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Spanning Trails and Spanning Trees

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ABSTRACT

Spanning Trails and Spanning Trees

Meng Zhang

There are two major parts in my dissertation. One is based on spanning trail, the other one is comparing spanning tree packing and covering.

The results of the spanning trail in my dissertation are motivated by Thomassen's Conjecture that every 4-connected line graph is hamiltonian. Harary and Nash-Williams showed that the line graph $L(G)$ is hamiltonian if and only if the graph G has a dominating eulerian subgraph. Also, motivated by the Chinese Postman Problem, Boesch et al. introduced supereulerian graphs which contain spanning closed trails. In the spanning trail part of my dissertation, I proved some results based on supereulerian graphs and, a more general case, spanning trails.

Let $\alpha(G)$, $\alpha'(G)$, $\kappa(G)$ and $\kappa'(G)$ denote the independence number, the matching number, connectivity and edge connectivity of a graph G, respectively. First, we discuss the 3-edge-connected graphs with bounded edge-cuts of size 3, and prove that any 3-edge-connected graph with at most 11 edge cuts of size 3 is supereulerian, which improves Catlin's result. Second, having the idea from Chvátal-Erdös Theorem which states that every graph G with $\kappa(G) \geq \alpha(G)$ is hamiltonian, we find families of finite graphs $\mathcal{F}_1$ and $\mathcal{F}_2$ such that if a connected graph G satisfies $\kappa'(G) \geq \alpha(G) - 1$ (resp. $\kappa'(G) \geq 3$ and $\alpha'(G) \leq 7$), then G has a spanning closed trail if and only if G is not contractible to a member of $\mathcal{F}_1$ (resp. $\mathcal{F}_2$). Third, by solving a conjecture posed in [Discrete Math. 306 (2006) 87-98], we prove if G is essentially 4-edge-connected, then for any edge subset $X_0 \subseteq E(G)$ with $|X_0| \leq 3$ and any distinct edges $e, e' \in E(G)$, G has a spanning (e, e')-trail containing all edges in X_0.

The results on spanning trees in my dissertation concern spanning tree packing and covering. We find a characterization of spanning tree packing and covering based on degree sequence. Let $\tau(G)$ be the maximum number of edge-disjoint spanning trees in G, $a(G)$ be the minimum number of spanning trees whose union covers $E(G)$. We prove that, given a graphic sequence $d = (d_1, d_2, \cdots, d_n)$ $(d_1 \geq d_2 \geq \cdots \geq d_n)$ and integers $k_2 \geq k_1 > 0$, there exists a simple graph G with degree sequence d satisfying $k_1 \leq \tau(G) \leq a(G) \leq k_2$ if and only if $d_n \geq k_1$ and $2k_1(n-1) \leq \sum_{i=1}^{n} d_i \leq 2k_2(n - |I| - 1) + 2\sum_{i \in I} d_i$, where $I = \{i : d_i < k_2\}$.

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Contents

Chapter 1

Introduction of Spanning Trails

1.1   Background

In this thesis, graphs considered are finite and loopless. We follow [6] for undefined terms and notation. Let $\kappa(G)$ and $\kappa'(G)$ represent the connectivity and the edge connectivity of a graph G, respectively. A graph G is **nontrivial** if $|E(G)| > 0$, and we write $H \subseteq G$ to mean that H is a **subgraph** of G. Let $A \subseteq V(G)$ (resp. $B \subseteq E(G)$). Denote $G[A]$ (resp. $G[B]$) be the induced subgraph in G based on A (resp. B). An edge cut X of a graph G is **essential** if both components of $G - X$ are nontrivial. And G is **essentially k-edge-connected** if G is connected and does not have an essential edge cut of size less than k. For a graph G, the **line graph** of G, denoted by $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent. A trail T is called a **spanning trail** if $V(T) = V(G)$. If $u, v \in V(G)$ (resp. $u, v \in E(G)$), then T is called a **(u, v)-trail** if T starts with u and ends with v. Let $g(G)$ be the girth of G, which is the smallest circuit in G. Let K_n be the complete graph with n vertices, $K_{s,t}$ be the complete bipartite graph with vertex bipartition $|P_1| = s$ and $|P_2| = t$ and $P(10)$ be the Petersen Graph. For a graph G and integer $i \geq 1$, let $D_i(G) = \{v \mid d_G(v) = i, \ v \in V(G)\}$ and $d_i(G) = |D_i(G)|$. When G is understood, we write d_i for $d_i(G)$. Let $E_G(v) = \{uv \mid uv \in E(G), u \in V(G)\}$ and $N_G(v) = \{u \mid uv \in E(G), u \in V(G)\}$. If $U \subseteq V(G)$, then $N_G(U) = \bigcup_{v \in U} N_G(v) - U$. If K is a subgraph of G, then we also write $N_G(K)$ for $N_G(V(K))$. When G is understood, we often omit the subscript G in these notations.

By solving the famous Seven Bridge of Königsberg problem in 1736, Leonhard Euler introduced **eulerian trails** which contain each edge in a graph exactly once. The study of **eulerian graphs** which contain closed eulerian trail as a subgraph always be a hot topic in graph theory. On the other hand, there is another hot research direction called **hamiltonian path** which contains

each vertex in a graph exactly once. We call a graph **hamiltonian** if it has a closed hamiltonian path. In 1965, Harary and Nash-Williams [23] found a strong relationship between dominating eulerian subgraphs and Hamiltonian cycles in line graphs.

Theorem 1.1.1 *(Harary and Nash-Williams, [23]) The line graph $L(G)$ of a connected graph G is hamiltonian if and only if G has a dominating eulerian subgraph and $G \notin \{K_1, K_2, K_{1,2}\}$.*

And in 1986, Thomassen ([40]) posed a famous conjecture.

Conjecture 1.1.2 *(Thomassen, [40]) Every 4-connected line graph is hamiltonian.*

This conjecture remains open and many researchers worked on it. By definition, the line graph $L(G)$ is k-connected if and only if G is essentially k-edge-connected when $L(G) \ncong K_n$. So Conjecture 1.1.2 can be proved if we could prove every essential 4-edge-connected graph has a dominating eulerian subgraph.

In 1977, motivated by the Chinese Postman Problem, Boesch, Suffel and Tindell [4] introduced **supereulerian graphs** which contain spanning eulerian subgraphs, i.e. spanning closed trails. One of the main supereulerian problems is to determine what kinds of graphs are supereulerian. Boesch et al. indicated that this might be a difficult problem. Even when the graph are restricted to planar graphs with maximum degree at most 3, Pulleyblank [38] showed that this problem is NP-complete. After Boesch et al. posed supereulerian problem, many papers related to supereulerian have been published. In 1979, Jeager [24] proved that

Theorem 1.1.3 *(Jeager [24]) Every 4-edge-connected graph is supereulerian.*

And since supereulerian graphs always have dominating eulerian subgraph, by using Theorem 1.1.1, supereulerian graphs can be applied to the study of Conjecture 1.1.2. So in the following first three chapters of this thesis, we discuss the supereulerian graph and, a more general case, graphs with spanning trails. The main method we will use is the Catlin's Reduction Method.

1.2   Catlin's Reduction Method

Let H be a subgraph of G. The **contraction** G/H is the graph obtained from G by identifying the two ends of each edge in H and deleting the resulting loops. Let v_H denote the vertex in G/H to which H is contracted. Then H is the **preimage** of v_H, denoted by $H(v_H)$.

Catlin [8] introduced the collapsible graphs and a strong relationship between collapsible graphs and supereulerian graphs. Let $O(G) = \{v \in V(G) \mid d_G(v) \text{ is odd}\}$. A subgraph H of graph G is called **collapsible** if for any $R \subseteq V(H)$ with $|R|$ is even, then there exists a spanning subgraph

S_R such that $O(S_R) = R$. The **reduction** of G is the graph G' by contracting all maximal collapsible subgraph of G. A graph G is **reduced** if $G' = G$. By definition, each collapsible graph is supereulerian. Catlin [8] improve Theorem 1.1.3.

Theorem 1.2.1 *(Catlin [8]) Every 4-edge-connected graph is collapsible.*

Theorem 1.2.2 *(Catlin [8]) Let G be a connected graph, H be a collapsible subgraph of G and let G' be the reduction of G. Each of the following holds.*
(i) G is collapsible if and only if G/H is collapsible.
(ii) G is supereulerian if and only if G/H is supereulerian.
(iii) G has a spanning trail if and only if G/H has a spanning trail.
(iv) Any subgraph of a reduced graph is reduced.

By Theorem 1.2.2, finding collapsible subgraph and the reduction of the graph become important in the study of supereulerian problem.

In [7] and [10], Catlin et al. proved some useful results which can help us to find the graph which is not reduced. Let $F(G)$ be the minimum number of extra edges that must be added to G so that the resulting graph has two edge-disjoint spanning trees (hence collapsible ([7])).

Theorem 1.2.3 *Let G be a connected reduced graph. Then*
(i) (Catlin [7]) If $|V(G)| \geq 3$, then $F(G) = 2|V(G)| - |E(G)| - 2$.
(ii) (Catlin [7]) Every cycle of G has length at least 4.
(iii) (Catlin [7]) $\delta(G) \leq 3$.
(iv) (Catlin, et al. [10]) Either $G \in \{K_1, K_2\} \cup \{K_{2,t} | t \geq 1\}$ or $F(G) \geq 3$ and $|E(G)| \leq 2|V(G)| - 5$.

1.3 Main Results in Spanning Trail

In Chapter 3, we improve one of the Catlin's result ([11]) to every 3-edge-connected graph G with at most 11 edge-cuts of size 3 is supereulerian if and only if the reduction of G is not $P(10)$.

In Chapter 4, we prove several results based on the supereulerian graphs with bounded independence number or bounded matching number.

In Chapter 5, we prove a result based on spanning trail. That is if G is 4-edge-connected, then for any edge subset $X_0 \subseteq E(G)$ with $|X_0| \leq 3$ and any distinct edges $e, e' \in E(G)$, G has a spanning (e, e')-trail containing all edges in X_0, which solves a conjecture posed in ([34]).

Chapter 2

On 3-edge-connected Supereulerian Graphs

2.1 Prerequisites

By Theorem 1.2.1, efforts to characterize supereulerian graphs have been within families of 3-edge-connected graphs. Caltin and Lai ([11]) considered 3-edge-connected graphs with limited number of 3-edge cuts. They proved the following:

Theorem 2.1.1 *(Catlin and Lai [11]) Let G be a 3-edge-connected graph. If G has at most 10 edge-cuts of size 3, then exactly one of the following holds.*
(i) G is supereulerian;
(ii) The reduction of G is P(10).

Theorem 2.1.2 *(Catlin and Lai [11]) Let G be a 3-edge-connected graph. If G has at most 11 edge-cuts of size 3, then exactly one of the following holds.*
(i) G is supereulerian.
(ii) The reduction of G is P(10).
(iii) The reduction of G is a nonsupereulerian graph of order between 17 and 19, with girth at least 5, with exactly 11 vertices of degree 3 and 1 vertex of degree 5, and with the remaining vertices independent and of degree 4.

It has been a question whether graphs stated in Theorem 2.1.2 (iii) exist or not. In this chapter, we settle this problem by showing that no such graphs exist.

Theorem 2.1.3 *Let G be a 3-edge-connected graph. If G has at most 11 edge-cuts of size 3, then the following are equivalent:*
(i) G is supereulerian.
(ii) The reduction of G is not P(10).

Our proof depends on a new sufficient condition for a graph to be supereulerian. Let $\mathcal{F}$ denote the collection of all connected graphs satisfying each of the following.

(F1) $d_5(G) = 1$, $d_3(G) = 11$,

(F2) $3 \leq \delta(G) \leq \Delta(G) \leq 5$,

(F3) $g(G) \geq 5$,

(F4) no edge of G joins two vertices of even degree in G.

The following associate result plays an important role in our proof of Theorem 2.1.3.

Theorem 2.1.4 *If $G \in \mathcal{F}$, then G is supereulerian.*

2.2   The Proof of Main Theorem

Before proving Theorem 2.1.3, we will provide some useful theorems. By Theorem 1.2.3, we can prove that

Corollary 2.2.1 *If G is a connected reduced graph, then $2F(G) = 3d_1 + 2d_2 + d_3 - \sum_{j \geq 5}(j-4)d_j - 4$.*

Proof. As $|V(G)| = \sum_{j \geq 1} d_j$ and $2|E(G)| = \sum_{j \geq 1} jd_j$, by Theorem 1.2.3, we have $2F(G) = 3d_1 + 2d_2 + d_3 - \sum_{j \geq 5}(j-4)d_j - 4$. □

Theorem 2.2.2 *(Catlin and Lai [11]) Let G be a 3-edge-connected graph with $F(G) = 3$. If G is nonsupereulerian and reduced, then each of the following holds.*

(i) G has no edge joining two vertices of even degree;

(ii) $g(G) \geq 5$;

(iii) G has no subgraph H with $\kappa'(H) \geq 2$ and $F(H) = 2$.

Lemma 2.2.3 *Let G be a 3-edge-connected nonsupereulerian reduced graph with $F(G) = 3$. Then every edge-cut of size 3 is not an essential edge-cut (i.e. the number of edge-cut of size 3 is equal to $d_3(G)$).*

Proof. Let $X \subseteq E(G)$ be an edge-cut of size 3, and H_1 and H_2 the two components of $G - X$. By Theorem 1.2.2 (iv), H_1 and H_2 both are reduced. Then by Theorem 1.2.3,

$$
\begin{aligned}
F(G) &= 2|V(G)| - |E(G)| - 2 \\
&= 2(|V(H_1)| + |V(H_2)|) - (|E(H_1)| + |E(H_2)| + |X|) - 2 \\
&= 2|V(H_1)| - |E(H_1)| - 2 + 2|V(H_2)| - |E(H_2)| - 3 \\
&= F(H_1) + F(H_2) - 1
\end{aligned}
$$

and so $F(G) + 1 = F(H_1) + F(H_2)$. Since $F(G) = 3$, $\min\{F(H_1), F(H_2)\} \leq 2$, (say $F(H_1) \leq 2$). By Theorem 1.2.3, $H_1 \in \{K_1, K_2, K_{2,t}(t \geq 1)\}$. If $H_1 = K_1$, then X is not an essential edge-cut. If

5

$H_1 = K_2$ or $H_1 = K_{2,1}$, then vertex of degree 2 will appear, contrary to $\kappa'(G) \geq 3$. Hence $H_1 = K_{2,t}$ ($t \geq 2$). Since $K_{2,t}$ ($t \geq 2$) contains C_4, this is contrary to Theorem 2.2.2(ii). This completes the proof of lemma. $\square$

Now we are able to prove Theorem 2.1.3.

Proof of Theorem 2.1.3. Let G' be the reduction of G. By Theorem 1.2.2, it suffices to show that G' either is supereulerian or is $P(10)$. We shall show that G is contractible to $P(10)$ with the following assumption:

$$G' \text{ is not supereulerian.} \tag{2.1}$$

Since G has at most 11 edge cut of size 3, G' has at most 11 edge cut of size 3. Thus $d_3(G') \leq 11$. Since $\kappa'(G') \geq \kappa'(G) \geq 3$, $d_1(G') = d_2(G') = 0$. By Corollary 2.2.1, we have

$$2F(G') = 3d_1(G') + 2d_2(G') + d_3(G') - \sum_{j \geq 5}(j-4)d_j(G') - 4 = d_3(G') - \sum_{j \geq 5}(j-4)d_j(G') - 4. \tag{2.2}$$

By (2.2) and by $d_3(G') \leq 11$, $F(G') \leq 3$. If $F(G') \leq 2$, then by Theorem 1.2.3, $G' \in \{K_1, K_2\} \bigcup \{K_{2,t} \mid t \geq 1\}$. By (2.1), $G' \neq K_1$, and so $G' \in \{K_2\} \bigcup \{K_{2,t} \mid t \geq 1\}$, contrary to the fact that $\kappa'(G') \geq 3$. Hence $F(G') = 3$.

In the rest of the proof, we will write d_j for $d_j(G')$, $j \geq 1$. By (2.2) and by $F(G') = 3$,

$$10 = d_3 - \sum_{j \geq 5}(j-4)d_j. \tag{2.3}$$

Thus $11 \geq d_3 \geq 10$. If $d_3 = 10$, by Lemma 2.2.3, G' has exactly 10 edge-cuts of size 3. Hence by Theorem 2.1.1, $G' \cong P(10)$. If $d_3 = 11$, then by (2.3), $d_5 = 1$, $d_j = 0$ and $j \geq 6$. Thus $V(G') = D_3(G') \cup D_4(G') \cup D_5(G')$. Then by Theorem 2.2.2, $G' \in \mathcal{F}$. Thus by Theorem 2.1.4, G' is supereulerian, contrary to (2.1). This completes the proof of Theorem 1.3. $\square$

2.3 Graph Classification

Let $G \in \mathcal{F}$ be a graph. Throughout this section, we always use $w \in V(G)$ to denote the unique vertex of degree 5. Let H be the subgraph induced by the vertices of distance at least 2 from w in G and $G_0 = G - E(H)$. Define $S = N(w) \cap D_4(G)$, $T = N(w) \cap D_3(G)$, $S_1 = \cup_{u \in S} N(u) - w$, $T_1 = (\cup_{v \in T} N(v)) \cap D_3(G)$ and $T_2 = (\cup_{v \in T} N(v)) \cap D_4(G)$. Let $W = V(H) - (S_1 \cup T_1 \cup T_2)$, and let $a = |D_3(G) \cap W|$ and $b = |D_4(G) \cap W|$.

Lemma 2.3.1 *With the notations above, each of the following holds.*
(i) $N(w) = S \cup T$.

(ii) $V(G_0) = V(G)$ and $E(G_0) = \cup_{u \in S \cup T} E(u)$.

(iii) $\forall u, v \in S \cup T$ with $u \neq v$, $N(u) \cap N(v) - w = \emptyset$.

(iv) G_0 is acyclic.

(v) $(S_1 \cup T_1 \cup T_2) \subseteq V(H)$ and $S_1 \subseteq D_3(G)$.

(vi) $|S_1| = 3|S|$ and $|T_1| + |T_2| = 2|T|$.

(vii) $d_3(G) = |S_1| + |T| + |T_1| + a$ and $d_4(G) = |S| + |T_2| + b$.

(viii) $|E(H[V(H) \cap D_3(G)])| = \frac{1}{2}((3a + 2(|S_1| + |T_1|)) - (4b + 3|T_2|))$, and $4b + 3|T_2| \leq 3a + 2(|S_1| + |T_1|)$.

Proof. (i) follows from (F1) and (F2). The definition of H implies (ii). (iii) and (iv) follow from (F3) and (v) follows from (F4). Since $S \subseteq D_4(G)$ and $T \subseteq D_3(G)$, for every $u \in S$, $|N(u) \cap V(H)| = 3$ and for every $v \in T$, $|N(v) \cap V(H)| = 2$. These imply (vi).

