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On Flows of Graphs

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Dissertation submitted to the
Eberly College of Arts and Sciences
at West Virginia University
in partial fulfillment of the requirements
for the degree of

Doctor of Philosophy
in
Mathematics

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2004

Keywords: Graph, Integer Flow, Circular Flow, Bidirected Flow

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ABSTRACT

Tutte's 3-flow Conjecture, 4-flow Conjecture, and 5-flow Conjecture are among the most fascinating problems in graph theory. In this dissertation, we mainly focus on the nowhere-zero integer flow of graphs, the circular flow of graphs and the bidirected flow of graphs. We confirm Tutte's 3-flow Conjecture for the family of squares of graphs and the family of triangularly connected graphs. In fact, we obtain much stronger results on this conjecture in terms of group connectivity and get the complete characterization of such graphs in those families which do not admit nowhere-zero 3-flows. For the circular flows of graphs, we establish some sufficient conditions for a graph to have circular flow index less than 4, which generalize a new known result to a large family of graphs. For the Bidirected Flow Conjecture, we prove it to be true for 6-edge connected graphs.

Dedication

This dissertation is dedicated to my daughter, my wife, my parents and my whole family
for their love, understanding, encouragement and support.

Acknowledgments

I sincerely express my appreciation to my thesis advisor, Dr. Cun-Quan Zhang, for his excellent guidance, advice and continual support and encouragement. It has been my great pleasure to work under his supervision. Without him, this piece of work would have never come to completion.

I would like to thank the other committee members: Dr. Elaine M. Eschen, Dr. John L. Goldwasser, Dr. Hong-Jian Lai and Dr. Jerzy Wojciechowski for their support and help during my study here .

I would also like to thank the Department of Mathematics at West Virginia University for providing me with an excellent study environment during my four years study as a graduate student. My thanks also goes to all the professors who have given me support and help in my study and in my daily life.

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Chapter 1

Introduction

The map coloring problem is considered one of the major catalysts of the tremendous development of graph theory in its 260-year history. It is, thus, not surprising that graph theory coloring and its related problems have always been in the main line of graph theory research. It was observed by Tutte that the problem of the face-coloring of an embedded (planar) graph can be formulated in terms of integer flows in the graph. Since then, the topic of integer flow has been one of the most attractive in graph theory.

Tutte had three famous fascinating flow conjectures: 3-flow conjecture, 4-flow conjecture and 5-flow conjecture. There are a lot of partial results for these three conjectures (see the survey papers by Jackson [22], Jaeger [25] and Younger [52] and the book by Zhang [53]). All three of these conjectures are still open.

The concept of circular flow, introduced by Goddyn, Tarsi and Zhang in [16], is a generalization of integer flows. With the introduction of circular flow, the concept of integer flow was successfully extended to real values. By the relation between the circular flow index and the flow index, circular flows can be considered as a refinement of integer flows. Therefore, for each integer flow conjecture, there is a corresponding circular flow conjecture.

The concept of bidirected flow, introduced by Bouchet [7], is another generalization of integer flows. It is also motivated by the relation between the flow and the tension. For a graph drawn on an orientable surface of higher genus, flows are not dual to colorings, but to local-tensions. By Seymour's theorem, every graph on an orientable surface without the obvious obstruction has a nowhere-zero 6-local-tension. Bouchet conjectured that the same holds true on non-orientable surfaces and conjectured that every bidirected graph without the obvious obstruction should have a nowhere-zero bidirected 6-flow. This is known as the bidirected 6-flow conjecture.

In this dissertation, we mainly focus on the 3-flow conjecture, the circular flow problem and the bidirected flow conjecture.

In Chapter 2, we study the 3-flows of a family of graphs: squares of graphs. Squares of graphs can be used to model atom-bond networks with bond-angle forces, which arise in the study of glasses. Also, the problem of coloring squares of graphs has been studied recently for its applications to frequency allocation. Many properties have been studied for the family of squares of graphs by other authors, for example, the coloring number [1], the edge clique cover number [27], the existence of factors [37, 2] and the Hamiltonian property [13, 40]. Here, we prove that the Tutte's 3-flow conjecture is true for this family of graphs. In fact, we give a complete characterization of those graphs in this family which admit nowhere-zero 3-flows.

In Chapter 3, we study another family of graphs: triangularly connected graphs. The motivation of this work is the weak 3-flow conjecture, which states that the 3-flow conjecture is true for graphs with each edge contained in a triangle. Note that these graphs can be obtained by attaching together triangularly connected graphs. Therefore, if we can get the structure result about nowhere-zero 3-flows of triangularly connected graphs, it will help to completely solve this weak flow conjecture. Here, we prove that Tutte's 3-flow conjecture is true for this family of graphs. In fact, we give a complete characterization of those graphs in this family which admit nowhere-zero 3-flows. Furthermore, we get more general results

in terms of Z_3 -connectivity for this family of graphs.

In Chapter 4, we get some sufficient conditions for graphs with circular flow index less than 4. As an immediate corollary, we give a simple proof of a result obtained recently by Galluccio and Goddyn (*Combinatorica*, 2002), and obtain a larger family of such graphs.

In Chapter 5, we focus on the bidirected flows of graphs and prove that Bouchet's bidirected 6-flow conjecture is true for 6-edge connected graphs.

Chapter 2

Nowhere-zero 3-flows in squares of graphs

It was conjectured by Tutte [6] that every 4-edge connected graph admits a nowhere-zero 3-flow. In this paper, we give a complete characterization of graphs whose squares admit nowhere-zero 3-flows. As a corollary, we confirm Tutte's 3-flow conjecture for the family of squares of graphs.

2.1 Introduction

All graphs considered in this chapter are finite and simple. Let $G = (V, E)$ be a graph with vertex set V and edge set E . For any $v \in V(G)$, we use $d_G(v)$, $N_G(v)$ to denote the degree and the neighbor set of v in G , respectively. The minimal degree of a vertex of G is denoted by $\delta(G)$. We use K_m for a complete graph on m vertices, P_t for a path of length t and W_4 for a graph obtained from a 4-circuit by adding a new vertex x and edges joining x to all the vertices on the circuit. We call x the center of this W_4 and each edge with x as one endpoint is called a center edge. Let D be an orientation of G . Then the set of all edges with tails

(or heads) at a vertex v is denoted by $E^+(v)$ (or $E^-(v)$). If an edge uv is oriented from u to v under D , then we say $D(uv) = u \rightarrow v$. The square of G , denoted by G^2 , is the graph obtained from G by adding all the edges that join distance 2 vertices in G . We refer the reader to [6] for terminology not defined in this chapter.

Definition 2.1.1 *Let D be an orientation of G and f be a function: $E(G) \mapsto Z$. Then*

(1). *The ordered pair (D, f) is called a **k -flow** of G if $-k + 1 \leq f(e) \leq k - 1$ for every edge $e \in E(G)$ and $\sum_{e \in E^+(v)} f(e) = \sum_{e \in E^-(v)} f(e)$ for every $v \in V(G)$.*

(2). *The ordered pair (D, f) is called a **Modular k -flow** of G if for every $v \in V(G)$, $\sum_{e \in E^+(v)} f(e) \equiv \sum_{e \in E^-(v)} f(e) \pmod{k}$.*

*The **support** of a k -flow (modular k -flow) (D, f) of G is the set of edges of G with $f(e) \neq 0$ ($f(e) \not\equiv 0 \pmod{k}$), and is denoted by $\text{supp}(f)$. A k -flow (D, f) (Modular k -flow) of G is **nowhere-zero** if $\text{supp}(f) = E(G)$.*

Definition 2.1.2 *Let Γ be an Abelian group (an additive group with "0" as the identity). A Γ -flow of G is a flow (D, f) where f is a function: $E(G) \mapsto \Gamma$.*

For convenience, a nowhere-zero k -flow (Γ -flow) is abbreviated as a **k -NZF** (**Γ -NZF**). The concept of integer-flow was introduced by Tutte([45, 46], also see [52, 25]) as a refinement and generalization of face-coloring and edge-3-coloring problems. The following three conjectures are the most well-known open problems in this subject.

The 5-flow Conjecture (Tutte [46]) Every 2-edge connected graph admits a 5-NZF.

The 4-flow Conjecture (Tutte [48]) Every 2-edge connected graph containing no subdivision of the Petersen graph admits a 4-NZF.

The 3-flow Conjecture (Tutte, unsolved problem 48 in [6]) Every 4-edge connected graph admits a 3-NZF.

There are many partial results on these three conjectures, but all of them are still open.

For the k -NZF and Γ -NZF, Tutte obtained the following result.

Theorem 2.1.3 (Tutte [46]) *Let Γ be an Abelian group of order k . A graph G admits a k -NZF if and only if G admits a Γ -NZF.*

For the 5-flow conjecture, Jaeger and Kilpatrick obtained the following result independently.

Theorem 2.1.4 (Jaeger [23], Kilpatrick [29]) *Every 2-edge connected graph admits a 8-NZF.*

The strongest partial result for the 5-flow conjecture is due to Seymour.

Theorem 2.1.5 (Seymour [41]) *Every 2-edge connected graph admits a 6-NZF.*

The 4-flow conjecture has been confirmed for planar graphs. The result is also known as the Four Color Theorem.

Theorem 2.1.6 (Appel and Haken [3, 4, 5]) *The 4-flow conjecture is true for planar graphs.*

For the 3-flow conjecture, Jaeger obtained the best approach and Grötzsch proved it to be true for planar graphs.

Theorem 2.1.7 (Jaeger [23]) *Every 4-edge connected graph admits a 4-NZF.*

Theorem 2.1.8 (Grötzsch [17]) *Every 4-edge connected planar graph is face-3-colorable, equivalently, admits a 3-NZF.*

Jaeger proposed the following weaker version of the 3-flow conjecture.

The Weak 3-flow Conjecture (Jaeger [25]) There is an integer k such that every k -edge connected graph admits a 3-NZF.

The following is one approach to this conjecture.

Theorem 2.1.9 (Lai and Zhang [31]) *Let G be a k -edge connected graph with t odd vertices. If $k \geq 4\lceil \log_2^t \rceil$, then G admits a 3-NZF.*

In this chapter, squares of graphs admitting 3-NZF's will be characterized. The following families of graphs are the exceptions in the main theorem.

Definition 2.1.10 $\mathcal{T}_{1,3} = \{T \mid T \text{ is a tree and } d_T(v) = 1 \text{ or } 3 \text{ for every } v \in V(T)\}$

Definition 2.1.11 $\bar{\mathcal{T}}_{1,3} = \{T \mid T \in \mathcal{T}_{1,3} \text{ or } T \text{ is a 4-circuit or } T \text{ can be obtained from some } T' \in \mathcal{T}_{1,3} \text{ by adding some edges each of which joins a pair of distance 2 leaves of } T'\}$

The following is the main result of this chapter.

Theorem 2.1.12 *Let G be a connected simple graph. Then G^2 admits a 3-NZF if and only if $G \notin \bar{\mathcal{T}}_{1,3}$.*

An immediate corollary of Theorem 2.1.12 is the following partial result for Tutte's 3-flow conjecture.

Corollary 2.1.13 *Let G be a graph. If $\delta(G^2) \geq 4$ then G^2 admits a 3-NZF.*

This research is motivated by the 3-flow conjecture and the following open problem:

Conjecture 2.1.14 (Zhang [54]) *If every edge of a 4-edge connected graph G is contained in a 3-circuit, then G admits a 3-NZF.*

Theorem 2.1.12 and the following early results are partial results for Conjecture 2.1.14.

Theorem 2.1.15 (Catlin [9]) *If every edge of a graph G is contained in a circuit of length at most 4, then G admits a 4-NZF.*

Theorem 2.1.16 (Lai [32]) *Every 2-edge connected, locally 3-edge connected graph admits a 3-NZF.*

Theorem 2.1.17 (Imrich and Skrekovski [20]) *Let G and H be two graphs. Then $G \times H$ admits a 3-NZF if both G and H are bipartite.*

2.2 Splitting operation, flow extension and lemmas

Definition 2.2.1 (A special splitting operation) *Let G be a graph and $e = xy \in E(G)$. The graph G_{*e} is obtained from G by deleting the edge e and adding two new vertices x' and y' and adding two new edges, e_x and e_y , joining x and y' , y and x' , respectively.*

Definition 2.2.2 *Let G be a graph, let (D, f) be a 3-flow of G and let $F \subseteq E(G) \setminus \text{supp}(f)$. A 3-flow (D', f') of G is called an (F, f) -changer if $F \cup \text{supp}(f) \subseteq \text{supp}(f')$.*

Lemma 2.2.3 ([45]) *A graph G admits a k -flow (D, f_1) if and only if G admits a Modular k -flow (D, f_2) such that $f_1(e) \equiv f_2(e) \pmod{k}$ for each $e \in E(G)$.*

An orientation of a graph G is called a modular 3-orientation if $|E^+(v)| \equiv |E^-(v)| \pmod{3}$, for every $v \in V(G)$. The following result appears in [25, 43, 52], but by Lemma 2.2.3, we can attribute it to Tutte.

