta(x) + k & \text{if } d(x) - \beta(x) \text{ is odd.} \end{cases}$$

So $d(x) - \beta'(x)$ is even for every vertex $x \in V(G)$ since k is odd. Moreover we have that $\sum_{x \in V(G)} \beta'(x) \equiv 0 \pmod{2k}$ since $\sum_{x \in V(G)} \beta'(x) \equiv \sum_{x \in V(G)} \beta(x) \equiv 0 \pmod{k}$ and $\sum_{x \in V(G)} \beta'(x) \equiv \sum_{x \in V(G)} d(x) \equiv 2|E(G)| \equiv 0 \pmod{2}$. Therefore the mapping β' is a Z_{2k}-boundary of G.

Let $A \subset V(G)$ be an arbitrary vertex subset. We have that $\beta'(A) \equiv \sum_{x \in A} \beta'(x) \equiv \sum_{x \in A} \beta(x) \equiv \beta(A) \pmod{k}$. So $k + \beta(A) \geq \beta'(A)$ and G is $(4(2k) + \beta')$-edge-connected since $9k + \beta(A) \geq 4(2k) + \beta'(A)$ for every vertex subset $A \subset V(G)$. By the proof of previous part, G has an orientation D such that $\forall x \in V(G)$,

$$d_D^+(x) - d_D^-(x) \equiv \beta'(x) \pmod{2k}.$$

So,

$$d_D^+(x) - d_D^-(x) \equiv \beta'(x) \equiv \beta(x) \pmod{k}.$$

which means that D is also a β-orientation of G. ■

Theorems 3.1.7 and Theorems 3.1.8 are immediate corollaries of Corollary 3.3.2 since a $5k$-edge-connected graph (respectively $10k$-edge-connected graph) is $(4k + \beta)$-edge-connected (respectively $(9k + \beta)$-edge-connected) for any Z_k-boundary β, and, a $9k$-edge-connected graph is $(9k + \beta)$-edge-connected for the special Z_k-boundary $\beta = 0$.

3.4 Proof of Theorem 3.3.1

Proof. The proof is by induction. We assume (reductio ad absurdum) that G is a counterexample such that $|E(G)|$ is minimum.

We are going to apply liftings, pre-directings and/or contractions. By Prososition 3.2.1, a modified boundary (after such operations) remains a Z_k-boundary (satisfying necessary conditions (C1) and (C2)). For convenience, we will not repeatedly mention it in the proof.

Claim 14 *$e(x, y) < \frac{k-2}{2}$ for any two vertices $x, y \in V(G) - z_0$.*

Suppose $e(x, y) \geq \frac{k-2}{2}$.

Let $G' = G/E(x, y)$, and let w be the new vertex from the contraction. So, G' with the modified boundary $\beta(w) \equiv \beta(x) + \beta(y) \pmod{k}$ satisfies Condition (iv) for the new vertex w.

If $x \in V_0$ or $y \in V_0$, then by Condition (iv) $d(w) \geq 4k + \beta(w)$. We have that $V_0 = \emptyset$ for the graph G'.

If $|V(G)| > 3$, then $|V(G')| \geq 3$ and by induction G' has an orientation D' satisfying the theorem. If $|V(G)| = 3$, we let $D' = D_{z_0}$.

Extend the orientation D' to G by orienting each edge of $E(x, y)$ from x to y. Let

$$[d^+_G(x) - d^-_G(x)] - \beta(x) \;\equiv\; \eta \;\in \{0, 1, \ldots, k-1\} = Z_k.$$

Note that $[d^+(x) - d^-(x)] - \beta(x) = (d(x) - \beta(x)) - 2d^-(x)$ is even by the necessary condition (C2) on a Z_k-boundary. So $\eta \in \{0, 2, \ldots, k-2\}$. Reverse the orientations of the edges $E(x, y)$ one by one (while η is decreasing two by two) until x satisfies the conclusion of the theorem. Then y satisfies the conclusion as well. $\Box$

Claim 15 *If A is a vertex subset not containing z_0 such that $|A| > 1$ and $|V(G)\backslash A| > 1$, then*

$$d(A) \geq \begin{cases} 6k & \text{if } |A| = 2 \\ 7k & \text{if } |A| > 2 \end{cases} \tag{3.1}$$

<u>Case 1</u>. $|A| = 2$. Suppose $A = \{x, y\}$. By Claim 14, $2e(x, y) < k - 2 < k$. So, $d(A) = d(x) + d(y) - 2e(x, y) > 3k + 4k - k = 6k$.

