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Edge colorings of graphs on surfaces and star edge colorings of sparse graphs

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ABSTRACT

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In my dissertation, I present results on two types of edge coloring problems for graphs.

For each surface Σ, we define $\Delta(\Sigma) = \max\{\Delta(G) | \ G$ is a class two graph with maximum degree $\Delta(G)$ that can be embedded in $\Sigma\}$. Hence Vizing's Planar Graph Conjecture can be restated as $\Delta(\Sigma) = 5$ if Σ is a sphere. For a surface Σ with characteristic $\chi(\Sigma) \leq 0$, it is known $\Delta(\Sigma) \geq H(\chi(\Sigma)) - 1$, where $H(\chi(\Sigma))$ is the Heawood number of the surface, and if the Euler characteristic $\chi(\Sigma) \in \{-7, -6, \ldots, -1, 0\}$, $\Delta(\Sigma)$ is already known. I study critical graphs on general surfaces and show that (1) if G is a critical graph embeddable on a surface Σ with Euler characteristic $\chi(\Sigma) \in \{-6, -7\}$, then $\Delta(\Sigma) = 10$, and (2) if G is a critical graph embeddable on a surface Σ with Euler characteristic $\chi(\Sigma) \leq -8$, then $\Delta(G) \leq H(\chi(\Sigma))$ (or $H(\chi(\Sigma)) + 1$) for some special families of graphs, namely if the minimum degree is at most 11 or if Δ is very large et al. As applications, we show that $\Delta(\Sigma) \leq H(\chi(\Sigma))$ if $\chi(\Sigma) \in \{-22, -21, -20, -18, -17, -15, \ldots, -8\}$ and $\Delta(\Sigma) \leq H(\chi(\Sigma)) + 1$ if $\chi(\Sigma) \in \{-53, \ldots, 23, -19, -16\}$. Combining this with [19], it follows that if $\chi(\Sigma) = -12$ and Σ is orientable, then $\Delta(\Sigma) = H(\chi(\Sigma))$.

A *star k-edge-coloring* is a proper k-edge-coloring such that every connected bicolored subgraph is a path of length at most 3. The star chromatic index $\chi'_{st}(G)$ of a graph G is the smallest integer k such that G has a star k-edge-coloring. The list star chromatic index $ch'_{st}(G)$ is defined analogously. Bezegova et al. and Deng et al. independently proved that $\chi'_{st}(T) \leq \frac{3\Delta}{2}$ for any tree T with maximum degree Δ. Here, we study the list star edge coloring and give tree-like bounds for (list) star chromatic index of sparse graphs. We show that if $mad(G) < 2.4$, then $\chi'_{st}(G) \leq \frac{3\Delta}{2} + 2$ and if $mad(G) < \frac{15}{7}$, then $ch'_{st}(G) \leq \frac{3\Delta}{2} + 1$. We also show that for every $\varepsilon > 0$ there exists a constant $c(\varepsilon)$ such that if $mad(G) < \frac{8}{3} - \varepsilon$, then $ch'_{st}(G) \leq \frac{3\Delta}{2} + c(\varepsilon)$. We also find guaranteed substructures of graph with $mad(G) < \frac{3\Delta}{2} - \varepsilon$ which may be of interest in other problems for sparse graphs.

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Contents

Chapter 1

Preliminaries

1.1 Introduction and History

The interesting problems of graph coloring originate in a question concerning the coloring of regions on a map. Francis Guthrie, a student of Augustus de Morgan, made the observation that for every map he encountered, he could, using only 4 colors, color the regions so that no adjacent regions had the same color. This led to a straight-forward question: is this true for every map, or is there some map that requires at least 5 colors? Though Guthrie's question is simple, it required more than one hundred years, as well as the invention of the computer, to finally turn the Four Color Conjecture into the Four Color Theorem. Along the way, there were many failed "proofs", including, famously, by Alfred Kempe in 1879 [20] and by Peter Guthrie Tait in 1880 [40]. The theorem was finally proved in 1976 by Appel, Haken, and Koch [1, 2].

It is worth mentioning the Four Color Theorem here not only for its long and interesting history, but because the problem sparked the invention of tools still in use today. Firstly, the concept of edge coloring originates in the early attempts to turn conjecture into theorem. Tait realized that the problem of 4-coloring the regions of a map (vertices of a graph that is planar, cubic, and bridgeless) was equivalent to 3-coloring the region boundaries (the edges). Further, many graph theoretic techniques were invented in attempts to solve the Four Color Conjecture. In Kempe's attempted proof, he introduced the idea of the Kempe chain. Given a valid, proper coloring $f : V(G) \to \{1, \cdots, k\}$, let $V_i = \{v \in V(G) : f(v) = i\}$. Consider the graph induced by two color classes, $G[V_i \cup V_j]$. Let G' be a connected component of $G[V_i \cup V_j]$. Consider the coloring f' created by switching colors at vertices in G', i.e., $f'(v) = j$ if $v \in V(G')$ and $f(v) = i$, $f'(v) = i$ if $v \in V(G')$ and $f(v) = j$, and $f'(v) = f(v)$ if $v \in V(G) \setminus V(G')$. This new coloring is then a proper vertex coloring. Although Kempe's "proof" was eventually determined to be flawed, his technique could be modified to show that for a planar graph G, $\chi(G) \leq 5$. Moreover, his idea applies for the edge-coloring problem. In edge coloring, we consider alternating paths:

given two (edge) color classes, the subgraph induced has connected components that are either paths or even cycles. By examining the result of switching the colors along one or more of the maximal paths, we can eliminate certain graph structures in a minimal counterexample. Another graph theoretic tool was introduced in 1904, when Wernicke [46] used the discharging method to prove a lemma for planar graphs. In discharging, the elements of a closed system are assigned an initial charge; then, that charge is redistributed by some discharging rules. Clearly, the total charge of the system is unchanged. Wernicke assigned to each vertex and face of an embedded planar graph an initial charge, recognizing that he could use Euler's formula for polyhedra to calculate total charge. Knowing that the total much remain constant, he could make conclusions about substructures of the graph.

The proof of Appel and Haken (as well as a later stream-lined proof by Robertson et al. [35]) used both Kempe chains and the discharging method. These tools in tandem have proven useful in questions of edge coloring. In my results, I use lemmas derived from Kempe chains in conjunction with the discharging method to give bounds on proper edge coloring. I also give bounds on a variation of edge coloring, called star edge coloring. As noted in Chapter 3, the tools of Kempe chains does not apply, but the star edge coloring bounds make use of the discharging method.

1.2 Main Results

All graphs considered here are simple, i.e., no loops or multiple edges. An *edge coloring* of a graph is a function assigning values (colors) to the edges of the graph in such a way that any two adjacent edges receive different colors. A graph is *edge k-colorable* if there is an edge coloring of the graph with colors from $\{1,\dots,k\}$. The chromatic index of a graph, $\chi'(G)$ is the minimum k such that G is edge k-colorable. Vizing [42] showed that $\Delta(G) \le \chi'(G) \le \Delta(G)+1$.

I give the following partial characterizations of graphs that require only the minimum $\Delta(G)$ colors to properly edge color, from the parameter of graph embeddings.

Let G be a simple graph with maximum degree $\Delta(G)$. Let $H(\chi)$ be the Heawood number for a surface Σ with Euler characteristic χ.

