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Eulerian Subgraphs and Hamiltonicity of Claw-free Graphs

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ABSTRACT

Eulerian Subgraphs and Hamiltonicity of Claw-free Graphs

Mingquan Zhan

Let $C(l,k)$ denote the class of 2-edge-connected graphs of order n such that a graph $G \in C(l,k)$ if and only if for every edge cut $S \subseteq E(G)$ with $|S| \leq 3$, each component of $G - S$ has order at least $\dfrac{n-k}{l}$. We prove that If $G \in C(6,0)$, then G is supereulerian if and only if G cannot be contracted to $K_{2,3}$, $K_{2,5}$ or $K_{2,3}(e)$, where $e \in E(K_{2,3})$ and $K_{2,3}(e)$ stands for a graph obtained from $K_{2,3}$ by replacing e by a path of length 2. Previous results by Catlin and Li, and by Broersma and Xiong are extended.

We also investigate the supereulerian graph problems within planar graphs, and we prove that if a 2-edge-connected planar graph G is at most three edges short of having two edge-disjoint spanning trees, then G is supereulerian except a few classes of graphs. This is applied to show the existence of spanning Eulerian subgraphs in planar graphs with small edge cut conditions. We determine several extremal bounds for planar graphs to be supereulerian.

Kuipers and Veldman conjectured that any 3-connected claw-free graph with order n and minimum degree $\delta \geq \frac{n+6}{10}$ is Hamiltonian for n sufficiently large. We prove that if H is a 3-connected claw-free graph with sufficiently large order n, and if $\delta(H) \geq \frac{n+5}{10}$, then either H is hamiltonian, or $\delta(H) = \frac{n+5}{10}$ and the Ryjáček's closure $cl(H)$ of H is the line graph of a graph obtained from the Petersen graph P_{10} by adding $\frac{n-15}{10}$ pendant edges at each vertex of P_{10}.

Acknowledgements

DEDICATION

To

my parents

and

my wife

Contents

Chapter 1

Definitions and Notations

We use [1] for terminology and notations not defined here and consider finite, undirected graphs. We allow graphs to have multiple edges but not loops. A graph with at least two vertices is called a **nontrivial** graph.

Let G be a graph. We denote an n-cycle by C_n. We use $\kappa(G)$, $\kappa'(G)$ to denote the connectivity and the edge-connectivity of G, respectively. The **degree** and **neighborhood** of a vertex x of G are respectively denoted by $d_G(x)$ and $N_G(x)$, and the **minimum degree** of G is denoted by $\delta(G)$. An edge $e = uv$ is called a **pendant edge** if either $d_G(u) = 1$ or $d_G(v) = 1$. If $S \subseteq V(G)$ and $X_1 \subseteq E(G)$, $G[S]$ and $G[X_1]$ are the subgraphs induced in G by S and X_1, respectively. A vertex $v \in G$ is called a **locally connected vertex** if $G[N_G(v)]$ is connected. For $H \subseteq G$, $x \in V(G)$ and $A, B \subseteq V(G)$ with $A \cap B = \emptyset$, let $N_H(x) = N_G(x) \cap V(H)$, $d_H(x) = |N_H(x)|$, $N_H(A) = \bigcup_{v \in A} N_H(v)$, $[A, B]_G = \{uv \in E(G) \mid u \in A, v \in B\}$, and $G - A = G[V(G) - A]$. When $A = \{v\}$, we use $G - v$ for $G - \{v\}$. If $H \subseteq G$, then for an edge subset $X \subseteq E(G) - E(H)$, we write $H + X$ for $G[E(H) \cup X]$.

A graph G is **essentially k-edge-connected** if $|E(G)| \geq k + 1$ and if for every $E_0 \subseteq E(G)$ with $|E_0| < k$, $G - E_0$ has exactly one component H with $E(H) \neq \emptyset$. The greatest integer k such that G is essentially k-edge-connected is the **essential edge-connectivity** $\kappa_e(G)$ of G. For each $i = 0, 1, 2, \cdots$, let $D_i(G) = \{v \in V(G) | d_G(v) = i\}$,

and for any integer $t \geq 1$, let $D_t^*(G) = \bigcup_{i \geq t} D_i(G)$. The **edge arboricity** $a_1(G)$ of G is the minimum number of edge-disjoint forests whose union equals G. The **girth** of G, denoted by $g(G)$, is the length of a shortest cycle of G, or ∞ if G is acyclic.

Let $O(G)$ denote the set of odd degree vertices of G. A graph G is **Eulerian** if $O(G) = \emptyset$ and G is connected. A graph G is **supereulerian** if G has a spanning Eulerian subgraph. In particular, K_1 is both eulerian and supereulerian. Denote by $\mathcal{SL}$ the family of all supereulerian graphs.

Let $X \subseteq E(G)$. The **contraction** G/X is the graph obtained from G by identifying the two ends of each edge in X and then deleting the resulting loops. We define $G/\emptyset = G$. If K is a subgraph of G, then we write G/K for $G/E(K)$. Note that even if G is a simple graph, contracting some edges of G may result in a graph with multiple edges. If K is a connected subgraph of G, and if v_K is the vertex in G/K onto which K is contracted, then K is called the **preimage** of v_K, and is denoted by $PI(v_K)$. A vertex v in a contraction of G is **nontrivial** if $PI(v)$ has at least one edge.

The **line graph** of a graph G, denote by $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent.

A subgraph H of a graph G is **dominating** if $G - V(H)$ is edgeless. Let $v_0, v_k \in V(G)$. A $(\mathbf{v_0}, \mathbf{v_k})-$**trail** of G is a vertex-edge alternating sequence

$$v_0, e_1, v_1, e_2, \cdots, e_k, v_k$$

such that all the e_i's are distinct and such that for each $i = 1, 2, \cdots, k$, e_i is incident with both v_{i-1} and v_i. With the notation above, this (v_0, v_k)-trail is also called an $(\mathbf{e_1}, \mathbf{e_k})$-**trail**. All the vertices in $v_1, v_2, \cdots, v_{k-1}$ are internal vertices of trail. A **dominating** $(\mathbf{e_1}, \mathbf{e_k})$-**trail** T of G is an (e_1, e_k)-trail such that every edge of G is incident with an internal vertex of T. A **spanning** $(\mathbf{e_1}, \mathbf{e_k})$-**trail** of G is a dominating (e_1, e_k)-trail such that $V(T) = V(G)$. There is a close relationship between dominating eulerian subgraphs in graphs and Hamilton cycles in $L(G)$.

Theorem 1.0.1 *(Harary and Nash-Williams, [16])* *Let G be a graph with $|E(G)| \geq 3$. Then $L(G)$ is hamiltonian if and only if G has a dominating eulerian subgraph.*

A graph G is Hamilton-connected if for each $u, v \in V(G)$ ($u \neq v$), there exists a (u, v)-path containing all vertices of G. With a similar argument in the proof of Theorem 1.0.1, one can obtain a theorem for Hamilton-connected line graphs.

Theorem 1.0.2 *Let G be a graph with $|E(G)| \geq 3$. Then $L(G)$ is Hamilton-connected if and only if for any pair of edges $e_1, e_2 \in E(G)$, G has a dominating (e_1, e_2)-trail.*

We say that an edge $e \in E(G)$ is **subdivided** when it is replaced by a path of length 2 whose internal vertex, $v(e)$, has degree 2 in the resulting graph. The process of taking an edge e and replacing it by the length 2 path is called **subdividing e**. For a graph G and edges $e_1, e_2 \in E(G)$, let $G(e_1)$ denote the graph obtained from G by subdividing e_1, and let $G(e_1, e_2)$ denote the graph obtained from G by subdividing both e_1 and e_2. Thus

$$V(G(e_1, e_2)) - V(G) = \{v(e_1), v(e_2)\}.$$

From the definitions, one immediately has the following observation.

Proposition 1.0.3 *Let G be a graph and $e_1, e_2 \in E(G)$. If $G(e_1, e_2)$ has a spanning $(v(e_1), v(e_2))$-trail, then G has a spanning $(e_1, e_2)-trail$.*

Chapter 2

Catlin's Reduction Method

In [4] Catlin defined collapsible graphs. For $R \subseteq V(G)$, a subgraph Γ of G is called an **R−subgraph** if both $O(\Gamma) = R$ and $G − E(\Gamma)$ is connected. A graph is **collapsible** if G has an R−subgraph for every even set $R \subseteq V(G)$. In particular, K_1 is collapsible. Let $\mathcal{CL}$ denote the family of all collapsible graphs. For a graph G and its connected subgraph H, G/H denotes the graph obtained from G by contracting H, i.e. by replacing H by a vertex v_H such that the numbers of edges in G/H joining any $v \in V(G) − V(H)$ to v_H in G/H equals the number of edges joining v in G to H. A graph G is **contractible** to a graph G' if G contains pairwise vertex-disjoint connected subgraphs H_1, H_2, $\cdots$, H_k with $\bigcup_{i=1}^{k} V(H_i) = V(G)$ such that G' is obtained from G by successively contracting H_1, H_2, $\cdots$, H_k. The subgraph H_i of G is called the **pre-image** of the vertex v_{H_i} of G'. Catlin [4] showed that every graph G has a unique collection of pairwise vertex-disjoint maximal collapsible subgraphs H_1, H_2, $\cdots$, H_k such that $\bigcup_{i=1}^{k} V(H_i) = V(G)$. The **reduction** of G is the graph obtained from G by successively contracting H_1, H_2, $\cdots$, H_k. A graph is **reduced** if it is the reduction of some graph. A **nontrivial vertex** in G' is a vertex that is the contraction image of a nontrivial connected subgraph of G. Note that if G has an $O(G)$-subgraph Γ, then $G − E(\Gamma)$ is a spanning Eulerian subgraph of G. Therefore, every collapsible graph is supereulerian.

Theorem 2.0.4 *(Catlin, [4]) Let G be a connected graph.*

(i) If G has a spanning tree T such that each edge of T is in a collapsible subgraph of G, then G is collapsible.

(ii) If G is reduced, then G is a simple graph and has no cycle of length less than four.

(iii) G is reduced if and only if G has no nontrivial collapsible subgraphs.

(iv) Let G' be the reduction of G. Then $G \in \mathcal{SL}$ if and only if $G' \in \mathcal{SL}$, and $G \in \mathcal{CL}$ if and only if $G' = K_1$.

Jaeger in [17] showed that if G has two edge-disjoint spanning trees, then G is supereulerian. Letting $F(G)$ be the minimum number of additional edges that must be added to G so that the resulting graph has two edge-disjoint spanning trees, Catlin [4] and Catlin *et al.* [8] improved Jaeger's result. We combine these results in the following theorem.

Theorem 2.0.5 *Let G be a graph. Then each of the following holds.*

(i)(Jaeger, [17]) If $F(G) = 0$, then G is supereulerian.

(ii)(Catlin, [4]) If $F(G) \le 1$ and if G is connected, then G is collapsible if and only if G is not contractible to a K_2.

(iii)(Catlin, Han and Lai, [8]) If $F(G) \le 2$ and if G is connected, then either G is collapsible, or the reduction of G is a K_2 or a $K_{2,t}$ for some integer $t \ge 1$.

Theorem 2.0.6 *(Catlin, Han and Lai, [8]) Let G be a connected reduced graph. If $F(G) \le 2$, then exactly one of following holds:*

(i) $G \in \mathcal{SL}$;

(ii) G has a cut edge;

(iii) G is $K_{2,s}$ for some odd integer $s \ge 3$.

In 1987, Catlin [6] introduced a reduction technique of the 4-cycle. Let G be a graph and let $C = v_1 v_2 v_3 v_4 v_1$ be a 4-cycle of G. Let G_π denote the graph obtained from $G - E(C)$ by identifying v_1 and v_3 to form a single vertex w_1, and by identifying v_2 and v_4 to form a single vertex w_2, and by joining w_1 and w_2 with a new edge e_π.

Theorem 2.0.7 *Each of the following holds:*

(i) (Catlin, [6]) If G_π is collapsible, then G is collapsible; if G_π is supereulerian, then G is supereulerian.

(ii) (Catlin, Han and Lai, [8]) If G is reduced, then $F(G_\pi) = F(G) - 1$.

(iii) (Catlin, [6]) Let $K_{3,3} - e$ denote the graph obtained from $K_{3,3}$ by removing an edge. Then $K_{3,3} - e, K_n(n \geq 3)$ and C_2 are collapsible.

(iv) (Catlin, [6]) If G is reduced, then $a_1(G) \leq 2$; if $a_1(G) \leq 2$, then $F(G) = 2|V(G)| - |E(G)| - 2$.

Applying Theorem 2.0.7(i), we have the following lemma.

Lemma 2.0.8 *Let G_1, G_2, G_3 be the graphs given in Figure 2.1. Then G_1, G_2 and G_3 are collapsible.*

$$G_1 \qquad\qquad G_2 \qquad\qquad G_3$$
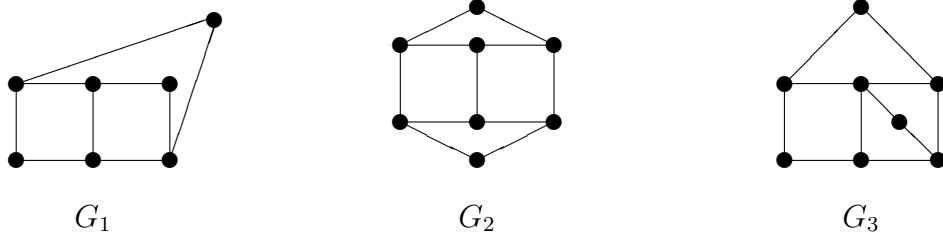
Figure 2.1

Let t, s_1, s_2, s_3 be nonnegative integers. Denote $K'_{2,t}$, $K''_{2,t}$, $K_{1,3}(s_1, s_2, s_3)$, $S(s_1, s_2)$, $J(s_1, s_2)$, $J'(s_1, s_2)$ to be graphs shown in Figure 2.2, where the s_i $(i = 1, 2, 3)$ vertices and the two vertices connected by the two lines shown in each of the graphs forms a K_{2,s_i} graph.

$$K_{1,3}(s_1, s_2, s_3) \qquad S(s_1, s_2) \qquad J(s_1, s_2) \qquad J'(s_1, s_2)$$
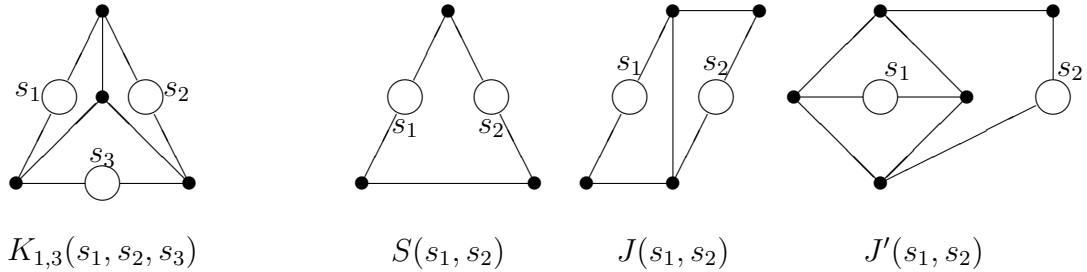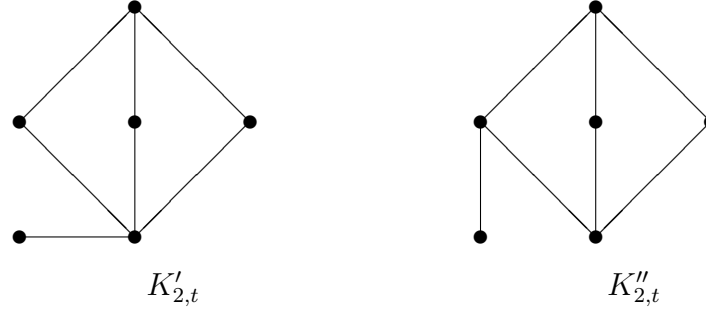
$$K'_{2,t} \qquad\qquad K''_{2,t}$$

Figure 2.2

Let

$$\mathcal{F} = \{K_1,\ K_2,\ K_{2,t},\ K'_{2,t},\ K''_{2,t},\ K_{1,3}(s_1, s_2, s_3),\ S(s_1, s_2),\ J(s_1, s_2),\ J'(s_1, s_2),\ P\},$$

where P denotes the Petersen graph.

