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Degree Sequences

by

Rong Luo

THESIS

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Of
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in partial fulfillment of the requirements
for the degree of

Master of Science
in
Computer Science

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Contents

Abstract

A nonincreasing sequence $\pi = (d_1,\ d_2, \cdots,\ d_n)$ of nonnegative integers is said to be *graphic* if it is the degree sequence of a simple graph of order n and such graph is referred to as a realization of π. Let H be a simple graph. A graphic sequence π is said to be *potentially H-graphic* if it has a realization G containing H as its subgraph. In this paper, we characterize the potentially C_k graphic sequence for $k = 3, 4, 5$. These characterizations imply theorems due to P. Erdös, M. S. Jacobson and J. Lehel [3], R. J. Gould, M. S. Jacobson and J. Lehel[5], and C. H. Lai [16] [17].

Chapter 1

Introduction

1.1 Notations and Definitions

In this thesis, all the graphs are simple. A nonincreasing sequence $\pi = (d_1, d_2, \cdots, d_n)$ of nonnegative integers is said to be *graphic* if it is the degree sequence of a simple graph of order n and such graph is referred to as a realization of π. We use $G(\pi)$ to denote the set of all non-isomorphic realizations of π.

For $\pi = (d_1, d_2, \cdots, d_n)$, $\sigma(\pi) = d_1 + d_2 + \cdots + d_n$ the degree sum of π. The *residual sequence* π' obtained by laying off d_k is defined as follows:

$$\pi' = \begin{cases} (d_1 - 1, \cdots, d_{k-1} - 1, d_{k+1} - 1, \cdots, d_{d_k+1} - 1, d_{d_k+2}, \cdots, d_n), & if \ d_k \geq k, \\ (d_1 - 1, \cdots, d_{d_k} - 1, d_{d_k+1}, \cdots, d_{k-1}, d_{k+1}, \cdots, d_n), & if \ d_k \leq k - 1. \end{cases}$$

It is easy to see that any two isomorphic graphs have the same degree sequence. But the converse is not true. Therefore, the degree sequence is an invariant of graphs under isomorphisms and it is a rather weak invariant. Hence, degree sequences partly reflect some features of graphs, and the extent of this reflection deserves studying.

An interesting and important question on graphic sequences is: *"how to characterize the graphic sequences?"*. This question and its various generalizations were settled by several graph theorists. Sierksma-Hoogeveen[31] lists seven characterizations. In Section 2, we will present an algorithmic characterization due to Senion[30], Havel[7], Hakim[6] and Kleitman and Wang[8].

1.2 An Algorithmic Characterization

In this section, we present an algorithmic characterization due to Senion[30], Havel[7], Hakim[6] and Kleitman and Wang[8]. The proof is due to Kleitman and Wang[8].

Theorem 1.2.1 *The sequence π is graphic if and only if the residual sequence π' is also graphic. Moreover, a realization G of π can be obtained from a realization G' of π' by adding a new vertex v_k to G' and joining it to the vertices whose degrees are reduced by one in going from π to π'.*

Proof. Suppose that π is graphic. It is sufficient to prove that there exists a realization G of π with the vertex set $V(G) = \{v_1, \cdots, v_n\}$ such that v_k is adjacent to the first d_k vertices other than itself and $d_i = d(v_i)$. If not, we may choose a graph $G \in G(\pi)$ such that the number of vertices adjacent to v_k among the first d_k vertices other than v_k is maximum. Let v_i be a vertex among these first d_k vertices that is not adjacent to v_k, and let v_j be a vertex not among these first d_k vertices that is adjacent to v_k. We may assume that $d_i > d_j$, otherwise, the order of i and j can be interchanged. Hence there is a vertex v_t, $t \neq i, j$, such that $v_i v_t \in E(G)$ and $v_t v_j \notin E(G)$. Clearly, $G^* = G - v_j v_k - v_i v_t + v_i v_k + v_j v_t \in G(\pi)$, and the number of vertices adjacent to v_k among the first d_k vertices other than v_k in G^* is larger than that of G. This contradicts the choice of G.

Conversely, suppose that π' is graphic. Let G' be a realization of π'. Adding a new vertex v_k to G' and joining it to the vertices whose degrees are decreased by one, the resulting graph G is a realization of π.

■

Theorem 1.2.1 is the basis of the following algorithm for determining whether a nonnegative integer sequence is graphic, and if it is, constructing a realization.

Algorithm: Determine whether a nonnegative integer sequence $\pi = (d_1, d_2, \cdots, d_n\}$ with $d_1 \geq d_2 \geq \cdots \geq d_n$ is graphic.

(a) If $d_1 \geq n$, then π is not graphic. Otherwise, remove any zero terms from the sequence and then go to (b).

(b) If the resulting sequence is of length zero, then the sequence is graphic. If the resulting sequence is of length 1, then the sequence is not graphic. Otherwise, go to (c).

(c) Lay off the first number from the sequence.

(d) Rearrange the order of integers in the sequence so that it is nonincreasing and set $n = n - 1$ and go to (a).

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1.3 Potentially P-graphic Sequences

Let P be a graph-theoretic property. Following Nash-Williams[27], we say that a graphic sequence π is *potentially P-graphic* if there exists a simple graph $G \in G(\pi)$ such that G has the property P.

The first question on potentially P-graphic sequences was solved by Edmonds[1], where P is k-edge connectedness.

Theorem 1.3.1 *(Edmonds[1]) A graphic sequence $\pi = (d_1,\ d_2, \cdots, d_n)$ is potentially k-edge connected if and only if*

(1) $d_i \geq k$ for all i, and ,
(2) $\sum_{i=1}^{n} d_i \geq 2(n-1)$ if $k = 1$.

A graphic sequence is potentially k-connected if it has a k-connected realization. S. B. Rao and A. R. Rao[29] first investigated the problem concerning the charaterization of the potentially 3-connected graphic sequences. Wang and Kleitman[32] solved this problem for general k by using the laying off operation.