Theorem 1.2.1. *Let G be a graph embedded on a surface Σ with Euler characteristic $\chi(\Sigma) \in \{-6,-7\}$. If $\Delta(G) \ge 11$, then $\chi'(G) = \Delta(G)$.*

Theorem 1.2.2. *Let G be a graph with $\delta(G) \le \frac{\Delta(G)+1}{2}$ which can be embedded in a surface Σ with Euler characteristic $\chi(\Sigma) \le -8$. Assume that $\delta(G) \le 7$, $\Delta(G) \ge H(\chi(\Sigma))+1$, and $\chi'(G-e) \le \Delta(G)$ for any $e \in E(G)$. Then $\chi'(G) = \Delta(G)$.*

Theorem 1.2.3. *Let G be a graph with $6 \le \delta(G) \le 10$ or $\delta(G) = 11 \le \frac{\Delta(G)+1}{2}$ which can be embedded in a surface Σ with Euler characteristic $\chi(\Sigma) \le -8$. Assume that $\Delta(G) \ge H(\chi(\Sigma))+2$*

and $\chi'(G-e) \le \Delta(G)$ for any $e \in E(G)$. Then $\chi'(G) = \Delta(G)$.

Theorem 1.2.4. *Let Σ be a surface with characteristic $\chi(\Sigma) \in \{-53, \ldots, -8\}$. Let G be a graph embedded in Σ. If*

$$\Delta(G) > \begin{cases} H(\chi(\Sigma)), & \text{if } \chi(\Sigma) \in \{-22, -21, -20, -18, -17, -15, \ldots, -8\} \\ H(\chi(\Sigma)) + 1, & \text{if } \chi(\Sigma) \in \{-53, \cdots, -23, -19, -16\}. \end{cases}$$

then $\chi'(G) = \Delta(G)$.

In conjunction with [19], we then have the following corollary.

Corollary 1.2.5. *Let G be embedded in Σ, where Σ is an orientable surface with $\chi(\Sigma) = -12$. If $\Delta(G) > H(\chi(\Sigma)$, then $\chi'(G) = \Delta(G)$.*

A *star edge coloring* of a graph is a function assigning values (colors) to the edges of the graph in such a way that any two adjacent edges receive different colors and the graph contains no bicolored path or cycle on four edges. A graph is *star edge k-colorable* if there is an star edge coloring of the graph with colors from $\{1, \ldots, k\}$. The star chromatic index of a graph, $\chi'_{st}(G)$ is the minimum k such that G is star edge k-colorable. This definition can be easily extended to the list case: the edge star choosability of a graph, $ch'_{st}(G)$, is the minimum k such every k-list assignment L for $E(G)$ admits a star edge coloring.

I study star edge coloring for sparse graphs. My parameter for sparseness is maximum average degree, $mad(G) = \max\{\frac{2|E(H)|}{|V(H)|} : H \subseteq G\}$.

Theorem 1.2.6. *If $mad(G) \le 2.4$, then, $\chi'_{st}(G) \le \lfloor \frac{3}{2}\Delta(G) \rfloor + 2$.*

Theorem 1.2.7. *If $mad(G) < \frac{15}{7}$, then, $ch'_{st}(G) \le \lfloor \frac{3}{2}\Delta(G) \rfloor + 1$.*

Theorem 1.2.8. *Let $\varepsilon > 0$ be a real number and $d = 2\lceil \frac{8-3\varepsilon}{9\varepsilon} \rceil$. If $mad(G) < \frac{8}{3} - \varepsilon$, then*

$$ch'_{st}(G) \le \max\{\frac{3}{2}\Delta(G) + \frac{d}{2} + 2, \Delta(G) + 2d + 1\}.$$

Chapter 2

Chromatic index of embedded graphs

2.1 Introduction

An *edge coloring* of a graph is a function assigning values (colors) to the edges of the graph in such a way that any two adjacent edges receive different colors. A graph is *edge k-colorable* if there is an edge coloring of the graph with colors from $\{1, \ldots, k\}$. The chromatic index of a graph, $\chi'(G)$ is the minimum k such that G is edge k-colorable. Although the study of edge coloring began with the Four Color Theorem, it has since been studied in part because of its connections to the problems of matching and scheduling.

In determining bounds for chromatic index, one parameter of obvious interest is maximum degree. Clearly, for any graph G with maximum degree Δ, we have $\chi'(G) \geq \Delta$. The following are early theorems regarding chromatic index.

Theorem 2.1.1 (König, 1916 [24]). *Let G be a bipartite graph. Then, $\chi'(G) = \Delta(G)$.*

Theorem 2.1.2 (Shannon, 1949 [38]). *Let G be a (multi)graph. Then, $\chi'(G) \leq \frac{3\Delta(G)}{2}$.*

Shannon's bounds are sharp in general when considering multigraphs. The "fat triangle" (a triangle with each pair of vertices connected by $\frac{\Delta}{2}$ multiple edges) obtains the upper bound.

Vizing, in 1964, improved this bound for simple graphs.

Theorem 2.1.3 (Vizing's Theorem [42]). *Let G be a simple graph with maximum degree Δ. Then, $\chi'(G) \leq \Delta + 1$.*

It is easy to find simple graphs that require $\Delta+1$ colors to properly edge color, by considering odd cycles. Further the complete graph on an odd number of vertices, K_{2l+1} can be shown to be overfull and thus requires $2l + 1 = \Delta(K_{2l+1}) + 1$ colors to properly edge color. Since no simple graph requires more than one additional color, each simple graph may be characterized as Class 1 if $\chi'(G) = \Delta$ and Class 2 otherwise. In 1981, Holyer [18] showed that the problem of classification of Class 1 and Class 2 graphs is NP-complete, even for cubic graphs.

A Class 2 graph G is called **critical** if G is Class 2 but every proper subgraph is Class 1. In 1965, Vizing [43] proposed several conjectures concerning Class 2 and critical graphs, all of which remain open today.

Conjecture 2.1.1. *Let G be a planar graph with maximum degree $\Delta \geq 6$. Then, G is Class 1.*

Conjecture 2.1.2. *Let G be a critical graph with maximum degree Δ and $|V(G)| = n$. Then, the average degree of G is at least $\Delta - 1 + \frac{3}{n}$.*

Conjecture 2.1.3. *Let G be a critical graph with maximum degree Δ and $|V(G)| = n$. Then, $\alpha(G) \leq \frac{n}{2}$.*

Conjecture 2.1.4. *Let G be a critical graph. Then, G has a 2-factor.*

Clearly, the statement that every Class 2 planar graph has maximum degree at most 5 is equivalent to the Planar Graph Conjecture. In general, given a closed surface (or surface in short), one may try to determine the maximum of the maximum degrees of all simple Class 2 graphs that can be embedded in the surface. We use $\Delta(G)$ (or simply Δ) to denote the maximum degree of a graph G. For a surface Σ, we define

$$\Delta(\Sigma) = \max \{\Delta(G) | \ G \text{ is a class two graph that can be embedded in } \Sigma\}.$$

If we use S to denote the sphere, then by the above definition, Vizing's Planar Graph Conjecture is equivalent to the claim that $\Delta(S) = 5$. In 1965, Vizing proved that $5 \leq \Delta(S) \leq 7$, and Zhang [48] and, independently, Sanders and Zhao [36] proved that $\Delta(S) \leq 6$.