Theorem 2.0.9 *(Chen and Lai, [12]) If G is connected reduced graph with $|V(G)| \leq 11$ and $F(G) \leq 3$, then $G \in \mathcal{F}$.*

Theorem 2.0.10 *(Chen, [10]) Let G be a reduced graph with $n \leq 11$ vertices, and $\kappa'(G) \geq 3$. Then G is either K_1 or the Petersen graph.*

Lemma 2.0.11 *Let G be a connected simple graph with $n \leq 8$ vertices and with $D_1(G) = \emptyset$, $|D_2(G)| \leq 2$. Then either G is one of three graphs in Figure 2.3, or the reduction of G is K_1 or K_2.*
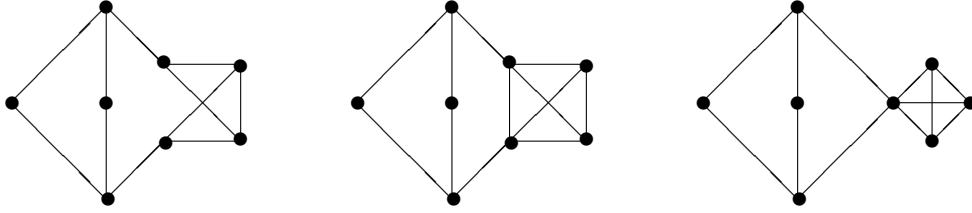
Figure 2.3

Proof. By Theorem 2.0.10, we may assume $\kappa'(G) \in \{1, 2\}$. Suppose first that G has a cut-edge e and that G_1 and G_2 are two components of $G - e$. Let $n_i = |V(G_i)|$, $1 \le i \le 2$ and assume, without loss of generality, $n_2 \ge n_1$. Since G is simple and $D_1(G) = \emptyset$, $n_1 \ge 3$ and equality holds if and only if $G_1 = K_3$. If both $n_2 = n_1 = 4$, then each of G_1 and G_2 must be K_4 or $K_4 - e$ where $e \in E(K_4)$. Since K_3 is collapsible and by Theorem 2.0.4(i), both G_1 and G_2 are collapsible, and so the reduction of G is K_2. Since $n_2 = 3$ will force $|D_2(G)| \ge 3$, we assume that $4 \le n_2 \le 5$ and $n_1 = 3$. If G_2 is not collapsible, then $G_2 \in \{C_4, C_5, K_{2,3}\}$ and $|D_2(G)| \ge 3$. So G_2 must be collapsible. Hence the reduction of G is K_2.

Now we assume that G is 2-edge-connected and G' is the reduction of G with $n' = |V(G')| \ge 2$. Then G' is 2-edge-connected and nontrivial. Let C_m be a longest cycle in G'. Then $m \ge 4$ by Theorem 2.0.4(ii).

If $n' = 8$ or 7, then $G = G'$. As $|D_2(G)| \le 2$, we have $F(G') \le 3$ by Theorem 2.0.7(iv). By Theorem 2.0.9 and $\kappa'(G') \ge 2$, $|D_2(G)| \ge 3$, a contradiction. If $n' = 6$, then either $G = G'$ or the pre-image of a vertex in G' is a triangle and the pre-images of the other vertices in G' are themselves. Thus $|D_2(G')| \le 2$. By Theorem 2.0.7(iv), we have $F(G') \le 2$. Therefore $|D_2(G')| \ge 3$ by Theorem 2.0.5(iii), a contradiction. If $n' = 4$, then $G' = C_4$. Note that the size of the pre-image of each vertex is either 1 or at least 3. Thus $|D_2(G)| \ge 3$. It contradicts the hypothesis that $|D_2(G)| \le 2$. So $n' = 5$. Note again that the size of the pre-image of each vertex is either 1 or at least 3. By $|D_2(G)| \le 2$, $|D_3(G')| \ne 0$. Thus $F(G') \le 2$. By Theorem 2.0.5(iii), $G' = K_{2,3}$. As $n \le 8$ and $|D_2(G)| \le 2$, the pre-image of a vertex having degree 2 in G' is either a K_4 or a K_4 minus an edge, and the pre-images of the other vertices in G' are themselves. Thus G is

one of the graphs in Figure 2.3. $\square$

Lemma 2.0.12 *If G is collapsible, then for any pair of vertices $u, v \in V(G)$, G has a spanning (u, v)-trail.*

Proof. Let $R = (O(G) \cup \{u, v\}) - (O(G) \cap \{u, v\})$. Then $|R|$ is even. Let Γ_R be an R−subgraph of G. Then $G - E(\Gamma_R)$ is a spanning (u, v)-trail of G. $\square$

Let G be a graph and let $S \subseteq V(G)$ be a vertex subset. A eulerian subgraph H of G is called an *S*-**eulerian subgraph** if $S \subseteq V(H)$. Let $K_{2,3}, K_{2,5}, W'_3, W'_4, L_1, L_2$ and L_3 be the labelled graphs defined in Figures 2.4-2.6, and let $\mathcal{F} = \{K_{2,3}, K_{2,5}, W'_3, W'_4, L_1, L_2, L_3\}$. Using the labels in Figures 2.4-2.6, for each $L \in \mathcal{F}$, we define $B(L)$, the bad set of L, to be the vertex subset of $V(L)$ that are labeled with the $b'_i s$.
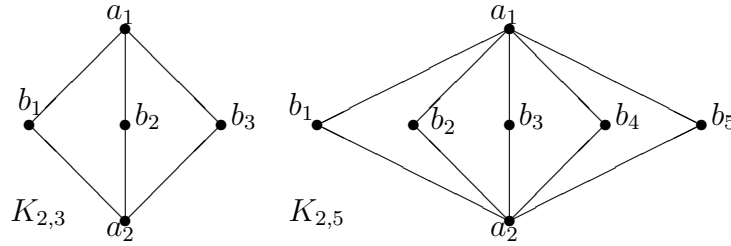
Figure 2.4 The graphs $K_{2,3}$ and $K_{2,5}$
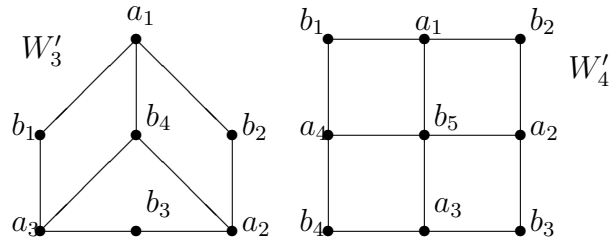
Figure 2.5 The graphs W'_3 and W'_4
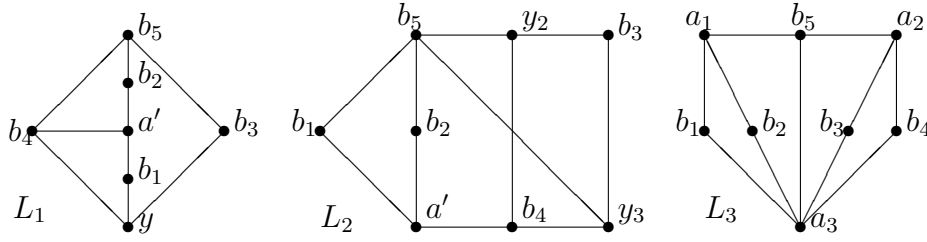
Figure 2.6 The graphs L_1, L_2 and L_3

Theorem 2.0.13 *(Lai, [20]) Let G be a 2-edge-connected graph and let $S \subseteq V(G)$ with $|S| \leq 5$. If $G - S$ is edgeless, and if G does not have an S-eulerian subgraph, then G is contractible to a member $L \in \mathcal{F}$ such that S intersects the preimage of every vertex in $B(L)$.*

Lemma 2.0.14 *Suppose that G is a simple graph which does not contain $K_4 - e$ as a subgraph. Then the following statements hold.*

(i) If $|V(G)| = 3$, then $|E(G)| \leq 3$.
(ii) If $|V(G)| = 4$, then $|E(G)| \leq 4$.
(iii) If $|V(G)| = 5$, then $|E(G)| \leq 6$.
(iv) If $|V(G)| = 6$, then $|E(G)| \leq 9$.
(v) If $|V(G)| = 7$, then $|E(G)| \leq 12$.

Proof. If $|V(G)| = 3$, then $|E(G)| \leq 3$. If $|V(G)| = 4$, then $|E(G)| \leq 4$ since G does not contain $K_4 - e$ as its subgraph. Note that the proofs for (iii),(iv) and (v) are similar. So we only present the proof for (v).

Let $V(G) = \{u_1, u_2, \cdots, u_7\}$. Without loss of generality, we assume that $d_G(u_7) = \Delta(G)$. If $d_G(u_7) \leq 3$, then it is clear that (v) holds. Suppose that $d_G(u_7) = 6$. Then the degree of $u_i (i = 1, 2, \cdots, 6)$ is at most 1 in $G - u_7$ since G does not contain $K_4 - e$ as its subgraph. Thus $|E(G - u_7)| \leq 3$ and $|E(G)| \leq 6 + 3 = 9$.

Suppose that $d_G(u_7) = 5$ and $u_7 u_6 \notin E(G)$. Then the degree of $u_i (i = 1, 2, \cdots, 5)$ is at most 1 in $G - \{u_7, u_6\}$. Let $p_1 = |E(G - \{u_6, u_7\})|$. Then $d_G(u_6) \leq 5 - p_1$ since G does not contain $K_4 - e$ as its subgraph. Thus $|E(G)| = d_G(u_6) + d_G(u_7) + |E(G - \{u_6, u_7\})| \leq 5 + (5 - p_1) + p_1 = 10$.

Suppose that $d_G(u_7) = 4$ and $u_7 u_6, u_7 u_5 \notin E(G)$. Then the degree of $u_i (i = 1, 2, 3, 4)$ is at most 1 in $G - \{u_7, u_6, u_5\}$. Let $p_2 = |E(G - \{u_5, u_6, u_7\})|$. Then $|[\{u_1, u_2, u_3, u_4\}, \{u_i\}]_G|$ $\leq 4 - p_2$ $(i = 5, 6)$ and $p_2 \leq 2$ since G does not contain $K_4 - e$ as its subgraph. If $p_2 = 0$ and $u_5 u_6 \in E(G)$, then $|[\{u_1, u_2, u_3, u_4\}, \{u_5, u_6\}]_G| \leq 5$. Thus $|E(G)| = d_G(u_7) + |[\{u_1, u_2, u_3, u_4\}, \{u_5, u_6\}]_G| + |E(G - \{u_5, u_6, u_7\})| + |E(G[\{u_5, u_6\}])| \leq 4 + 5 + 0 + 1 = 10$. Otherwise, we have $|E(G)| = d_G(u_7) + |[\{u_1, u_2, u_3, u_4\}, \{u_5\}]_G| + |[\{u_1, u_2, u_3, u_4\}, \{u_6\}]_G| + |E(G - \{u_5, u_6, u_7\})| + |E(G[\{u_5, u_6\}])| \leq 4 + (4 - p_2) + (4 - p_2) + p_2 + |E(G[\{u_5, u_6\}])| = 12 - p_2 + |E(G[\{u_5, u_6\}])| \leq 12$. $\square$

Lemma 2.0.15 *Suppose that G is a 2-edge-connected graph with at most 10 vertices, and that G does not contain $K_4 - e$ as a subgraph. If $|E(G)| \geq 17$, then G is collapsible.*

Proof. By Theorem 2.0.5(ii), if H is a simple collapsible subgraph of G with $|V(H)| = 4$, then H must contain contain $K_4 - e$ as a subgraph. Thus we have

if H is a simple collapsible subgraph of G, then $|V(H)| \geq 3$ and $|V(H)| \neq 4$. $\qquad$ (2.1)

Let G' be the reduction of G. Note that G is collapsible if and only if $G' = K_1$. Suppose, by contradiction, that $G' \neq K_1$. Then $\kappa'(G') \geq 2$ and $|V(G')| \leq 10$. If $\kappa'(G') \geq 3$, then by Theorem 2.0.10 and by $|E(G)| \geq 17$, we must have $G' = K_1$, a contradiction. So we may assume that $\kappa'(G') = 2$ and let $X \subseteq E(G')$ be an edge cut of G' with $|X| = 2$. Pick an $e \in X$, and denote $[e] = \{e' \in E(G') \mid \{e, e'\}$ is an edge cut of $G'\} \cup \{e\}$. Then for any $\{e_1, e_2\} \subseteq [e]$, $\{e_1, e_2\}$ is also an edge cut of G'. Let $|[e]| = k \geq 2$. Then G' has k connected subgraphs $G_1, G_2, \cdots, G_k$ such that G_i, G_{i+1} $(i = 1, 2, \cdots, k-1)$ and G_1, G_k are joined by exactly one edge in $[e]$ (see Figure 2.7), and each G_i $(i = 1, 2, \cdots, k)$ is either a K_1 or a nontrivial 2-edge-connected graph. Thus, if G_i is nontrivial, then $|V(G_i)| \geq 4$, and $F(G_i) \geq 2$ by Theorem 2.0.5(ii). If follows by Theorem 2.0.7(iv) that $|E(G_i)| \leq 2|V(G_i)| - 4$. Let

$$t = |\{G_i | G_i \text{ is nontrivial}, 1 \leq i \leq k\}|.$$

Since $|V(G)| \leq 10$, we have $0 \leq t \leq 2$.
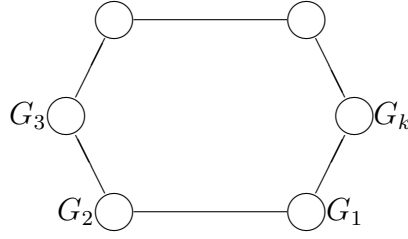
Figure 2.7

Case 1. $t = 0$.

Let $V(G_i) = \{v_i\}$ $(i = 1, 2, \cdots, k)$ and let s be the number of nontrivial vertices in $\{v_1, v_2, \cdots, v_k\}$. Assume that for each $i = 1, \cdots, s$, v_i is a nontrivial vertex and $|V(PI(v_1))| \leq |V(PI(v_2))| \leq \cdots |V(PI(v_s))|$. Since G' is the reduced graph, we have $k \geq 4$. Since $|V(G)| \leq 10$, $s \leq 3$. By (2.1), $|V(PI(v_i))| \neq 4$. If $s = 3$, then $|V(PI(v_i))| = 3$ and $k = 4$. Thus $17 \leq |E(G)| = \sum_{i=1}^{3} |E(PI(v_i))| + k = 9 + 4 = 13$ by Lemma 2.0.14(i), a contradiction. If $s = 2$, then $|V(PI(v_1))| = 3$, and either $|V(PI(v_2))| = 3$ or $|V(PI(v_2))| = 5$. Thus

$$17 \leq |E(G)| = \sum_{i=1}^{2} |E(PI(v_i))| + k = \begin{cases} 3 + 3 + k \leq 6 + 6 = 12, & \text{if } |V(PI(v_2))| = 3 \\ 3 + 6 + k \leq 9 + 4 = 13, & \text{if } |V(PI(v_2))| = 5 \end{cases}, \text{ a}$$

contradiction. If $s = 1$, then $|V(PI(v_1))| = 3, 5, 6, 7$. Thus $|E(G)| = |E(PI(v_1))| +$

$$k = \begin{cases} 3 + k \leq 3 + 8 = 11, & \text{if } |V(PI(v_1))| = 3 \\ 6 + k \leq 6 + 6 = 12, & \text{if } |V(PI(v_1))| = 5 \\ 9 + k \leq 9 + 5 = 14, & \text{if } |V(PI(v_1))| = 6 \\ 12 + k \leq 12 + 4 = 16, & \text{if } |V(PI(v_1))| = 7 \end{cases}, \text{ contrary to the assumption that}$$

$|E(G)| \geq 17$.

