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Dynamic Coloring of Graphs

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in
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ABSTRACT

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Bruce Montgomery

In this dissertation, we introduce and study the idea of a dynamic coloring of a graph, a coloring in which any multiple-degree vertex of the graph must be adjacent to at least two color classes.

As parts of the overall research, we study (for some interesting subjects of colorings) the corresponding subjects of dynamic colorings, we compare the chromatic number and dynamic chromatic number, and we study some problems unique to dynamic colorings. Also, we introduce and briefly study a generalization of dynamic coloring.

The interesting subjects of colorings we consider are the chromatic number of important graphs, upper bounds of the chromatic number, vertex-critical graphs, and stable graphs. For these first three subjects, we prove theorems for dynamic colorings that are similar to important theorems known for proper colorings, while we show no such theorems exist for stable graphs.

We make an extensive comparison of the two chromatic numbers that includes a description of graphs for which the two chromatic numbers are equal, that presents a class of graphs for which the per-graph differences in the two chromatic numbers is unbounded, that shows the difference is at most two for any $K_{1,3}$ -free graph, and that studies the difference for regular graphs.

In our study of some unique problems of dynamic colorings, we characterize the graphs for which the dynamic chromatic number equals the number of vertices, we characterize the graphs for which the dynamic chromatic number

equals one less than the number of vertices, we characterize the graphs for which the deletion of some vertex causes the dynamic chromatic number to decrease by more than one, and we obtain strong results describing graphs for which the removal of any vertex causes the dynamic chromatic number to increase.

Finally, we introduce and briefly study a generalization of dynamic coloring.

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Contents

1	Introduction	1
1.1	General Idea	1
1.2	Definitions, Terminology, and Notation	2
2	Dynamic Chromatic Number of Particular Graphs	3
3	Upper Bounds	6
3.1	$\chi_d \leq \Delta + 3$	6
3.2	$\chi_d \leq \alpha(G)$	10
4	Vertex-Critical Graphs	12
4.1	Proper Colorings	12
4.2	Dynamic Colorings	13
5	Stable Graphs	16
5.1	Proper Colorings	16
5.2	Dynamic Colorings	19
6	Comparison of χ and χ_d	22
6.1	Normal Graphs	22
6.2	$\chi_d - \chi$ is Unbounded	24
6.3	$K_{1,3}$ -Free Graphs	25

6.4 Regular Graphs	28
7 Graphs with $\chi_d(G) = n$ or $n - 1$	30
8 $\chi_d(G - v) = \chi_d(G) - 2$	36
9 Gainer Graphs	39
10 A Generalization of Dynamic Coloring	43

List of Figures

Figure 5.1	Petersen Graph
Figure 5.2	G_1
Figure 5.3	G_2
Figure 5.4	G_3
Figure 7.1	$\chi_d(G) = n - 1$ for $n = 5$
Figure 9.1	G_1
Figure 9.2	G for $l = 4$

List of Symbols

c	coloring
$c(v)$	color of vertex v
$d(v)$	degree of vertex v
$E(G)$	edge set of G
G	graph
K_n	complete graph on n vertices
K_{i_1, \dots, i_l}	complete l -partite graph
n	number of vertices in a graph
$N(v)$	neighborhood of vertex v
P_k	path of length k
v	vertex
$V(G)$	vertex set of G
$\alpha(G)$	maximum length of cycles in G
$\Delta(G)$	maximum vertex degree
$\delta(G)$	minimum vertex degree
χ	chromatic number
χ_d	dynamic chromatic number
χ_r	r-dynamic chromatic number
\overline{G}	complement
$G - e$	deletion of an edge
$G - v$	deletion of a vertex

Chapter 1

Introduction

1.1 General Idea

This dissertation proposes the idea of a *dynamic coloring* of a graph, which we now motivate and define.

A graph coloring assigns distinct colors to adjacent vertices. A coloring of the graph in which a typical vertex is adjacent to more than one color class represents a situation in which the typical individual has a greater variety in the types of relationships. Thus, the overall interactions would not be so limited but more dynamic.

Hence, a *dynamic coloring* is defined as a proper coloring for which any vertex of degree at least two is adjacent to more than one color class. As in the case of graph colorings, a chromatic number for dynamic colorings may similarly be defined.

Of course, any interesting problems of proper colorings may be reconsidered for dynamic colorings—the additional condition gives a fresh perspective. And the new condition creates significant problems that are unique for dynamic colorings.

1.2 Definitions, Terminology, and Notation

In this section, the definitions, terminology, and notation used throughout are described. Terminology especially relevant to a particular section but not to previous sections will first be described in that section. With few exceptions, the definitions, terminology, and notation used here is consistent with that found in [2].

All graphs are finite and simple graphs. A graph with just one vertex is called *trivial*. Any graph is assumed to be connected, although the removal of edges or vertices may result in a graph that is not connected.

A graph is typically denoted by G with vertex set $V = V(G)$ and edge set $E = E(G)$, with n denoting $|V|$. The *neighbor set* $N(v)$ of a vertex v is the set of vertices adjacent to v . The degree of v is denoted by $d(v)$. The minimum vertex degree is denoted by $\delta = \delta(G)$, and the maximum vertex degree is denoted by $\Delta = \Delta(G)$. A k -*coloring* of G is an assignment of k colors $1, 2, \dots, k$ to the vertices of G . The k -coloring is *proper* if no two adjacent vertices are assigned the same color. The *chromatic number* $\chi = \chi(G)$ is the minimum k for which G has a proper k -coloring. A *coloring* of G is typically a χ -coloring of G .

A *dynamic coloring* is defined as a proper coloring in which any multiple-degree vertex is adjacent to more than one color class. A dynamic coloring is thus a map c from V to the set of colors such that

- (C1) If $uv \in E(G)$, then $c(u) \neq c(v)$, and
- (C2) For each vertex $v \in V(G)$, $|c(N(v))| \geq \min\{2, d(v)\}$

We call the first condition, which characterizes proper colorings, the *adjacency condition*, and we call the second condition the *double-adjacency condition*. The *dynamic chromatic number* $\chi_d = \chi_d(G)$ is the minimum k for which G has a dynamic k -coloring. The term dynamic coloring at times denotes a dynamic χ_d -coloring, which should be clear from context.

Chapter 2

Dynamic Chromatic Number of Particular Graphs

In this chapter, the dynamic chromatic number is obtained for complete graphs, trees, complete bipartite graphs, and cycles.

First note the simple fact that any graph has a dynamic coloring, since a coloring for which each vertex is colored differently satisfies the adjacency and double-adjacency conditions, and so minimizing over a nonempty set determines a dynamic coloring. Thus, $\chi_d(G) \leq n$ for any graph G .

Since any dynamic coloring is a proper coloring, $\chi_d(G) \geq \chi(G)$ for any graph G . Thus, $\chi_d(K_n) = n$.

Theorem 1 *Let G be a connected, nontrivial graph. Then $\chi_d(K_2) = 2$ and $\chi_d(G) \geq 3$ otherwise.*

Proof: If $n = 2$, then $G = K_2$ and $\chi_d(K_2) = 2$.

Otherwise, $n \geq 3$ and since G is connected, then G has a path of length two as a subgraph. Hence, some vertex v on the path has two or more neighbors, and in a dynamic coloring at least two of them have different colors

by the double-adjacency condition. Since those colors must be different from the color assigned to v , then $\chi_d(G) \geq 3$. \square

The path of n vertices is denoted by P_n .

Theorem 2 *For any tree G other than K_1 or K_2 , $\chi_d(G) = 3$.*

Proof: The proof is by induction on n .

For $n = 1$ or 2 , $G = K_n$ and $\chi_d(K_n) = n$.

For $n = 3$, $G = P_3$ and $\chi_d(P_3) = 3$.

For a tree G of $n \geq 4$ vertices, the tree $G - v$, where $d(v) = 1$, satisfies $\chi_d(G - v) = 3$ by the induction hypothesis. Given a dynamic three-coloring c of $G - v$, color v the third color distinct from $c(u)$ and $c(w)$, where u is the vertex adjacent to v and w is a vertex other than v adjacent to u . This yields a dynamic three-coloring of G , and so $\chi_d(G) = 3$, since $\chi_d(G) \geq 3$ by *Theorem 1*. \square

The *complete bipartite graph* $K_{i,j}$ is the bipartite graph that has $V(K_{i,j})$ partitioned as $X_1 \cup X_2$ with $|X_1| = i$, $|X_2| = j$ and has each vertex in X_1 adjacent to each vertex in X_2 .

Theorem 3 *If $i, j \geq 2$, then $\chi_d(K_{i,j}) = 4$.*

Proof: Since $d(v) \geq 2$ for any vertex, then by the double-adjacency condition, at least two colors must be used to color X_1 and X_2 individually. Since by the adjacency condition, the vertices of X_1 and X_2 may share no common color, then at least $2 + 2 = 4$ colors are needed in any dynamic coloring of $K_{i,j}$. Clearly, a coloring which colors vertices of X_1 from $\{1, 2\}$ and vertices of X_2 from $\{3, 4\}$, and which uses all four colors, is a dynamic coloring. \square

The *complete l -partite graph* K_{i_1, \dots, i_l} is defined analogously for $l \geq 3$. Since $\chi(K_{i_1, \dots, i_l}) = l$ and the proper coloring c having $c(X_j) = j$ is clearly a dynamic coloring, then $\chi_d(K_{i_1, \dots, i_l}) = l$ if $l \geq 3$.

An n -cycle is a cycle of n vertices and is denoted by C_n .

Theorem 4

$$\chi_d(C_n) = \begin{cases} 5 & \text{if } n = 5 \\ 3 & \text{if } n = 3k, k \geq 1 \\ 4 & \text{otherwise} \end{cases}$$

Proof: Let $v_1v_2 \dots v_nv_1$ denote an n -cycle of C_n . Note that any three consecutive vertices of the cycle must be colored differently in any dynamic coloring of C_n , since the first and third vertices are the only neighbors of the second vertex and must be colored differently (by the double-adjacency condition) and also differently from the second vertex.

Then $\chi_d(C_{3k}) = 3$ if $k \geq 1$, since $\chi_d(G) \geq 3$ and since the coloring $1, 2, 3, \dots, 1, 2, 3, 1$ of consecutive vertices is a dynamic three-coloring.

Since any three consecutive vertices must be colored differently, then $c(v_1), c(v_2)$, and $c(v_3)$ must be three different colors in a dynamic coloring c of C_5 . Then $c(v_4) \neq c(v_1), c(v_2), c(v_3)$ by the double-adjacency, double-adjacency, and adjacency conditions, respectively. Finally, $c(v_5) \neq c(v_1), c(v_2), c(v_3), c(v_4)$ by the adjacency, double-adjacency, double-adjacency, and adjacency conditions, respectively. Thus, all five vertices of C_5 must be colored differently, and so $\chi_d(C_5) = 5$.

Let $k \geq 1$. Suppose $\chi_d(C_{3k+1}) = 3$. Since any three consecutive vertices must be colored differently, then, beginning with $1, 2, 3$, only the coloring $1, 2, 3, 1, 2, 3, \dots, 1, 2, 3, c(v_{3k+1}), 1$ results. However, $c(v_{3k+1}) \neq 1, 2, 3$ by the adjacency, double-adjacency, and adjacency conditions, respectively. Thus, $\chi_d(C_{3k+1}) \geq 4$, and the coloring $1, 2, 3, 1, 2, 3, \dots, 1, 2, 3, 4, 1$ shows $\chi_d(C_{3k+1}) = 4$.

Let $k \geq 2$. If $\chi_d(C_{3k+2}) = 3$, then as argued before, only the coloring $1, 2, 3, 1, 2, 3 \dots 1, 2, 3, 1, c(v_{3k+2}), 1$ may be possible. However, $c(v_{3k+2}) \neq 1, 2, 3$ by the adjacency, double-adjacency, and double-adjacency conditions, respectively. Thus, $\chi_d(C_{3k+2}) \geq 4$, and the coloring $1, 2, 3, 1, 2, 3, \dots, 4, 2, 1, 4, 3, 1$ shows $\chi_d(C_{3k+2}) = 4$. \square

Chapter 3

Upper Bounds

This chapter considers three of the most fundamental and important upper bounds known for $\chi(G)$. In each case, the upper bound is a function of a basic graph parameter. Thus, a natural question is whether an upper bound for $\chi_d(G)$ exists, in terms of the same parameter, that is otherwise similar to the upper bound for $\chi(G)$.

3.1 $\chi_d \leq \Delta + 3$

The next theorem gives an upper bound for $\chi_d(G)$ that corresponds to the upper bound for $\chi(G)$ given by Brook's theorem [1] ($\chi(G) \leq \Delta + 1$, with equality holding only for complete graphs and odd cycles).

