2008

Pierce-Engel hybrid expansions

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Dissertation submitted to the
Eberly College of Arts and Sciences
at West Virginia University
in partial fulfillment of the requirements
for the degree of

Doctor of Philosophy
in
Mathematics

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Morgantown, West Virginia
2008

Keywords: Pierce expansion, Engel expansion

ABSTRACT

Pierce-Engel Hybrid Expansions

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Pierce and Engel expansions are representations of numbers between 0 and 1 as sums of unitary fractions (of alternating signs in the case of Pierce) whose denominators are built multiplicatively, choosing the successive factors greedily. We show some results for Pierce expansions, and investigate the idea of hybrid expansions, which are built similarly but without regard to the signs of the terms.

Dedication

Trust in the Lord with all your heart and lean not on your own understanding: in all your ways acknowledge Him, and He will make your paths straight. Proverbs 2: 5-6

First and foremost, I thank God for not only opening paths for me, but giving me the strength to walk them.

Many thanks to Dr. Mays, whose time and insights have proven immensely valuable. I would also like to thank the other members of my committee for their comments and support.

I would be remiss not to extend my gratitude to Professor Michael Berry:
PB - YFAGHMMTMTYWEK.

Finally, my eternal thanks to my husband Mark and to my family, who have lovingly and patiently supported me through this endeavor.

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1 Pierce and Engel Expansions

1.1 Introduction

The Pierce expansion of a number $0 < x \leq 1$ is the unique way of writing

$$x = \frac{1}{q_1} - \frac{1}{q_1 q_2} + \frac{1}{q_1 q_2 q_3} - \cdots,$$

where the sequence q_n is a strictly increasing sequence of positive integers. Similarly, the Engel expansion is the unique representation

$$x = \frac{1}{q_1} + \frac{1}{q_1 q_2} + \frac{1}{q_1 q_2 q_3} + \cdots,$$

where q_n is an increasing sequence of positive integers.

The terms of the sequences q_n for the respective expansions of a rational number $\frac{a}{b}$ can be found using variations on the Euclidean algorithm, or division algorithm. Without loss of generality, hereafter we will assume that $a \leq b$. The well-known algorithm is as follows:

THE EUCLIDEAN ALGORITHM: Given a pair of positive integers, b, a, there exist a unique quotient q and a unique remainder r with $0 \leq r < a$ such that

$$b = aq + r.$$

Iterating the division algorithm, we get a sequence of expressions

$$b = aq_1 + r_1$$
$$a = r_1 q_2 + r_2$$

$$r_1 = r_2 q_3 + r_3$$
$$\vdots$$
$$r_{n-2} = r_{n-1} q_n + 0,$$

where each r_i satisfies $0 \leq r_i < r_{i-1} < a$. (The greatest common divisor of (b, a) arises as the last non-zero remainder, namely r_{n-1}). The quotients produced by this algorithm also generate the continued fraction expansion of $\frac{a}{b}$.

By changing the way that we iterate this algorithm, we find a tool for devising the quotients necessary for the Pierce expansion of $\frac{a}{b}$, iterating for each pair (b, r_i) as follows:

THE PIERCE ALGORITHM:

$$b = aq_1 + r_1$$
$$b = r_1 q_2 + r_2$$
$$b = r_2 q_3 + r_3$$
$$\vdots$$
$$b = r_{n-1} q_n + 0,$$

where each r_i satisfies $0 \leq r_i < r_{i-1} < a$. [3]

The sequence q_i produces the unique Pierce expansion for the quotient $\frac{a}{b}$, given a pair of positive integers b, a. This sequence terminates if and only if $\frac{a}{b}$ is rational. [4]

The algorithm for finding the Engel expansion of a rational number is also closely related to the division algorithm, with the exception that the quotients are chosen so that the remainder is negative (with magnitude less than that of a.)

THE ENGEL ALGORITHM:

$$
\begin{aligned}
b &= aq_1 - r_1 \\
b &= r_1 q_2 - r_2 \\
b &= r_2 q_3 - r_3 \\
&\vdots \\
b &= r_{n-1} q_n - 0,
\end{aligned}
$$

where each r_i satisfies $0 \leq r_i < r_{i-1} < a$.

It should be immediately clear that for either of these algorithms, any two equivalent fractions $\frac{a}{b}$ and $\frac{ka}{kb}$ will yield precisely the same sequence of quotients q_i, and thus have the same expansion. As a consequence, we may assume that $\frac{a}{b}$ is in lowest terms.

1.2 Notable Results

Pierce expansions, while not unexplored, still hold a certain mystery. Perhaps the most intriguing question revolves around predicting the length of the expansion. As in the work of Erdös and Shallit [1], we define a function $P(b, a)$ such that for each pair b, a, $P(b, a)$ is equal to the number of terms in the Pierce expansion (or, equivalently, the number of steps for the Pierce algorithm to terminate). As an example, consider $b = 7, a = 4$:

$$7 = 4(1) + 3$$

$$7 = 3(2) + 1$$

$$7 = 1(7) + 0.$$

Since the algorithm terminates after three steps, we say $P(7,4) = 3$. The expansion also has three terms: $\frac{4}{7} = \frac{1}{1} - \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 7} = \frac{1}{1} - \frac{1}{2} + \frac{1}{14}$.

Below is a short table of values for $P(b,a)$ for $1 \leq a \leq b \leq 25$.

$b\|a$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1	1																								
2	1	1																							
3	1	2	1																						
4	1	1	2	1																					
5	1	2	3	2	1																				
6	1	1	1	2	2	1																			
7	1	2	2	3	3	2	1																		
8	1	1	2	1	3	2	2	1																	
9	1	2	1	2	3	2	3	2	1																
10	1	1	2	2	1	3	3	2	2	1															
11	1	2	3	4	2	3	5	4	3	2	1														
12	1	1	1	1	2	1	3	2	2	2	2	1													
13	1	2	2	2	3	2	3	4	3	3	3	2	1												
14	1	1	2	2	3	2	1	3	4	3	3	2	2	1											
15	1	2	1	2	1	2	2	3	3	2	3	2	3	2	1										
16	1	1	2	1	2	2	2	1	3	3	3	2	3	2	2	1									
17	1	2	3	2	3	4	4	2	3	5	5	4	3	4	3	2	1								
18	1	1	1	2	2	1	3	2	1	3	4	2	3	3	2	2	2	1							
19	1	2	2	3	4	2	5	3	2	3	4	6	3	5	4	3	3	2	1						
20	1	1	2	1	1	2	3	2	2	1	3	3	4	3	2	2	3	2	2	1					
21	1	2	1	2	2	2	1	3	2	2	3	3	4	2	3	3	3	2	3	2	1				
22	1	1	2	2	2	3	2	4	3	2	1	3	4	5	3	4	3	3	3	2	2	1			
23	1	2	3	4	4	5	3	4	5	4	2	3	5	6	5	4	6	5	5	4	3	2	1		
24	1	1	1	1	2	1	2	1	2	2	2	1	3	3	3	2	3	2	3	2	2	2	2	1	
25	1	2	2	2	1	2	3	2	4	2	3	2	3	4	3	5	3	4	3	2	3	3	3	2	1

Some natural first questions are to wonder if it is possible to use known values of $P(b,a)$ to predict the value for certain other pairs, to consider the properties of the sequence $P(b,a)$, and so forth. The following observations are relevant in responding to such inquiries.

Theorem 1.1. *For fixed* $b > 0$, *if* $max_c P(b,c) = n$, *then for each* $j \leq n, \exists a$ *such that* $P(b,a) = j$.

Proof. Since $max_c P(b,c) = n, \exists d$ such that $P(b,d) = n$. Then applying the Pierce algorithm, we have

$$b = dq_1 + r_1$$

$$b = r_1 q_2 + r_2$$

$$b = r_2 q_3 + r_3$$

$$\vdots$$

$$b = r_{n-1} q_n + 0,$$

from which it is clear that

$$P(b, r_1) = n - 1,$$

$$P(b, r_2) = n - 2,$$

$$\vdots$$

$$P(b, r_{n-1}) = 1.$$

□

Theorem 1.2. *If $P(b, a) = n$ where $a < b - a$ (or equivalently, $a < \frac{b}{2}$), then $P(b, b - a) = n + 1$.*

Proof. Apply the Pierce algorithm to the pair $(b, b - a)$:

$$b = (b - a) \cdot \lfloor \frac{b}{b - a} \rfloor + a.$$

(Note that $\lfloor \frac{b}{b-a} \rfloor = 1$.) This is merely the first step. To compute $P(b, b-a)$, the problem falls to finding $P(b, a)$. Thus $P(b, b-a) = n + 1$. $\square$

Recalling that equivalent fractions share a Pierce expansion, the two theorems below consider some simple congruence classes.

Theorem 1.3. *For $b > 3$,*

(i) $P(b, \frac{b}{2}) = 1$ if b is even.

(ii) $P(b, \lfloor \frac{b}{2} \rfloor) = 2$ if b is odd.

(iii) $P(b, \lceil \frac{b}{2} \rceil) = 3$ if b is odd.