Case 2. $t = 1$.

Without loss of generality, we assume that G_1 is the nontrivial subgraph in G'. Then $|E(G_1)| \leq 2|V(G_1)| - 4$. Since $|V(G_1)| \geq 4$ and $|V(G)| \leq 10$, we have $2 \leq k \leq 7$.

Case 2.1. $8 \leq |V(G_1)| \leq 9$

In this case, $G_1 \subseteq G$, $k \leq 11 - |V(G_1)|$ and for each $v \in V(G') - V(G_1)$, v must be trivial. Thus $17 \leq |E(G)| = |E(G_1)| + k \leq (2|V(G_1)| - 4) + (11 - |V(G_1)|) = 7 + |V(G_1)| \leq$

16, a contradiction.

Case 2.2. $6 \le |V(G_1)| \le 7$

If v is trivial for each $v \in V(G')$, then $k \le 11 - |V(G_1)|$ and $17 \le |E(G)| = |E(G_1)| + k \le 7 + |V(G_1)| \le 14$, a contradiction. So there exists some vertex $v \in V(G')$ such that v is not trivial. By (2.1), $|V(PI(v))| \ne 4$ and $|V(PI(v))| \ge 3$. Note that $|V(G)| = |V(G_1)| + |V(PI(v))| + (k - 2) \le 10$ and $k \ge 2$. Then we have $|V(PI(v))| = 3$. By Lemma 2.0.14(i), $|E(PI(v))| = 3$. Thus $17 \le |E(G)| = |E(G_1)| + |E(PI(v))| + k \le (2|V(G_1)| - 4) + 3 + (12 - |V(G_1)| - |V(PI(v))|) = 8 + |V(G_1)| \le 15$, a contradiction.

Case 2.3. $|V(G_1)| = 5$

If v is trivial for each $v \in V(G')$, then $k \le 6$ and $|E(G)| = |E(G_1)| + k \le (2|V(G_1)| - 4) + k \le 6 + 6 = 12$, a contradiction. So $V(G')$ must have a nontrivial vertex. Let $q_1 = |\{v \in V(G') \mid v \text{ is a nontrivial vertex}\}|$.

Suppose that $q_1 = 1$ and v is a nontrivial vertex. Then $|V(PI(v))| = 3$ or 5 by (2.1). If $|V(PI(v))| = 3$, then $k \le 4$. Thus $17 \le |E(G)| = |E(G_1)| + |E(PI(v))| + k \le (2|V(G_1)| - 4) + 3 + 4 = 6 + 7 = 13$, a contradiction. If $|V(PI(v))| = 5$, then $k = 2$. Thus $17 \le |E(G)| = |E(G_1)| + |E(PI(v))| + k \le (2|V(G_1)| - 4) + 6 + 2 = 6 + 8 = 14$, a contradiction.

Suppose that $q_1 \ge 2$ and that $v_2, \cdots, v_{q_1+1}$ are nontrivial vertices. By (2.1), we have $k = 2$, $q_1 = 2$, $|V(PI(v_i))| = 3$ $(i = 2, 3)$. Thus $17 \le |E(G)| = |E(G_1)| + |E(PI(v_2))| + |E(PI(v_3))| + k \le (2|V(G_1)| - 4) + 3 + 3 + 2 = 6 + 8 = 14$, a contradiction.

Case 2.4. $|V(G_1)| = 4$

If v is trivial for each $v \in V(G')$, then $k \le 7$ and $17 \le |E(G)| = |E(G_1)| + k \le (2|V(G_1)| - 4) + k = 4 + 7 = 11$, a contradiction. So $V(G')$ must have a nontrivial vertex. Let $q_2 = |\{v \in V(G') \mid v \text{ is a nontrivial vertex }\}|$.

Suppose that $q_2 = 1$ and v is a nontrivial vertex. Then $|V(PI(v))| = 3, 5, 6$ by (2.1). If $|V(PI(v))| = 3$, then $k \le 5$. Thus $17 \le |E(G)| = |E(G_1)| + |E(PI(v))| + k \le (2|V(G_1)| - 4) + 3 + 5 = 4 + 8 = 12$, a contradiction. If $|V(PI(v))| = 5$, then $k \le 3$.

Thus $17 \leq |E(G)| = |E(G_1)| + |E(PI(v))| + k \leq (2|V(G_1)| - 4) + 6 + 3 = 4 + 9 = 13$, a contradiction. If $|V(PI(v))| = 6$, then $k = 2$. Thus $17 \leq |E(G)| = |E(G_1)| + |E(PI(v))| + k \leq (2|V(G_1)| - 4) + 9 + 2 = 4 + 11 = 15$, a contradiction.

Suppose that $q_2 \geq 2$ and that $v_2, \cdots, v_{q_2+1}$ are nontrivial vertices. By (2.1), we have $k \leq 3$, $q_2 = 2$ and $|V(PI(v_i))| = 3$ $(i = 2, 3)$. Thus $17 \leq |E(G)| = |E(G_1)| + |E(PI(v_1))| + |E(PI(v_2))| + k \leq (2|V(G_1)| - 4) + 3 + 3 + 3 = 4 + 9 = 13$, a contradiction.

Case 3. $t = 2$.

Without loss of generality, we assume that G_1, G_2 are the nontrivial subgraphs in G'. If there exists $v \in V(G_1) \cup V(G_2)$ such that v is a nontrivial vertex in G', then $k = 2$, $V(G') = V(G_1) \cup V(G_2)$, $|V(G_1)| = |V(G_2)| = 4$ and $|PI(v)| = 3$. Thus we have $17 \leq |E(G)| = |E(G_1)| + |E(G_2)| + |E(PI(v))| + k \leq 4 + 4 + 3 + 2 = 13$, a contradiction. So $V(G_i) \subseteq V(G)(i = 1, 2)$. Note that $|V(G_i)| \geq 4$, we have $k \leq 4$. Since $|V(G_1)| + |V(G_2)| \leq 10$ and $|E(G_i)| \leq 2|V(G_i)| - 4$, we have $|E(G_1)| + |E(G_2)| \leq 12$. Thus $17 \leq |E(G)| = |E(G_1)| + |E(G_2)| + k \leq 12 + 4 = 16$, a contradiction. $\square$

Chapter 3

Eulerian subgraphs and Hamilton-connected line graphs

3.1 Introduction

For integers $k \geq 0$ and $l > 0$, let $C(l, k)$ denote the class of 2-edge-connected graphs of order n such that $G \in C(l, k)$ if and only if for every edge cut $S \subseteq E(G)$ with $|S| \leq 3$, each component of $G - S$ has order at least $\dfrac{n - k}{l}$.

Catlin and Li, and Broersma and Xiong proved the following results concerning when a graph in the family $C(l, k)$ is supereulerian.

Theorem 3.1.1 *(Catlin and Li, [9]) If $G \in C(5, 0)$, then $G \in \mathcal{SL}$ if and only if G is not contractible to $K_{2,3}$.*

Theorem 3.1.2 *(Broersma and Xiong, [2]) If $G \in C(5, 2)$ and $n \geq 13$, then $G \in \mathcal{SL}$ if and only if G is not contractible to $K_{2,3}$ or $K_{2,5}$.*

In this chapter, we further study the distribution of the small degree vertices in the

reduction of a graph, and sharpen Theorems 3.1.1 and 3.1.2, as shown in Theorem 3.1.3 and Corollary 3.1.4.

Theorem 3.1.3 *If $G \in C(6, 0)$, then $G \in \mathcal{SL}$ if and only if G is not contractible to $K_{2,3}$, $K_{2,5}$ or $K_{2,3}(e)$, where $e \in E(K_{2,3})$.*

Note that when $n \geq 6k + 1$, $C(5, k) \subseteq C(6, 0)$. So we have

Corollary 3.1.4 *If $G \in C(5, k)$ and $n \geq 6k + 1$, then $G \in \mathcal{SL}$ if and only if G is not contractible to $K_{2,3}$ or $K_{2,5}$.*

We investigate the Hamilton-connectedness of line graphs of graphs in $C(l, k)$ and get the following.

Theorem 3.1.5 *If $G \in C(6, 0)$ and $n \geq 7$, then $L(G)$ is Hamilton-connected if and only if $\kappa(L(G)) \geq 3$.*

Corollary 3.1.6 *If $G \in C(5, k)$ and $n \geq \max\{6k + 1, 6\}$, then $L(G)$ is Hamilton-connected if and only if $\kappa(L(G)) \geq 3$.*

3.2 Proofs of Theorems 3.1.3 and 3.1.5

Proof of Theorem 3.1.3. Let G' be the reduction of G. If $G' = K_1$, then $G \in \mathcal{SL}$ by Theorem 2.0.4(iv). Next we suppose that $G' \neq K_1$. Then G' is 2-edge-connected and nontrivial. Denote $d_i = |D_i(G')|(i \geq 2)$.

If $d_2 + d_3 \geq 7$, then we assume that $v_1, v_2, \cdots, v_7$ are the vertices of $V(G')$ in $D_2(G') \cup D_3(G')$, i.e. $d_{G'}(v_i) \leq 3$ for each i, and the corresponding pre-images are $H_1, H_2, \cdots, H_7$.

Each H_i is joined to the rest of G by an edge cut consisting of $d_{G'}(v_i) \leq 3$ edges. By the hypothesis of Theorem 3.1.3, $|V(H_i)| \geq \dfrac{n}{6}$, and

$$n = |V(G)| \geq \sum_{i=1}^{7} |V(H_i)| \geq \frac{7n}{6},$$

a contraction. Therefore we assume $d_2 + d_3 \leq 6$, and when $d_2 + d_3 = 6$, $V(G') = D_2(G') \cup D_3(G')$. We break the proof into two cases.

Case 1. $F(G') \leq 2$.

By $\kappa'(G') \geq 2$ and by Theorem 2.0.6, $G' \in \mathcal{SL}$ or $G' = K_{2,s}$ where $s \geq 3$ is an odd integer. In the former case, $G \in \mathcal{SL}$ by Theorem 2.0.4(iv). In the latter, $s = 3$ or $s = 5$ by $d_2 + d_3 \leq 6$.

Case 2. $F(G') \geq 3$.

Note that $|V(G')| = \sum_{i \geq 2} d_i$, $2|E(G')| = \sum_{i \geq 2} i d_i$. By Theorem 2.0.7(iv), we have the following

$$2d_2 + d_3 \geq 10 + \sum_{i \geq 5}(i - 4)d_i. \tag{3.1}$$

Since $d_2 + d_3 \leq 6$, $d_2 \geq 4$. We distinguish two cases to complete the proof.

Case 2.1. $d_2 = 4$.

By (3.1) and $d_2 + d_3 \leq 6$, $d_3 = 2$. Thus $V(G') = D_2(G') \cup D_3(G')$. Let $D_2(G') = \{u_1, u_2, u_3, u_4\}$ and $D_3(G') = \{v_1, v_2\}$. If $v_1 v_2 \in E(G')$, then $E(G') = \{u_1 u_2, u_2 v_2, v_2 u_3, u_3 u_4, u_4 v_1, v_1 u_1, v_1 v_2\}$ by Theorem 2.0.4(ii). Thus $G' \in \mathcal{SL}$ and so $G \in \mathcal{SL}$. If $v_1 v_2 \notin E(G')$, then $G' = K_{2,3}(e)$, where $e \in E(K_{2,3})$.

Case 2.2. $d_2 = 5$ or 6.

If $d_2 = 5$, then $d_3 = 0$ and $d_i = 0 (i \geq 5)$ by (3.1). If $d_2 = 6$, then $d_i = 0 (i \geq 3)$. Thus $G' \in \mathcal{SL}$ and so $G \in \mathcal{SL}$.

Proof of Theorem 3.1.5. It is trivial that $\kappa(L(G)) \geq 3$ if $L(G)$ is Hamilton-connected.

So we only need to prove that $L(G)$ is Hamilton-connected when $\kappa(L(G)) \geq 3$.

Let $e_1, e_2 \in E(G)$. By Theorem 1.0.2, Proposition 1.0.3 and Lemma 2.0.12, we need to prove $G(e_1, e_2) \in \mathcal{CL}$. Let G' be the reduction of $G(e_1, e_2)$. By Theorem 2.0.4(iv), it suffices to prove that $G' = K_1$. Suppose that $G' \neq K_1$. Then G' is 2-edge-connected and nontrivial. Denote $d_i = |D_i(G')|(i \geq 2)$.

If $d_2 \geq 3$, then there exists $v \in D_2(G') - \{v(e_1), v(e_2)\}$ such that $d_{G'}(v) = 2$. Let H be the pre-image of v in $G(e_1, e_2)$. Then H is joined to the rest of $G(e_1, e_2)$, therefore of G, by an edge-cut consisting of $d_{G'}(v) = 2$ edges. By the hypothesis of Theorem 3.1.5, $|V(H)| \geq \dfrac{n}{6} > 1$. Thus $\kappa(L(G)) \leq 2$, a contradiction. So $d_2 \leq 2$.

If $d_2 + d_3 \geq 9$, then $|D_2(G') \cup D_3(G') - \{v(e_1), v(e_2)\}| \geq 7$. We assume that $v_1, v_2, \cdots, v_7$ are the vertices of $V(G')$ in $D_2(G') \cup D_3(G') - \{v(e_1), v(e_2)\}$, i.e. $d_{G'}(v_i) \leq 3$ for each i, and the corresponding pre-images are $H_1, H_2, \cdots, H_7$. Each H_i is joined to the rest of $G(e_1, e_2)$, therefore to the rest of G, by an edge cut consisting of $d_{G'}(v_i) \leq 3$ edges. By the hypothesis of Theorem 3.1.5, $|V(H_i)| \geq \dfrac{n}{6}$, and

$$n = |V(G)| \geq \sum_{i=1}^{7} |V(H_i)| \geq \frac{7n}{6},$$

a contraction. So $d_2 + d_3 \leq 8$, and when $d_2 + d_3 = 8$, $V(G') = D_2(G') \cup D_3(G')$.

Suppose that $F(G') \geq 3$, i.e., $2|V(G')| - |E(G')| \geq 5$ by Theorem 2.0.7(iv). Note that $|V(G')| = \sum_{i \geq 2} d_i$, $2|E(G')| = \sum_{i \geq 2} i d_i$, we have the following

$$2d_2 + d_3 \geq 10 + \sum_{i \geq 5} (i - 4)d_i. \tag{3.2}$$

By (3.2) and $d_2 + d_3 \leq 8$, $d_2 = 2$ and $d_3 = 6$. Thus $|V(G')| = 8$. As G' is 2-edge-connected nontrivial reduced graph, we have $G' = K_{2,3}$ by Lemma 2.0.11, a contradiction. So $F(G') \leq 2$.

As G' is 2-edge-connected and $d_2 \leq 2$, $G \neq K_2, K_{2,t}(t \geq 1)$. Thus by Theorem 2.0.5(iii), $G' = K_1$, a contradiction.

3.3 Applications

Theorem 3.1.3 and Corollary 3.1.6 have a number of applications.

Theorem 3.3.1 *(Zhan, [27]) Let G be a graph. If $\kappa'(G) \geq 4$, then $L(G)$ is Hamilton-connected.*

Theorem 3.3.2 *(Broersma and Xiong, [2]) Let G be a 2-edge-connected simple graph with $n \geq 13$ vertices. If $\delta(G) \geq 4$ and if*

$$\min\{\max\{d(x), d(y)\}|xy \in E(G)\} \geq \frac{n-2}{5} - 1,$$

then $G \in \mathcal{SL}$ if and only if G is not contractible to $K_{2,3}$ or $K_{2,5}$.