By the definitions of S_1, T_1 and T_2 and by (F3), it is mutually disjoint between S_1, T_1 and T_2. Then direct computation yields (vii). By the definition of H, $|V(H)| = a + b + |S_1| + |T_1| + |T_2|$. Let $H_1 = H[V(H) \cap D_3(G)]$. Then counting $\sum_{v \in V(H_1)} d_G(v)$ in two different ways, we obtain

$$3a + 3(|S_1| + |T_1|) = \sum_{v \in V(H_1)} d_G(v) = 2|E(H_1)| + |S_1| + |T_1| + 4b + 3|T_2|,$$

and so (viii) follows. $\qquad \square$

By (F1), $11 = d_3(G) = 3|S| + |T| + |T_1| + a \geq 3|S| + |T| = 3|S| + 5 - |S|$, and so

$$|S| \leq 3, \text{ where } |S| = 3 \text{ only if } |T_1| + a = 0. \tag{2.4}$$

Throughout this section, let

$$S = \{u_1, u_2, ..., u_{|S|}\} \tag{2.5}$$

$$N(u_i) \cap V(H) = \{w_{3i-2}, w_{3i-1}, w_{3i}\}, \text{ where } 1 \leq i \leq |S|$$

$$T = \{v_1, v_2, ..., v_{5-|S|}\}$$

$$N(v_i) \cap V(H) = \{w_{3|S|+2j-1}, w_{3|S|+2j}\}, \text{ where } 1 \leq j \leq 5 - |S| = |T|.$$

As $|S| \leq 3$, $3|S| + 2(5 - |S|) \leq 13$. By (F3),

$$w_i \neq w_j \text{ if and only if } i \neq j \text{ for } 1 \leq i, j \leq 13. \tag{2.6}$$

Lemma 2.3.2 *G must be one of 8 possible graphs.*

Proof. By (2.4), $|S| \leq 3$ and so we can analyze cases when $|S|$ takes different values.

Case 1 $|S| = 3$.

Then $|T| = 2$. By (2.4), $|T_1| + a = 0$. As $d_3(G) = 11$, $D_3(G) = T \cup S_1$ and $|T_2| = 2|T| - |T_1| = 4$. By Lemma 2.3.1 (viii), $0 \leq b \leq 1$. If $b = 1$, then $V(G) \cap W \cap D_4(G)$ has a vertex z. Since

7

G	n	S	S_1	T	$|T_1|$	$T_1 \cup T_2$	a	b
G_1	20	$\{u_1, u_2\}$	$\{w_1, w_2, \cdots, w_6\}$	$\{v_1, v_2, v_3\}$	0	$\{w_7, w_8, \cdots, w_{12}\}$	2	0
G_2	19	$\{u_1, u_2, u_3\}$	$\{w_1, w_2, \cdots, w_9\}$	$\{v_1, v_2\}$	0	$\{w_{10}, w_{11}, w_{12}, w_{13}\}$	0	0
G_3	19	$\{u_1, u_2\}$	$\{w_1, w_2, \cdots, w_6\}$	$\{v_1, v_2, v_3\}$	2	$\{w_7, w_8, \cdots, w_{12}\}$	0	1
G_4	19	$\{u_1, u_2\}$	$\{w_1, w_2, \cdots, w_6\}$	$\{v_1, v_2, v_3\}$	1	$\{w_7, w_8, \cdots, w_{12}\}$	1	0
G_5	18	$\{u_1, u_2\}$	$\{w_1, w_2, \cdots, w_6\}$	$\{v_1, v_2, v_3\}$	2	$\{w_7, w_8, \cdots, w_{12}\}$	0	0
G_6	18	$\{u_1\}$	$\{w_1, w_2, w_3\}$	$\{v_1, v_2, v_3, v_4\}$	3	$\{w_4, w_5, \cdots, w_{11}\}$	1	0
G_7	17	$\{u_1\}$	$\{w_1, w_2, w_3\}$	$\{v_1, v_2, v_3, v_4\}$	4	$\{w_4, w_5, \cdots, w_{11}\}$	0	0
G_8	16	$\emptyset$	$\emptyset$	$\{v_1, v_2, v_3, v_4, v_5\}$	6	$\{w_1, w_2, \cdots, w_{10}\}$	0	0

Table 2.1: The graphs G_i, $(1 \le i \le 8)$

$|T_1| = 0$ and by (F4), $N(z) \subseteq S_1$. Since $d_G(z) = 4$, $|N(z) \cap N(u_i)| \ge 2$ for some $i \in \{1, 2, 3\}$, whence $G[(N(z) \cap N(u_i)) \cup \{z, u_1\}]$ induces a C_4, contrary to (F3). Therefore in Case 1, $b = 0$, and so there is only one possible graph, called G_2, as presented in Table 2.1.

Case 2 $|S| = 2$.

As $d_3(G) = 11$, $|T_1| = 11 - |T| - |S_1| - a = 2 - a$ and $|T_2| = 6 - (2 - a) = 4 + a$. Then by Lemma 2.3.1 (viii), $4b + 3(4 + a) \le 3a + 2(6 + 2 - a)$, and so, $a + 2b \le 2$. Therefore, there will be 4 different possible graphs in this case. Let G_1, G_3, G_4, G_5 denote such a graph when $a = 2$ and $b = 0$, or when $a = 0$ and $b = 1$, or when $a = 1$ and $b = 0$, or $a = 0$ and $b = 0$, respectively, as presented in Table 2.1.

Case 3 $|S| = 1$.

In this case, $|T_1| = 11 - |T| - |S_1| - a = 4 - a$ and $|T_2| = 8 - (4 - a) = 4 + a$. By Lemma 2.3.1 (viii), $4b + 3(4 + a) \le 3a + 2(3 + 4 - a)$, and so $a + 2b \le 1$. Let G_6, G_7 denote such a graph when $a = 1$ and $b = 0$, or when $a = 0$ and $b = 0$, respectively, as presented in Table 2.1.

Case 4 $|S| = 0$.

Then $S = S_1 = \emptyset$. Again by $d_3(G) = 11$, $|T_1| = 11 - |T| - |S_1| - a = 6 - a$ and $|T_2| = 10 - (6 - a) = 4 + a$. Then by Lemma 2.3.1 (viii), $4b + 3(4 + a) \le 3a + 2(0 + 6 - a)$, and so $a = 0$ and $b = 0$. Thus there is one such graph, denoted by G_8, as presented in Table 2.1.

Summing up, we list the 8 possibilities of G in the following Table 2.1, with $n = |V(G)|$. This proves the lemma. $\square$

Throughout the rest of this section, the graphs G_i $(1 \le i \le 8)$, will be these graphs defined in Table 2.1.

Lemma 2.3.3 *If* $G \in \{G_1, G_3, G_6, G_8\}$, *then* $|E(H[V(H) \cap D_3(G)])| = 0$ *and* $4b + 3|T_2| = 3a +$

$2(|S_1| + |T_1|)$.

Proof. By Lemma 2.3.1 (viii), it suffices to show that $4b + 3|T_2| = 3a + 2(|S_1| + |T_1|)$.

If $G = G_1$, then $a = 2$, $b = 0$, $|T_1| = 0$ and $|S_1| = 6$. By Lemma 2.3.1(vi), $|T_2| = 2|T| = 6$. Thus $4b + 3|T_2| = 18 = 3a + 2(|S_1| + |T_1|)$. If $G = G_3$, then $a = 0$, $b = 1$, $|T_1| = 2$, $|T_2| = 4$ and $|S_1| = 6$. Thus $4b + 3|T_2| = 16 = 3a + 2(|S_1| + |T_1|)$. If $G = G_6$, then $a = 1$, $b = 0$, $|T_1| = 3$, $|T_2| = 5$ and $|S_1| = 3$. Thus $4b + 3|T_2| = 15 = 3a + 2(|S_1| + |T_1|)$. If $G = G_8$, then $a = b = 0$, $|T_1| = 6$, $|T_2| = 4$ and $|S_1| = 0$. Thus $4b + 3|T_2| = 12 = 3a + 2(|S_1| + |T_1|)$. $\qquad\square$

Lemma 2.3.4 $G \neq G_3$.

Proof. Suppose $G = G_3$. Then as shown in Table 2.1, G_0 is isomorphic to G_3' in Figure 1. Thus $S = \{u_1, u_2\}$, $S_1 = \{w_1, w_2, w_3, w_4, w_5, w_6\}$, $T = \{v_1, v_2, v_3\}$, $a = 0$, $b = 1$, $|T_1| = 2$ and $|T_2| = 4$. Denote the vertex of degree 4 in $V(G_3) \cap W$ by x. If the two vertices in T_1 have one common neighbor in T (say $v_1 \in N(w_7) \cap N(w_8)$, and so $T_1 = \{w_7, w_8\}$), then by (F4), $N(x) \subseteq S_1 \cup T_1$. Since $|N(x)| = 4$, either $T_1 \subseteq N(x)$, whence $G[\{v_1, x\} \cup T_1]$ contains a 4-cycle, contrary to (F3); or for some $i = 1, 2$, $|N(x) \cap N(u_i)| \geq 2$, whence $G[(N(x) \cap N(u_i)) \cup \{x, u_i\}]$ has a 4-cycle, contrary to (F3). Hence by symmetry, we may assume that $T_1 = \{w_7, w_9\}$. By (F3) and (F4), $w_7, w_9 \in N(x)$, $w_8 \in N(w_9)$ and $w_{10} \in N(w_7)$, and so $N_H(w_{11}) \subseteq S_1 = N(u_1) \cup N(u_2)$. Since $w_{11} \in D_3(H)$, then for some $i \in \{1, 2\}$, $|N_H(w_{11}) \cap N(u_i)| \geq 2$, and so $G\{w_{11}, u_i\} \cup (N_H(w_{11}) \cap N(u_i))$ contains a 4-cycle, contrary to (F3). $\qquad\square$

Lemma 2.3.5 $G \neq G_6$.

Proof. Suppose $G = G_6$. Then as shown in Table 2.1, G_0 is isomorphic to G_6' in Figure 1. Thus we have $S = \{u_1\}$, $S_1 = \{w_1, w_2, w_3\}$, $T = \{v_1, v_2, v_3, v_4\}$, $a = 1$ and $b = 0$. Then $|T_1| = 3$ and $|T_2| = 5$. Denote the vertex of degree 3 in $V(G_6) \cap W$ by x. By Lemma 2.3.3, $|E(H[V(H) \cap D_3(G)])| = |E(G_6[S_1 \cup T_1])| = 0$, and so $N_H(w_1) \cup N_H(w_2) \cup N_H(w_3) \subseteq T_2$.

By (F3), $N_H(w_i) \cap N_H(w_j) = \emptyset$ for all $i \neq j, 1 \leq i \leq 3, 1 \leq j \leq 3$, and so $|N_H(w_1) \cup N_H(w_2) \cup N_H(w_3)| = 6$, contrary to the fact that $|T_2| = 5$. $\qquad\square$

Lemma 2.3.6 $G \neq G_7$.

Proof. Assume that $G = G_7$. Then as shown in Table 2.1, G_0 is isomorphic to G_7' in Figure 1. Thus we have $S = \{u_1\}$, $S_1 = \{w_1, w_2, w_3\}$, $T = \{v_1, v_2, v_3, v_4\}$ and $a = b = 0$. Then $|T_1| = 4$ and $|T_2| = 4$. By (F4) and Lemma 2.3.1(viii), $|E(G_7[S_1] \cup T_1)| = |E(H[V(H) \cap D_3(G)])| = \frac{1}{2}((3a + 2(|S_1| + |T_1|)) - (4b + 3|T_2|)) = 1$. By (F3), $N_H(w_i) \cap N_H(w_j) = \emptyset$ for any $i \neq j$ $(i, j \in \{1, 2, 3\})$. As in this case, $\{w_1, w_2, w_3\} \subseteq D_2(H)$, and so $|N_H(w_1) \cup N_H(w_2) \cup N_H(w_3)| = 6$. Since $|E(G_7[S_1] \cup T_1)| = 1$, we have $|(N_H(w_1) \cup N_H(w_2) \cup N_H(w_3)) \cap T_1| \leq 1$, and so $|(N_H(w_1) \cup N_H(w_2) \cup N_H(w_3)) \cap T_2| \geq 5$ by $N_H(w_1) \cup N_H(w_2) \cup N_H(w_3) \subseteq T_1 \cup T_2$, contrary to the fact that $|T_2| = 4$. $\qquad\square$

9

Lemma 2.3.7 *If $G = G_1$, then G is supereulerian.*

Proof. Suppose $G = G_1$. We use the notation in Table 2.1 for G_1. As $a = 2$, let $D_3(G) \cap W = \{x, y\}$. By Lemma 2.3.3, $E(G[S_1 \cup \{x, y\}]) = \emptyset$. Hence $N(x) \cup N(y) \subseteq T_2 = \{w_7, w_8, w_9, w_{10}, w_{11}, w_{12}\}$.

If $N(x) \cap N(y) \neq \emptyset$, then there is a vertex in T_2 (say w_7) which is adjacent to neither x nor y. Hence $N_H(w_7) \subseteq S_1$. Since vertex w_7 has degree 3 in H, C_4 must be induced. Therefore $N(x) \cap N(y) = \emptyset$.

Without loss of generality, by (F3) we may assume that $x \in N(w_7) \cap N(w_9) \cap N(w_{11})$, and $y \in N(w_8) \cap N(w_{10}) \cap N(w_{12})$. Thus $|N(w_7) \cap S_1| = |N(w_8) \cap S_1| = 2$. By (F3), without loss of generality, we may assume that $w_7 \in N(w_1) \cap N(w_4)$ and $w_8 \in N(w_2) \cap N(w_5)$. Hence $|N(w_3) \cap \{w_9, w_{10}, w_{11}, w_{12}\}| = |N(w_6) \cap \{w_9, w_{10}, w_{11}, w_{12}\}| = 2$. By symmetry and by (F3), we may also assume that $w_3 \in N(w_9) \cap N(w_{11})$ and $w_6 \in N(w_{10}) \cap N(w_{12})$.

By the assumptions above, we got a graph $G_1' = G[E(G_0) \cup \{xw_7, xw_9, xw_{11}, yw_8, yw_{10}, yw_{12}, w_1w_7, w_4w_7, w_2w_8, w_5w_8, w_3w_9, w_3w_{11}, w_6w_{10}, w_6w_{12}\}]$ (see Figure 1). Then G_1' is a spanning subgraph of G. Since $G_1' - \{wv_1, wv_2, wv_3, w_3w_{11}, w_6w_{12}, xw_9, yw_{10}\}$ is a spanning eulerian subgraph of G_1', G is supereulerian. $\qquad \square$

Lemma 2.3.8 *If $G = G_2$, Then G is supereulerian.*

Proof. Suppose $G = G_2$. We use the notation in Table 2.1 for G_2. Then $T_1 = \emptyset$, and so by Lemma 2.3.1(vi), $T_2 = \{w_{10}, w_{11}, w_{12}, w_{13}\}$. As $a = b = 0$, $3a + 2(|S_1| + |T_1|) - 4b + 3|T_2| = 18 - 12 = 6$, and so by Lemma 2.3.1 (viii) and by (F4), $|E(G[S_1])| = 3$. Let $H_1 = H - E(G[S_1])$.

By (F3), $g(G) \geq 5$, and so $N_{H_1}(w_{10}) \cap N_{H_1}(w_{11}) = \emptyset$ and $N_{H_1}(w_{12}) \cap N_{H_1}(w_{13}) = \emptyset$. Let $P = N_{H_1}(w_{10}) \cup N_{H_1}(w_{11})$ and $Q = N_{H_1}(w_{12}) \cup N_{H_1}(w_{13})$. Then by (F4),

$$P \cup Q \subseteq S_1. \tag{2.7}$$

As $\{w_{10}, w_{11}, w_{12}.w_{13}\} \subseteq D_3(H_1)$, $|N_{H_1}(w_{10})| = |N_{H_1}(w_{11})| = |N_{H_1}(w_{12})| = |N_{H_1}(w_{13})| = 3$. Thus $|P| = |Q| = 6$. If $|P \cap Q| \geq 5$, then $N_{H_1}(w_{10}) \subseteq (P \cap Q)$ or $N_{H_1}(w_{11}) \subseteq (P \cap Q)$. We suppose $N_{H_1}(w_{10}) \subseteq (P \cap Q)$. By $|N_{H_1}(w_{10})| = 3$, w_{10} has two neighbors in some member of $\{N_{H_1}(w_{12}), N_{H_1}(w_{13})\}$, say in $N_{H_1}(w_{12})$. Thus the two neighbors and $\{w_{10}, w_{12}\}$ together induce a 4-cycle in G, contrary to (F3). If $|P \cap Q| \leq 2$, then $|P \cup Q| \geq 10 > 9 = |S_1|$, contrary to (2.7). Hence $3 \leq |P \cap Q| \leq 4$.

Case 1 $|P \cap Q| = 4$.

Since $|P \cap Q| = 4$ and $|S| = 3$, for some $u_i \in S$, $|(P \cap Q) \cap N(u_i)| \geq 2$. Hence we may assume that $w_1, w_2 \in (P \cap Q) \cap N(u_1)$. By (F3), $N_{H_1}(w_1) \cap N_{H_1}(w_2) = \emptyset$. As $\{w_1, w_2\} \subseteq (P \cap Q) \cap D_2(H)$, we have $|N_{H_1}(w_1) \cap \{w_{10}, w_{11}\}| = |N_{H_1}(w_1) \cap \{w_{12}, w_{13}\}| = 1$ and $|N_{H_1}(w_2) \cap \{w_{10}, w_{11}\}| = |N_{H_1}(w_2) \cap \{w_{12}, w_{13}\}| = 1$. Hence by $N_{H_1}(w_1) \cap N_{H_1}(w_2) = \emptyset$, $\{w_{10}, w_{11}, w_{12}, w_{13}\} \subseteq N_{H_1}(w_1) \cup N_{H_1}(w_2)$. Without

10

loss of generality, assume that $\{w_1w_{10}, w_1w_{12}, w_2w_{11}, w_2w_{13}\} \subseteq E(G_2)$. By symmetry and (F3), we may further assume $\{w_{10}w_4, w_{10}w_7, w_{11}w_5, w_{11}w_8\} \subseteq E(G_2)$. As $|P \cup Q| = |P| + |Q| - |P \cap Q| = 8 < 9 = |S_1|$ and by (2.7), $|S_1 - P \cup Q| = 1$. If $w_3 \in P \cup Q$, then by $\{w_{10}, w_{11}, w_{12}, w_{13}\} \subseteq N_{H_1}(w_1) \cup N_{H_1}(w_2)$, for some $i \in \{10, 11, 12, 13\}$, $|N(w_i) \cap N_{H_1}(u_1)| \geq 2$, say $|N(w_{10}) \cap N_{H_1}(u_1)| \geq 2$. Then $G[\{u_1, w_{10}\} \cup (N(w_{10}) \cap N_{H_1}(u_1))]$ contains a 4-cycle, contrary to (F3). Therefore $w_3 \notin P \cup Q$.

It follows that either $w_6 \in N(w_{12})$ and $w_9 \in N(w_{13})$ or $w_6 \in N(w_{13})$ and $w_9 \in N(w_{12})$. By symmetry, we assume $w_6 \in N(w_{12})$ and $w_9 \in N(w_{13})$. Thus w_3 must be adjacent to one of vertices w_4, w_5 and w_6. The proofs for each of these subcases will be similar, and so we shall only prove the case when $w_3w_4 \in E(G)$ and omit the others.

Let $G_2' = G_0 + \{w_1w_{10}, w_1w_{12}, w_2w_{11}, w_2w_{13}, w_{10}w_4, w_{10}w_7, w_{11}w_5, w_{11}w_8, w_6w_{12}, w_9w_{13}, w_3w_4\}$ (see Figure 1). Then G_2' is a spanning subgraph of G. As $G_2' - \{wv_2, v_1w_{10}, w_1w_{12}, w_2w_{13}\}$ is a spanning eulerian subgraph of G_2', G is supereulrian.