Proof. (i) If b is even, the result is obvious: $b = \frac{b}{2} \cdot 2 + 0$, so $P(b, \frac{b}{2}) = 1$. If b is odd, then

$$b = \lfloor \frac{b}{2} \rfloor \cdot 2 + 1$$

$$b = 1(b) + 0.$$

and

$$b = \lceil \frac{b}{2} \rceil \cdot 1 + \frac{b-1}{2}$$

$$b = \frac{b-1}{2} \cdot 2 + 1$$

$$b = 1(b) + 0.$$

$\square$

Theorem 1.4. *If $b \equiv 1 \pmod{k}$ and $\frac{b}{k} \geq 2$, then $P(b, \lfloor \frac{b}{k} \rfloor) = 2$.*

Proof. $b \equiv 1 \pmod{k}$ implies that $b = km + 1$ for some non-negative integer m. Then $b = \lfloor \frac{km+1}{k} \rfloor \cdot k + 1$, or $b = 1(b) + 0$ Thus $P(b, \lfloor \frac{b}{k} \rfloor) = 2$. $\square$

Theorem 1.5. *If $b \equiv 2 \pmod{k}$ and $\frac{b}{k} \geq 3$, then $P(b, \lfloor \frac{b}{k} \rfloor) = 2$ if b is even, 3 if b is odd.*

Proof. $b \equiv 2 \pmod{k}$ implies that $b = km + 2$ for some non-negative integer m. Then

$$b = \lfloor \frac{km+2}{k} \rfloor \cdot m + 2$$

$$b = 2 \cdot \lfloor \frac{b}{2} \rfloor + r_2.$$

Now if b is even, $r_2 = 0$, so $P(b, \lfloor \frac{b}{k} \rfloor) = 2$. If b is odd, $r_2 = 1$, and so $r_3 = 0$: thus $P(b, \lfloor \frac{b}{k} \rfloor) = 3$. □

These results describe the behavior of certain diagonals in the table of Pierce values given previously. For example, for $k = 3$, we could take the values $b = 11, 14, 17, 20, ...$ (all congruent to 2 mod 3), and a diagonal of Pierce values alternating between 2 and 3 when we consider $P(b, \lfloor \frac{b}{k} \rfloor)$. The restrictions on b, k are severely limiting, as they require the diagonals to lie within the leftmost third of the table.

For a general base b, we have the following result:

Theorem 1.6. *$P(b^m - 1, b^{m-k}) = 2 + P(b^m - 1, b^k - 1)$ for $0 < k < \frac{m}{2}$, $m > 2, b > 2$.*

Proof.

$$b^m - 1 = b^{m-k}(b^k - 1) + (b^{m-k} - 1)$$

$$b^m - 1 = (b^{m-k} - 1)(b^k) + (b^k - 1)$$

And the remaining steps of the algorithm fall to computing $P(b^m - 1, b^k - 1)$. □

Considering the possibilities for b's congruence modulo a, we can arrive at several tedious and impractical results, which we omit here. Clearly, $P(b,a)$ can attain its maximum value if the remainders decrease as slowly as possible, as noted in [3]. To do this, b should satisfy the following system of congruences.

Theorem 1.7. *If* $b \equiv k \pmod{a}, b \equiv k-1 \pmod{k}, b \equiv k-2 \pmod{k-1}, \ldots, b \equiv 1 \pmod{2}$*, then* $P(b,a) = k+1$.

Proof. Since the sequence of remainders will decrease by 1 at each step, this process has $k+1$ steps, so $P(b,a) = k+1$ as desired. $\square$

This may be useful in constructing values of b, using a, to achieve the maximum $P(b,a)$. The simplest examples are to take $b = a! - 1$ or $b = lcm(1,2,\ldots,a) - 1$. By the previous theorem, $P(b,a) = a$ for either choice. However, both force b to be significantly larger than the "champion" for $P(x,y) = a$ – that is, the smallest pair (x,y) such that $P(x,y) = a$. [3] To find a champion, then, we would like to find a pair b, a with a small b that produces a large first remainder, and thus a large possible value for $P(b,a)$. We also want to choose b so that it will NOT satisfy all these congruences! Rather, we want to keep the remainders large, but not quite large enough to satisfy the congruences above. This paradoxical convolution makes finding champions quite difficult. Numerical results have shown the champion pairs b, a for $P(b,a) = n, 1 \leq n \leq 49$, requiring denominators as large as 1,371,719. [2] As n marches toward infinity, it is also known that champion pairs satisfy $\lim \frac{b_n}{a_n} = \frac{e}{e-1}$. [3]

Analyzing specific systems of congruences as we did above finds a sequence of $P(b,a)$ that is constant, or has period 1. In fact, the lengths of the Pierce expansions for well-chosen sequences of $\frac{a}{b}$ is always periodic:

Lemma 1.8. *For any rational* $\frac{c}{d} = \frac{1}{q_1} - \frac{1}{q_1 \cdot q_2} + \ldots + \frac{(-1)^{n+1}}{q_1 \cdot q_2 \cdots q_n} = \frac{q_2 \cdot q_3 \cdots q_n - q_3 \cdot q_4 \cdots q_n + \ldots + (-1)^{n+1}}{q_1 \cdot q_2 \cdots q_n}$*, and any* (b,a) *pair whose Pierce expansion agrees up through* q_n*, the Pierce expansions of each of the expressions in the sequence* $\{\frac{a+cz}{b+dz} : z \in Z^+\}$ *all begin with the given expansion for* $\frac{c}{d}$*.*

Proof. Since $\frac{a}{b}$ agrees with $\frac{c}{d}$ through q_n, we have

$$b = aq_1 + r_1$$

$$b = r_i q_{i+1} + r_{i+1},$$

where each $r_i < r_{i-1}, r_0 = a$, for $0 \leq i < n+k$ for $P(b,a) = n+k$. Then

$$\frac{a+c}{b+d} = \frac{q_2 \cdot q_3 \cdots q_{n+k-1} - q_3 \cdot q_4 \cdots q_{n+k-1} + \cdots + (-1)^{n+k} + c}{q_1 q_2 \cdots q_{n+k} + q_1 q_2 \cdots q_n}$$

$$\frac{a+c}{b+d} = \frac{c(q_{n+1}q_{n+2} \cdots q_{n+k-1} + 1)}{d(q_{n+1}q_{n+2} \cdots q_{n+k} + 1)}.$$

We can bound this rational by $\frac{cq_{n+1}q_{n+2}\cdots q_{n+k-1}}{d(q_{n+1}q_{n+2}\cdots q_{n+k}+1)} < \frac{a+c}{b+d} < \frac{c(q_{n+1}q_{n+2}\cdots q_{n+k-1}+2)}{d(q_{n+1}q_{n+2}\cdots q_{n+k}+1)}$.

The left bound is a bit smaller than $\frac{c}{d}$, the right a bit larger. The difference between the two bounds is smaller than $\frac{2}{q_1 q_2 \cdots q_{n+k}}$, and thus the expansion for $\frac{a+c}{b+d}$ must agree with the expansion for $\frac{c}{d}$ through the nth quotient q_n.

Since $\frac{cz}{dz} = \frac{c}{d}$ for all positive integers z, they may be written with the same expansion – that is, they share the same quotient sequence q_i – and so all Pierce expansions for pairs $(b + dz, a + cz)$ agree up to q_n.

□

Further, if the first n quotients are fixed by our choice of $\frac{c}{d}$, then this causes the nth remainder found when calculating $P(b + d, a + c)$ to be r_n, the same remainder required by the nth step of calculation for $P(b,a)$.

Lemma 1.9. *For a,b,c,d as above, with $b = r_{i-1}q_i + r_i, a = r_0$, $b+d = r'_{i-1}q'_i + r'_i, a+c = r'_0$, and for $i \leq n, q_i = q'_i$, we have $r_n = r'_n$.*

Proof. Calculating $P(b+d, a+c)$, we find

$$b + d = (a+c)q_1 + r'_1 = (aq_1 + r_1) + d.$$

Solving the rightmost equation for r'_1, we have $r'_1 = r_1 + d - cq_1$.

In the next step of the calculation,

$$b + d = r'_1 q_2 + r'_2 = (r_1 q_2 + r_2) + d.$$

So $r'_2 = r_2 + d - dq_2 + cq_1q_2$. Continuing this process, we find that in general,

$$r'_i = r_i + d - dq_i + dq_{i-1}q_i - dq_{i-2}q_{i-1}q_i + \ldots + (-1)^i cq_1q_2 \cdots q_i$$

$$r'_i = r_i + d(1 - q_i + q_{i-1}q_i - \ldots + (-1)^i c).$$

In particular, for $i = n$, $r'_n = r_n + d(1 - q_n + q_{n-1}q_i - \ldots + (-1)^n c) = r_n + d(c - c) = r_n$.

$\square$

Theorem 1.10. *Let a,b,c,d be as in the lemmas above. Then $\{P(b+dz, a+cz), z \in Z^+\}$ is periodic, of period $p = lcm\{1, 2, \ldots r_n\}$ (or some divisor thereof).*

Proof. Consider $P(b+dp, a+cp)$. As in the lemmas, $q_1, q_2, \cdots, q_n$ are fixed, as is r_n:

$$b + dp = (a + cp)q_1 + r_1'$$

$$b + dp = r_1'(q_2) + r_2'$$

$$\vdots$$

$$b + dp = r_{n-1}'q_n + r_n.$$

The next step of the process is as follows:

$$b + dp = r_n q_{n+1}' + r_{n+1} + dp = r_n(q_{n+1}' + d\frac{p}{r_n}) + r_{n+1}$$

where $r_{n+1} < r_n$.

$$b + dp = r_{n+1}q_{n+2}' + r_{n+2} + dp = r_{n+1}(q_{n+2}' + \frac{dp}{r_{n+1}}) + r_{n+2}$$

where $r_{n+2} < r_{n+1} < r_n$, implying also that $\frac{dp}{r_{n+1}}$ is an integer. The process continues this way, so that the remainder at the ith step (for $i > n$) is r_i. Since $r_{n+k} = 0$ when calculating $P(b, a)$, we find that calculating $P(b+dp, a+cp)$ requires $n+k$ steps also. Thus $P(b+dp, a+cp) = n+k = P(b, a)$. Now for $0 < z < p$, we may consider $b+dz = b', a+cz = a'$ and apply the logic above, noting that the lemmas ensure that all quantities r_n, q_i for $i \leq n$ are independent of our choice of z. Further, each remainder produced after the nth step must be smaller than r_n, and will divide p. Thus the sequence $\{P(b+dz, a+cz), z \in Z^+\}$ is periodic, of period p. $\square$

The algorithms used for finding Pierce and Engel expansions are defined for positive integers *a*,*b* only. However, if either is negative, allowing q to be a negative integer and modifying the requirement for the remainder so that $0 \leq r < |a|$, we get algorithms which accommodate negative numbers. On the other hand, if a or b is a rational (non-integer) number, we simply allow r to be rational (non-integer) as well – naturally, its denominator must be the same as that of a. Using this modified algorithm, we can investigate analogous expansions. Note, however, that the sign patterns we observe in the Pierce and Engel expansions will, of course, be altered when working with negative numbers. Consider the brief example $b = \frac{3}{4}, a = -\frac{1}{5}$:

$$\frac{3}{4} = -\frac{1}{5}(-3) + \frac{3}{20}$$
$$\frac{3}{4} = \frac{3}{20}(5) + 0.$$

Thus we find the expansion $\frac{a}{b} = \frac{1}{-3} + \frac{1}{-3 \cdot 5} = \frac{1}{-3} - \frac{1}{-15}$. Using this expanded division algorithm, we can investigate $P(b, a)$ for negative pairs.