Theorem 3.3.3 *Let G be a 2-edge-connected simple graph. If $\delta(G) \geq 4$ and if*

$$\min\{\max\{d(x), d(y)\}|xy \in E(G)\} \geq \frac{n}{6} - 1,$$

then $G \in \mathcal{SL}$ if and only if G is not contractible to $K_{2,3}$, $K_{2,5}$ or $K_{2,3}(e)$.

Proof. Let S be an edge cut of G with $|S| \leq 3$, and let G_1 and G_2 be the two components of $G - S$ with $|V(G_1)| \leq |V(G_2)|$. It is sufficient to prove that $|V(G_1)| \geq \frac{n}{6}$ by Theorem 3.1.3. Since $\delta(G) \geq 4$, G_1 has at least an edge, say uv, such that both of u, v are not incident with any edges of S. By the hypothesis of Theorem 3.3.3,

$$|V(G_1)| \geq \max\{d(u), d(v)\} + 1 \geq \frac{n}{6}$$

Thus Theorem 3.3.3 follows from Theorem 3.1.3.

Obviously, Theorem 3.3.3 improves Theorem 3.3.2 and the following results.

Corollary 3.3.4 *Let G be a 2-edge-connected simple graph with $n \geq 6k + 1$ vertices. If $\delta(G) \geq 4$ and if*
$$\min\{\max\{d(x), d(y)\}|xy \in E(G)\} \geq \frac{n-k}{5} - 1,$$
then $G \in \mathcal{SL}$ if and only if G is not contractible to $K_{2,3}$ or $K_{2,5}$.

Corollary 3.3.5 *G be a 2-edge-connected simple graph. If $\delta(G) \geq 4$ and if every edge $uv \in E(G)$ satisfies*

$$d(u) + d(v) \geq \frac{n}{3} - 2,$$

then $G \in \mathcal{SL}$ if and only if G is not contractible to $K_{2,3}$, $K_{2,5}$ or $K_{2,3}(e)$.

Corollary 3.3.6 *Let G be a 2-edge-connected simple graph of order $n > 24$. If*

$$\delta(G) \geq \frac{n}{6} - 1,$$

then $G \in \mathcal{SL}$ if and only if G is not contractible to $K_{2,3}$, $K_{2,5}$ or $K_{2,3}(e)$.

Chapter 4

Supereulerian planar graphs

4.1 Introduction

In 1989, Catlin [5] conjectured that if $F(G) \leq 3$ and if G is 3-edge-connected, then G is collapsible if and only if the reduction of G is not the Petersen graph. Noting that planar graphs cannot be contracted to the Petersen graph, we in this chapter prove the following.

Theorem 4.1.1 *Let G be a 3-edge-connected planar graph. If $F(G) \leq 3$, then G is collapsible.*

$$K_4(s_1, s_2, s_3) \qquad T(s_1, s_2) \qquad C_3(s_1, s_2) \qquad S(s_1, s_2) \qquad K_{2,3}(s_1, s_2)$$
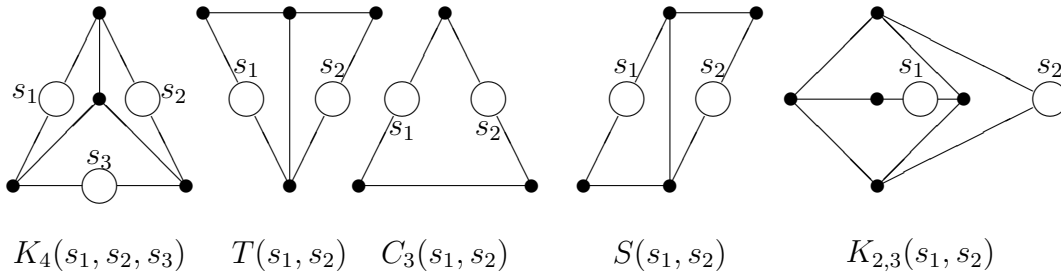
Figure 4.1

Let $s_i \geq 1 (i = 1, 2, 3)$ be integers. Denote $K_4(s_1, s_2, s_3)$, $T(s_1, s_2)$, $C_3(s_1, s_2)$, $S(s_1, s_2)$ and $K_{2,3}(s_1, s_2)$ to be the graphs depicted in Figure 4.1, where the s_i $(i = 1, 2, 3)$ vertices

and the two vertices connected by the two lines shown in each of the graphs forms a K_{2,s_i} graph. Denote

$$\mathcal{F}_1 = \left\{ K_4(s_1, s_2, s_3), T(s_1, s_2), C_3(s_1, s_2), S(s_1, s_2), K_{2,3}(s_1, s_2) \middle| s_i \geq 1 \, (i = 1, 2, 3) \text{ is an integer} \right\}$$

and $\mathcal{F} = \mathcal{F}_1 \cup \{K_{2,t} \mid t \geq 2\}$. Clearly, each graph G in $\mathcal{F}$ is reduced and $F(G) \leq 3$.

Theorem 4.1.2 *Let G be a 2-edge-connected planar graph. If $F(G) \leq 3$, then either G is collapsible or the reduction of G is a graph in $\mathcal{F}$.*

Theorem 4.1.3 *Let G be an essentially 3-edge-connected planar reduced graph with $\kappa'(G) \geq 2$. If $F(G) \leq 5$, then G has a dominating Eulerian subgraph H such that $D_3^*(G) \subseteq H$.*

Applying Theorem 4.1.3, we can improve Theorems 3.1.1 and 3.1.2 within planar graphs.

Theorem 4.1.4 *Let G be a simple planar graph of order n with $\kappa'(G) \geq 3$. If $G \in C(16, 0)$, then G is supereulerian.*

In [3], Cai considered this problem: Find the best possible bound $f(n)$ for a simple graph G with n vertices such that if $|E(G)| \geq f(n)$, then G is supereulerian. Let Q_3 denote the cube $(K_2 \times C_4)$ and $Q_3 - v$ denote the cube minus a vertex. Cai proved the following result.

Theorem 4.1.5 *(Cai, [3]) Let G be a simple graph of order $n \geq 5$ and $\kappa'(G) \geq 2$. If*

$$|E(G)| \geq \binom{n-4}{2} + 6,$$

then exactly one of the following holds:

    (i) G is supereulerian.
    (ii) G can be contracted to a $K_{2,3}$.
    (iii) G is the graph $K_{2,5}$ or $Q_3 - v$.

In [3], Cai conjectured that when restricted to the 3-edge-connected simple graphs, the lower bound can be improved. In [7] and [11], Catlin and Chen settled this conjecture.

Theorem 4.1.6 *(Catlin and Chen, [7], [11]) Let G be a simple graph with $n \geq 11$ vertices and with $\kappa'(G) \geq 3$. If*

$$|E(G)| \geq \binom{n-9}{2} + 16,$$

then G is collapsible.

Graphs contractible to the Petersen graph indicate the sharpness of this result. In this chapter, we prove the following related results among planar graphs.

Theorem 4.1.7 *Let G be a planar graph with n vertices, with $\kappa'(G) \geq 3$ and $g(G) \geq 4$. If $|E(G)| \geq 2n - 5$, then G is collapsible.*

Theorem 4.1.7 cannot be relaxed to 2-edge-connected planar graphs since $K_{2,t}$ is not collapsible.

Theorem 4.1.8 *Let G be a planar graph with $\kappa'(G) \geq 3$ such that every edge of G is in a face of degree at most 6. If either G has at most two faces of degree 5 and no faces of degree bigger than 5, or G has exactly one face of degree 6 and no other faces of degree bigger than 4, then G is collapsible.*

Theorem 4.1.8 is related to a former conjecture of Paulraja ([23], [24]): If G is a 2-connected graph with $\delta(G) \geq 3$ such that every edge of G lies in a cycle of length at most 4, then G is supereulerian. This conjecture was proved in [19].

Theorem 4.1.9 *If G is a 2-edge-connected simple planar graph with order $n \geq 6$ and $|E(G)| \geq 3n - 8$, then $F(G) = 0$.*

Theorem 4.1.10 *If G is a 2-edge-connected simple planar graph with $n \geq 9$ vertices and $|E(G)| \geq 3n - 12$ edges, then exactly one of the following holds:*

   (i) G is supereulerian.

   (ii) G has a maximal collapsible subgraph H with order $n - 4$ such that G/H is a $K_{2,3}$.

In this chapter, we present the proofs of Theorem 4.1.1 and Theorem 4.1.2 in Section 4.2. Theorems 4.1.2 and 4.1.1 will be applied to prove Theorems 4.1.3 and 4.1.4 in Section 4.3, to prove Theorems 4.1.7 and 4.1.8 in Section 4.4, respectively. The proofs for Theorems 4.1.9 and 4.1.10 are in Section 4.5.

4.2   Proofs of Theorem 4.1.1 and Theorem 4.1.2

Proof of Theorem 4.1.1.   Let G be a 3-edge-connected planar graph with $F(G) \leq 3$ and let G' denote its reduction. By Theorem 2.0.4(iv), we only need to show that $G' = K_1$.

By contradiction. Suppose that G' is nontrivial. Since G is 3-edge-connected and planar, G' is also 3-edge-connected and planar. Since every spanning tree of G will become a connected spanning subgraph in any contraction of G, $F(G') \leq F(G) \leq 3$. We may assume that G' is embedded on the plane.

By Theorem 2.0.4(ii) and (iii), G' is reduced and cannot have any cycles of length 2 or 3, and so G' is a simple plane graph each of whose face has degree at least 4. Let f denote the number of faces of G' and let f_i denote the number of faces of G' having degree i, where $i \geq 1$ is an integer. Note that $f_1 = f_2 = f_3 = 0$, and so we have

$$4f + \sum_{i=5}^{\infty}(i-4)f_i = \sum_{i=4}^{\infty} i f_i = 2|E(G')|. \tag{4.1}$$

By (4.1) and by Euler's formula,

$$2|E(G')| = 2|V(G')| + 2f - 4 = 2|V(G')| + |E(G')| - 4 - \frac{1}{2}\sum_{i=5}^{\infty}(i-4)f_i. \tag{4.2}$$

Thus $|E(G')| = 2|V(G')| - 4 - \frac{1}{2}\sum_{i=5}^{\infty}(i-4)f_i$. On the other hand, since $F(G') \leq 3$ and by Theorem 2.0.7(iv), $|E(G')| \geq 2|V(G')| - 5$. It follows that

$$2|V(G')| - 5 \leq |E(G')| = 2|V(G')| - 4 - \frac{1}{2}\sum_{i=5}^{\infty}(i-4)f_i. \tag{4.3}$$

If $|E(G')| = 2|V(G')| - 4$, then by Theorem 2.0.7(iv), $F(G') = 2$, and so by Theorem 2.0.5(iii) and by the fact that G' is 3-edge-connected, G' must be a collapsible graph. Thus $G' = K_1$ by Theorem 2.0.4(iv). This contradicts the assumption that G' is nontrivial. Therefore, to obtain our final contradiction, we only need to show that $|E(G')| = 2|V(G')| - 5$ is impossible.

If $|E(G')| = 2|V(G')| - 5$, then by (4.3), we have

$$\text{either } f_4 = f - 2 \text{ and } f_5 = 2, \text{ or } f_4 = f - 1 \text{ and } f_6 = 1. \tag{4.4}$$

Since $\kappa'(G') \geq 3$, we must have $f_4 \geq 1$. Let $C = v_1v_2v_3v_4v_1$ denote a 4-cycle of G' and consider G'_π. Since $\kappa'(G') \geq 3$, G'_π is connected. Moreover,

$$\text{if } \kappa'(G'_\pi) \leq 2, \text{ then } e_\pi = w_1w_2 \text{ is in an edge cut of size at most 2 in } G'_\pi. \tag{4.5}$$

Suppose first that $\kappa'(G'_\pi) \geq 2$. Then by Theorem 2.0.7(ii), $F(G'_\pi) \leq 2$. It follows by Theorem 2.0.5(iii) that either G'_π is collapsible, whence G' is collapsible by Theorem 2.0.7(i), contrary to the assumption that G' is reduced; or the reduction of G'_π is a $K_{2,t}$ for some integer $t \geq 2$, whence G' has an edge cut of size 2, contrary to the fact that $\kappa'(G') \geq 3$.

Therefore by (4.5), e_π must be the only cut edge of G'_π. Let G'_1 and G'_2 be the two components of $G'_\pi - e_\pi$ with $w_1 \in V(G'_1)$ and $w_2 \in V(G'_2)$. Then $G' - E(C)$ has two components G_1 and G_2 with $v_1, v_3 \in V(G_1)$ and $v_2, v_4 \in V(G_2)$ such that G'_1 can be obtained from G_1 by identifying v_1 and v_3, and G'_2 can be obtained from G_2 by identifying v_2 and v_4. Since G' is reduced, both G_1 and G_2 are reduced.

If $F(G_1) \leq 2$, then by Theorem 2.0.5(iii) and by the fact that G_1 is reduced, G_1 is either a K_2 or a $K_{2,t}$, for some integer $t \geq 1$. If $G_1 = K_2$, then G' has a 3-cycle, contrary to Theorem 2.0.4(ii); if $G_1 = K_{2,t}$, then G' has an edge cut of two edges, contrary to the assumption that $\kappa'(G') \geq 3$. Therefore $F(G_1) \geq 3$. Similarly, $F(G_2) \geq 3$.

By Theorem 2.0.7(iv), $|E(G_i)| \leq 2|V(G_i)| - 5$ for both $i = 1$ and $i = 2$. It follows by $F(G') \leq 3$ and by Theorem 2.0.7(iv) that

$$2|V(G')| - 9 \;=\; |E(G')| - 4 = \sum_{i=1}^{2} |E(G_i)|$$

$$\leq 2(\sum_{i=1}^{2} |V(G_i)| - 5) = 2|V(G')| - 10,$$

a contradiction. Thus G' must be a K_1 and so G is collapsible. $\square$

Proof of Theorem 4.1.2. Suppose that G is not collapsible, and G' is the reduction of G. Then $G' \neq K_1$ and $F(G') \leq 3$. By Theorem 4.1.1, $\kappa'(G') = 2$. We apply induction on $n = |V(G')|$ to prove $G' \in \mathcal{F}$.

Clearly, $n \geq 4$. If $n = 4$, then $G = K_{2,2}$ and the result holds. We suppose that the result holds for fewer vertices.

Note that $\kappa'(G') = 2$. Let $X \subseteq E(G')$ be an edge cut of G' with $|X| = 2$. Pick an $e \in X$, and denote $[e] = \{e' \in E(G') | \{e, e'\}$ is an edge cut of $G'\} \cup \{e\}$. Then for any $\{e_1, e_2\} \subseteq [e]$, $\{e_1, e_2\}$ is also an edge cut of G. Let $|[e]| = k \geq 2$. Then there are k connected subgraphs $H_1, H_2, \cdots, H_k$ such that H_i, H_{i+1} ($i = 1, 2, \cdots, k - 1$) and H_1, H_k are joined by one edge in $[e]$(see Figure 4.2), and each H_i ($i = 1, 2, \cdots, k$) is either a K_1 or 2-edge-connected.
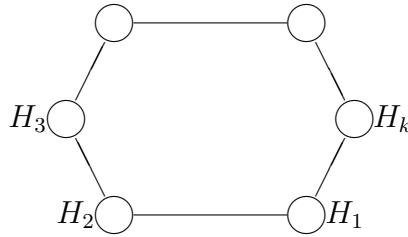
Figure 4.2

Thus, we have

$$
\begin{aligned}
\sum_{i=1}^{k} F(H_i) &= \sum_{i=1}^{k} (2|V(H_i)| - |E(H_i)| - 2) \\
&= 2 \sum_{i=1}^{k} |V(H_i)| - \sum_{i=1}^{k} |E(H_i)| - 2k \\
&= 2|V(G')| - (|E(G')| - k) - 2k = 2|V(G')| - |E(G')| - k \\
&= F(G') - k + 2.
\end{aligned}
$$

We break it into four cases.