Lemma 2.2.4 ([45]) *Let G be a graph. Then G admits a 3-NZF if and only if G has a modular 3-orientation.*

A **partial 3-orientation** D of G is an orientation of some edges of G satisfying $|E^+(v)| \equiv |E^-(v)| \pmod{3}$, for any $v \in V(G)$. The support of D is the set of edges oriented under D and is denoted by $\text{supp}(D)$. Clearly the partial orientation obtained by reversing every oriented edge of a partial 3-orientation is also a partial 3-orientation.

Let D be a partial 3-orientation of G and let $C = v_0v_1 \cdots v_{k-1}v_0$ be a circuit of G . A **circuit-operation** along C is defined as following: For $0 \leq i \leq k-1$, if $D(v_iv_{i+1}) = v_i \rightarrow v_{i+1} \pmod{k}$, then reverse the direction of this edge; if $(v_iv_{i+1}) \pmod{k}$ is not oriented under D , then orient it as $v_i \rightarrow v_{i+1}$; if $D(v_iv_{i+1}) = v_{i+1} \rightarrow v_i \pmod{k}$ then v_iv_{i+1} loses its orientation.

Lemma 2.2.5 *Let G be a graph, (D, f) be a 3-flow of G and H be a subgraph of G*

(1). *If $H \cong W_4$ and $e \in E(H) \setminus \text{supp}(f)$ is a center edge, then an $(\{e\}, f)$ -changer exists.*

(2). *If H is a circuit of length 3 with $E(H) \cap \text{supp}(f) = \{e\}$, then an $(E(H) \setminus \{e\}, f)$ -changer exists.*

Proof. (1). Since $H \cong W_4$, let x be the center of H and let $u_1u_2u_3u_4u_1$ be the 4-circuit $H \setminus x$. Since G has a 3-flow (D, f) , then G has a partial 3-orientation D^* with $\text{supp}(D^*) = \text{supp}(f)$. We need only to find a partial 3-orientation D' such that $\text{supp}(D^*) \cup \{e\} \subseteq \text{supp}(D')$. Since e is a center edge, without loss of generality, assume that $e = xu_1$.

First we assume $E(H) \setminus \{e\} \subseteq \text{supp}(D^*)$. Without loss of generality, assume $D^*(u_1u_2) = u_1 \rightarrow u_2$. Then $D^*(u_2x) = x \rightarrow u_2$. Otherwise, we do a circuit-operation along $u_1u_2xu_1$ and then get a needed partial 3-orientation D' of G . For the same reason, u_4 must be the tail (or head) of both u_1u_4 and xu_4 . By symmetry, we consider the following two cases.

Case 1. $D^*(u_1u_4) = u_1 \rightarrow u_4$ and $D^*(xu_4) = x \rightarrow u_4$.

We may assume that u_3 is the tail (or head) of all edges incident with it in H . Otherwise, there exists a directed 2-path xu_3u_i (or u_iu_3x) for some $i \in \{2, 4\}$. Then we do circuit-operations along xu_3u_ix (or $u_iu_3xu_i$) and along $u_1u_ixu_1$. Therefore, we get a needed partial 3-orientation of D' of G .

If all edges in H have u_3 as a tail, then we do circuit-operations along xu_1u_4x , along $u_4xu_3u_4$, along xu_3u_2x and along $u_2xu_1u_2$; If all edges in H have u_3 as a head, then we do circuit-operations along $u_1u_2u_3xu_1$ and along $u_3xu_4u_3$. In both cases, we get a needed partial 3-orientation D' of G .

Case 2. $D^*(u_1u_4) = u_4 \rightarrow u_1$ and $D^*(xu_4) = u_4 \rightarrow x$.

Similar to Case 1, we may assume u_3 be the tail (or head) of all edges incident with it in H . If all edges in H have u_3 as a tail, then we do circuit-operations along xu_1u_4x , along $u_3u_4u_1u_2u_3$ and along $u_3xu_2u_3$; If all edges in H have u_3 as a head, then we do circuit-operations along $u_1xu_2u_1$, along $u_4u_1u_2u_3u_4$ and along $u_4xu_3u_4$. In both cases, we get a needed partial 3-orientation D' of G .

If $\text{supp}(D^*)$ misses some other edges of $E(H)$, say $e^* = ab \in E(H) \setminus \text{supp}(D^*)$, then we define $D^*(ab) = a \rightarrow b$ or $b \rightarrow a$, by the proof of Case 1 and Case 2, we can find a needed D' of G .

(2). it is trivial. ■

Lemma 2.2.6 *For each $G \in \bar{\mathcal{T}}_{1,3}$ and each $e_0 \in E(G)$, the graph G^2 admits a 3-flow (D, f) such that $\text{supp}(f) = E(G^2) \setminus \{e_0\}$*

Proof. Induction on $|E(G)|$. It is obviously true for graphs G with $G^2 = K_4$ (including $G = C_4$, the circuit of length 4). So, assume that $|V(G)| \geq 5$ and let D be any fixed orientation of G^2 .

Let $e = xy$ with $d_G(x) = d_G(y) = 3$. Then G_{*e} consists of two components, say G_1 and G_2 . Clearly, $G_1, G_2 \in \bar{\mathcal{T}}_{1,3}$. Without loss of generality, let $e_0 \in E(G_1)$. By induction, G_1^2 admits a 3-flow (D, f_1) such that $\text{supp}(f_1) = E(G_1^2) \setminus \{e_0\}$ and G_2^2 admits a 3-flow (D, f_2) that $\text{supp}(f_2) = E(G_2^2) \setminus \{e\}$.

Then, identifying the split vertices and edges, back to G , $(D, f_1 + f_2)$ is a 3-flow (D, f) with $\text{supp}(f) = E(G^2) \setminus \{e_0\}$. ■

Lemma 2.2.7 (1). *Let G be a k -path with $k \geq 2$ or an m -circuit with $m = 3$ or $m \geq 5$. Then G^2 admits a 3-NZF.*

(2). *Let G be a graph obtained from an r -circuit $x_0x_1 \cdots x_{r-1}x_0$ by attaching an edge x_iv_i at each x_i for $0 \leq i \leq r-1$, where $v_i \neq v_j$ if $i \neq j$. Then G^2 admits a 3-NZF.*

(3). *Let G be a graph obtained from an m -circuit $x_0x_1 \cdots x_{m-1}x_0$ by attaching an edge $x_{m-1}v$ at x_{m-1} alone, where $m \geq 5$. Then G^2 admits a 3-NZF.*

Proof. (1). If G is an m -circuit with $m = 3$ or $m \geq 5$, then G^2 is a cycle (every vertex is of even degree) and G^2 admits 2-NZF. If G is a k -path with $k \geq 2$, by induction on k and using Lemma 2.2.5-(2), G^2 admits a 3-NZF.

(2). For $r \geq 5$ (or $r = 3$): let D be an orientation such that v_i ($0 \leq i \leq r-1$) is the tail of every edge of G^2 incident with it and all the other edges are oriented as $x_i \rightarrow x_{i+1}$, $x_i \rightarrow x_{i+2} \pmod{r}$ (or $x_i \rightarrow x_{i+1} \pmod{3}$ only for $r = 3$). Obviously, D is a modular 3-orientation of G^2 .

For $r = 4$: let D be the orientation such that v_0 and v_2 be the tail of every edge of G^2 incident with it, v_1 and v_3 be the head of every edge of G^2 incident with it, $x_0x_1x_3x_2x_0$ as a directed circuit and other edges are oriented as $x_3 \rightarrow x_0$, $x_1 \rightarrow x_2$. Obviously, D is a modular 3-orientation of G^2 .

(3). Orient all the edges as $x_i \rightarrow x_{i+1}$, $x_i \rightarrow x_{i+2} \pmod{m}$ for $0 \leq i \leq m-1$ and let v be the tail of every edge of G^2 incident with it. Then reverse the direction of the following edges: x_0x_{m-1}, x_0x_{m-2} . Clearly, this orientation is a modular 3-orientation of G^2 . ■

2.3 Proof of the main theorem

Proof. \Rightarrow By contradiction. Suppose $G \in \bar{\mathcal{T}}_{1,3}$. Let G be a counterexample with $|V(G)| + |E(G)|$ as small as possible. Clearly $|V(G)| \geq 5$ and G contains no circuits. So $G \in \mathcal{T}_{1,3}$. Let $v \in V(G)$ be a degree 3 vertex such that $N_G(v) = \{v_1, v_2, v_3\}$, $d_G(v_1) = d_G(v_2) = 1$. Clearly, $G_1 = G \setminus \{v_1, v_2\} \in \mathcal{T}_{1,3}$. Since G^2 has a modular 3-orientation D and both v_1 and v_2 are degree 3 vertices in G^2 , then this orientation restricted to the edge set of G_1^2 will generate a modular 3-orientation of G_1^2 . Therefore, G_1^2 admits a 3-NZF, a contradiction.

\Leftarrow Let G be a counterexample to the theorem such that

- (i). $|E(G)| - |V(G)|$ is as small as possible,
- (ii). subject to (i), $|E(G)|$ is as small as possible.

Note that $|E(G)| - |V(G)| + 1$ is the rank of the cycle space of G .

Claim 1. Let $e_0 = xy \in E(G)$. If $d_G(x) \geq 3$ and $d_G(y) \geq 2$, then xy is not a cut edge of G .

If e_0 is a cut-edge, then at least one component of G_{*e_0} is not in $\bar{\mathcal{T}}_{1,3}$, say, G_1 is not, while G_2 might be. By induction, let (D, f_i) be a 3-flow of G_i^2 for each $i = 1, 2$ such that f_1 is nowhere-zero, f_2 might miss only one edge e_x (that is a copy of e_0). Without loss of generality, assume that $f_1(e_y) + f_2(e_x) \not\equiv 0 \pmod{3}$. Then, identifying the split vertices and edges, back to G , $(D, f_1 + f_2)$ is a nowhere-zero Modular 3-flow of G^2 . By Lemma 2.2.3, G^2 admits a 3-NZF, a contradiction. □

Claim 2. $d_G(x) \leq 3$ for any $x \in V(G)$.

Otherwise, assume that $d_G(x) \geq 4$ for some vertex $x \in V(G)$. Clearly $G \not\cong K_{1,m}$ for $m \geq 4$ since $K_{1,m}$ is not a counterexample. So there exists $e_0 = xy \in E(G)$ with $d_G(y) \geq 2$. By Claim 1, e_0 is not a cut edge of G and $G_1 = G_{*e_0} \notin \bar{\mathcal{T}}_{1,3}$. Then by (i), G_1^2 admits a 3-NZF.

In G_1^2 , identify x and x' , y and y' , and use one edge to replace two parallel edges, by Lemma 2.2.3, we will get G^2 and a Modular 3-flow (D, f) of G^2 such that $E(G^2) \setminus \text{supp}(f) \subseteq \{xv \text{ or } yw \mid v \in N_G(y), w \in N_G(x)\}$. Let $C(x) = G^2[N_G(x) \cup \{x\}]$. Then $C(x)$ is a clique of order at least 5. We are to adjust (D, f) so that the resulting Modular 3-flow (D, f') of G^2 misses only edges of $\{uv \mid u, v \in V(C(x))\}$. For each edge xv which is missed by $\text{supp}(f)$ and $xv \notin E(C(x))$, $xyvx$ must be a circuit of G^2 , so let (D, f_{xv}) be a 3-flow of G^2 with $\text{supp}(f_{xv}) = \{xy, yv, xv\}$ and $f_{xv}(yv) + f(yv) \not\equiv 0 \pmod{3}$. Now $(D, f + f_{xv})$ is a Modular 3-flow of G^2 whose support contains xv, yv , but may miss xy . Repeat this adjustment and do the similar adjustment for the edges yw not in the support until we get a Modular 3-flow (D, f') of G^2 such that $E(G^2) \setminus \text{supp}(f') \subseteq E(C(x))$. Since each edge in $C(x)$ is contained in some K_5 and thus is a center edge in some W_4 , by Lemma 2.2.3 and Lemma 2.2.5-(1), G^2 admits a 3-NZF, a contradiction. \square

Claim 3. No degree 2 vertex is contained in a 3-circuit.