Theorem 1.11. *For any positive integers a, b, c, d, $P(\frac{a}{b}, \frac{c}{d}) = P(ad, bc)$.*

Proof.

$$\frac{a}{b} = \frac{c}{d}q_1 + r_1, 0 \leq r_1 < \frac{c}{d}.$$

Multiplying both sides by bd, we get the equivalent statement

$$ad = cbq_1 + bdr_1.$$

Therefore $P(\frac{a}{b}, \frac{c}{d}) = P(ad, bc)$. □

Theorem 1.12. *Let b, a be positive integers. $P(-b, -a) = P(b, a)$, and $P(b, -a) = P(-b, a) = E(b, a)$, where $E(b, a)$ is the function which counts the terms in the Engel expansion for $\frac{a}{b}$.*

Proof. For the first half of the statement, simply note that $\frac{a}{b} = \frac{-a}{-b}$. We must have precisely the same sequence of quotients (and thus remainders), so multiplying both sides by -1 is the only way to achieve this result.

$$b = aq + r$$

$$-b = (-a)q - r$$

Note that the negative remainder looks more like the algorithm required for the Engel expansion.

To prove the latter half of the statement, note that since $\frac{-a}{b} = \frac{a}{-b}$, we have $P(b, -a) = P(-b, a)$. Now suppose $E(b, a) = n$. Then

$$\frac{a}{b} = \frac{1}{q_1} + \frac{1}{q_1 q_2} + \frac{1}{q_1 q_2 q_3} + \ldots + \frac{1}{q_1 q_2 \cdots q_n}$$

$$\frac{-a}{b} = (-1)(\frac{1}{q_1} + \frac{1}{q_1 q_2} + \frac{1}{q_1 q_2 q_3} + \ldots + \frac{1}{q_1 q_2 \cdots q_n})$$

$$\frac{a}{-b} = \frac{1}{-q_1} - \frac{1}{(-q_1)(-q_2)} + \frac{1}{(-q_1)(-q_2)(-q_3)} + \ldots + (-1)^{n-1}\frac{1}{(-q_1)(-q_2)\cdots(-q_n)}$$

which is precisely the Pierce expansion for $\frac{-a}{b}$, hence their lengths are the same. Thus, $P(b, -a) = P(-b, a) = E(b, a)$. It is also relevant to note that if $q_1 = \lfloor \frac{b}{a} \rfloor$ (in other words, it is the first quotient produced by algorithm 6), then $-q_1 = \lceil \frac{-b}{a} \rceil$, the quotient required by the Engel algorithm (and vice versa). Thus allowing negative quotients effectively switches the algorithm required to find the desired sign pattern. $\square$

Hence, we can graphically represent $P(b, a)$ on the plane as shown in Figure 1. Here, we have used gray scale to indicate the functional value for a pair b, a, with the vertical axis representative of a values.

Our graphical representation has several notable features. In each quadrant, an empty region corresponds to improper fractions $\frac{a}{b}$. (If this flat region does not appeal to you, it can be made more interesting by allowing the first term $\lceil \frac{a}{b} \rceil$ for the expansion of an improper

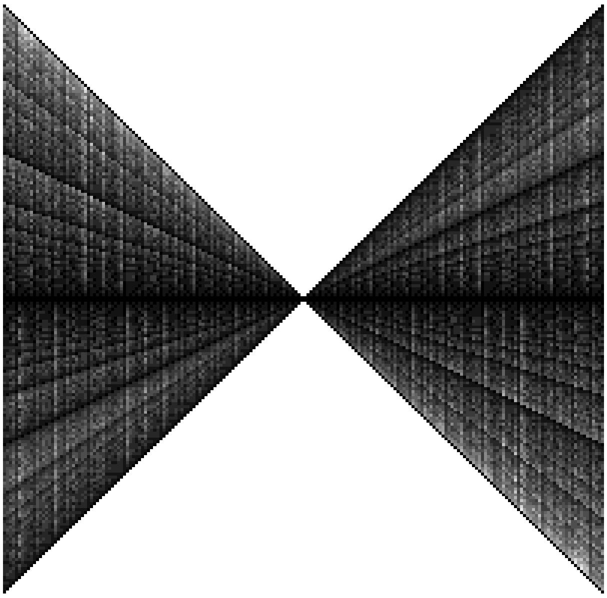

Figure 1: $P(b, a)$ for $-1000 \leq b \leq 1000$, $-b \leq a \leq b$

fraction.) Prominent diagonal stripes are caused by reducing rational numbers. The lighter vertical stripes correspond to choosing prime b. Naturally, prime $b > 2$ means $b \pm 1$ is even, so many of the $\frac{a}{b \pm 1}$ reduce, giving a shorter expansion and thus a darker vertical line.

2 Hybrid Expansions

2.1 Introduction

Given the connection between the Pierce and Engel expansions, it becomes natural to wonder what other sign patterns we could use for expansions, and whether we have similar results for these new expansions. Since Pierce expansions have terms with alternating signs, we next consider expansions in which the signs cycle two at a time. For example, suppose we would like an expansion for $\frac{7}{9}$ with sign pattern cycling ++ - - ++ - -... The first term of the expansion needs to be less than $\frac{7}{9}$, but adding the second term must produce a number larger than $\frac{7}{9}$ to force the third term to be negative. Thus we want to choose q_1 a bit too large – in contrast to the division algorithm, this produces a negative remainder. (However, the remainder should still be smaller than a in magnitude.) The second term is then found as usual.

$$9 = 7(2) - 5$$

$$9 = 5(1) + 4$$

$$9 = 4(3) - 3$$

$$9 = 3(3) + 0$$

Then $\frac{7}{9} = \frac{1}{2} + \frac{1}{2\cdot 1} - \frac{1}{2\cdot 1\cdot 3} - \frac{1}{2\cdot 1\cdot 3\cdot 3}$.

A modified expansion for π with this sign pattern may be found as sequence A015884 in the Online Encyclopedia of Integer Sequences [6]. Modifications necessary include allowing remainders in our algorithm to be non-integer, as well as taking the first term of the expansion as the integer part of the number being represented.

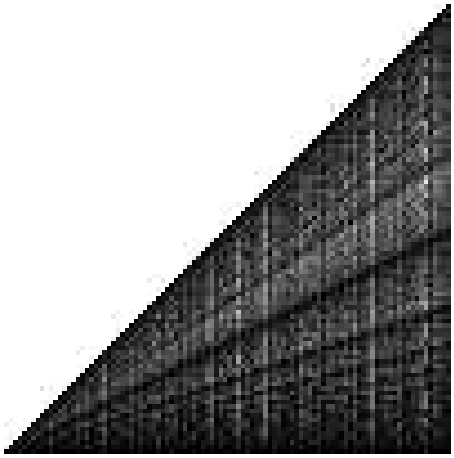

Figure 2: $P(b, a)$ for $0 < a \leq b$, $0 < b \leq 100$

Observe that these expansions produce a sequence $\{q_n\}$ which is not necessarily increasing. Decreases (of one) may occur in the sequence of resultant quotients only when the process takes a negative remainder r_i followed by a positive remainder r_{i+1}.

As long as we keep to positive remainders, the process is akin to finding a Pierce expansion, which means that subsequent quotients will be strictly increasing. In order to choose a negative remainder at some point, the quotient will necessarily increase by at least two, as it must increase by at least one for the Pierce algorithm equivalent, with positive remainder, and the quotient required for a negative remainder will be one greater than that. Hence, after any decrease, the quotient sequence must "right itself" by increasing by at least two before any additional decreases may occur.

Not surprisingly, the length of an expansion of this type for a given pair b, a cannot be generalized using the length of the Pierce expansion. However, below are first-quadrant graphs relating (again via gray scale) a pair (b, a) to the length of its expansions: Figure 2 shows Pierce lengths, Figure 3 expansions which cycle ++ - - ++ - -..., and Figure 4 shows expansions which cycle + - - ++ - - +.... The results are surprisingly similar, due largely to divisibility.