Case 2.1. $|A| > 2$ and $V_0 \not\subseteq A$. If $d(A) < 7k$, then we first get an extension of D_{z_0} to the contracted graph G/A by induction. Then all edges of the edge-cut $[A, A^c]$, where $A^c = V(G)\backslash A$, are oriented in this extension. We then contract A^c into a single vertex as a new z_0, and again we use induction to extend the orientation $[A, A^c]$ to the edges in $G[A]$.

Case 2.2. $|A| > 2$ and $V_0 \subseteq A$. If $d(A) < 7k$, similar to the proof of Case 2.1 we get extensions of D_{z_0} of the contracted graph G/A (with $V_0 = \emptyset$) and then G/A^c (with $|V_0| = 1$) by inductions. The only additional requirement is that we need verify $e(z_0, V_0)$ of Condition (iii) for the new z_0 of the graph G/A^c if $V_0 \neq \emptyset$. By the conclusions of Case 1 and Case 2.1 we have that $d(A\backslash V_0) \geq 6k$ since $|A\backslash V_0| \geq 2$. So,

$$e(z_0, V_0) = \frac{d(V_0) + d(A) - d(A\backslash V_0)}{2} < \frac{d(V_0) + 7k - 6k}{2} = \frac{d(V_0) + k}{2} \leq d(V_0) - k$$

(by Condition (iii)). □

Claim 16

$$|V(G)| > 4.$$

Case 1. $|V_0| = 1$. Let x_0 be the vertex of V_0. By Condition (iii) (that $d(x_0) - e(x_0, z_0) \geq k$) and Claim 14 (that $e(x_0, y) < \frac{k-2}{2}$ for every $y \neq z_0$), we have

$$|N(x_0) - z_0| > \frac{d(x_0) - e(x_0, z_0)}{(k-2)/2} \geq \frac{2k}{k-2} > 2.$$

So, $|V(G)| > 4$.

Case 2. $|V_0| = 0$. By Conditions (i), we have $|V(G) - z_0| \geq 2$ and then there exist a vertex $x \in V(G) - z_0$ such that $e(x, z_0) \leq \frac{d(z_0)}{2} < \frac{7k}{2}$. So, by Condition (iv) and Claim 14,

$$|N(x) - z_0| > \frac{2d(x) - 7k}{k-2} \geq \frac{k}{k-2} > 1.$$

Thus, $|V(G) - z_0| \geq 3$. Again, there exist a vertex $y \in V(G) - z_0$ such that $e(y, z_0) \leq \frac{d(z_0)}{3} < 3k$ and

$$|N(x) - z_0| > \frac{2d(x) - 6k}{k-2} \geq \frac{2k}{k-2} > 2$$

(by Claim 14). So, $|V(G)| > 4$. □

Claim 17 *$e(x, z_0) < \frac{d(x)}{2}$ for every vertex $x \in V(G) - z_0$.*

Let $A = V(G) - x - z_0$. Then, by Claim 15, $d(A) \geq 7k$ since $|V(G)\backslash A| = 2 > 1$ and, by Claim 16, $|A| > 2$. So,

$$e(x, z_0) = \frac{d(x) + d(z_0) - d(A)}{2} < \frac{d(x)}{2}$$

since $d(z_0) < 7k$. □

Claim 18 *$\beta(x) \neq 0$ for every vertex $x \in V(G) - z_0$.*

Suppose $\beta(x) = 0$ for some vertex x. Thus, $d(x) \geq 3k$ by Conditions (iii) and (iv) and $d(x)$ is even by the necessary condition (C2). We consider two cases $d(x) \geq 4k + 2$ and $3k \leq d(x) \leq 4k$, separately.

Case 1. $d(x) \geq 4k + 2$.

By Claim 14 and Claim 17, x has at least two neighbors distinct from z_0. We lift one pair of edges incident with x and apply induction to the resulting graph G' with $d'(x) \geq 4k = 4k + \beta'(x)$. If $V_0 \neq \emptyset$, then $e'(z_0, V_0) \leq e(z_0, V_0) + 1$ and the equality holds only if the new edge produced by lifting is between z_0 and V_0. By Claim 17, $e(z_0, V_0) < \frac{d(V_0)}{2} = \frac{d'(V_0)}{2}$. So, $e'(z_0, V_0) < \frac{d'(V_0)}{2} + 1 \leq d'(V_0) - k$ and Condition (iii) still holds.

For any single vertex of $V(G) - z_0$, Conditions (iv) clearly hold. And for any non-trivial vertex subset A described in Condition (iv), $d(A)$ remains the same or decreases by two. By Claim 15, it still satisfies Condition (iv).

Applying induction to the smaller graph G', an extension of D_{z_0} exists, and, it can be considered as an extension of the original graph G.

Case 2. $3k \leq d(x) \leq 4k$.