A graphic sequence π is said to be *unigraphic* if every two realizations of π are isomorphic. Many authors (for example, Hakimi[6], Johnson[9][10][11], Koren[13][14][15] and S. Y. R. Li[24]) studied the unigraphic sequences. Various charaterizations of unigraphic sequences have been obtained in Kleitman and S. Y. R. Li[12], S. Y. R. Li[24] and Koren[14]. The following is one of the most important and interesting results on the unigraphic sequences.

Theorem 1.3.2 *(Koren[15]) A graphic sequence $\pi = (d - 1,\ d_2, \cdots, d_n)$ with $d_2 = d_{n-1}$ is unigraphic if and only if one of the following conditions holds:*

(1) $d_1 = d_n = k$, where $k = 1,\ n - 1,\ n - 2$;
(2) $d_1 = d_n = 2$ and $n = 5$;
(3) $d_1 > d_2 = d_{n-1} = 1$;
(4) $d_1 > d_2 = d_n = 2$ where $d_1 = n - 1$, or, $n - 2$;
(5) $n - 2 = d_1 = d_{n-1} > d_n$;
(6) $n - 3 = d_1 = d_{n-1} > d_n = 1$;
(7) $n - 1 = d_1 > d_2 = d_n = 3$, $n = 6$;
(8) $n - 1 = d_1 > d_2 = d_n = n - 2$.

Some other potentially P-graphic sequences have also been studied such as potentially $(k_1, \cdots, k_n)$-factorable graphic sequences, potentially planar-graphic sequences, potentially Hamiltonian-graphic sequences and so on.

Chapter 2

Potentially $H-$graphic Sequences

2.1 Introduction

Let H be a simple graph. A graphic sequence π is said to be *potentially H-graphic* if it has a realization G containing H as its subgraph.

In 1991, Erdös, Jacobson and Lehel[3] considered the following problem about potentially K_k-graphic sequences: determine the smallest positive even number $\sigma(k, n)$ such that every n-term graphic sequence $\pi = (d_1, d_2, \cdots, d_n)$ without zero terms and with degree sum $\sigma(\pi) = d_1 + d_2 + \cdots + d_n$ at least $\sigma(k, n)$ is potentially K_k-graphic. They gave a lower bound for $\sigma(k, n)$ by the example $\pi_0 = ((n-1)^{k-1}, (k-1)^{n-k+1})$, i.e., $\sigma(k, n) \geq (k-1)(2n-k)+2$, and they further conjectured that the lower bound is the exact value of $\sigma(k, n)$. They also proved the conjecture is true for $k = 2$ and $n \geq 6$, i.e., $\sigma(2, n) = 2n$ for $n \geq 6$. The conjecture is confirmed in [5], [21], [22] and [23] for any $k \geq 4$ and for n sufficiently large. Li, Song and Luo[23] and Mubayi [26] also determined the values $\sigma(k, 2k + 2)$ independently. Recently, Li and Yin[20] determine the other values $\sigma(k, n)$.

In [5], Gould, Jacobson and Lehel generalized the problem: for a given simple graph H, determine the smallest positive even number $\sigma(H, n)$ such that every n-term graphic sequence $\pi = (d_1, d_2, \cdots, d_n)$ without zero terms and with degree sum $\sigma(\pi) = d_1 + d_2 + \cdots + d_n$ at least $\sigma(H, n)$ is potentially H-graphic. They determined the values $\sigma(pK_2, n)$ and $\sigma(C_4, n)$ where pK_2 is the matching consisted of p edges and C_4 is the cycle of length 4.

In [16] and [17], C. H. Lai determined the values of $\sigma(C_k, n)$ for $k \geq 5$.

A graphic sequence is potentially $_3C_l$-graphic if it has a realization containing a cycle of length r for each r, $3 \leq r \leq l$.

In [18] and [19], Li and Luo determined the values $\sigma(3C_l, n)$ for $l = 4, 5, \cdots, 9$ and $n \geq l$.

Motivated by the above problems, we consider the problem: characterize the potentially H-graphic sequences without zero terms.

In [28], Niu characterizes the potentially $(K_k - e)$-graphic sequences for $k = 4, 5$. Recently, Luo and Warner[25] characterized the potentially K_4-graphic sequences.

In this paper, we characterize the potentially C_k-graphic sequences for $k = 3, 4, 5$. By these characterizations, the values $\sigma(K_3, n)$ and $\sigma(C_k, n)$ for $k = 4, 5$ are straightforward.

2.2 Lemmas

In order to prove our main results, we need the following results.

Let $\pi = (d_1, d_2, \cdots, d_n)$ be a nonincreasing positive integer sequence. Denote

$$\pi' = (d_1 - 1, d_2 - 1, \cdots, d_{d_n} - 1, d_{d_n+1}, \cdots, d_{n-1})$$

From here on, π' will denote the residual sequence obtained by laying off d_n from π and all the graphic sequences have no zero terms.

The following theorem will be used repeatedly implicitly in the proof of our main theorem. We use $G \subseteq H$ to indicate that G is a subgraph of H.

Theorem 2.2.1 *(Gould, Jacobson and Lehel[5]) Let π be a graphic sequence. If H is a realization of π with $G \subseteq H$, then there is a realization H' of π with $G \subseteq H'$ so that the vertices of G have the largest degrees of π.*

Lemma 2.2.2 *(D. J. Kleitman and D. L. Wang [8] and Hakimi[6]) π is graphic if and only if π' is graphic.*

The following corollary is obvious.

Corollary 2.2.3 *Let H be a simple graph. If π' is potentially H-graphic, then π is potentially H-graphic.*

We will use Corollary 2.2.3 repeatedly in our proofs.

2.3 Potentially C_3-graphic Sequences

The main result of this section is the following theorem.