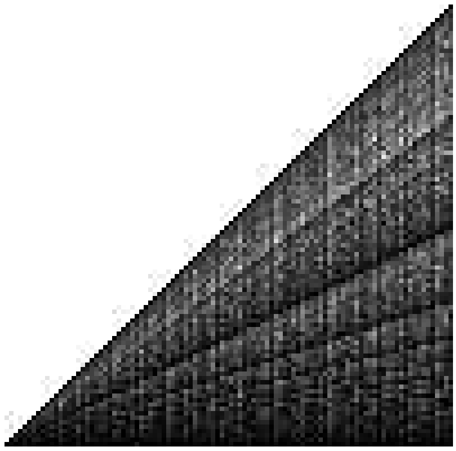

Figure 3: Expansions with pattern ++ - - for $0 < a \leq b$, $0 < b \leq 100$

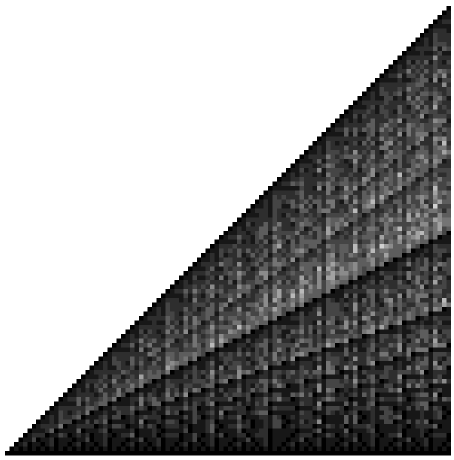

Figure 4: Expansions with pattern + - - + for $0 < a \leq b$, $0 < b \leq 100$

Define a Hybrid Expansion for rational number $\frac{a}{b}$ as

$$\frac{a}{b} = \frac{1}{q_1} \pm \frac{1}{q_1 q_2} \pm \ldots \pm \frac{1}{q_1 q_2 \ldots q_n},$$

where q_i values are chosen at each step by either undershooting or overshooting the amount left to approximate $\frac{a}{b}$. (Or taking the algorithmic approach, by allowing remainders to be either positive or negative, but requiring that their magnitude be less than b. In this case, if the remainder is positive, the next term in the expansion should have sign opposite to the previous term. If negative, the sign should stay the same.) For example, we previously noted that $\frac{7}{9} = \frac{1}{2} + \frac{1}{2 \cdot 1} - \frac{1}{2 \cdot 1 \cdot 3} - \frac{1}{2 \cdot 1 \cdot 3 \cdot 3}$ is one expansion. Further, since

$$9 = 7(2) - 5$$

$$9 = 5(2) - 1$$

$$9 = 1(9) - 0,$$

we find $\frac{7}{9} = \frac{1}{2} + \frac{1}{2 \cdot 2} + \frac{1}{2 \cdot 2 \cdot 9}$ is also a hybrid expansion. In all, $\frac{7}{9}$ has 5 expansions, shown below:

$$
\begin{array}{lllllll}
 & & 9=4\cdot 2+1 & \rightarrow & 9=9\cdot 1 & & (\frac{7}{9}=1-\frac{1}{4}+\frac{1}{36}) \\
 & \nearrow & & & & & \\
9=1\cdot 7+2 & & & & & & \\
 & \searrow & & & & & \\
 & & 9=5\cdot 2-1 & \rightarrow & 9=9\cdot 1 & & (\frac{7}{9}=1-\frac{1}{5}-\frac{1}{45}) \\
 & & & & 9=2\cdot 4+1 \rightarrow 9=9\cdot 1 & & (\frac{7}{9}=\frac{1}{2}+\frac{1}{2}-\frac{1}{4}+\frac{1}{36}) \\
 & & & \nearrow & & & \\
 & & 9=1\cdot 5+4 & & & & \\
 & \nearrow & & \searrow & & & \\
9=2\cdot 7-5 & & & & 9=3\cdot 4-3 \rightarrow 9=3\cdot 3 & & (\frac{7}{9}=\frac{1}{2}+\frac{1}{2}-\frac{1}{6}-\frac{1}{18}) \\
 & \searrow & & & & & \\
 & & 9=2\cdot 5-1 & \rightarrow & 9=9\cdot 1 & & (\frac{7}{9}=\frac{1}{2}+\frac{1}{4}+\frac{1}{36})
\end{array}
$$

Table 1: *The 5 Hybrid Expansions of* $\dfrac{7}{9}$

Allow $H(b, a)$ to represent the number of hybrid expansions for a given pair b, a of non-zero integers. Note that negative values are irrelevant here: expansions for (b, a) and $(-b, -a)$ are clearly the same (as before), and any expansion for (b, a) can be transformed to an expansion for $(-b, a)$ or $(b, -a)$ simply by multiplying the expansion by -1. (Thus there is a 1-1 correspondence among these expansions.) A key observation to counting these hybrid expansions is to note that, if $b = aq + r$, then we could calculate $H(b, a)$ by the simple recurrence relation $H(b, a) = H(b, r) + H(b, a - r)$.

How many different expansions are possible for a given pair of integers b, a? There will always be a finite number: since the remainders produced by the iterations form a strictly decreasing sequence of remainders, the iterations will always terminate in a finite number of steps. It is noteworthy, however, that if we remove the requirement that $r < |a|$, any pair b, a has infinitely many expansions. To see this fact, suppose $\frac{1}{n}$ is the first term of any expansion of $\frac{a}{b}$. If we drop the restrictions that the remainders must decrease, then we can form infinitely many expansions simply by replacing $\frac{1}{n}$ with $\frac{1}{2n} + \frac{1}{2n}$ or $\frac{1}{3n} + \frac{1}{3n} + \frac{1}{3n}$, etc. Clearly then, this restriction on the remainders is necessary to keep our problem interesting! However, as a consequence of this restriction, some very natural-looking expansions are not considered hybrids. For instance, $\frac{2}{9} = \frac{1}{3} - \frac{1}{6} + \frac{1}{18}$ appears at first glance to be a hybrid expansion, since each denominator is a multiple of the last. However, if we write out the corresponding division algorithm statements to generate the indicated quotients, we have

$$9 = 2(3) + 3$$

$$9 = 3(2) + 3$$

$$9 = 3(3) + 0.$$

The first two steps of the iteration violate our division algorithms by allowing remainders that are too large, and so this is not a proper expansion.

Figure 5 shows a graph representing the values of $H(b, a)$ for $0 < a \leq b, 0 < b \leq 100$: again, we use the convention that lighter pixels represent higher values, meaning more expansions. Limited values for $H(b, a)$ are listed in Appendix B, as well as in sequences A135511, A135513, and A135514 in the Online Encyclopedia of Integer Sequences. [6]

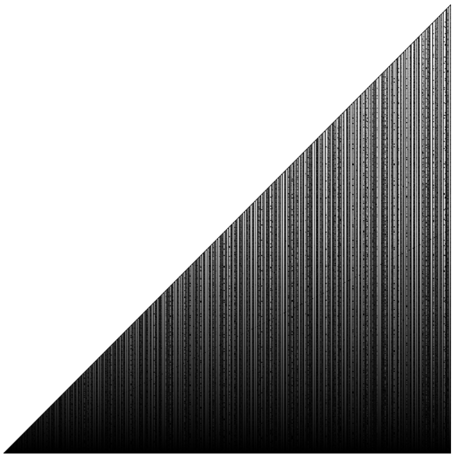

Figure 5: $H(b,a)$ for $0 < a \leq b$, $0 < b \leq 500$

2.2 Notable Results

It is simple to place an upper bound on the number of proper hybrid expansions possible:

Theorem 2.1. $H(b,a) \leq a$

Proof. If $a = 1$, this is trivial. So assume that $H(b,a) \leq a$ for all $a < n$. Consider $H(b,n)$, which we may assume is in lowest terms. There are two choices for the first term of the expansion:

$$\frac{n}{b} = \frac{1}{\lceil \frac{b}{n} \rceil} + S_1$$

or

$$\frac{n}{b} = \frac{1}{\lfloor \frac{b}{n} \rfloor} - S_2,$$

where S_1 is an expansion of $\frac{n\lceil \frac{b}{n} \rceil - b}{b\lceil \frac{b}{n} \rceil}$ and S_2 is an expansion of $\frac{n\lfloor \frac{b}{n} \rfloor - b}{b\lfloor \frac{b}{n} \rfloor}$. However, these expansions must have a common term of $\frac{1}{\lceil \frac{b}{n} \rceil}$ and $\frac{1}{\lfloor \frac{b}{n} \rfloor}$, respectively. Factoring these out, we desire expansions for $\frac{n\lceil \frac{b}{n} \rceil - b}{b}$ and $\frac{n\lfloor \frac{b}{n} \rfloor - b}{b}$. Because the latter is negative, we instead consider $\frac{b - n\lfloor \frac{b}{n} \rfloor}{b}$, which has precisely the same number of expansions, as previously noted.

$$(n\lceil\frac{b}{n}\rceil - b) + (b - n\lfloor\frac{b}{n}\rfloor) = n$$

Since n does not divide b, both of these terms are nontrivial, so each is strictly less than n. Applying the induction hypothesis to each, S_1 has at most $n\lceil\frac{b}{n}\rceil - b$ expansions, S_2 at most $b - n\lfloor\frac{b}{n}\rfloor$, and thus $H(b,n) \leq (n\lceil\frac{b}{n}\rceil - b) + (b - n\lfloor\frac{b}{n}\rfloor) = n$. $\square$

Special choices for a, b also lend predictability to our function H:

Theorem 2.2. *For p prime with $1 \leq n < p$, $H(p,n) = n$.*

Proof. Certainly it is true for $n = 1$. As an induction hypothesis, suppose that it is true for $1 \leq k < n$.

Suppose $p \equiv i \pmod{n}$. Then $\frac{n}{p} = \frac{1}{\lceil\frac{p}{n}\rceil} + S_1$ or $\frac{n}{p} = \frac{1}{\lfloor\frac{p}{n}\rfloor} - S_2$ with S_1 representing the expansion for

$$\frac{n\lceil\frac{p}{n}\rceil - p}{\lceil\frac{p}{n}\rceil p} = \frac{n(\frac{p}{n} + \frac{n-i}{n}) - p}{\lceil\frac{p}{n}\rceil p} = \frac{n-i}{\lceil\frac{p}{n}\rceil p}.$$

Factoring out the common term $\frac{1}{\lceil\frac{p}{n}\rceil}$ which must occur in each term, we find the number of possible expansions is exactly the number of expansions of $\frac{n-i}{p}$, which has $n-i$ expansions by induction.

Similarly, S_2 represents the expansions for

$$\frac{n\lfloor\frac{p}{n}\rfloor - p}{\lfloor\frac{p}{n}\rfloor p} = \frac{n(\frac{p}{n} - \frac{i}{n}) - p}{\lfloor\frac{p}{n}\rfloor p} = \frac{-i}{\lfloor\frac{p}{n}\rfloor p}.$$

Factoring out the common term $\frac{1}{\lfloor\frac{p}{n}\rfloor}$ which must occur in each term, we find the number of possible expansions is exactly the number of expansions of $\frac{i}{p}$, which has i expansions by induction. Thus $H(p,n) = (n-i) + i = n$ expansions. $\square$

Theorem 2.3. $H(2^n, 2^n - 1) = n$.