Case 1. $k \geq 5$.

Note that $F(G') \leq 3$, we have $F(H_i) = 0$ $(i = 1, 2, \cdots, k)$ and $k = 5$. Thus $H_i = K_1$ $(i = 1, 2, \cdots, k)$ and $G' = C_5 = C_3(1, 1)$.

Case 2. $k = 4$.

Then $\sum_{i=1}^{4} F(H_i) = F(G') - 2$. If $F(G') \leq 2$, then $F(H_i) = 0$ $(i = 1, 2, 3, 4)$ and $G' = C_4 = K_{2,2}$. If $F(G') = 3$, then $\sum_{i=1}^{4} F(H_i) = 1$. Without loss of generality, we assume that $F(H_1) = 1$. By Theorem 2.0.5(ii), either H_1 is nontrivial and collapsible, contrary to the fact that G' is reduced, or $H_1 = K_2$, contrary to the assumption that $\kappa'(G') \geq 2$ or $k = 4$.

Case 3. $k = 3$.

Then $\sum_{i=1}^{3} F(H_i) = F(G') - 1$. Note that a triangle is collapsible, we have $F(G') = 3$, and there doesn't exist some H_i such that $F(H_i) = 1$. Without loss of generality, we let $F(H_1) = 2$ and $F(H_2) = F(H_3) = 0$. Then $H_2 = H_3 = K_1$. Note that H_1 is 2-edge-connected, we have $H_1 = K_{2,t}$ $(t \geq 2)$ by Theorem 2.0.5(iii). Thus G' must be one of the following graphs shown in Figure 4.3.

$$G' = C_3(t, 1)$$

$$G' = C_3(2, 1)(\text{when } t = 2)$$
$$G' = K_{2,3}(1, t - 2)(\text{when } t \geq 3)$$

$$G' = S(t - 1, 1)$$
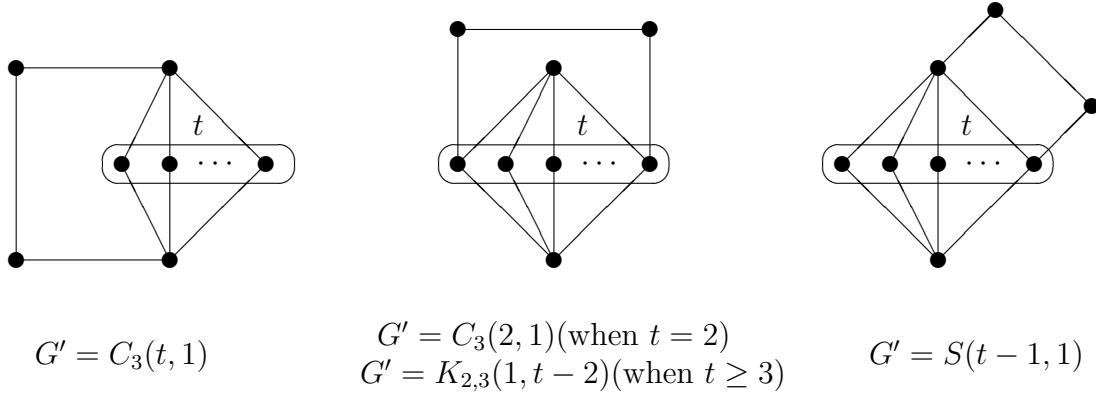
Figure 4.3

Case 4 $k = 2$.

Let H_1, H_2 be the two components of $G' - [e]$ and we assume that $F(H_1) \leq F(H_2)$. Since $\sum_{i=1}^{2} F(H_i) = F(G') \leq 3$, $F(H_1) \leq 1$. Note that for $i = 1, 2$, either $H_i = K_1$ or H_i is 2-edge-connected, we have $H_1 = K_1$ by Theorem 2.0.5(ii), and $H_2 \neq K_1$ since C_2 is collapsible. Therefore $F(H_2) \leq 3$ and $\kappa'(H_2) \geq 2$.

Note that H_2 is reduced, we have $H_2 \in \mathcal{F}$ by induction. Thus there exists a vertex $v \in V(G')$ such that $d_{G'}(v) = 2$ and v is one vertex of a 4-cycle of G'. Let $G_1 = G' - v$. Then $G_1 \in \mathcal{F}$ by induction.

When $G_1 = K_4(s_1, s_2, s_3)$, there are 4 possible way for v to join G_1(see Figure 4.4). Let $\{s_i, s_j, s_k\} = \{s_1, s_2, s_3\}$. Then for the first graph in Figure 4.4, in which case $G' = K_4(s_i + 1, s_j, s_k)$. By Lemma 2.0.8 and Theorem 2.0.7(iii), each of the other graphs in Figure 4.4 contains a collapsible graph, and so G' could not be these three graphs.
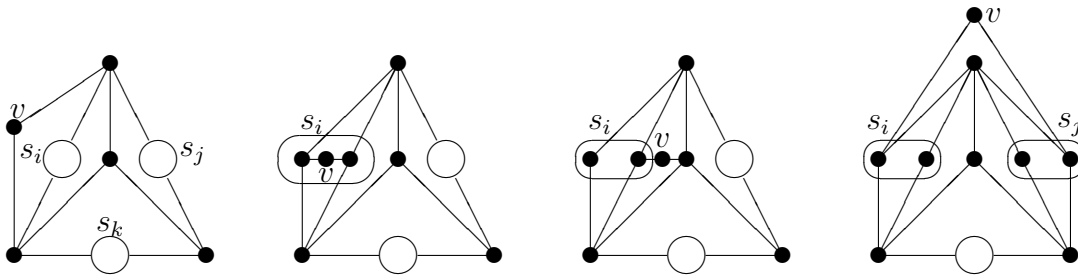
Figure 4.4

Similarly, we can check other 5 cases. If $G_1 = K_{2,t}$ $(t \geq 2)$, then $G' = K_{2,t+1}$. If $G_1 = T(s_1, s_2)$, then $G' \in \{T(s_1 + 1, s_2), T(s_1, s_2 + 1), K_4(s_1, s_2, 1), S(2, s_1), S(2, s_2)\}$. If $G_1 = C_3(s_1, s_2)$, then $G' \in \{C_3(s_1 + 1, s_2), C_3(s_1, s_2 + 1), K_{2,3}(s_1, 1), K_{2,3}(s_2, 1)\}$. If $G_1 = S(s_1, s_2)$, then $G' \in \{S(s_1+1, s_2), S(s_1, s_2+1), T(2, s_1), T(2, s_2), K_4(1, 1, s_1), K_4(1, 1, s_2)\}$. If $G_1 = K_{2,3}(s_1, s_2)$, then $G' \in \{K_{2,3}(s_1 + 1, s_2), K_{2,3}(s_1, s_2 + 1)\}$. $\square$

4.3 Proofs of Theorems 4.1.3 and 4.1.4

The following theorem and lemma are needed in the proof of Theorem 4.1.3.

Theorem 4.3.1 *(Chen et al., [13]) If G is a 3-edge-connected planar graph with $|V(G)| \leq 23$, then G is supereulerian.*

Lemma 4.3.2 *Let $C = v_1 v_2 v_3 v_4 v_1$ be a cycle of a graph G with $N(v_i) - V(C) = \{u_i\}$ $(i = 1, 2, 3, 4)$, and with either $u_1 \neq u_2$ or $u_1 \neq u_3$. If $a_1(G) \leq 2$, then $a_1(G_\pi) \leq 2$.*

Proof.   Let (E_1, E_2) be a partition of $E(G)$ such that each $G[E_i]$ $(i = 1, 2)$ is acyclic, and let

$$E = E(C) \cup \{u_1 v_1, u_2 v_2, u_3 v_3, u_4 v_4\}, \ E'_1 = E_1 - E, \ E'_2 = E_2 - E$$

$$E_{11} = E'_1 \cup \{u_1 v_1, u_4 v_4, v_1 v_2, v_2 v_3\}, \quad E_{12} = E'_2 \cup \{u_2 v_2, u_3 v_3, v_3 v_4, v_4 v_1\}$$

$$E_{21} = E'_1 \cup \{u_1 v_1, u_2 v_2, v_2 v_3, v_3 v_4\}, \quad E_{22} = E'_2 \cup \{u_3 v_3, u_4 v_4, v_1 v_2, v_4 v_1\}$$

$$E_{31} = E'_1 \cup \{u_2 v_2, u_3 v_3, v_3 v_4, v_4 v_1\}, \quad E_{32} = E'_2 \cup \{u_1 v_1, u_4 v_4, v_1 v_2, v_2 v_3\}$$

$$E_{41} = E'_1 \cup \{u_3 v_3, u_4 v_4, v_1 v_2, v_4 v_1\}, \quad E_{42} = E'_2 \cup \{u_1 v_1, u_2 v_2, v_2 v_3, v_3 v_4\}$$

Then for each $i = 1, 2, 3, 4$, (E_{i1}, E_{i2}) is also a partition of $E(G)$ such that each $G[E_{ij}]$ $(j = 1, 2)$ is acyclic. Let $e_\pi = w_1 w_2$ be the new edge in G_π, and for $i = 1, 2, 3, 4$, let

$$E'_{i1} = E_{i1} - E(C), \quad E'_{i2} = (E_{i2} - E(C)) \cup \{e_\pi\}$$

Then (E'_{i1}, E'_{i2}) is a partition of $E(G_\pi)$. Suppose that $a_1(G_\pi) \geq 3$. Note that $G_\pi[E'_{i1}]$ is acyclic by the construction of (E_{i1}, E_{i2}), $G_\pi[E'_{12}]$, $G_\pi[E'_{22}]$, $G_\pi[E'_{32}]$, $G_\pi[E'_{42}]$ contain

cycles $u_2 P_1 u_3 w_1 w_2 u_2$, $u_3 P_2 u_4 w_2 w_1 u_3$, $u_4 P_3 u_1 w_1 w_2 u_4$, $u_1 P_4 u_2 w_2 w_1 u_1$, respectively, where P_1, P_2, P_3, P_4 respectively are (u_2, u_3)-path, (u_3, u_4)-path, (u_4, u_1)-path and (u_1, u_2)-path in G. As either $u_1 \neq u_2$ or $u_1 \neq u_3$, $G[E_2]$ contains a cycle $C \subseteq P_1 \cup P_2 \cup P_3 \cup P_4$, a contradiction. $\square$

Proof of Theorem 4.1.3.   By contradiction, suppose that G is a smallest counterexample. Then G is reduced.

Claim 1. $\kappa'(G) \geq 3$.

Since $\kappa_e(G) \geq 3$, that is, G is essentially 3-edge-connected, we only need to prove that $d_G(v) \geq 3$ for any $v \in V(G)$. Suppose that there exists $v \in V(G)$ such that $d_G(v) = 2$. Let $e_1 = vu_1$, $e_2 = vu_2$. Note that G is reduced, G doesn't contain triangle. Thus $u_1 u_2 \notin E(G)$. Let $G_1 = G/e_1$. Then $\kappa'(G_1) \geq 2$, $\kappa_e(G_1) \geq 3$ and $F(G_1) \leq 5$. Note that G is smallest, there exists a dominating Eulerian subgraph H' in G_1 such that $D_3^*(G_1) \subseteq V(H')$. By $u_1 u_2 \notin E(G)$ and $\kappa_e(G) \geq 3$, we have $d_G(u_1) \geq 3$ and $d_G(u_2) \geq 3$. Thus $d_{G_1}(u_i) \geq 3$ $(i = 1, 2)$ and $u_1, u_2 \in H$. Let
$$H = \begin{cases} H', & \text{if } u_1 u_1 \notin E(H') \\ G[(E(H') - \{u_1 u_2\}) \cup \{vu_1, vu_2\}], & \text{if } u_1 u_2 \in E(H') \end{cases}. \text{ Then } H \text{ is a dominating}$$
Eulerian subgraph of G such that $D_3^*(G) \subseteq H$, a contradiction.

Claim 2. $\kappa_e(G) \geq 4$.

Suppose that S is a 3-edge cut and G_1, G_2 are two components of $G - S$ with $F(G_1) \leq F(G_2)$ and $E(G_1) \neq \emptyset, E(G_2) \neq \emptyset$. Then $F(G_1) + F(G_2) = F(G) + 1 \leq 6$ by Theorem 2.0.7(iv). Thus we have $F(G_1) \leq 3$.

If G_1 has an cut edge e, let H_1 and H_2 be two components of $G_1 - e$ and H_2 be the component adjacent to at least two edges of S. Then either $[V(H_1), V(G) - V(H_1)]_G = e$ or $[V(H_1), V(G) - V(H_1)]_G$ is a 2-edge cut in G, contrary to Claim 1. So we have $\kappa'(G_1) \geq 2$. Note that G_1 is reduced and $|V(G_1)| \geq 2$, by Theorem 4.1.2, $G_1 \in \mathcal{F}$. Since $|D_2(G_1)| \leq 3$ and by planarness of G, $G_1 = K_4(1, 1, 1)$. Similarly, $G_2 = K_4(1, 1, 1)$. So G must be the graph shown in Figure 4.5. Clearly, G is supereulerian, a contradiction.
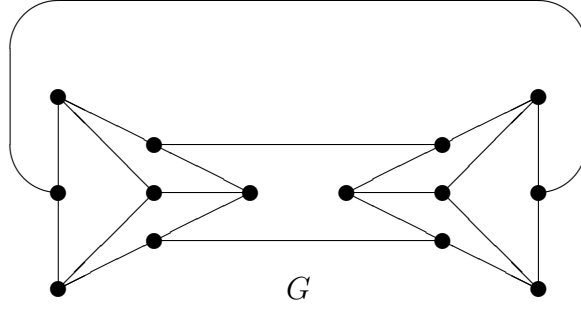
Figure 4.5

Claim 3. G has at least 13 4-faces.

By Claim 1 and Theorem 4.3.1, we have $n = |V(G)| \geq 24$. Let x be the number of 4-faces, m the number of the edges, f the number of faces. Then $2m \geq 4x + 5(f - x)$. Thus $2m \geq 5f - x$. Note that $n - m + f = 2$, we have $5n - 3m \geq 10 - x$. Since $F(G) = 2n - m - 2 \leq 5$, we have $x \geq n - 11 \geq 13$. Thus G has at least 13 4-faces.

Claim 4. No two 4-faces $C_1 = v_1v_2v_3v_4v_1$ and $C_2 = v_1v_2v_5v_6v_1$ in G satisfy $d_G(v_i) = 3(i = 1, 2 \cdots, 6)$.