By contradiction. Assume $xyzx$ is a circuit of G with $d_G(x) = 2$. If $d_G(y) = 2$, then we must have $d_G(z) = 3$. Therefore $G_1 = G \setminus \{xy\} \notin \bar{\mathcal{T}}_{1,3}$ and $G_1^2 = G^2$, contradicting (ii). So $d_G(y) = d_G(z) = 3$.

Let $N_G(y) = \{x, y', z\}$ and $N_G(z) = \{x, y, z'\}$. Let $G_1 = G - \{x\}$. Since $(N_G(y) \cap N_G(z)) \setminus \{x\} = \emptyset$ (otherwise, let $G_2 = G \setminus \{yz\}$, then $G_2^2 = G^2$, $G_2 \notin \bar{\mathcal{T}}_{1,3}$, contradicting (ii)) and $d_{G_1}(y) = 2$, then $G_1 \notin \bar{\mathcal{T}}_{1,3}$. So G_1^2 admits a 3-NZF. Since $E(G^2) \setminus E(G_1^2) = \{xy, xy', xz, xz'\}$, by Lemma 2.2.5-(2), G^2 admits a 3-NZF, a contradiction. \square

Claim 4. No degree 2 vertex of G is contained in a 4-circuit.

Assume $C = xu_1u_2u_3x$ is a 4-circuit of G and $d_G(x) = 2$. By Claim 3, $u_1u_3 \notin E(G)$. Let u'_i be the adjacent vertex of u_i which is not in $V(C)$ if $d_G(u_i) = 3$ for some $i \in \{1, 2, 3\}$. Let $G_1 = G \setminus \{x\}$. We consider the following 3 cases.

Case 1. $d_G(u_1) = d_G(u_3) = 2$.

Then $d_G(u_2) = 3$ and $d_G(u'_2) \geq 2$ (if $d_G(u'_2) = 1$, it's easy to show G^2 admits a 3-NZF). Clearly, $u_2u'_2$ is a cut edge, contradicting Claim 1.

Case 2. Exactly one of u_1, u_3 has degree 3.

Assume $d_G(u_1) = 3$ and $d_G(u_3) = 2$. Since $d_{G_1}(u_1) = 2$, if $d_{G_1}(u'_1) = 2$ then u'_1 is not contained in a 3-circuit in G (by Claim 3), and so $G_1 \notin \bar{T}_{1,3}$. By induction, G_1^2 admits a 3-NZF. Since $E(G^2) \setminus E(G_1^2) = \{xu'_1, xu_1, xu_2, xu_3\}$ and $G^2[V(C) \cup \{u'_1\}]$ contains a W_4 with x as its center, by Lemma 2.2.5-(1), G^2 admits a 3-NZF, a contradiction.

Case 3. $d_G(u_1) = d_G(u_3) = 3$.

If $u'_1 = u'_3$, then $u'_1u_1u_2u_3$ is a 3-path, otherwise $u'_1u_1u_2u_3u'_3$ is 4-path. In both cases G_1^2 admits a 3-NZF. Since $E(G^2) \setminus E(G_1^2) = \{xu'_1, xu_1, xu_2, xu_3, xu'_3\}$ and each edge xu_i or xu'_j is contained in some W_4 in G^2 as a center edge for $1 \leq i \leq 3$ and $j = 1, 3$, by Lemma 2.2.5-(1), G^2 admits a 3-NZF. a contradiction. \square

Claim 5. For any $v \in V(G)$, $d_G(v) \neq 2$.

Otherwise, if there exists $v \in V(G)$ such that $d_G(v) = 2$, then by Claim 3-4, v is not contained in any circuits of length 3 or 4. By Lemma 2.2.7-(1), G cannot be a k -path with $k \geq 2$ or an m -circuit with $m = 3$ or $m \geq 5$. Let us consider the following cases.

Case 1. There exists a path $P_m = v_1v_2 \cdots v_m$ such that $m \geq 3$, $v = v_t$ for some $2 \leq t \leq m-1$, $d_G(v_k) = 2$ for $2 \leq k \leq m-1$ and $d_G(v_1) \neq 2$, $d_G(v_m) \neq 2$.

Clearly, at least one of v_1, v_m has degree 3. If $d_G(v_i) = 3$ for $i = 1$, or m , let $N_G(v_i) \setminus V(P_m) = \{v'_i, v''_i\}$. Clearly, $G_1 = G \setminus \{v_2, v_3, \dots, v_{m-1}\} \notin \bar{\mathcal{T}}_{1,3}$ (because by Claim 3, degree 2 vertices are not contained in any 3-circuits of G). By Claim 1, G_1 is connected. So G_1^2 admits a 3-NZF (D, f_1) . By Lemma 2.2.7-(1), P_m^2 admits a 3-NZF (D, f_2) . Then G^2 admits a 3-flow (D, f) with $\text{supp}(f) = \text{supp}(f_1) \cup \text{supp}(f_2)$. By Claim 3-4, $E(G^2) \setminus \text{supp}(f) = \{v_2v'_1, v_2v''_1, v_{m-1}v'_m, v_{m-1}v''_m\}$, then by Lemma 2.2.5-(2), G^2 admits a 3-NZF, a contradiction.

Case 2. There exists a m -circuit $C = v_1v_2 \cdots v_mv_1$ with $m \geq 5$, $d_G(v_i) = 2$ for $1 \leq i \leq m-1$, $d_G(v_m) = 3$ and $v = v_t$ for some $1 \leq t \leq m-1$.

Suppose that $v_0 \in N_G(v_m) \setminus V(C)$. By Claim 1, $d_G(v_0) = 1$. So by Lemma 2.2.7-(3), G^2 admits a 3-NZF, a contradiction. \square

Claim 6. Let $e = xy \in E(G)$ with $d_G(x) = d_G(y) = 3$. Then e is contained in a circuit of length 3 or 4.

By contradiction. Let G_1 be the graph obtained from G by deleting the edge e and adding a new vertex y' and a new edge xy' . Since G contains no degree 2 vertices and $d_{G_1}(y) = 2$, then $G_1 \notin \bar{\mathcal{T}}_{1,3}$. By Claim 1, e is not a cut edge of G , then by (i), G_1^2 admits a 3-NZF (D, f_1) . Identify y and y' , the resulting 3-flow (D, f_2) in G^2 misses only two edges y_1x and y_2x where $N(y) = \{y_1, y_2, x\}$ (since xy is not contained a circuit of length 3 or 4). By Lemma 2.2.5-(2), G^2 admits a 3-NZF, a contradiction. \square

Claim 7. For each $x \in V(G)$ with $d_G(x) = 3$, $|N_G(x) \cap V_3| \leq 2$, where V_3 is the set of all the degree 3 vertices of G .

By contradiction. Assume that $U = \{u_1, u_2, u_3\} = N_G(x) \cap V_3$. Let $G_1 = G \setminus \{x\}$. By Claim 1, G_1 is connected. Since G contains no degree 2 vertices, $G_1 \notin \bar{\mathcal{T}}_{1,3}$ and G_1^2 admits a 3-NZF (D, f) . By Claim 6, each xu_i ($1 \leq i \leq 3$) is contained a circuit of length at most 4. We consider the following 3 cases.

Case 1. $G[U]$ contains at least 2 edges.

Suppose that $u_1u_2, u_2u_3 \in E(G)$. Let $u'_i \in N_G(u_i) \setminus U$ for $i = 1, 3$. If $u'_1 = u'_3$, then $G^2[U \cup \{u'_1, x\}] \cong K_5$, by Lemma 2.2.5-(1), we can get a 3-NZF of G^2 , a contradiction. If $u'_1 \neq u'_3$, then $G[u'_1u_1u_2u_3u'_3]$ is a 4-path, by Lemma 2.2.5-(1) (similar to Case 3 of Claim 4), we can get a 3-NZF of G^2 , a contradiction.

Case 2. $G[U]$ contains exactly 1 edge.

Assume that $u_1u_2 \in E(G)$. By Claim 6, each edge xu_i ($i = 1, 2, 3$) is contained in a circuit of length 3 or 4. So we may assume $z \in (N_G(u_2) \cap N_G(u_3)) \setminus \{x\}$. Clearly, $G^* = G^2[U \cup \{x, z\}] \cong K_5$. Let $u'_i \in N_G(u_i) \setminus (U \cup \{z\})$ for $i = 1, 3$. Clearly, $E(G^2) \setminus \text{supp}(f) \subseteq E(G^*) \cup \{xu'_1, xu'_3\}$. Since $xu_ju'_jx$ ($j = 1, 3$) is a circuit of G^2 , we can get a 3-flow (D, f_1) such that $E(G^2) \setminus \text{supp}(f_1) \subseteq E(G^*)$. By Lemma 2.2.5-(1), we can get a 3-NZF of G^2 , a contradiction.

Case 3. $G[U]$ contains no edges.

Assume that $z_1 \in (N_G(u_1) \cap N_G(u_2)) \setminus \{x\}$ and $z_2 \in (N_G(u_1) \cap N_G(u_3)) \setminus \{x\}$. Let $G_2 = G \setminus \{xu_1\}$, then $G_2 \notin \bar{\mathcal{T}}_{1,3}$ and G_2^2 admits a 3-NZF (D, f_1) . Clearly, $E(G^2) \setminus \text{supp}(f_1) = \{xu_1\}$. Since xu_1 is contained in a W_4 which is contained in the graph induced by $\{u_1, z_1, u_2, u_3, x\}$ in G^2 with x as center, by Lemma 2.2.5-(1), we can get a 3-NZF of G^2 , a contradiction. \square

Final Step. By Claim 2, Claim 5 and Claim 7, all vertices of G have degree 1 or 3 and each degree 3 vertex is adjacent to at most 2 degree 3 vertices. So $G[V_3]$ is a path or a circuit, hence G must be a graph obtained from an r -circuit $x_0x_1 \cdots x_{r-1}x_0$ by attaching an edge x_iv_i at each x_i for $0 \leq i \leq r-1$, where $v_i \neq v_j$ if $i \neq j$, or a path $x_0x_1 \cdots x_p$ by attaching an edge v_ix_i ($1 \leq i \leq p-1$) at each x_i , where $v_i \neq v_j$ if $i \neq j$. Clearly the latter case is a graph in $\bar{\mathcal{T}}_{1,3}$. By Lemma 2.2.7-(2), G^2 admits a 3-NZF, a contradiction. \blacksquare

Chapter 3

Group connectivity of triangularly connected graphs

Tutte [6] conjectured that every 4-edge connected graph admits a nowhere-zero Z_3 -flow and Jeager *et al* [26] conjectured that every 5-edge connected graph is Z_3 -connected. In this paper, we characterize the triangularly connected graphs that are A -connected for any Abelian group A with $|A| \geq 3$. As a corollary, we also characterize the triangularly connected graphs that admit nowhere-zero Z_3 -flows, confirming these two conjectures for triangularly connected graphs.

3.1 Introduction

For a subset $X \subseteq E(G)$, the *contraction* G/X is the graph obtained from G by identifying the two ends of each edge in X and then deleting the resulting loops. Note that even when G is simple, G/X may have multiple edges. For convenience, we write G/e for $G/\{e\}$, where $e \in E(G)$. If H is a subgraph of G , then G/H denotes $G/E(H)$.

We use P_k for a path of length k . A nontrivial 2-regular connected graph is called a circuit and a circuit with k edges is called a k -circuit. A wheel W_k is a graph obtained from a k -circuit by adding a new vertex v and then joining this new vertex to all the vertices on the circuit. This new vertex v is called the center of W_k . A wheel W_k is an even wheel if k is even and an odd wheel otherwise. We use $\langle W_{odd} \rangle$ to denote the set of all odd wheels.

A graph G is called a Fan-graph, if it can be obtained from a path $P_k = v_1v_2 \cdots v_k$ ($k \geq 2$) by adding a new vertex v and joining v to all the vertices on the path. v is called the center of G , vv_1, vv_k are called the end edges of G and v_1, v_k are called the end points of G . The set of Fan-graphs are denoted by $\langle F \rangle$.

Let G be a graph with $u'v' \in E(G)$ and H be a graph with $uv \in E(H)$. We use $G \oplus H$ to denote a new graph obtained from the disjoint union of $G - \{u'v'\}$ and H by identifying u' and u and identifying v' and v . This operation is called attaching G on H over the edge uv and the resulting graph is denoted by $G \oplus H$.