Theorem 5 *For any graph G , $\chi_d(G) \leq \Delta + 3$, and equality holds only for $G = C_5$.*

Proof: Throughout the proof, $C(j)$ denotes a set of j colors used by a typical coloring c of j colors ($C(j) \supset C(i)$ if $i < j$).

Case 1: $\Delta = 0$. Then G is K_1 , and so $\chi_d(G) = \Delta + 1$.

Case 2: $\Delta = 1$. Then G is K_2 , and so $\chi_d(G) = \Delta + 1$.

Case 3: $\Delta = 2$. Then if G is acyclic, G is a tree and so $\chi_d(G) = 3 = \Delta + 1$. If G contains a cycle, then G is a cycle, and so $\chi_d(G) \leq 5 = \Delta + 3$, with equality only for $G = C_5$.

Case 4: $\Delta = 3$. We show that $\chi_d(G) \leq 5$.

The proof is by induction on n . The result is clearly true if $n \leq 5$, so suppose $n \geq 6$. In each case, a dynamic k -coloring c , where $k \leq 5$ is the dynamic chromatic number of the indicated graph, is extended to a dynamic k' -coloring of the larger graph G , with $k \leq k' \leq 5$.

Case 4A: $\delta = 1$. Then if $d(v) = 1$, extend a coloring c of $G - v$ by choosing $c(v) \in C(5) - \{c(u), c(w)\}$, where u is adjacent to v , and $w \neq v$ is adjacent to u .

Case 4B: $\delta = 2$. Let u and v be neighbors and $d(v) = 2$. Suppose first that $d(u) = 2$. Suppose v is adjacent to $v' \neq u$, and u is adjacent to $u' \neq v$. Let $u'' \neq u$ be some neighbor of u' and $v'' \neq v$ be some neighbor of v' . Extend a coloring c of $G - \{u, v\}$ to a coloring of G by choosing $c(u) \in C(5) - \{c(v'), c(u'), c(u'')\}$ and $c(v) \in C(5) - \{c(u), c(u'), c(v'), c(v'')\}$.

Suppose next that $d(u) = 3$ and u is adjacent to u'_1, u'_2 , and v . If $d(u'_1) = d(u'_2) = 3$, extend a coloring of $G - \{u, v\}$ by choosing $c(u) \in C(5) - \{c(u'_1), c(u'_2), c(v')\}$ and $c(v) \in C(5) - \{c(u'_1), c(u), c(v'), c(v'')\}$. If $d(u'_1) = 2$ or $d(u'_2) = 2$, then extend a coloring c of $G - \{v', v\}$ by choosing $c(v) \in C(5) - \{c(u), c(v'), c(v'')\}$ and $c(v') \in C(5) - \{c(u), c(v), c(v''), c(v''')\}$, where $v''' \neq v'$ is adjacent to v'' . This completes the $\delta(G) = 2$ case.

Case 4C: $\delta(G) = \Delta(G) = 3$. Then extend a coloring c of $G - v$ for any $v \in V$ by choosing $c(v) \in C(5) - \{c(u), c(w), c(x)\}$, where v is adjacent to u, w , and x . This completes the $\Delta = 3$ case.

Case 5: $\Delta \geq 4$. We show $\chi_d(G) \leq \Delta + 2$.

Let $V(G) = \{v_1, \dots, v_n\}$, and let $G_i = G(\{v_1, \dots, v_i\})$. Note that any G_i for $i \neq 1$ may not be connected.

We now show by induction on $\Delta \leq i \leq n - 1$ that

(1) G_{i+1} has a $(\Delta + 2)$ -coloring if $\Delta + 2 - d_{G_{i+1}}(v_{i+1}) \geq 2$, in which v_{i+1} can be colored with any of two or more colors if $\Delta + 2 - d_{G_{i+1}}(v_{i+1}) \geq 3$.

We denote as follows these various neighbor sets of v_{i+1} in G_{i+1} : those of degree at least two in G_i as $X = \{x_1, \dots, x_{k_1}\}$, those of degree one in G_i as $Y = \{y_1, \dots, y_{k_2}\}$, and those of degree zero in G_i as $Z = \{z_1, \dots, z_{k_3}\}$. Let $Y' = \{y'_1, \dots, y'_{k_2}\}$, where y'_j is the neighbor of y_j in G_i .

We shall denote $\Delta + 2 - d_{G_{i+1}}(v_{i+1})$ by l . Note that $l \leq |C(\Delta + 2) - (c(X) \cup c(Y) \cup c(Z))|$.

Case 5A: $i = \Delta$. Then (1) holds, since if $c(v_i) = \sigma_i$ for $1 \leq i \leq \Delta$, then $c(v_{\Delta+1}) = \sigma_{\Delta+1}$ or $c(v_{\Delta+1}) = \sigma_{\Delta+2}$ each yield a dynamic coloring of $G_{\Delta+1}$.

Case 5B: $i \geq \Delta + 1$. Let the current coloring of G_i be denoted by c' . Let $S = \{\sigma_1, \dots, \sigma_t\}$, where $\sigma_1, \dots, \sigma_t$ are the distinct colors in $c'(Y)$ and $\{\sigma_{t+1}, \dots, \sigma_m\} = C(\Delta + 2) - c'(X) \cup c'(Z) \cup c'(Y')$.

We obtain a new coloring c of G_i by recoloring y_1, \dots, y_{k_2} (for any other vertex of G_i , c is identical to c'). Let $c(y_j) = \sigma_{i_j}$ such that $\sigma_{i_j} \neq c'(y'_j)$ and the coloring of y_j with σ_{i_j} ensures that y'_j satisfies the double-adjacency condition.

Case 5B1: $|c'(Y')| \geq 3$. Then at most one color β' from $c(Y')$ is not in $c(Y)$, since $\beta \in c(Y')$ and $\beta \notin c(Y)$ implies, for each j , $1 \leq j \leq k_2$, $c(y'_j) = \beta$ or $c(y''_j) = \beta$ for every $y''_j \neq y_j$ adjacent to y'_j in G_i . This could occur simultaneously for two colors β but not for three or more.

Since $l \geq 2$, then there exists at least one choice to color v_{i+1} from $C(\Delta + 2) - (c(X) \cup c(Y) \cup c(Z) \cup \{\beta'\})$ if $l = 2$ and at least two choices if $l \geq 3$. Each yields a dynamic coloring since, in particular, $|c(N_{G_{i+1}}(v_{i+1}))| \geq |c(Y)| \geq |c'(Y')| \geq 3$ and v_{i+1} colored differently from any y'_j ensures that each y_j satisfies the double-adjacency condition.

Case 5B2: $|c'(Y')| = 2$. Then at most one color $\beta_1 \in c'(Y')$ is not in $c(Y)$, or neither of the colors β_1 and β_2 in $c'(Y')$ is in $c(Y)$ (which occurs when, for each j , $1 \leq j \leq k_2$, y'_j is adjacent in G_i to only y_j and y''_j , and either

$c'(y'_j) = \beta_1$ and $c'(y''_j) = \beta_2$, or $c'(y'_j) = \beta_2$ and $c'(y''_j) = \beta_1$.

First consider $k_1 = k_3 = 0$. Among the four or more colors in $C(\Delta + 2) - \{\beta_1, \beta_2\}$, we use two to color the vertices in Y , so that at least two choices remain to color v_{i+1} .

If $k_1 > 0$ or $k_3 > 0$, then among the $l + k_2$ or more colors in $C(\Delta + 2) - (c(X) \cup c(Z))$, use one not in $\{\beta_1, \beta_2\}$ to color the vertices in Y , so that at least two choices remain to color v_{i+1} if $l \geq 3$ and at least one choice if $l = 2$.

Case 5B3: $|c'(Y')| = 1$. If $|c(X) \cup c(Z)| \geq 1$, then, as in the last case, use a color not in $c'(Y')$ to color the vertices in Y , so that at least two choices remain to color v_{i+1} .

Case 5B4: $|c'(Y')| = 0$. Thus $k_2 = 0$. Suppose $k_3 > 0$.

If $|c(X) \cup c(Z)| \geq 2$, then v_{i+1} can be colored with any of the $l \geq 2$ or more colors in $C(\Delta + 2) - (c(X) \cup c(Z))$ to obtain a $(\Delta + 2)$ -coloring of G_{i+1} .

If $|c(X) \cup c(Z)| = 1$, then of the five colors in $C(\Delta + 2) - (c(X) \cup c(Z))$, recolor z_1 with one and color v_{i+1} with any of the 4 other colors to obtain a $(\Delta + 2)$ -coloring of G_{i+1} .

Finally, suppose $k_3 = 0$.

If $|c(X)| \geq 2$, v_{i+1} can be colored with any of the $l \geq 2$ or more colors in $C(\Delta + 2) - c(X)$ to obtain a $(\Delta + 2)$ -coloring of G_{i+1} .

If $|c(X)| = 1$, let $\{x_1, \dots, x_{k_1}\} = \{v_{i_1}, \dots, v_{i_{k_1}}\}$, with $i_1 < \dots < i_{k_1} = i$. Since v_i is adjacent to v_{i+1} , then $\Delta + 2 - d_{G_i}(v_i) \geq 3$, and so by induction we can recolor v_i to obtain the $|c(X)| \geq 2$ case. This completes the proof of the $\Delta(G) \geq 4$ case, and thus completes the proof of $\chi_d(G) \leq \Delta(G) + 3$ for any graph G .

The bound is achieved with equality only for $G = C_5$, since we have shown that if $\Delta(G) \neq 2$, $\chi_d(G) \leq \Delta + 2$, and if $\Delta(G) = 2$, $\chi_d(G) = 5$ only for $G = C_5$. \square

3.2 $\chi_d \leq \alpha(G)$

A well-known upper bound for $\chi(G)$ in terms of the length $\alpha(G)$ of a longest cycle in G is (Corollary 8.8 of [3], p. 226): $\chi(G) \leq \alpha(G)$. This bound holds with equality for K_n , since $\chi(K_n) = n = \alpha(K_n)$ for any $n \geq 3$.

Theorem 6 *If G has a cycle, then $\chi_d(G) \leq \alpha(G)$.*

Proof: The proof is by induction on n .

The theorem is clearly true for $n \leq 3$, so suppose $n \geq 4$ and G has a cycle. Let v be a terminal vertex of a longest path P in G , say $P = vv_1v_2 \dots v_s$, where $s \geq \alpha(G) - 1$. Then $G - v$ is also connected. If $G - v$ has no cycles for every such choice of v , then G is a cycle, and so $\chi_d(G) \leq \alpha(G)$. So, suppose $G - v$ has a cycle. Thus, $\chi_d(G - v) \leq \alpha(G - v) \leq \alpha(G)$.

Note that v may be adjacent only to vertices on P (since otherwise G has a longer path than P) and in particular only to vertices in $A = \{v_1, \dots, v_{\alpha(G)-1}\}$ (since otherwise G has a cycle of length greater than $\alpha(G)$). By the same argument, $N_G(W) \subset A$ for $W = N_G(v_1) - \{v, v_2, \dots, v_{\alpha(G)}\}$.

Suppose c_1 is a dynamic coloring of $G - v$ of at most $\alpha(G)$ colors from $C(\alpha(G))$. Let c be a coloring of G defined by $c(u) = c_1(u)$ if $u \notin W \cup \{v, v_1\}$; $c(w) = c(v)$ is chosen from $C(\alpha(G)) - c(\{v_2, \dots, v_{\alpha(G)-1}\})$ for any $w \in W$, taken to be $c(v_{\alpha(G)})$ if one such choice; and $c(v_1)$ is chosen from $C(\alpha(G)) - c(\{v, v_2, \dots, v_{\alpha(G)}\})$ (nonempty because $c(v) = c(v_i)$ for some i , $2 \leq i \leq \alpha(G)$). Thus, at most $\alpha(G)$ colors are necessary to define c . Note that $c(w), c(v) \notin c(A)$ for any $w \in W$.

We now show that c is a dynamic coloring. Note that for any $w \in W$, A contains $N_G(v) \cup N_G(w)$ and A contains a neighbor of x for each vertex $x \in N_G(v) \cup N_G(w)$. Thus, v and w satisfy the adjacency condition, since $c(w) = c(v) \notin c(A)$; v and w satisfy the double-adjacency condition, since v and w are adjacent to v_1 while $c(v_1) \notin c(A - \{v_1\})$ and $A \supset N_G(v), N_G(w)$.

Also, any neighbor x of v or w satisfies the double-adjacency condition, since $c(w) = c(v) \notin c(A)$ and A contains a neighbor of x .

Note that $B = \{v, v_2, \dots, v_{\alpha(G)}\} \cup W$ contains $N_G(v_1)$. Thus, v_1 satisfies the adjacency condition, since $c(v_1) \notin c(B)$; v_1 satisfies the double-adjacency condition, since v_1 is adjacent to v and v_2 but $c(v) \neq c(v_2)$. Note that B contains, for each vertex $x \neq v, w$ in $N_G(v_1)$, a neighbor of x . Thus, any neighbor $x \neq v, w$ of v_1 satisfies the double-adjacency condition, since $c(v_1) \notin c(B)$ and B contains a neighbor of x .