Let $\chi \leq 2$ be an integer. Let $H(\chi) = \lfloor \frac{7+\sqrt{49-24\chi}}{2} \rfloor$ (where $H(\chi)$ is the Heawood number of a surface of Euler characteristic χ). It is noticed in [39] that for a surface Σ with characteristic $\chi(\Sigma) \leq 1$ except the Klein bottle, $\Delta(\Sigma) \geq H(\chi(\Sigma)) - 1$. Vizing's Average Degree Conjecture [43] implies the following:

Proposition 2.1.1. *If Vizing's Average Degree Conjecture is true, then $\Delta(\Sigma) \leq \lfloor 3 + \sqrt{13 - 6\chi} \rfloor$.*

Proof. Let G be a critical graph embedded in a surface Σ with Euler characteristic χ. Let $|V(G)| = n$ and $|E(G)| = m$. Let d be the average degree of G. By Euler's formula, $-6\chi \geq \sum_{v \in V(G)} (d(v) - 6) = -6n + \sum_{v \in V(G)} d(v)$. Note that $\sum_{v \in V(G)} d(v) = 2m = d \times n \geq (\Delta - 1)n + 3$. Further, $n \geq \Delta + 1$. Thus, $-6\chi \geq (\Delta - 7)(\Delta + 1) + 3$. $\square$

One can easily check that either $\lfloor 3 + \sqrt{13 - 6\chi} \rfloor = H(\chi)$ or $\lfloor 3 + \sqrt{13 - 6\chi} \rfloor = H(\chi) - 1$.

In 1970, Mel'nikov [33] proved the following bounds:

Theorem 2.1.4. $\Delta(\Sigma) \leq max\ \{\lfloor\frac{11+\sqrt{25-24\chi(\Sigma)}}{2}\rfloor, \lfloor\frac{8+2\sqrt{52-18\chi(\Sigma)}}{3}\rfloor\}$.

In addition to the above general bounds for $\Delta(\Sigma)$, the exact values for some $\Delta(\Sigma)$ are known if $-7 \leq \chi(\Sigma) \leq 0$. Sanders and Zhao [37] proved $\Delta(\Sigma) = 6$ if $\chi(\Sigma) = 0$. Luo and Zhao [29, 30] proved that $\Delta(\Sigma) = 7$ if $\chi(\Sigma) = -1$, $\Delta(\Sigma) = 8$ if $\chi(\Sigma) \in \{-2, -3\}$, $\Delta(\Sigma) = 9$ if $\chi(\Sigma) = -5$. Luo, Miao, and Zhao [31] proved that $\Delta(\Sigma) = 8$ if $\chi(\Sigma) = -4$.

The results in my dissertation include exact bounds given surfaces of Euler characteristic -6 or -7, as well as improved bounds for graphs with small minimum degree or low Euler characteristic. We show that if G is a critical graph embedded on a surface Σ with Euler characteristic $\chi(\Sigma) \in \{-6, -7\}$, then $\Delta(G) \leq 10$. We also study critical graphs on general surfaces and show that if G is a critical graph embeddable on a surface Σ, then $\Delta(G) \leq H(\chi(\Sigma))$ (or $H(\chi(\Sigma)) + 1$) for some special cases, namely if the minimum degree is at most 11. We believe that those results and the ideas used may shed light to eventually improve Melnikov's bound in general. As a corollary, we show that $\Delta(\Sigma) \leq H(\chi(\Sigma))$ if $\chi(\Sigma) \in \{-22, -21, -20, -18, -17, -15, \ldots, -8\}$ and $\Delta(\Sigma) \leq H(\chi(\Sigma)) + 1$ if $\chi(\Sigma) \in \{-53, \ldots, 23, -19, -16\}$. Furthermore, we show that if $\chi(\Sigma) = -12$ and Σ is orientable, then $\Delta(\Sigma) = H(\chi(\Sigma))$.

Before proceeding, we introduce the following notation. A *critical graph* G is a connected graph such that G is class two and $G - e$ is class one for each edge e of G. A critical graph of maximum degree Δ is called a Δ-*critical graph*. Given an embedded graph G, let $V(G)$, $E(G)$, and $F(G)$ be the vertex set, edge set, and face set of G, respectively. The *order* of G is $|V(G)|$ and the *size* of G is $|E(G)|$. A *k-vertex*, k^+*-vertex* or k^-*-vertex* is a vertex of degree k, at least k or at most k, respectively. Let $N(x)$ be the set of all neighbors of x, $N(x, y) = N(x) \cup N(y)$, and $N[x] = N(x) \cup \{x\}$. We denote the set of all k-vertices of G by V_k and let $n_k = |V_k|$. We use $d(x)$ to denote the degree of x for each element $x \in V(G) \cup F(G)$. We call k-vertices adjacent to x *k-neighbors* of x and define $d_k(x)$ to be the number of k-neighbors of x. Similarly, we define k^+*-neighbors,* k^-*-neighbors*, $d_{k^+}(x)$, and $d_{k^-}(x)$. Let $\delta(G)$ (or simply δ) be the minimum degree of G.

2.1.1 Useful Lemmas

Let G be a Δ-critical graph with $xy \in E(G)$ and let f be a Δ-coloring of $G - xy$. For each vertex u, denote $c_f(u)$ the set of colors used by the edges incident with u and $\overline{c}_f(u) = \{1, 2, \ldots, \Delta\} \setminus c_f(u)$, the complement of $c_f(u)$. If an edge incident with u is colored i, then we say that u *sees* i. Otherwise, we say that u *misses* i. For $i, j \in \{1, \cdots, k\}$, an i-j *edge chain* is a chain of edges colored alternately i and j. Clearly, each maximal i-j edge chain is either a path or an even cycle, and any two maximal i-j edge chains are either disjoint or identical. We denote the maximal i-j edge chain containing u by $L_{i,j}(u)$. If u sees only one of the two colors, i and j, then $L_{i,j}(u)$ is a path where u is one of its endvertices. We use the notation that $A \oplus B = (A - B) \cup (B - A)$,

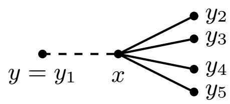

Figure 1.

where A, B are sets. Let $X \subseteq V(G)$. X is called elementary with respect to f if $\overline{c}_f(u) \cap \overline{c}_f(v) = \emptyset$ for every two distinct vertices $u, v \in X$.

The following definition of a fan and theorem are a slight modification of those found in [39].

Definition 2.1.1. *Let G be a Δ-critical graph with edge $e = xy$, and let f be an edge Δ-coloring of $G - e$. A **fan** (See Figure 1.) at x with respect to e and f is a sequence $F = (e_1, y_1, \cdots, e_p, y_p)$ with $p \geq$ consisting of edges $e_1, \cdots, e_p$ and vertices $y_1, \cdots, y_p$ satisfying the following:*

(F1) *The edges $e_1, \cdots, e_p$ are distinct, $e_1 = e$, and e_i is incident to x for $i = 1, \cdots, p$*

(F2) *For every edge e_i with $2 \leq i \leq p$, there is a vertex y_j with $1 \leq j < i$ with $f(e_i) \in \overline{c}_f(y_j)$.*

Theorem 2.1.5. *Let G be a Δ-critical graph with edge $e = xy$. Let f be an edge Δ-coloring of $G - e$ and let $F = (e_1, y_1, \cdots, e_p, y_p)$ be a fan at x w.r.t. e and f. Then the following hold:*

(a) $\{x, y_1, \cdots, y_p\}$ is an elementary set

(b) If $\alpha \in \overline{c}_f(x)$ and $\beta \in \overline{c}_f(y_i)$ for $1 \leq i \leq p$, then $L_{\alpha,\beta}(x)$ is a path with endpoints x and y_i.

(c) If F is a maximal fan, then $|V(F)| \geq 2$ and $\sum_{z \in V(F)}(d(z) + 1 - \Delta) = 2$.

Lemma 2.1.6. *(Luo and Zhao [30]) Let G be a Δ-critical graph and $xy \in E(G)$. Let $u \neq y$ be a neighbor of x and $v \notin \{x, y\}$ be a neighbor of u. Let f be an edge Δ-coloring of $G - xy$ with $f(ux) = k$ and $f(uv) = l$. If $k \notin c_f(y)$, then we have the following facts:*

(1) u sees every color in $c_f(x) \oplus c_f(y)$;

(2) If $l \in c_f(x) \oplus c_f(y)$, then v sees every color in $c_f(y) \oplus c_f(x)$, if $d(x) < \Delta$.