By contradiction. Suppose that there exist two 4-faces $C_1 = v_1v_2v_3v_4v_1$ and $C_2 = v_1v_2v_5v_6v_1$ in G such that $d_G(v_i) = 3$ $(i = 1, 2 \cdots, 6)$. Let $H = G - \{v_1, v_2, \cdots, v_6\}$. Then we can get the new graph $(G_\pi)_\pi$ by using π−collapsible 2 times (see the graphs in Figure 4.6).
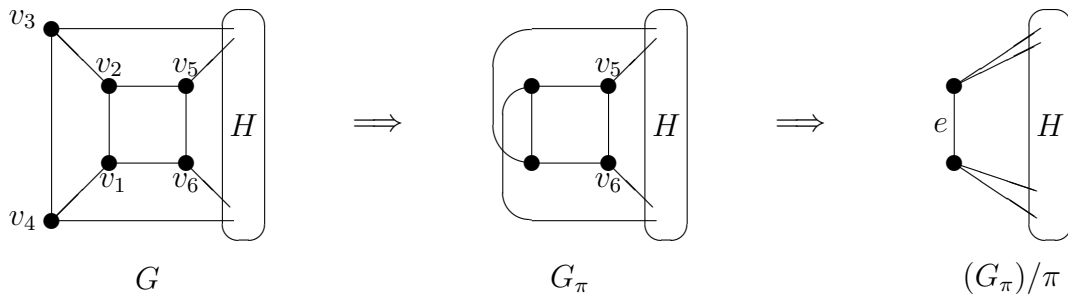
Figure 4.6

Let $e = w_1w_2$ denote the new edge in $(G_\pi)_\pi$. Clearly, $(G_\pi)_\pi$ is 2-edge-connected, otherwise, e is a cut edge of $(G_\pi)_\pi$ and $X = \{v_1v_2, v_3v_4, v_5v_6\}$ is a 3-edge cut in G in which both sides of $G - X$ have edges, contrary to Claim 2. Next we want to prove that G is 3-edge-connected. Suppose that G is not 3-edge-connected. Then $\{e, e_1\}$ is a 2-edge cut, where $e_1 = w_3w_4 \in E(H)$. Let H_1, H_2 be two components of $(G_\pi)_\pi - \{e, e_1\}$, and $w_1, w_3 \in V(H_1)$, $w_2, w_4 \in V(H_2)$. Then $E(H_1 - \{w_1\}) = \emptyset$ and $E(H_2 - \{w_2\}) = \emptyset$ by Claim 2. Thus $N_{(G_\pi)_\pi}(w_1) = w_3$ and $N_{(G_\pi)_\pi}(w_2) = w_4$. Hence $V(H) = \{w_3, w_4\}$, $N_G(v_3) \cap H = N_G(v_5) \cap H = \{w_3\}$ and $N_G(v_6) \cap H = N_G(v_4) \cap H = \{w_4\}$. So G is supereulerian, a contradiction. Thus G is 3-edge-connected. Clearly, $(G_\pi)_\pi$ is planar with $F((G_\pi)_\pi) = F(G) - 2 \leq 3$ by Lemma 4.3.2 and Theorem 2.0.7(iv). By Theorem 4.1.2, $(G_\pi)_\pi$ is supereulerian. Thus G is supereulerian by Theorem 2.0.7(i), a contradiction. So Claim 4 holds.

Claim 5. Suppose that $C = v_1v_2v_3v_4v_1$ is a 4-face of G. Then for $i = 1, 2, 3, 4$, $d_G(v_i) = 3$.

Let $G_1 = (G - \{v_1v_2, v_3v_4\})/\{v_1v_4, v_2v_3\}$ and $G_2 = (G - \{v_1v_4, v_2v_3\})/\{v_1v_2, v_3v_4\}$. First we prove that either $\kappa_e(G_1) \geq 3$ or $\kappa_e(G_2) \geq 3$. Suppose that $\kappa_e(G_i) = 2$ for $i = 1, 2$. Then G must have the structure in Figure 4.7.

G

Figure 4.7

For $i = 1, 2, 3$, $|[V(H_i'), V(H_{i+1}')]_G| = 1$ and $|[V(H_4'), V(H_1')]_G| = 1$. Let $H_i = H_1' \cup \{v_i\}$ $(i = 1, 2, 3, 4)$. Then

$$\sum_{i=1}^{4} F(H_i) = \sum_{i=1}^{4} (2|V(H_i)| - |E(H_i)| - 2)$$
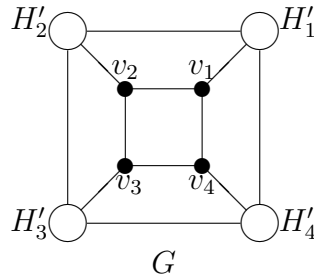$$= 2n - (m - 8) - 8 = 2n - m = F(G) + 2 \leq 7.$$

Note that $|V(H_i)| \geq 2$, there are at least three of these H_i's, say H_1, H_2, H_3, such that $F(H_i) \leq 2$ $(i = 1, 2, 3)$. Since $\kappa'(G) \geq 3$, we have $H_i \neq K_{2,t}$ $(i = 1, 2, 3)$. Thus $H_1 = H_2 = H_3 = K_2$ by Theorem 2.0.5(iii). This contradicts Claim 4. So without loss generality, we assume that $\kappa_e(G_1) \geq 3$ and w_1, w_2 are two new vertices. By the assumption of G, G_1 has a dominating Eulerian subgraph H' such that $D_3^*(G_1) \subseteq H'$. If either $w_1 \in H'$ or $w_2 \in H'$, then we can always get a dominating Eulerian subgraph H of G such that $D_3^*(G) \subseteq H$, it is impossible. Thus $w_1, w_2 \notin H'$. Therefore Claim 5 holds.

By Claim 3, let C_1, C_2, C_3, C_4, C_5 be five 4-faces of G. By Claims 4,5, no two of these 4-faces have common vertices or edges. Applying $\pi-$collapsible to each of these five 4-faces, we get the graph G_3 with $F(G_3) = 0$ by Lemma 4.3.2 and Theorem 2.0.7(iv). Note that G_3 is connected, G_3 is supereulerian by Theorem 2.0.5(i). Thus G is also supereulerian, a contradiction. $\square$

Proof of Theorem 4.1.4.   Let G' be the reduction of G. If $G' = K_1$, then G is supereulerian. Next we suppose that $G' \neq K_1$. Then G' is 3-edge-connected and nontrivial. Denote $d_i = |D_i(G')|$ $(i \geq 3)$.

If $d_3 \geq 17$, then we assume that $v_1, v_2, \cdots, v_{17}$ are the vertices of $V(G')$ in $D_3(G')$, i.e. $d_{G'}(v_i) = 3$ for each i, and the corresponding pre-images are $H_1, H_2, \cdots, H_{17}$. Each H_i is joined to the rest of G by an edge cut consisting of $d_{G'}(v_i) = 3$ edges. By the hypothesis of Theorem 4.1.4, $|V(H_i)| \geq \dfrac{n}{16}$, and

$$n = |V(G)| \geq \sum_{i=1}^{17} |V(H_i)| \geq \frac{17n}{16},$$

a contradiction. So $d_3 \leq 16$.

By Theorem 4.1.3, we only need to consider $F(G') \geq 6$. Note that $|V(G')| = \sum\limits_{i \geq 3} d_i$, $2|E(G')| = \sum\limits_{v \in V(G)} d_G(v) = \sum\limits_{i \geq 3} i d_i$, and $F(G') = 2|V(G')| - |E(G')| - 2$, we have the following

$$d_3 \geq 16 + \sum_{i \geq 5} (i - 4) d_i.$$

Thus $n = d_3 = 16$. By Theorem 4.3.1, G is supereulerian. $\square$

4.4   Proofs of Theorems 4.1.7 and 4.1.8

We shall apply Theorem 4.1.1 to prove both Theorems 4.1.7 and 4.1.8. First, we need one more lemma in this section.

Lemma 4.4.1 *Let G be a planar graph such that every face of G has degree at least 4. Then $|E(G)| \leq 2|V(G)| - 4$.*

Proof.   Let f denote the number of faces of G. Since every face of G has degree at least 4, $4f \leq 2|E(G)|$ and so the lemma follows from Euler's formula. $\square$

Proof of Theorem 4.1.7.   Let G be a planar graph with $\kappa'(G) \geq 3$ and with n vertices. Assume that $|E(G)| \geq 2n - 5$. Let G' denote the reduction of G. By Theorem 2.0.4(iv), it suffices to show $G' = K_1$.

By contradiction, assume that G' is nontrivial. Then by Theorem 2.0.4(ii), G' also has girth at least 4. Note that G' is planar and $\kappa'(G') \geq 3$, let $H_1, \cdots H_l$ denote the nontrivial maximal collapsible subgraphs of G and let p denote the number of vertices of G'. Then by Lemma 4.4.1, each $|E(H_i)| \leq 2|V(H_i)| - 4$,

$$2n - 5 \leq |E(G)| = \sum_{i=1}^{l} |E(H_i)| + |E(G')| \leq 2 \sum_{i=1}^{l} |V(H_i)| - 4l + |E(G')|,$$

and so $|E(G')| \geq 2p - 5 + 2l$. By Theorem 2.0.5(iii) and $\kappa'(G') \geq 3$, we have $F(G') \geq 3$. Thus, by Theorem 2.0.7(iv), $2p - 5 \geq |E(G')| \geq 2p - 5 + 2l$, and so $l = 0$ and $F(G') = 3$.

Therefore by Theorem 4.1.1, G' must be collapsible, contrary to the fact that G' does not have nontrivial collapsible subgraphs. This proves Theorem 4.1.7. □

Proof of Theorem 4.1.8. Let G' be the reduction of G. Again we argue by contradiction and assume that $G' \neq K_1$. Note that $\kappa'(G') \geq 3$, by Euler's formula, by Theorem 2.0.7(iv) and by (4.1) in the process of the proof of Theorem 4.1.1, one concludes that $F(G') = 3$ and so Theorem 4.1.8 follows from Theorem 4.1.1. □

4.5   Proofs of Theorems 4.1.9 and 4.1.10

A few more lemmas and a former theorem of Nash-Williams and Tutte are needed in the proofs in this section.

Theorem 4.5.1 *(Nash-Williams [21] and Tutte [26]) A graph $G = (V, E)$ contains l edge-disjoint spanning trees if and only if for each partition $(V_1, V_2, \cdots, V_k)$ of V, the number of edges which have end in different parts of the partition is at least $l(k-1)$.*

Proof of Theorem 4.1.9. Suppose $V = (V_1, V_2, \cdots, V_k)$ is any partition of V. Without loss of generality, let $|V_1| = |V_2| = \cdots = |V_l| = 1,\quad |V_{l+1}| = \cdots = |V_{l+m}| = 2$, and for $l + m + 1 \leq j \leq k$, $|V_j| \geq 3$. Since G is a 2-connected simple planar graph, $|E(G[V_j])| \leq 3|V_j| - 6$ for $l + m + 1 \leq j \leq k$; $|E(G[V_i])| \leq 1$ for $l + 1 \leq i \leq l + m$; and $E(G[V_j]) = \emptyset$ for $1 \leq j \leq l$. Then, we have

$$
\begin{aligned}
\sum_{1 \leq i < j \leq k} |[V_i, V_j]_G| \ &= |E(G)| - \sum_{j=1}^{k} |E(G[V_j])| \\
&\geq 3n - 8 - m - [3(n - 2m - l) - 6(k - l - m)] \\
&= 6k - 3l - m - 8 \\
&= 2(k - 1) + 4k - 3l - m - 6.
\end{aligned}
$$

We consider the following cases.

Case 1. $l = 0$.

By $n \geq 6$ and $k \geq 2$, $4k - 3l - m - 6 = 3k + (k - m) - 6 \geq 0$.

Case 2. $m = 0$.

If $l = k$, then $4k - 3l - m - 6 = k - 6 \geq 0$ since $k = n \geq 6$. If $l < k$, then $4k - 3l - m - 6 = 3(k - l) + k - 6 \geq 0$ except for $l = 1, k = 2$. But when $l = 1, k = 2$, since G is a 2-edge-connected simple graph, $|[V_1, V_2]_G| \geq 2 = 2(k - 1)$.

Case 3. $l > 0$ and $m > 0$.

If $k = l + m$, then $4k - 3l - m - 6 = 3m + l - 6 = (2m + l) + m - 6 = n + m - 6 \geq 0$ since $n = 2m + l \geq 6$. If $k > l + m$, then $k - l \geq m + 1 \geq 2$ and $k - m \geq 0$. It follows that $4k - 3l - m - 6 = 3(k - l) + (k - m) - 6 \geq 0$.

Therefore, in any case, $\sum\limits_{1 \leq i < j \leq k} |[V_i, V_j]_G| \geq 2(k - 1)$, and so G must have two edge-disjoint spanning trees by Theorem 4.5.1. $\square$

We shall prove a stronger result than Theorem 4.1.10, as stated below.

Theorem 4.5.2 *If G is a 2-edge-connected simple planar graph with $n \geq 9$ vertices and with $|E(G)| \geq 3n - 12$ edges, then exactly one of the following holds:*

(i) G is collapsible.

(ii) The reduction of G is a 4-cycle.

(iii) The reduction of G is isomorphic to $K_{2,3}$ with exactly one nontrivial vertex whose pre-image is a maximal planar graph of $n - 4$ vertices.

We need two more lemmas.

Lemma 4.5.3 *If G is a simple planar graph with $n \geq 9$ vertices and with $|E(G)| \geq 3n - 12$, then G is not reduced.*

Proof. If G is reduced, then by Theorem 2.0.4(ii) and Lemma 4.4.1, $2n - 4 \geq |E(G)| \geq 3n - 12$, whence $n \leq 8$, contrary to the assumption that $n \geq 9$. Therefore, G is not reduced. $\square$

Lemma 4.5.4 *Let $C > 0$ be a constant, and let G be a simple planar graph with n vertices and with l nontrivial maximal collapsible subgraphs. Let G' denote the reduction of G. If $|E(G)| \geq 3n - C$, then*

$$|E(G')| \geq 3|V(G')| - C + 3l.$$

Proof. Let $H_1, \cdots H_l$ be nontrivial maximal collapsible subgraphs of G. Let $G' = G/(\bigcup\limits_{i=1}^{l} E(H_i))$ be the reduction of G with n' vertices. Since each H_i is a nontrivial planar graph for each i with $1 \leq i \leq l$,

$$|E(H_i)| \leq 3|V(H_i)| - 6. \tag{4.6}$$

Note that $|V(G')| = n - \sum\limits_{i=1}^{l} |V(H_i)| + l$. It follows from (4.6) that

$$
\begin{aligned}
|E(G')| &= |E(G)| - \sum_{i=1}^{l} |E(H_i)| \geq |E(G)| - \sum_{i=1}^{l} (3|V(H_i)| - 6) \\
&\geq 3n - C - 3(n - |V(G')| + l) + 6l = 3|V(G')| - C + 3l.
\end{aligned}
$$

This proves the lemma. $\square$

Proof of Theorem 4.5.2. Since the 4-cycle and $K_{2,3}$ are not collapsible, (i),(ii) and (iii) are mutually exclusive. We assume that both Theorem 4.5.2(i) and Theorem 4.5.2(ii) are false, and want to prove that Theorem 4.5.2(iii) must hold.

By Lemma 4.5.3, G is not reduced. Let $H_1, \cdots H_l$ be the nontrivial maximal collapsible subgraphs of G. Let $G' = G/(\bigcup\limits_{i=1}^{l} E(H_i))$ be the reduction of G with n' vertices. By Theorem 2.0.4(iii), G' is reduced and so by Lemma 4.5.4 with $C = 12$ and $l \geq 1$, we have

$$|E(G')| \geq 3|V(G')| - 12 + 3l. \tag{4.7}$$

By Theorem 2.0.4(ii) and Lemma 4.4.1,

$$2|V(G')| - 4 \geq |E(G')|. \tag{4.8}$$

It follows from (4.7) and (4.8) that

$$2|V(G')| - 4 \geq |E(G')| \geq 3|V(G')| - 12 + 3l. \tag{4.9}$$

By (4.9), $|V(G')| \leq 8 - 3l$. Since any 2-edge-connected graph with 3 vertices is collapsible, and since any simple graph with 4 vertices is either collapsible or is isomorphic to the 4-cycle, we must have $l = 1$ and $|V(G')| = 5$. Therefore, equalities must hold everywhere in (4.9), and so by Theorem 2.0.5(iii) and by $|V(G')| = 5$, $G' \cong K_{2,3}$ with exactly one nontrivial vertex, named by H.

Note that $|E(H)| \geq |E(G)| - 6 \geq 3n - 18$ and that $|V(H)| = n - 4$. It follows that $|E(H)| \geq 3(n-4) - 6 = 3|V(H)| - 6$, and so H must be a maximal planar graph with $n - 4$ vertices. $\square$

Proof of Theorem 4.1.10.   Note that 4-cycles are supereulerian, and so Theorem 4.1.9 follows from Theorem 2.0.4(iv) and Theorem 4.5.2. $\square$

Chapter 5

Hamiltonicity in 3-connected claw-free graphs

5.1 Introduction

Let H be the line graph $L(G)$ of a graph G. Then the order $\nu(H)$ of H is equal to the number $m(G)$ of edges of G, and $\delta(H) = \min\{d_G(x) + d_G(y) - 2 | xy \in E(G)\}$. If $L(G)$ is k-connected, then G is **essentially k-edge-connected**, which means that the only edge-cut sets of G having less than k edges are the sets of edges incident with some vertex of G.