Finally, since c_1 is a dynamic coloring of $G-v$, the conditions of a dynamic coloring for vertices other than v, v_1, w , and their neighbors hold also for c .
 \square

The bound holds with equality for such graphs as C_4, C_5 , and K_n for all $n \geq 3$.

A well-known upper bound for $\chi(G)$ in terms of the length $l(G)$ of a longest path in G is given in [3]: $\chi(G) \leq l(G) + 1$. The bound holds with equality for K_n for any $n \geq 2$.

As a corollary of the previous theorem, we obtain an identical upper bound for $\chi_d(G)$.

Corollary 1 *For any graph G , $\chi_d(G) \leq l(G) + 1$.*

Proof: The result is clearly true for $n = 1$ and $n = 2$. So suppose $n \geq 3$. If G is acyclic, then G is a tree, and so $\chi_d(G) = 3 \leq l(G) + 1$, since a tree has a path of length 2 if $n \geq 3$. If G contains a cycle, then since any cycle contains a path with length one less than the cycle length, $\chi_d(G) \leq \alpha(G) \leq l(G) + 1$.
 \square

The bound holds with equality for such graphs as C_4, C_5 , and K_n for all $n \geq 2$.

Chapter 4

Vertex-Critical Graphs

4.1 Proper Colorings

A *vertex-critical graph for proper colorings* is defined to be a graph G such that $\chi(G - v) < \chi(G)$ for any vertex v of G . Graphs having much symmetry are expected to be among the best candidates. So, it is perhaps not surprising to find that all complete graphs and all odd cycles are vertex-critical graphs, since $\chi(K_n - v) = n - 1 = \chi(K_n) - 1$ for any $n \geq 1$ and $\chi(C_n - v) = 2 = \chi(C_n) - 1$ for any odd $n \geq 3$.

Any graph $G = K_{2k-1} - E(C_{2k-1})$ for $\chi(G) = k \geq 3$ is also a vertex-critical graph, since for $k \geq 4$ the consecutive vertices of C_{2k-1} may be colored as $1, 1, 2, 2, \dots, k-1, k-1, k$ in an optimal coloring of G , and the consecutive vertices of P_{2k-2} of $G - v = K_{2k-2} - E(P_{2k-2})$ may be colored as $1, 1, 2, 2, \dots, k-1, k-1$. (For $k = 3$, $G = K_5 - E(C_5) = C_5$ is an odd cycle.)

It is now shown that complete graphs and odd cycles are sufficient to characterize the vertex-critical graphs if $\chi(G) = 2, 3$, or n .

Theorem 7 *Let G be a graph of n vertices. Then G is a vertex-critical graph with $\chi(G) = 2$ if and only if $G = K_2$, G is a vertex-critical graph with*

$\chi(G) = 3$ if and only if G is an odd cycle, and G is a vertex-critical graph with $\chi(G) = n$ if and only if $G = K_n$ and $n \geq 1$.

Proof: Let G be a vertex-critical graph with $\chi(G) = 2$. Since $\chi(G-v) = 1$, then $G-v$ consists of one or more isolated vertices, so that $G = K_{1,n-1}$. But $\chi(K_{1,j} - v) = 1$ only for v the central vertex, so that $G = K_{1,1} = K_2$. Of course, $\chi(K_2) = 2$.

Let G be a vertex-critical graph with $\chi(G) = 3$. Thus, G has an odd cycle C . If there exists a vertex v not on C , then since $G-v$ contains C , an odd cycle, then $\chi(G-v) \neq 2$, and G would not be vertex critical. Thus, there is no vertex not on C , i. e., $G = C_n$ and n is odd. The converse was previously noted.

Let G be a vertex-critical graph with $\chi(G) = n$. Since $\chi(G) = n$, any two vertices are adjacent, since otherwise a pair of nonadjacent vertices could be colored the same color different from each of the $n-2$ colors used to uniquely color each of the remaining vertices. Thus, $G = K_n$. The converse was previously noted. \square

4.2 Dynamic Colorings

A graph G is defined to be a *vertex-critical graph for dynamic colorings* if $\chi_d(G-v) < \chi_d(G)$ for any vertex v of G .

It is also true for dynamic colorings that all complete graphs are vertex critical. However, as is readily checked from $\chi_d(C_n)$, C_n is a vertex-critical graph if and only if n is not a multiple of three, so that a significantly higher fraction (two-thirds) of these graphs are vertex critical in the dynamic case as compared to the proper case (one-half).

Any graph $G = K_{2k-1} - E(C_{2k-1})$ for $k = \chi_d(G) \geq 4$ is a vertex-critical graph for dynamic colorings, since the proper coloring specified for G and $G-v$ is in each case also a dynamic coloring.

The Petersen graph ($\chi_d(G) = 4, \chi(G) = 3$) is vertex critical for dynamic colorings but not for proper colorings.

An example of a class of graphs vertex critical for dynamic colorings but not for proper colorings is $G = SK_k$ for $k = \chi_d \geq 4$. G is a bipartite graph with $V(G)$ partitioned as $V(G) = X \cup Y$. Each of the $\binom{k}{2}$ vertices in Y is adjacent to a unique pair of the k vertices in $X = \{x_1, \dots, x_k\}$. Thus, $\chi_d(G) \geq k$, since any two vertices in X must be colored differently, being the only two vertices adjacent to some vertex in Y . Let $c(x_i) = i$ and, if y in Y is adjacent to x_i and x_j in X , let $c(y) \in \{1, \dots, k\}$ such that $c(y) \neq i, j$. Thus, no x_i is adjacent to only vertices of the same color l , since some y in Y is adjacent to x_i and x_l and hence has $c(y) \neq i, l$. Hence, c is clearly a dynamic coloring of G , and so $\chi_d(SK_k) = k$. Let $x_m \in X$. Consider a coloring c that is a 1-1 map of $X - \{x_m\}$ onto $\{1, \dots, k - 1\}$ and for which $c(y) \in \{1, \dots, k - 1\}$ and $c(y) \neq c(x_i), c(x_j)$ if y is adjacent to x_i and x_j , $c(y) \neq c(x_i)$ if y is adjacent only to x_i . Then c is a dynamic coloring of $G - x_m$. Thus, $\chi_d(G - x_m) = k - 1$, since any two of the $k - 1$ vertices of $X - x_m$ must be colored differently, being the only two vertices adjacent to some vertex in Y . Also, $\chi_d(G - z) = k - 1$, where z in Y is adjacent to x_i and x_j in X : first, a coloring c is a dynamic coloring of $G - z$ if (except for $c(x_i) = c(x_j)$) c is a 1-1 map of X onto $\{1, 2, \dots, k - 1\}$ and has $c(y) \neq c(x_i), c(x_m)$ if y in $Y - \{z\}$ is adjacent to x_i and x_m ; second, at least $k - 1$ colors are required, since any two vertices of $X - x_i$ must be colored differently, being the only two vertices adjacent to some vertex in $Y - \{z\}$.

As was true for proper colorings, it is also possible to characterize the vertex-critical graphs for dynamic colorings if $\chi_d(G) = 2, 3$, or n .

Theorem 8 *Let G be a graph of n vertices. Then G is a vertex-critical graph with $\chi_d(G) = 2$ if and only if $G = K_2$, G is a vertex-critical graph with $\chi_d(G) = 3$ if and only if $G = K_3$ or P_3 , and G is a vertex-critical graph with*

$\chi_d(G) = n$ if and only if $G = P_3, C_4, C_5,$ or K_n .

Proof: Since $\chi_d(G) = 2$ if and only if $G = K_2$ and since $\chi_d(K_2 - v) = 1$ for each vertex v , then the first part is shown.

Clearly, for $n \leq 3$, $G = K_3$ and $G = P_3$ are the only vertex-critical graphs with $\chi_d(G) = 3$. Let $n \geq 4$ and let G be a vertex-critical graph with $\chi_d(G) = 3$. Since $\chi_d(G - v) = 2$ for any vertex v , then any component of $G - v$ is K_1 or K_2 . However, if some component is $K_1 = \{u\}$, then $d(u) = 1$ and thus $\chi_d(G - u) \geq 3$, since $G - u$ is a connected graph of at least three vertices. So, G has minimum degree of at least two. Thus, each component of $G - v$ is K_2 and v is adjacent to each vertex, so that G consists of triangles with v a vertex of each triangle and otherwise no two triangles sharing a vertex. However, any vertex u other than v is not a cut vertex of G , and so $\chi_d(G - u) \geq 3$ and hence G is not vertex-critical.

Finally, by *Theorem 17*, $\chi_d(G) = n$ if and only if $G = P_3, C_4, C_5,$ or K_n , and each has been shown to be vertex critical for dynamic colorings. \square

The vertex-critical graphs for dynamic colorings with $\chi_d(G) = 4$ may be quite difficult to characterize since already they have been shown to include such a wide variety of graphs as C_n (n not equal to 5 and n not a multiple of 3), $K_4, SK_4, K_9 - E(C_9)$, and the Petersen graph. However, one property that can be proved is that $\delta(G) \geq 2$. (Suppose $\delta(G) = 1$ and let $d(v) = 1$. Since $\chi_d(G - v) = 3$, color v differently from its adjacent vertex u and differently from some vertex $w \neq v$ adjacent to u to extend a dynamic 3-coloring of $G - v$ and obtain a dynamic 3-coloring of G , a contradiction.)

Chapter 5

Stable Graphs

A graph G is said to be *stable for proper colorings* if $\chi(G - v) = \chi(G)$ for any v , and is said to be *stable for dynamic colorings* if $\chi_d(G - v) = \chi_d(G)$ for any v . Thus, in either case, a stable graph is the antithesis of a vertex-critical graph.

5.1 Proper Colorings

We first consider stable graphs for proper colorings. Some examples of stable graphs when $\chi(G) = 2$ are the graphs C_n if and only if n is even, and any tree if and only if $n \geq 4$ and other than $K_{1,n-1}$. (Since any tree of at least two vertices has chromatic number two, then a tree G with $n \geq 2$ is not stable if and only if for some v , all the components of $G - v$ are single vertices, which holds if and only if v is adjacent to all other vertices of G , i. e., $G = K_{1,n-1}$. For $n = 2, 3$, the only tree is $K_{1,n-1}$, and for $n = 1$, the only tree is K_1 .)

Any graph $G = K_{2k} - E(C_{2k})$ is a stable graph, since the consecutive vertices of C_{2k} colored as $1, 1, 2, 2, \dots, k, k$ give a coloring of G and, for $G - v = K_{2k-1} - E(P_{2k-1})$, the consecutive vertices of P_{2k-1} colored as $1, 1, 2, 2, \dots, k - 1, k - 1, k$ give a coloring of $G - v$.

The following theorem characterizes the stable graphs having chromatic number three.

Theorem 9 *Suppose $\chi(G) = 3$. Then G is stable if and only if no vertex is in all odd cycles.*

Proof: Suppose $\chi(G) = 3$. Then $\chi(G - v) = 3$ for any v in V if and only if $G - v$ is not bipartite for any v in V if and only if $G - v$ has an odd cycle for any v in V if and only if no vertex v is in all odd cycles. \square

Thus, the Petersen graph (Figure 5.1) is stable by *Theorem 9*, since it has chromatic number three and has two vertex-disjoint 5-cycles—the outer, pentagon-shaped cycle and the inner, star-shaped cycle.

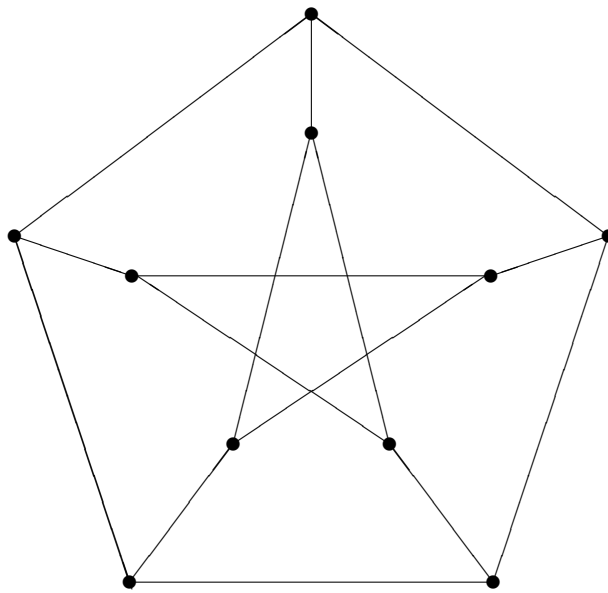


Figure 5.1: Petersen Graph

The following lemma is useful in proving properties of stable graphs.