Lemma 2.1.7. *(Vizing Adjacency Lemma [42]) Let G be a Δ-critical graph. Then $d(u)+d(v) \geq \Delta+2$ for any two adjacent vertices u and v and $d_\Delta(x) \geq \max\{2, \Delta-k+1\}$ if x has a k-neighbor.*

Taking $d(x) = \Delta$ and $k = \delta$ in the above lemma, we have the following proposition.

Proposition 2.1.2. *Let G be a Δ-critical graph with minimum degree δ. Then, $n_\Delta \geq \Delta - \delta + 2$.*

Lemma 2.1.8. *(Luo, Miao, and Zhao [27]) Let G be a Δ-critical graph with $\Delta \geq 5$ and x be a 3-vertex. Then x has at least two Δ-neighbors which are not adjacent to any $(\Delta - 2)^-$-vertices except x.*

Lemma 2.1.9. *(Luo, Miao, and Zhao [31]) Let G be a Δ-critical graph with $\Delta \geq 6$ and x be a 3-vertex. Then x has a Δ-neighbor which is adjacent to at least $\Delta - 4 - \lfloor \frac{\Delta-1}{3} \rfloor$ Δ-vertices z with $d_{(\Delta-3)^-}(z) = 0$.*

The following lemma is a special case of Corollary 2.14 in [31].

Lemma 2.1.10. *Let G be a Δ-critical graph and x, y be two adjacent vertices with $d(x) = 5$ and $d(y) = \Delta - 2$. Then x is adjacent to three $(\Delta - 1)^+$-vertices z such that $d_{(\Delta-2)^+}(z) \geq \Delta - 2$.*

Lemma 2.1.11. *(Sanders and Zhao [37] and Zhang [48]) Let G be critical and $xy \in E(G)$ so that $d(x) + d(y) = \Delta + 2$. Then the following hold:*

(1) every vertex of $N(x, y) \setminus \{x, y\}$ is a Δ-vertex;

(2) every vertex of $N(N(x, y)) \setminus \{x, y\}$ is of degree at least $\Delta - 1$;

(3) if $d(x), d(y) < \Delta$, then every vertex of $N(N(x, y)) \setminus \{x, y\}$ is a Δ-vertex.

Corollary 2.1.12. *Let G be a Δ-critical graph and $xy \in E(G)$ so that $d(x) + d(y) = \Delta + 2$. Then if $d(x) \neq d(y)$, $n_\Delta \geq \Delta$; otherwise, $n_\Delta \geq \Delta - 1$.*

Proof. We first assume $d(x) \neq d(y)$. Without loss of generality, assume $d(x) < d(y)$. If $d(x) = 2$, then $d(y) = \Delta$ and $N[y] \setminus \{x\} \subseteq V_\Delta$. Then, $n_\Delta \geq |N[y] \setminus \{x\}| = \Delta$. If $3 \leq d(x) < d(y)$, then there is a neighbor z of y that is not adjacent to x. By Lemma 2.1.11, $d(z) = \Delta$ and since $d(y) \leq \Delta - 1$, $N[z] \setminus \{y\} \subseteq V_\Delta$. Thus $n_\Delta \geq \Delta$.

Now assume $d(x) = d(y) = \frac{\Delta+2}{2}$. In this case, it is possible that $N[x] = N[y]$ and so for $z \in N(y) \setminus \{x\}$, $d_\Delta(z) = \Delta - 2$. Thus including z we have $n_\Delta \geq \Delta - 1$. $\square$

Lemma 2.1.13. *(Sanders and Zhao [36]) Let G be a Δ-critical graph, $x \in V(G)$ and $y, z \in N(x)$. If xz is in at least $d(x)+d(y)-\Delta-2$ triangles not containing y, then $d(x)+d(y)+d(z) \geq 2\Delta + 2$.*

Lemma 2.1.14. *(i) ([4], [5], [6], [9]) There are no critical graphs of even order at most 14;*
(ii) ([6]) There are only two critical graphs of order 11 with size at most 5Δ, both of which are 3-critical;
(iii) ([5])There are only three critical graphs of order 13 with size at most 6Δ, which are 3-critical.

Lemma 2.1.15. *(Sanders and Zhao [36]) Let G be a Δ-critical graph and xyz be a path in G. If $d(x)+d(y)+d(z) \leq 2\Delta+1$ and x, z are not adjacent, then $|N(z) \cap N(y)| \leq d(x)+d(y)-\Delta-3$ and thus $|N(z) \setminus N[y]| \geq d(z) - d(x) - d(y) + \Delta + 2$.*

Lemma 2.1.16. *(Luo and Zhao [30]) Let G be a Δ-critical graph. Let $xy \in E(G)$ with $4 \leq d(x) \leq \Delta-2$ and $4 \leq d(y)$. If for an integer $k \geq 0$, either y is adjacent to $d(x)+d(y)-\Delta-2-k$ $(2\Delta - (d(x) + d(y)) + 1)^-$-vertices $u \neq x$, where $d(x) + d(y) - \Delta - 2 - k \geq 1$; or y is adjacent to $d(x)+d(y)-\Delta-2-k$ $(2\Delta-(d(x)+d(y))+2)^-$-vertices $u \neq x$, where $d(x)+d(y)-\Delta-2-k \geq 2$, then*

(1) there exist at least $\Delta - d(x) + 1$ neighbors $y' \neq x$ of y satisfying the following:

(i) $d(y') \geq \Delta - k$;

(ii) if $d(y) \neq \Delta$, then y' is adjacent to at least $\Delta - k - 2$ vertices distinct from x, y of degree at least $\Delta - k$;

(iii) if $d(y) \neq \Delta$ and y' is not adjacent to x, then y' is adjacent to at least $\Delta - k - 1$ vertices distinct from y of degree at least $\Delta - k$;

(iv) if $d(y) = \Delta$, then replace at least $\Delta - k$ in (ii) and (iii) by at least $\Delta - k - 1$.

(2) there exist at least $d(x) - k - 1$ neighbors $x' \neq y$ of x satisfying the following:

(i) $d(x') \geq \Delta - k$;

(ii) if $d(y) \neq \Delta$, then x' is adjacent to at least $\Delta - k - 2$ vertices distinct from x, y of degree at least $\Delta - k$;

(iii) if $d(y) \neq \Delta$ and x' is not adjacent to y, then x' is adjacent to at least $\Delta - k - 1$ vertices distinct from x of degree at least $\Delta - k$;

(iv) if $d(y) = \Delta$, then replace at least $\Delta - k$ in (ii) and (iii) by at least $\Delta - k - 1$.

When $\Delta \geq 2d(x) - 1$, $\Delta - d(x) + 1 > d(x) - 1$ and thus there is one neighbor of y described in (1) of the above lemma which is not adjacent to x. By Lemma 2.1.16-(1) (ii) and (iv), we have the following.

Lemma 2.1.17. *Let G be a Δ-critical graph. Let $xy \in E(G)$ with $4 \leq d(x) \leq \frac{\Delta+1}{2}$. Assume for an integer $k \geq 0$, either y is adjacent to $d(x) + d(y) - \Delta - 2 - k$ $(2\Delta - (d(x) + d(y)) + 1)^-$-vertices $u \neq x$, where $d(x) + d(y) - \Delta - 2 - k \geq 1$; or y is adjacent to $d(x) + d(y) - \Delta - 2 - k$ $(2\Delta - (d(x) + d(y)) + 2)^-$-vertices $u \neq x$, where $d(x) + d(y) - \Delta - 2 - k \geq 2$. We have the following:*

(1) If $d(y) \geq \Delta - k$, then there are at least $\Delta - k + 1$ vertices with degree at least $\Delta - k - 1$ (or with degree at least $\Delta - k$ if $d(y) \neq \Delta$);

(2) If $d(y) < \Delta - k$, then there are at least $\Delta - k$ vertices with degree at least $\Delta - k$.