A graph H is **claw-free** if it does not contain $K_{1,3}$ as an induced subgraph. In [25], Ryjáček defined the **closure** $cl(H)$ of a claw-free graph H to be one obtained by recursively adding edges to join two nonadjacent vertices in the neighborhood of any locally connected vertex of H, as long as this is possible.

Theorem 5.1.1 *(Ryjáček, [25]) Let H be a claw-free graph and $cl(H)$ its closure. Then*
(i) $cl(H)$ is well-defined, and $\kappa(cl(H)) \geq \kappa(H)$,
(ii) there is triangle-free graph G such that $cl(H) = L(G)$,
(iii) both graphs H and $cl(H)$ have the same circumference.

As a corollary of Theorem 5.1.1, a claw-free graph H is hamiltonian if and only if $cl(H)$ is hamiltonian, and so H is said to be **closed** if $H = cl(H)$.

Many works have been done to give sufficient conditions for a claw-free graph H to be hamiltonian in terms of its minimum degree $\delta(H)$. These conditions depend on the connectivity $\kappa(H)$. If $\kappa(H) = 4$, Matthews and Sumner [22] conjectured that H is hamiltonian and this conjecture is still open. When $\kappa(H) = 2$, Kuipers and Veldman [18], and independently Favaron, Flandrin, Li and Ryjáček [14], proved that if H is a 2-connected claw-free graph with sufficiently large order ν, and if $\delta(H) \geq \frac{\nu+c}{6}$ (where c is a constant), then H is hamiltonian except a member of ten well-defined families of graphs. When $\kappa(H) = 3$, the following have been proved and proposed.

Theorem 5.1.2 *(Kuipers and Veldman, [18]) If H is a 3-connected claw-free simple graph with sufficiently large order ν, and if $\delta(H) \geq \frac{\nu+29}{8}$, then H is hamiltonian.*

Theorem 5.1.3 *(Favaron and Fraisse, [15]) If H is a 3-connected claw-free simple graph with order ν, and if $\delta(H) \geq \frac{\nu+37}{10}$, then H is hamiltonian.*

Conjecture 5.1.4 *(Kuipers and Veldman, [18], see also [15]) Let H be a 3-connected claw-free simple graph of order ν with $\delta(H) \geq \frac{\nu+6}{10}$. If ν is sufficiently large, then H is hamiltonian.*

The main purpose of this chapter is to prove Conjecture 5.1.4. In fact, we proved a somewhat stronger result.

Theorem 5.1.5 *If H is 3-connected claw-free simple graph with $\nu \geq 196$, and if $\delta(H) \geq \frac{\nu+5}{10}$, then either H is hamiltonian, or $\delta(H) = \frac{\nu+5}{10}$ and $cl(H)$ is the line graph of G obtained from the Petersen graph P_{10} by adding $\frac{\nu-15}{10}$ pendant edges at each vertex of P_{10}.*

5.2   Proof of Theorem 5.1.5

The proof of Theorem 5.1.5 needs the following Theorem and Lemma.

Theorem 5.2.1 *(Chen, Lai, Li, Li and Mao, [13]) Let G be a 3-edge-connected graph and let $S \subseteq V(G)$ be a vertex subset such that $|S| \leq 12$. Then either G has a eulerian subgraph C such that $S \subseteq V(C)$, or G can be contracted to the Petersen graph in such a way that the preimage of each vertex of the Petersen graph contains at least one vertex in S.*

Lemma 5.2.2 *(Favaron and Fraisse, [15]) Let S be a set of vertices of a graph G contained in a eulerian subgraph of G and let C be a maximal eulerian subgraph of G containing S. Assume that some component A of $G - V(C)$ is not an isolated vertex and is related to C by at least r edges. Then*
(i). G contains a matching T of $r + 1$ edges such that at most $2r$ edges of G are adjacent to two distinct edges of T.
(ii). The number $m(G)$ of edges of G is related to the minimum degree $\delta(H)$ of the line graph H of G by $m(G) \geq (r + 1)\delta(H) - r + 1$.

Portion of the proof of Theorem 5.1.5 (the treatment to deal with Claims 1 and 2) is a modification of Favaron and Fraisse's proof for Theorem 1 in [15], with Theorem 5.2.1 being utilized in our proof.

Proof of Theorem 5.1.5. By Theorem 5.1.1, the graph H is hamiltonian if and only if its closure $cl(H)$ is hamiltonian. As $\nu(cl(H)) = \nu(H)$, $\delta(cl(H)) \geq \delta(H)$, and $cl(H)$ is 3-connected, the graph $cl(H)$ satisfies the same hypotheses as H. Hence it suffices to prove Theorem 5.1.5 for closed claw-free graphs.

By Theorem 5.1.1, we may assume that H is the line graph of a triangle-free graph G (i.e., $H = L(G)$), and suppose that H is 3-connected and satisfies $\delta(H) \geq \frac{\nu(H)+5}{10}$. Assume by contradiction that neither of the conclusions of Theorem 5.1.5 holds. By Theorem 1.0.1, G does not contain a dominating eulerian graph.

Let $B = \{v \in V(G)|d_G(v) = 1, 2\}$. Since H is 3-connected, the sum of degrees of the two ends of each edge in G is at least 5 and thus the set B is independent. Let $X_0 = N_G(B)$. We name the vertices of X_0 as $x_1, x_2, \cdots, x_p$ in the following way. Assume the vertices $x_1, \cdots, x_i$ are already defined or else put $i = 0$. Let y_{i+1} denote

a vertex of B which is adjacent to some vertex of $X_0 - \{x_1, \cdots, x_i\}$. Either y_{i+1} has exactly one neighbor in $X_0 - \{x_1, \cdots, x_i\}$ and we name it x_{i+1}, or y_{i+1} has exactly two neighbors in $X_0 - \{x_1, \cdots, x_i\}$ and we name them x_{i+1} and x_{i+2} and put $y_{i+2} = y_{i+1}$. Let $Y_0 = \{y_1, \cdots, y_p\}$. We note that if $1 \le i < j \le p$, then $y_i y_j \notin E(G)$ and $y_i x_j \notin E(G)$, except for the edges $y_i x_{i+1}$ when $y_i = y_{i+1}$; and that the components of the subgraph induced by the edges $x_i y_i, 1 \le i \le p$, are paths of length 1 or 2.

Consider now a matching M of G formed by $q - p$ edges $x_i y_i$ of G, $p + 1 \le i \le q$, considered in this order and such that
(i) the sets $X_0, Y_0, X = \{x_{p+1}, \cdots, x_q\}$ and $Y = \{y_{p+1}, \cdots, y_q\}$ are pairwise disjoint,
(ii) for $p + 1 \le i < j \le q$, $y_i y_j, y_i x_j \notin E(G)$.

We choose this matching as large as possible subject to the conditions (i) and (ii). Note that by the definition of X_0 and Y_0, the whole set B is disjoint from $X \cup Y$ and that Property (ii) holds for any i and j with $1 \le i < j \le q$.

Let J be the set of indices j between $p + 1$ and q such that y_j is adjacent to some vertex $z \notin X_0 \cup Y_0 \cup X \cup Y$ with $y_k z \notin E(G)$ for $1 \le k < j$. For each $j \in J$ we choose such a vertex z_j and we put $I = \{p + 1, \cdots, q\} - J$. Let $X_I = \{x_i \in X | i \in I\}$, $X_J = \{x_i \in X | i \in J\}$, $Y_I = \{y_i \in Y | i \in I\}$ and $Y_J = \{y_i \in Y | i \in J\}$.

Claim 1. (Favaron and Fraisse, [15]) The set $S = X_0 \cup X_I \cup Y_J$ is not contained in any eulerian subgraph of G.

Proof. Suppose Claim 1 is false and let C be a maximal eulerian subgraph of G containing $S = X_0 \cup X_I \cup Y_J$ and $R = V(G) - V(C)$. By the assumption that G has no dominating eulerian subgraphs, at least one component A of $G[R]$ is not a single vertex. This component A is disjoint from Y_0 since the vertices of Y_0 are isolated in $G[R]$.

Suppose first that every vertex of A has a neighbor in C. Then, if uv is an edge of A and if s denotes the number of edges between A and C, $s \ge d_C(u) + d_C(v) + |A| - 2$. Since G is triangle-free, $d_A(u) + d_A(v) \le |A|$ and thus $d_G(u) + d_G(v) = d_C(u) + d_C(v) + d_A(u) + d_A(v) \le d_C(u) + d_C(v) + |A|$. Hence $s \ge d_G(u) + d_G(v) - 2 \ge \delta(H)$. Apply Lemma 5.2.2 with $r = \delta(H)$ to conclude that the number of edges of G satisfies $m(G) \ge \delta^2(G) + 1$. Since $\delta(H) \ge \frac{\nu(H)+5}{10}$, then $m(G) = \nu(H) \le 10\delta(H) - 5$, and so $\delta^2(H) + 1 \le 10\delta(H) - 5$,

contrary to the hypothesis that $\nu(H) \geq 196$.

Therefore A contains a vertex z such that $N_G(z) \subseteq A$. Then $z \notin X_0 \cup Y_0 \cup X \cup Y$ and the neighbors of z are all in $Y_I \cup X_J \cup (R - (Y_0 \cup Y_I \cup X_J))$.

If z has a neighbor in Y_I, let i be the least index such that $y_i \in Y_i$ and $zy_i \in E(G)$. Since z has no neighbor in Y_J, $zy_k \notin E(G)$ for all $k < i$, in contradiction to the definition of I. Hence z has no neighbor in Y_I, and thus in Y.

If z has a neighbor in X_J, let x_j be the vertex of $N_G(z) \cap X_J$ with the largest index. Consider the ordered sets $X' = \{x_{p+1}, \cdots, x_{j-1}, x_j, z_j, x_{j+1}, \cdots, x_q\}$ and $Y' = \{y_{p+1}, \cdots, y_{j-1}, z, y_j, y_{j+1}, \cdots, y_q\}$. Then vertex z is neither adjacent to any x_k with $k > j$ by the definition of x_j and since z has no neighbor in X_I, nor to any vertex of Y as said above. The vertex z_j is not adjacent to any vertex y_k with $k < j$ by the choice of z_j. If $zz_j \notin E(G)$, then the sets X' and Y' define a matching M' which satisfies (i) and (ii), and thus which contradicts the maximality of M. If $zz_j \in E(G)$, then the eulerian subgraph $G[(E(C) - E(C')) \cup (E(C') - E(C))]$, with $C' = y_j z_j z x_j y_j$, satisfies $V(C) \cap V(C') = \{y_j\}$ since z has no neighbor in C, and thus contradicts the maximality of C. Hence $N_G(z) \cap X_J = \emptyset$ and z has no neighbor in X.

Finally if z has a neighbor t in $R - (Y_0 \cup Y_I \cup X_J)$, then the matching M'' corresponding to the ordered sets $X'' = \{t, x_{p+1}, \cdots, x_q\}$ and $Y'' = \{z, y_{p+1}, \cdots, y_q\}$ satisfies the conditions (i) and (ii) since z has no neighbor in $X \cup Y$. This contradicts the maximality of M and achieves the proof of Claim 1. $\square$

Claim 2. (Favaron and Fraisse, [15]) G must be contractible to the Petersen graph.

Proof. By contradiction. Suppose that G can not be contracted to the Petersen graph. Let G^1 be the graph or multigraph obtained from G by deleting the vertices of degree 1 or 2 and replacing each path ayb where $d_G(y) = 2$ by the edge ab. Since G is essentially 3-edge-connected, G^1 is 3-edge-connected. Moreover, to each eulerian subgraph C of G^1 corresponds a eulerian subgraph of G containing $V(C)$. Since $S \cap B = \emptyset$, the set S is contained in $V(G^1)$. Since S is not contained in any eulerian subgraph of G by Claim 1, S is not contained in any eulerian subgraph of G^1. By Theorem 5.2.1, $|S| \geq 13$. Let $F = \{x_i y_i | 1 \leq i \leq 13\}$, $P = \{x_i | 1 \leq i \leq 13\}$ and $Q = \{y_i | 1 \leq i \leq 13\}$. We suppose that

F consists of l paths of length 2 with $0 \leq l \leq 6$ and $13 - 2l$ edges of a matching. Then $|P| = 13$ and $|Q| = 13 - l$. We know that Q is independent, that $y_i x_j \notin E(G) - F$ for any $y_i \in Q$ and $x_j \in P$ with $1 \leq i < j \leq 13$, and that G is triangle-free. Hence, two different edges of F are joined by at most one edge of G which is of type $x_i x_j$ or $x_i y_j$ with $1 \leq i < j \leq 13$. More precisely, we can give an upper bound on the number μ of edges of G which are adjacent to two different edges of F. For a given value of l, this number can be maximum if the l paths of F occur with smaller indices than those of the $13 - 2l$ edges of the matching. This is due to the fact that the l vertices y_i belonging to paths of length 2 have degree 2 and thus they cannot be adjacent by an edge not in F to any vertex x_i with $i < j$. When this condition is fulfilled, there are at most l^2 edges between the vertices $x_1, x_2, \cdots, x_{2l}$ (since the number of edges of a triangle-free graph of order $2l$ is at most $(2l)^2/4$), $2l(13 - 2l)$ edges of type $x_i y_j$ between the sets $\{x_1, x_2, \cdots, x_{2l}\}$ and $\{y_{2l+1}, y_{2l+2}, \cdots, y_{13}\}$, and $\frac{(13-2l)(13-2l-1)}{2}$ edges of type $x_i x_j$ or $x_i y_j$ with $i < j$ between the vertices of the set $\{x_{2l+1}, \cdots, x_{13}, y_{2l+1}, \cdots, y_{13}\}$. Then $\mu \leq l^2 + 2l(13 - 2l) + \frac{(13-2l)(13-2l-1)}{2} = l - l^2 + 78$. Counting the edges of $G - F$ adjacent to some edge of F, we find at least $(13 - 2l)\delta(H)$ edges adjacent to an edge of a matching of F and $2l(\delta(H) - 1)$ edges adjacent to an edge of a path of length 2 (since each vertex y_i on such a path has degree 2 in G). At most $l - l^2 + 78$ of these edges have their two endvertices in $P \cup Q$ and are thus counted twice. Hence $m(G) \geq (13 - 2l)\delta(H) + 2l(\delta(H) - 1) - (l - l^2 + 78) + 13$, that is $\nu(H) = m(G) \geq 13\delta(H) + l^2 - 3l - 65 \geq 13\delta(H) - 67 \geq 10\delta(H) - 4$ since l is an integer between 0 and 6 and $\nu(H) \geq 196$. This contradicts to the hypothesis that $\delta(H) \geq \frac{\nu(H)+5}{10}$, and so Claim 2 must hold. $\square$

By Claim 2, G can be contracted to the Petersen graph P_{10}. Let $v_1, v_2, \cdots, v_{10}$ be the ten vertices of the Petersen graph P_{10}, and W_i be the preimage of v_i ($i = 1, 2, \cdots, 10$). Denote $\mathcal{SV} = \{v \in V(G) | d_G(v) \geq 12\}$. Since $d_G(u) + d_G(v) - 2 \geq \delta(H) \geq 21$ for every edge $e = uv \in E(G)$, we have either $d_G(u) \geq 12$ or $d_G(v) \geq 12$. So we have

$$\text{for every edge } e = uv \in E(G), \text{ either } u \in \mathcal{SV} \text{ or } v \in \mathcal{SV}. \tag{5.1}$$

Moreover, if $u, v \notin \mathcal{SV}$, then $uv \notin E(G)$. By the hypothesis of Theorem 5.1.5 that H is 3-connected, we have

$$G \text{ is essentially 3-edge-connected.} \tag{5.2}$$

Let $W \in \{W_i \mid 1 \leq i \leq 10\}$. Note that G is contracted to P_{10}. Then $|N_W(V(G) - V(W))| = 3$. If for any two vertices $w_1, w_2 \in N_W(V(G) - V(W))$, there is a dominating (w_1, w_2)-trail in W, then say W is **dominatable**.