It follows from Claims 14 and Claim 17 that x has at least two neighbors distinct from z_0. Moreover, no edge-multiplicity of an edge incident with x is greater than the sum of others incident with x, and, therefore, we can successively lift the edges incident with x and keep (at each stage) that property.

Let G' be the resulting graph with $\beta'(y) = \beta(y)$ for every vertex $y \in V(G')$, where $V(G') = V(G) - x$ and $|V'(G)| \geq 4$(By Claim 16).

If $V_0 \neq \emptyset$, then $e(x, V_0) \leq \frac{k-2}{2}$ by Claim 14, and $e(z_0, V_0) < \frac{d(V_0)}{2} = \frac{d'(V_0)}{2}$ by Claim 17. So, we have that

$$e'(z_0, V_0) \leq e(z_0, V_0) + e(x, V_0) < \frac{d'(V_0)}{2} + \frac{k}{2} \leq d'(V_0) - k$$

and Condition (iii) is still satisfied.

For any single vertex of G' not contained in V_0 and other than z_0, Conditions (iv) clearly hold. Now Condition (iv) is to be verified for any non-trivial vertex subset A of G' described in Condition (iv).

If $|A| = 2$, say $A = \{x_1, x_2\}$, then $d'(A) \geq d(A) - e(x, x_1) - e(x, x_2)$. By Claim 14 and Claim 15, we have that $d'(A) \geq 6k - \frac{k-2}{2} - \frac{k-2}{2} > 5k > 4k + \beta'(A)$.

If $|A| > 2$, then, by Claim 15, $d(A) \geq 7k$ and $d(A + x) \geq 7k$ since $|V(G) \backslash (A + x)| = |V(G') \backslash A| > 1$.

We have $(d(A) - d'(A)) + (d(A + x) - d'(A)) \leq d(x)$.

So,

$$d'(A) \geq 7k - \frac{d(x)}{2} = 5k > 4k + \beta'(A).$$

Since both Conditions (iii) and (iv) are verified, we can apply induction on the smaller graph G': an extension of D_{z_0} exists for G'. And, it can be further considered as an extension of the original graph G. □

The final step.

By Claim 18,

$$\beta(v) > 0$$

for every vertex v distinct from z_0.

If $V_0 = \emptyset$, then we choose an arbitrary vertex x other than z_0. And if $V_0 \neq \emptyset$, then we choose x be the unique element of V_0. We are to apply pre-directing and/or lifting operations to edges of $E(x)$.

Let $y \in N(x) - z_0$ be a neighbor of x. We pre-direct xy from y to x. Let G' and β' be the resulting graph and the modified boundary with $V'_0 = \{x\}$. Here, $\beta'(y) = \beta(y) - 1$. So, for any single vertex distinct from x and z_0, Condition (iv) still holds.

Now consider a non-trivial vertex subset A of G' described in Condition (iv). It is obvious that, by Claim 15, we have

$$d'(A) \geq 6k - 1 > 5k > 4k + \beta'(A).$$

By Claim 17 we have

$$e'(z_0, x) = e(z_0, x) < \frac{d(x)}{2} = \frac{d'(x) + 1}{2} \leq d'(x) - k.$$

Note that $\beta(x) > 0$ and $d(x) \geq 4k - \beta(x)$ by Condition (iii) and Claim 18. So, if $0 < \beta(x) \leq k - 2$, then

$$\begin{cases} \beta'(x) &= \beta(x) + 1 > 0 \\ d'(x) &= d(x) - 1 \geq 4k - \beta(x) - 1 = 4k - \beta'(x) \end{cases}$$

and, if $\beta(x) = k - 1$, then

$$\begin{cases} \beta'(x) &= 0 \\ d'(x) &= d(x) - 1 \geq 4k - \beta(x) - 1 = 3k \end{cases}$$

Condition (iii) is therefore verified.

This contradicts that G is a counterexample since an extension of D_{z_0} to G' can be considered as an extension to the entire graph G. ■

3.5 Remarks

Definition 3.5.1 *Let G be a graph, $k > 0$ be an integer and $\theta : V(G) \mapsto Z_k$ be a function such that $\sum_{v \in V(G)} \theta(x) \equiv |E(G)| \pmod{k}$. An orientation D of G is a θ-orientation such that, for every vertex $x \in V(G)$,*

$$d^+(x) \equiv \theta(x) \pmod{k}.$$

Note that for a Z_k-boundary β of G with odd integer k, a θ-orientation of G is exactly a β-orientation if $2\theta(x) \equiv d(x) + \beta(x) \pmod k$ for every vertex $x \in V(G)$.