Proof. The Engel expansion has n terms:

$$\frac{1}{2} + \frac{1}{2 \cdot 2} + \frac{1}{2 \cdot 2 \cdot 2} + \ldots + \frac{1}{2^n}.$$

When this expansion is generated (algorithmically), each quotient is 2:

$$2^n = (2^n - 1)(2) - (2^n - 2)$$

$$2^n = (2^n - 2)(2) - (2^n - 4)$$

and so forth. If we choose to change the sign of the $m+1$st term of the expansion, it requires the mth step of the algorithm to use a smaller quotient, namely 1, and have a nonnegative remainder. However, this will always end the expansion immediately, since we already have

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^m} + \frac{1}{2^m}$$

which is equivalent to

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^{m-1}} + \frac{2}{2^m}$$

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^{m-1}} + \frac{1}{2^{m-1}}$$

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{2}{2^{m-1}}$$

$$\vdots$$

$$\frac{2^m - 1}{2^m}.$$

Therefore, we have only as many additional expansions as signs that we may change. Since we cannot change the first term's sign, there are n possible: the Engel expansion plus the $n-1$ expansions made possible by changing signs. Thus $H(2^n, 2^n - 1) = n$. $\square$

Theorem 2.4. *For a prime p with $a \neq p$, $H(2p, a) = \lceil \frac{a}{2} \rceil$.*

Proof. If a is even, then the fraction $\frac{a}{2p}$ will reduce to $\frac{\frac{a}{2}}{p}$. Since the two fractions are equal, they have the same number of expansions: by Theorem 2.2, we see that they must have $\frac{a}{2}$ expansions.

If, on the other hand, a is odd, then we can proceed by induction: It is certainly true for $a = 1$, as $\frac{1}{2p}$ has only 1 expansion, itself. Assume the result holds for a with $1 < a < 2n + 1$. Then

$$\frac{2n+1}{2p} = \frac{1}{\lceil \frac{2p}{2n+1} \rceil} + S_1$$

or

$$\frac{2n+1}{2p} = \frac{1}{\lfloor \frac{2p}{2n+1} \rfloor} - S_2.$$

For all expansions arising from the first possibility, factoring out the common term $\frac{1}{\lceil \frac{2p}{2n+1} \rceil}$ which must occur in each term, we find the number of possible expansions is exactly the number of expansions of

$$\frac{(2n+1)\lceil \frac{2p}{2n+1} \rceil - 2p}{(2p)}.$$

If we write $2p = m(2n + 1) + i$ with $0 < i < a = 2n + 1$, then

$$\begin{aligned} &\frac{(2n+1)(m+1) - 2p}{2p} \\ &= \frac{2p - i + (2n+1) - 2p}{2p} \\ &= \frac{2n+1-i}{2p}. \end{aligned}$$

Now apply the induction hypothesis: there are $\lceil \frac{2n+1-i}{2} \rceil$ expansions resulting from S_1.

Similarly, expansions arising from the second possibility have the common term $\frac{1}{\lfloor \frac{2p}{2n+1} \rfloor}$. Factoring it out, we find the number of possible expansions is exactly the number of expansions of

$$\frac{(2n+1)\lfloor\frac{2p}{2n+1}\rfloor - 2p}{(2p)}.$$

Again writing $2p = m(2n+1) + i$, then

$$\frac{(2n+1)(m) - 2p}{2p}$$

$$= \frac{-i}{2p}.$$

Since negative values are irrelevant, this has the same number of expansions as $\frac{i}{2}$. Now since $i < 2p$, apply the induction hypothesis: there are $\lceil\frac{i}{2}\rceil$ expansions resulting from S_2, and thus

$$H(2p, 2n+1) = \lceil\frac{2n+1-i}{2}\rceil + \lceil\frac{i}{2}\rceil$$

$$= n + \lceil\frac{1-i}{2}\rceil + \lceil\frac{i}{2}\rceil$$

$$= n + 1 = \lceil\frac{a}{2}\rceil.$$

□

Just as we found a periodicity result for Pierce expansions, it is no surprise that periodicity is also found for hybrid expansions. In fact, more can be said for hybrids: the period consists of 2 palindromic subsequences of predictable lengths.

Lemma 2.5. *For fixed integer a, let $L = lcm\{1, 2, \ldots, a\}$. The sequence $\{H(b, a)\}_{b=a}^{\infty}$ is periodic, of period L.*

Proof. Note that if $b = aq \pm r$, then $b + L = a(q + n) \pm r$, for $n = \frac{L}{a}$. Also, since for any $0 < r < a$, r divides L also, this argument applies to each stage of the algorithm. Further, if $r = 0$, then a divides b, and trivially, a divides $b + L$. Hence, $H(b, a) = H(b + L, a) = H(b + zL, a)$ for any integer z: the sequence is periodic of period L. □

Theorem 2.6. *(Short palindrome) For fixed integer $a > 2$, $L = lcm\{1, 2, \ldots, a\}$, the sequence $\{H(b, a)\}_{b=L-a}^{L+a}$ is palindromic.*

Proof. First, we note that $H(L, a) = 1$, since a divides L trivially. For any $0 < r < a$, performing the algorithm to find the hybrid expansions, we have

$L + r = aq + r$ and $L + r = a(q + 1) - (a - r)$.

On the other hand, $L - r = a(q - 1) + (a - r)$ and $L - r = aq - r$.

Observe that the remainders found for $L + r, L - r$ are the same (though reversed). Continuing the algorithm, note that both $r, a - r < a$ and so both divide L. The branches that had remainder r terminate in the next step, while the branches with remainder $a - r$ continue:

$$L + r = (a - r)q_2 + r_2, \qquad L + r = (a - r)(q_2 + 1) - (a - r - r_2)$$

$$L - r = (a - r)(q_2 - 1) + (a - r - r_2), \qquad L - r = (a - r)(q_2) - r_2.$$

In general, suppose an arbitrary remainder $i < a$ is encountered (along with its appropriate complementary remainder in the other branch of the algorithm) . If i divides r, then it divides $L \pm r$ and the algorithm terminates for one branch of both $L + r, L - r$, leaving the branches with the complementary remainders, which we consider as our new i.

Suppose then that i does not divide r. Then $b = L + r$ gives some

$L + r = iq + j, \qquad L + r = i(q + 1) - (i - j),$

and since i divides L, it must be that $r \equiv j \mod i$. Hence, for $b = L - r$,

$L - r = iq + j - 2r = iQ + j - 2j = iQ - j$ for some integer Q, and

$L - r = i(Q - 1) + (i - j)$, accordingly.

Since the remainders will be exactly the same at each stage, the expansions will end after precisely the same number of steps. Therefore, $H(L - r, a) = H(L + r, a)$ for any $0 < r < a$. $\square$

Theorem 2.7. *(Long palindrome) For fixed integer $a > 2$, $L = lcm\{1, 2, \dots, a\}$, the sequence $\{H(b, a)\}_{b=a}^{L-a}$ is palindromic.*

Proof. If the proposed sequence is palindromic, its center is at $b = \frac{L}{2}$. We proceed as in the last proof, considering $H(\frac{L}{2} \pm i, a)$ for $0 < i < \frac{L}{2} - a$. Note that if $a \neq 2^n$ for some integer $n > 1$, then $\frac{L}{2}$ is divisible by a. (If not, then a contains the highest power of 2 in L, say 2^m, and so $a = 2^m q$ for some $q > 2$. But $q > 2$ means that $2^{m+1} < a$, so 2^{m+1} divides L, a contradiction.) Suppose then that $\frac{L}{2}$ is divisible by a, and take $0 < i < \frac{L}{2} - a$. Then

$\frac{L}{2} + i = a\frac{L}{2a} + i$ and $\frac{L}{2} + i = a(\frac{L}{2a} + 1) - a + i = a(\frac{L}{2a} + 1) - (a - i)$

$\frac{L}{2} - i = a(\frac{L}{2a}) - i$ and also $\frac{L}{2} - i = a(\frac{L}{2a} - 1) + (a - i)$.

The remainder terms above are not guaranteed to be smaller than a as required algorithmically. However, this presents no problem: if the matching remainders are too large, the corresponding quotients will be increased/decreased by the same amounts, and the remainders, now reduced modulo a, will still be precisely the same. Subsequently, we encounter the same remainders for $\frac{L}{2} \pm i$. As above, this continues as long as the remainders we find divide $\frac{L}{2}$, which is true unless the remainder is the largest power of 2 which divides L.

What happens when a (or a remainder) is this largest power of 2?

If $i = \frac{a}{2}$, then $\frac{L}{2}$ is divisible by i, so $\frac{L}{2} \pm i = aQ + 0$ for appropriate Q.

If $i < \frac{a}{2}$:

$\frac{L}{2} + i = a\lfloor \frac{L}{2a} \rfloor + (\frac{a}{2} + i)$

$\frac{L}{2} + i = a\lceil \frac{L}{2a} \rceil + i - \frac{a}{2} = a\lceil \frac{L}{2a} \rceil - (\frac{a}{2} - i)$

$\frac{L}{2} - i = a\lfloor \frac{L}{2a} \rfloor - i + \frac{a}{2} = a\lfloor \frac{L}{2a} \rfloor + (\frac{a}{2} - i)$

$\frac{L}{2} - i = a\lceil \frac{L}{2a} \rceil - \frac{a}{2} - i = a\lceil \frac{L}{2a} \rceil - (\frac{a}{2} + i)$.

If $i > \frac{a}{2}$:

$\frac{L}{2} + i = a(\lfloor \frac{L}{2a} \rfloor + 1) + (i - \frac{a}{2})$

$\frac{L}{2} + i = a(\lceil \frac{L}{2a} \rceil + 1) + i - \frac{3a}{2} = a(\lceil \frac{L}{2a} \rceil + 1) - (\frac{3a}{2} - i)$

$\frac{L}{2} - i = a\lfloor \frac{L}{2a} \rfloor - i + \frac{a}{2} = a(\lfloor \frac{L}{2a} \rfloor - 1) + (\frac{3a}{2} - i)$

$\frac{L}{2} - i = a\lceil \frac{L}{2a} \rceil - \frac{a}{2} - i = a\lfloor \frac{L}{2a} \rfloor - (i - \frac{a}{2})$.