Lemma 1 *A graph G is stable if and only if for any $\chi(G)$ -coloring of G , every color class has at least two vertices.*

Proof: If some $\chi(G)$ -coloring c of G has a color class with only one vertex v , then $\chi(G - v) = \chi(G) - 1$ as shown by the $(\chi(G) - 1)$ -coloring of c restricted to $G - v$, and so G is not stable. Conversely, if G is not stable, then $\chi(G - v) = \chi(G) - 1$ for some vertex v . Then any $(\chi(G) - 1)$ -coloring of $G - v$ combined with a new color for v yields a $\chi(G)$ -coloring of G having a color class $\{v\}$ with only one vertex. \square

Theorem 10 *Suppose G is a stable graph with n vertices. Then $\chi(G) \leq \lfloor n/2 \rfloor$.*

Proof: By Lemma 1, for a $\chi(G)$ -coloring of G , any color class has at least two vertices. Thus, $n \geq 2\chi(G)$, so that $\chi(G) \leq \lfloor n/2 \rfloor$. \square

For any $k \geq 2$, we may easily construct infinitely many stable graphs G with $\chi(G) = k$, as shown by the following theorem.

Theorem 11 *A graph G is stable if G has vertex-disjoint subgraphs H_1 and H_2 with $\chi(H_1) = \chi(H_2) = \chi(G)$.*

Proof: For any vertex v , $\chi(G - v) \geq \chi(H_i) = \chi(G)$, where H_i does not contain v . Hence, $\chi(G - v) = \chi(G)$, i. e., G is stable. \square

Note that Theorem 11 also suffices to prove (with H_1 and H_2 as 5-cycles) that the Petersen graph is stable.

Thus, Theorem 11 shows that joining two vertex-disjoint graphs of chromatic number k with edges (an edge possible between any pair of vertices having different colors) always forms a stable graph G with $\chi(G) = k$ if $k \geq 2$. Hence, if the asterisk is used to denote such a joining of graphs, then if G_1 is stable, $G_{i+1} = G_i * G_i$, $i \geq 1$, is an infinite class of graphs with each $\chi(G_i) = \chi(G_1)$.

The converse of *Theorem 11* is not true, as shown for $\chi(G) = 3$ by the following example. The graph G_1 contains the triangle $v_1v_2v_3$ (colored 1,2,3) with v_4 (colored 3) adjacent to v_1 and v_2 , v_5 (colored 1) adjacent to v_2 and v_3 , and v_6 (colored 2) adjacent to v_1 and v_3 (Figure 5.2).

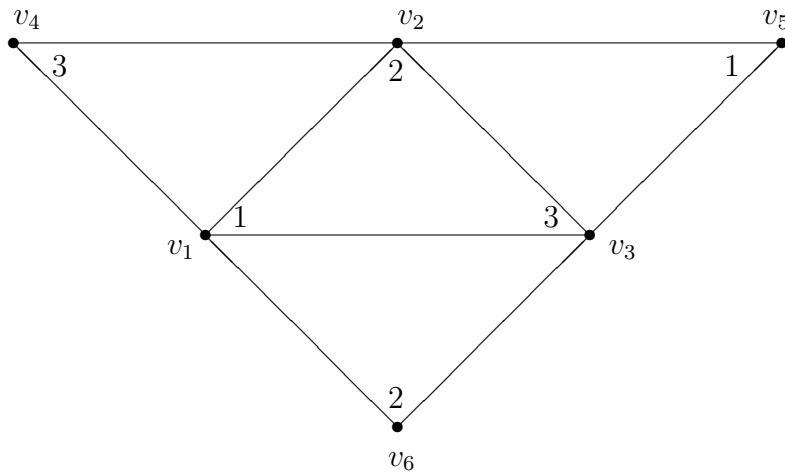


Figure 5.2: G_1

Since no vertex is in all triangles, then G_1 is stable by *Theorem 9*. Since no two triangles of G_1 are vertex disjoint, then the converse of *Theorem 11* does not hold for G_1 .

5.2 Dynamic Colorings

We now consider stable graphs for dynamic colorings.

Examples of stable graphs (for $\chi_d(G) = 3$) are C_n if and only if $n = 3j$, $j \geq 2$, and any tree (other than $K_{1,n-1}$ or the tree obtained by subdividing j edges of $K_{1,n-1-j}$ for any $j \leq \lfloor (n-1)/2 \rfloor$) if and only if $n \geq 6$. (Since any tree of at least three vertices has dynamic chromatic number three, then a tree with $n \geq 3$ is not stable if and only if for some v , $G - v$ has no components other than K_1 and K_2 , i.e., if and only if $G = K_{1,n-1}$ or G is obtained by

subdividing some of the edges in $K_{1,i}$. Since any tree with $3 \leq n \leq 5$ is easily seen to be of the excluded form, and since the only tree for $n = 1, 2$ is K_n , then the assertion is true as stated.)

Any graph $G = K_{2k} - E(C_{2k})$ for $k = \chi_d(G) \geq 3$ is a stable graph, since the proper colorings given for G and $G - v$ are also dynamic colorings.

A stable graph with $\chi_d(G) \geq 4$, $\chi_d(G) \neq \chi(G)$, is obtained by joining two vertex-disjoint graphs C_4 with an edge ($\chi_d(G) = 4$, $\chi(G) = 2$).

The proof of *Theorem 9* uses facts of regular colorings that are not true for dynamic colorings. Both implications of the theorem are false for dynamic colorings: $\chi_d(G_1 - v_5 - v_6) = 3$ and $G_1 - v_5 - v_6$ is stable, but v_1 is in all odd cycles; $\chi_d(G_2) = 3$ and no vertex is in all odd cycles, but G_2 is not stable, since $\chi_d(G_2 - v) = \chi_d(C_5) = 5$ (Figure 5.3).

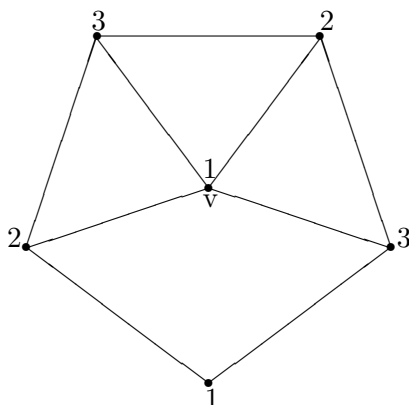


Figure 5.3: G_2

Because a dynamic coloring of G need not be a dynamic coloring of $G - v$, then the proof of *Lemma 1* does not hold for dynamic colorings. In fact, *Lemma 1* and *Theorem 10* (which was proved using *Lemma 1*) are false for dynamic colorings, as shown by the following examples. The diamond graph $G_1 - v_5 - v_6$ and $G_1 - v_6$ are each dynamic stable graphs with the colorings given for G_1 . Each coloring has the color class for the color 2 equal to $\{v_2\}$.

Also, $\chi_d(G_1 - v_5 - v_6) = 3 > 2 = \lfloor \frac{4}{2} \rfloor$ and $\chi_d(G_1 - v_6) = 3 > 2 = \lfloor \frac{5}{2} \rfloor$.

Examples are known of dynamic stable graphs with $\chi_d(G) \geq 4$ and having a color class with only one vertex. But these examples satisfy $\chi_d(G) \leq \lfloor n/2 \rfloor$, and so whether $\chi_d(G) \leq \lfloor n/2 \rfloor$ for dynamic stable graphs with $\chi_d(G) \geq 4$ remains a topic of investigation.

Theorem 11 is also false for dynamic colorings, as shown by the graph G_3 (Figure 5.4). The subgraphs $H_1 = G_3(v_1, v_2, v_3, v_4)$ and $H_2 = G_3(v_5, v_6, v_7)$ are each paths, which have 3-colorings identical to the one shown for G_3 . However, $\chi_d(G_3 - v_1) = 4$ (change the color for v_3 to 4), so that G_3 is not a stable graph.

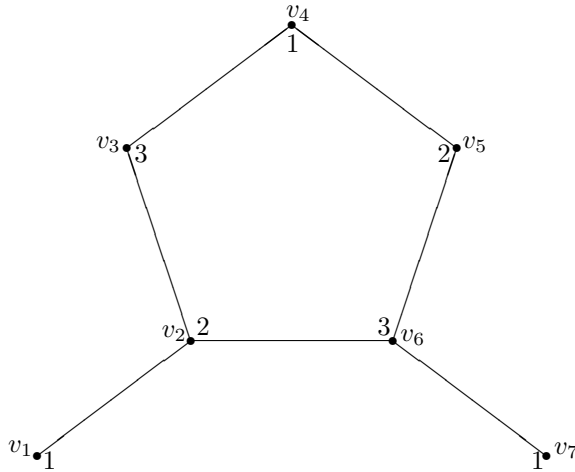


Figure 5.4: G_3

Chapter 6

Comparison of χ and χ_d

A natural question to ask is how much the difference between $\chi_d(G)$ and $\chi(G)$ varies, and the problem is investigated in this chapter. Graphs for which the two chromatic numbers are identical are the subject of the first section. The second section investigates whether the difference is bounded for all graphs. The final two sections consider classes of graphs for which the difference is small between any two graphs in the class.

6.1 Normal Graphs

A graph G is defined to be *normal* if $\chi_d(G) = \chi(G)$.

If n is odd and a multiple of three, then C_n is normal; any other cycle is not normal. Any complete graph is normal. The only trees that are normal are K_1 and K_2 .

Theorem 12 *If any vertex of degree greater than one is in a triangle, then G is normal.*

Proof: If a vertex is in a triangle, then its two neighbors in the triangle are adjacent and by the adjacency condition must be colored differently in

any proper coloring of G . Thus, any proper coloring of G is also a dynamic coloring of G , and so $\chi_d(G) = \chi(G)$. \square

The condition presented in *Theorem 12*, while sufficient for a graph to be normal, is not necessary. This is demonstrated by the following theorem, in which a method used to construct triangle-free graphs ([2], Theorem 8.7, page 129) is shown to also produce normal graphs when the initial graph is a normal graph.

Theorem 13 *For every $k \geq 1$, there exists a normal, triangle-free, k -chromatic graph.*

Proof: Let G_k denote the normal, triangle-free, k -chromatic graph. Let $G_1 = K_1$, $G_2 = K_2$, and $G_3 = C_9$.

Construct G_{k+1} from G_k by adding $n + 1$ vertices $\{u_1, \dots, u_n, v\}$ to the vertices $\{v_1, \dots, v_n\}$ of G_k and by joining u_i to each vertex v_j to which v_i is adjacent; v is joined to each u_i .

Assume a proper k -coloring of G_k is given. Then color u_i the same as v_i and color v a $(k + 1)$ st color. Then the proof that G_{k+1} is triangle free and $\chi(G_{k+1}) = k + 1$ is the same as the proof given in [2] when $G_3 = C_5$, since C_9 is also triangle free (G_{k+1} clearly has no triangles. For, since $\{u_1, u_2, \dots, u_n\}$ is an independent set in G_{k+1} , no triangles can contain more than one u_i ; and if $u_i v_j v_k u_i$ were a triangle in G_{k+1} , then $v_i v_j v_k v_i$ would be a triangle in G_k , contrary to assumption. We now show that G_{k+1} is $(k + 1)$ -chromatic. Note, first, that G_{k+1} is certainly $(k + 1)$ -colorable, since any k -coloring of G_k can be extended to a $(k + 1)$ -coloring of G_{k+1} as described above. Therefore it remains to show G_{k+1} is not k -colorable. If possible, consider a k -coloring of G_{k+1} in which, without loss of generality, v is assigned color k . Clearly, no u_i can also have color k . Now recolor each vertex v_i of color k with the color assigned to u_i . This results in a $(k - 1)$ -coloring of the k -chromatic graph G_k . Therefore G_{k+1} is indeed $(k + 1)$ -chromatic.)

Suppose that each vertex v_i of G_k has some neighbors of different colors, where $k \geq 3$. Since the neighbors of v_i are also neighbors of u_i , then u_i has some neighbors of different colors in G_{k+1} . Since each u_i is colored the same as v_i , which are not all colored the same, then v , being adjacent to each u_i , has some neighbors of different colors in G_{k+1} . Thus, by using a dynamic coloring of $G_3 = C_9$, the k -coloring of G_k is guaranteed to satisfy the double-adjacency condition for all $k \geq 3$. Since $\chi_d(C_9) = \chi(C_9)$, then by induction $\chi_d(G_k) = \chi(G_k) = k$ for $k \geq 3$. \square

Theorem 14 *Let $n \geq 3$. If $\delta(G) > \lfloor n/2 \rfloor$, then G is normal. This bound on δ cannot be lowered.*

Proof: Suppose $n \geq 3$. For any vertex v , a neighbor of v not adjacent to another neighbor of v would be adjacent to at most $n - \delta(G) \leq \lceil n/2 \rceil - 1 < \delta(G)$ vertices. Thus, any two adjacent vertices are in a triangle. Hence, by *Theorem 12*, G is normal.