Theorem 2.3.1 *Let $\pi = (d_1, d_2, \cdots, d_n)$ be a graphic sequence with $n \geq 3$. Then π is potentially C_3-graphic if and only if $d_3 \geq 2$ except for 2 cases: $\pi = (2^4)$ and $\pi = (2^5)$.*

Before we prove Theorem 2.3.1, we first apply the theorem to give a simple proof of the following theorem due to P. Erdös et al.

Theorem 2.3.2 *(P. Erdös, M. S. Jacobson and J. Lehel [3])*
  $\sigma(K_3, n) = 2n$ *for $n \geq 6$.*

Proof. In [3], Erdös et al. gave the lower bound by the extremal example $D_k = ((n-1)^{k-1}, (k-1)^{n-k+1})$, i.e., $\sigma(K_k, n) \geq (k-1)(2n-k) + 2$. Therefore, $\sigma(K_3, n) \geq 2n$. We only need to show that $\sigma(K_3, n) \leq 2n$. It is enough to show that any graphic sequence $\pi = (d_1, d_2, \cdots, d_n)$ with $\sigma(\pi) \geq 2n$ is potentially K_3-graphic. Since $\sigma(\pi) \geq 2n$, we must have that $d_3 \geq 2$. By Theorem 2.3.1, π is potentially C_3-graphic. ∎

In order to prove Theorem 2.3.1, we need the following lemma.

Lemma 2.3.3 *Let $\pi = (d_1, d_2, \cdots, d_n)$ be a graphic sequence with $d_n \geq 2$ and $n \geq 6$. Then π is potentially C_3-graphic.*

Proof. Since $\sigma(\pi) \geq 2n$, every realization of π must contain a cycle. Let G be a realization of π with minimal girth $g(G)$. Then $3 \leq g(G) \leq n$. We only need to show that $g(G) = 3$.

By way of contradiction, we assume that $g(G) \geq 4$.

We consider the following 3 cases.

Case 1: $g(G) \geq 6$.

Let $v_1 v_2 v_3 \cdots v_{g(G)} v_1$ be a cycle of length $g(G)$. Then $v_1 v_4, v_1 v_5, v_2 v_4, v_2 v_5 \notin E(G)$. Therefore, $G' = G - v_1 v_2 - v_4 v_5 + v_1 v_5 + v_2 v_4$ is a realization of π and $v_2 v_3 v_4 v_2$ is a 3-cycle in G'. This contradicts the minimality of $g(G)$.

Case 2: $g(G) = 5$. Then G has a cycle $C = v_1 v_2 v_3 v_4 v_5 v_1$ of length 5.

Subcase 2.1: C is not a connected component of G.

By the assumption, there exits a vertex $u \neq v_i, i = 1, 2, \cdots, 5$ so that u is adjacent to some vertex in C.

Without loss of generality, we may assume that u is adjacent to v_1. Since $g(G) = 5$, $uv_i \notin E(G)$, $i = 2, 3, 4, 5$. Also $v_1v_4 \notin E(G)$, otherwise there is a cycle of length 3 or 4 in G. Therefore, $G' = G - uv_1 - v_3v_4 + v_1v_4 + uv_3$ is also a realization of π and G' contains the cycle $v_1v_5v_4v_1$, which contradicts the minimality of $g(G)$.

Subcase 2.2: C is a connected component of G.

By the assumption, for each v_i, $i = 1, 2, 3, 4, 5$, is not adjacent to any vertices outside of C.

Since $n \geq 6$ and $d_n \geq 2$, there exists an edge uv not on the cycle C. Therefore, $G' = G - v_1v_2 - v_3v_4 - uv + v_1v_4 + v_2u + v_3v$ is also a realization of π with $g(G') = 3$ since $v_1v_4v_5v_1$ is a 3-cycle in G', which is a contradiction.

Case 3: $g(G) = 4$. Let $C = v_1v_2v_3v_4v_1$ be a 4-cycle in G.

This case is similar to Case 2, we consider the following two subcases.

Subcase 3.1: C is not a connected component of G.

By the assumption, there exists a vertex u not in C such that u is adjacent to some v_i. Without loss of generality, we assume that u is adjacent to v_1. Since $g(G) = 4$, we must have that $v_1v_3, uv_2 \notin E(G)$. Therefore, $G' = G - v_3v_2 - uv_1 + v_1v_3 + uv_2$ is also a realization of π and the girth of G' is 3 since $v_1v_3v_4v_1$ is a 3-cycle in G', which is a contradiction.

Subcase 3.2: C is a connected component of G.

Since $n \geq 6$ and $d_n \geq 2$, there is an edge uv not on the cycle C. Therefore, $G' = G - v_1v_2 - v_2v_3 - uv + v_1v_3 + v_2u + v_2v$ is also a realization of π. Since $v_1v_3v_4v_1$ is a 3-cycle in G', the girth of G' is 3, which is a contradiction.

Combining Case 1, 2 and 3, $g(G) = 3$. ∎

Proof of Theorem 2.3.1 The necessary condition is obvious. Therefore, we only need to prove the sufficient condition. Let $\pi = (d_1, d_2, \cdots, d_n)$ be a graphic sequence satisfying the conditions of the theorem. It suffices to show that π has a realization containing C_3 as its subgraph. We consider the following cases:

Case 1: $n = 3$. This case is obvious.