Theorem 3.1.6 is a reformulation of the following, which is Theorem 2 in [56]:

Theorem 3.5.2 (Thomassen [56]) *Let G be a graph, $k \geq 3$ be an integer and $\theta : V(G) \mapsto Z_k$ be a function such that $\sum_{v \in V(G)} \theta(x) \equiv |E(G)| \pmod k$. If G is $(2k^2 + k)$-edge-connected, then G has a θ-orientation.*

Theorem 3.5.2 also implies the following tree-decomposition conjecture proposed by Barát and Thomassen when restricted to stars.

Conjecture 3.5.3 (Barát and Thomassen [4]) *For each tree T, there exists an integer k_T such that if G is k_T-edge-connected and $|E(T)|$ divides $E(G)$, then G has a T-decomposition.*

Denote $\mathcal{M}_k$ be the collection of all graphs having a β-orientation for every Z_k-boundary β of G. And denote $\mathcal{N}_k$ be the collection of all graphs having a θ-orientation for every function $\theta : V(G) \mapsto Z_k$ such that $\sum_{x \in V(G)} \theta(x) \equiv |E(G)| \pmod k$.

An easy calculation shows the following relation between $\mathcal{N}_k$ and $\mathcal{M}_k$.

Proposition 3.5.4 *Let G be a graph and $k > 0$ be an integer.*

(i)

$$G \in \mathcal{N}_k \iff G \in \mathcal{M}_{2k}.$$

(i) If k is odd, then

$$G \in \mathcal{N}_k \iff G \in \mathcal{M}_k.$$

Theorem 3.1.6 follows from Theorem 3.5.2 and Proposition 3.5.4. With Proposition 3.5.4 and Theorem 3.1.7, we have the following strengthening of Theorem 3.5.2.

Corollary 3.5.5 *Let G be a graph, let k be an integer, $k \geq 3$, and let $\theta : V(G) \mapsto Z_k$ be a function such that $\sum_{v \in V(G)} \theta(x) \equiv |E(G)| \pmod k$. If G is $10k$-edge-connected, then G has a θ-orientation.*

Chapter 4

Final Remarks

By Proposition 3.5.4-(i) we have $G \in \mathcal{N}_3 \iff G \in \mathcal{M}_6$ and by Proposition 3.5.4-(ii) we have $G \in \mathcal{N}_3 \iff G \in \mathcal{M}_3$. So $G \in \mathcal{M}_6 \iff G \in \mathcal{M}_3$, that is, a graph G is Z_3-connected (particularly G admits a nowhere-zero 3-flow) if and only if it has a β-orientation for every Z_6-boundary β. As we mentioned in Section 2.5 we proved that a graph is Z_3-connected if $d(A) \geq 3 + |\tau(A)|$ for every non-empty, proper vertex subset A of G but here the constant 3 can not be lowered to constant 2 since there are counterexamples which are 4-edge-connected. However it is easy to check that those counterexamples with Z_6 boundary $\beta = 4$ are not $(2 + \beta)$-edge-connected. Therefore we give the following conjecture.

Conjecture 4.0.6 *Let G be a graph, and $\beta : V(G) \mapsto \{0, 1, \cdots, 5\}$ be a Z_6-boundary of G. If G is $(2 + \beta)$-edge-connected, then it has a β-orientation.*

Let G be a graph with no 1-edge-cut and no 3-edge-cut, and let β be a Z_6 boundary of G such that $\beta(v) = 0$ if $d(v)$ is even and $\beta(v) = 3$ if $d(v)$ is odd. It is easy to check that G is $(2 + \beta)$-edge connected. G admits a nowhere-zero 3-flow is equivalent that G has a β-orientation. So, if Conjecture 4.0.6 is true, then Tutte's 3-flow conjecture follows.

As we can see in the proofs of of Theorem 2.3.1 and Theorem 3.3.1 we apply lifting or/and pre-directing operations which are local reductions and the resulting graphs still satisfy certain edge connectivity corresponding to τ or β. But local reductions may not works to prove Conjecture 4.0.6. Consider a 5-regular, 5-edge-connected graph G such that the Z_3 boundary β is a constant 3 for every vertex. G is $(2+\beta)$-edge-connected but the resulting graphs are not after any local reductions.

We say a proper vertex subset A of G is trivial if $|A| = 1$ or $|V(G)\backslash A| = 1$. A graph G with Z_6 boundary β is called essential β-edge-connected if $d(v) \geq 2 + \beta(v)$ or $d(V(G) - v) \geq 2 + \beta(V(G) - v)$ for any vertex v and $d(A) \geq 2 + \beta(A)$ for any non-trivial proper vertex subset A of G. We can give a stronger conjecture which local reductions may still work to prove.

Conjecture 4.0.7 *Let G be a graph, and β be a Z_6-boundary of G. If G is essential $(2+\beta)$-edge-connected, then it has a β-orientation.*

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