The following result on the sizes of independent sets was proved in [28].

Lemma 2.1.18. *(Luo, Miao, and Zhao [28]) Let G be a Δ-critical graph. Let S be an independent set that contains no Δ-vertices. Then $|S| < \frac{|V(G)|}{2}$.*

The following lemma, which is applied numerous times in the following proofs, is a simple variation of Euler's formula $|V(G)| - |E(G)| + |F(G)| \geq \chi(\Sigma)$ for a graph embedded in a surface Σ.

Lemma 2.1.19. *For a graph G, with order n, minimum degree δ, and maximum degree Δ, embedded in a surface Σ, we have the following:*

(1) $-6\chi(\Sigma) \geq \sum_{v \in V(G)}(d(v) - 6) = n(\delta - 6) + (\Delta - \delta)n_\Delta + (\Delta - 1 - \delta)n_{\Delta-1} + \cdots + 2n_{\delta+2} + 1n_{\delta+1}$.

(2) If $\Delta^2 - 7\Delta \leq -6\chi$ or $\Delta^2 - 7\Delta \leq \sum_{v \in V(G)}(d(v) - 6)$, then $\Delta \leq H(\chi(\Sigma))$.

(3) If $\Delta^2 - 9\Delta + 8 \leq -6\chi$ or $\Delta^2 - 9\Delta + 8 \leq \sum_{v \in V(G)}(d(v) - 6)$, then $\Delta \leq H(\chi(\Sigma)) + 1$.

2.1.2 New Lemmas

In addition to the previously existing lemmas above, we need some new lemmas for our results.

By applying Theorem 2.1.5-(c), we prove the lemma below.

Lemma 2.1.20. *Let G be a Δ-critical graph and $xy \in E(G)$ such that $d(x)+d(y)=\Delta+3$ and $5 \le d(x), d(y) \le \Delta-2$. Suppose that y has no $(\Delta-2)^-$-neighbors other than x. Then one of the following holds:*

(1) except the neighbor x, y has $d(y)-2$ Δ-neighbors, each of which has at most one $(\Delta-2)^-$-neighbor distinct from x or y and one $(\Delta-1)^+$-neighbor adjacent to $\Delta-d(y)+1$ Δ-vertices;

(2) except the neighbor x, y has $d(y)-3$ Δ-neighbors, each of which has at most one $(\Delta-2)^-$-neighbor distinct from x or y, one $(\Delta-1)$-neighbor having no $(\Delta-2)$-neighbors except x or y, and one Δ-neighbor adjacent to $\Delta-d(y)+1$ Δ-vertices.

Proof. Let $d(x)=d$, $d(y)=s$, $x_i \in N(x)$ for $i \in \{1,\ldots,d\}$ and $y_j \in N(y)$ for $j \in \{d-2, d-1, \ldots, \Delta\}$, where $y=x_d$ and $x=y_{d-2}$. Consider $G-xy$. Since G is critical, $G-xy$ is Δ-colorable. Let f be a Δ-edge coloring of $G-xy$. Without loss of generality, we assume that $f(xx_i)=i$ for $i \in \{1,\ldots,d-1\}$ and $f(yy_j)=j$ for $j \in \{d-1,\ldots,\Delta\}$. Then either $F=\{yx, x, yy_d, y_d, \ldots, y_\Delta\}$ or $F=\{yx, x, yy_{d-1}, y_{d-1}, yy_d, y_d, \ldots, y_\Delta\}$ is a maximal fan at y with respect to yx and f.

If $F=\{yx, x, yy_d, y_d, \ldots, y_\Delta\}$, then by the fan equation $\sum_{z\in V(F)}(d(z)+1-\Delta)=2$, one can conclude that $d(y_j)=\Delta$ for $j \in \{d,\ldots,\Delta\}$. By Lemma 2.1.6, y_j is adjacent to at most one neighbor of degree at most $(\Delta-2)$ distinct from x or y. By our condition and Lemma 2.1.7, we have that y_{d-1} is a $(\Delta-1)^+$ vertex that is adjacent to at least $\Delta-d(y)+1$ Δ-vertices. Since $d(y)+d=\Delta+3$, $\Delta-d+1=d(y)-2$. Thus (1) holds in this case.

If $F=\{yx, x, yy_{d-1}, y_{d-1}, yy_d, y_d, \ldots, y_\Delta\}$, then by the fan equation $\sum_{z\in V(F)}(d(z)+1-\Delta)=2$, one can conclude that $d(y_j)=\Delta$ for $j \in \{d-1,\ldots,\Delta\}-\{i\}$ and $d(y_i)=\Delta-1$ for some $i \in \{d,\ldots,\Delta\}$. By Lemma 2.1.6, y_j is adjacent to at most one neighbor of degree at most $(\Delta-2)$ distinct from x or y and y_i is adjacent to no $(\Delta-2)^-$-neighbors except x or y. By Lemma 2.1.7, y_{d-1} is adjacent to at least $\Delta-d(y)+1$ Δ-vertices. Thus (2) holds in this case. □

A useful tool similar to the fans in Definition 2.1.1 is the broom (see Figure 2.). Let $xy \in E(G)$ and let $v \in N(y)$ such that $f(yv) \in \overline{c}_f(x)$. Let $B_f(v)=\{w \in N(v) : f(vw) \in \overline{c}_f(x) \cup \overline{c}_f(y)\} \setminus \{x,y\}$. The **broom** at x with respect to y and v is $B_f(v) \cup \{x,y\}$. The following lemma appears in [8], which shows that $B_f(v) \cup \{x,y,v\}$ is elementary under certain conditions.

Lemma 2.1.21. *(Chen, Chen, and Zhao [8]) Let $v \in B_f(y)$. If $min\{d(y), d(v)\} < \Delta$ and $|\overline{c}_f(x) \cup \overline{c}_f(y)| \ge 4$, then $B_f(v) \cup \{x,y,v\}$ is elementary.*

We further expand the broom to include fans. For each $u \in \{x,y\}$, let $B_f(u)=\{w \in N(u) : f(uw) \in \overline{c}_f(x) \cup \overline{c}_f(y)\}$ and for each $v \in B_f(u)$ denote $B_f(v)=\{w \in N(v) : f(vw) \in$

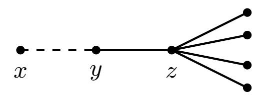

Figure 2.

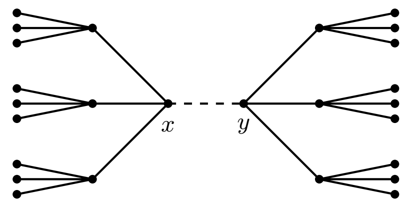

Figure 3.

$\overline{c}_f(x) \cup \overline{c}_f(y)\} \setminus \{x, y\}$. Let $B = [\bigcup_{w \in B_f(x) \cup B_f(y)} B_f(w)] \cup B_f(x) \cup B_f(y)$. We call $B \cup \{x, y\}$ the **extended broom** at xy (see Figure 3.).

The following lemma extends the above lemma.

Lemma 2.1.22. *Let G be a Δ-critical graph with $xy \in E(G)$ and $\max\{d(x), d(y)\} < \Delta$. Let f be a Δ-coloring of $G - xy$.*

(1) If $|\overline{c}_f(x) \cup \overline{c}_f(y)| \geq 4$, then $B_f(x) \cup B_f(y) \cup B_f(w) \cup \{x, y\}$ is elementary for each $w \in B_f(x) \cup B_f(y)$.