We find that the remainders continue to match. Hence, as before, the algorithms will end after precisely the same number of steps, so $H(\frac{L}{2} + i, a) = H(\frac{L}{2} - i, a)$. $\square$

It is trivial to find a b, a pair with a small number of expansions by choosing a small (in particular, $a = 1$ is an obvious choice). The theorems above outline some special cases where the number of expansions for a b, a pair will be predictable. However, as yet, we have left untouched the question of which b, a pairs will have the most expansions. Clearly, they must be larger than $\frac{1}{2}$, as any smaller b, a pair has fewer expansions than $b, b-a$ by using a simple remainder argument. What other conclusions can be drawn about the most likely pairs to have the greatest number of hybrid expansions?

Theorem 2.8. *For a fixed b, any choice of a that results in the largest number of hybrid expansions is of the form $a = b - (2n-1)$, where $0 < n < \frac{b}{4}$. That is, champion pairs have numerators with parity opposite that of the denominator.*

Proof. For $a = b - m$ with $m < \frac{b}{2}$, we have $b = (b-m)(2) - (b-2m)$ as one of the valid steps of the hybrid algorithm, as $0 < b - 2m < b - m$. Thus we can encounter any suitably large integer with the same parity as b among the remainders of iterations for other pairs, preventing these same-parity numbers from being champions. Hence, any numerator that results in the largest number of hybrid expansions for some denominator has the opposite parity.

□

Numerical evidence suggests that $b-1$ and $b-3$ are common numerators for champions, while denominators that are round numbers (in the sense of having many small divisors) are likely to have a large number of numerators which all share the status of champion.

2.3 Infinite Expansions

We next turn our attention to extending our understanding of hybrid expansions to include not just rational numbers, but irrationals as well. Clearly, the algorithm outlined for rational $\frac{a}{b}$ can be applied to irrational numbers as well by requiring only the quotients chosen to be integers. The process would be non-terminating since each expansion is, by necessity, nonterminating for irrationals. Consequently, we will have an infinite number of hybrid expansions for any irrational number. If we specify an irrational number as well as an infinite pattern of signs we would like to satisfy, an appropriate infinite expansion exists: for each term, we simply choose to (as nearly as possible) overshoot or undershoot the quotient necessary to accommodate the desired sign.

Expansions for irrationals, being by nature infinitely long and infinitely many, leave us no interesting questions pertaining to "how many". However, if we instead reverse the question, and choose an infinite family of quotients, we could ask which numbers are representable using the chosen family of quotients. Obviously, if the sequence of quotients chosen were periodic, choosing a periodic sign pattern as well would result in a series with a rational sum. However, any rational should have finitely many hybrid expansions, and so we term such an occurrence an improper expansion.

Consider, on the other hand, expansions whose quotients are built from the sequence $\{3, 6, 12, 24, ...\}$, but whose sign pattern for each term (from the second on - we require the first term to be positive since we assume for the time being that we are writing expansions of nonnegative numbers) could be any possible combination of positive and negative ones. The smallest number resulting from such an expansion would be $\frac{1}{3} - \frac{1}{3\cdot 6} - \frac{1}{3\cdot 6\cdot 12} ... = m$, and the largest is the Engel expansion, $E = \frac{1}{3} + \frac{1}{3\cdot 6} + \frac{1}{3\cdot 6\cdot 12} ...$.

Which numbers inside the interval (m, E) are representable via such expansions? Truncating the expansion after some number n of terms, we could use these partial sums with n terms, P_i $(1 \leq i \leq 2^n)$, to approximate the results. If we call the sum of the terms lost in truncation S_n, we could feasibly add or subtract this difference to obtain numbers within $\pm S_n$ of each P_i. This defines small intervals of numbers that are still possible to reach at each step. As we consider larger and larger n, these bands will winnow down, defining ever smaller intervals.

Truncating infinite expansions after some finite number of steps produces intervals of feasibility - the result of the infinite expansion must lie within this region. Moreover, *any* expansion, finite or not, that matches the truncated expansion up to this point will lie in the feasibility region. This makes it simple to find the family of rational numbers whose expansions begin with this sequence.

Suppose instead that we select a sequence from which to build denominators and allow either positive, negative or zero terms, so that we may effectively skip any term of the resulting series if we choose. (However, a zero numerator still means that the denominator contributes a factor to the ever-increasing denominators of the expansion.) The smallest possible number obtainable (again requiring any first term to be nonnegative) is 0, and the maximum is the Engel expansion. However, making this small change can result in a set of positive measure for certain sequences.

Theorem 2.9. *For the sequence $\{q_i\}$ where each $q_i \in \{2, 3\}$, the set of numbers attainable by an infinite sum $\sum_{i=1...\infty} \frac{a_i}{\prod_{j=1..i} q_j}$, where each $a_i \in \{-1, 0, 1\}$, $a_1 \geq 0$, has positive measure. In particular, the set is the interval $[0, 1]$ if the sequence has $q_i = 2 \quad \forall i$; otherwise, it is the interval $[0, E]$, where $E = \sum_{i=1..\infty} \frac{1}{\prod_{j=1...i} q_j}$, the largest (Engel) expansion.*

Proof. We will first consider the result for the sequences $S = \{2, 2, 2, ...\}$ and $T = \{3, 3, 3, ...\}$.

Note that $\sum \frac{1}{2^i} = 1$, and $\sum \frac{1}{3^i} = 1/2$.

For S, we only need to observe that every number in $[0,1]$ can be written in binary, which translates directly to the representations described above: a 0 in the jth position in the binary representation indicates $a_j = 0$, a 1 in the jth position indicates $a_j = 1$. (Hence, we do not even need expansions with negative terms in this case.)

For T, consider the ternary representation of all numbers in $[0, \frac{1}{2}]$. Clearly, any representation consisting solely of 0's and 1's can be represented. Suppose a number has a ternary representation $\{t_j\}$ with a 2 first occurring in the jth position. Then

$$\frac{t_1}{3} + \frac{t_2}{3^2} + \cdots + \frac{2}{3^j} + \sum \frac{t_i}{3^i} = \frac{t_1}{3} + \frac{t_2}{3^2} + \cdots + \frac{t_{j-1}+1}{3^{j-1}} - \frac{1}{3^j} + \sum \frac{t_i}{3^i}.$$

Therefore, if a number has t_j as its first 2, increase a_{j-1} by $1 \pmod 3$ and set $a_j = -1$. If the increase creates a new 2 in position $j-1$, simply repeat the procedure, as this position is now home to the new leading 2. Note that it is not possible that this procedure is nonterminating, as that would indicate that all the first j positions are already 1 in ternary, which means the current number is larger than $.1111\ldots2$, and so greater than $\frac{1}{2}$, thus outside our domain.

Therefore, any number between 0 and the Engel expansion is representable for a sequence of strictly 2's or strictly 3's. In a mixed sequence, we must have either infinitely many 2's or infinitely many 3's (or both). Suppose we have infinitely many 3's. Then $[0, E]$ is representable, as above. Consider the insertion of a 2 into the jth position of the sequence. Without loss of generality, suppose a 2 is inserted into the first position. The first term in any resultant expansion is either 0 or $\frac{1}{2}$, and the rest of the sequence is composed by either adding or subtracting $\frac{1}{2} \cdot T'$, for T' some expansion using the sequence of 3's. As noted previously, these expansions cover all of $[0, \frac{1}{2} \cdot \frac{1}{2}]$. Adding these sequences when choosing 0 as the first term gives $[0, \frac{1}{4}]$, subtracting when choosing $\frac{1}{2}$ as first term gives $[\frac{1}{4}, \frac{1}{2}]$, and adding when choosing $\frac{1}{2}$ as first term gives $[\frac{1}{2}, \frac{3}{4}]$: together, these cover $[0, E]$ for E the Engel expansion for $\{2, 3, 3, 3, \ldots\}$.

By repeating this argument for each 2 to be inserted into a sequence with infinitely many 3's, the desired result is achieved. A similar argument holds if we consider inserting a 3 into a sequence with infinitely many 2's.

$\square$

Recall that the Cantor set is a fractal built on the interval $[0, 1]$ by removing the middle third of the interval, and iterating the process to infinity by removing the middle third of each remaining interval. This construction forms an infinite set of points which is totally disconnected. Furthermore, this set has measure zero, since the sum of all the intervals removed is 1. The set has some well known variations: a fat Cantor set is built by removing middle subintervals of some fixed ratio less than one third. It has positive measure, since the sum of the lengths of the intervals removed is necessarily less than 1. Similarly, thin Cantor sets can be built by removing middle subintervals of length at least one third. Both fat and thin Cantor sets are homeomorphic to the Cantor set, since they share the same structure, despite the fact that measure is not preserved. Of course, they are not the only sets homeomorphic to the Cantor set: any nonempty, perfect, compact metric space which is totally disconnected is homeomorphic to the Cantor set. [7] We say that a set S is a Cantor set if it is homeomorphic to the Cantor set.

Theorem 2.10. *For a sequence q_i with $q_i > 1 \quad \forall i$, and infinitely many $q_n \geq 3$, the set $S = \{\sum_{i=1..\infty} \frac{a_i}{\prod_{j=1..i} q_i}\}$, where $a_i \in \{-1, 1\}$ is a Cantor set.*

Proof. Without loss of generality, suppose the first $q_1 \geq 3$.

The Cantor set may be constructed from the interval $[0, 1]$ by removing the middle third of the interval and repeating the process on the smaller intervals created. At the ith step, 2^{i+1} endpoints of 2^i intervals are created, each with length $\frac{1}{3^i}$. Order these endpoints as $C_{i,j}$, where i indicates the current step number and j indicates that $C_{i,j}$ is the jth smallest endpoint.