Case 2 $|P \cap Q| = 3$.

By (2.7) and $|P \cup Q| = |P| + |Q| - |P \cap Q| = 9 = |S_1|$, $P \cup Q = S_1$, and so $\Delta(G_2[S_1]) = 1$. Let $P \cup Q = \{z_1, z_2, z_3\}$. Hence $\{N_{H_1}(z_1), N_{H_1}(z_2), N_{H_1}(z_3)\} \subset \{\{w_{10}, w_{12}\}, \{w_{10}, w_{13}\}, \{w_{11}, w_{12}\}, \{w_{11}, w_{13}\}\}$. By symmetry, we may assume $N_{H_1}(z_1) = \{w_{10}, w_{12}\}$, $N_{H_1}(z_2) = \{w_{10}, w_{13}\}$ and $N_{H_1}(z_3) = \{w_{11}, w_{12}\}$. Let $G_2'' = G_0 + E(H_1)$. Then G_2'' is a spanning subgraph of G. (An example with $z_1 = w_1, z_2 = w_4, z_3 = w_7$ is shown in Figure 1). By $|E(G_2[S_1])| = 3$ and $\Delta(G_2[S_1]) = 1$, $O(G_2'') = \{w, v_1, v_2, z_1, z_2, z_3\}$. It follows that $G_2'' - \{wv_1, z_1w_{10}, z_2w_{10}, z_3w_{12}, v_2w_{12}\}$ is a spanning eulerian subgraph of G_2'', and so G is supereulrian. □

Lemma 2.3.9 *If $G = G_4$, then G is supereulerian.*

Proof. Suppose $G = G_4$. We use the notation in Table 2.1 for G_4. As $a = 1$, let $D_3(G) \cap W = \{x\}$. Since $|T_1| = 1$, by Lemma 2.3.1(vi), $|T_2| = 2|T| - |T_1| = 5$. Without loss of generality, let $T_1 = \{w_7\}$ and so $T_2 = \{w_8, w_9, w_{10}, w_{11}, w_{12}\}$. By Lemma 2.3.1(viii), $|E(G_4[S_1 \cup \{w_7, x\}])| = 3a + 2(|S_1| + |T_1|) - 4b + 3|T_2| = 1$. Let $E(G_4[S_1 \cup \{w_7, x\}]) = \{e\}$.

Case 1 x is not incident with e.

Since $E(G_4[S_1 \cup \{w_7, x\}]) = \{e\}$, x is an isolated vertex in $G_4[S_1 \cup \{w_7, x\}]$ and so $N(x) \subseteq T_2$. If $N(x) \subseteq T_2 - \{w_8\}$, then by $|N(x)| = 3$, for some $i \in \{2, 3\}$, $|N(v_i) \cap N(x)| \geq 2$, and so $G[\{x, v_i\} \cup (N(v_i) \cap N(x))]$ has a 4-cycle, contrary to (F3). Hence $x \in N(w_8)$. Without loss of generality, we may assume $x \in N(w_9) \cap N(w_{11})$.

Thus by (F4), $N_H(w_{10}) \subseteq S_1 \cup \{w_7\}$. If $N_H(w_{10}) \subseteq S_1$, then as $|N_H(w_{10})| = 3$, for some $i \in \{1, 2\}$, $|N(u_i) \cap N_H(w_{10})| \geq 2$, and so $G[\{u_i, w_{10}\} \cup (N(u_i) \cap N_H(w_{10}))]$ has a 4-cycle, contrary to (F3). Hence $w_{10} \in N(w_7)$. Similarly, $w_{12} \in N(w_7)$. Since $|N_H(w_8)| = 3$, $w_8 \notin N(w_7)$ and $x \in N_H(w_8)$, we

$N_H(w_3)$ is in	G has a 4-cycle in
w_9, w_{10}	$G[\{w_3, w_9, w_{10}, v_2\}]$
w_9, w_{11}	$G[\{w_3, w_9, w_{11}, x\}]$
w_9, w_{12}	$G[\{w_3, w_9, w_{12}, w_2\}]$
w_{10}, w_{11}	$G[\{w_3, w_{10}, w_{11}, w_5\}]$
w_{10}, w_{12}	$G[\{w_3, w_{10}, w_{12}, w_7\}]$
w_{11}, w_{12}	$G[\{w_3, w_{11}, w_{12}, v_3\}]$

Table 2.2: Possible 4-cycles in G

have $|N_H(w_8) \cap S_1| = 2$. Then by (F3), w_8 must be adjacent to one vertex in $\{w_1, w_2, w_3\}$ and to one vertex in $\{w_4, w_5, w_6\}$. Thus we may assume $w_8 \in N(w_1) \cap N(w_4)$. Since e cannot be incident with two vertices in $\{w_1, w_2, w_3\}$, with $w_8 \in N(w_1)$, one of $\{w_2, w_3\}$ must be adjacent to two vertices in $\{w_9, w_{10}, w_{11}, w_{12}\}$. Similarly, one of $\{w_5, w_6\}$ must be adjacent to two vertices in $\{w_9, w_{10}, w_{11}, w_{12}\}$. Without loss of generality, let $|N(w_2) \cap \{w_9, w_{10}, w_{11}, w_{12}\}| = |N(w_5) \cap \{w_9, w_{10}, w_{11}, w_{12}\}| = 2$. By (F3), $\{N_H(w_2), N_H(w_5)\} = \{\{w_9, w_{12}\}, \{w_{10}, w_{11}\}\}$ and $N_H(w_2) \neq N_H(w_5)$. By symmetry, we assume $\{w_2 w_9, w_2 w_{12}, w_5 w_{10}, w_5 w_{11}\} \subseteq E(G_4)$. Note that $N_H(w_3) \cap \{w_7, w_8\} = \emptyset$ and $N_H(w_6) \cap \{w_7, w_8\} = \emptyset$.

Under these assumptions, we shall show $e = w_3 w_6$. If $N_H(w_3) \subseteq \{w_9, w_{10}, w_{11}, w_{12}\}$, then $N_H(w_3) \in \{\{w_9, w_{10}\}, \{w_9, w_{11}\}, \{w_9, w_{12}\}, \{w_{10}, w_{11}\}, \{w_{10}, w_{12}\}, \{w_{11}, w_{12}\}\}$ by $|N_H(w_3)| = 2$. In any case, G would have a 4-cycle (see Table 2.2), contrary to (F3). Hence, by $N_H(w_3) \cap \{w_7, w_8\} = \emptyset$, $|N_H(w_3) \cap \{w_4, w_5, w_6\}| \geq 1$. If $|N_H(w_3) \cap \{w_4, w_5, w_6\}| \geq 2$, then $G[N(w_3) \cap N(u_2) \cup \{w_3\}]$ contains a 4-cycle, contrary to (F3). Hence $|N_H(w_3) \cap \{w_4, w_5, w_6\}| = 1$. By symmetry, $|N_H(w_6) \cap \{w_1, w_2, w_3\}| = 1$. As $\{e\} = E(G_1[S_1 \cup \{w_7, x\}])$, we have $e = w_3 w_6$.

Let $G_4' = G_0 + \{xw_8, xw_9, xw_{11}, w_7 w_{10}, w_7 w_{12}, w_1 w_8, w_4 w_8, w_2 w_9, w_2 w_{12}, w_5 w_{10}, w_5 w_{11}, w_3 w_6\}$. Thus we obtained a spanning subgraph G_4' of G_4 (see Figure 1). Since $G_4' - \{wv_1, wv_2, wv_3, w_7 w_{10}, w_5 w_{11}, w_2 w_{12}, xw_9\}$ is a spanning eulerian subgraph of G_4', G_4 is supereulerian.

Case 2 x is incident with e.

If $e = xw_7$, then as $|E(G_1[S_1 \cup \{w_7, x\}])| = 1$, $N_H(w_1) \cup N_H(w_2) \cup N_H(w_3) \subseteq \{w_8, w_9, w_{10}, w_{11}, w_{12}\}$ and by $g(G_4) \geq 5$, $N_H(w_i) \cap N_H(w_j) = \emptyset$ ($i, j \in \{1, 2, 3\}$ and $i \neq j$). Hence $|N_H(w_1) \cup N_H(w_2) \cup N_H(w_3)| = 6$, contrary to $N_H(w_1) \cup N_H(w_2) \cup N_H(w_3) \subseteq \{w_8, w_9, w_{10}, w_{11}, w_{12}\}$. Therefore $x \notin N(w_7)$ and so $|N(x) \cap S_1| = 1$. Thus by (F3), for every $v \in \{w_8, w_9, w_{10}, w_{11}, w_{12}\}$, $N_H(v) \cap \{x, w_7\} \neq \emptyset$. Therefore, $\{w_8, w_9, w_{10}, w_{11}, w_{12}\} \subseteq N_H(x) \cup N_H(w_7)$, and so $|N_H(x) \cup N_H(w_7) \cap \{w_8, w_9, w_{10}, w_{11}, w_{12}\}| \geq 5$. But as $d_H(x) = 3$, $d_H(w_7) = 2$ and $|N_H(x) \cap S_1| = 1$, $|(N_H(w_7) \cup N_H(x)) \cap \{w_8, w_9, w_{10}, w_{11}, w_{12}\}| \leq 4$, contrary to $|N_H(x) \cup N_H(w_7) \cap \{w_8, w_9, w_{10}, w_{11}, w_{12}| \geq 5$. $\square$

Lemma 2.3.10 *If $G = G_5$, then G is supereulerian.*

Proof. Suppose $G = G_5$. We use the notation in Table 2.1 for G_5, and so $S = \{u_1, u_2\}$, $S_1 = \{w_1, w_2, w_3, w_4, w_5, w_6\}$, $T = \{v_1, v_2, v_3\}$ and $a = b = 0$. Since $|T_1| = 2$, by Lemma 2.3.1(vi), $|T_2| = 2|T| - |T_1| = 4$. By Lemma 2.3.1(viii), $|E(G_5[S_1 \cup T_1])| = 3a + 2(|S_1| + |T_1|) - 4b + 3|T_2| = 2$. Denote $E(G_5[S_1 \cup T_1]) = \{e_1, e_2\}$. As $|T_1| = 2$, we may assume that $T_1 = \{w_7, w'\}$, for some $w' \in \{w_8, w_9, ..., w_{12}\}$.

Case 1 $w' \in N(v_1)$. Then $w' = w_8$.

Without loss of generality, we may assume that w_1, w_4 and $w_7 \in N(w_9)$, and that w_2, w_5 and $w_8 \in N(w_{10})$. Then each of w_{11} and w_{12} must be adjacent to one in $\{w_7, w_8\}$. By symmetry, assume $w_7w_{11}, w_8w_{12} \in E(G_5)$. As $w_9, w_{11} \in N(w_7)$ and as $w_{10}, w_{12} \in N(w_8)$, both of e_1 and e_2 can only be adjacent to vertices in S_1. By (F3), $g(G_5) \geq 5$, and so e_1 is not adjacent to e_2. Since $N_H(w_9) = \{w_1, w_4, w_7\}$ and $N_H(w_{10}) = \{w_2, w_5, w_8\}$, each of w_3 and w_6 is adjacent to at least one in $\{w_{11}, w_{12}\}$. Thus we may assume that $w_3w_{11}, w_6w_{12} \in E(G_5)$ (The proofs for the other cases $w_3w_{12}, w_6w_{11} \in E(G_5)$ or $w_3w_{11}, w_6w_{11} \in E(G_5)$ or $w_3w_{12}, w_6w_{12} \in E(G_5)$ are similar).

Let $G_5' = G_0 + \{w_1w_9, w_4w_9, w_7w_9, w_2w_{10}, w_5w_{10}, w_8w_{10}, w_7w_{11}, w_8w_{12}, w_3w_{11}, w_6w_{12}\}$. Then G_5' is a spanning subgraph of G_5 (see Figure 1). Since $G_5' - \{wv_1, wv_2, wv_3, w_7w_{11}, w_8w_{12}\}$ is a spanning eulerian subgraph of G_5', G_5 is supeuelerian.

Case 2 $w' \notin N(v_1)$. Thus we may assume that $w' = w_9$.

Then by (F3), w_8w_9, $w_{10}w_7 \in E(G_5)$. By symmetry, each of w_{11} and w_{12} must be adjacent to one in $\{w_7, w_9\}$, to one in $\{w_1, w_2, w_3\}$ and one in $\{w_4, w_5, w_6\}$. Without loss of generality, we assume vertex w_1, w_4 and $w_7 \in N(w_{11})$, and w_2, w_5 and $w_9 \in N(w_{12})$. Let $G_5'' = G_0 + \{w_8w_9, w_7w_{10}, w_1w_{11}, w_4w_{11}, w_7w_{11}, w_2w_{12}, w_5w_{12}, w_9w_{12}\}$. Thus G_5'' is a spanning subgraph G_5 (see Figure 1).

As $N_H(w_7) = \{w_{10}, w_{11}\}$ and $N_H(w_9) = \{w_8, w_{12}\}$, $E(G_5[S_1] \cup T_1) = E(G_5[S_1])$. By (F3), $\Delta(G_5[S_1]) = 1$. Since $N_H(w_{11}) = \{w_1, w_4, w_7\}$ and $N_H(w_{12}) = \{w_2, w_5, w_9\}$, each of w_3 and w_6 is adjacent to w_8 or w_{10}. If $\{w_3w_{10}, w_6w_8\} \subset E(G_5)$, (or similarly, $\{w_3w_8, w_6w_{10}\} \subset E(G_5)$), then $G_5'' + \{w_3w_{10}, w_6w_8\} - \{wv_1, wv_2, wv_3, w_8w_9, w_7w_{10}\}$ is an eulerian subgraph of $G_5'' + \{w_3w_{10}, w_6w_8\}$ which spans G_5, and so G_5 is supereulerian.

If $\{w_3w_8, w_6w_8\} \subset E(G_5)$, (or similarly, $\{w_3w_{10}, w_6w_{10}\} \subset E(G_5)$), then $G_5'' + \{w_3w_8, w_6w_8\} - \{wv_3, v_1w_7, v_2w_9\}$ is a spanning eulerian subgraph of $G_5'' + \{w_3w_8, w_6w_8\}$ that spans G_5, and so G_5 must be supereulerian. $\square$

Lemma 2.3.11 *If $G = G_8$, then G is supereulerian.*

Proof. Suppose $G = G_8$. We use the notation in Table 2.1, so $S = \emptyset$ and $T = \{v_1, v_2, v_3, v_4, v_5\}$. By Lemma 2.3.3, $|E(H[V(H) \cap D_3(G)])| = \emptyset$, and so

$$H \text{ is a bipartite graph with a vertex bipartition } (T_1, T_2). \tag{2.8}$$

By (F3), for any i with $1 \leq i \leq 5$,

$$N_H(w_{2i-1}) \cap N_H(w_{2i}) = \emptyset. \tag{2.9}$$

Without loss of generality, assume that $w_8, w_{10} \in T_2$. Define

$$T' = \{v \in T : N_H(v) \subseteq T_2\}.$$

If $|T'| \geq 2$, as $|T_1| = 6$ and $|T_2| = 4$, we may assume $\{w_7, w_8, w_9, w_{10}\} = T_2$. By (F3) and (2.8), $N_H(w_7) \cup N_H(w_8) = \{w_1, w_2, w_3, w_4, w_5, w_6\} = N_H(w_9) \cup N_H(w_{10})$, $|N_H(w_7)| = |N_H(w_8)| = |N_H(w_9)| = |N_H(w_{10})| = 3$ and $N_H(w_7) \cap N_H(w_8) = N_H(w_9) \cap N_H(w_{10}) = \emptyset$. It follows that either $|N_H(w_7) \cap N_H(w_9)| \geq 2$ or $|N_H(w_7) \cap N_H(w_{10})| \geq 2$, forcing G_8 to has a 4-cycle, contrary to (F3). Hence $|T'| \leq 1$.

Case 1. $|T'| = 1$.

We may assume that $T' = \{v_5\}$, and so by symmetry, assume that $T_2 = \{w_6, w_8, w_9, w_{10}\}$. By (2.9) and (F3), we have that $N_H(w_9) \cup N_H(w_{10}) = \{w_1, w_2, w_3, w_4, w_5, w_7\}$. By symmetry, let $\{w_1 w_9, w_3 w_9, w_5 w_9\} \subset E(G_8)$, it follows $\{w_2 w_{10}, w_4 w_{10}, w_7 w_{10}\} \subset E(G_8)$. By (F3), $w_6 w_7, w_5 w_8 \in E(G_8)$. Let $G'_8 = G_0 + \{w_1 w_9, w_3 w_9, w_5 w_9, w_2 w_{10}, w_4 w_{10}, w_7 w_{10}, w_6 w_7, w_5 w_8\}$. Thus G'_8 is a spanning subgraph of G_8 (see Figure 1). Since $G'_8 - \{wv_1, wv_2, wv_3, w_5 w_9, w_9 v_5, w_7 v_4\}$ is eulerian, G_8 is supereulerian.

Case 2. $|T'| = 0$.