(2) If $|\overline{c}_f(x) \cup \overline{c}_f(y)| \geq 5$, then $B \cup \{x, y\}$ is elementary.

Proof. Suppose not. Thus, there exist $u, v \in B_f(x) \cup B_f(y) \cup B_f(w) \cup \{x, y\}$ for (1) (or in $B \cup \{x, y\}$ for (2)) with $u \neq v$ and $\sigma \in \overline{c}_f(u) \cap \overline{c}_f(v)$. Note that $u, v \notin \{x, y\}$, and, by Lemma 2.1.6, we have $\sigma \in c_f(x) \cap c_f(y)$. Then, by Lemma 2.1.21 and by symmetry, we have the following possible configurations for u, v (see Figure 4.).

(1) $u \in B_f(x)$ and $v \in B_f(y)$;

(2) $u \in B_f(x)$ and $v \in B_f(w)$ for some $w \in B_f(y) \setminus \{u\}$;

(3) $u, v \in B_f(y)$;

(4) $u \in B_f(y)$ and $v \in B_f(w)$ for some $w \in B_f(y) \setminus \{u\}$;

(5) $u \in B_f(w)$ for some $w \in B_f(y)$ and $v \in B_f(z)$ for some $z \in B_f(y) \setminus \{w\}$;

(6) $u \in B_f(w)$ for some $w \in B_f(x)$ and $v \in B_f(z)$ for some $z \in B_f(x) \setminus \{w\}$.

Note that the conditions $|\overline{c}_f(x) \cup \overline{c}_f(y)| \geq 4$ in (1) and $|\overline{c}_f(x) \cup \overline{c}_f(y)| \geq 5$ in (2) guarantee that in each case, there is some $\gamma \in \overline{c}_f(x) \cup \overline{c}_f(y)$ not otherwise labeled at a relevant vertex. WLOG, assume that $\gamma \in \overline{c}_f(y)$. Note that $\gamma \in c_f(w)$ for every $w \in B$. Consider $L_{\gamma,\sigma}(y)$: since this is a path with y as an endvertex, it cannot contain both u and v, and thus by switching

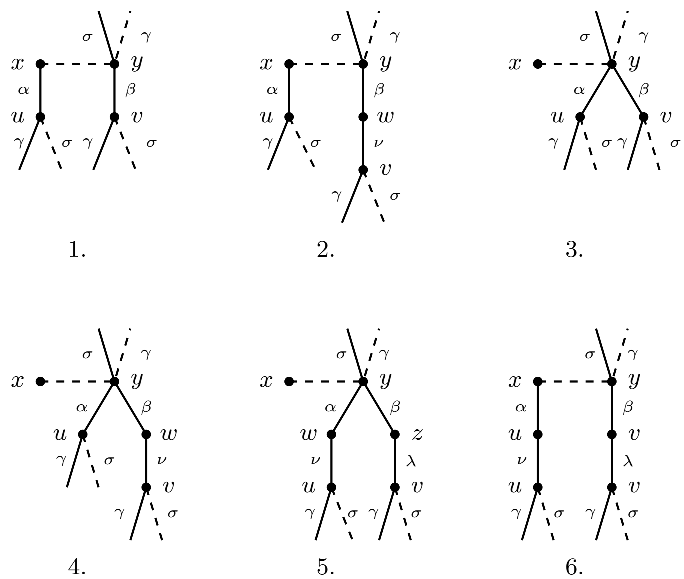

Figure 4.

colors along this path we obtain a new coloring with σ not seen at y and a vertex in B, a contradiction to Lemma 2.1.6. $\square$

We have the following result as a direct consequence of Lemma **??**.

Corollary 2.1.23. *Let G be a Δ-critical graph and xy be an edge of G with $d(x)+d(y) = \Delta+2+\eta$ and $\max\{d(x), d(y)\} \leq \Delta - 1$. Let f be a Δ-coloring of $G - xy$.*

(1) If $\eta \leq \Delta - 4$, then for each $u \in B_f(y)$, $|\cup_{w \in B_f(u) \cup \{u\}} \overline{c}_f(w)| \leq \eta$.

(2) If $\eta \leq \Delta - 5$, then $|\bigcup_{w \in B} \overline{c}_f(w)| \leq \eta$.

Proof. The conclusion follows from Lemma **??** with the facts that a subset of an elementary set is elementary and $|\overline{c}_f(x) \cup \overline{c}_f(y)| = 2\Delta - (d(x) + d(y)) + 2 = \Delta - \eta$. $\square$

Corollary 2.1.24. *Let G be a Δ-critical graph with minimum degree $\delta \geq 6$ and $|V| \geq \Delta + 4$. Let xy be an edge in G with $d(x) + d(y) = \Delta + 2 + \eta$ and $\max\{d(x), d(y)\} < \Delta$ where $\eta \leq \Delta - 4$. Then we have the following.*

(1) $\sum_{w \in V}(d(w) - 6) \geq \Delta^2 - (\eta + 6)\Delta + (\eta + 3)\delta - 22$.

(2) Furthermore, if G can be embedded in a surface with Euler characteristic χ, then

$$\eta \geq \lceil \frac{\Delta^2 - 6\Delta + 3\delta - 22 + 6\chi}{\Delta - \delta} \rceil.$$

Proof. Let f be a Δ-coloring of $G - xy$.

(1) Let $v \in B_f(y)$ and consider $B_f(v) \cup \{x, y, v\}$. By Lemma 2.1.6, $\overline{c}_f(x) \cup \overline{c}_f(y) \subseteq c_f(w)$ for each $w \in B_f(v) \cup \{v\}$. Note that $|\overline{c}_f(x) \cup \overline{c}_f(y)| = \Delta - \eta$. Since $B_f(v)$ doesn't include x or y, we have $|B_f(v) \cup \{v\}| \geq \Delta - \eta - 2 + 1 = \Delta - \eta - 1$.

By Corollary 2.1.23, we have that $|\bigcup_{w\in B_f(v)\cup\{v\}}\overline{c}_f(w)| = \sum_{w\in B_f(v)\cup\{v\}}(\Delta - d(w)) \leq \eta$. From this, we have

$$\sum_{w\in B_f(v)\cup\{v\}}(d(w)-\delta) = \sum_{w\in B_f(v)\cup\{v\}}(d(w)-\Delta+\Delta-\delta) \geq (\Delta-\eta-1)(\Delta-\delta)-\eta.$$

Hence by Lemma 2.1.19-(1),

$$\begin{aligned}\sum_{u\in V}(d(u)-6) &\geq (\Delta+4)(\delta-6)+d(x)+d(y)-2\delta+\sum_{w\in B_f(v)\cup\{v\}}(d(w)-\delta)\\ &\geq (\Delta+4)(\delta-6)+\Delta+\eta+2-2\delta+(\Delta-\eta-1)\Delta-\eta\\ &= \Delta^2-(6+\eta)\Delta+(3+\eta)\delta-22.\end{aligned}$$

(2) If G is embedded in a surface of Euler characteristic χ, then by Lemma 2.1.19 again, we have

$$-6\chi \geq \Delta^2-(6+\eta)\Delta+(3+\eta)\delta-22.$$

The bound for η follows. □

The following lemma is an easy application of Corollary 2.1.24 together with $-6\chi \geq \sum_{u\in V}(d(u)-6)$.

Lemma 2.1.25. *Let G be a Δ-critical graph with minimum degree $\delta \geq 6$ and $|V| \geq \Delta+4$ which can be embedded in a surface with Euler characteristic χ. Let xy be an edge in G with $d(x)+d(y) = \Delta+2+\eta$ where $\eta \leq \Delta-4$. Then we have the following.*

(a) If $\eta \leq 1$, then $\Delta \leq H(\chi)$.