Case 2: $n = 4$. Since $\pi \neq (2^4)$, $d_3 \geq 2$ and $\sigma(\pi)$, we must have that $d_1 = 3$. Then π must be one of the following sequences:

$$(3, 2, 2, 1), (3^2, 2^2), (3^4).$$

It is easy to see that all of these are potentially C_3-graphic

Case 3. $n = 5$. If $d_1 = 2$, since $\pi \neq (2^5)$, $d_3 \geq 2$ and $\sigma(\pi)$ is even, we must have $\pi = (2^3, 1^2)$. It is easy to see that it is potentially C_3-graphic. If $d_1 = 3$, then π is one of the following:

$$(3^4, 2), \ (3^2, 2^3), \ (3^3, 2, 1), \ (3^2, 2, 1^2), \ (3, 2^3, 1).$$

It is easy to see that all of the above sequences are potentially C_3-graphic. If $d_1 = 4$, let G be a realization of π with vertices set $\{v_1, v_2, v_3, v_4, v_5\}$ where $d(v_i) = d_i$, $i = 1, 2, \cdots, 5$. Since $d(v_1) = 4$, v_1 is adjacent to every other vertex. Since $d(v_3) \geq 2$, v_3 is adjacent to an vertex other than v_1, say v_i. Then $v_1 v_3 v_i$ forms a C_3.

Case 4: $n \geq 6$.
We use induction to prove the sufficient condition.

First we prove the sufficient condition for $n = 6$. Assume that $d_3 \geq 2$.

If $d_6 \geq 2$, by Lemma 2.3.3, π is potentially C_3-graphic. Now we assume that $d_6 = 1$. Let $\pi' = (d_1', d_2', d_3', d_4', d_5')$ be the residual sequence obtained by laying off d_6. If $d_1 \geq 4$, then $d_3' \geq min\{d_1 - 1, d_2, d_3\} \geq 2$ and $d_1' \geq 3$. By Case 3, π' is potentially C_3-graphic. Therefore, π is potentially C_3-graphic by Corollary 2.2.3. Thus, we may assume that $2 \leq d_1 \leq 3$. If $d_1 = 2$, then $d_1 = d_2 = d_3 = 2$. Since $\sigma(\pi)$ is even, we must have that $d_5 = 1$ and $d_4 = 2$. Therefore, $\pi = (2^4, 1^2)$. It is easy to see that π is potentially C_3-graphic. Now assume that $d_1 = 3$. If $d_2 = 3$, then $d_1' = d_2 = 3$ and $d_3' \geq min\{d_1 - 1, d_2, d_3\} \geq 2$. Thus, by case 3, π' is potentially C_3-graphic. Therefore, π is potentially C_3-graphic by Corollary 2.2.3. Thus, we may further assume that $\pi = (3, 2, 2, d_4, d_5, 1)$. Since $\sigma(\pi)$ is even, $d_4 + d_5$ must be even. Therefore, $\pi = (3, 2^4, 1)$ or $\pi = (3, 2^2, 1^3)$. It is easy to see that these both have realizations containing C_3. Therefore, π is potentially C_3-graphic for $n = 6$.

Now we suppose that the sufficient condition is true for $n - 1 \geq 6$. We prove that it is also true for n. It suffices to prove that the graphic sequence $\pi = (d_1, d_2, \cdots, d_n)$ with $d_3 \geq 2$ is potentially C_3-graphic by the definition.

If $d_n \geq 2$, by Lemma 2.3.3, π is potentially C_3-graphic. Therefore, we may assume that $d_n = 1$. Let $\pi' = (d_1 - 1, d_2, d_3, \cdots, d_{n-1}) = (d_1' d_2', \cdots, d_{n-1}')$ be the residual sequence obtained by lay ingoff $d_n = 1$ where $d_1' \geq d_2' \geq \cdots \geq d_{n-1}'$. If $d_3' \leq 1$, then $d_1 = d_2 = d_3 = 2$ and $d_4 = d_5 = \cdots = d_n = 1$. It follows that $\pi = (2^3, 1^{n-3})$. Since $\sigma(\pi)$ is even, $n - 3$ must be even. It follows that π has a realization consisting of a C_3 and $(n - 3)/2$ disjoint edges. If $d_3' \geq 2$, by induction hypothesis, π' is potentially C_3-graphic. Therefore, π is potentially C_3-graphic by Corollary 2.2.3.

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This completes the proof.

2.4 Potentially C_4-graphic Sequences

The main result of this section is the following:

Theorem 2.4.1 *Let $\pi = (d_1, d_2, \cdots, d_n)$ be a graphic sequence. Then π is potentially C_4-graphic if and only if the following conditions hold:*
(1) $d_4 \geq 2$.
(2) $d_1 = n - 1$ implies $d_2 \geq 3$.
(3) If $n = 5, 6$, then $\pi \neq (2^n)$.

Proof. First we assume that π is potentially C_4-graphic. Then (1) and (3) are obvious. Assume that $d_1 = n - 1$. Let G be a realization of π that contains a C_4 as its subgraph and let v_1 be the vertex of G with degree $d(v_1) = d_1$. Then $G - v_1$ contains a path with length at least 2. Thus, $G - v_1$ has a vertex with degree at least 2. Since $d(v_1) = n - 1$, we must have $d_2 \geq 3$.

Now we prove the sufficient condition.

We consider the following Cases:

Case 1. $n = 4$.

If $d_1 = 2$, then $\pi = (2^4)$, which is potentially C_4-graphic. Now assume that $d_1 = 3$. Then $d_2 = 3$. Therefore, $\pi = (3^2, 2^2)$ or $\pi = (3^4)$. Obviously, both of these are potentially C_4-graphic..

Case 2. $n = 5$.

If $d_5 \geq 3$, then π' satisfies the assumption. Thus, π' is potentially C_4-graphic. Therefore, π is potentially C_4-graphic by Corollary 2.2.3. Now we assume that $d_5 \leq 2$. If $d_5 = 2$, consider π'. If $d_2 \geq 3$, then π' satisfies the assumption. Thus, π' is potentially C_4-graphic. If $d_2 = 2$, then $\pi = (d_1, 2^4)$. Since $\sigma(\pi)$ is even, we have that $d_1 = 2, 4$, which is impossible. If $d_5 = 1$, then $d_1 \geq 3$ since $d_4 \geq 2$. Therefore, π' satisfies the assumption. Thus, π is potentially C_4-graphic by Corollary 2.2.3.

Case 3. $n = 6$.