Let K_{n_1, n_2} denote the complete bipartite graph, defined as having vertex set V partitioned as $X_1 \cup X_2$, $|X_i| = n_i$ for $i = 1, 2$, and having any vertex of X_1 adjacent to any vertex of X_2 as its vertex adjacencies. Let $G = K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ for $n \geq 3$. Then $\delta(G) = \lfloor n/2 \rfloor$. Since $\chi(G) = 2$ and $\chi_d(G) \geq 3$, then G is not normal. \square

6.2 $\chi_d - \chi$ is Unbounded

In this section, we consider whether there exists a constant upper bound for $\chi_d(G) - \chi(G)$ that holds for all graphs G .

One class of graphs important in showing the difference between $\chi_d(G)$ and $\chi(G)$ is the class SK_k of bipartite graphs defined as follows for $k \geq 3$. The graph SK_k has vertex set V partitioned as $X \cup Y$, where $|X| = k$ and

$|Y| = \binom{k}{2}$. Each vertex in Y is adjacent to two vertices in X , and distinct vertices in Y are adjacent to distinct pairs of vertices in X . Thus, $d(v) = k-1$ for each $v \in X$. Note that SK_k may be obtained from K_k by subdividing each edge of K_k . Then $X = V(K_k)$ and Y is the set of vertices used to subdivide the edges.

In a dynamic coloring of SK_k , any two vertices of X must be assigned different colors, since otherwise some vertex of Y would be adjacent to exactly two vertices, both assigned the same color. Hence, $\chi_d(SK_k) \geq k$. If $X = \{x_1, \dots, x_k\}$, then the coloring $c(x_i) = i$, $c(y) \in \{1, \dots, k\}$ and $c(y) \neq i, j$ if $y \in Y$ is adjacent to x_i and x_j , is a dynamic coloring of SK_k . Thus, $\chi_d(SK_k) = k$.

Thus, the difference between χ_d and χ is unbounded.

6.3 $K_{1,3}$ -Free Graphs

For $k \geq 4$, SK_k contains $K_{1,3}$, one of the smallest and simplest graphs G for which $\chi_d(G)$ and $\chi(G)$ differ, as an induced subgraph at each vertex of V . This suggests considering $K_{1,3}$ -free graphs as a possible class of graphs for which $\chi_d(G) - \chi(G)$ is bounded.

Lemma 2 *Suppose G is connected and G is $K_{1,3}$ -free. If $\chi(G) = 2$, then $\chi_d(G) \leq 4$ with $\chi_d(G) = 4$ only if G is a cycle of even length not a multiple of 3.*

Proof: Suppose $\chi(G) = 2$ and G is $K_{1,3}$ -free. Then $\Delta(G) \leq 2$, since otherwise any vertex of degree at least 3 is contained in K_3 , and so $\chi(G) \geq 3$.

If each vertex has degree 2, then G is an even cycle, since $\chi(G) = 2$. Thus, $\chi_d(G) \leq 4$, and so $\chi_d(G) = 4$ only if the cycle also has length not a multiple of 3.

Otherwise, each vertex has degree 1 or degree 2, so that G is a path. Thus, $\chi_d(G) = 2$ or $\chi_d(G) = 3$. \square

The length of a cycle is the number of vertices in the cycle, and the length of a path is one less than the number of vertices in the path.

Theorem 15 *If G is $K_{1,3}$ -free, then $\chi_d(G) \leq \chi(G) + 2$, and equality holds if and only if G is a cycle of length 5 or of even length not a multiple of 3.*

Proof: The upper bound holds as stated for any cycle C_n .

Assume henceforth that G is not a cycle.

Define an arc to be a path for which all the internal vertices have degree two. Let l denote the maximum length of an arc in G , and let u and v typically denote the endvertices of such an arc $P_{u,v}$. Let G' denote the subgraph of G induced by $V(G) - V(P_{u,v}) \cup \{u, v\}$. Let the neighborhood N_u of u be the subgraph of G induced by u and its adjacent vertices in G' . Since G is $K_{1,3}$ -free, then if $l \geq 3$, N_u and N_v are complete, which must also hold for $l = 2$ if u and v are nonadjacent.

The proof is by induction on n . The result is easily verified for $n \leq 3$.

Suppose $l = 1$. Then G has no arcs of length at least two and hence no vertices of degree two. Thus, any vertex of degree greater than one is in some K_3 , since G is $K_{1,3}$ -free. Hence, any proper coloring is a dynamic coloring, and so $\chi_d(G) = \chi(G)$.

Suppose $l = 2$.

If $\chi(G') = 1$, then G' consists of the disjoint vertices u and v , so that $G = P_{uv}$ and $\chi_d(G) = 3 = \chi(G) + 1$.

Suppose $\chi(G') = 2$. If G' is not connected, then its two components, if nontrivial, could by the lemma only be paths or even cycles, which would violate G $K_{1,3}$ -free (at u or v .) So, $G = P_{u,v}$. If G' is connected, G' is a path or even cycle. Since G is $K_{1,3}$ -free, then u and v are adjacent in the path

that is G' , and the path may have u or v as an endvertex or may extend one or two vertices past u or v . Then $\chi_d(G) = \chi(G) = 3$.

Suppose $\chi(G') \geq 3$. Hence, $\chi(G') = \chi(G)$.

Suppose some optimal dynamic coloring c of G' has $c(u) \neq c(v)$. Then $\chi_d(G) = \chi_d(G')$. Since $l = 2$, G' is not a cycle of length greater than three, hence also not an even cycle. Thus, $\chi_d(G) = \chi_d(G') \leq \chi(G') + 1 = \chi(G) + 1$, so that $\chi_d(G) \leq \chi(G) + 1$.

Suppose any optimal dynamic coloring c of G' has $c(u) = c(v)$. Since $N(u), N(v)$ are complete subgraphs and $|N(u)| \geq 3$ or $|N(v)| \geq 3$, then $|N(u)| = \chi_d(G')$ or $|N(v)| = \chi_d(G')$, since otherwise G' could be recolored by recoloring u to be in $c(G') - c(N(u))$ or, respectively, by recoloring v to be in $c(G') - c(N(v))$, yielding an optimal dynamic coloring c' of G' having $c'(u) \neq c'(v)$. Thus, $\chi_d(G') = \omega(G')$ and, since $\omega \leq \chi \leq \chi_d$ for any graph, then $\chi_d(G') = \chi(G')$. Since $\chi_d(G) \leq \chi_d(G') + 1$, then $\chi_d(G) \leq \chi(G') + 1 = \chi(G) + 1$.

Suppose $l \geq 3$.

Suppose $\chi_d(G') \leq 3$. Since the remaining vertices of P_{uv} may be colored with four colors including $c(G')$ to extend any optimal dynamic coloring c of G' to a dynamic 4-coloring of G , then $\chi_d(G) \leq 4$. By *Lemma 15*, $\chi(G) = 2$ when $\chi_d(G) = 4$ only if G is a cycle of even length not a multiple of three.

Suppose $\chi_d(G') \geq 4$. Then $\chi(G') = 1$ is not possible, since then $G = P_{uv}$ and $\chi_d(P_{uv}) = 3$. Consider $\chi(G') = 2$. If not connected, G' has two components which, by the lemma, would be incomplete graphs if nontrivial; $G = P_{uv}$ if trivial. If G' is connected and hence a path of length at least three or a cycle of length at least four, then N_u and N_v are incomplete.

Consider $\chi(G') \geq 3$. Then $\chi(G) = \chi(G')$. Also, $\chi_d(G) \leq \chi_d(G')$, since any dynamic coloring c of G' can be extended to a dynamic coloring of G by coloring the remaining vertices of P_{uv} with colors of $c(G')$ so that at most four colors of $c(G')$ color P_{uv} . If G' is a cycle, then $G' = K_3$ to ensure N_u

and N_v are complete; in this case, $\chi_d(G) = 4$ and $\chi(G) = 3$. Otherwise, $\chi_d(G') \leq \chi(G') + 1$ by the induction hypothesis. Thus, $\chi_d(G) \leq \chi_d(G') \leq \chi(G') + 1 = \chi(G) + 1$, and so $\chi_d(G) \leq \chi(G) + 1$. \square

6.4 Regular Graphs

A graph is said to be *k-regular* if each vertex of G has degree k . A *regular* graph is a graph that is k -regular for some $k \geq 1$.

Because of this uniformity of vertex degree and the symmetry it may cause in a graph, it is conjectured that $\chi_d(G) - \chi(G)$ is typically small for regular graphs. In fact, we conjecture that $\chi_d(G) - \chi(G) \leq 2$ for any regular graph.

If $k > \lfloor n/2 \rfloor$, then $\delta > \lfloor n/2 \rfloor$, and thus $\chi_d(G) - \chi(G) = 0$ is *Theorem 14* previously shown.

If $k = 1$, then $G = K_2$, and $\chi_d(G) - \chi(G) = 2 - 2 = 0$.

If $k = 2$, then G is a cycle, and $\chi_d(G) - \chi(G) = 0, 1$, or 2 for any cycle, as previously shown.

If $k = 3$, then $\chi_d(G) \leq 5$ by *Theorem 5*, and so $\chi_d(G) - \chi(G) \leq 2$ except possibly when G is bipartite.

The Petersen graph is a 3-regular graph, and $\chi_d(G) - \chi(G) = 4 - 3 = 1$.

We now investigate the case of k -regular bipartite graphs.

Theorem 16 *Suppose G is a k -regular bipartite graph for $k \geq 3$. If $n < 2^k$, then $\chi_d(G) \leq 4$.*

Proof: Since G is bipartite and k -regular, then $V(G) = (U, W)$ and $|U| = |W| = n/2$. Let $m = n/2$, $U = \{u_1, \dots, u_m\}$, and $W = \{w_1, \dots, w_m\}$.

For any particular i , $1 \leq i \leq m$, $N(w_i)$ is contained in $\binom{m-k}{\lfloor m/2 \rfloor - k}$

$\lceil m/2 \rceil$ -subsets S of U . Thus, there are at most $m \binom{m-k}{\lceil m/2 \rceil - k}$ $\lceil m/2 \rceil$ -subsets S of U that contain at least one $N(w_i)$, $1 \leq i \leq m$, so that there are at most $2m \binom{m-k}{\lceil m/2 \rceil - k}$ $\lceil m/2 \rceil$ -subsets S of U such that S or $U - S$ contains at least one $N(w_i)$.

Since $2m \binom{m-k}{\lceil m/2 \rceil - k} < \binom{m}{\lceil m/2 \rceil}$ if and only if

$$\frac{2m(m-k)!}{\lceil m/2 \rceil! (\lceil m/2 \rceil - k)!} < \frac{m!}{\lceil m/2 \rceil! \lceil m/2 \rceil!}$$

if and only if

$$2m < \frac{m(m-1) \cdots (m-k+1)}{\lceil m/2 \rceil \lceil m/2 \rceil (\lceil m/2 \rceil - 1) \cdots (\lceil m/2 \rceil - k + 1)}$$

if $n < 2^k$, then if $n < 2^k$, there is a $\lceil m/2 \rceil$ -subset S of U such that both S and $U - S$ contain no $N(w_i)$. Similarly, if $n < 2^k$, there is a $\lceil m/2 \rceil$ -subset T of W such that both T and $W - T$ contain no $N(u_i)$.

Hence, if each vertex in S , $U - S$, T , or $W - T$ is colored 1, 2, 3, or 4, respectively, then the double-adjacency condition is satisfied, since no w_i is adjacent only to vertices colored 1 or only to vertices colored 2 and no u_i is adjacent only to vertices colored 3 or only to vertices colored 4. The adjacency condition is satisfied, since G is bipartite and partition sets U and W are colored differently. \square

Chapter 7

Graphs with $\chi_d(G) = n$ or $n - 1$

We first consider graphs for which the chromatic number equals n or $n - 1$, and then consider the graphs for which the dynamic chromatic number equals n or $n - 1$.

There are only two (limited) types of graphs for which the chromatic number equals n or $n - 1$, which is now discussed.

If two vertices of G are nonadjacent, then they may be colored alike in an $(n - 1)$ -coloring of G ; hence $\chi(G) = n$ only for $G = K_n$.

Suppose $\chi(G) = n - 1$. Then \overline{G} has no clique $H = K_j$ of $j \geq 3$ vertices, since then G has a proper (not necessarily minimal) coloring c with $|c(V(H))| = 1$, showing $\chi(G) \leq |c(V(G))| \leq n - 2$. Also, \overline{G} has no “parallel edges” (i. e., $x_1x_2, y_1y_2 \in E(V(\overline{G}))$ and $|\{x_1, x_2, y_1, y_2\}| = 4$), since then G has a proper (not necessarily minimal) coloring c with $c(x_1) = c(x_2)$, $c(y_1) = c(y_2)$, showing $\chi(G) \leq |c(V(G))| \leq n - 2$.