We define $\langle WF \rangle = \{H | H = H_1 \oplus H_2 \oplus \cdots \oplus H_k, \text{ where } H_i \in \langle F \rangle \cup \langle W_{odd} \rangle \text{ for } 1 \leq i \leq k\}$. We call H_i ($1 \leq i \leq k$) is a WF-component of H . An end-WF-component of H is a WF-component H_i such that H_i shares at most one edge with the union of all the other WF-components of H .

Let G be a digraph, A be a nontrivial additive Abelian group and A^* be the set of nonzero elements in A . We define

$$F(G, A) = \{f \mid f : E(G) \mapsto A\} \quad \text{and} \quad F^*(G, A) = \{f \mid f : E(G) \mapsto A^*\}.$$

For each $f \in F(G, A)$, the **boundary** of f is a function $\partial f : V(G) \mapsto A$ defined by $\partial f = \sum_{e \in E^+(v)} f(e) - \sum_{e \in E^-(v)} f(e)$, where " \sum " refers to the addition in A . We define $Z(G, A) = \{b \mid b : V(G) \mapsto A \text{ with } \sum_{v \in V(G)} b(v) = 0\}$.

An undirected graph G is **A -connected**, if G has an orientation G' such that for every function $b \in Z(G, A)$, there is a function $f \in F^*(G', A)$ such that $\partial f = b$. For an Abelian

group A , let $\langle A \rangle$ denote the family of graphs that are A -connected. It has been observed in [26] that $G \in \langle A \rangle$ is independent of the orientation of G .

Clearly, an **A -nowhere-zero-flow** (abbreviated as A -NZF) in a digraph G is a function $f \in F^*(G, A)$ such that $\partial f = 0$.

The concept of A -connectivity was introduced by Jaeger *et al* in [26], which is a generalization to the A -NZF.

Jaeger proposed several conjectures on group connectivity of graphs.

Conjecture 3.1.1 (Jaeger *et al* [26]) *Every 3-edge connected graph is Z_5 -connected.*

Conjecture 3.1.2 (Jaeger *et al* [26]) *Every 5-edge connected graph is Z_3 -connected.*

A weaker version of the Conjecture 3.1.2 was also proposed:

Conjecture 3.1.3 (Jaeger *et al* [26]) *There is an integer k such that every k -edge connected graph is Z_3 -connected.*

Note that Conjecture 3.1.1 is stronger than the 5-flow conjecture, Conjecture 3.1.2 is stronger than the 3-flow conjecture and Conjecture 3.1.3 is equivalent to the weak 3-flow conjecture (see [26]). For these conjectures, Jaeger got some partial results.

Theorem 3.1.4 (Jaeger *et al* [26]) *Every graph which contains 2 edge-disjoint spanning trees (in particular, every 4-edge connected graph) is A -connected for every Abelian group A with $|A| \geq 4$.*

Theorem 3.1.5 (Jaeger *et al* [26]) *Every 3-edge connected graph is A -connected for every Abelian group A with $|A| \geq 6$.*

Theorem 3.1.4 and Theorem 3.1.5 are generalizations to Theorem 2.1.7 and Theorem 2.1.5, respectively.

A connected graph G is *locally connected* if $G[N(v)]$ is connected for every $v \in V(G)$. A graph G is *triangularly connected* if for every $e_1, e_2 \in E(G)$, there exists a sequence of circuits C_1, C_2, \dots, C_k such that $e_1 \in E(C_1)$, $e_2 \in E(C_k)$, $|E(C_i)| \leq 3$ for $1 \leq i \leq k$, and such that $E(C_j) \cap E(C_{j+1}) \neq \emptyset$ for $1 \leq j \leq k-1$. Note that every locally connected graph on at least three vertices is also triangularly connected.

Motivated by the 3-flow conjecture and the conjecture above for Z_3 -connectivity, we study the family of triangularly connected graphs and confirm these two conjectures for this family of graphs.

The following are the main results of this chapter.

Theorem 3.1.6 *Let G be a triangularly connected graph. Then $G \in \langle A \rangle$ for all additive Abelian group A with $|A| \geq 3$ if and only if $G \notin \langle WF \rangle$.*

As the corollary of Theorem 3.1.6, we get the following result.

Theorem 3.1.7 *Let G be a triangularly connected graph. Then G admits a Z_3 -NZF if and only if $G \neq G_1 \oplus G_2 \oplus \dots \oplus G_k$ such that $G_i \in \langle W_{\text{odd}} \rangle$.*

Theorem 3.1.7 is a generalization of Theorem 2.1.16.

Corollary 3.1.8 *The 3-flow conjecture is true for the family of triangularly connected graphs.*

Proof. By the structure of forbidden graphs in Theorem 3.1.7, every such graph has at least one small degree (less than 4) vertex. Therefore, it is not 4-edge connected. Then by Theorem 3.1.7, the 3-flow conjecture is true for the family of triangularly connected graphs.

■

3.2 Lemmas

Lemma 3.2.1 (*Chen et al [11]*) (1) Let H be a subgraph of G and $H \in \langle Z_3 \rangle$, then $G/H \in \langle Z_3 \rangle$ if and only if $G \in \langle Z_3 \rangle$.

(2) If $G \in \langle Z_3 \rangle$, then for any $e \in E(G^c)$, $G + e \in \langle Z_3 \rangle$, where G^c is the complement of G .

Lemma 3.2.2 (*Lai [32]*) (1) Let C_n be a n -circuit with $n \geq 2$ and let A be an Abelian group. Then, $C_n \in \langle A \rangle$ if and only if $|A| \geq n + 1$.

(2) Let T be a connected spanning subgraph of G . If for each edge $e \in E(T)$, G has a subgraph $H_e \in \langle A \rangle$ with $e \in H_e$, then $G \in \langle A \rangle$.

Lemma 3.2.3 Let $G \cong W_n$ for some integer $n \geq 3$. Then G admit a Z_3 -NZF if n is even and G admits no Z_3 -NZF if n is odd.

Proof. By Lemma 2.2.4, we need only to show that any even wheel has a modular 3-orientation while any odd wheel does not.

Suppose that G is an even wheel with circuit $v_1, v_2, \dots, v_{2k}v_1$ and center v_0 . Since it is an even wheel, we orient the edges of G such that v_i is the head of all the edges incident with it for i is odd. And then orient remaining edges such that v_0 is the head. It is clear, the resulting orientation is a modular 3-orientation of G .

Now suppose that G is an odd wheel with circuit $v_1, v_2, \dots, v_{2k+1}v_1$ and center v_0 . Since $d_G(v_1) = d_G(v_2) = \dots = d_G(v_{2k+1}) = 3$, if there is a modular 3-orientation of G , then each vertex v_i ($1 \leq i \leq 2k + 1$) is either the head of all the 3 edges incident with it or the tail of the 3 edges incident with it. But it is impossible because $v_1, v_2, \dots, v_{2k+1}v_1$ is a circuit with odd number of vertices. Therefore, G admits no Z_3 -NZF. ■

Lemma 3.2.4 (1). Let $G \cong W_{2n}$ for some integer $n \geq 1$. Then $G \in \langle Z_3 \rangle$.

(2). Let $G \cong W_{2n+1}$ be a digraph and $b \in Z(G, Z_3)$. Then there exists an $f \in F^*(G, Z_3)$ with $\partial f = b$ if and only if $b \neq \bar{0}$.

Proof. We prove (1) and (2) at the same time by induction on n . If $n = 1$, then W_{2n} can be obtained from a 3-circuit by adding a parallel edge. By Lemma 3.2.2-(1) and Lemma 3.2.1-(1), it is Z_3 -connected.

For the sake of convenience, let $v_1, v_2 \cdots, v_m v_1$ be the circuit of W_m and v_0 be its center. Edges are oriented so that $v_1 v_2 \cdots v_m v_1$ is a directed circuit and all the other edges with v_0 as the head.

By induction hypothesis, we may assume that Lemma 3.2.4 holds for W_k with $k \leq 2n$.

(I). Let us consider $G \cong W_{2n+1}$ first.

\implies By contradiction. Suppose that $b = \bar{0}$. There exists an $f \in F^*(G, Z_3)$ with $\partial f = b = \bar{0}$. This means $G \cong W_{2n+1}$ admits a Z_3 -NZF, which is impossible by Lemma 3.2.3. So $b \neq \bar{0}$.

\Leftarrow Suppose that $b \neq \bar{0}$. Without loss of generality, we may assume $b(v_{2n+1}) \neq 0$. Then let $G' = G \setminus \{v_0 v_{2n+1}\}$ and define

$$b'(v) = \begin{cases} b(v) & \text{if } v \notin \{v_0, v_{2n+1}\}, \\ b(v) + b(v_{2n+1}) & \text{if } v = v_0, \\ 0 & \text{if } v = v_{2n+1}. \end{cases}$$

Let $G'' = G' \setminus \{v_1 v_{2n+1}, v_{2n+1} v_{2n}\} \cup \{v_1 v_{2n}\}$ and orient the edge $v_1 v_{2n}$ as $v_{2n} \rightarrow v_1$. Let b'' be the restriction of b' to $V(G'')$. Clearly, $b'' \in Z(G'', Z_3)$ and $G'' \cong W_{2n}$. By induction hypothesis, there exists an $f'' \in F^*(G'', Z_3)$ with $\partial f'' = b''$. Now let us define

$$f'(e) = \begin{cases} f''(e) & \text{if } e \neq v_1 v_{2n}, \\ f''(v_1 v_{2n}) & \text{if } e \in \{v_{2n} v_{2n+1}, v_{2n+1} v_1\}. \end{cases}$$

By the relation between G'' and G' , we have $f' \in F^*(G', Z_3)$ and $\partial f' = b'$. Clearly this f' can be extended to an $f \in F^*(G, Z_3)$ with $\partial f = b$ by defining

$$f(e) = \begin{cases} f'(e) & \text{if } e \neq v_0 v_{2n+1}, \\ b(v_{2n+1}) & \text{if } e = v_0 v_{2n+1}. \end{cases}$$

(II). Now let us consider $G \cong W_{2n+2}$. Let $b \in Z(G, Z_3)$.

Case 1. $b = \bar{0}$.

By Lemma 3.2.3, W_{2n+2} admits a Z_3 -NZF, then there exists an $f \in F^*(G, Z_3)$ with $\partial f = \bar{0}$.

Case 2. Exactly one of $b(v_i)$ ($1 \leq i \leq 2n+2$) is not 0.

Suppose that $b(v_1) = i \neq 0$, then $b(v_0) = -i$ and $b(v_j) = 0$ for $2 \leq j \leq 2n+2$. Let us first reorient the edges to get a digraph G' as follows: Orient $v_0 v_1$ as $v_0 \rightarrow v_1$ and let v_k be the tail of all the other edges incident with it if k is odd, let v_k be the head of all the edges incident with it if k is even and not zero. Define $f' : E(G') \mapsto Z_3$ as $f'(e) = i$ for each $e \in E(G')$. It is easy to verify that $f' \in F^*(G', Z_3)$ with $\partial f' = b$. Now, we can get an $f \in F^*(G, Z_3)$ with $\partial f = b$ by defining

$$f(e) = \begin{cases} f'(e) & \text{if } e \text{ has the same orientation in } G \text{ as in } G', \\ -f'(e) & \text{otherwise.} \end{cases}$$

Case 3. At least two of $b(v_i)$ ($1 \leq i \leq 2n+2$) are not 0.