Let $x_{i,j} = \sum_{n=1...i} \frac{a_n}{\prod_{m=1..i} q_m} \pm \sum_{n=i+1...\infty} \frac{1}{\prod_{m=1..n} q_m}$, ordered so that for fixed i, $x_{i,j}$ is the jth smallest such number. For example, $x_{i,1} = -E$ for all i, where E is the Engel expansion built from the sequence given. If we consider the specific sequence $\{3, 3, ..\}$, we can choose to begin an expansion with $\pm\frac{1}{3}$, and the remaining terms can cover an additional distance of $\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$, we have four $x_{1,j}$ values:

$x_{1,1} = -\frac{1}{3} - \frac{1}{6} = -\frac{1}{2} = -E$,

$x_{1,2} = -\frac{1}{3} + \sum_{i=2..\infty} \frac{a_i}{\prod_{n=1..i} q_n} = -\frac{1}{3} + \frac{1}{6} = -\frac{1}{6}$,

$x_{1,3} = \frac{1}{3} - \sum_{i=2..\infty} \frac{a_i}{\prod_{n=1..i} q_n} = \frac{1}{3} - \frac{1}{6} = \frac{1}{6}$, and

$x_{1,4} = \frac{1}{3} + \frac{1}{6} = \frac{1}{2} = E$.

It is clear that for any i, we have exactly 2^{i+1} $x_{i,j}$: this was shown for $i = 1$ above, and proceeding by induction, each $x_{i,j}$, we can form $x_{i+1,2j}$ and $x_{i+1,2j+1}$ by choosing a_{i+1} to be -1 and 1 respectively, and changing the limits on the sums and products appropriately. Since there were 2^{i+1} $x_{i,j}$, we form $2 \cdot 2^{i+1}$ $x_{i+1,j}$, or 2^{i+2} $x_{i+1,j}$. Therefore the $C_{i,j}, x_{i,j}$ are in a one to one correspondence.

Furthermore, from the construction of the $x_{i,j}$ (as a partial sum on i terms plus or minus the remaining distance that may be covered with the remaining terms), it is clear that any number representable as a hybrid expansion with denominators built from S must lie inside $[x_{i,2j-1}, x_{i,2j}]$ for all i, for some $1 \leq j \leq 2^{i+1}$.

Define a continuous function f from $[-E, E]$ to $[0, 1]$, where E is the maximum Engel expansion for the sequence S, so that $f(x_{i,j}) = C_{i,j}$. Defining such a continuous map is clearly possible since the order of the points is preserved. Furthermore, we shall define f to be a bijection, noting that our supposition that there is a $q_n \geq 3$ for $n > i$ guarantees that $x_{i,j} \neq x_{i,j+1}$.

For any x not representable by an expansion built from S, x must fail to be between $x_{i,2j-1}, x_{i,2j}$ for some i. Then under f, its image will fail to fall between some $C_{i,2j-1}, C_{i,2j}$, and thus is one of the intervals eliminated during the construction of the Cantor set. On the other hand, a representable x can never fall inside $(x_{i,2j-1}, x_{i,2j})$, as this indicates an insurmountable difference from a limit point of the sequence after the ith term. Under the function f, such an x maps to a number lying between some $C_{i,2j-1}, C_{i,2j}$, and so has been eliminated from membership in the Cantor set.

Thus the Cantor set and the set of numbers representable as a hybrid expansion built from S as prescribed are homeomorphic.

$\square$

For sequences which introduce even a single term larger than 3, the resultant set of representable numbers is not a single interval: in fact, it may not even contain an interval of nonzero length.

Consider a variation S on the Cantor set: beginning with some closed interval, split the interval into 5 equal pieces and remove the 2nd and 4th pieces. Iterate this process on the

3 remaining pieces. At each stage, we remove $\frac{2}{5}$ of each interval. As this process continues to infinity, we can calculate the amount removed with a simple series.

$$\frac{2}{5}\sum_{k=0}^{\infty}(\frac{3}{5})^k = 1$$

As such, the remaining set has measure zero. However, the set S is not empty, as the endpoints of each interval are never removed.

We can also conclude that S, like the Cantor set, is uncountable, either by using a diagonalization argument or by noting that our set contains a thin Cantor set (ie, removing the middle $\frac{3}{5}$), which is of course uncountable. Our set S is closed, as its complement is a union of open intervals, and thus it is a complete metric space. S is also bounded, and so is compact.

Set S also has no isolated points: let s be an element of our set. For any neighborhood N about s, s is contained in an interval I of length $\frac{1}{5^k}$ for some k such that I is contained in N. As the endpoints of I are never removed from our set, we have found additional points of S within our arbitrary neighborhood, and so S is perfect.

Finally, S is totally disconnected. For any 2 distinct points x, y of the set, $\{x, y\}$ is disconnected: a removal must take place between the two at some level of the iteration. (Though this could occur earlier, it has certainly happened by the nth round of iteration, where n is such that $\frac{1}{5^n} < |x - y|$.) After this removal, the points x and y are contained in complementary closed intervals, and so $\{x, y\}$ is disconnected.

However, any nonempty, perfect, compact metric space which is totally disconnected must be homeomorphic to the Cantor set, and so S is homeomorphic to the Cantor set. [7]

Theorem 2.11. *For a sequence q_i with $q_i > 1, a_i \in \{-1, 0, 1\}$ for all i, and infinitely many*

$q_n > 3$, *the set* $T = \{\sum_{i=1..\infty} \frac{a_i}{\prod_{j=1..i} q_i}\}$ *is a Cantor set.*

Proof. Without loss of generality, suppose $q_1 > 3$.

Let $x_{i,j} = \sum_{n=1...i} \frac{a_n}{\prod_{m=1..i} q_m} \pm \sum_{n=i+1...\infty} \frac{1}{\prod_{m=1..n} q_m}$, ordered so that for fixed i, $x_{i,j}$ is the jth smallest such number.

Let $s_{i,j}$ be, similarly, the jth smallest endpoint of some interval remaining after the ith round of deletions in constructing set S as above.

It is clear that for any i, we have exactly $2 \cdot 3^i$ $x_{i,j}$. Therefore the $s_{i,j}, x_{i,j}$ are in a one to one correspondence.

Define a continuous function f from $[-E, E]$ to $[-1, 1]$, where E is the maximum Engel expansion for the sequence T, so that $f(x_{i,j}) = s_{i,j}$. Defining such a continuous map is clearly possible since the order of the points is preserved. Furthermore, we shall define f to be a bijection, noting that our supposition that there are $q_n > 3$ for $n > i$ guarantees that $x_{i,j} \neq x_{i,j+1}$.

Note that for any x not representable by an expansion built from T, x must be between $x_{i,j}, x_{i,j+1}$ for some i, and for $j \in \{2, 4\} \mod 5$. Then under f, its image will fall between $s_{i,j}, s_{i,j+1}$, and thus is one of the intervals eliminated during the construction of our Cantor-like set S. On the other hand, a representable x can never fall inside $(x_{i,j}, x_{i,j+1})$ for $j \in \{2, 4\}$ mod 5, as this indicates an insurmountable difference from a limit point of the sequence after the ith term. Under the function f, such an x maps to a number lying between $s_{i,j}, s_{i,j+1}$, and so has been eliminated from membership in the Cantor set.

Restricting our function f to the set of numbers representable as an expansion from T, we find this set is homeomorphic to the Cantor-like set S, and thus, to the Cantor set.

$\square$

There are many questions remaining about these hybrid expansions, both finite and infinite. In the future, we hope to narrow the possibilities for champion pairs of b, a with respect to hybrid expansions, and to give better insight to those expansions which have a decreasing quotient.

3 Appendix A: Maple Code

The following code was written in Maple 7. First, we define a procedure which will determine the list of quotients necessary to build a hybrid expansion. Output is a list of signed integers: the integers are the quotients necessary to build the expansions, while the signs denote the overall sign of the current term in the expansion. For example, $\frac{3}{5}$ has three lines of output representing its three expansions:

2 5; 1 -3 -5; 1 -2 5;