Thus, \overline{G} has no cliques of three or more vertices and \overline{G} has no parallel edges. Hence, \overline{G} has no edges (impossible, since then $G = K_n$) or all the edges of \overline{G} have a common vertex v but otherwise any pair of edges in \overline{G} is nonadjacent (i. e., the subgraph of edges in \overline{G} is $K_{1,j}$ for some $j \geq 1$, yet $j \leq n - 2$ for v to not be an isolated vertex in the connected graph G). So,

$G = K_n - E(K_{1,j}) = K_n - vu_1 - \dots - vu_j$ for any $1 \leq j \leq n - 2$, and the $(n - 1)$ -colorings of G are the colorings c_i , $1 \leq i \leq j$, with $c_i(v) = c_i(u_i)$ and otherwise $c_i(x) \neq c_i(y)$ if $x \neq y$, i. e., $\{v, u_i\}$ is the only color class of c_i of two or more vertices.

We now investigate the graphs for which the dynamic chromatic number equals n or $n - 1$, and see that this set of graphs contains those just obtained but is more extensive and complex.

For what graphs must a different color be assigned to each vertex in a minimal coloring? If $n \geq 2$, then $G \subset K_n - e$ for an edge e implies $\chi(G) \leq \chi(K_n - e) = n - 1$. But such reasoning is not valid for dynamic colorings, since $H \subset G$ does not imply $\chi_d(H) \leq \chi_d(G)$ (for example, $\chi_d(C_5) = 5 > 4 = \chi_d(G)$ if G consists of C_5 and an additional vertex adjacent to one vertex of C_5).

Lemma 3 *Suppose $\chi_d(G) = n$. Then if u and v are not neighbors, u and v are the only neighbors of some vertex x (i.e., $u \notin N(v)$ implies $N(x) = \{u, v\}$ for some $x \neq u, v$).*

Proof: Since u and v are nonadjacent vertices, an $(n - 1)$ -coloring of G in which only u and v receive the same color must satisfy both the adjacency and double-adjacency conditions unless the double-adjacency condition is not satisfied for some vertex x adjacent to u and v . \square

We denote the path of n vertices by P_n .

Theorem 17 *The dynamic chromatic number $\chi_d(G) = n$ if and only if $G = P_3, C_4, C_5$, or K_n .*

Proof: The theorem is clearly true for $n \leq 3$, and the specified graphs satisfy $\chi_d(G) = n$ for any n .

Suppose $n \geq 4$ and $\chi_d(G) = n$. If $G \neq K_n$, suppose u_1 and u_2 are not neighbors. Hence, by Lemma 3, $N(v_1) = \{u_1, u_2\}$ for some $v_1 \neq u_1, u_2$. Since

$n \geq 4$, then $v_1 \notin N(v_2)$ for some vertex $v_2 \neq u_1, u_2, v_1$. Hence, by *Lemma 3*, $N(w) = \{v_1, v_2\}$ for some $w \neq v_1, v_2$. Since $N(v_1) = \{u_1, u_2\}$, then $w = u_1$ or $w = u_2$, say $w = u_2$ (without loss of generality), which gives $N(u_2) = \{v_1, v_2\}$

If $u_1v_2 \in E(G)$, then if $n = 4$, all possible edges of the graph have been determined, so that $G = u_1v_1u_2v_2u_1 = C_4$. If $n > 4$, then since $N(v_1) = \{u_1, u_2\}$, $v_1 \notin N(x)$ for any $x \neq u_1, u_2, v_2$. Hence, by *Lemma 3*, $N(y) = \{v_1, x\}$ for some $y \neq v_1, x$. Since $N(v_1) = \{u_1, u_2\}$, then $y = u_1$ or $y = u_2$, which is impossible since $v_2 \in N(u_1)$ and $v_2 \in N(u_2)$ (but $x \neq v_2$).

If $u_1v_2 \notin E(G)$, then $N(z) = \{u_1, v_2\}$ for some $z \neq u_1, v_2$. Also, $z \neq v_1, u_2$, since $v_1u_2 \in E(G)$. If $n = 5$, then all possible edges have been determined, so that $G = u_1v_1u_2v_2zu_1 = C_5$. If $n \geq 6$, then since $N(v_1) = \{u_1, u_2\}$, $v_1 \notin N(x)$ for any $x \neq u_1, u_2, v_2, z$. Hence, by *Lemma 3*, $N(y) = \{v_1, x\}$ for some $y \neq v_1, x$. Since $N(v_1) = \{u_1, u_2\}$, then $y = u_1$ or $y = u_2$, which is impossible, since $z \in N(u_1)$ but $z \neq v_1, x$, and since $v_2 \in N(u_2)$ but $v_2 \neq v_1, x$. \square

Now consider the case $\chi_d(G) = n - 1$. Two lemmas are useful. The first follows directly from a result in [4] showing that the upper bound of $\chi_d(G) \leq \Delta + 2$ (if $G \neq C_5$) in *Theorem 5* is achieved with equality only for certain cycles.

Lemma 4 *If $\Delta = 2$, then $\chi_d(G) \leq 5$; if $\Delta \geq 3$, then $\chi_d(G) \leq \Delta + 1$.*

Lemma 5 *Suppose $\chi_d(G) = n - 1$. Then $G = P_4$ if $\Delta = 2$, and $\Delta \geq n - 2$ if $\Delta \geq 3$.*

Proof: By *Lemma 4*, $\chi_d(G) \leq 5$ if $\Delta = 2$, so that if $\chi_d(G) = n - 1$, then $n \leq 6$. The only graph satisfying these conditions is P_4 . \square

Theorem 18 *The dynamic chromatic number $\chi_d(G) = n - 1$ if and only if G is one of the following graphs: P_4 ; $K_n - E(K_{1,j})$, $1 \leq j \leq n - 2$, for $n \geq 4$; $K_{n-1} - xy$, $n \geq 6$, and an additional vertex adjacent only to x and y ; or the four graphs shown in *Figure 7.1*.*

Proof: If $\Delta = 1$, then $G = K_2$ but $\chi_d(K_2) = 2$.

If $\Delta = 2$, then by *Lemma 5*, $G = P_4$ if $\chi_d(G) = n - 1$. Also, $\chi_d(P_4) = 3 = n - 1$.

If $\Delta \geq 3$, then by *Lemma 5*, $\Delta \geq n - 2$.

Suppose $\Delta = n - 1$. For $n \leq 3$, no graphs satisfy $\chi_d(G) = n - 1$. Let $n \geq 4$ and let $d(v) = n - 1$. Then any proper coloring is easily seen to be a dynamic coloring if $\chi_d(G) = n - 1$. First, note that v satisfies the double-adjacency condition since $G \neq K_{1,n-1}$. If $d(x) \geq 2$ and $x \neq v$, then $xy \in E$ for some $y \neq v$ implies $xyvx$ is in G , so that x satisfies the double-adjacency condition.

Hence, if $\Delta = n - 1$, then since $\chi_d(G) = \chi(G)$ for $n \geq 4$, then the graphs that satisfy $\chi_d(G) = n - 1$ are the graphs that satisfy $\chi(G) = n - 1$, i. e., $G = K_n - E(K_{1,j})$ for some j , where $1 \leq j \leq n - 2$.

Suppose $\Delta = n - 2$. For $n \leq 4$, the only graph G with $\chi_d(G) = n - 1$ is $G = P_4$. So suppose $n \geq 5$. Let $d(v) = n - 2$ and let v' be the only vertex not adjacent to v .

Since $d(v') \geq 1$, then there is a vertex x adjacent to v' that, since $\Delta = n - 2$, is not adjacent to some vertex $y \neq v', v$. Thus, a coloring of G with v, v' colored alike, x, y colored alike, and otherwise with vertices colored distinctly, is a dynamic coloring of $n - 2$ colors unless some vertex u is adjacent only to x and y (or some vertex u' is adjacent only to v and v'). Then $u = v'$, since any $u \neq v, v'$ is adjacent also to v . Since v' is the only possibility for such a vertex u , then x, y is the only such nonadjacent pair of vertices, so that G consists of $K_{n-1} - xy$ and a final vertex $u = v'$, which is adjacent precisely to x and y .

Suppose some vertex u' is adjacent only to v and v' . Consider first $n = 5$. Then the remaining two vertices, w and w' , are both adjacent to v ; vertices u', v , and v' must be colored differently.

If v' is not adjacent to w and w' , then $c(w) = c(u')$ and $c(w') = c(v')$

describes a 3-coloring of G .

If v' is adjacent to at least one of w and w' , then whichever of w and w' is adjacent to v' cannot be colored $c(v)$ or $c(v')$, and cannot be colored $c(u')$ if it alone is adjacent; also, both w and w' cannot be colored $c(u')$ if both are adjacent to v' . So, one of w, w' must be colored distinctly from $u', v,$ and v' ; coloring the other the same as u' describes a 4-coloring of G . Thus, $\chi_d(G) = 4$ for these four graphs with $n = 5$ (Figure 7.1).

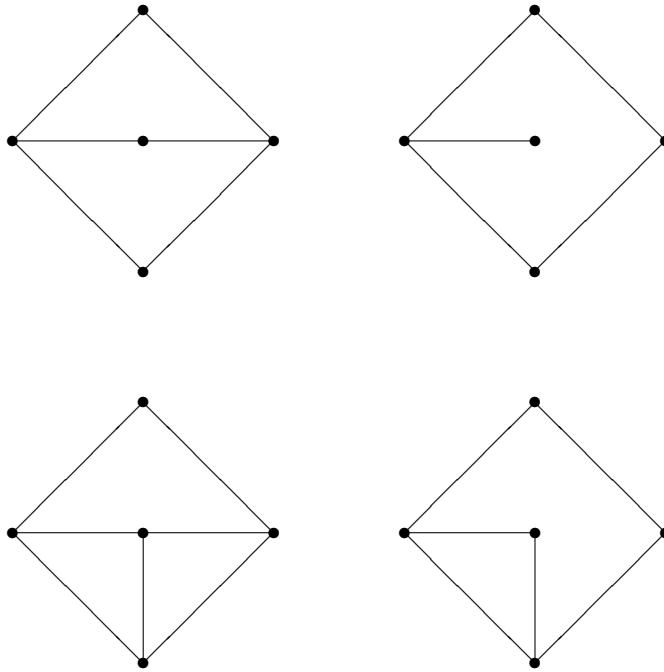


Figure 7.1: $\chi_d(G) = n - 1$ for $n = 5$

Consider $n \geq 6$. We wish to show that $G(V - u') = K_{n-1} - vv'$, where u' is adjacent only to v and v' . Suppose $G(V - u') \neq K_{n-1} - vv'$. Thus, since v is adjacent to every vertex except v' , we can suppose $G(V - \{u', v\}) \neq K_{n-2}$, and consider for $U = V - \{u', v, v'\}$ the two cases $G(U) = K_{n-3}$ and $G(U) \neq K_{n-3}$.

In each case that follows, the coloring describes an $(n - 2)$ -dynamic coloring of G . All other vertices are colored distinctly.

Suppose $G(U) = K_{n-3}$. Since v' is not adjacent to some $w \in U$, then color w the same as v' and color another vertex, which is not adjacent to v' or is not the only vertex adjacent to v' , the same color as u' .

Suppose $G(U) \neq K_{n-3}$. If v' is adjacent to no vertex in U or more than one vertex in U , color two of the nonadjacent vertices in U the same but differently from u', v , and v' ; also color another vertex in U the same as u' .

If v' is adjacent to exactly one vertex w in U , color w differently from u', v , and v' ; if w is not adjacent to some w' in U , then color w' the same as w and color another vertex the same as u' , otherwise color a nonadjacent pair of vertices the same as u' . \square

Chapter 8

$$\chi_d(G - v) = \chi_d(G) - 2$$

For any graph G , $\chi(G - v) \geq \chi(G) - 1$ for any vertex v of G , since otherwise a $(\chi(G) - 2)$ -coloring of $G - v$ with an additional color to color v would give a $(\chi(G) - 1)$ -coloring of G .

However, for $G = C_5$ or $G = P_3$, $\chi_d(G - v) = \chi_d(G) - 2$ for at least one vertex v of G . Thus, it is of interest to know what is the maximum possible decrease in the dynamic chromatic number that can occur with the removal of a vertex, and to know the graphs for which the maximum decrease can occur. These questions are answered completely by the following theorem.