Without loss of generality, we may assume $b(v_{2n+2}) \neq 0$. Then let $G' = G \setminus \{v_0 v_{2n+2}\}$ and define

$$b'(v) = \begin{cases} b(v) & \text{if } v \notin \{v_0, v_{2n+2}\}, \\ b(v) + b(v_{2n+2}) & \text{if } v = v_0, \\ 0 & \text{if } v = v_{2n+2}. \end{cases}$$

Let $G'' = G' \setminus \{v_1 v_{2n+2}, v_{2n+2} v_{2n+1}\} \cup \{v_1 v_{2n+1}\}$ and orient the edge $v_1 v_{2n+1}$ as $v_{2n+1} \rightarrow v_1$. Let b'' be the restriction of b' to $V(G'')$. Then $b'' \in Z(G'', Z_3)$, $G'' \cong W_{2n+1}$ and $b'' \neq \bar{0}$. By

(I) there exists an $f'' \in F^*(G'', Z_3)$ with $\partial f'' = b''$. Similar to the proof in (I), we can get an $f \in F^*(G, Z_3)$ with $\partial f = b$. ■

Lemma 3.2.5 *Let G be a triangularly connected graph. If H is a non-trivial subgraph of G and $H \in \langle Z_3 \rangle$, then $G \in \langle Z_3 \rangle$.*

Proof. Let H' be a subgraph of G with $E(H) \subseteq E(H')$ such that $H' \in \langle Z_3 \rangle$ and $|E(H')|$ as large as possible. We are to show that $H' = G$. Assume not, let $e \in E(G) \setminus E(H')$ and $e' \in E(H')$. Let C_1, C_2, \dots, C_k be a triangle chain joining e' and e in G with $e' \in C_1$ and $e \in C_k$. Let i be the smallest subscribe such that $C_i \not\subseteq E(H')$. Note that C_i shares at least one edge with H' . Therefore, if we let $H'' = H' \cup C_i$, then H''/H' is a 2-circuit a loop which is Z_3 -connected. Then by Lemma 3.2.1-(1), $H'' \in \langle Z_3 \rangle$, contrary to the choice of H' . ■

Lemma 3.2.6 *Let G be a graph. Then $G \in \langle Z_3 \rangle$ if and only if $G \oplus W_{2n+1} \in \langle Z_3 \rangle$ for any integer $n \geq 1$.*

Proof. If $G \in \langle Z_3 \rangle$, then $G \oplus W_{2n+1}/G$ is a triangularly connected graph containing a 2-circuit. By Lemma 3.2.5, $G \oplus W_{2n+1}/G \in \langle Z_3 \rangle$. Therefore by Lemma 3.2.1-(1), $G \oplus W_{2n+1} \in \langle Z_3 \rangle$.

Conversely, for any $b \in Z(G, Z_3)$, let us define $b^* \in Z(G \oplus W_{2n+1}, Z_3)$ as follows:

$$b^*(v) = \begin{cases} b(v) & \text{if } v \in V(G), \\ 0 & \text{otherwise} \end{cases}$$

Since $G \oplus W_{2n+1} \in \langle Z_3 \rangle$, then there exists $f^* \in F^*(G \oplus W_{2n+1}, Z_3)$ such that $\partial f^* = b^*$. Let f be the restriction of f^* to $E(G)$. We will show that $f \in F^*(G, Z_3)$ and $\partial f = b$.

Let v_0 be the center of W_{2n+1} and $v_1 v_2 \in E(G) \cap E(W_{2n+1})$. Without loss of generality, let $v_0 v_2, v_1 v_2, v_3 v_2$ be oriented so that v_2 is the head of these three edges.

If $f^*(v_0v_2) + f^*(v_1v_2) + f^*(v_3v_2) = \alpha \in Z_3 \setminus \{0\}$ then change $f(v_1v_2)$ to be α . It is easy to see that $f \in F^*(G, Z_3)$ and $\partial f = b$. So assume that $\alpha = 0 \in Z_3$. In this case, it is also easy to see that the restriction of f^* on W_{2n+1} is a 3-NZF which contradicts Lemma 3.2.4-(2). Therefore $G \in \langle Z_3 \rangle$. ■

Lemma 3.2.7 (Devos et al [12]) *Let G_1, G_2 be graphs and let $H = G_1 \oplus G_2$. If neither G_1 or G_2 is Z_3 -connected, then H is not Z_3 -connected.*

Lemma 3.2.8 *Let $H \in \langle F \rangle$. Then $H \notin \langle Z_3 \rangle$.*

Proof. Since $H \in \langle F \rangle$, H is a nontrivial subgraph of W_{2n+1} for some integer n . By Lemma 3.2.4-(2), $W_{2n+1} \notin \langle Z_3 \rangle$. Then by Lemma 3.2.5, $H \notin \langle Z_3 \rangle$. ■

Lemma 3.2.9 *Let $G = G_1 \cup G_2$ be a graph so that G_1 and G_2 are two distinct wheels. If $|E(G_1) \cap E(G_2)| \geq 2$, then $G \in \langle Z_3 \rangle$.*

Proof. Clearly, G is a triangularly connected. If one of G_i is an even wheel, then by Lemma 3.2.4-(1) and Lemma 3.2.5, $G \in \langle Z_3 \rangle$.

So we may assume both G_1 and G_2 are odd wheels. Since $|E(G_1) \cap E(G_2)| \geq 2$, let $e_1 = u_1u_2, e_2 = w_1w_2$ be two edges in $E(G_1) \cap E(G_2)$. If the edge induced graph $G[\{e_1, e_2\}]$ has a degree one vertex which is not the center of neither G_1 nor G_2 , then let v be that vertex; Otherwise, e_1, e_2 must share one vertex, say $u_2 = w_2$, and u_1, w_1 are centers of G_1 and G_2 respectively. Then $u_1w_1 \in E(G_1) \cap E(G_2)$. For this case, let $v = u_2$. Thus for all these two cases, $G_1 \setminus v$ and $G_2 \setminus v$ still share at least one edge. Clearly, both $G_1 \setminus v$ and $G_2 \setminus v$ are triangularly connected. Therefore $G \setminus v$ is triangularly connected.

Clearly, $d_G(v) \geq 4$. Let $N_G(v) = \{v_1, v_2, \dots, v_k\}$ with $k \geq 4$. Since G is triangularly connected, v must be contained in some triangle, say vv_1v_2v . Let G^* be a new graph obtained

from G by deleting v and adding a parallel edge v_1v_2 . Because $G \setminus v$ is triangularly connected, G^* remains triangularly connected and contains a 2-circuit. By Lemma 3.2.5, $G^* \in \langle Z_3 \rangle$. By Lemma 3.2.2-(1) and Lemma 3.2.1-(1), $G^* \cup \{vv_3, vv_4, \dots, vv_k\} \in \langle Z_3 \rangle$. Then it follows $G \in \langle Z_3 \rangle$. ■

3.3 Proof of the theorem

Proof of Theorem 3.1.6. Note that since G is a triangularly connected graph, each edge of G is contained in at least one circuit C with $|E(C)| \leq 3$, then by Lemma 3.2.2-(1) and (2), $G \in \langle A \rangle$ for any Abelian group A with $|A| \geq 4$. So we need only to prove the theorem for $A = Z_3$.

\implies By contradiction. Suppose that $G \in \langle WF \rangle$. By Lemma 3.2.8, for any $H \in \langle F \rangle$, we have $H \notin \langle Z_3 \rangle$ and by Lemma 3.2.4-(2), we have $W_{2n+1} \notin \langle Z_3 \rangle$. Therefore by Lemma 3.2.7, $G \notin \langle Z_3 \rangle$, a contradiction.

\Leftarrow By contradiction. Let G be a graph such that

(1) G is a counterexample to Theorem 3.1.6, i.e. G is triangularly connected, $G \notin \langle WF \rangle$ and $G \notin \langle Z_3 \rangle$.

(2) Subjected to (1), $|E(G)|$ is minimized.

Claim 1. The girth of G is 3 and $\delta(G) \geq 3$.

Clearly, $G \not\cong K_3$. It is easy to see that G contains no 2-circuits. Otherwise, if G contains a 2-circuit C , then by Lemma 3.2.2-(1) and Lemma 3.2.5, $G \in \langle Z_3 \rangle$, contrary to (1).

If there exists $v \in V(G)$ such that $d_G(v) = 2$, then let $C = vxyv$ be a triangle containing the vertex v . Since G contains no 2-circuits, C is the only triangle containing vx (and vy as well). Therefore $G \setminus v$ remains triangularly connected.

If $G \setminus v \in \langle Z_3 \rangle$, then it follows $G \in \langle Z_3 \rangle$, contrary to (1); If $G \setminus v \notin \langle Z_3 \rangle$, then either $G \setminus v \in \langle WF \rangle$ or it is contrary to (2). Suppose that $G \setminus v \in \langle WF \rangle$. Then $G = (G \setminus v) \oplus C$, that is, $G \in \langle WF \rangle$, contrary to (1). So $\delta(G) \geq 3$. \square

Claim 2. Let H be a maximal triangularly connected subgraph of G such that $|E(H)| < |E(G)|$. Then $|E(G)| - |E(H)| = 1$ and $V(G) = V(H)$.

A proper subgraph of G that is triangularly connected exists since any triangle of G is a subgraph of this kind.

If $|E(G)| - |E(H)|$ has more than two edges, then there is an edge $e \in H$ that is contained in a triangle C such that $E(C) \setminus E(H) \neq \emptyset$ since G is triangularly connected.

So, adding $E(C) \setminus E(H) \neq \emptyset$, which consists of at most two edges, into H . Repeat this procedure, we are able to obtain the maximal one H . Clearly, $|E(G)| - |E(H)| \leq 2$.

If $|E(G)| - |E(H)| = 2$, the two edges in $E(G) \setminus E(H)$ must share one vertex, we may assume these two edges are vx, vy . Clearly, $xy \in E(H)$. If $v \in V(G) \setminus V(H)$, then $d_G(v) = 2$, contrary to Claim 1. So $V(H) = V(G)$. Clearly, $d_G(v) \geq 4$. Delete vx, vy from G and add a parallel edge to xy , the resulting graph, call it H^* , is triangularly connected and contains a 2-circuit. Then by Lemma 3.2.2-(1) and Lemma 3.2.5, $H^* \in \langle Z_3 \rangle$. Therefore $G \in \langle Z_3 \rangle$, contrary to (1).

Then $|E(G)| - |E(H)| = 1$. By Claim 1, $\delta(G) \geq 3$, then clearly $V(G) = V(H)$. \square

The last one edge of $E(G) \setminus E(H)$ will add the last triangle (say, $C = xyzx$) into H . For convenience, we may assume that $e = xz \in E(G) \setminus E(H)$.

Claim 3. $H \in \langle WF \rangle$.

Otherwise, by the choice of G , $H \in \langle Z_3 \rangle$. Then G/H is a loop which is Z_3 -connected. By Lemma 3.2.1-(1), $G \in \langle Z_3 \rangle$, contrary to the choice of G . Therefore $H \in \langle WF \rangle$. \square

Claim 4. H contains exactly two end-WF-components and one end-WF-component con-

tains yx , the other one contains yz .

If H contains exactly one WF-component, then H is either a Fan-graph or an odd wheel. Suppose that H is a Fan-graph. Since $\delta(G) \geq 3$, then x and z will be the end points of H , therefore G will be a wheel, contrary to (1). If G is an odd wheel, then $G = H \cup e$ contains an even wheel. By Lemma 3.2.4-(1) and Lemma 3.2.5, $G \in \langle Z_3 \rangle$, contrary to (1).

Then we will show that H contains exactly two end-WF-components. Otherwise, since $E(G) \setminus E(H) = \{xz\}$, $yx, yz \in E(H)$ and $\delta(G) \geq 3$, then there exists an end-WF-component $H' \in \langle W_{odd} \rangle$ such that $H = H^* \oplus H'$ with $H^* \in \langle WF \rangle$ and $\{x, y, z\} \cap (V(H') \setminus V(H^*)) = \emptyset$. Therefore $G = (H^* \cup e) \oplus H'$. Since $G \notin \langle WF \rangle$, $H^* \cup e \notin \langle WF \rangle$. Since $\{x, y, z\} \subseteq V(H^*)$ and $H^* \in \langle WF \rangle$, then $H^* \cup e$ is triangularly connected. By Lemma 3.2.6, $H^* \cup e$ is a smaller counterexample, contrary to (2).

The WF-component of H containing yx must be an end-WF-component. Otherwise, since there are two end-WF-components of H , then at least one of them, say H_1 , has the similar property as H' mentioned above. By the same argument, we can get a contradiction. Similarly, the WF-component of H containing yz is the other end-WF-component. \square

Final Step. By Claim 4, let H_1, H_2 be the two end-WF-components of H . We may assume that $yx \in E(H_1)$, $yz \in E(H_2)$. Clearly, H_1 will not share yx with other WF-components of H . Otherwise, H_1 would have the similar property as H' in Claim 4. Similarly, H_2 will not share yz with other WF-components of H .

Since H_1, H_2 are the only two end-WF-components of H , both containing vertex y . Then every other WF-component of H must contain y , too. Clearly, no edges are contained in more than two WF-components of H (otherwise, H would have more than two end-WF-components) and if an edge is contained in two WF-components of H , then it must have y as one end.

Since $\delta(G) \geq 3$, then each WF-component which belongs to $\langle F \rangle$ must have y as its center.