Then we may assume that $T_2 = \{w_4, w_6, w_8, w_{10}\}$. By (2.9) and by symmetry, we may assume that $\{w_1 w_4, w_1 w_6, w_2 w_8, w_2 w_{10}\} \subset E(G_8)$. As $w_8, w_{10} \in N(w_2)$, by (F3), $w_3 \notin N(w_8) \cap N(w_{10})$, and so $w_3 w_6 \in E(G_8)$. Similarly, $w_4 w_5, w_7 w_{10}, w_8 w_9 \in E(G_8)$. Let $G''_8 = G_0 + \{w_1 w_4, w_1 w_6, w_2 w_8, w_2 w_{10}, w_3 w_6, w_4 w_5, w_7 w_{10}, w_8 w_9\}$. Thus G''_8 is a spanning subgraph of G_8 (see Figure 1). As $wv_3 w_5 w_4 v_2 w_3 w_6 w_1 v_1 w_2 w_{10} w_7 v_4 w_8 w_9 v_5 w$ is a Hamilton cycle of G''_8, G_8 is supereulerian. $\square$

Both Theorem 2.1.1 (Theorem 3.12 of [11]) and Theorem 2.1.3 in this paper raise the following a question: if G is a 3-edge-connected graph and if the number of 3-edge-cuts of G is k, what is the largest value of k such that every 3-edge-connected graph G with at most k edge-cuts of size 3 is supereulerian if and only if G cannot be contracted to $P(10)$? Theorem 2.1.1 says that $k \geq 10$ and in this paper we prove $k \geq 11$. However, since either of the two Blanusa snarks (see [3] or [18]) is 3-edge-connected and nonsupereulerian, has exactly 18 edge-cuts of size 3, and cannot be contracted to $P(10)$, we kave $k \leq 17$. We conclude this section by conjecturing that $k = 17$.
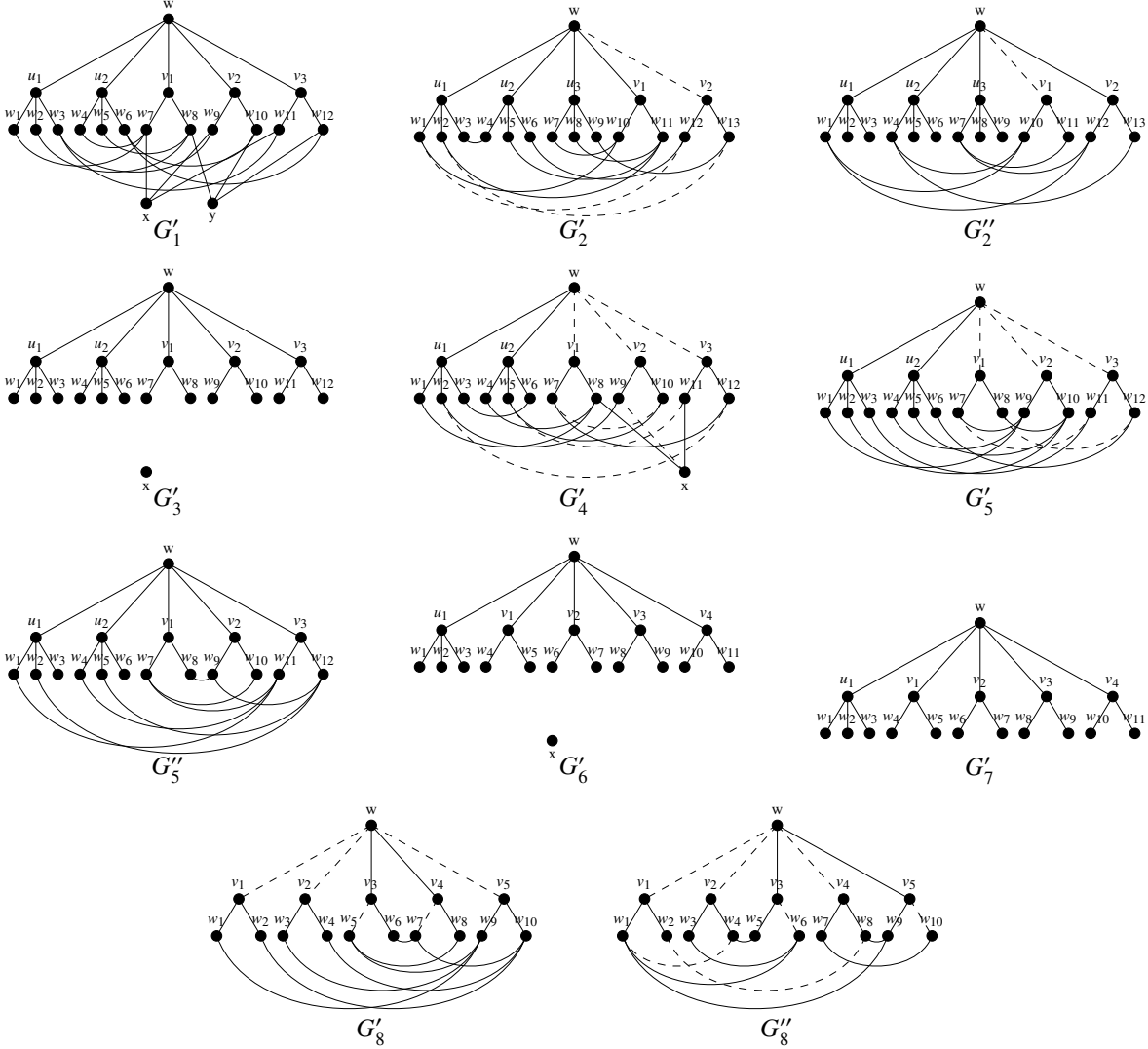
14

Figure 1. $G'_1, G'_2, G''_2, G'_3, G'_4, G'_5, G''_5, G'_6, G'_7, G'_8, G''_8$

15

Chapter 3

Chvátal-Erdös Like Conditions

3.1 Prerequisites

Let $\alpha(G)$, $\alpha'(G)$ be the independence number, the matching number of a graph G, respectively. Motivated by a well-known result of Chvátal and Erdös ([19]) that every graph G with $\kappa(G) \geq \alpha(G)$ is Hamiltonian, there have been researches on conditions analogous to this Chvátal-Erdös Theorem to assure the existence of spanning trials in a graph utilizing relationship among independence number $\alpha(G)$, matching number $\alpha'(G)$ with edge-connectivity $\kappa'(G)$ and connectivity $\kappa(G)$. See [1], [22] and [27], among others. Let $K_{2,3}(1, 2, 2)$, $S_{1,2}$, $K'_{2,3}$ be the graphs depicted in Figure 2. The following are proved.

Theorem 3.1.1 *(Han, Lai, Xiong and Yan [22]) Let G be a simple graph with $\kappa(G) \geq 2$. If $\kappa(G) \geq \alpha(G) - 1$, then exactly one of the following holds.*
(i) G is supereulerian.
(ii) $G \in \{P(10), K_{2,3}, K_{2,3}(1, 2, 2), S_{1,2}, K'_{2,3}\}$.
(iii) G is a 2-connected graph obtained from $K_{2,3}$ (resp. $S_{1,2}$) by replacing a vertex whose neighbors have degree three in $K_{2,3}$ (resp. $S_{1,2}$) with a complete graph of order at least three.
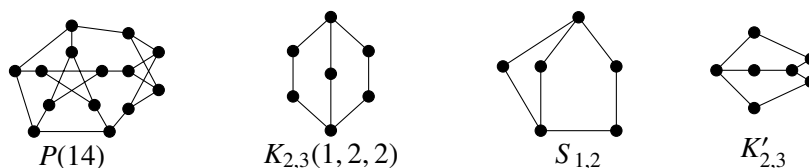

Figure 2. $P(14)$ and some graphs in Theorem 3.1.1

The supereulerian property for graphs G with $\alpha'(G) \leq 2$ and $\kappa'(G) \geq 2$ have been completely determined in [1] and [27].

The purpose of this chapter is to investigate the existence of spanning trails in graphs with given relationship between edge-connectivity and independence number, or matching number.

3.2 Bounded Independence Numbers

In this section, we investigate the relationship between minimum degree and independence number that assures supereulerian property.

Lemma 3.2.1 *Let G be a reduced graph with $\delta(G) \geq 2$ and $\alpha(G) \leq 3$. Then G is supereulerian if and only if $G \notin \{K_{2,3}, K_{2,3}(1, 2, 2), S_{1,2}\}$.*

Proof. Since G is reduced, by Theorem 1.2.3(ii), G is simple and K_3-free. Thus, $\Delta(G) \leq \alpha(G) \leq 3$. Assume that G has a cut vertex u. Since $\Delta(G) \leq 3$, at least one of the edges incident with u is a cut edge of G. Let uv be this cut edge. Suppose G_1 and G_2 are two connected components in $G - uv$. Since $\delta(G) \geq 2$, $|D_1(G_i)| \leq 1$ ($i = 1, 2$). Since G is K_3-free, the girth of G_i is at least 4. Hence we may assume that, for $1 \leq i \leq 2$, $G_i - \{u, v\}$ has two vertices u_i, v_i with $u_i v_i \notin E(G_i)$. It follows that $\{u_1, v_1, u_2, v_2\}$ is an independent set in G, contrary to the assumption $\alpha(G) \leq 3$. Thus we may assume that $\kappa(G) \geq 2$ and so $\kappa(G) \geq \alpha(G) - 1$. Since G is reduced with $\alpha(G) \leq 3$, by Theorem 3.1.1 and $\alpha(P(10)) = 4$, either G is supereulerian or $G \in \{K_{2,3}, K_{2,3}(1, 2, 2), S_{1,2}\}$. $\square$

Theorem 3.2.2 *(Chen [13], Chen and Lai [16]) Let G be a connected graph with $|V(G)| \leq 11$ and $\kappa'(G) \geq 3$, and G' be the reduction of G. Then either G is collapsible or $G' \cong P(10)$.*

Corollary 3.2.3 *Let G be a connected reduced graph with $|V(G)| \leq 11$ and $\delta(G) \geq 3$. Then $G \cong P(10)$.*

Proof. By contradiction, we assume that G is a nontrivial reduced graph with at most 11 vertices and with $\delta(G) \geq 3$, but $G \not\cong P(10)$. By Theorem 3.2.2, G must have an edge cut X with $|X| \leq 2$. Let G_1 and G_2 be the two components in $G - X$ with $|V(G_1)| \leq |V(G_2)|$. Since $|V(G)| \leq 11$, we have $|V(G_1)| \leq 5$. Since $|X| \leq 2$ and $\delta(G) \geq 3$,

$$\text{either } |D_2(G_1)| \leq 2 \text{ and } |D_1(G_1)| = 0 \text{ or } |D_2(G_1)| = 0 \text{ and } |D_1(G_1)| = 1. \qquad (3.1)$$

Hence $|E(G_1)| \geq \frac{|D_1(G_1)| + 2|D_2(G_1)| + 3(|V(G_1)| - |D_1(G_1)| - |D_2(G_1)|)}{2} > 6$. By Theorem 1.2.3, $F(G_1) < 2$, and so $G_1 \in \{K_1, K_2\}$, contrary to (3.1). Then $G \cong P(10)$. $\square$

Theorem 3.2.4 *(Chen [15]) Let G be a connected reduced graph with order n.*
(i) If $\alpha(G) = 2$, then $n \leq 5$.
(ii) If $\alpha(G) = 3$, then $n \leq 8$.
(iii) If $\alpha(G) \geq 4$, then $\dfrac{\delta(G)\alpha(G) + 4}{2} \leq n \leq 4\alpha(G) - 5$.

Let $k \geq 1$ be an integer, and let u denote a vertex of degree 3 in given $K_{2,3}$. Let P^n be a path of order n. Define

$$\mathcal{F}_1 = \{K_2, P^3, P^4, K_{2,3}, K_{2,3}(1, 2, 2), S_{1,2}, P(10)\}.$$

The results above can be applied to prove our main theorem of this section, as stated below.

Theorem 3.2.5 *Let G be a connected reduced graph with $\delta(G) \geq \alpha(G) - 1$. Then G is supereulerian if and only if $G \notin \mathcal{F}_1$.*

Proof. It is routine to verify that every graph in $\mathcal{F}_1$ is not supereulerian. It suffices to prove the necessity. Since K_1 is supereulerian, $|V(G)| \geq 2$. By Theorem 1.2.3(iii), $\delta(G) \leq 3$. If $\delta(G) = 3$, then $\alpha(G) \leq 4$. By Theorem 3.2.4, $|V(G)| \leq 11$. By Corollary 3.2.3, we have $G \cong P(10)$. If $\delta(G) = 2$, then $\alpha(G) \leq 3$. By Lemma 3.2.1, we have $G \in \{K_{2,3}, K_{2,3}(1, 2, 2), S_{1,2}\}$. If $\delta(G) = 1$, and so $\alpha(G) \leq 2$. Since G is K_3-free, $\Delta(G) \leq \alpha(G) \leq 2$. Thus, G must be a path with length at most 4. Thus, $G \in \{K_2, P^3, P^4\} \subseteq \mathcal{F}_1$. Theorem 3.2.5 is proved. $\square$

Corollary 3.2.6 *Let G be a connected graph with $\kappa'(G) \geq \alpha(G) - 1$. Then G is supereulerian if and only if $G' \notin \mathcal{F}_1$.*

Proof. Assume that G is a graph satisfying $\kappa'(G) \geq \alpha(G) - 1$. Let G' be the reduction of G. By the definition of graph contractions, we have $\kappa'(G') \geq \kappa'(G) \geq \alpha(G) - 1 \geq \alpha(G') - 1$. By Theorem 3.2.5, G' is supereulerian if and only if $G' \notin \mathcal{F}_1$. $\square$

3.3 Bounded Matching Numbers

In this section, we will investigate supereulerian graphs with a bounded size of maximum matchings. A component H of G is an **odd component** if $|V(H)| \equiv 1 \pmod 2$. Let $q(G) = |\{Q \mid Q$ is an odd component of $G\}|$. Tutte [41] and Berge [2] proved the following theorem.

Theorem 3.3.1 *(Berge [2], Tutte [41]) Let G be a graph with n vertices. If*

$$t = \max_{S \subset V(G)} \{q(G - S) - |S|\}, \tag{3.2}$$

then $\alpha'(G) = (n - t)/2$.

For reduced graphs, Chen and Lai [14] have found a lower bound of the size of matching number.

Theorem 3.3.2 *(Chen and Lai [14]) Let G be a reduced graph with n vertices and $\delta(G) \geq 3$. Then $\alpha'(G) \geq \min\{\frac{n-1}{2}, \frac{n+4}{3}\}$.*

Following a similar idea in [14], the lower bond in Theorem 3.3.2 can be slightly improved as shown in Theorem 3.3.4 below. We start with a lemma on reduced graphs.

Lemma 3.3.3 *Let G be a connected reduced graph with $\delta(G) \geq 3$ and $\alpha'(G) = c$. Suppose that $S \subseteq V(G)$ is a vertex subset attaining the maximum in (3.2) with $|S| > 0$, $m = q(G - S)$ and that $G_1, G_2, \cdots, G_m$ are the components in $G - S$ with odd number of vertices such that $|V(G_1)| \leq |V(G_2)| \leq \cdots \leq |V(G_m)|$. Define*

$$X = \{G_i \mid |V(G_i)| = 1, 1 \leq i \leq m\}, Y = \{G_i \mid |V(G_i)| = 3, 1 \leq i \leq m\}, x = |X|, y = |Y|.$$

$$V^* = \bigcup_{k=1}^{x+y} V(G_k), \ G^* = G[V^* \bigcup S^*], \ S^* = \{s \in S \mid v^*s \in E(G), \ v^* \in V^*\} \ and \ s^* = |S^*|. \quad (3.3)$$

Thus G^* is spanned by a bipartite subgraph with (V^*, S^*) being its vertex bipartition with $|V^*| = x + y \geq 1$. By the definition of x, V^* contains x isolated vertices in $G^*[V^*]$. Then each of the following holds.

(i) $n \geq \sum_{i=1}^{m} |V(G_i)| + |S| \geq m|V(G_1)| + |S|$.

(ii) If $x > 0$, then $s^* \geq 3$.

(iii) $m \leq \frac{n+4x+2y-|S|}{5}$.

(iv) $G^* \notin \{K_1, K_2, K_{1,2}, K_{2,2}\}$.

(v) $|E(G^*)| \geq 3x + 7y$.

Proof. Statement (i) follows from the definition of m and G_i. If $x > 0$, by $\delta(G) \geq 3$, there must be at least 3 vertices in S^* adjacent to the only vertex in G_1, and so $s^* \geq 3$. This justifies (ii). By (3.3), we have $n \geq |S| + x + 3y + 5(m - x - y)$, and so (iii) follows. As $\delta(G) \geq 3$, every vertex in V^* must have degree at least 3 in G^*, and so (iv) must hold. Since $\delta(G) \geq 3$ and G does not contain a 3-cycle, every vertex in $\bigcup_{G_i \in X} V(G_i)$ is incident with at least 3 edges in G^*; and every component in $G^*[\bigcup_{G_i \in Y} V(G_i)]$ is a $K_{1,2}$ and is incident with at least 5 edges with one end in S^* plus two edges in $E(G_i)$. Hence $|E(G^*)| \geq 3x + 7y$. This proves (v). $\qquad\square$

Theorem 3.3.4 *Let G be a connected reduced graph with n vertices and $\delta(G) \geq 3$. Then $\alpha'(G) \geq \min\{\frac{n}{2}, \frac{n+5}{3}\}$.*

Proof. Let t be defined as in (3.2). By Theorem 3.3.1, if $t = 0$, $\alpha'(G) = \frac{n}{2} \geq \min\{\frac{n}{2}, \frac{n+5}{3}\}$. Hence we assume that $t \geq 1$. If $n \leq 11$, then since $\delta(G) \geq 3$, by Corollary 3.2.3, $G \cong P(10)$. As $\alpha'(P(10)) = 5 = \frac{10}{2}$, Theorem 3.3.4 holds when $n \leq 11$.

Hence we assume that $n \geq 12$, and so $\frac{n+5}{3} < \frac{n}{2}$. By Theorem 3.3.1, in order to prove Theorem 3.3.4, it suffices to show that

$$\frac{n-t}{2} \geq \frac{n+5}{3}, \text{ or equivalently, } t \leq \frac{n-10}{3}. \quad (3.4)$$

19

In the rest of the proof, we shall show that (3.4) always holds in any case, which implies the validity of Theorem 3.3.4. Define S, m, $G_1, G_2, \cdots, G_m$, V^*, S^*, s^* and G^* as in Lemma 3.3.3. Since G is reduced, by Theorem 1.2.3 (ii), G is simple and K_3-free. If $|S| = 0$, since G is connected and $n \geq 12$, we have $t = 1$ and so $|V(G)|$ is odd and $n \geq 13$. By Theorem 3.3.1 and as $n \geq 13$, $\alpha'(G) \geq \frac{n-1}{2} \geq \frac{n+5}{3}$, and so (3.4) holds. Hence we assume that $|S| \geq 1$.

Case 1. $x = 0$, i.e. $|V(G_1)| \geq 3$.

Subcase 1.1. $|V(G_1)| = 3$.

Since G is K_3-free, $G_1 \cong K_{1,2}$. By $\delta(G) \geq 3$, we have $|S| \geq 3$. It follows by $n \geq 3m + |S|$ that

$$t = m - |S| \leq \frac{n - |S|}{3} - |S| = \frac{n - 4|S|}{3} \leq \frac{n - 12}{3},$$

and so (3.4) must hold.

Subcase 1.2. $|V(G_1)| = 5$.

If $|S| = 1$, then as G is K_3-free and $\delta(G) \geq 3$, we have $|E(G[V(G_1) \cup S])| \geq \frac{15}{2} > 7$. On the other hand, by Theorem 1.2.3 (i) and (iv), we have $|E(G[V(G_1) \cup S])| \leq 2(5 + 1) - 5 = 7$. A contradiction is obtained. Hence we assume that $|S| \geq 2$. As $n \geq 5m + |S|$, we have $m \leq \frac{n-|S|}{5}$. It follows by $n \geq 12$ and $t = m - |S| \geq 1$ that

$$t = m - |S| \leq \frac{n - |S|}{5} - |S| = \frac{n - 6|S|}{5} \leq \frac{n - 12}{5} < \frac{n - 10}{3},$$

and so (3.4) must hold.

Subcase 1.3. $|V(G_1)| \geq 7$.

Since $t = m - |S| \geq 1$, we have $m \geq 2$. Then $n \geq 7m + |S| \geq 15$ and $m \leq \frac{n-|S|}{7}$. It follows that

$$t = m - |S| \leq \frac{n - |S|}{7} - |S| = \frac{n - 8|S|}{7} \leq \frac{n - 8}{7} < \frac{n - 10}{3},$$

and so (3.4) must hold.

Case 2. $x \geq 1$.

By Lemma 3.3.3 (iv), G^* is not in $\{K_1, K_2, K_{1,2}, K_{2,2}\}$, and so by Theorem 1.2.3 (iv), either for some integer $\ell \geq 3$, $G^* \cong K_{2,\ell}$ or $F(G^*) \geq 3$.

Subcase 2.1. For some integer $\ell \geq 3$, $G^* \cong K_{2,\ell}$.

Since $\delta(G) \geq 3$, every vertex in V^* must have degree at least 3 in G^*. Then $|V^*| = x = 2$ and $s^* = \ell \geq 3$. By the definition of y, we must have $y = 0$ and $|S| \geq |S^*|$. It follows by Lemma 3.3.3 (iii) that $1 \leq t = m - |S| \leq \frac{n+8+2y-|S|}{5} - |S| \leq \frac{n+8-6s^*}{5}$, and so $n \geq 6s^* - 3$. As $s^* \geq 3$, we have

$n \geq 6s^* - 3 \geq 15 \geq 32 - 9s^*$, or $5(n - 10) \geq 3(n + 8 - 6s^*)$. Hence

$$t \leq \frac{n + 8 - 6s^*}{5} < \frac{n - 10}{3},$$

and so (3.4) must hold.