(b) If $\eta \leq 3$, then $\Delta \leq H(\chi)+1$.

Lemma 2.1.26. *Let G be a Δ-critical graph with minimum degree $\delta \geq 6$ and $|V| \geq \Delta+4$ which can be embedded in a surface with Euler characteristic χ. Let $xy \in E$ with $d(x) \leq \Delta-2$ so that there is an integer $k \geq 0$ such that either y is adjacent to $d(x)+d(y)-\Delta-2-k$ $(2\Delta-(d(x)+d(y))+1)^-$-vertices $u \neq x$, where $d(x)+d(y)-\Delta-2-k \geq 1$; or y is adjacent to $d(x)+d(y)-\Delta-2-k$ $(2\Delta-(d(x)+d(y))+2)^-$-vertices $u \neq x$, where $d(x)+d(y)-\Delta-2-k \geq 2$. Then we have the following:*

(i) If $k \leq 1$ and $\Delta \geq 8$, then $\Delta \leq H(\chi)$.

(ii) If $k = 2, d(x) \leq \frac{\Delta+1}{2}$, and either $\delta \geq 8$ or $d(y) < \Delta$, then $\Delta \leq H(\chi)$.

(iii) If $k \leq 3$ or $k = 4$ and $d(x) \leq \frac{\Delta+1}{2}$, then $\Delta \leq H(\chi)+1$.

Proof. By Lemma 2.1.25, we may assume that $d(x)+d(y) \geq \Delta+4$. Define s as $s = 0$ if $d(x) \geq \frac{\Delta+2}{2}$ and $s = 1$ if $d(x) \leq \frac{\Delta+1}{2}$ and define t as $t = 0$ if $d(y) = \Delta$ and $t = 1$ if $d(y) < \Delta$.

Then, by Lemma 2.1.16-(1), y has a neighbor $y' \neq x$ such that $d(y') \geq \Delta-k$, and y' has at least $\Delta-k-2+s$ neighbors distinct from y of degree at least $\Delta-k-1+t$. Thus

$n_{\Delta-k-1+t} \geq \Delta - k - 2 + s + 1 + (1-t)$. Note that the lower bound doesn't count x and if $d(y) < \Delta$, neither does it count y.

Thus by Lemma 2.1.19, we have

$$-6\chi \geq (\Delta+4)(\delta-6)+(\Delta-\delta+2)(\Delta-\delta)+[(\Delta-k+s-t)-(\Delta-\delta+2)](\Delta-k-1+t-\delta)+t(\Delta+4-2\delta).$$

Hence

$$-6\chi \geq \Delta^2 - (k+6-s)\Delta + k^2 + (3-s)k + (3-s)\delta + s(t-1) - t^2 + 3t - 22.$$

It is easy to check all cases and our conclusion holds. □

2.2 $\Delta(\Sigma)$ for $\chi(\Sigma) = -6, -7$

2.2.1 Discharging Rules

Let G be an 11-critical graph that is embedded in a surface Σ of characteristic $\chi(\Sigma) \in \{-6, -7\}$.

For each vertex x, let $M(x) = d(x) - 8.63$ be the *initial charge* of x and for each face f, let $M(f) = 2(d(f) - 3)$ be the *initial charge* of f. We redistribute the initial charge of each element in $V(G) \cup F(G)$ according to the following rules.
(R1) Each 9^+-vertex x sends $\frac{M(x)}{d_{8^-}(x)} = \frac{d(x)-8.63}{d_{8^-}(x)}$ to each adjacent 8^--vertex if $d_{8^-}(x) \neq 0$.
(R2) Each 4^+-face f sends $\frac{2(d(f)-3)}{d_2(f)}$ to each incident 2-vertex if $d_2(f) \neq 0$.

Let $M'(x)$ denote the *new charge* of x for each $x \in V(G) \cup F(G)$. We have the following observations:

(I) Let x be a vertex with $d(x) \in \{2, 9, 10, 11\}$. Then $M'(x) \geq 0$. Moreover if $d(x) \geq 9$ and $d_{8^-}(x) = 0$, then $M'(x) = M(x) = d(x) - 8.63$.

By (R1), (I) is obvious if $d(x) \geq 9$.

If $d(x) = 2$, then by Lemma 2.1.11, each neighbor of x is an 11-vertex which has only one 8^--neighbor. Thus x receives 2.37 from each of its neighbors. Since G is simple, x is adjacent to a 4^+-face which, by Lemma 2.1.11, is adjacent to at most $\lfloor \frac{d(f)}{4} \rfloor$ 2-vertices, and thus x receives at least $\frac{2(d(f)-3)}{\lfloor \frac{d(f)}{4} \rfloor} \geq 2$ from each adjacent 4^+-face f. Therefore $M'(x) \geq 2 - 8.63 + 2 \times 2.37 + 2 > 0$.

(II) Let x and y be two adjacent vertices with $3 \leq d(x) \leq d(y)$ and $d(x) + d(y) = 13 = \Delta + 2$. If $d(y) \geq 9$, then $M'(x) + M'(y) = M'(x) \geq 0.48$ and y has at least $d(y) - d(x) \geq 5$ 11-neighbors z with $M'(z) = M(z) = 2.37$; otherwise $M'(x) + M'(y) > 2.55$ and $M'(w) \geq 1.11$ for each $w \in \{x, y\}$.

By Lemma 2.1.11, all vertices in $N(x, y) \cup N(N(x, y))$ except x and y have degree 11.

If $d(y) \geq 9$, then $d(x) \leq 4$ and y has $d(y) - d(x) \geq 5$ 11-neighbors not adjacent to x and thus having no 8^--neighbors. By (R1), $M'(x) + M'(y) = M'(x) \geq d(x) - 8.63 + d(y) - 8.63 + 2.37(d(x) - 1) = 2.37(d(x) - 1) - 4.26 \geq 0.48$.

Assume $d(y) \leq 8$. Then $d(y) \geq 7$, $|N(x, y) \setminus \{x, y\}| \geq d(y) - 1$, and each vertex in $N(x, y)$ has no 8^--neighbor other than x or y. Thus $M'(x) + M'(y) \geq d(x) - 8.63 + d(y) - 8.63 + \frac{2.37}{2}(d(y) - 1) > 2.55$ since $d(y) \geq 7$. Further, for $w \in \{x, y\}, M'(w) \geq d(w) - 8.63 + (d(w) - 1) \times \frac{2.37}{2} \geq 1.11$.

(III) Let x be a 3-vertex. Then $M'(x) > 0$.

By (II), if x has a 10-neighbor, then $M'(x) \geq 0.48$. If x is adjacent to three 11-vertices, then by Lemma 2.1.7, each 11-neighbor is adjacent to at most two 8^--vertices and by Lemma 2.1.8, x has two 11-neighbors, each of which is adjacent to only one 8^--vertex. Thus $M'(x) \geq 3 - 8.63 + 2.37 \times 2 + \frac{2.37}{2} > 0$.

(IV) Let x be a 4-vertex with four 10^+-neighbors. Then $M'(x) \geq 0.11$.

If x has a neighbor y adjacent to $d(y)-11+2$ 8^--vertices other than x, then by Lemma 2.1.16 (taking $k = 0, d = 4$, and $s \in \{10, 11\}$), x has three 11-neighbors other than y, each of which is adjacent to only one 8^--vertex. Thus

$$M'(x) \geq -4.63 + \min\{1.37/2 + 2.37 \times 3, 2.37/3 + 2.37 \times 3\} = 3.165 > 2.85.$$

If each neighbor y of x is adjacent to at most $d(y) - 11 + 1$ 8^--vertices other than x, then each 10-neighbor sends 1.37 to x and each 11-neighbor sends at least $\frac{2.37}{2} = 1.185$ to x. Since $1.185 < 1.37$, $M'(x) \geq -4.63 + 4 \times 2.37/2 = 0.11$.