If $d_6 \geq 4$, then π' is potentially C_4-graphic. If $d_6 = 3$, then $\pi' = (d_1 - 1, d_2 - 1, d_3 - 1, d_4, d_5)$ satisfies the assumption. Thus, π' is potentially C_4-graphic. Therefore, π is poten-

tially C_4-graphic by Corollary 2.2.3. If $d_6 = 2$, then $\pi' = (d_1 - 1, d_2 - 1, d_3, d_4, d_5)$ satisfies that $d'_4 \geq 2$ by condition (3). If $\pi' = (2^5)$, then $\pi = (3^2, 2^4)$ and it is easy to see that π is potentially C_4-graphic. If $d'_1 = 4$ and $d'_2 \leq 2$, then $d'_2 = 2$. Therefore, $\pi' = (4, 2^k, 1^{5-k-1})$ where $k \geq 3$. Since $\sigma(\pi')$ is even, $k = 4$. Thus, $\pi' = (4, 2^4)$. Therefore, $\pi = (5, 3, 2^4)$ and it is easy to see that π is potentially C_4-graphic. If $d_6 = 1$, then $\pi' = (d_1 - 1, d_2, d_3, d_4, d_5)$. If $d'_4 = 1$, then $\pi = (2^4, 1^2)$. Obviously, π is potentially C_4-graphic. Now we assume that $d'_4 \geq 2$. If $\pi' \neq (2^5)$ or $(4, 2^4)$, then π' is potentially C_4-graphic. Therefore, π is potentially C_4-graphic by Corollary 2.2.3. If $\pi' = (2^5)$, then $\pi = (3, 2^4, 1)$. Clearly π is potentially C_4-graphic. If $\pi' = (4, 2^4)$, then $\pi = (5, 2^4, 1)$, which is impossible by (2).

Case 4: $n \geq 7$.

We use induction to prove this case.

First we show the case for $n = 7$. If $d_7 \geq 3$, by Case 3, we are done. We assume that $d_7 \leq 2$. If $d_7 = 2$, then $d'_4 \geq 2$. If $d_7 = 1$, then $d'_4 \geq 2$ otherwise $\pi = (2^4, 1^3)$. Therefore, $d'_4 \geq 2$ if $d_7 \leq 2$. If $d'_1 = 5$ and $d'_2 = 2$, then $\pi' = (5, 2^4, 1)$. Therefore, $d_1 = 6$ and $d_2 = 2$, which contradicts the assumption. Thus, if $\pi' \neq (2^6)$, then π' is potentially C_4-graphic. Therefore, π is potentially C_4-graphic by Corollary 2.2.3. If $\pi' = (2^6)$, then $\pi = (3^2, 2^5)$ or $\pi = (3, 2^5, 1)$. Clearly they are potentially C_4-graphic. It follows that the sufficient condition is true for $n = 7$

Now we assume that the sufficient condition is true for $n-1 \geq 7$. We prove that it is true for n. Let $\pi = (d_1, \cdots, d_n)$ be a graphic sequence with n terms that satisfies the conditions (1) and (2). We only need to show that π is potentially C_4-graphic. If $d_n \geq 3$, then π' satisfies the assumption. By induction hypothesis, π' is potentially C_4-graphic. Therefore, π is potentially C_4-graphic by Corollary 2.2.3. If $d_n = 2$, then $\pi' = (d_1 - 1, d_2 - 1, d_3, \cdots, d_{n-1})$. Then $d'_4 \geq 2$. If $d'_1 = n - 2$, then either $d_1 = n - 1$ or $d_1 = d_2 = d_3 = n - 2$. In the later case, π' satisfies the assumption. By the induction hypothesis, π' is potentially C_4-graphic. Therefore, π is potentially C_4-graphic by Corollary 2.2.3. If $d_1 = n - 1$, then either π' satisfies the assumption or $\pi = (n - 1, 3, 2^{n-2})$. It is easy to see that $(n - 1, 3, 2^{n-2})$ is potentially C_4-graphic. Now we assume that $d_n = 1$. If $d'_4 = 1$, then $\pi = (2^4, 1^{n-4})$. Then n is even and it is easy to see π is potentially C_4-graphic. Now assume that $d'_4 \geq 2$. If $d'_1 = n - 2$ and $d'_2 = 2$, then $d_1 = n - 1$ and $d_2 = 2$, which contradicts the assumption. Therefore, π' is potentially C_4-graphic. Hence π is potentially C_4-graphic by Corollary 2.2.3. ∎

By Theorem 2.4.1 , we give a simple proof of the following theorem due to R.J. Gould et al.:

Theorem 2.4.2 *(R. J. Gould, M. S. Jacobson, J. Lehel[5])*
For $n \geq 4$,

$$\sigma(C_4, n) = \begin{cases} 3n - 1, & \textit{if } n \textit{ is odd,} \\ \\ 3n - 2, & \textit{if } n \textit{ is even.} \end{cases}$$

Proof. In [5], by taking the extremal examples $\pi = ((2k)^1, 2^{2k})$ when $n = 2k + 1$ and $\pi = ((2k + 1)^1, 2^{2k}, 1)$ when $n = 2k$, R. J. Gould et al. presented a lower bound for $\sigma(C_4, n)$, i.e.,

$$\sigma(C_4, n) \geq \begin{cases} 3n - 1, & \textit{if } n \textit{ is odd,} \\ \\ 3n - 2, & \textit{if } n \textit{ is even.} \end{cases}$$

We show that

$$\sigma(C_4, n) \leq \begin{cases} 3n - 1, & \textit{if } n \textit{ is odd,} \\ \\ 3n - 2, & \textit{if } n \textit{ is even.} \end{cases}$$