Subcase 2.2. $F(G^*) \geq 3$.

By Theorem 1.2.3 (i) and by Lemma 3.3.3 (v), $2x + 7y \leq |E(G^*)| \leq 2(|V(G^*)| - 1) - 3 = 2(x + 3y + |S|) - 5$. This implies that $|S| \geq \frac{x+y+5}{2}$, and so $n \geq x + 3y + |S| \geq \frac{3x+7y+5}{2}$. It follows that

$$t = m - |S| \leq \frac{n + 4x + 2y - |S|}{5} - |S| \leq \frac{n + x - y - 15}{5} \leq \frac{n - 10}{3},$$

and so (3.4) must hold. This completes the proof of the theorem. $\square$

In [12], W. Chen and Z. Chen characterized 3-edge-connected supereulerian graphs with order at most 15. Define $\mathcal{F}_2 = \{P(10), P(14)\}$.

Theorem 3.3.5 *(W. Chen and Z. Chen [12]) Let G be a 3-edge-connected graph and G' be the reduction of G.*
(i) If $|V(G)| \leq 13$, then either G is supereulerian or $G' \cong P(10)$.
(ii) If $|V(G)| \leq 14$, then either G is supereulerian or $G' \in \mathcal{F}_2$.
(iii) If $|V(G)| = 15$, G is not supereulerian and $G' \notin \mathcal{F}_2$, then G is an essentially 4-edge-connected reduced graph with girth at least 5, $\kappa(G) \geq 2$ with $V(G) = D_3(G) \cup D_4(G)$ where $|D_4(G)| = 3$.

A few more former results are needed in the proof of the main theorem in this section.

Theorem 3.3.6 *(Reiman [39], Bollobás [5]) Let G be a connected bipartite C_4-free graph with vertex bipartition $\{X, Y\}$, where $|X| \leq |Y|$. Then*

$$|E(G)| \leq \sqrt{|Y| \cdot |X|(|X| - 1) + \frac{|Y|^2}{4}} + \frac{|Y|}{2}.$$

Lemma 3.3.7 *Every 3-edge-connected reduced graph G with order $n \geq 15$ and $\alpha'(G) \leq 7$ is supereulerian.*

Proof. By contradiction, assume that G is not supereulerian. As $\alpha'(G) \leq 7$ and $n \geq 15$, by Theorem 3.3.4,

$$15 \leq n \leq 3\alpha'(G) - 5 \leq 16. \tag{3.5}$$

Let t be the integer satisfying (3.2) in Theorem 3.3.1. Then $\alpha'(G) = \frac{n-t}{2}$. By (3.5) and Theorem 3.3.4, we have $7 \geq \alpha'(G) = \frac{n-t}{2} \geq \frac{n+5}{3}$. Thus $\frac{n-10}{3} \geq t \geq n - 14$. By (3.5), we have $t \geq 1$ when $n = 15$ and $t \geq 2$ when $n = 16$. We shall show that each case can occur to reach a contradiction to the assumption that G is not supereulerian, thereby proving the theorem. Define $S, m, G_1, G_2, \cdots, G_m$, V^*, S^*, s^* and G^* as in Lemma 3.3.3.

21

Claim 1. $G^* \notin \{K_1, K_2\} \cup \{K_{2,\ell}, \ell \geq 1\}$.

Proof of Claim 1. By Lemma 3.3.3 (iv), $G^* \notin \{K_1, K_2, K_{1,2}, K_{2,2}\}$. Suppose that $G^* \cong K_{2,\ell}$, for some $\ell \geq 3$. By the definition of G^*, we have $x = 2$. This implies that $y = 0$ and $|S| = s^* = \ell \geq 3$. By Lemma 3.3.3 (i), $n \geq |S| + x + 5(m - x) = |S| + 5m - 4x$. As $|S| \geq 3$, $n \in \{15, 16\}$, $x = 2$ and $m = |S| + t$, we have $16 \geq n \geq 6|S| + 5t - 8 \geq 18 - 5t - 8 = 10 - 5t$, and so $t \leq \frac{6}{5} < 2$. Thus, $t = 1$ and $n = 15$. By Theorem 3.3.5(iii), G does not have cycles of length at most 4, contrary to the assumption that $G^* \cong K_{2,\ell}$. This proves Claim 1.

Claim 2. Each of the following holds.
(i) $|S| \geq \frac{x+y+5}{2}$.
(ii) $x - y \geq 5t + 15 - n$.

Proof of Claim 2. By Claim 1, $G^* \notin \{K_1, K_2\} \cup \{K_{2,\ell}, \ell \geq 1\}$. By Theorem 1.2.3 (iv), $F(G^*) \geq 3$. As G is reduced, G^* is also reduced. And by Theorem 1.2.3(i) and Lemma 3.3.3 (v), $2x + 7y \leq |E(G^*)| \leq 2(x + 3y + |S|) - 5$. Hence (i) must hold.

By Lemma 3.3.3 (iii) and by $m - |S| = t$, we have $\frac{n+4x+2y-|S|}{5} - |S| \geq t$. It follows by Claim 2 (i) that

$$\frac{n + 4x + 2y - \frac{x+y+5}{2}}{5} - \frac{x + y + 5}{2} \geq \frac{n + 4x + 2y - |S|}{5} - |S| \geq t$$

which implies $x - y \geq 5t + 15 - n$. Hence (ii) holds as well. This proves Claim 2.

Case 1. $t \geq 1$ when $n = 15$.

By Claim 2 (ii) with $n = 15$, $x \geq 5 + y \geq 5$. Assume that $|S| \geq x + 1$. By the choice of S, we have $1 \leq t = m - |S|$, and so $m = |S| + 1 \geq x + 2$. By Lemma 3.3.3 (i) and by $|S| \geq x + 1$, $n \geq \sum_{i=1}^{m} |V(G_i)| + |S| \geq x + |V(G_{x+1})| + |V(G_{x+2})| + |S| \geq x + 3 + 3 + (x + 1) \geq 17$, contrary to $n = 15$. Hence $|S| \leq x$. Let $E^+ = \{uv \in E(G) \mid u \in [\cup_{G_i \in X} V(G_i)], v \in S\}$ and $G^+ = G[E^+]$. By (3.3) and the definition of G^+, G^+ is a bipartite graph with a vertex bipartition $\{\cup_{G_i \in X} V(G_i), S\}$. Since $\delta(G) \geq 3$, $|E(G^+)| \geq 3x$. Since $|V(G^+)| \leq n = 15$, by Theorem 3.3.5, G^+ is C_4-free. Since $|S| \leq x$, by Theorem 3.3.6,

$$3x \leq |E(G^+)| \leq \sqrt{x \cdot |S|(|S| - 1) + \frac{x^2}{4}} + \frac{x}{2} \leq \sqrt{x^2(x - 1) + \frac{x^2}{4}} + \frac{x}{2} \qquad (3.6)$$

Solving (3.6) for x to get $\frac{25x^2}{4} \leq x^3 - x^2 + \frac{x^2}{4}$, and so $x \geq 7$. In particular, when $x = 7$, the equality in (3.6) holds. Thus, if $x = 7$, $|S| = x = 7$ and so $m = |S| + t = 8$. By Lemma 3.3.3(i), $|S| \leq 15 - x - \sum_{i=x+1}^{m} |V(G_i)| \leq 15 - 7 - 3 = 5$, contrary to that $|S| = 7$. Hence we must have $x \geq 8$. As $n = 15$ and $x - |S| \leq m - |S| = t = 1$, we have $|S| = 7$ and $x = 8$. By Theorem 3.3.6, $|E(G^+)| < 23$. As $\delta(G) \geq 3$, $23 > |E(G^+)| \geq 3x = 24$, a contradiction. This proves that Case 1 does not occur.

Case 2. $t \geq 2$ when $n = 16$.

22

By Claim 2(ii), $x \geq 9 + y$. By Claim 2(i), $|S| \geq 7 + y$. Since $n = 16$, we must have $x = 9 = |V^*|$, $|S| = 7$ and $V(G^*) = V(G)$. As $\delta(G) \geq 3$, we have $|E(G)| \geq |E(G^*)| \geq 3x = 27$. By Theorem 1.2.3 (i) and (iv), $|E(G)| \leq 2|V(G)| - 5 = 27$. Therefore, $|E(G)| = 27$, $F(G) = 3$ and $G^* \cong G$ is a bipartite graph with bipartition $D_3(G) = V^*$ and S. By Theorem 2.2.2, $g(G) \geq 5$. By Theorem 3.3.6, $|E(G)| \leq 24$, contrary to $|E(G)| = 27$. This proves that Case 2 does not occur as well, and completes the proof. $\qquad\square$

Theorem 3.3.8 *Let G be a connected graph with n vertices and $\kappa'(G) \geq 3$, and G' be the reduction of G. If $\alpha'(G) \leq 7$, then G is supereulerian if and only if $G' \notin \mathcal{F}_2$.*

Proof. As $P(10)$ and $P(14)$ are not supereulerian, the necessity is clear. By Theorem 1.2.2, G is supereulerian if and only if G' is supereulerian. By the definition of contractions, we have $\kappa(G') \geq \kappa'(G) \geq 3$ and $\alpha'(G') \leq \alpha'(G) \leq 7$. Hence we only need to prove that

$$\text{if a reduced graph } G \text{ is not supereulerian, then } G \in \mathcal{F}_2. \qquad (3.7)$$

By Lemma 3.3.7, (3.7) holds if $|V(G)| \geq 15$. By Theorem 3.3.5, (3.7) holds if $|V(G)| \leq 14$. This completes the proof of the theorem. $\qquad\square$

Corollary 3.3.9 *Let G be a connected graph. If $|V(G)| \leq 15$ and $\kappa'(G) \geq 3$, then G is supereulerian if and only if the reduction of G is not in $\mathcal{F}_2$.*

Proof. If $|V(G)| \leq 15$, then $\alpha'(G) \leq \frac{15}{2}$. So $\alpha'(G) \leq 7$. By Theorem 3.3.8, this corollary holds. $\qquad\square$

Corollary 3.3.10 *Let G be a connected reduced graph. Each of the following holds.*
(i) If $|V(G)| \leq 15$ and $\delta(G) \geq 3$, then G is supereulerian if and only if $G \notin \mathcal{F}_2$.
(ii) If $\delta(G) \geq 3$ and $\alpha(G) \leq 5$, then G is supereulerian if and only if $G \neq P(10)$.

Proof. First we prove (i). Suppose that $\kappa'(G) \leq 2$. Let X be a minimal edge cut in G with $|X| \leq 2$. Let G_1 and G_2 be the two components in $G - X$ with $|V(G_1)| \leq |V(G_2)|$. Since $|V(G)| \leq 15$, $|V(G_1)| \leq 7$. Since $|X| \leq 2$ and $\delta(G) \geq 3$, $|D_1(G_1)| = 0$ and $|D_2(G_1)| \leq 2$. By Theorem 1.2.3, $|E(G_1)| \leq 2|V(G_1)| - 5$. Since $\delta(G) \geq 3$, $|E(G_1)| \geq \frac{4+3(|V(G_1)|-2)}{2}$. Therefore, $\frac{4+3(|V(G_1)|-2)}{2} \leq |E(G_1)| \leq 2|V(G_1)| - 5$. Then $|V(G_1)| \geq 8$, contrary to that $|V(G_1)| \leq 7$. Thus, $\kappa'(G) \geq 3$. Statement (i) follows from Corollary 3.3.9.

Now we prove (ii). If $\alpha(G) \leq 5$, by Theorem 3.2.4, $|V(G)| \leq 15$. Since $\alpha(P(14)) = 6$, the statement follows from (i) above. $\qquad\square$

Chapter 4

Spanning trails Containing Given Edges

4.1 Prerequisites

By the definition of essentially k-edge-connected, we have the following proposition:

Proposition 4.1.1 *Let G be an essentially k-edge-connected graph with the minimum degree $\delta(G)$ and the edge-connectivity $\kappa'(G)$. Then $\kappa'(G) = \min\{\delta(G), k\}$.*

As shown in [34], Theorem 1.1.3 can be improved in the sense that a 4-edge-connected graph can have spanning closed trail containing some fixed edges. In [34], Luo et al. called a graph G **r-edge-Eulerian-connected** if for any edge subset $X \subseteq E(G)$ with $|X| \le r$ and any distinct edges $e, e' \in E(G)$, G has a spanning (e, e')-trail containing all edges in X. Define $\xi(r)$ to be the smallest integer k such that every k-edge-connected graph is r-edge-Eulerian-connected. They proved the following:

Theorem 4.1.2 *(Luo, Chen and Chen [34]) Let $r > 0$ be an integer. Then*

$$\xi(r) = \begin{cases} 4, & 0 \le r \le 2, \\ r + 1, & r \ge 4. \end{cases}$$

For $r = 3$, Luo et al [34] indicated that $4 \le \xi(3) \le 5$, and conjectured $\xi(3) = 4$.

In this chapter, we introduce a reduction method on essentially 4-edge-connected graphs and investigate spanning trails in essentially 4-edge-connected graphs. As an application, we prove the following.

Theorem 4.1.3 *If G is a 4-edge-connected graph, then for any $X_0 \subseteq E(G)$ with $|X_0| \le 3$ and any distinct edges $e, e' \in E(G)$, G has a spanning (e, e')-trail T such that $X_0 \subseteq E(T)$. Thus, G is 3-edge-Eulerian-connected and so $\xi(3) = 4$.*

Theorem 4.1.3 confirms the conjecture above, and so all the values of $\xi(r)$ are determined for all integer $r \geq 0$.

4.2   Reductions of Essentially 4-edge-connected Graphs

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For vertex disjoint subsets $V_1, V_2 \subseteq V(G)$, let $[V_1, V_2]_G$ denotes the set of all edges in G with one end in V_1 and the other in V_2. For vertex disjoint subgraphs H, L of G, we write $[H, L]_G = [V(H), V(L)]_G$, and define $\partial_G(H) = [V(H), V(G) - V(H)]_G$, called the **boundary** of H in G. When $H = K_1$ is a single vertex v, we denote $\partial_G(v)$ as $\partial_G(H)$ and $|\partial_G(v)| = d_G(v)$. We often omit the subscript G in these notations when G is understood.

Let $z \in D_2(G)$ with $N_G(z) = \{z_1, z_2\}$ such that $z_1 \in D_4(G)$ and $N_G(z_1) = \{z, w_1, w_2, w_3\}$. For $i \in \{1, 2, 3\}$, if $w_i \in D_2(G)$, then let $N_G(w_i) = \{z_1, w_i'\}$. For $j \in \{1, 2\}$, let $G_j^- = (G - \{z_1\}) + \{zw_j, w_{3-j}w_3\}$, and $W(G_j^-) = \{e = xy \in E(G_j^-) : x, y \in D_2(G_j^-)\}$. Define

$$G_j = G_j^- / W(G_j^-). \tag{4.1}$$

For an essentially 4-edge-connected graph G, if $w_i \in D_2(G)$, then $N_G(w_i) = \{z_1, w_i'\} \cap D_2(G) = \emptyset$. Thus, if an edge $e \in W(G_j^-)$, then $e \in \{zw_j, w_{3-j}w_3\}$.
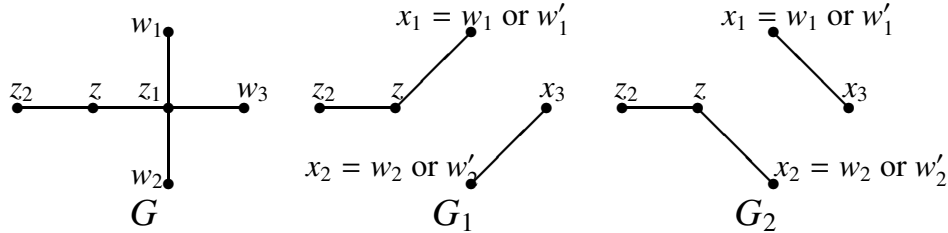

Figure 3: the graphs G_1 and G_2 from G in Theorem 4.2.1

Theorem 4.2.1 *Let G be an essentially 4-edge-connected graph with $\delta(G) \geq 2$ and $D_3(G) = \emptyset$. Let $z \in D_2(G)$ with $N_G(z) = \{z_1, z_2\}$ such that $z_1 \in D_4(G)$ and $N_G(z_1) = \{z, w_1, w_2, w_3\}$. For $i \in \{1, 2, 3\}$, if $w_i \in D_2(G)$, then let $N_G(w_i) = \{z_1, w_i'\}$. Let G_1 and G_2 be the graphs defined by (4.1) above. Then either G_1 or G_2 is also essentially 4-edge-connected and $\delta(G_j) \geq 2$ and $D_3(G_j) = \emptyset$ (j=1,2).*

Proof. Since G is essentially 4-edge-connected with $\delta(G) \geq 2$, by Proposition 4.1.1, G is 2-edge-connected. Then by the definition of G_j (j=1,2), G_j is connected with $\delta(G_j) \geq 2$ and $D_3(G_j) = \emptyset$. It suffices to show that either G_1 or G_2 is essentially 4-edge-connected. For $j \in \{1, 2\}$, by (4.1), when $w_{3-j}w_3 \in W(G_j^-)$, we shall use w_{3-j} to denote the vertex $\theta(w_{3-j}w_3)$ in G_j; and when $w_j \in D_2(G)$, use z to denote the vertex $\theta(zw_j)$ in G_j. Let x_1, x_2 and x_3 denote the vertices in G_1 and G_2 such that

$$x_1 = \begin{cases} w_1 & \text{if } w_1 \notin D_2(G) \\ w_1' & \text{if } w_1 \in D_2(G) \end{cases}, \quad x_2 = \begin{cases} w_2 & \text{if } w_2 \notin D_2(G) \\ w_2' & \text{if } w_2 \in D_2(G) \end{cases}, \tag{4.2}$$

25

and

$$x_3 = \begin{cases} w_3 & \text{if } w_{3-j} \notin D_2(G) \text{ in } G_j, \ j \in \{1,2\} \\ w_2 & \text{if } w_2 \in D_2(G) \text{ in } G_1 \\ w_1 & \text{if } w_1 \in D_2(G) \text{ in } G_2 \end{cases}. \tag{4.3}$$

The notation x_3 in (4.3) is for the convenience in our discussion below for G_1 and G_2, respectively. In G_1, if $w_2 \in D_2(G)$, then (4.3) defines $x_3 = w_2$ in G_1; if $w_2 \notin D_2(G)$, then (4.3) defines $x_3 = w_3$ (See Figure 4 below for G_1). Similarly, one can find what x_3 is in G_2 from (4.3).
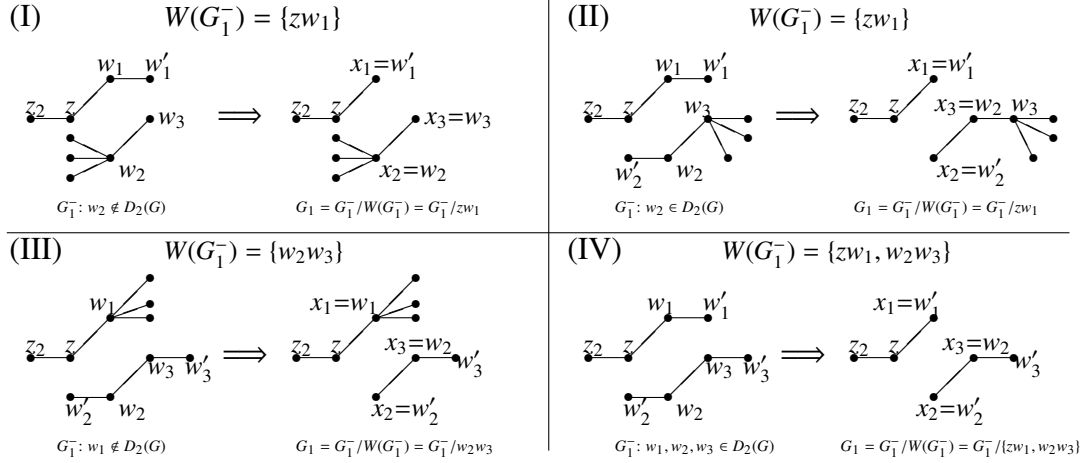

Figure 4: All the cases of G_1 with labels $x_1, x_2,$ and x_3 from G_1^- with $W(G_1^-) \neq \emptyset$

Since G is essentially 4-edge-connected, by $D_3(G) = \emptyset$ and by (4.2),

$$d_G(x_i) \geq 4, \text{ if } 1 \leq i \leq 2. \tag{4.4}$$

By way of contradiction, suppose both G_1 and G_2 are not essentially 4-edge-connected. Then G_1 and G_2 have minimum essential edge cuts X and Y, respectively, such that $2 \leq |X| \leq 3$ and $2 \leq |Y| \leq 3$.