Claim 3. Let W' be a graph obtained from W by deleting the vertices of degree 1. If $E(W') \neq \emptyset$, then W' is 2-edge-connected. Therefore W' contains some cycle.

Proof. Since G is contracted to the P_{10} and W is the preimage of some vertex v_i, we may assume that $[V(W), V(G) - V(W)]_G = \{e_1, e_2, e_3\}$, where e_1, e_2, e_3 are edges adjacent to v_i in P_{10}. Suppose that W' contains a cut-edge $e = z_1 z_2$. Then e is also a cut-edge of W. Let (U_1, V_1) be the partition of $V(W)$ such that $[U_1, V_1]_W = \{e\}$ and $z_1 \in U_1$ and $z_2 \in V_1$. Since $z_1, z_2 \in V(W')$, we have $d_W(z_1) \geq 2$ and $d_W(z_2) \geq 2$. Thus $E(G[U_1]) \neq \emptyset$ and $E(G[V_1]) \neq \emptyset$. Note that $[V(W), V(G) - V(W)]_G = \{e_1, e_2, e_3\}$. We may assume that the number of edge joining U_1 and $V(G) - V(W)$ is 1, say e_1. Then $\{e_1, e\}$ is an essential edge-cut in G, contrary to (5.2). So Claim 3 holds. $\square$

Claim 4. If $\alpha'(W) = 1$, then $W = K_{1,p}$ for some $p \geq 1$. Therefore all three edges in $[V(W), V(G) - V(W)]_G$ must be incident with the vertex of $K_{1,p}$ with degree p, and so H_1 is dominatable.

Proof. Since W is a connected triangle-free graph and $\alpha'(W) = 1$, G is acyclic. By Claim 3 and $\alpha'(W) = 1$, $W = K_{1,p}$ for some $p \geq 1$. $\square$

Claim 5. Suppose that $\alpha'(W) = t \in \{2, 3, \cdots, 5\}$ and $\{u_1 a_1, u_2 a_2, \cdots, u_t a_t\}$ is a matching in W. Suppose that $u_i \in \mathcal{SV}$ $(i = 1, 2, \cdots, t)$. Then $V(W) \cap \mathcal{SV} = \{u_1, u_2, \cdots, u_t\}$ and $E(W - \{u_1, u_2, \cdots, u_t\}) = \emptyset$.

Proof. Let $A = \{u_1, \cdots, u_t, a_1, \cdots, a_t\}$, $A_1 = A - u_i$ and $A_2 = A - a_i$. By $\alpha'(W) = t$, $E(W - A) = \emptyset$. Note that G is triangle-free and $\mathcal{SV} = \{v \in V(G) | d_G(v) \geq 12\}$. For each $z \in V(W) - A$, $d_W(z) \leq 5$ and so $d_G(z) \leq 8$. Thus $z \notin \mathcal{SV}$.

Since G does not contain a triangle and $\alpha'(W) = t \leq 5$, by $d_G(u_i) \geq 12$, we have $N_W(u_i) - A_1 \neq \emptyset$. Thus $N_W(a_i) \subseteq A_2$ (otherwise, $\{u_1 a_1, \cdots, u_{i-1} a_{i-1}, u_{i+1} a_{i+1}, \cdots, u_t a_t, u_i u, a_i a\}$ is a matching of W, where $u \in N_W(u_i) - A_1$ and $a \in N_W(a_i) - A_2$, contrary to the assumption that $\alpha'(W) = t$). Since G is triangle-free, we have $d_W(a_i) \leq 5$,

and so $d_G(a_i) \leq 8$. Thus $a_i \notin \mathcal{SV}$. Therefore $\mathcal{SV} \cap V(W) = \{u_1, u_2, \cdots, u_t\}$, and $E(W - \{u_1, u_2, \cdots, u_t\}) = \emptyset$. $\square$

Claim 6. If $\alpha'(W) = t \in \{2, 3, 4\}$, then W is dominatable.

Proof. Suppose that $\alpha'(W) = t$ and $\{u_1a_1, \cdots, u_ta_t\}$ is a matching in W. Without loss of generality, we assume that $u_i \in \mathcal{SV}(i = 1, 2, \cdots, t)$ by (5.1). By Claim 5, $\mathcal{SV} \cap V(W) = \{u_1, u_2, \cdots, u_t\}$, and $E(W - \{u_1, u_2, \cdots, u_t\}) = \emptyset$. Let $w_1, w_2, w_3 \in N_W(V(G) - V(W))$ and $w_1z_1, w_2z_2, z_3w_3 \in [V(W), V(G) - V(W)]_G$. If $w_1 = w_2$ and $d_W(w_1) = 1$, then $\{z_3w_3, w_1x\}$ is an essential edge-cut in G for some $x \in N_W(w_1)$, contrary to (5.2). So we have $d_W(w_1) \geq 2$ if $w_1 = w_2$.

Suppose, by contradiction, that W does not exist a dominating (w_1, w_2)-trail. If $w_1 \neq w_2$, we let $K_1 = W + \{w_1w, w_2w\}$ where w is a new vertex; if $w_1 = w_2$, we let $K_1 = W$ and $w = w_1$. Let $K = K_1 - D_1(K_1)$. Then $u_1, \cdots, u_t, w \in V(K)$, and K is 2-edge-connected by Claim 3. Let $S = \{u_1, \cdots, u_t\} \cup \{w\}$. Then $K - S$ is edgeless, and K does not have an S-eulerian subgraph. By Theorem 2.0.13, K is contractible to a member $L \in \mathcal{F}$ (see Figures 2.4, 2.5, 2.6) such that S intersects the preimage of every vertex in $B(L)$. Note that for each $L \in \mathcal{F}$, $d_L(b_i) = 2$ $(i = 1, 2, 3)$ and the set of degree 2 vertices is independent. Without loss of generality, we assume that the preiamges of b_1, b_2 do not contain w.

Note that $[V(W), V(G) - V(W)]_G = \{w_1z_1, w_2z_2, z_3w_3\}$. Suppose that $w \in V(L)$. Then $w_1, w_2 \in V(L)$. If $w_1 \neq w_2$, then $d_L(w) = 2$. Thus $w_1, w_2 \notin \{b_1, b_2\}$. If $w_1 = w_2$, then $w_1 = w_2 = w$. Thus $w_1, w_2 \notin \{b_1, b_2\}$ still hold. Since either $w_3 \notin V(PI(b_1))$ or $w_3 \notin V(PI(b_2))$, we may assume that $w_3 \notin V(PI(b_1))$. Thus $[V(PI(b_1)), V(G) - V(W)] = \emptyset$ and the set of two edges adjacent to $V(PI(b_1))$ is an essential edge-cut of G, contrary to (5.2). So $w \notin V(L)$. We assume that the preimage of some $b_i(\notin \{b_1, b_2\})$ contains w. Thus $w_1, w_2 \notin V(PI(b_i))(i = 1, 2)$. Therefore either $|[V(PI(b_1)), V(G) - V(W)]| = 0$ or $|[V(PI(b_2)), V(G) - V(W)]| = 0$. Without loss of generality, we assume that $|[V(PI(b_1)), V(G) - V(W)]| = 0$. Then the set of two edges adjacent to $V(PI(b_1))$ is an essential edge-cut of G, contrary to (5.2). $\square$

Claim 7. If $\alpha'(W) = t \geq 1$, then $|E(W)| \geq t\delta(H) + 2t - t^2 - 3$.

Proof. Let $\{u_1v_1, \cdots, u_tv_t\}$ be a matching in W. Then $E(W - \{u_1, \cdots, u_t, v_1, \cdots, v_t\}) = \emptyset$, and for any pair of $u_iv_i, u_jv_j (i \neq j)$, $|[\{u_i, v_i\}, \{u_j, v_j\}]_W| \leq 2$ since W does not contain a triangle. Since for $\sum\limits_{v \in V(W)} d_W(v)$, the edges of u_iv_i and the edges in $[\{u_i, v_i\}, \{u_j, v_j\}]_W$ are counted twice, and since $|[V(W), V(G) - V(W)]_G| = 3$, we have $|E(W)| = \sum\limits_{v \in V(W)} d_W(v) - |\{u_1v_1, u_2v_2, \cdots, u_tv_t\}| - \sum\limits_{i \neq j} |[\{u_i, v_i\}, \{u_j, v_j\}]_W| \geq (\sum\limits_{v \in V(W)} d_G(v) - 3) - t - 2\binom{t}{2}$. Since $\delta(H) \leq d_G(u_i) + d_G(v_i) - 2$ for each u_iv_i, we have $|E(W)| \geq t(\delta(H) + 2) - 3 - t - 2\binom{t}{2} = t\delta(H) + 2t - t^2 - 3$. $\square$

Now we finish the proof of Theorem 5.1.5. Let $|\{v_i | v_i \text{ is a trivial vertex in } P_{10}\}| = s$. By (5.1), the set of all trivial vertices in P_{10} is independent. Since $\alpha'(P_{10}) = 4$, we have $0 \leq s \leq 4$. If $s = 0$, then each v_i is nontrivial vertex. Thus $|E(W_i)| \geq \delta(H) - 2$ by Claim 7. Therefore $m(G) = \sum\limits_{i=1}^{10} |E(W_i)| + 15 \geq 10(\delta(H) - 2) + 15 = 10\delta(H) - 5$. By the hypothesis of Theorem 5.1.5, we have $\delta(H) = \frac{\nu(H)+5}{10}$, $|E(W_i)| = \delta(H) - 2$, $\alpha'(W_i) = 1$ and $W_i = K_{1,p}$, where $p = \delta(H) - 2 = \frac{\nu(H)-15}{10}$.

If $s \geq 1$, without loss of generality, we assume that v_1 is trivial. Since $P_{10} - v_1$ has a spanning cycle, there exists a W_i, say W_{10}, such that $\alpha'(W_{10}) \geq 5$ by Claims 4 and 6. If $s \leq 3$, then $m(G) = \sum\limits_{i=1}^{10} |E(W_i)| + 15 \geq (10 - s - 1)(\delta(H) - 2) + (5\delta(H) - 18) + 15 \geq 6(\delta(H)-2) + 5\delta(H) - 3 = 11\delta(H) - 15 \geq 10\delta(H) - 4$. Thus $\delta(H) \leq \frac{\nu(H)+4}{10}$, a contradiction. So $s = 4$. By Claims 3, 6 and $\delta(H) \geq \frac{\nu(H)+5}{10}$, $\alpha'(W_{10}) = 5$. If there exists some $W_j(j \neq 10)$ such that $\alpha'(W_j) \geq 2$, then $m(G) = \sum\limits_{i=1}^{10} |E(W_i)| + 15 \geq |E(W_{10})| + |E(W_j)| + 4(\delta(H) - 2) + 15 = (5\delta(H) - 18) + (2\delta(H) - 3) + 4\delta(H) + 7 = 11\delta(H) - 17 \geq 10\delta(H) + 4$, a contradiction. So the number of W_i with $\alpha'(W_i) = 1$ is 5. Without loss of generality, we assume that $\alpha'(W_i) = 1(i = 5, 6, 7, 8, 9)$ and $\alpha'(W_{10}) = 5$. Let $\{e_1f_1, e_2f_2, e_3f_3, e_4f_4, e_5f_5\}$ be a matching of W_{10} and $B = \{e_1, \cdots, e_5, f_1, \cdots, f_5\}$ and $Z = W_{10}[B]$. By (2), we assume that $e_i \in \mathcal{SV}(i = 1, 2, \cdots, 5)$. By Claim 5, $\mathcal{SV} \cap V(W_{10}) = \{e_1, e_2, \cdots, e_5\}$, and $E(W_{10} - \{e_1, e_2, \cdots, e_5\}) = \emptyset$.

If $|E(Z)| \leq 16$, then $|E(W_{10})| = \sum\limits_{v \in B} d_G(v) - |E(Z)| - 3 \geq 5(\delta(H) + 2) - 16 - 3 = 5\delta(H) - 9$. Thus $m(G) = \sum\limits_{i=5}^{9} |E(W_i)| + |E(W_{10})| + 15 \geq 5(\delta(H) - 2) + (5\delta(H) - 9) + 15 = 10\delta(H) - 4$, and so $\delta(H) \leq \frac{\nu(H)+4}{10}$, a contradiction. So we have

$$|E(Z)| \geq 17. \tag{5.3}$$

If Z is collapsible, then $W_{10} - D_1(W_{10})$ is collapsible by Theorem 2.0.7(iii). Thus for any pair of vertices $u, v \in W_{10} - D_1(W_{10})$, $W_{10} - D_1(W_{10})$ has a spanning $(u, v)-$trail by Lemma 2.0.12. Then for any pair of vertices $u, v \in V(W_{10})$, W_{10} has a dominating $(u, v)-$trail, and so W_{10} is dominatable. Since each $W_i(i = 1, 2, 3, 4)$ is a trivial graph, since each $W_i(i = 5, 6, \cdots, 9)$ is dominatable, and since $P_{10} - v_1$ has a spanning cycle, G has a dominating eulerian subgraph, a contradiction. So Z is not collapsible. Moreover,

$$W_{10} - D_1(W_{10}) \text{ is not collapsible.} \tag{5.4}$$

Therefore Z is not 2-edge-connected by Lemma 2.0.15.

Let $K \subseteq Z$ with $|V(K)| = 8$. Suppose that $|E(K)| \geq 14$. Then K is 2-edge-connected by Lemma 2.0.14. If $|D_2(K)| \geq 2$, then $|E(K)| \leq 2 + 2 + 9 = 13$ by Lemma 2.0.14(iv), a contradiction. So $|D_2(K)| \leq 1$. By Lemma 2.0.11 and by the fact that G is triangle-free, K is collapsible. By Claim 3 and Theorem 2.0.7(iii), $M_{10} - D_1(M_{10})$ is collapsible, contrary to (5.4). So

$$|E(K)| \leq 13. \tag{5.5}$$

Suppose that Z is not connected and Z_1 is a component of Z. Then $|V(Z_1)| \in \{2, 4, 6, 8\}$. By Lemmas 2.0.14(ii),(iv) and (5.3), $|V(Z_1)|$ is either 2 or 8. We may assume that $|V(Z_1)| = 2$ and $Z_2 = Z - V(Z_1)$. Then $|E(Z_1)| = 1$, $|V(Z_2)| = 8$ and $|E(Z_2)| \geq 16$, contrary to (5.5). So Z is connected. Let X be a cut-edge of Z and Z_3, Z_4 are components of $Z - X$ with $|V(Z_3)| \leq |V(Z_4)|$. By Lemma 2.0.14 and (5.3), $|V(Z_3)|$ is either 1 or 2. If $|V(Z_3)| = 2$, then $|E(Z_4)| \geq 17 - 2 = 15$, contrary to (5.5). So $|V(Z_3)| = 1, |V(Z_4)| = 9$, $|[V(Z_3), V(Z_4)]_Z| = 1$ and $|E(Z_4)| \geq 16$.

By (5.5) and Lemma 2.0.14, Z_4 is 3-edge-connected. Let Z_4' be the reduction of Z_4. Then Z_4' is still 3-edge-connected and $|V(Z_4')| \leq 9$. Thus $Z_4' = K_1$ by Theorem 2.0.11,

that is, Z_4 is collpasible. By Claim 3 and Theorem 2.0.7(iii), $W_{10} - D_1(W_{10})$ is collapsible, contrary to (5.4). $\square$

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