Let π be a graphic sequence with $\sigma(\pi) \geq 3n - 2$. It suffices to show that π is potentially C_4-graphic. If $d_4 = 1$, then $\sigma(\pi) = d_1 + d_2 + d_3 + (n-3)$ and $d_1 + d_2 + d_3 \leq 6 + (n-3) = n+3$. Therefore, $\sigma(\pi) \leq 2n \leq 3n - 2$. Thus, $d_4 \geq 2$. Clearly, if $n = 5, 6$, $\pi \neq (2^n)$. Assume that $d_1 = n - 1$. If $d_2 \leq 2$, then $\sigma(\pi) \leq n - 1 + 2(n-1) = 3n - 3 \leq 3n - 2$. Therefore, π satisfies the conditions (1)-(3) in Theorem 2.4.1, and thus, π is potentially C_4-graphic. ■

2.5 Potentially C_5-graphic Sequences

The main result of this section is the following:

Theorem 2.5.1 *Graphic sequence $\pi = (d_1, d_2, \cdots, d_n)$ is potentially C_5-graphic if and only if π satisfies the following conditions:*
(1) $d_5 \geq 2$ and $\pi \neq (2^n)$ for $n = 6, 7$.
(2) For $i = 1, 2$, $d_1 = n - i$ implies $d_{4-i} \geq 3$.
(3) If $\pi = (d_1, d_2, 2^k, 1^{n-k-2})$, then $d_1 + d_2 \leq n + k - 2$.

Proof. We assume that π is potentially C_5-graphic. It is obvious that (1) holds. Assume $\pi = (d_1, d_2, 2^k, 1^{n-k-2})$ with $d_1 + d_2 \geq n + k$. Since π is graphic, $d_1 + d_2 \leq 2k + n - k - 2 + 2 = n + k$. Therefore, $d_1 + d_2 = n + k$. Hence π has a unique realization. It is easy to see that the realization is of girth 4. Hence (3) holds. Let G be a realization of π that contains a

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cycle C_5 and let v_1 be the vertex of G with degree $d(v_1) = d_1$. Then $G - v_1$ contains a path with length at least 3. Thus, $G - v_1$ contains at least two vertices with degree at least 2. Therefore, $d_1 = n - i, i = 1, 2$ implies $d_{4-i} \geq 3$. Thus, (2) holds.

Now we prove the sufficient condition. It is enough to show the following three lemmas.
∎

Lemma 2.5.2 *Graphic sequence $\pi = (d_1, d_2, \cdots, d_5)$ is potentially C_5-graphic if π satisfies the following conditions:*

(1) $d_5 \geq 2$;

(2) If $d_1 = 4$, then $d_3 \geq 3$.

Proof. If $d_5 \geq 3$, then, by Theorem 2.4.1, π' is potentially C_4-graphic. Since $d_5 \geq 3$, π is potentially C_5-graphic. Now assume that $d_5 = 2$. If $d_1 = 2$, then $\pi = (2^5)$. Obviously this is potentially C_5-graphic. If $d_1 = 3$, then $\pi = (3^2, 2^3)$ or $\pi = (3^4, 2)$. It is easy to see that these are potentially C_5-graphic. Now assume that $d_1 = 4$, then $d_3 \geq 3$ by (2). Since $d_5 = 2$, we have $d_3 = 3$. Therefore, π is $(4^2, 3^3, 2)$ or $(4, 3^2, 2^2)$. It is easy to see that these are potentially C_5-graphic. ∎

Lemma 2.5.3 *For $n = 6, 7$, $\pi = (d_1, d_2, \cdots, d_n)$ is potentially C_5-graphic if π satisfies the following conditions:*

(1) $d_5 \geq 2$ and $\pi \neq (2^n)$.

(2) For $i = 1, 2$, $d_1 = n - i$ implies $d_{4-i} \geq 3$.

(3) $\pi = (d_1, d_2, 2^k, 1^{n-k-2})$ implies $d_1 + d_2 \leq n + k - 2$.

Proof. We first prove the case for $n = 6$. Notice that if $d_6 \geq 3$, then π' satisfies the conditions (1) and (2) in Lemma 2.5.2. Hence by Lemma 2.5.2, π' is potentially C_5-graphic. Therefore, π is potentially C_5-graphic by Corollary 2.2.3. Thus, we may assume that $d_6 \leq 2$.

We consider the following two cases:

Case 1: $d_6 = 1$. In this case, $d_1 \geq 3$ otherwise $\sigma(\pi)$ is odd. If $d_3 \geq 3$, then by Lemma 2.5.2, π' is potentially C_5-graphic. Now we assume that $d_3 = 2$. Then $\pi = (d_1, d_2, 2^3, 1)$. Then by conditions (3) and (2), $d_1 \leq n - 2 = 4$ and $d_1 + d_2 \leq n + k - 2 = 6 + 3 - 2 = 7$. Since $d_1 \geq 3$, $5 \leq d_1 + d_2 \leq 7$. Since $\sigma(\pi)$ is even, $d_1 + d_2 = 5$ or 7. Thus, $\pi = (3, 2^4, 1)$ or $(4, 3, 2^3, 1)$ and it is easy to see that they are potentially C_5-graphic.

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Case 2: $d_6 = 2$. By (1), $d_1 \geq 3$. If $d_5 \geq 3$ or $d_3 \geq 4$, then by Lemma 2.5.2, π' is potentially C_5-graphic. Therefore, π is potentially C_5-graphic by Corollary 2.2.3. Thus, we may assume that $d_5 = 2$ and $d_3 \leq 3$. If $d_3 = 2$, then $\pi = (d_1, d_2, 2^4)$. Hence by (2) and (3), we have that $d_1 + d_2 \leq n + k - 2 = 6 + 4 - 2 = 8$ and $d_1 \leq 4$. Thus, $\pi = (4^2, 2^4)$ or $(3^2, 2^4)$. It is easy to see both of these are potentially C_5-graphic. If $d_3 = 3$, then we may further assume that $d_2 = 3$ otherwise by Lemma 2.5.2, π' is potentially C_5-graphic. Then π is one of the following sequences:

$$(5, 3^3, 2^2), (4, 3^2, 2^2), (4, 3^4, 2), (3^4, 2^2).$$

It is easy to see that these are potentially C_5-graphic.