Claim 1. For any essential edge cuts X in G_1 and Y in G_2 with $2 \leq |X| \leq 3$ and $2 \leq |Y| \leq 3$, $X \cap \{zx_1, x_2x_3\} = \emptyset$, and $Y \cap \{zx_2, x_1x_3\} = \emptyset$.

We will prove the case for X only. The proof for Y is similar and hence omitted. By way of contradiction, suppose X contains either zx_1 or x_2x_3, (we may, without lose of generality, assume that z and x_2 are in the same component of $G_1 - X$), then define

$$X' = \begin{cases} (X - zx_1) \cup \{z_1w_1\} & \text{if } zx_1 \in X \text{ and } x_2x_3 \notin X \\ (X - x_2x_3) \cup \{z_1w_3\} & \text{if } x_2x_3 \in X \text{ and } zx_1 \notin X \\ (X - \{zx_1, x_2x_3\}) \cup \{z_1w_1, z_1w_3\} & \text{if } x_2x_3 \in X \text{ and } zx_1 \in X \end{cases}.$$

Thus, X' is an essential edge cut of G with $|X'| = |X|$, contrary to the assumption that G is essentially 4-edge-connected. Claim 1 is proved.

Since $X \cap \{zx_1, x_2x_3\} = \emptyset$, zx_1 and x_2x_3 must be in distinct components of $G_1 - X$. Let A_1 and A_2 be the two components of $G_1 - X$ with $zx_1 \in E(A_1)$ and $x_2x_3 \in E(A_2)$.

Similarly, since $\{zx_2, x_1x_3\} \cap Y = \emptyset$, zx_2 and x_1x_3 are in distinct components of $G_2 - Y$. Let B_1 and B_2 be the two components of $G_2 - Y$ such that $zx_2 \in E(B_1)$ and $x_1x_3 \in E(B_2)$. Hence

$$|\partial_{G_1}(A_1)| = |\partial_{G_1}(A_2)| = |X| \leq 3 \text{ and } |\partial_{G_2}(B_1)| = |\partial_{G_2}(B_2)| = |Y| \leq 3. \tag{4.5}$$

By the definition of G_1 and G_2, $A_1 \cap B_1$, $A_1 \cap B_2$, $A_2 \cap B_1$ and $A_2 \cap B_2$ are subgraphs of G. Furthermore, we may assume that $z \in V(A_1 \cap B_1)$, $x_1 \in V(A_1 \cap B_2)$ and $x_2 \in V(A_2 \cap B_1)$.

Claim 2. $|\partial_G(A_1 \cap B_2)| \geq 4$ and $|\partial_G(A_2 \cap B_1)| \geq 4$.

By symmetry, we prove $|\partial_G(A_1 \cap B_2)| \geq 4$ only. By contradiction, suppose $|\partial_G(A_1 \cap B_2)| \leq 3$. Since G is 2-edge-connected and essentially 4-edge-connected with $D_3(G) = \emptyset$, we must have $|\partial_G(A_1 \cap B_2)| = 2$ and so $|V(A_1 \cap B_2)| = 1$. Hence $V(A_1 \cap B_2) = \{x_1\}$, contrary to (4.4). This proves Claim 2.

In the following, we define $\alpha_1 = |[A_1 \cap B_2, A_2 \cap B_2]|$, $\alpha_2 = |[A_1 \cap B_2, A_2 \cap B_1]|$, $\alpha_3 = |[A_1 \cap B_1, A_2 \cap B_1]|$, $\beta_1 = |[A_1 \cap B_1, A_1 \cap B_2]|$, $\beta_2 = |[A_1 \cap B_1, A_2 \cap B_2]|$, $\beta_3 = |[A_2 \cap B_1, A_2 \cap B_2]|$. Thus by (4.5),

$$\textstyle\sum_{i=1}^{3} \alpha_i + \beta_2 = |X| \leq 3 \text{ and } \sum_{i=1}^{3} \beta_i + \alpha_2 = |Y| \leq 3$$

and so

$$\alpha_1 + \alpha_2 + \alpha_3 \leq 3 - \beta_2 \text{ and } \beta_1 + \beta_3 + \alpha_2 \leq 3 - \beta_2. \tag{4.6}$$

Note that

$$\partial_G(A_1 \cap B_2) \subseteq [A_1 \cap B_2, A_1 \cap B_1] \cup [A_1 \cap B_2, A_2 \cap B_1] \cup [A_1 \cap B_2, A_2 \cap B_2],$$

$$\partial_G(A_2 \cap B_1) \subseteq [A_2 \cap B_1, A_2 \cap B_2] \cup [A_2 \cap B_1, A_1 \cap B_1] \cup [A_2 \cap B_1, A_1 \cap B_2].$$

By Claim 2, we have

$$4 \leq |\partial_G(A_1 \cap B_2)| \leq \beta_1 + \alpha_2 + \alpha_1, \text{ and } 4 \leq |\partial_G(A_2 \cap B_1)| \leq \beta_3 + \alpha_3 + \alpha_2. \tag{4.7}$$

By (4.7) and (4.6),

$$8 \leq \beta_1 + \beta_3 + \alpha_2 + \alpha_1 + \alpha_2 + \alpha_3 \leq 3 - \beta_2 + 3 - \beta_2 = 6 - 2\beta_2 \leq 6.$$

This contradiction establishes the theorem. $\qquad\square$

4.3 Spanning Trails in Essentially 4-edge-connected Graphs

For a reduced graph G with $\delta(G) \geq 2$, let $d_i = |D_i(G)|$, then $|V(G)| = \sum_{i \geq 2} d_i$ and $2|E(G)| = \sum_{i \geq 2} id_i$. By Theorem 1.2.3(i),

$$2F(G) = 4 \sum_{i \geq 2} d_i - \sum_{i \geq 2} id_i - 4. \tag{4.8}$$

Hence, if $F(G) \geq 3$, then (4.8) implies

$$\sum_{i \geq 5} (i - 4)d_i + 10 \leq 2d_2 + d_3. \tag{4.9}$$

We are now ready to prove the main result of this section, which will be needed to prove the conjecture $\xi(3) = 4$ in next section.

Theorem 4.3.1 *Let G be an essentially 4-edge-connected graph with $\delta(G) \geq 2$ and $|D_2(G) \cup D_3(G)| \leq 5$. Then each of the following holds.*
(i) If $|D_2(G)| \leq 3$, then G is collapsible.
(ii) Either G is supereulerian or the reduction of G is $K_{2,5}$ such that all the vertices of degree 2 in the reduction are trivial.
(iii) If $|D_2(G)| \geq 2$, then for any pair of distinct vertices $u, v \in D_2(G)$, G has a spanning (u, v)-trail.

Proof. Since G is an essentially 4-edge-connected graph with $\delta(G) \geq 2$, by Proposition 4.1.1, $\kappa'(G) \geq 2$. We argue by contradiction and assume that

$$G \text{ is a counterexample with } |V(G)| \text{ minimized.} \tag{4.10}$$

If G is collapsible, then Theorem 4.3.1(i) holds. Hence we may assume that G is not collapsible. Let G' be the reduction of G. Then $G' \neq K_1$ and $\kappa'(G') \geq 2$. If $F(G') \leq 2$, then by Theorem 1.2.3(iv) G' is a $K_{2,t}$ for some $t \geq 2$. Since G is essentially 4-edge-connected, we must have $t \in \{4, 5\}$ and any vertex in $D_2(G')$ must be a trivial contraction, and so we can view $D_2(G') \subseteq D_2(G)$. Thus, $|D_2(G)| \geq |D_2(G')| = t \geq 4$. If $t = 4$, then $K_{2,t} = K_{2,4}$ is eulerian and so by Theorem 1.2.2(ii) G is supereulerian. If G is not supereulerian, then the reduction of G must be $K_{2,5}$, and so Theorem 4.3.1(ii) must holds. Moreover, by inspection, if $u \in D_2(K_{2,t})$ and $v \in V(K_{2,t} - u)$, then $K_{2,t}$ always has a spanning (u, v)-trail, and so by Theorem 1.2.2 (iii), Theorem 4.3.1(iii) must hold. Hence we may assume that

$$\text{the reduction of } G \text{ is not a } K_{2,t} \text{ for any integer } t \geq 2. \tag{4.11}$$

Thus by Theorem 1.2.3(iv), $F(G') \geq 3$. By (4.10), we may assume that G is reduced. Thus, $G = G'$. By (4.9), $d_2 + d_3 \leq 5$. It follows from (4.9) that we must have $d_2 = 5$, $d_3 = 0$ and

$$V(G) = D_2(G) \cup D_4(G). \tag{4.12}$$

Hence, G must be eulerian, and we are done for the proof of Theorem 4.3.1(i) and (ii). It remains to prove Theorem 4.3.1(iii).

We introduce the following notations in our argument. For each vertex $z \in D_2(G)$, let $N_G(z) = \{z_1, z_2\}$. As G is essentially 4-edge-connected, $z_1, z_2 \in D_4(G)$. Let $N_G(z_1) = \{w_1, w_2, w_3, z\}$ and $N_G(z_2) = \{w_1^*, w_2^*, w_3^*, z\}$. Define

$$M(z) = \{w_1, w_2, w_3\} \text{ and } M^*(z) = \{w_1^*, w_2^*, w_3^*\}.$$
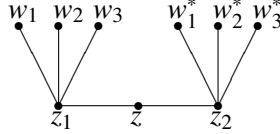

Figure 5: $M(z)$ and $M^*(z)$ in G

By (4.10), there exist a pair of distinct vertices $u, v \in D_2(G)$ such that

$$G \text{ has not spanning } (u, v)\text{-trails.} \tag{4.13}$$

We proceed our proof by verifying the following claims and let $D_2(G) = \{a, b, c, u, v\}$.

Claim 1. For any $z \in \{a, b, c\} = D_2(G) - \{u, v\}$,
(a). $|M(z) \cap D_2(G)| \geq 2$ and $|M^*(z) \cap D_2(G)| \geq 2$
(b). $|M(z) \cap \{u, v\}| \geq 1$ and $|M^*(z) \cap \{u, v\}| \geq 1$.

Proof of Claim 1 (a): By symmetry, it suffices to show that $|M(z) \cap D_2(G)| \geq 2$. By contradiction, suppose $|M(z) \cap D_2(G)| \leq 1$. Then we may assume $M(z) \cap D_2(G) \subseteq \{w_3\}$.

Using the reduction method and the same notations in Theorem 4.2.1, we obtain two graphs G_1 and G_2 from G with $\delta(G_i) \geq 2$ and $D_3(G_i) = \emptyset$ (i=1,2). By Theorem 4.2.1, we may assume that G_1 is essentially 4-edge-connected. Since $M(z) \cap D_2(G) \subseteq \{w_3\}$, $w_1, w_2 \notin D_2(G)$, and by (4.1), we have $G_1 = (G - \{z_1\}) + \{zw_1, w_2w_3\}$, $x_1 = w_1$, $x_2 = w_2$ and $x_3 = w_3$. Thus we may view $D_2(G_1) = D_2(G)$. By (4.10), G_1 has a spanning (u, v)-trail H_1'. Since z has degree 2 in G_1 and $z \notin \{u, v\}$, $zx_1 \in E(H_1')$. Define

$$H_1 = \begin{cases} G[E(H_1' - zx_1) \cup \{zz_1, z_1w_1\}] & \text{if } x_2x_3 \notin E(H_1') \\ G[E(H_1' - \{zx_1, x_2x_3\}) \cup \{zz_1, z_1w_1, w_2z_1, z_1w_3\}] & \text{if } x_2x_3 \in E(H_1') \end{cases}.$$

Then H_1 is a spanning (u, v)-trial of G, contrary to (4.13). This proves Claim 1(a).

Proof of Claim 1(b): By way of contradiction, suppose Claim 1(b) is not true. Let z be a vertex in $\{a, b, c\}$ such that $M(z) \cap \{u, v\} = \emptyset$. We may assume that $z = a$. By Claim 1(a), $|M(z) \cap D_2(G)| \geq 2$. Since $z = a \notin M(z)$ and $M(z) \cap \{u, v\} = \emptyset$, $M(z) \cap D_2(G) = D_2(G) - \{a, u, v\} = \{b, c\}$. We may assume that $w_1 = b$, and $w_2 = c$, and so $d_G(w_1) = d_G(w_2) = 2$ and $d_G(w_3) = 4$. Let $N_G(w_i) = \{z, w_i'\}$ ($i =$

29

1, 2). Again using the reduction method on G as in Theorem 2.2, we obtained two graphs G_1 and G_2 with $\delta(G_i) \geq 2$ and $D_3(G_i) = \emptyset$ ($i=1,2$). By Theorem 2.2, we may assume that G_1 is essentially 4-edge-connected. Then since $d_G(z) = d_G(w_1) = 2$, and $d_G(w_3) = 4$, $G_1^- = (G - \{z_1\}) + \{zw_1, w_2w_3\}$ with $W(G_1^-) = \{zw_1\} = \{zb\}$, and so $G_1 = G_1^-/zw_1$ with $z = \theta(zw_1)$ and $zw_1' \in E(G_1)$, and with $x_1 = w_1'$, $x_2 = w_2'$ and $x_3 = w_2 = c$ (See Figure 4 (II) for G_1). Thus, by (4.10), G_1 has a spanning (u, v)-trail H_0.

Since $\{z, x_3\} = \{a, c\} \subseteq D_2(G_1) - \{u, v\}$, $zx_1 = zw_1'$ and $x_2x_3 = w_2'w_2$ are both in $E(H_0)$. Since $d_{G_1}(w_2) = d_{G_1}(c) = d_G(c) = 2$ and $c \notin \{u, v\}$, w_2w_3 is also in $E(H_0)$. Define

$$H_1 = (H_0 - \{zx_1, w_2w_3\}) + \{zz_1, z_1w_1, w_1w_1', z_1w_2, z_1w_3\}.$$

Then H_1 is a spanning (u, v)-trail in G, a contradiction. Thus, Claim 1(b) is proved.

Claim 2. For any $z \in D_2(G)$, $|D_2(G) \cap M(z) \cap M^*(z)| \leq 1$.

By the definition of $M(z)$ and $M^*(z)$, $|D_2(G) \cap M(z) \cap M^*(z)| \leq 3$, where equality holds if and only if $G = K_{2,4}$. Since $|D_2(G)| = d_2 = 5$, $G \neq K_{2,4}$, and so $|D_2(G) \cap M(z) \cap M^*(z)| \leq 2$. If $|D_2(G) \cap M(z) \cap M^*(z)| = 2$, then we may assume that $w_1 = w_1^*$ and $w_2 = w_2^*$ in $D_2(G)$. Then $\{z_1w_3, z_2w_3^*\}$ is an essential edge cut of G, contrary to that G is essentially 4-edge-connected. This proves Claim 2.

Claim 3. For all $y \in \{u, v\}$, $M(y) \cap M^*(y) \cap \{a, b, c\} = \emptyset$.

Without loss of generality, we may assume $y = u$. By way of contradiction, suppose there is a vertex z in $\{a, b, c\}$ such that $z \in M(u) \cap M^*(u)$. Let $N_G(u) = \{u_1, u_2\}$. Then zu_1 and zu_2 are the two edges incident with z. Let $G_0 = G/zu_2$ with $u_2 = \theta(zu_2)$. Then $u_1u_2 \in E(G_0)$. Note G_0 has the same essentially edge-connectivity as G and $\delta(G_0) \geq 2$ with $|V(G_0)| < |V(G)|$. Therefore, by (4.10), G_0 has a spanning (u, v)-trail H_0.

If $u_1u_2 \in E(H_0)$, then $H = H_0 - u_1u_2 + \{u_1z, zu_2\}$ is a spanning (u, v)-trail in G, contrary to (4.13). If $u_1u_2 \notin E(H_0)$, then since H_0 is a spanning (u, v)-trail in G_0, one and only one of uu_1 or uu_2 (say uu_1) is in H_0, then $H = H_0 - uu_1 + \{uu_2, u_2z, zu_1\}$ is a spanning (u, v)-trail in G, a contradiction again. Claim 3 is proved.

Figure 6: (A)   (B) $b \in M^*(a)$ (C) $b \notin M^*(a)$
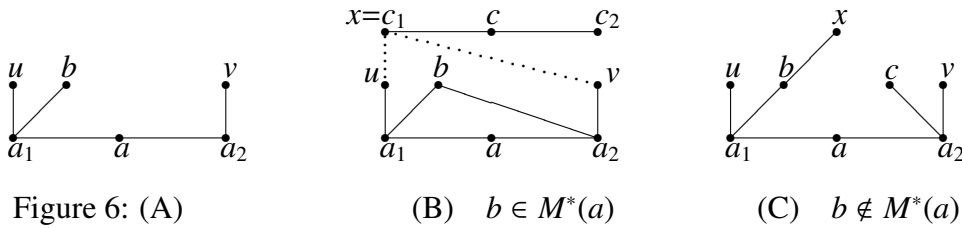
For $\{a, b, c\} = D_2(G) - \{u, v\}$, let $N_G(a) = \{a_1, a_2\}$, $N_G(b) = \{b_1, b_2\}$, and $N_G(c) = \{c_1, c_2\}$. Then since G is essentially 4-edge-connected and by (4.12), $d(a_i) = d(b_i) = d(c_i) = 4$ where $i = 1, 2$.

Let $S = N_G(a) \cup N_G(b) \cup N_G(c)$. If $|S| = 2$, then $S = N_G(a) = N_G(b) = N_G(c)$, contrary to Claim 2. Thus, $|S| \geq 3$. In the following, we assume $N_G(a) = \{a_1, a_2\} \subseteq S$ and let $x \in S - \{a_1, a_2\}$. Thus,

$$S = \{a_1, a_2, x, \cdots\}.$$

By Claim 1(a) and (b), $|M(a) \cap D_2(G)| \geq 2$, $|M^*(a) \cap D_2(G)| \geq 2$, $|M(a) \cap \{u, v\}| \geq 1$ and $|M^*(a) \cap \{u, v\}| \geq 1$. We may assume that $b \in M(a) = N_G(a_1) - \{a\}$, $u \in M(a) = N_G(a_1) - \{a\}$, and by Claim 3 $v \in M^*(a) = N_G(a_2) - \{a\}$ and so $u \notin M^*(a)$ and $v \notin M(a)$ (See the Figure 6(A)).