By the structure of H , there is a path in $N_H(y)$ connecting x and z , say, $P = xu_1u_2 \cdots u_kz$, such that P contains at least one edge of each WF-component of H . Then $H[V(P) \cup \{y\}]$ contains a subgraph $H_4 \in \langle F \rangle$ with y as its center and yx, yz as its end edges. Therefore, $H_5 = H_4 \cup \{xz\}$ is a wheel. Clearly H_5 contains at least two edges of each WF-component of H because P contains at least one edge of each WF-component.

We claim that H contains at least one odd wheel, say H_6 , as its WF-component. Otherwise, each WF-component belongs to $\langle F \rangle$ and y is the center of each WF-component. Then $H \in \langle F \rangle$ with y as its center and H has only one WF-component, a contradiction.

Clearly H_5 and H_6 are two distinct wheels sharing at least two edges, by Lemma 3.2.9, $H_5 \cup H_6 \in \langle Z_3 \rangle$. By Lemma 3.2.5, $G \in \langle Z_3 \rangle$, contrary to (1). And this ends the proof of the Theorem 3.1.6. ■

Proof of Theorem 3.1.7. Similarly to the proof of Lemma 3.2.6, we can get the following result:

A graph H admits a Z_3 -NZF if and only if $H \oplus W_{2k+1}$ admits a Z_3 -NZF for any integer $k \geq 1$.

Also it is easy to see that for any $H \in \langle F \rangle$, H admits a Z_3 -NZF, therefore this theorem is a corollary of Theorem 3.1.6. ■

Chapter 4

On circular flows of graphs

In this chapter, we obtain some sufficient conditions for graphs with circular flow index less than 4. As an immediate corollary, we give a simple proof of a result obtained recently by Galluccio and Goddyn (*Combinatorica*, 2002), and obtain a larger family of such graphs.

4.1 Introduction

The concept of circular flow, a dual version of circular coloring, was introduced by Goddyn et. al. [16]. (Note that the circular flow problem is not a fractional version of the integer flow problem since it is a real line extension instead of a linear programming relaxation).

Definition 4.1.1 *Let $G = (V, E)$ be a finite 2-edge connected graph. The circular flow index $\phi_c(G)$ of G is defined by*

$$\phi_c(G) = \min_D \max_{\emptyset \neq X \subset V} \frac{|\delta X|}{|\delta^+ X|}$$

where D ranges over the strong orientations of G , $\delta^+ X$ denotes the set of arcs from X to $V - X$ and $\delta X = \delta^+ X \cup \delta^+(V - X)$.

The smallest integer k such that G admits a k -NZF is called the flow index of G , denoted by $\phi(G)$. Note that $\phi(G) = \lceil \phi_c(G) \rceil$ (see [16]). Therefore, circular flow index can be considered as a refinement of flow index and all the flow conjectures can be restated as follows:

The 5-flow Conjecture (Tutte [46]) Let G be a 2-edge connected graph, then $\phi_c(G) \leq 5$.

The 4-flow Conjecture (Tutte [48]) Let G be a 2-edge connected graph containing no subdivision of the Petersen graph, then $\phi_c(G) \leq 4$.

The 3-flow Conjecture (Tutte, unsolved problem 48 in [6]) Let G be a 4-edge-connected graph, then $\phi_c(G) \leq 3$.

Recently, Galluccio and Goddyn ([15]) proved the following result which is a new approach to the 3-flow conjecture in the direction of circular flows.

Theorem 4.1.2 (Galluccio and Goddyn [15]) *Let G be a 6-edge connected graph. Then $\phi_C(G) < 4$.*

Linear programming methods were applied in the original proof of Theorem 4.1.2 ([15]). In this chapter, we will prove some sufficient conditions for graphs with $\phi_C(G) < 4$. These results (Theorem 4.3.1 and 4.3.2) are to be proved in purely combinatorial/structural methods and Theorem 4.1.2 will follow immediately. The proof of Theorem 4.1.2 that we will provide here is simpler than the original one in [15].

Now let us introduce some notations and terminology for this chapter. Let G be a graph and H be a subgraph of G . For a vertex $v \in V(G)$, the number of edges of H incident with v is denoted by $d_H(v)$.

A *cycle* C is a subgraph G such that $d_C(v) \equiv 0 \pmod{2}$. A *parity subgraph* P of G is a subgraph of G such that $d_P(v) \equiv d_G(v) \pmod{2}$ for every vertex v of G . A decomposition of a graph G is called a *parity subgraph decomposition* provided each member of the decomposition

is a parity subgraph of G . A parity subgraph decomposition of G is called trivial if it has only one member (the graph G itself).

Let $T \subseteq V(G)$. A T -join in G is a subset $J \subseteq E(G)$ such that T is the set of all odd-degree vertices in the induced subgraph $G[J]$.

A graph H is *collapsible* if and only if, for every T with $|T| \equiv 0 \pmod{2}$, H contains a connected and spanning T -join J in H .

We refer readers to [6] or [53] for terminology and notation not defined in this chapter.

4.2 Lemmas

Lemma 4.2.1 (Hoffman [19]) *Let G be a 2-edge connected graph, D be an orientation of G and a, b be two positive integers ($a \leq b$). The following two statements are equivalent.*

(1)

$$\frac{a}{b} \leq \frac{|\delta^+ X|}{|\delta^+(V(G) - X)|} \leq \frac{b}{a}$$

for each $\emptyset \subset X \subset V(G)$;

(2) G admits a nowhere-zero integer flow (D, f) such that $a \leq f(e) \leq b$ for each $e \in E(G)$.

Lemma 4.2.2 (Nash-Williams [35], Tutte [47], or see [18], [30], [39]) *Every $2k$ -edge connected graph contains k edge-disjoint spanning trees.*

Lemma 4.2.3 (Itai and Rodeh [21]) *Every spanning tree contains a parity subgraph.*

4.3 Main results

The following are the main results of this chapter. With these results, we will give a combinatorial, simple proof of Theorem 4.1.2 and find a large family of graphs with circular flow index less than 4.

Theorem 4.3.1 *Let G be a graph and (D, f) be a nowhere-zero 4-flow of G . Let $E_{f=i} = \{e \in E(G) : f(e) = i\}$ for $i = \pm 1, \pm 2, \pm 3$. If $\delta(X) \cap E_{f=\pm 2} \neq \emptyset$ for every $X : \emptyset \subset X \subset V(G)$ with $|\delta(X)| \equiv 0 \pmod{4}$, then $\phi_c(G) < 4$.*

Proof. It is sufficient to consider only positive flow since a positive flow can be obtained from a nowhere-zero flow by reversing orientations of some edges and changing the corresponding function value signs. For a positive 4-flow (D, f) of G , it is easy to verify that D is a strong orientation, that is, each edge is contained in a directed circuit. For each nonempty $X \subset V(G)$, since (D, f) is a positive 4-flow, by Lemma 4.2.1, we have $\delta(X)/\delta^+(X) \leq 4$. Suppose there exists $X \subseteq V(G)$ such that $\delta(X)/\delta^+(X) = 4$. Consider the distribution ratio of $\delta^+(X) \cap E_{f=i}$ and $\delta^+(V - X) \cap E_{f=i}$, for each $i \in \{1, 2, 3\}$ by applying the equation

$$\sum_{e \in \delta^+(X)} f(e) = \sum_{e \in \delta^+(V-X)} f(e),$$

then we must have that $\delta^+(X) \subseteq E_{f=3}$, $\delta^+(V - X) \subseteq E_{f=1}$ and $|\delta(X)| \equiv 0 \pmod{4}$. This contradicts with the fact $\delta(X) \cap E_{f=\pm 2} \neq \emptyset$ (for $i = 1, 2$). Hence, for each $X \subseteq V(G)$, $\delta(X)/\delta^+(X) < 4$ which implies $\phi_c(G) < 4$. ■

Clearly, similar result can be obtained for $\phi_c(G) < k$ for any integer $k > 3$.

Theorem 4.3.2 *Let G be a graph. If G has a nontrivial parity subgraph decomposition such that at least one of its members is connected and spanning, then $\phi_c(G) < 4$.*

Proof. Let $\{P_1, P_2, P_3, \dots\}$ be a non-trivial parity subgraph decomposition of G with P_3 connected and spanning. Let $C_1 = P_1 \cup P_2$, $C_2 = G \setminus E(P_1)$. Clearly, both C_1 and C_2 are

cycles with $E(C_1) \cup E(C_2) = E(G)$. Let (D, f_i) be a 2-flow of G with $\text{supp}(f_i) = E(C_i)$. Then $(D, f) = (D, f_1 + 2f_2)$ is a nowhere-zero 4-flow of G . For each subset X : $\emptyset \subset X \subset V(G)$, $\delta(X) \setminus E(C_1) \neq \emptyset$ since the spanning connected subgraph P_3 is contained in $C_2 - C_1$. Hence, $\delta(X) \cap E_{f=\pm 2} \neq \emptyset$. By Theorem 4.3.1, we have that $\phi_c(G) < 4$. ■

Proof of Theorem 4.1.2. Since G is 6-edge connected, by Lemma 4.2.2, let T_1, T_2, T_3 be three edge disjoint spanning trees of G . By Lemma 4.2.3, we can let P_i be a parity subgraph of T_i (for $i = 1, 2$ only). Then we have a parity subgraph decomposition $\{P_1, P_2, G \setminus (P_1 \cup P_2)\}$ described in Theorem 4.3.2, where $G \setminus (P_1 \cup P_2) \supseteq T_3$ is connected and spanning. ■

The concept of collapsible graphs and its corresponding reduction methods were introduced by Catlin ([8]). Note that collapsible graphs contain *spanning, connected T -joins* (including parity subgraphs). Many families of collapsible graphs have been discovered. For example (see [8], [10]),

- (1). Graphs containing two edge-disjoint spanning trees,
- (2). 2-edge connected graphs containing two edge-disjoint spanning forests F_1 and F_2 such that F_1 is a tree and $|E(F_2)| \geq |V(G)| - 2$,
- (3). 2-edge connected graphs containing two edge-disjoint spanning forests F_1 and F_2 such that $|E(F_1)| + |E(F_2)| \geq 2(|V(G)| - 2)$ and G is not contractible to $K_{2,t}$ for some $t \geq 3$,
- (4). Graphs in which every edge is contained in a triangle.

The family of graphs described in the following corollary is bigger than that in Theorem 4.1.2.

Corollary 4.3.3 *If a graph contains two edge-disjoint subgraphs P and H such that P is a parity subgraph and H is a connected, spanning collapsible subgraph of G , then $\phi_C(G) < 4$.*

Proof. Let $H' = G - E(P)$. Here $H' (\supseteq H)$ is a cycle and remains connected, spanning and collapsible in G . Let P' be a connected, spanning parity subgraph of G contained in H' .

Then we obtain a parity subgraph decomposition $\{H' - P', P, P'\}$ described in Theorem 4.3.2.

■

Chapter 5

On flows of bidirected graphs

Bouchet [7] conjectured that every bidirected graph which admits a nowhere-zero bidirected flow also admits a nowhere-zero bidirected 6-flow. He proved that it is true with 6 replaced by 216. Zyka [56] proved in his Ph.D dissertation that this conjecture is true with 6 replaced by 30. Khelladi [28] proved it is true with 6 replaced by 18 for 4-connected graph. In this chapter, we prove that Bouchet's conjecture is true for 6-edge connected bidirected graphs.

5.1 Introduction

A bidirected graph G is a graph with vertex set $V(G)$ and edge set $E(G)$ such that each edge is oriented as one of the four possibilities: $\bullet \rightarrow \rightarrow \bullet$, $\bullet \leftarrow \leftarrow \bullet$, $\bullet \rightarrow \leftarrow \bullet$, $\bullet \leftarrow \rightarrow \bullet$.

An edge with orientation $\bullet \rightarrow \rightarrow \bullet$ (resp. $\bullet \leftarrow \leftarrow \bullet$) is called an *in-edge* (resp. *out-edge*).

Edges with other orientations are called *ordinary edges*. The set of in(out)-edges is denoted by $E_i(G)$ ($E_o(G)$, respectively). We define $O(G) = |E_i(G)| + |E_o(G)|$.

For an ordinary edge e , reversing the orientation of e means the natural way to change its orientation. For any non-ordinary edge $e \in E(G)$, reversing the orientation of e means

changing e from an in(out)-edge to an out(in)-edge. Note that after reversing the orientation of an edge of G , $O(G)$ remains the same.