Now we prove the case for $n = 7$.

Similar to the $n = 6$ case, we may assume that $d_7 \leq 2$. If $d_3 \geq 4$ or $d_5 \geq 3$, then by the case for $n = 6$, π' is potentially C_5-graphic. Therefore, we may further assume that $d_3 \leq 3$ and $d_5 = 2$. We consider the following two cases:

Case 1. $d_7 = 1$. We may assume that $d_4 = 2$, otherwise π' is potentially C_5-graphic. If $d_3 = 3$, then we may assume that $d_1 = d_2 = d_3 = 3$, otherwise π' is potentially C_5-graphic. Therefore, $\pi = (3^3, 2^3, 1)$ and it is potentially C_5-graphic. If $d_3 = 2$, then by (2) and (3), $d_1 \leq 5$ and $d_1 + d_2 \leq n + k - 2 \leq 7 + 4 - 2 = 9$. Then π is one of the following sequences:

$$(5, 4, 2^4, 1), (4, 3, 2^4, 1), (3, 2^5, 1), (5, 3, 2^3, 1^2),$$

$$(4^2, 2^3, 1^2), (4, 2^4, 1^2), (3^2, 2^3, 1^2), (2^5, 1^2).$$

It is easy to see that these are all potentially C_5-graphic.

Case 2: $d_7 = 2$. In this case $d_1 \geq 3$. If $d_3 = 3$, then similar to Case 1, we may assume that $d_2 = d_3 = 3$. Therefore, π is one of the following sequences: $(3^4, 2^3), (5, 3^3, 2^3), (6, 3^2, 2^4), (4, 3^2, 2^4)$ and it is easy to see that these are potentially C_5-graphic. If $d_3 = 2$, then by (2) and (3), $d_1 \leq 5$ and $d_1 + d_2 \leq n + k - 2 = 7 + 5 - 2 = 10$. Then π is one of the following sequences: $(5^2, 2^5), (5, 3, 2^5), (4^2, 2^5), (4, 2^6), (3^2, 2^5)$ and it is easy to see that these are potentially C_5-graphic. $\blacksquare$

Lemma 2.5.4 *For $n \geq 8$, $\pi = (d_1, d_2, \cdots, d_n)$ is potentially C_5-graphic if π satisfies the following conditions:*

(1)$d_5 \geq 2$;

(2) For $i = 1, 2$, $d_1 = n - i$ implies $d_{4-i} \geq 3$.

(3) $\pi = (d_1, d_2, 2^k, 1^{n-k-2})$ implies $d_1 + d_2 \leq n + k - 2$.

Proof. We are going to prove this lemma by induction on n.

We first prove the lamma for $n = 8$. We may assume that $d_8 \leq 2$, otherwise, by Lemma 2.5.3, π' is potentially C_5-graphic. Similar to the proof of Lemma 2.5.3, we may further assume that $d_3 \leq 3$ and $d_5 = 2$. We consider the following two cases:

Case 1. $d_8 = 1$. If $d_3 = 3$, then we may assume that $d_1 = d_2 = d_3 = 3$ and $d_4 = 2$ otherwise π' is potentially C_5-graphic by Lemma 2.5.3. Therefore, $\pi = (3^3, 2^4, 1)$ or $(3^3, 2^2, 1^3)$ and it is easy to see that these are potentially C_5-graphic. Now we assume that $d_3 = 2$. Then $\pi = (d_1, d_2, 2^k, 1^{8-k-2})$. Then by (2) and (3), $d_1 \leq 6$ and $d_1 + d_2 \leq n + k - 2$. If $d_1 = 2$, then $\pi = (2^6, 1^2)$. It is easy to see that this is potentially C_5-graphic. Thus, we may further assume that $d_1 \geq 3$. Then $\pi' = (d_1 - 1, d_2, 2^k, 1^{n-1-k-2})$ and $d_1 - 1 + d_2 \leq n + k - 2 - 1 = n - 1 + k - 2$. If $\pi' = (2^7)$, then $\pi = (3, 2^6, 1)$ and it is easy to see that this is potentially C_5-graphic. Now we assume that $\pi' \neq (2^7)$. Since $d_1 \leq 6$, π' satisfies the condition (2) in Lemma 2.5.3. If $\pi' \neq (2^7)$, then it satisfies the conditions (1), (2) and (3) in Lemma 2.5.3. By Lemma 2.5.3, π' is potentially C_5-graphic. Therefore, π is potentially C_k-graphic by Corollary 2.2.3.

Case 2. $d_8 = 2$. If $d_3 = 2$, then $\pi = (d_1, d_2, 2^6)$. By (2) and (3), $d_1 \leq 6$ and $d_1 + d_2 \leq n + k - 2 = 8 + 6 - 2 = 12$. If $d_2 = 2$, then $\pi = (4, 2^7)$ or (2^8). It is easy to see that these are potentially C_5-graphic. Thus, we may further assume that $d_2 \geq 3$. Then $\pi' = (d_1 - 1, d_2 - 1, 2^5)$. If $\pi' = (2^7)$, then $\pi = (3^2, 2^6)$. It is easy to see that this is potentially C_5-graphic. Therefore, we may assume that $\pi' \neq (2^7)$. Thus, π' satisfies the condition (1) in Lemma 2.5.3. Since $d_1 \leq 6$, π' satisfies the condition (2) in Lemma 2.5.3. Since $d_1' + d_2' = d_1 - 1 + d_2 - 1 \leq 12 - 2 = 10 = 7 + 5 - 2$, π' satisfies the condition (3) in Lemma 2.5.3. Therefore, by Lemma 2.5.3, π' is potentially C_5-graphic. Therefore, π is potentially C_k-graphic by Corollary2.2.3. If $d_3 = 3$, then we may assume that $d_2 = 3$, otherwise π' is potentially C_5-graphic. Thus, π is one of the following sequences:

$$(3^4, 2^4), (5, 3^3, 2^4), (7, 3^3, 2^4), (6, 3^2, 2^5), (4, 3^2, 2^5).$$

It is easy to see that these are all potentially C_5-graphic.