Theorem 19 *For any graph G , $\chi_d(G - v) \geq \chi_d(G) - 2$ for any vertex. The only graphs for which $\chi_d(G - v) = \chi_d(G) - 2$ for at least one vertex are $K_{1,n-1}$, $n \geq 3$, and C_5 .*

Proof: Let c_1 be an optimal dynamic coloring of $G - v$ from $\{1, 2, \dots, m\}$, where $m = \chi_d(G - v)$. Let $c_2(v) = m + 1$ and $c_2(x) = c_1(x)$ if $x \neq v$. This yields a dynamic coloring c_2 of G unless $d(v) \geq 2$ and $c_1(u)$ is the same for all u in $N(v)$. In that case, c_2 is defined as before except $c_2(u_1) = m + 2$ for one vertex u_1 in $N(v)$ to obtain a dynamic coloring c_2 of G of $m + 2$ colors. Thus, $\chi_d(G) \leq m + 2 = \chi_d(G - v) + 2$.

Now an argument is begun to characterize the graphs G for which $\chi_d(G - v) = \chi_d(G) - 2$ for some vertex v of G .

Note that, as shown above, if $d(v) = 1$ or not all vertices of $N(v)$ are colored the same by some optimal dynamic coloring of $G - v$, then $\chi_d(G - v) \geq \chi_d(G) - 1$. Thus, we assume henceforth that $d(v) \geq 2$ and that $c_1(u) = 1$ for all u in $N(v)$ for any optimal dynamic coloring c_1 of $G - v$. This implies that no two vertices of $N(v)$ are adjacent, which is also assumed henceforth.

If $d(u) = 1$ for each u in $N(v)$, then $G = K_{1,n-1}$, $n \geq 3$, which satisfies $\chi_d(G - v) = \chi_d(G) - 2$.

If $d(u) = 1$ for some but not all u in $N(v)$, then a coloring c_1 of $G - v$ with $c_1(u) = c_1(w)$ for any u in $N(v)$ with $d(u) = 1$, where $w \neq v$ is a vertex adjacent to some u_1 in $N(v)$ with $d(u_1) \geq 2$, has not all vertices in $N(v)$ colored the same.

If $d(u) \geq 3$ for all u in $N(v)$, then since $c_1(u) = 1$ for all u in $N(v)$, then an $(m + 1)$ -coloring c_2 of G is given by specifying for one vertex u_1 in $N(v)$ that $c_2(u_1) = m + 1$, $c_2(v) = 2$, and $c_2(z) = c_1(z)$ otherwise.

So assume henceforth that $d(u) \geq 2$ for all u in $N(v)$ and $d(u) = 2$ for some u in $N(v)$. Let $U = \{u \in N(v) : d(u) = 2\} = \{u_i : 1 \leq i \leq |U|\}$ and let $U' = \{u \in N(v) : d(u) \geq 3\}$. Let w_i be the vertex adjacent to u_i in $G - v$ and let $W = \{w_i : 1 \leq i \leq |U|\}$ ($w_i = w_j$ if $i \neq j$ is not precluded).

Consider the following condition (to be referenced throughout the remainder of the proof):

There exists x , not in U, U', W , and adjacent to some w_i in W (*)

If (*) holds, then some w_i is adjacent to some vertex x not colored 1, say colored $b \neq 1$. For one such i , let $c_2(x) = m + 1$, $c_2(u_i) = b$ for one u_i adjacent to w_i , $c_2(v) = m + 1$, and $c_2(z) = c_1(z)$ otherwise. Then c_2 is an $(m + 1)$ -coloring of G . Thus, if (*) holds, $\chi_d(G - v) \geq \chi_d(G) - 1$.

Since G is connected, then $G - v$ is connected (otherwise, there exists an m -coloring c_1 of $G - v$ such that $c_1(u) = 2$ for all u in $N(v)$ in one

component and $c_1(u) = 1$ for all u in $N(v)$ in all other components). Let l be the minimum of all the lengths (number of edges) of shortest nontrivial paths in $G - v$ from vertices in U to vertices in $N(v)$. Thus, $l \geq 2$.

If $l = 2$, then there exists a path $u_i w_i u$ for some u in $N(v)$. Thus, by the double-adjacency condition, w_i is adjacent to some vertex $y \neq u_i, u$ satisfying $c_1(y) \neq 1$. Recoloring u_i as $c_1(y)$ gives a new m -coloring of $G - v$ with not all vertices in $N(v)$ colored the same.

If $l \geq 4$, then (*) is satisfied. So, suppose finally that $l = 3$.

If $|U'| \geq 1$, then a shortest path in $G - v$ from U to U' begins $u_i w_i x$ for some u_i in U and some x not in U, U', W . Thus, x satisfies (*).

So, suppose U' is empty, and suppose, since $d(v) \geq 2$, that $|U| \geq 2$. Then, since $l = 3$, (*) is not satisfied only if each vertex of $G - v - U$ is adjacent to precisely one vertex of U , which we henceforth assume.

If $|U| = 2$, then $l = 3$ implies that $G - v = P_3$. Then $G = C_5$, for which $\chi_d(G - v) = \chi_d(G) - 2$.

If $|U| \geq 3$, then if some $d(w_i) = 2$, note that since $G - v - u_i$ is connected, then w_i is adjacent only to u_i and to some w_j in W with $d(w_j) \geq 3$. Thus, let $c_2(u_i) = c_1(w_i)$, $c_2(w_i) = 1$, $c_2(v) = m + 1$, and $c_2(z) = c_1(z)$, otherwise, yielding an $(m + 1)$ -coloring c_2 of G . If $d(w_i) \geq 3$ for all w_i in W , then the coloring c_2 will suffice for any choice of i . \square

Chapter 9

Gainer Graphs

For any graph G , $\chi(G - v) \leq \chi(G)$, since any coloring of G is also a coloring of $G - v$. However, there exist graphs G , which we shall call *gainer graphs*, for which $\chi_d(G - v) > \chi_d(G)$ holds for at least one vertex v of G , called a *gainer vertex*. For example, consider a graph G_1 that consists of a 5-cycle $C = v_1v_2v_3v_4v_5v_1$ and a vertex v_6 adjacent only to v_1 and v_2 (Figure 9.1).

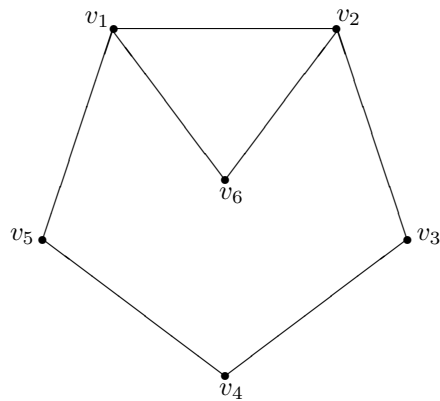


Figure 9.1: G_1

A dynamic coloring of G_1 is obtained by coloring C as 1,2,1,3,2,1 and coloring v_6 as 3. Thus, $\chi_d(G) = 3$ and, since $G_1 - v$ is C_5 , $\chi_d(G_1 - v) = 5$.

It has been previously shown that for any graph G , $\chi_d(G-v) \geq \chi_d(G)-2$, i. e., the removal of a vertex can cause the dynamic chromatic number to decrease by at most two. Thus, we first ask whether, for some positive integer k such as 2, $\chi_d(G-v) \leq \chi_d(G) + k$ holds for all graphs G and all vertices of G or, instead, whether for any given positive integer k , there exists a graph G such that $\chi_d(G-v) \geq \chi_d(G) + k$ for some vertex v of G .

The following example shows that no constant upper bound k independent of the graph can be placed on the increase in dynamic chromatic number when a vertex is removed from the graph.

Let G consist of the bipartite graph $SK_l = (X, Y)$, $l = |X| \geq 4$, for which any pair of vertices in X has a unique vertex in Y adjacent precisely to those two vertices (described in Section 4.2), with another vertex v adjacent to all the vertices in Y (which are the vertices of degree two of SK_l). Assign a color of 4 to v , a color of 3 to each vertex x_i , $1 \leq i \leq l$, of X , and a color of $((i+j) \bmod 2)+1$ to y in Y if y is adjacent to x_i and x_j . Figure 9.2 shows this graph and coloring for $l = 4$.

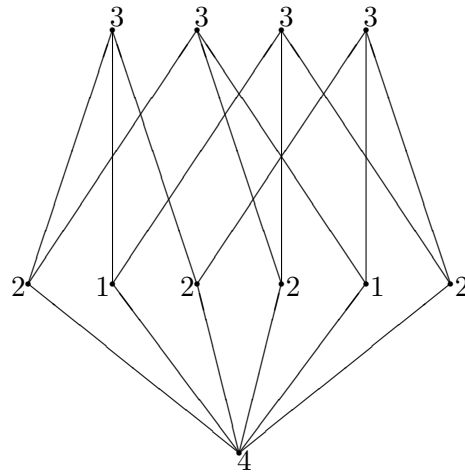


Figure 9.2: G for $l = 4$

The coloring is dynamic, so that $\chi_d(G) \leq 4$. Since $\chi_d(G-v) = \chi_d(SK_l) =$

l , then $\chi_d(G - v) \geq \chi_d(G) + k$ when $l = k + 4$.

Although no constant upper bound independent of the graph may be placed on $\chi_d(G - v) - \chi_d(G)$ for all graphs, the following theorem shows that an upper bound exists in terms of common graph parameters.

Theorem 20 *For any graph G and any vertex v of G ,*

$$\chi_d(G - v) \leq \chi_d(G) + t(v),$$

where $t(v)$ is the number of neighbors of v of degree at least 3. Thus,

$$\chi_d(G - v) \leq \chi_d(G) + d(v) \leq \chi_d(G) + \Delta.$$

Proof: Any dynamic coloring c of G of $\chi_d(G)$ colors will be a dynamic coloring of $G - v$ except that some of the $t(v)$ neighbors u of v having degree at least three in G may not satisfy the double-adjacency condition. To remedy this, use at most $t(v)$ new colors in a coloring of $G - v$ that recolors (in the coloring c induced on $G - v$) a neighbor of each u a new color. Since any new color is used only once, then the double-adjacency condition, as well as the adjacency condition, now holds for all vertices in a coloring of $G - v$ of at most $\chi_d(G) + t(v)$ colors. \square

The graph G obtained from SK_l does not satisfy with equality the upper bound stated in the theorem since $\chi_d(G - v) = \chi_d(G) + l - 4$ and $t(v) = \binom{l}{2}$. The graph G_1 is an example of a graph that satisfies with equality the upper bound for $t(v) = d(v) = 2$.

For each of the two examples of gainer graphs, only one vertex of the graph is a gainer vertex. The following theorem determines whether graphs exist for which every vertex is a gainer vertex.

Theorem 21 *A graph exists for which all vertices are gainer vertices. Such graphs do not exist if $\chi_d > 4$.*

Proof: The case for $n \leq 3$ is clear, so let $n \geq 4$ and suppose all vertices of a graph G are gainer vertices. Let c be a dynamic coloring of G . Since any v

is a gainer vertex, then any v is adjacent to some u such that $d(u) \geq 3$ and all $x \neq v$ in $N(u)$ have the same color $c(x) \neq c(v)$. Let any such pair u, v be an arc (v, u) in the *partially directed* graph G_p (denoted $(v, u) \in A(G_p)$), which also has $xy \in E(G_p)$ if $xy \in E(G)$ and $(x, y), (y, x) \notin A(G)$, and has $V(G_p) = V(G)$.

Since any v is a gainer vertex, then for any v , $(v, u) \in A(G_p)$ for at least one u . Thus, $\sum_{V(G_p)} d^+(v) \geq n$.

A vertex u cannot have more than one incoming arc, since if (v, u) , $(w, u) \in A(G_p)$, then since $(v, u) \in A(G_p)$ and $w \in N(u)$, then $c(w) = c(x)$ for all $x \neq v$ in $N(u)$, a contradiction of $(w, u) \in A(G_p)$. Thus, $\sum_{V(G_p)} d^-(v) \leq n$.

Since $\sum_{V(G_p)} d^-(v) = |A(G_p)| = \sum_{V(G_p)} d^+(v)$, then $\sum_{V(G_p)} d^-(v) = n = \sum_{V(G_p)} d^+(v)$. Thus, every vertex has precisely one incoming arc and precisely one outgoing arc, so that the arcs and vertices of G_p form one or more vertex-disjoint directed cycles.

Suppose $\chi_d(G) \geq 5$. Let $C = v_1v_2 \dots v_kv_1$ be one of the vertex-disjoint cycles. The only vertex of $G - v_i$ that may not satisfy the double-adjacency condition is v_{i+1} (addition for indices is mod k). Recolor v_{i+2} to be different from $c(v_{i+1})$, different from $c(v)$ for any $v \in N(v_{i+1}) - \{v_{i+2}\}$, different from $c(v)$ for any $v \in N(v_{i+2}) - \{v_{i+1}\}$ and, if $|C| \geq 4$, different from $c(v)$ for any $v \in N(v_{i+3}) - \{v_{i+2}\}$. For each of the last three cases, $c(v)$ is the same for all v in the specified set. This coloring is easily seen to be a dynamic coloring of $G - v_i$, and since $\chi_d(G) \geq 5$, a color does exist to recolor v_{i+2} among the $\chi_d(G)$ colors that were used to color G . Thus, $\chi_d(G - v_i) \leq \chi_d(G)$, a contradiction of v_i being a gainer vertex of G .