Let G be a bidirected graph, for any $v \in V(G)$, the set of all edges with tails (or heads) at v is denoted by $E^+(v)$ (or $E^-(v)$) and we define $E(v) = E^+(v) \cup E^-(v)$ (For general graph G , we use $E(v)$ to denote the set of edges which are incident with v in G). Readers are referred to [6] for terminology not defined in this chapter.

Definition 5.1.1 *Let G be a bidirected graph and f be a function: $E(G) \mapsto \mathbb{Z}$. Then*

(1) f is called a **bidirected k -flow** of G if $-k + 1 \leq f(e) \leq k - 1$ for every edge $e \in E(G)$ and $\sum_{e \in E^+(v)} f(e) = \sum_{e \in E^-(v)} f(e)$ for every $v \in V(G)$;

(2) f is called a **bidirected modular k -flow** of G if $0 \leq f(e) \leq k - 1$ for every edge $e \in E(G)$ and $\sum_{e \in E^+(v)} f(e) \equiv \sum_{e \in E^-(v)} f(e) \pmod{k}$ for every $v \in V(G)$;

(3) The **support** of a bidirected k -flow (resp. modular k -flow) f of G is the set of edges of G with $f(e) \neq 0$ (resp. $f(e) \not\equiv 0 \pmod{k}$), and is denoted by $\text{supp}(f)$. A bidirected k -flow (modular k -flow) of G is **nowhere-zero** if $\text{supp}(f) = E(G)$.

For nowhere-zero bidirected integer flows, Bouchet [7] proposed the following conjecture (see also Toft and Jensen's book [44]).

Conjecture 5.1.2 *Every bidirected graph which admits a nowhere-zero bidirected flow also admits a nowhere-zero bidirected 6-flow.*

Note that the value 6 in Conjecture 5.1.2 is best possible: see [7] for details. The following is a list of some partial results for Conjecture 5.1.2.

Theorem 5.1.3 *Let G be a bidirected graph admitting a nowhere-zero bidirected flow. Then*

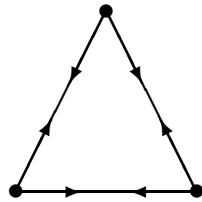
(1) (Bouchet [7]) G admits a nowhere-zero bidirected 216-flow;

- (2) (Zyka [56], or see [28]) G admits a nowhere-zero bidirected 30-flow;
- (3) (Khelladi [28]) G admits a nowhere-zero bidirected 18-flow if G is 4-connected.

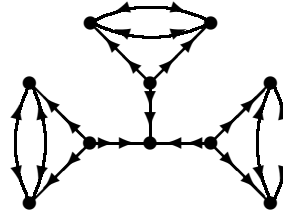
Bidirected flow is a generalization of the concept of integer flow. This is because a directed graph G can be considered as a bidirected graph G^* with $O(G^*) = 0$. However, bidirected flows and integer flows can be quite different due to the existence of those in-edges and out-edges. Some results for integer flows can be generalized to bidirected flows while some other results cannot. The following observation for bidirected 2-flows is a generalization of Tutte's 2-flow characterization. Though the proof for integer 2-flows is very straightforward, the corresponding result for bidirected 2-flow does need a few more steps in the proof.

Proposition 5.1.4 *Let G be a connected bidirected graph. Then G admits a nowhere-zero bidirected 2-flow if and only if G is an eulerian graph and $O(G)$ is even.*

Integer flows and modular flows are proved to be equivalent for general graphs (see [45, 46] or [53]). However, for bidirected graphs, they are not equivalent to each other in many cases. The followings are some examples that a nowhere-zero modular 2-flow in Example 1 and a nowhere-zero modular 3-flow in Example 2 cannot be converted to a nowhere-zero bidirected 2-flow (or 3-flow). For nowhere-zero bidirected modular 3-flows, we are able to establish their equivalent relation for 2-edge connected bidirected graphs.



Example 1



Example 2

The following is one of our main results in this chapter which provides a major step in the proof of Theorem 5.1.6.

Theorem 5.1.5 *Let G be a 2-edge connected bidirected graph. Then G admits a nowhere-zero bidirected 3-flow if and only if G admits a nowhere-zero modular bidirected 3-flow.*

By applying Theorem 5.1.5, we are able to verify Conjecture 5.1.2 for 6-edge connected bidirected graphs.

Theorem 5.1.6 *Let G be a 6-edge connected bidirected graph. If G admits a nowhere-zero bidirected flow, then it admits a nowhere-zero bidirected 6-flow.*

5.2 Proof of Proposition 5.1.4 and its applications

A *circuit* is a connected 2-regular graph and a *cycle* is a graph such that every vertex is of even degree. Let G be a bidirected graph. A circuit C of G is said to be *balanced* provided that $O(C)$ is even and *unbalanced* otherwise. A cycle P of G is said to be *balanced* provided that for each of its component P' , $O(P')$ is even. A collection of cycles $F = \{P_1, P_2, \dots, P_r\}$ of G is called a *proper r -cycle cover* of G if every P_i is balanced and $\bigcup_{i=1}^r E(P_i) = E(G)$.

Proof of Proposition 5.1.4.

“ \implies ”. Note that if f is a nowhere-zero bidirected 2-flow of G and we reverse the orientation of some edge e_1 , then the resulting graph G_1 admits a nowhere-zero bidirected 2-flow f_1 such that $f_1(e) = f(e)$ for any $e \neq e_1$ and $f_1(e_1) = -f(e_1)$. Also, $O(G) = O(G_1)$.

So we may assume that G admits a nowhere-zero bidirected 2-flow f such that $f(e) = 1$ for every edge $e \in E(G)$. Therefore, for any $v \in V(G)$, $N^+(v) = N^-(v)$ which implies G is an eulerian graph. Since $\sum_{v \in V(G)} N^+(v) = \sum_{v \in V(G)} N^-(v)$ and each ordinary edge contributes 1 to both $\sum_{v \in V(G)} N^+(v)$ and $\sum_{v \in V(G)} N^-(v)$, the total contribution of non-ordinary edges to $\sum_{v \in V(G)} N^+(v)$ must be the same as that to $\sum_{v \in V(G)} N^-(v)$. Therefore, we get $|E_i(G)| = |E_o(G)|$ and consequently, $O(G)$ is even.

“ \Leftarrow ”. Since G is an eulerian graph, let $F = \{C_1, C_2, \dots, C_k\}$ be a circuit decomposition of $E(G)$. We will prove it by induction on k .

If $k = 1$, clearly, it is true. For $k \geq 2$, since G is connected, then there exists C_i such that the induced subgraph $G^* = G[E(G) \setminus E(C_i)]$ is connected. Note that $O(C_i)$ is even if and only if $O(G^*)$ is even.

Assume that $O(C_i)$ is even. By induction hypothesis, both C_i and G^* admits nowhere-zero bidirected 2-flows, therefore G admits a nowhere-zero bidirected 2-flow. So, we may assume that $O(C_i)$ is odd ($O(G^*)$ is odd as well). Let v be any vertex in $V(C_i) \cap V(G^*)$. Suppose $v \in C_j$ for some $j \neq i$. Let C'_i be the new circuit obtained from C_i by splitting an edge $e_i = vv_i$ away from v (becomes $e_i = v'_i v_i$, where v'_i is a new vertex) and adding a new in-edge between v and v'_i (this operation is called “expending v in C_i to an in-edge $e_i = vv'_i$ ”). Similarly, we get a new graph G' from G^* by expending v only in C_j to an out-edge $e_j = vv'_j$. Clearly, both $O(C'_i)$ and $O(G')$ are even. By induction hypothesis, C'_i admits a nowhere-zero bidirected 2-flow f_1 and G' admits a nowhere-zero bidirected 2-flow f_2 . Note that $-f_1$ is also a nowhere-zero bidirected 2-flow of C'_i . So, we can assume $f_1(e_i) = f_2(e_j)$ (otherwise since $-f_1(e_i) = f_2(e_j)$ we will use $-f_1$ instead of f_1). Therefore let us define

$$f(e) = \begin{cases} f_1(e) & \text{if } e \in E(C_i), \\ f_2(e) & \text{if } e \in E(G) \setminus E(C_i) \end{cases}$$

Clearly, f is a nowhere-zero bidirected 2-flow of G . ■

The following result is a generalization of a theorem about integer flows.

Proposition 5.2.1 *Let G be a bidirected graph. If G has a proper r -cycle cover, then G admits a nowhere-zero bidirected 2^r -flow.*

Proof. Let $F = \{P_1, P_2, \dots, P_r\}$ be a proper r -cycle cover of G . By Proposition 5.1.4, each P_i admits a nowhere-zero bidirected 2-flow f_i . It is easy to verify that $\sum_{i=1}^r 2^{r-1} f_i$ is a nowhere-zero bidirected 2^r -flow. ■

5.3 Proof of Theorem 5.1.5

Let us first introduce a useful lemma whose proof is very straightforward.

Lemma 5.3.1 *Let G be a bidirected graph, E_0 be a subset of E and G_{E_0} be the bidirected graph obtained from G by reversing the orientation of each edge in E_0 . Then we have*

(1) *G admits a bidirected k -flow if and only if G_{E_0} admits a bidirected k -flow with the same support.*

(2) *G admits a bidirected modular k -flow if and only if G_{E_0} admits a bidirected modular k -flow with the same support.*

Proof. (1) Suppose that f is a bidirected k -flow of G . Let

$$f'(e) = \begin{cases} f(e) & \text{if } e \notin E_0, \\ -f(e) & \text{if } e \in E_0 \end{cases}$$

Clearly f' is a bidirected k -flow of G_{E_0} with the same support.

(2) Suppose that f is a bidirected modular k -flow of G . Let

$$f'(e) = \begin{cases} k - f(e) & \text{if } e \in E_0 \text{ and } f(e) \neq 0 \\ f(e) & \text{otherwise} \end{cases}$$

Clearly f' is a bidirected modular k -flow of G_{E_0} with the same support. ■

Lemma 5.3.2 *Let G be a bidirected cubic graph. Then G admits a nowhere-zero bidirected 3-flow if and only if G admits a nowhere-zero bidirected modular 3-flow and G has a perfect matching.*

Proof. “ \implies ”. Let f be a nowhere-zero bidirected 3-flow of G and $E_0 = \{e \mid f(e) = -1 \text{ or } -2\}$. By the proof of Lemma 5.3.1-(1), G_{E_0} admits a positive bidirected 3-flow f' . Clearly, f'

is also a nowhere-zero bidirected modular 3-flow of G_{E_0} . By Lemma 5.3.1-(2), G admits a nowhere-zero bidirected modular 3-flow. Because for every vertex $v \in V(G)$, there is exactly one incident edge e such that $f'(e) = 2$, then $E' = \{e | f'(e) = 2\}$ is a perfect matching of G .

“ \Leftarrow ”. Let f be a nowhere-zero bidirected modular 3-flow of G and $E_0 = \{e | f(e) = 2\}$. By Lemma 5.3.1-(2), $G^* = G_{E_0}$ admits a nowhere-zero bidirected modular 3-flow f' such that $f'(e) = 1$ for each edge e . Since G^* is a cubic graph, then for any vertex v either $E(v) = E^+(v)$ or $E(v) = E^-(v)$. Let M be a perfect matching of G^* . Then reverse the direction of each edge e of M and change the value $f'(e)$ to be 2. The resulting nowhere-zero bidirected modular 3-flow is also a nowhere-zero bidirected 3-flow of G_M^* . By Lemma 5.3.1-(1), G^* , and therefore G admit nowhere-zero bidirected 3-flows. ■

Definition 5.3.3 *Let G be a graph and v be a vertex of G . Suppose that $F \subset E(v)$, then $G_{[v;F]}$ is the graph obtained from G by splitting the edges of F away from v , that is, adding a new vertex v' and change the end v of the edges in F to be v' .*

Lemma 5.3.4 (Nash-Williams [36]) *Let k be an even integer and G be a k -edge connected graph and $v \in V(G)$. Let a be an integer such that $k \leq a$ and $k \leq d_G(v) - a$. Then there is an edge set $F \subset E(v)$ such that $|F| = a$ and $G_{[v;F]}$ is k -edge connected.*

Lemma 5.3.5 (Fleischner [14]) *Let G be a 2-edge connected graph. Suppose that v is a vertex of G with $d_G(v) \geq 4$ and let $e_0, e_1, e_2 \in E(v)$. Then either $G_{[v;\{e_0, e_1\}]}$ or $G_{[v;\{e_0, e_2\}]}$ is 2-edge connected or $G_{[v;\{e_0, e_1, e_2\}]}$ has more components than G .*

Lemma 5.3.6 *Let G be a 2-edge connected graph. Suppose that v is vertex of G with $d_G(v) \geq 4$ and let $e_0, e_1, e_2, e_3 \in E(v)$. Then*

- (1) *at least one of $G_{[v;\{e_0, e_i\}]}$ ($i = 1, 2, 3$) is 2-edge connected;*
- (2) *in the case of $d_G(v) = 4$, at least one of $G_{[v;\{e_0, e_i\}]}$ ($i = 1, 2$) is 2-edge connected.*

Proof. By Lemma 5.3.5, it suffices to show that at least one of $G_{[v;\{e_0,e_1,e_2\}]}$ and $G_{[v;\{e_0,e_1,e_3\}]}$ is connected. For convenience, let $e_i = vv_i$ ($i = 0, 1, 2, 3$). If $d_G(v) = 4$, then $G_{[v;\{e_0,e_1,e_2\}]}$ is connected. Otherwise e_3 will be an edge cut of G which is impossible. So, we may assume that $d_G(v) \geq 5$. Suppose that $G_{[v;\{e_0,e_1,e_2\}]}$ is disconnected. Since G is 2-edge connected then there is a path P in $G_{[v;\{e_0,e_1,e_2\}]}$ which connects v_2 to v_0 without using e_2 . Therefore $G_{[v;\{e_0,e_1,e_3\}]}$ is connected. ■

Now we are ready to prove Theorem 5.1.5.