Now we assume that the lemma is true for $n - 1 \geq 8$. We show it is true for n. Let $\pi = (d_1, d_2, \cdots, d_n)$ be a graphic sequence satisfying the conditions (1)-(3). We need only

show that π is potentially C_5-graphic. If $d_n \geq 3$, then, by induction hypothesis, π' is potentially C_5-graphic and therefore, π is potentially C_5-graphic by Corollary 2.2.3. Thus, we may assume that $d_n \leq 2$. Similarly, we may further assume that $d_{d_n+3} = 2$ and $d_3 \leq 3$. We consider the following two cases:

Case 1. $d_n = 1$. In this case $d_4 = 2$. If $d_1 = 2$, then $\pi = (2^k, 1^{n-k})$. Since $k \geq 5$ and $n \geq 9$, π is potentially C_5-graphic. Thus, we further assume that $d_1 \geq 3$. If $d_3 = 3$, then we may assume that $d_1 = d_2 = d_3 = 3$. Then $\pi' = (3^2, 2^{k+1}, 1^{n-k-4})$ satisfies (1),(2) and (3) and therefore, π' is potentially C_5-graphic and hence π is potentially C_5-graphic. If $d_3 = 2$, then $\pi = (d_1, d_2, 2^k, 1^{n-k-2})$. Therefore, $\pi' = (d_1 - 1, d_2, 2^k, 1^{n-k-2})$. By (3), $d_1 + d_2 \leq n + k - 2$. Thus, $d_1 - 1 + d_2 \leq n - 1 + k - 2$. Obviously, π' satisfies (1)-(3). So π' is potentially C_5-graphic and hence π is potentially C_5-graphic.

Case 2. $d_n = 2$. In this case $d_5 = 2$. Therfore $\pi = (d_1, d_2, d_3, d_4, 2^{n-4})$. If $d_3 = 2$, then $\pi = (d_1, d_2, 2^{n-2})$. By (2) and (3), $d_1 + d_2 \leq n + n - 2 - 2 = 2(n - 2)$ and $d_1 \leq n - 2$. If $d_2 \geq 4$ or $d_1 \leq n - 3$ and $d_2 \geq 3$, then π' satisfies (1)-(3) and therefore, by induction hypothesis it is potentially C_5-graphic and hence π is potentially C_5-graphic. If $\pi = (n - 2, 3, 2^{n-2})$, it is easy to see that π is potentially C_5-graphic. Now assume that $d_2 = 2$. If $d_1 = 2$, then π is potentially C_5-graphic since $n \geq 9$. If $d_1 \geq 3$, then by (3)and (2), $d_1 - 1 + 2 \leq n - 3 + 2 - 2 = n - 3 \leq n - 1 + n - 3 - 2 = 2n - 5$ and $d_1 - 1 \leq n - 3 - 1 = (n-1) - 3$. It follows that π' is potentially C_5-graphic and hence π is potentially C_5-graphic. Now assume that $d_3 = 3$. If $d_4 = 3$, then we may assume that $d_1 = 3$ otherwise π' is potentially C_5 graphic. Thus, $\pi = (3^4, 2^{n-4})$ and is potentially C_5-graphic since $n \geq 9$. Now we assume that $d_4 = 2$. Then $\pi = (d_1, d_2, 3, 2^{n-3})$. Similarly we may assume that $d_2 = 3$. If $d_1 = n - 1$, then π is potentially C_5-graphic. If $d_1 \leq n - 2$, then $d_1' = d_1 - 1 \leq (n - 1) - 2$ and $d_1' + d_2' = d_1 - 1 + 3 \leq n - 2 - 1 + 3 = n \leq n - 1 + n - 3 - 2 = 2n - 6$. Thus, π' is potentially C_5-graphic and therefore, π is potentially C_5-graphic.

Therefore, π is potentially C_5-graphic. ∎

Now we use Theorem 2.5.1 to give a simple proof of a theorem due to C.H. Lai.

Theorem 2.5.5 *(C.H. Lai [16])* $\sigma(C_5, n) = 4n - 4$ *for* $n \geq 5$.

Proof. Take $\pi = ((n - 1)^2, 2^{(n-2)})$. Then π has unique realization and this realization doesn't contain a cycle of length 5. Therefore, $\sigma(C_5, n) \geq \sigma(\pi) + 2 = 4n - 4$. Now we will show that $\sigma(C_5, n) \leq 4n - 4$. Let π be a graphic sequence with n terms and with $\sigma(\pi) \geq 4n - 4$. It suffices to show that π is potentially C_5-graphic. If $d_5 = 1$, then

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$d_1 + d_2 + d_3 + d_4 \leq 4 \times 3 + n - 4 = n + 8$. Therefore, $\sigma(\pi) \leq n + 8 + (n-4) = 2n + 4 \leq 4n - 4$, since $n \geq 5$, which is a contradiction. Thus, $d_5 \geq 2$. If $d_3 = 2$, then $4n - 4 \leq \sigma(\pi) \leq d_1 + d_2 + d_3(n-2) \leq 2(n-1) + 2(n-2) = 4n - 6$. The contradiction shows that $d_3 \geq 3$. Therefore, π satisfies the conditions in Theorem 2.5.1. By Theorem 2.5.1, π is potentially C_4-graphic. Thus, $\sigma(C_5, n) = 4n - 4$ for $n \geq 5$. ∎

2.6 Conclusion

It would be nice to characterize potentially C_k-graphic sequences for general k. This question seems very difficult. The special case, characterizing potentially Hamiltonian graphic sequences, has been open for several decades. We believe that new techniques are needed to slove this problem.

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