Case 1. $b \in M^*(a)$ (See Figure 6(B)).
Then $N_G(b) = \{a_1, a_2\} = N_G(a)$. Since $N_G(c) \subseteq S$, c must be adjacent to x, and so $x \in N_G(c)$. We may assume that $x = c_1$ and $M(c) = N_G(c_1) - \{c\}$. By Claim 1(a), c_1 must be adjacent to another two degree 2 vertices in addition to c. Hence, since $N_G(a) = N_G(b)$, u and v must be the two vertices adjacent to c_1, and so $N_G(u) = \{a_1, c_1\}$ and $N_G(v) = \{a_2, c_1\}$. Therefore, another vertex c_2 in $N_G(c)$ is not in $\{a_1, a_2\}$. Otherwise, $c \in M(u) \cap M^*(u)$ or $c \in M(v) \cap M^*(v)$, contrary to Claim 3. Note $M^*(c) = N_G(c_2) - \{c\}$. Thus,

$$D_2(G) \cap M^*(c) = \{a, b, u, v, c\} \cap M^*(c) = \emptyset,$$

contrary to Claim 1(a) that $|M^*(c) \cap D_2(G)| \geq 2$.

Case 2. $b \notin M^*(a)$ (See Figure 6(C)).
Then by Claim 1(a), $M^*(a) = N_G(a_2) - \{a\}$ must have at least two degree 2 vertices, and so $c \in M^*(a) = N_G(a_2) - \{a\}$. Since $b \notin M^*(a)$, $N_G(b) \cap S \neq \emptyset$, and so we may assume $x \in N_G(b) - \{a_1\}$ (See Figure 6(C)). Then since both u and b are adjacent to a_1, by Claim 3 u is not adjacent to x. By Claim 1(a), $M^*(b) = N_G(x) - \{b\}$ must have at least two degree 2 vertices other than b and u. Thus, v and c must be in $M^*(b) = N_G(x) - \{b\}$. Therefore, $N_G(v) = \{a_2, x\} = N_G(c)$, contrary to Claim 3.

We have a contradiction for each case above, and so the statement (4.13) is false. The theorem is proved. $\square$

Lemma 4.3.2 *(Lai, Shao, Yu and Zhan [25]) If G is collapsible, then for any $u, v \in V(G)$, G has a (u, v)-trail.*

In Theorem 3.12 of [10], Catlin and Lai proved that if a 3-edge-connected graph G has at most 9 edge cuts of size 3, then G is supereulerian. For an essentially 4-edge-connected graph G with $\delta(G) \geq 3$, we have the following:

Theorem 4.3.3 *If G is an essentially 4-edge-connected graph with $\delta(G) \geq 3$ and $|D_3(G)| < 10$, then G is collapsible and has a spanning (u, v)-trail for any $u, v \in V(G)$.*

Proof. Since G is essentially 4-edge-connected with $\delta(G) \geq 3$, by Proposition 4.1.1, $\kappa'(G) \geq 3$. Let G' be the reduction of G. By way of contradiction, suppose G is not collapsible. Then $G' \neq K_1$ and

$\kappa'(G') \geq 3$. Let $d_i = |D_i(G')|$. Then since $\kappa'(G') \geq 3$, $d_1 = d_2 = 0$. Since G is essentially 4-edge-connected, G does not have an essential edge cut of size 3, and so $d_3 = |D_3(G')| \leq |D_3(G)| < 10$. If $F(G') \leq 2$, then by Theorem 1.2.3(iv), $G' \in \{K_1, K_{2,t}\}$ $(t \geq 2)$, contrary to $G' \neq K_1$ and $\kappa'(G') \geq 3$. Hence, $F(G') \geq 3$, then by (4.9) and $d_2 = 0$,

$$\sum_{i \geq 5} (i - 4)d_i + 10 \leq 2d_2 + d_3;$$
$$10 \leq d_3 < 10,$$

a contradiction. Thus, G must be collapsible. By Lemma 4.3.2, for any $u, v \in V(G)$, G has a spanning (u, v)-trail. The theorem is proved. $\qquad\square$

4.4 Graphs That Are 3-Edge-Eulerian-Connected

In this section, we shall investigate what graphs are 3-edge-Eulerian-connected. First, we prove the following theorem, as stated in Theorem 4.1.3, which proves the conjecture posed in [34].

Theorem 4.4.1 *If G is a 4-edge-connected graph, then G is 3-edge-Eulerian-connected. And so $\xi(3) = 4$.*

Proof. Let G be a graph with $\kappa'(G) \geq 4$, and let $X \subseteq E(G)$ be an edge set with $|X| = 3$. Pick any pair of edges $e', e'' \in E(G) - X$. Let L be the graph obtained from G by subdividing each edge $e \in X \cup \{e', e''\}$ exactly once. (That is, for each edge $e = a_e b_e \in X \cup \{e', e''\}$, we replace e by a path $a_e v_e b_e$ by inserting a new vertex v_e). Then $D_2(L)$ is the set of the five degree 2 vertices generated by the subdivision, and L is 2-edge-connected and essentially 4-edge-connected. By Theorem 4.3.1(iii), L has a spanning $(v_{e'}, v_{e''})$-trail. This implies that G has a spanning (e', e'')-trail containing X, and so by definition, G is 3-edge-Eulerian-connected. $\qquad\square$

As we know many 3-edge-connected graphs such as $P(10)$ have no spanning closed trail, the edge-connectivity in Theorem 4.4.1 cannot be lowered to 3-edge-connected. However, a 3-edge-Eulerian-connected graph is not necessarily 4-edge-connected. For example, let G be a graph obtained from K_n $(n \geq 8)$ and a vertex v by joining v to v_1 and v_2 with two edges vv_1 and vv_2, where $v_1, v_2 \in V(K_n)$ and $v \notin V(K_n)$. Then G is a 3-edge-Eulerian connected graph with $d(v) = 2$. We have the following necessary conditions for 3-edge-Eulerian-connected graphs.

Proposition 4.4.2 *Let G be a 3-edge-Eulerian-connected graph with $|E(G)| \geq 6$. Then G must be essentially 4-edge-connected with $D_3(G) = \emptyset$.*

Proof. We shall first show that G does not have an edge cut of size 3. By contradiction, assume that G an edge cut of G with $|X| = 3$. Let H_1 and H_2 be the two components of $G - X$ with $|E(H_1)| \leq$

$|E(H_2)|$. Since G is 3-edge-Eulerian-connected with $|E(G)| \geq 6$ and $|X| = 3$, we may assume that $|E(H_2)| \geq 2$. Let e_1 and e_2 be two distinct edges in $E(H_2)$. Then G has a spanning (e_1, e_2)-trail T with $X \subseteq E(T)$. Since both $e_1, e_2 \in E(H_2)$, $T' = T/(H_2 \cap T)$ is a spanning closed trail of G/H_2 that contains X. Since T' is a spanning closed trail and X is an edge cut, $|X| = |E(T') \cap X| \equiv 0 \pmod{2}$, contrary to that $|X| = 3$. Hence G does not have an edge cut of size 3 and so $D_3(G) = \emptyset$.

To show G is essentially 4-edge-connected, it suffices to show that G does not have an essential edge cut X' with $|X'| = 2$. By way of contradiction, suppose that such an edge cut X' exists and $G - X'$ has two components H'_1 and H'_2. Since X' is an essential edge cut, we can pick an edge $e'_i \in E(H'_i)$, $(1 \leq i \leq 2)$. Since $|X'| = 2 < 3$ and G is 3-edge-Eulerian-connected, G has a spanning (e'_1, e'_2)-trail T' such that $X' \subseteq E(T')$. Let e'' be an edge not in G joining the two end vertices of T'. Then $T' + e''$ is a spanning closed trail of $G + e''$, which contains a 3-edge-cut $X' \cup \{e''\}$ of $G + e''$. This yields a contradiction as the intersection of any close trail and any edge cut must have an even number of edges. $\square$
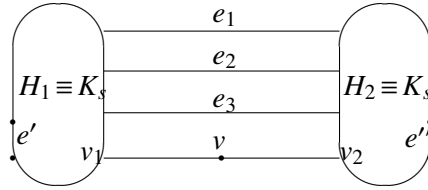

Figure 7

Let G be the graph shown in Figure 7 with $s \geq 6$, where v is a vertex of degree 2, $e' \in E(H_1)$ and $e'' \in E(H_2)$. Let $X = \{e_1, e_2, e_3\}$ be the set of the three edges shown in Figure 7. As we can see that a trail started from e_1 in H_1 must ended in H_1 after tracing through the three edges in X and vertex v. Hence, there is no spanning (e', e'')-trail T in G such that $X \subseteq E(T)$ and $V(T) = V(G)$. Thus, an essentially 4-edge-connected graph G with $D_3(G) = \emptyset$ may not be 3-edge-Eulerian connected. It remains a problem to completely characterize the structures of 3-edge-Eulerian connected graphs.

Let $G_0 = G - \{v\} + v_1 v_2$. Then G_0 is 4-edge-connected and $X_0 = \{e_1, e_2, e_3, v_1 v_2\}$ is an edge-cut of G_0. And G_0 has no spanning (e', e'')-trails containing X_0. This shows that Theorem 4.4.1 is best possible in the sense that 4-edge-connected graph G cannot be 4-edge-Eulerian connected.

Chapter 5

Spanning Tree Packing and Covering Degree Sequence Characterizations

5.1 Background

The problem of designing networks with n processors each of which has a given number of connections and with a certain level of expected network strength is often modeled as a problem of finding graph realizations with certain graphical properties for a given degree sequence. For more on the literature on the degree sequence realization with given properties, see a resourceful survey by Li [28].

Following [6], $c(G)$ denotes the number of components of a graph G. An integral sequence $d = (d_1, d_2, \cdots, d_n)$ is **graphic** if there is a simple graph G with degree sequence d. Let (d) denote the set of all simple graphs with degree sequence d. Any graph $G \in (d)$ is called a realization of d, or simply a d**-realization**. The **spanning tree packing number** of G (see [37]), denoted by $\tau(G)$, is the maximum number of edge-disjoint spanning trees in G. There have been many studies on the behavior of $\tau(G)$, see [20, 21, 32, 35, 42], among others. In a recent paper [26], the authors characterized the degree sequences d for which there exists a graph $G \in (d)$ with $\tau(G) \geq k$.

Theorem 5.1.1 *(Lai et al [26]) Let $k > 0$ be an integer. For a graphic sequence $d = (d_1, d_2, \cdots, d_n)$ with $d_1 \geq d_2 \geq \cdots \geq d_n$ with $n \geq 2$, there exists $G \in (d)$ such that $\tau(G) \geq k$ if and only if both $d_n \geq k$ and $\sum_{i=1}^{n} d_i \geq 2k(n-1)$.*

The **arboricity** of G, denoted by $a(G)$, is the minimum number of spanning trees whose union equals $E(G)$. By definition, $\tau(G) \leq a(G)$. The main result of this paper is the following. (Any empty summation is considered to have value zero).

Theorem 5.1.2 *Let $k_2 \geq k_1 \geq 0$ and $n > 1$ be integers. Let $d = (d_1, d_2, \cdots, d_n)$ with $d_1 \geq d_2 \geq \cdots \geq d_n$ be a graphic sequence and let $I = \{i : d_i < k_2\}$. Then there exists a graph $G \in (d)$ such that $k_2 \geq a(G) \geq \tau(G) \geq k_1$ if and only if each of the following holds.*
(i) $d_n \geq k_1$.
(ii) $2k_2(n - |I| - 1) + 2 \sum_{i \in I} d_i \geq \sum_{i=1}^{n} d_i \geq 2k_1(n - 1)$.

Theorem 5.1.2 has two immediate corollaries, as stated below, by letting $k_1 = 0$ and $k_2 = k$ in Corollary 5.1.3 and let $k_1 = k_2 = k$ in Corollary 5.1.4.

Corollary 5.1.3 *Let $n \geq 2$ and $k > 0$ be integers. For a graphic sequence $d = (d_1, d_2, \cdots, d_n)$ with $d_1 \geq d_2 \geq \cdots \geq d_n$, the following are equivalent.*
(i) There exists a d-realization G such that $a(G) \leq k$.
(ii) $\sum_{i=1}^{n} d_i \leq 2k(n - |I| - 1) + 2 \sum_{i \in I} d_i$, where $I = \{i : d_i < k\}$.

Corollary 5.1.4 *Let $n \geq 2$ and $k > 0$ be integers. For a graphic sequence $d = (d_1, d_2, \cdots, d_n)$ with $d_1 \geq d_2 \geq \cdots \geq d_n$, there exists $G \in (d)$ such that $a(G) = \tau(G) = k$ if and only if $d_n \geq k$ and $\sum_{i=1}^{n} d_i = 2k(n - 1)$.*

We shall utilize the properties related to uniformly dense graphs (see [9]) together with a decomposition (introduced in [32]) based on subgraph densities in the proofs of the main result. In the next section, we present the preliminaries on uniformly dense graphs and the related decomposition, which will be deployed in the proof arguments of our main result. The proof of Theorem 5.1.2 and the corollaries will be given in last section.

5.2 Prerequisites

In this section, we introduce some notations and results that will be needed in the proofs of our main results. For a vertex subset $V_1 \subseteq V(G)$, define $E[V_1] = \{uv \in E(G) : u, v \in V_1\}$. For an integer $r \geq 1$, let $\mathcal{T}_r$ denote the family of all graphs G with $\tau(G) \geq r$. Let G be a connected graph. A subgraph H of G is called **r-maximal** if $H \in \mathcal{T}_r$ and if there is no subgraph K of G, such that K contains H properly and $K \in \mathcal{T}_r$. An r-maximal subgraph H of G is called an **r-region** if $\tau(H) = r$. A subgraph H of G is a region if H is an r-region for some integer r. Define $\eta(G) = \max\{r | G \text{ has}$ a subgraph as an r-region\}.

Let H be a graph with $|V(H)| > 1$. The **density** of H is

$$d(H) = \frac{|E(H)|}{|V(H)| - 1}.$$

It should be indicated that when H is a graph, $d(H)$ denotes the density of H, but when $v \in V(G)$

35

is a vertex of a graph G, $d_G(v)$ denotes the degree of v in G. In the following, we list some known results which will be used in Section 3.

Theorem 5.2.1 *(Nash-Williams, [36]) Let G be a graph. Then*

$$a(G) = \max_{H \subseteq G} \lceil d(H) \rceil,$$

where the maximum is taken over all induced subgraphs H of G with $|V(H)| \geq 2$.

By definition, for a connected graph,

$$a(G) \geq d(G) = \frac{|E(G)|}{|V(G)| - 1} \geq \tau(G). \tag{5.1}$$

Theorem 5.2.2 *(Catlin et al, [9]) If $a(G) > \tau(G)$, then $d(G) > \tau(G)$.*

Theorem 5.2.3 *(Liu et al, [32]) Let G be a nontrivial connected graph. Then*
(i) there exists an integer $m \in \mathbb{N}$, and an m-tuple $(i_1, i_2, \cdots, i_m)$ of integers in $\mathbb{N}$ with $\tau(G) = i_1 < i_2 < \cdots < i_m = \eta(G)$, and a sequence of edge subsets $E_m \subset \cdots \subset E_2 \subset E_1 = E(G)$ such that each component of the spanning subgraph of G induced by E_j is an r-region of G for some $r \in \mathbb{N}$ with $r \geq i_j$ $(1 \leq j \leq m)$, and such that at least one component H in $G[E_j]$ is an i_j-region of G;
(ii) if H is a subgraph of G with $\tau(H) \geq i_j$, then $E(H) \subseteq E_j$;
(iii) the integer m and the sequences in (i) are uniquely determined by G.

Theorem 5.2.4 *(Liu et al, [32]) If G is a nontrivial connected graph, then $a(G) \geq \eta(G) \geq a(G) - 1$.*

Lemma 5.2.5 *(Lai et al, [26]) Let $k \geq 1$ be an integer, G be a graph with $\eta(G) \geq k$. Then each of the following statements holds.*
(i) The graph G has a unique edge subset $X_k \subseteq E(G)$, such that every component H of $G[X_k]$ is a maximal subgraph with $\tau(H) \geq k$. In particular, $G \notin \mathcal{T}_k$ if and only if $E(G) \neq X_k$.
(ii) If $G \notin \mathcal{T}_k$, then G/X_k contains no nontrivial subgraph H' with $\tau(H') \geq k$.
(iii) If $G \notin \mathcal{T}_k$, then $d(H') < k$ for any nontrivial subgraph H' of G/X_k.

By Theorem 5.2.4 and by $\eta(G) = i_m$, we deduce that the same conclusions of Lemma 5.2.5 also hold if the condition $\eta(G) \geq k$ in Lemma 5.2.5 is replaced by the condition $a(G) > k$.

Lemma 5.2.6 *(Lai et al, [26]) Let G be a graph with $d(G) \geq k$ and let $X_k \subset E(G)$ be the edge subset defined in Lemma 5.2.5(i). If $G[X_k]$ has at least two components, then for any nontrivial component H of $G[X_k]$, $d(H) \geq k$, and $G[X_k]$ has at least one component H with $d(H) > k$.*

Next, we shall show that the same conclusions of Lemma 5.2.6 hold if we replace the condition $d(G) \geq k$ in Lemma 5.2.6 by the condition $a(G) > k$. For this purpose, the following result is needed.

Theorem 5.2.7 *(Liu et al, [32]) Let G be a connected graph, and let r, r' be integers with $r' \geq r > 0$. Each of the following holds.*
(i) If $\tau(G) \geq r$, then for any $e \in E(G)$, $\tau(G/e) \geq r$.
(ii) If H is a subgraph of G with $\tau(H) \geq r'$, then $\tau(G/H) \geq r$ if and only if $\tau(G) \geq r$.

Theorem 5.2.8 *Let G be a graph with $a(G) > k$ and let $X_k \subset E(G)$ be the edge subset defined in Lemma 5.2.5(i). Then $G[X_k]$ has at least one component H with $d(H) > k$.*

Proof. Since $a(G) > k$, by Theorem 5.2.1, there exists $G_0 \subseteq G$ with $d(G_0) > k$.

Let $X'_k \subset E(G_0)$ be the edge subset defined in Lemma 5.2.5(i). If $G_0[X'_k]$ has only one component, then $G_0[X'_k] = G_0$ and $d(G_0[X'_k]) = d(G_0) > k$. If $G_0[X'_k]$ has at least two components, then by Lemma 5.2.6, $G_0[X'_k]$ has at least one component K with $d(K) > k$. In both cases, we use K to denote a component of $G_0[X'_k]$ with $\tau(K) \geq k$ and $d(K) > k$.