Proof of Theorem 5.1.5.

“ \implies ”. Suppose that G admits a nowhere-zero bidirected 3-flow. Similar to the proof of the first part of Lemma 5.3.2, we can get a nowhere-zero bidirected modular 3-flow of G .

“ \impliedby ”. Prove by contradiction. Let G be a smallest counterexample with respect to $|E(G)|$. By Lemma 5.3.1, we may assume that G admits a nowhere-zero bidirected modular 3-flow f such that $f(e) = 1$ for each edge e . Then for any vertex v with $d_G(v) = 3$ we have either $E(v) = E^+(v)$ or $E(v) = E^-(v)$.

Claim 1. $\delta(G) \geq 3$.

Otherwise, suppose that there exists $v \in V(G)$ such that $N_G(v) = \{v_1, v_2\}$. Since f is a nowhere-zero bidirected modular 3-flow of G and $f(v_1v) = f(vv_2) = 1$, we can delete the edges v_1v , vv_2 from G and add a new edge v_1v_2 (or a parallel edge if v_1v_2 exists). If $v_1v \in E^+(v_1)$ ($E^-(v_1)$), then orient this new edge such that this $v_1v_2 \in E^+(v_1)$ ($E^-(v_1)$); If $v_2v \in E^+(v_2)$ ($E^-(v_2)$), then orient this new edge such that this $v_1v_2 \in E^+(v_2)$ ($E^-(v_2)$). The resulting bidirected graph is denoted by G_0 . Then G_0 is 2-edge connected and admits a nowhere-zero bidirected modular 3-flow with $|E(G_0)| < |E(G)|$. By the choice of G , G_0 admits a nowhere-zero bidirected 3-flow. Clearly, from this flow we can easily get a nowhere-zero bidirected 3-flow of G , a contradiction. □

Claim 2. There is no $v \in V(G)$ such that $E(v) \cap E^+(v) \neq \emptyset$ and $E(v) \cap E^-(v) \neq \emptyset$.

Assume that there exists a vertex $v \in V(G)$ such that $E(v) \cap E^+(v) \neq \emptyset$ and $E(v) \cap E^-(v) \neq \emptyset$. Then $d_G(v) \geq 4$ since, for any degree 3 vertex, either $E(v) = E^+(v)$ or $E(v) = E^-(v)$.

If $|E^-(v)|$ or $|E^+(v)| \geq 3$, then we may apply Lemma 5.3.6-(1) to split two edges e', e'' away from the vertex v where $e' \in E^+(v)$ and $e'' \in E^-(v)$, and the resulting graph is still 2-edge connected and f remains as a nowhere-zero bidirected modular 3-flow. Clearly, this new graph has a degree two vertex and if this graph admits a nowhere-zero 3-flow, then so does G . Similar to the proof of Claim 1, we can get a contradiction.

If $0 < |E^-(v)|, |E^+(v)| \leq 2$, then $d_G(v) = 4$ since $|E^+(v)| \equiv |E^-(v)| \pmod{3}$. By Lemma 5.3.6-(2) and with the same argument as Case 1, we get a contradiction as well. \square

By Claim 1 and Claim 2, we have either $E(v) = E^+(v)$ or $E(v) = E^-(v)$ for any $v \in V(G)$. Therefore, $d_G(v) \equiv 0 \pmod{3}$ for any $v \in V(G)$. For any v with $d_G(v) \geq 6$, we may apply Lemma 5.3.4 to split this vertex into several degree 3 vertices and the resulting graph is still 2-edge connected and f remains as a nowhere-zero bidirected modular 3-flow. Also if the resulting graph admits a nowhere-zero bidirected 3-flow, then so does G . Recursively splitting high degree vertices, we obtain a 2-edge connected cubic graph G^* which admits a nowhere-zero bidirected modular 3-flow.

By Petersen Theorem [38], the 2-connected, cubic graph G^* has a perfect matching. Then by Lemma 5.3.2, G^* admits a nowhere-zero bidirected 3-flow. So, from this flow of G^* , we can get a nowhere-zero bidirected 3-flow of G , a contradiction. \blacksquare

5.4 Proof of Theorem 5.1.6

Product of flows is a method used in the proof of the theorems of 8-flow [23], 4-flow [24] and 6-flow [41]. The following lemma generalizes this method for bidirected flows of graphs and is to be used in the proof of Theorem 5.1.6.

Lemma 5.4.1 *Let G be a bidirected graph. If G admits a bidirected k_1 -flow f_1 and a bidirected k_2 -flow f_2 such that $\text{supp}(f_1) \cup \text{supp}(f_2) = E(G)$, then G admits a nowhere-zero bidirected $k_1 k_2$ -flow.*

Proof. Let $f(e) = f_1(e) + k_1 f_2(e)$ for every $e \in E(G)$, then it is easy to verify that f is a nowhere-zero bidirected $k_1 k_2$ -flow. ■

Let G be a bidirected graph. A subgraph C of G is called a *bidirected circuit* of G provided that either C is a balanced circuit of G or C is the union of two unbalanced circuits sharing exactly one vertex or C is the union of two vertex-disjoint unbalanced circuits and a simple path meeting each of the two circuits at exactly one of its end points.

We extend Seymour's closure operation [41] to bidirected graphs as follows:

For a positive integer k , if X is a subgraph of G , then the k -closure of X in G , denoted by $\langle X \rangle_k$, is the (unique) maximal subgraph of G of the form $X \cup C_1 \cup \dots \cup C_n$, where for every i , $1 \leq i \leq n$, C_i is a bidirected circuit of G and $|E(C_i) \setminus E(X \cup C_1 \cup \dots \cup C_{i-1})| \leq k$.

By the Theorem 4.3 in [28], we have the following:

Lemma 5.4.2 *Let G be a connected bidirected graph which admits a nowhere-zero bidirected flow. Let E' be a subset of $E(G)$ such that the induced graph $G[E']$ is connected and spanning. Then $\langle E' \rangle_2 = E(G)$.*

By the Proposition 2.2 in [28], we have the following:

Lemma 5.4.3 *Let G be a bidirected graph and $k \geq 3$ be a prime integer. Let E' be a subset of $E(G)$ such that $\langle E' \rangle_{k-1} = E(G)$. Then G admits a bidirected integer flow f such that $f(e) \not\equiv 0 \pmod{k}$ for every element e of $E(G) \setminus E'$.*

The bidirected flow f obtained in Lemma 5.4.3 can be considered as a bidirected modular k -flow with the same support though the value of f can be very large.

Definition 5.4.4 Let H_1 and H_2 be two subgraphs of a graph G . The **symmetric difference** of H_1 and H_2 , denoted by $H_1 \triangle H_2$, is the subgraph of G induced by the set of edges $[E(H_1) \cup E(H_2)] \setminus [E(H_1) \cap E(H_2)]$. The symmetric difference of finitely many subgraphs $\{H_1, \dots, H_t\}$ of G is defined recursively as

$$\triangle_{1 \leq i \leq t} H_i = H_1 \triangle \dots \triangle H_{t-1} \triangle H_t = [H_1 \triangle \dots \triangle H_{t-1}] \triangle H_t.$$

The following is a well-known fact:

Lemma 5.4.5 Let $\{H_1, \dots, H_t\}$ be a family of subgraphs of G . Let $S = H_1 \triangle \dots \triangle H_t$. Then

- (1) S is the subgraph of G induced by the edges contained in an odd number of H_i 's;
- (2) If H_1, \dots, H_t are all cycles of G , then S is also a cycle.

Now we are ready to prove Theorem 5.1.6.

Proof of Theorem 5.1.6.

Since G is 6-edge connected, by Lemma 4.2.2, G has three edge-disjoint spanning trees T_1, T_2 and T_3 . Because G admits a nowhere-zero bidirected flow, by Lemma 5.4.2, $\langle T_i \rangle_2 = E(G)$ for $1 \leq i \leq 3$.

For $1 \leq i \leq 3$, let $k = 3$, $E' = E(T_i)$ and apply Lemma 5.4.3, we get a bidirected integer flow f_i of G such that $f_i(e) \not\equiv 0 \pmod{3}$ for every element e of $E(G) \setminus E(T_i)$. Clearly $\text{supp}(f_1) \supseteq E(G) \setminus E(T_1)$. Since $G[\text{supp}(f_1)]$ contains two edge-disjoint spanning trees T_2 and T_3 , the subgraph $G[\text{supp}(f_1)]$ is 2-edge connected and spanning. Similarly, $\text{supp}(f_2) \supseteq E(G) \setminus E(T_2)$, $\text{supp}(f_3) \supseteq E(G) \setminus E(T_3)$ and both $G[\text{supp}(f_2)]$ and $G[\text{supp}(f_3)]$ are 2-edge connected and spanning. From each f_i ($1 \leq i \leq 3$), we can get a bidirected modular 3-flow f'_i such that $\text{supp}(f_i) = \text{supp}(f'_i)$. Then, by Theorem 5.1.5, we get a bidirected 3-flow f''_i such that $\text{supp}(f''_i) = \text{supp}(f_i)$ for $i = 1, 2, 3$.

Let T_i, T_j be two edge-disjoint spanning trees. For each $e \in T_j$, there is a unique circuit C_e of G contained in $T_i \cup \{e\}$. By Lemma 5.4.5-(2), $C_i^j = \Delta_{e \in E(T_j)} C_e$ is eulerian. By Lemma 5.4.5-(1), $(E(T_i) \cup E(T_j)) \supseteq E(C_i^j) \supseteq E(T_j)$, then C_i^j is also connected and spanning. Let us consider the following two cases.

Case 1. There exists one C_i^j such that $O(C_i^j)$ is even.

Without loss of generality, we may assume that $O(C_1^2)$ is even. Since C_1^2 is eulerian and connected, by Proposition 5.1.4, C_1^2 admits a nowhere-zero bidirected 2-flow. Therefore, G admits a bidirected 2-flow $f_{1,2}$ such that $\text{supp}(f_{1,2}) \supseteq E(T_2)$. Also f_2'' is bidirected 3-flow such that $\text{supp}(f_2'') \supseteq E(G) \setminus E(T_2)$. By Lemma 5.4.1, we get a nowhere-zero bidirected 6-flow of G .

Case 2. All $O(C_i^j)$'s are odd for $1 \leq i \neq j \leq 3$.

Since $O(C_1^3)$ is odd, the eulerian subgraph C_1^3 contains an unbalanced circuit c_1^3 . Clearly, $E(c_1^3) \subseteq (E(T_1) \cup E(T_3))$. Because $O(C_1^2)$ is odd and $E(T_2) \subseteq E(C_1^2) \subseteq E(T_1) \cup E(T_2)$, then $C^* = C_1^2 \Delta c_1^3$ is an eulerian, connected and spanning subgraph of G with $O(C^*)$ even. Clearly, $E(T_2) \subseteq E(C^*)$. By Proposition 5.1.4, C^* admits a nowhere-zero bidirected 2-flow f^* with $\text{supp}(f^*) \supseteq E(T_2)$. Similar to Case 1, by Lemma 5.4.1, we can get a nowhere-zero bidirected 6-flow of G . ■

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