Consider the graph G that consists of two vertex-disjoint three-cycles and three additional nonadjacent edges joining pairs of vertices in different cycles. Then $\chi_d(G) = 3$ and it is easily seen that any vertex of G is a gainer vertex. \square

Chapter 10

A Generalization of Dynamic Coloring

The concept of dynamic coloring may be generalized to *r-dynamic coloring* by generalizing the double-adjacency condition in the appropriate fashion.

An *r-dynamic coloring* of a graph G is thus a map c from V to the set of colors such that

(C1) If $uv \in E(G)$, then $c(u) \neq c(v)$, and

(C2) For each vertex $v \in V(G)$, $|c(N(v))| \geq \min\{r, d(v)\}$

The second condition is called the *multiple-adjacency condition*.

The minimum k for which G has an r -dynamic k -coloring is the *r-dynamic chromatic number* $\chi_r(G)$. Thus, $\chi_d(G)$, as denoted previously, is identical to what is now denoted as $\chi_2(G)$.

From (C1) and (C2), it immediately follows that $\chi_r(G) \geq \min\{r, \Delta\} + 1$.

Since any r -dynamic coloring of G is a t -dynamic coloring of G if $r > t \geq 1$, then $\chi_r(G) \geq \chi_t(G)$ if $r > t \geq 1$ for any graph G .

Thus, it is easily deduced that $\chi_r(K_n) = n$ for any $r \geq 2$ and that for any particular $n \geq 3$, the chromatic number of C_n is the same for all $r \geq 2$.

Theorem 22 *If $r \geq 2$, then*

$$\chi_r(C_n) = \begin{cases} 5 & \text{if } n = 5 \\ 3 & \text{if } n = 3k, k \geq 1 \\ 4 & \text{otherwise} \end{cases}$$

Proof: For any $r \geq 2$, $\chi_r(C_n) \geq \chi_2(C_n)$, and these values for $r = 2$ and each n (as stated and proved in *Theorem 4*) are as given above. Since in each case the proof of *Theorem 4* gives a coloring of the cycle having the indicated number of colors and this coloring is an r -dynamic coloring for any $r \geq 2$, then the same values are obtained for $\chi_r(C_n)$ for any $r \geq 2$. \square

For the case of trees, a more general result holds than the one previously obtained for $r = 2$.

Theorem 23 *For any tree, $\chi_r(G) = \min\{\Delta, r\} + 1$.*

Proof: The proof is by induction on n .

The result is clearly true for $n = 1$ or $n = 2$.

Let G be a tree of $n \geq 3$ vertices and let v be a vertex of degree one, adjacent to some vertex u . By the induction hypothesis, the tree $G - v$ has a coloring c of $\chi_r(G - v)$ colors. A coloring for G is now obtained by choosing an appropriate color for v to extend the coloring.

Suppose first that $\Delta(G) \leq r$. Choose a color for v that is different from $c(u)$ and the at most $\min\{d_G(u) - 1, \Delta(G - v)\}$ colors of the other neighbors of u . If $d_G(u) \leq \Delta(G) = \Delta(G - v)$, this gives a coloring of G of $\chi_r(G - v) = \Delta(G - v) + 1 = \Delta(G) + 1$ colors; if $d_G(u) = \Delta(G) = \Delta(G - v) + 1$, this gives a coloring of G of $\chi_r(G - v) + 1 = \Delta(G - v) + 2 = \Delta(G) + 1$ colors. Thus, a coloring of $\Delta(G) + 1$ colors has been obtained, which is minimal and satisfies the theorem, since $\chi_r(G) \geq \min\{r, \Delta(G)\} + 1 = \Delta(G) + 1$ in this case.

Suppose finally that $\Delta(G) \geq r + 1$. Then $\Delta(G - v) \geq r$ and so $\chi_r(G - v) = r + 1$ by the induction hypothesis. Choose a color for v different from $c(u)$

and the at most $\min\{r, d_{G-v}(u)\} - 1 \leq r - 1$ colors of the other neighbors of u to obtain a coloring of G of $r + 1$ colors, which is minimal and satisfies the theorem since $\chi_r(G) \geq \min\{r, \Delta(G)\} + 1 = r + 1$ in this case. \square

Theorem 24 *If $k \geq r + 1$, then $\chi_r(K_{i_1, \dots, i_k}) = k$ if each $i_j \geq 1$.*

Proof: The unique proper coloring of K_{i_1, \dots, i_k} of k colors is also an r -dynamic coloring and thus $\chi_r(K_{i_1, \dots, i_k}) = \chi(K_{i_1, \dots, i_k}) = k$. \square

A graph G is defined to be r -normal if $\chi_r(G) = \chi(G)$.

Any cycle C_n , where n is odd and a multiple of three, is r -normal for any $r \geq 2$. All other cycles are not r -normal for any $r \geq 2$. Any complete graph is r -normal for any $r \geq 2$. The only trees that are r -normal for at least one value of r are K_1 and K_2 .

Theorem 25 *If any vertex v of a graph G is contained in K_k for some $k \geq \min\{r, d(v)\} + 1$, then G is r -normal.*

Proof: For any proper coloring c of G , $|c(N(v))| \geq \min\{r, d(v)\}$ by the adjacency condition. Thus, c satisfies the multiple-adjacency condition and hence is also an r -dynamic coloring. Thus, $\chi_r(G) = \chi(G)$ and so G is r -normal. \square

As was shown for $r = 2$ in *Theorem 14*, if $\delta(G)$ is sufficiently large, then G is r -normal.

Theorem 26 *If $\delta \geq \lfloor (r - 1)n/r \rfloor + 1$, then $\chi_r(G) = \chi(G)$. The lower bound on δ is the best possible.*

Proof: It suffices by *Theorem 25* to show that any vertex is contained in a complete subgraph of $r + 1$ vertices. Suppose $\delta \geq \lfloor (r - 1)n/r \rfloor + 1$.

For any vertex v_1 , there exists a vertex v_2 not in the set S_{v_1} of vertices nonadjacent to v_1 , and in general there exists a vertex v_t not in the set $\bigcup_1^{t-1} S_{v_i}$

(so that each v_i in $\{v_1, \dots, v_t\}$ is adjacent to any other vertex in $\{v_1, \dots, v_t\}$) if $\sum_1^{t-1} |S_{v_i}| < n$. Since $|S_{v_i}| \leq n - (\lfloor (r-1)n/r \rfloor + 1) = \lfloor n/r \rfloor - 1$, then $\sum_1^r |S_{v_i}| \leq r(\lfloor n/r \rfloor - 1) < n$.

If $n \geq r + 2$, then $G = K_{i_1, \dots, i_r}$, where $i_1, \dots, i_j = \lfloor n/r \rfloor, i_{j+1}, \dots, i_r = \lceil n/r \rceil$ and $j = \lceil n/r \rceil r - n$, has $\delta(G) = \lfloor (r-1)n/r \rfloor$. Also, $\chi_r(G) \geq r + 1$ since, otherwise, $\chi_r(G) = \chi(G) = r$ and since G is colored uniquely with r color classes, then $|c(N(v))| = r - 1 < \min\{r, d(v)\}$ for any v , a contradiction. \square

We now consider the problem, treated previously for $r = 2$, of what graphs G satisfy $\chi_r(G) = n$ if $r \geq 2$.

Theorem 27 *For any $r \geq 2$, $\chi_r(G) = n$ if and only if any two nonadjacent vertices of G are adjacent to a vertex of degree at most r .*

Proof: Suppose $\chi_r(G) = n$. If the stated condition is not satisfied for vertices u and w , then a coloring c of G of $n - 1$ colors in which only u and w are colored the same clearly satisfies the adjacency condition; c also satisfies the multiple-adjacency condition, since for any v not adjacent to both u and w , $|c(N(v))| = |N(v)| = d(v) \geq \min\{r, d(v)\}$, and for any v adjacent to both u and w , $d(v) > r$, and so $|c(N(v))| = |N(v)| - 1 = d(v) - 1 \geq \min\{r, d(v)\}$. Thus, $\chi_r(G) \leq n - 1$, a contradiction.

Suppose the stated condition holds but $\chi_r(G) \leq n - 1$. Then some r -dynamic coloring c of G of $\chi_r(G)$ colors has $c(u) = c(w)$ for two nonadjacent vertices u and w . Thus, u and w are adjacent to some vertex v such that $d(v) \leq r$. Hence, $d(v) > |c(N(v))| \geq \min\{r, d(v)\} = d(v)$, a contradiction. \square

Theorem 27 can be useful for specifying particular graphs satisfying $\chi_r(G) = n$ for $r \geq 2$. For example, P_3, C_4, C_5 , and K_n are immediately seen to satisfy the condition of *Theorem 27*. Hence, they each satisfy $\chi_r(G) = n$ for $r \geq 2$; this also follows from *Theorem 18* and the fact $\chi_r(G) \geq \chi_2(G)$ if $r \geq 3$.

Since for all graphs of $n = 4$ vertices other than P_4 , any pair of vertices has a common neighbor (of degree at most $\Delta(G) \leq n - 1 = 3$), then precisely all five graphs of four vertices other than P_4 satisfy $\chi_r(G) = n$ for $r \geq 3$.

Similarly, for $n = 5$ and $r \geq 4$, the only graphs not satisfying $\chi_r(G) = n$ are precisely those in which some nonadjacent vertices have no common vertices, i. e., K_3 with an end of P_3 adjoined, C_4 with an end of P_2 adjoined, $C_4 + e$ with an end of P_2 adjoined to a low-degree vertex, and any tree other than $K_{1,4}$.

Theorem 27 allows us to deduce that the only trees satisfying $\chi_r(G) = n$ are $K_{1,n-1}$ for $n \leq r + 1$, and to deduce that $\chi_r(K_n - e) = n$ if and only if $n \leq r + 1$.

Summary

The number of colors in a minimal dynamic coloring is called the *dynamic chromatic number*.

One topic of interest was the value of the dynamic chromatic number for important graphs, which was investigated in Chapter 2.

For any of the important theorems involving the chromatic number, a natural question is whether similar theorems exist for the dynamic chromatic number. This was addressed in four different areas by Chapters 3-5. Chapter 3 presented three upper bounds on the dynamic chromatic number that are each quite similar to the corresponding upper bound on chromatic number. Chapter 4 presented results characterizing dynamic vertex-critical graphs for particular values of dynamic chromatic number, and these are similar to results for vertex-critical graphs. Chapter 5 showed that the important theorems for stable graphs for the case of proper colorings do not hold for the case of dynamic colorings.

The difference between the dynamic chromatic number and the chromatic number of graphs is of interest, especially identifying classes of graphs for which the difference is small and identifying whether the difference is bounded for all graphs. Chapter 6 addressed these topics.

The idea of a dynamic coloring is sufficiently different to introduce interesting new topics of graph coloring. Three such new topics were addressed in the next three chapters. Chapter 7 showed that the process of specify-

ing graphs for which the dynamic chromatic number equals the number of vertices (or one less) is much more complex than for the non-dynamic case. Chapter 8 obtained a characterization of the graphs whose dynamic chromatic number decreases by two with the removal of a vertex. Chapter 9 investigated graphs for which the removal of any vertex causes the dynamic chromatic number to increase.

Chapter 10 introduced a generalization of dynamic coloring. Some of the topics that have been investigated for dynamic coloring in previous chapters were investigated in this chapter for the more general case.

Recommendations

Three upper bounds on the dynamic chromatic number were proved in Chapter 3. Additional upper bounds should be investigated, such as an upper bound on the dynamic chromatic number in terms of the number of edges and an upper bound on the sum of the dynamic chromatic number and the dynamic chromatic number of the complement.

The vertex-critical graphs for dynamic chromatic number less than three or equal to the graph order were completely specified in Chapter 4. Describing the vertex-critical graphs for other dynamic chromatic numbers is recommended.

The difference between the dynamic chromatic number and chromatic number was shown in Chapter 6 to be small for certain regular bipartite graphs, and the case for any regular graph is recommended for investigation.

Gainer graphs were shown in Chapter 9 to exist for dynamic chromatic number equal to three and not to exist for dynamic chromatic number greater than four. The case for dynamic chromatic number equal to four is recommended for investigation.

The r -dynamic colorings were introduced and briefly investigated in Chapter 10, and additional investigation is recommended, especially in the areas studied for dynamic colorings.

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