Let $X_k \subset E(G)$ be the edge subset defined in Lemma 5.2.5(i). Then $X'_k \subseteq X_k$, and there exists a component H of $G[X_k]$ such that $K \subseteq H$ and $\tau(H) \geq k$. By Theorem 5.2.7, we have $\tau(H/K) \geq k$, and so $|E(H/K)| \geq k(|V(H/K)| - 1)$. Since $d(K) > k$, $|E(K)| > k(|V(K)| - 1)$. By $|V(H)| = |V(H/K)| + |V(K)| - 1$, $|E(H)| = |E(H/K)| + |E(K)|$, we have

$$
\begin{aligned}
d(H) &= \frac{|E(H)|}{|V(H)| - 1} = \frac{|E(H/K)| + |E(K)|}{|V(H/K)| + |V(K)| - 2} \\
&> \frac{k(|V(H/K)| - 1) + k(|V(K)| - 1)}{|V(H/K)| + |V(K)| - 2} = k.
\end{aligned}
$$

This completes the proof. □

The following lemma is useful in the proof of the main result. The related matroidal extensions can be found in [29] and [30].

Lemma 5.2.9 *(Lai et al, [26]) Let G be a graph and let $X_k \subset E(G)$ be the edge subset defined in Lemma 5.2.5(i). If H' and H'' are two components of $G[X_k]$, then each of the following holds.*
(i) $|[H', H'']| < k$.
(ii) If $d(H') > k$, then H' has a subgraph K such that $d(K) > k$ and $\tau(K - e) \geq k$ for any $e \in E(K)$.
(iii) If $d(H') > k$, then H' has an edge e' such that $\tau(H' - e') \geq k$, and $E(G) - X_k$ has at most one edge jointing the ends of e' to H''.

5.3 The Proofs

Throughout this section, suppose $k_1, k_2 > 0$ and $n > 1$ are integers and that $d = (d_1, d_2, \cdots, d_n)$ is a nonincreasing graphic sequence. For this degree sequence d, define $I = \{i : d_i < k_2\}$ and $t = |I|$.

For a graph $G \in (d)$, define $V_I = \{v : d(v) < k_2\}$ and $V_{II} = \{v : d(v) \geq k_2\}$. Thus $|V_I| = t$ and $|V_{II}| = n - t$.

Lemma 5.3.1 *If some $G \in (d)$ has $a(G) \leq k_2$, then*

$$\sum_{i=1}^{n} d_i \leq 2k_2(n - |I| - 1) + 2 \sum_{i \in I} d_i.$$

Proof. Since $a(G[V_{II}]) \leq a(G) \leq k_2$, by (5.1), $|E[V_{II}]| \leq k_2((n-t)-1)$. By counting the incidences of vertices in V_I, we have $|[V_I, V_{II}] \cup E[V_I]| \leq \sum_{i \in I} d_i$. It follows that $\sum_{i=1}^{n} d_i = 2|E[V_{II}]| + 2|[V_I, V_{II}] \cup E[V_I]| \leq 2k_2(n - t - 1) + 2\sum_{i \in I} d_i$, and so the lemma follows. $\qquad\square$

In the following, we assume that $d = (d_1, d_2, \cdots, d_n)$ satisfy Theorem 5.1.2 (i) and (ii).

Since d satisfies Theorem 5.1.2 (ii), by the definition of I, we have $\sum_{i \in I} d_i < \sum_{i \in I} k_2$. As $\sum_{i=1}^{n} d_i \leq 2k_2(n - |I| - 1) + 2\sum_{i \in I} d_i = 2k_2(n - 1) - 2(k_2|I| - \sum_{i \in I} d_i)$, it follows that

$$\sum_{i=1}^{n} d_i \leq 2k_2(n - 1). \tag{5.2}$$

The next lemma will be needed in the proof of Theorem 5.1.2.

Lemma 5.3.2 *Let $k' > k > 0$ and $r \geq k$ be integers. Let G be a graph with $a(G) \geq k'$ and $\tau(G) \geq k$, and let H be an r-region of G such that for some $e = uv \in E(H)$ with $\tau(H - e) \geq r$. For any edge $e' = xy \in E(G - H)$, if $f = ux, f' = vy \notin E(G)$, then*

$$\tau((G - \{e, e'\}) \cup \{f, f'\}) \geq k.$$

Proof. Let $G' = G/H$ and $G'' = (G - \{e, e'\}) \cup \{f, f'\}$. Since $\tau(G) \geq k$, by Theorem 5.2.7 (i), $\tau(G') \geq k$. Let $T'_1, T'_2, \cdots, T'_k$ be k edge-disjoint spanning trees of G'. Since $\tau(H - e) \geq r \geq k$, $H - e$ has k edge-disjoint spanning trees $L_1, L_2, \cdots, L_k$. Since $G' = G/H$ and since $e \in E(H)$, $e \notin E(G')$. For each i with $1 \leq i \leq k$, if $e' \notin E(T'_i)$, then $E(L_i) \cup E(T'_i) \subseteq E(G'')$. Let

$$T_i = G''[E(L_i) \cup E(T'_i)].$$

Then, T_i is a spanning tree of G''. In particular, if $e' \notin \bigcup_{i=1}^{k} E(T'_i)$, then $T_1, T_2, \cdots, T_k$ are k edge-disjoint spanning trees of G'', and so $G'' \in \mathcal{T}_k$.

Thus we assume that $e' \in E(T'_1)$. Let T'_{11} and T'_{12} be the two components of $T'_1 - e'$ in G'. We may assume that T'_{11} contains the vertex v_H in G' onto which the subgraph H is contracted. Since $e' = xy$, we may also assume that $x \in V(T'_{11})$ and $y \in V(T'_{12})$. Let $T''_1 = G''[E(L_1) \cup E(T'_1 - e') \cup \{f'\}]$. Then T''_1 is a spanning tree of G''. It follows that $T''_1, T_2, \cdots, T_k$ are k edge-disjoint spanning trees of G'', and so $G'' \in \mathcal{T}_k$. $\qquad\square$

Proof of Theorem 5.1.2. By Theorem 5.1.1 and Lemma 5.3.1, the necessity of Theorem 5.1.2 follows immediately. It remains to prove the sufficiency.

Let $(d)_1 = \{G \in (d) : \tau(G) \geq k_1\}$. By Theorem 5.1.1, $(d)_1 \neq \emptyset$. To prove the sufficiency, we argue by contradiction and assume that

$$\text{for any } G \in (d)_1, a(G) > k_2. \tag{5.3}$$

Thus by (5.2), for any $G \in (d)_1$,

$$\sum_{i=1}^{n} d_i \leq 2k_2(n-1) < 2a(G)(n-1). \tag{5.4}$$

By Theorem 5.2.3, there exists a sequence of positive integers $\tau(G) = i_1 < i_2 < \cdots < i_m = \eta(G)$.

Claim 1. For any $G \in (d)_1$, $m \geq 2$.

Proof of Claim 1. By contradiction, we assume that $m = 1$ for some $G \in (d)_1$. By Theorem 5.2.4, $\tau(G) = a(G)$ or $\tau(G) = a(G) - 1$. If $\tau(G) = a(G)$, then by (5.1), $2a(G) = 2|E(G)|/(n-1)$, and so $2a(G)(n-1) = \sum_{i=1}^{n} d_i \leq 2k_2(n-1)$, contrary to (5.4). Thus we must have $\tau(G) = a(G) - 1$. By Theorem 5.2.2, $2\tau(G) < 2d(G) = 2|E(G)|/(n-1)$, and so by (5.2), $2\tau(G)(n-1) < 2|E(G)| = \sum_{i=1}^{n} d_i \leq 2k_2(n-1)$. It follows that $\tau(G) \leq k_2 - 1$, and so $a(G) = \tau(G) + 1 \leq k_2$, contrary to (5.3). This proves Claim 1.

By Claim 1, $m \geq 2$. By (5.3) and by Theorem 5.2.3, there exists an m-tuple $(i_1, i_2, \cdots, i_m)$ of integers as stated in Theorem 5.2.3 with $k_2 \leq a(G) - 1 \leq i_m \leq a(G)$. Thus there exists a smallest index i_j such that $i_j \geq k_2$. By Theorem 5.2.3, G has a unique edge subset $E_{i_j} \subseteq E(G)$ such that each component of $G[E_{i_j}]$ is a k_2-maximal subgraph of G.

Claim 2. For any $G \in (d)_1$, $E_{i_j} \neq E(G)$.

Proof of Claim 2. By contradiction, assume that for some $G \in (d)_1$, $E_{i_j} = E(G)$. By Theorem 5.2.3, $E(G)$ has i_j edge-disjoint spanning trees, and so $2k_2(n-1) \leq 2i_j(n-1) \leq 2|E(G)| = \sum_{i=1}^{n} d_i$. By (5.2), we have $i_j = k_2$ and $|E(G)| = k_2(n-1)$. It follows that $E(G)$ is a disjoint union of k_2 spanning trees, and so by definition, $a(G) = k_2$, contrary to (5.3). This proves Claim 2.

By Theorem 5.2.3 with a given value k_2, for any $G \in (d)_1$, E_{i_j} is uniquely determined by G. Throughout this paper, we define $X(G) = E(G) - E_{i_j}$, and when G is understood from the context, we also use X for $X(G)$. By Claim 2, $X \neq \emptyset$. Let $c = c(G - X)$. (Thus $c = c(G[E_{i_j}])$ as well). Label the components of $G - X$ as $H_1, H_2, \cdots, H_c$ so that

$$d(H_1) \geq d(H_2) \geq \cdots \geq d(H_s) \geq i_j, \text{ and } H_{s+1} = \cdots = H_c = K_1. \tag{5.5}$$

39

Notice that $H_1, H_2, ..., H_s$ are all the nontrivial k_2-maximal subgraphs of G. Since $X = X(G)$ is uniquely determined by G, it follows that the components of $G - X$ and the value of $s = s(G)$ satisfying (5.5) above are also uniquely determined by G. Since $G - X$ is spanning in G and by Claim 2, we have $c \geq 2$. By (5.3) and by Theorem 5.2.8,

$$\text{for any } G \in (d)_1, \text{ we always have } d(H_1) \geq k_2. \tag{5.6}$$

Throughout the rest of the proof in this section, we choose $G \in (d)_1$ such that

$$c = c(G[E_{i_j}]) \text{ is minimized,} \tag{5.7}$$

and subject to (5.7),

$$|X(G)| \text{ is maximized.} \tag{5.8}$$

Claim 3. If $s \geq 2$, then $d(H_2) \leq k_2$.

Proof of Claim 3. Suppose that $s \geq 2$ and $d(H_2) > k_2$. By Lemma 5.2.9(iii), there exists $e_1 = u_1v_1 \in E(H_1)$ and $e_2 = u_2v_2 \in E(H_2)$ such that $\tau(H_1 - e_1) \geq k_2$ and $\tau(H_2 - e_2) \geq k_2$, and there exists at most one edge in X joining the ends of e_1 and e_2. Without loss of generality, assume $u_1u_2, v_1v_2 \notin E(G)$ and let

$$G_1' = (G - \{u_1v_1, u_2v_2\}) \cup \{u_1u_2, v_1v_2\} \text{ and } X_1 = X \cup \{u_1u_2, v_1v_2\}. \tag{5.9}$$

It follows from Lemma 5.3.2 that $G_1' \in (d)_1$. For each $i \in \{1, 2\}$, by the choice of $e_i = u_iv_i$, $H_i - u_iv_i$ is contained in a k_2-maximal subgraph of G_1'. It follows by (5.7) that $G_1' - X(G_1') = (H_1 - u_1v_1) \cup (H_2 - u_2v_2) \cup H_3 \cup \cdots \cup H_c$, and so $|X(G_1')| = |X(G)| + 2$, contrary to (5.8). This proves Claim 3.

By Claim 3 and by Lemma 5.2.8, there exises $G \in (d)_1$ such that

$$G \text{ has a unique } k_2\text{-maximal subgraph } H_1 \text{ with } d(H_1) > k_2. \tag{5.10}$$

Among all such graphs in $(d)_1$ satisfying (5.10), choose G so that

$$|V(H_1)| \text{ is maximized,} \tag{5.11}$$

and subject to (5.11),

$$|X(G)| \text{ is maximized.} \tag{5.12}$$

Throughout the rest of the proof, we shall assume that $G \in (d)_1$ satisfies (5.10), as well as (5.11) and (5.12).

Claim 4. $s = 1$.

40

Proof of Claim 4. Suppose $s \geq 2$. By Lemma 5.2.9, H_1 has an edge $e_1 = uv \in E(H_1)$ with

$$d(H_1 - e_1) \geq \tau(H_1 - e_1) \geq i_j \text{ and } \|[H_1[\{e_1\}], H_2]\| \leq 1. \tag{5.13}$$

By (5.13), H_2 has an edge $e_2 = xy$ such that $xu, yv \notin E(G)$. Let $G_1 = (G - \{xy, uv\}) \cup \{xu, yv\}$. By Lemma 5.3.2, $G_1 \in (d)_1$. By Claim 3, $d(H_2) = k_2$ and so $\tau(H_2 - e_2) < k_2$. Let $H_{2,1}, \cdots, H_{2,l}$ be the k_2-maximal subgraphs of $H_2 - e_2$. Thus for each $z \in \{1, 2, \ldots, l\}$, either $d(H_{2,z}) = k_2$ or $H_{2,z} = K_1$. By the choice of e_1, $\tau(H_1 - e_1) \geq k_2$. If $\tau(H_1 - e_1) = k_2$, then by Claim 3, and by the fact that either $d(H_{2,z}) = k_2$ or $H_{2,z} = K_1$, we must have $a(G_1) \leq k_2$, contrary to (5.3). Hence $d(H_1 - e_1) > k_2$, and so $H_1 - e_1, H_{2,1}, \cdots, H_{2,l}$ are the k_2-maximal subgraphs of $G_1[(H_1 - e_1) \cup (H_2 - e_2)]$. It follows that $X \subseteq X' - \{xu, yv\}$, and so $|X'| \geq |X| + 2$, contrary to (5.12). This proves Claim 4.

By Claim 4, $s = 1$. If $c = 2$, then $|V(H_1)| = n - 1$. Let $V(H_2) = \{x\}$. By the definition of $X(G)$ and i_j, $\tau(H_1) \geq i_j \geq k_2$. By Theorem 5.2.8, we have

$$\sum_{i=1}^{n} d_i = 2|E(H_1)| + 2|[\{x\}, H_1]| > 2k_2(n - 2) + 2d_G(x). \tag{5.14}$$

If $d_G(x) \geq k_2$, then $\sum_{i=1}^{n} d_i > 2k_2(n-1)$, contrary to (5.4). Hence $d_n \leq d_G(x) < k_2$. For any $v \in V(H_1)$, we have $d_G(v) \geq d_{H_1}(v) \geq \tau(H_1) \geq k_2$. It follows that $t = 1$, that is, there is a unique vertex whose degree is smaller than k_2. By (5.14), we have $\sum_{i=1}^{n} d_i = \sum_{i=1}^{n-1} d_i + d_n > 2k_2(n-t-1) + 2\sum_{i \in I} d_i$, contrary to Theorem 5.1.2 (ii).

Thus for the rest of the proof, we shall assume that $s = 1$ and $c > 2$. Since $s = 1$, for each i with $2 \leq i \leq c$, denote $V(H_i) = \{x_i\}$. Since $\tau(H_1) \geq k_2$, if for some i, $|N_G(x_i) \cap V(H_1)| \geq k_2$, then $G[V(H_1) \cup \{x_i\}]$ should have been in a k_2-maximal subgraph of G, contrary to the choice of E_{i_j}. Hence we have

$$\text{for any } i \text{ with } 2 \leq i \leq c, |N_G(x_i) \cap V(H_1)| < k_2. \tag{5.15}$$

Claim 5. For some $i \neq j$, $x_i x_j \in E(G)$.

Proof of Claim 5. By contradiction, we assume that $\{x_2, x_3, \cdots, x_c\}$ is an independent set of G. Then for any x_i with $i \geq 2$, $N_G(x_i) \subseteq V(H_1)$. By (5.15), $d_G(x_i) < k_2$. Since for any $v \in V(H_1)$, $d_G(v) \geq d_{H_1}(v) \geq \tau(H_1) \geq k_2$, it follows that $t = |I| = c - 1$ and

$$\begin{aligned}
\sum_{i=1}^{n} d_i &= 2|E(H_1)| + 2\sum_{i=2}^{c} |[\{x_i\}, V(H_1)]| \\
&> 2k_2[n - (c - 1) - 1] + 2\sum_{i \in I} d_i = 2k_2(n - t - 1) + 2\sum_{i \in I} d_i,
\end{aligned}$$

contrary to the assumption in Theorem 5.1.2 (ii). This proves Claim 5.

By Claim 5, we may assume $e' = x_2 x_3 \in E(G)$. By Lemma 5.2.9 (ii), H_1 has a subgraph K such that $d(K) > i_j$, $\tau(H) \geq i_j$, and such that $\tau(K - e) \geq i_j$ for any $e \in E(K)$. As G is a simple graph,

$$|V(K)| \geq i_j \geq k_2. \tag{5.16}$$

Claim 6. For any edge $e = uv \in E(K)$, if $ux_2 \notin E(G)$, then $vx_3 \in E(G)$; if $ux_3 \notin E(G)$, then $vx_2 \in E(G)$.

Proof of Claim 6. By contradiction, suppose for some edge $e = uv \in V(K)$ such that $ux_2, vx_3 \notin E(G)$. Define $G_2 = (G - \{uv, x_2 x_3\}) \cup \{ux_2, vx_3\}$. By Lemma 5.3.2, $G_2 \in (d)_1$. Since $\tau(K - e) \geq i_j \geq k_2$, it follows by Theorem 5.2.7 (ii) that $\tau(H_1 - e) \geq k_2$, and so $H_1 - e$ belongs to a k_2-maximal subgraph H_1'' of G_2. By Claim 4, H_1'' is the only nontrivial k_2-maximal subgraph of G_2. It follows by (5.11) that $V(H_1'') = V(H_1)$ and so $H_1'' = H_1 - e_1$. Hence $X(G) \subseteq X(G_2) - \{ux_2, vx_3\}$, and so $|X(G)| < |X(G_2)|$, contrary to (5.12). This proves Claim 6.

Define

$$
\begin{aligned}
S_1 &= N_G(x_2) \cap N_G(x_3) \cap V(K). \\
S_2 &= (N_G(x_2) - N_G(x_3)) \cap V(K). \\
S_3 &= (N_G(x_3) - N_G(x_2)) \cap V(K). \\
S_4 &= V(K) - (S_1 \cup S_2 \cup S_3).
\end{aligned}
$$

Claim 7. $S_4 \neq \emptyset$.

Proof of Claim 7. By (5.15) and by (5.16), we have $V(K) - N_G(x_3) \neq \emptyset$, and so $S_2 \cup S_4 \neq \emptyset$. Assume by contradiction that $S_4 = \emptyset$. Then $S_2 \neq \emptyset$. By Claim 6, $N_K(S_2) \subseteq S_1 \cup S_2$, and so $|S_1 \cup S_2| \geq |N_K(S_2)| \geq \delta(K) \geq i_j \geq k_2$. On the other hand, it follows by (5.15) that $|S_1 \cup S_2| = |E_G(V(K), \{x_2\})| \leq |N_G(x_2) \cap V(H_1)| < k_2$. This contradiction establishes Claim 7.

By Claim 7, $S_4 \neq \emptyset$. By Claim 6, $N_K(S_4) \subseteq S_1$. It follows that $|S_1| \geq |N_K(S_4)| \geq \delta(K) \geq i_j$. On the other hand, we have $|S_1| \leq |V(K) \cap N_G(x_2)| \leq |N_G(x_2) \cap V(H_1)| < i_j$. This contradiction proves the theorem. $\square$

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