These indicate the expansions $\frac{1}{2} + \frac{1}{2 \cdot 5}$, $\frac{1}{1} - \frac{1}{1 \cdot 3} - \frac{1}{1 \cdot 3 \cdot 5}$, $\frac{1}{1} - \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 5}$, respectively.

```
> WriteAllExpansions:=proc(c, string, currQuotient, outStream,
        loopCounter)
>       ### outStream is OPENED BEFORE calling writeAllExps to which
>       ### the expansions for a given fraction will be written.
>
>       local steps::integer,       ##length of Engel expansion
>       expansion::Array,           ##sequence of factors
>               quotients::Array,   ##quotients for expansion
>               copyC::integer,     ##a changable copy of c=a/b
>               q::integer,
>               signArray::Array,   ##keeps signs for an expansion
>               current::integer,   ##index of quotient modified
>                                   ##[(current+1)'s sign changes]
>               temp::float,        ##used to approximate copyC
>
>               loopCount::integer, ##helps keep signs
>               tempString::string: ##expansion found
>
>         steps:=0:                 ##initialize all variables
>         copyC:=c:
>         tempString:=string:
>         q:=1:
```

```
>         temp:=1/currQuotient:
>         for i from 0 to 500 do
>             quotients(i):=currQuotient:
>         end do:
>         expansion(0):=1:
>         loopCount:=loopCounter:
>         FAILSAFE:=0:             ##tracks length of expansion
>                                  ##if too high, must be a mistake.
>                                  ##failsafe allows a bail out.
> if (c>1)
> then copyC:=0;
>    quotients(1):=1/floor(c):
>    tempInteger:=floor(c)*(-1)^(loopCount):
>    tempString:=cat(string," (",tempInteger,")"):
>    WriteAllExpansions(c-1/quotients(1),  tempString,1, outStream,
          loopCount):
>    if not(1/quotients(1)=c) then
>       quotients(1):=1/( floor(c)+1):
>       tempInteger:=ceil(c)*(-1)^(loopCount):
>       tempString:=cat(string," (",tempInteger,")"):
>       WriteAllExpansions(abs(1/(quotients(1))-c),tempString,1,
           outStream, loopCount+1):
>
>    end if:
> else
>    ################## Find Engel Expansion first ################
>    while not(copyC=0) do
>           if (FAILSAFE<500000) then
>            steps:=steps+1:
>            signArray(steps):=0:
>            for j from 1 to 50000 while (evalf(copyC<(temp*1/j)))
                  do              ###always undershoot
>                q:=j+1:
>            end do:
>            temp:=temp*1/q:
>            expansion(steps):=q:
>            tempInteger:=q*(-1)^(loopCount):
>            tempString:=cat(tempString," ", tempInteger):
>            quotients(steps):=quotients(steps-1)*q:
>            copyC:= copyC-temp:
>            FAILSAFE:=FAILSAFE+1:
>           else copyC:=0:
>           end if:
>    end do:
>         if (FAILSAFE<500000) then
```

```
>           fprintf(outStream, "%s\n", "");
>           fprintf(outStream, "%s\n", tempString);
>        end if:
>    ### Now begin varying the patterns.
>    ###Notice that overshooting jth term means (j-1)th term will
>    ###be negative
>        if (FAILSAFE<500000)
>        then current:=steps-1:
>        else copyC:=0:
>        end if:
>        ### if Engel aborted, no basis to vary on... abort  ###
>    while (current>0) do
>        while (signArray(current+1)=1 and current>0) do
>            current:=current-1:   ##Find next sign change
>        end do:
>        copyC:=c:
>        tempString:=string:
>        for i from 1 to (current-1) do
>           tempInteger:=expansion(i)*(-1)^(signArray(i)+loopCount):
>           tempString:=cat(tempString," ",tempInteger):
>        end do:
>            if (current=0)        ##Then all possibilities are
>            then copyC:=0:        ##exhausted, so stop looking for
>            else                  ##new ones. Otherwise, overshoot
>               signArray(current+1):=1:
>                                  ##current term so next term will
>                                  ##be forced negative.
>               expansion(current):=expansion(current)-1:
>               tempInteger:=expansion(current)*
                       (-1)^(signArray(current)+loopCount):
>               tempString:=cat(tempString," ",tempInteger):
>            end if:
>            if (current>1)
>               then quotients(current):=quotients(current-1)*
                       expansion(current):
>               else quotients(current):=expansion(current)*
                       currQuotient:
>            end if:
>            if (copyC>0) then     ##reduce copyC by current
>                                  ##approximation, then find
>                                  ##next term in expansion for
>                                  ##this smaller problem.
>                temp:=sum('1/quotients(j)*(-1)^signArray(j)',
                       'j'=1..current):
>                if (copyC-temp<0) then copyC:=temp-copyC:
```

```
>                 else copyC:=copyC-temp:
>                 end if:
>                 #Search for next appropriate term of expansion.#
>                 q:=1:
>                 for i from 1 to 500 while
                    (copyC<1/(quotients(current)*q)) do
>                           q:=i:
>                 end do:
>                 expansion(current+1):=q:
>                 quotients(current+1):=quotients(current)*
>                       expansion(current+1):
>                 ####We have a new approximation of copyC. If it
>                 ####is exact, show it and increase numExps.
>                 ####If not, find all expansions for what we have
>                 ####left to approximate.
>                 copyC:=abs(copyC-1/quotients(current+1)):
>                 if (copyC=0) then
>                     fprintf(outStream, "%s\n", "");
>                     tempInteger:=expansion(current+1)*
                            (-1)^(loopCount+signArray(current+1)):
>                     tempString2:=cat(tempString," ",tempInteger):
>                     fprintf(outStream, "%s\n", tempString2);
>                    else
>                       tempInteger:=expansion(current+1)*
                            (-1)^(loopCount+signArray(current+1));
>                       tempString2:=cat(tempString," ",tempInteger):
>                       WriteAllExpansions(copyC, tempString2,
                            quotients(current)*q, outStream,
                            loopCount+1):
>                       tempInteger:=(expansion(current+1)-1)*
                            (-1)^(loopCount+signArray(current+1)):
>                       tempString2:=cat(tempString," ",tempInteger):
>                       copyC:=abs(c-(temp-1/(quotients(current)*
                            (q-1)))):
>                       WriteAllExpansions(copyC, tempString2,
                            quotients(current)*(q-1), outStream,
                            loopCount):
>                   end if:
>              end if:
>              copyC:=c:
>    end do:
>    end if:
>    end proc:
```

After defining this procedure, these lines of code will write the quotient lists to the file specified. (The variables a and b must be positive integers.)

```
> outStream:=fopen("c:\\destination\\folder\\yourfile.txt", WRITE);
> WriteAllExpansions(a/b,"",1, outStream,0);
>
> fclose(outStream);
```

4 Appendix B: Values of $H(b, a)$

$b\|a$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1	1																								
2	1	1																							
3	1	2	1																						
4	1	1	2	1																					
5	1	2	3	4	1																				
6	1	1	1	2	3	1																			
7	1	2	3	4	5	6	1																		
8	1	1	2	1	3	2	3	1																	
9	1	2	1	2	3	2	5	6	1																
10	1	1	2	2	1	3	4	4	5	1															
11	1	2	3	4	5	6	7	8	9	10	1														
12	1	1	1	1	2	1	3	2	2	3	4	1													
13	1	2	3	4	5	6	7	8	9	10	11	12	1												
14	1	1	2	2	3	3	1	4	5	5	6	6	7	1											
15	1	2	1	2	1	2	3	4	3	2	5	4	7	8	1										
16	1	1	2	1	2	2	3	1	4	3	4	2	5	3	4	1									
17	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	1								
18	1	1	1	2	2	1	3	2	1	3	5	2	4	5	3	6	7	1							
19	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	1						
20	1	1	2	1	1	2	3	2	4	1	5	3	5	4	2	4	6	5	6	1					
21	1	2	1	2	3	2	1	4	3	4	5	4	7	2	5	8	9	6	11	12	1				
22	1	1	2	2	3	3	4	4	5	5	1	6	7	7	8	8	9	9	10	10	11	1			
23	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	1		
24	1	1	1	1	2	1	2	1	2	2	3	1	4	3	3	2	4	2	5	3	3	4	5	1	
25	1	2	3	4	1	2	7	8	9	2	11	12	13	14	3	16	17	18	15	4	21	18	23	24	1
26	1	1	2	2	3	3	4	4	5	5	6	6	1	7	8	8	9	9	10	10	11	11	12	12	13
27	1	2	1	2	3	2	3	4	1	4	5	2	3	4	3	8	7	2	9	6	5	10	11	6	13
28	1	1	2	1	3	2	1	2	3	3	5	3	6	1	7	4	7	5	6	5	2	6	8	6	8
29	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
30	1	1	1	2	1	1	2	2	2	1	3	2	4	3	1	4	6	3	5	2	4	5	6	4	3
31	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
32	1	1	2	1	3	2	3	1	4	2	3	2	5	3	6	1	7	4	7	3	5	4	7	2	7
33	1	2	1	2	3	2	5	6	3	6	1	4	7	6	5	6	7	6	9	12	7	2	13	8	13
34	1	1	2	2	3	3	4	4	5	5	6	6	7	7	8	8	1	9	10	10	11	11	12	12	13
35	1	2	3	4	1	2	1	4	5	2	7	8	9	2	3	12	13	14	15	4	3	14	15	16	5
36	1	1	1	1	2	1	2	2	1	2	3	1	3	3	2	2	3	1	4	3	3	5	5	2	6
37	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
38	1	1	2	2	3	3	4	4	5	5	6	6	7	7	8	8	9	9	1	10	11	11	12	12	13
39	1	2	1	2	3	2	3	4	3	4	5	4	1	6	5	6	7	6	7	8	7	10	9	8	11
40	1	1	2	1	1	2	2	1	2	1	3	2	3	3	2	2	5	4	6	1	7	5	7	3	3
41	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
42	1	1	1	2	2	1	1	2	2	3	3	2	4	1	3	4	4	3	5	4	1	5	7	4	6
43	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
44	1	1	2	1	2	2	3	2	3	3	1	3	4	4	5	4	6	5	6	5	7	1	8	6	8
45	1	2	1	2	1	2	3	2	1	2	3	2	5	4	1	6	5	2	5	2	3	4	5	4	3
46	1	1	2	2	3	3	4	4	5	5	6	6	7	7	8	8	9	9	10	10	11	11	1	12	13
47	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
48	1	1	1	1	2	1	2	1	2	2	3	1	3	2	2	1	3	2	4	2	3	3	4	1	5
49	1	2	3	4	5	6	1	2	9	10	11	12	13	2	15	16	17	18	13	20	3	22	23	24	25
50	1	1	2	2	1	3	4	4	3	1	4	2	5	7	2	8	9	9	6	2	9	11	8	12	1

$b\|a$	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
26	1																								
27	14	1																							
28	7	8	1																						
29	26	27	28	1																					
30	7	5	8	9	1																				
31	26	27	28	29	30	1																			
32	5	7	3	7	4	5	1																		
33	14	9	16	15	10	17	18	1																	
34	13	14	14	15	15	16	16	17	1																
35	18	19	4	17	6	23	20	25	26	1															
36	4	2	5	7	3	6	6	4	7	8	1														
37	26	27	28	29	30	31	32	33	34	35	36	1													
38	13	14	14	15	15	16	16	17	17	18	18	19	1												
39	2	9	12	11	10	13	14	11	14	15	12	17	18	1											
40	5	6	4	7	2	7	4	7	6	3	5	8	6	7	1										
41	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	1									
42	7	5	2	8	5	7	8	6	9	3	6	10	11	7	12	13	1								
43	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	1							
44	7	9	7	9	8	9	8	2	9	10	9	11	10	11	10	12	11	12	1						
45	8	3	8	11	2	9	10	5	8	5	4	13	12	7	6	15	8	17	18	1					
46	13	14	14	15	15	16	16	17	17	18	18	19	19	20	20	21	21	22	22	23	1				
47	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	1			
48	4	4	3	6	3	5	2	4	4	6	2	7	5	5	3	6	3	7	4	4	5	6	1		
49	26	27	4	29	24	31	32	33	28	5	36	37	38	39	40	35	6	43	44	39	46	41	42	1	
50	13	10	14	13	3	8	16	17	17	4	18	17	15	18	4	19	21	22	18	5	23	20	24	25	1

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