(V) Let x and y be adjacent vertices with $5 \leq d(x) \leq d(y)$ and $d(x) + d(y) = 14 = \Delta + 3$. If $d(x) = 5$ and $d(y) = 9$, then $M'(x) + M'(y) = M'(x) > 0.75$, $n_9 \geq 1$, and $n_{10} + n_{11} \geq 10$; otherwise $M'(x) + M'(y) > 2.55$ and $M'(w) \geq 2.11$ for each $w \in \{x, y\}$.

First, assume $d(x) \geq 6$, and hence $d(y) \leq 8$.

Consider the case when $w \in \{x, y\}$ has another 9^--neighbor distinct from x, y. By Lemma 2.1.16 (taking $k = 0, d \in \{6, 7\}$, and $s \in \{7, 8\}$), all other neighbors of x and y must have degree 11 and have no 8^--neighbors distinct from x and y. Therefore

$$M'(x) + M'(y) \geq d(x) - 8.63 + d(y) - 8.63 + (d(y) - 2)\frac{2.37}{2} + (d(x) - 1)\frac{2.37}{2} > 2.55$$

Further, $M'(w) \geq d(w) - 8.63 + (d(w) - 2)\frac{2.37}{2} = 2.11$.

Now we assume that x and y have no other 9^--neighbors distinct from x or y. By Lemma 2.1.20, for each vertex $w \in \{x, y\}$, w is adjacent to $d(w) - 3$ 11-vertices z such that z has at most three 8^--neighbors and one 10^+-vertex u having at most $d(u)-10+2$ 8^--neighbors. Since $\frac{1.37}{2} < \frac{2.37}{3}$, w receives at least $(d(w)-3)\frac{2.37}{3} + \frac{1.37}{2}$ from its neighbors. Therefore

$$M'(x)+M'(y) \geq d(x)+d(y)-17.26+1.37+(d(x)+d(y)-6)\frac{2.37}{3} = -3.26+\frac{18.96}{3} > 2.55.$$

Now we assume $d(x)=5$ and $d(y)=9$.

If y is adjacent to a 9^--vertex other than x, then by Lemma 2.1.16 (taking $k=0, d=5, s=9$), x has four 11-neighbors, each of which is adjacent to ten 9^+-vertices. Hence we have

$$M'(x)+M'(y) = M'(x) \geq 5-8.63+\frac{0.37}{2}+4\times 2.37 \geq 5.85.$$

Now assume that y is adjacent to only one 8^-vertex, which is x. Since $d(y)=9$, by Lemma 2.1.20, except the neighbor y, either x has three 11-neighbors each of which is adjacent to at most two 8^--vertices and one additional 10^+-neighbor distinct from y with at least $11-5+1=7$ 11-neighbors; or x has two 11-neighbors each of which is adjacent to at most two 8^--vertices, one 10-neighbor which is adjacent to only one 8^--vertex, and one 11-neighbor which is adjacent to at least 7 11-vertices. Note $\min\{\frac{2.37}{2}, 1.37\} + \min\{\frac{2.37}{11-7}, \frac{1.37}{10-7}\} = \frac{2.37}{2} + \frac{1.37}{3}$. Therefore

$$M'(x)+M'(y) = M'(x) \geq 5-8.63+0.37+2\times\frac{2.37}{2}+\frac{2.37}{2}+\frac{1.37}{3} > 0.75.$$

It is obvious $n_9 \geq 1$. By Lemma 2.1.20, y has seven 11-neighbors each of which is adjacent to at most one 9^--vertex distinct from x and y. Since $d(x)=5$, there must be such an 11-vertex which is nonadjacent to x and thus has at least nine 10^+-neighbors. Therefore $n_{10}+n_{11} \geq 10$.

(VI) Let x be a 5-vertex with five 10^+-neighbors. Then $M'(x) \geq 1.295$.

Let $y \in N(x)$ such that $k = d(y)-d_{8^-}(y)-7$ is the minimum among all neighbors of x.

We first consider the case when $k \leq 1$. Then $d_{8^-}(y) \geq 2$ if $d(y)=10$ and $d_{8^-}(y) \geq 3$ if $d(y)=11$. Note $8 \leq 2\Delta-(5+\Delta)+2$. By Lemma 2.1.16 (taking $k \leq 1$, $d=5$ and $s=d(y)$), x has $5-k-1=4-k$ neighbors, each of which is adjacent to at least $10-k$ $(10-k)^+$-vertices. Denote the set that contains such neighbors of x by A_1 and let $A_2 = N(x)\setminus A_1$. Thus $|A_1| \geq 4-k$ and $|A_2| \leq 1+k$. Then we have

$$M'(x) \geq -3.63+\sum_{z\in A_1}\frac{d(z)-8.63}{d_{8^-}(z)}+\sum_{z\in A_2}\frac{d(z)-8.63}{d_{8^-}(z)}.$$

Let $z \in A_2$. Then $d_{8^-}(z) \leq d(z)-k-7$ by the choice of y. Since $k \leq 1$ and $d(z) \geq 10$, we have

$$\frac{d(z)-8.63}{d_{8^-}(z)} \geq \frac{d(z)-8.63}{d(z)-k-7} \geq \frac{1.37}{3-k}.$$

Let $z \in A_1$. Since $k \le 1$, we have

$$\frac{d(z)-8.63}{d_{8^-}(z)} \ge \frac{d(z)-8.63}{d(z)-10+k} \ge \frac{2.37}{1+k}.$$

Since $k \le 1$ and $\frac{2.37}{1+k} > \frac{1.37}{3-k}$, we have

$$M'(x) \ge -3.63 + (4-k)\frac{2.37}{1+k} + (1+k)\frac{1.37}{3-k} \ge 1.295.$$

Now we assume $k \ge 2$. Then by the choice of y, each 11-neighbor has at most two 8^--neighbors, each 10-neighbor has only one 8^--neighbor. Since $d_{11}(x) \ge 2$ and $d_{10}(x) + d_{11}(x) = 5$, we have

$$M'(x) \ge -3.63 + 1.37 d_{10}(x) + \frac{2.37}{2} d_{11}(x) \ge 2.295.$$

(VII) Let x be a 6-vertex with six 9^+-neighbors. Then

(VII-1) $M'(x) > 2$ if x has a neighbor z such that $d_{8^-}(z) \ge d(z) - 7$ if $d(z) \le 10$ or $d_{7^-}(z) \ge 4$ if $d(z) = 11$ or x has three 9-neighbors.

(VII-2) Otherwise $M'(x) \ge -2.63 + 0.37 d_9(x) + \frac{1.37}{2} d_{10}(x) + \frac{2.37}{5} d_{11}(x) \ge 0.006$. Moreover, if no neighbors of x is adjacent to an 8-vertex, then $M'(x) \ge 1.165$.

By Lemma 2.1.7, each neighbor u of x sends at least $\frac{d(u)-8.63}{d(u)-6}$ to x. Note $\frac{0.37}{3} < \frac{1.37}{4} < \frac{2.37}{5}$.

If x is adjacent to three 9-vertices, then by Lemma 2.1.16 (taking $k = 0$, $d = 9$ and $s = 6$), x has three 11-neighbors, each of which is adjacent to at least nine 11-vertices, and the 9-vertices have no 8^--neighbors other than x. Hence we have $M'(x) \ge -2.63 + 0.37 \times 3 + \frac{2.37}{2} \times 3 = 2.035$.

Now we assume that x has a neighbor z such that $d_{8^-}(z) \ge d(z) - 7$ if $d(z) \le 10$ or $d_{7^-}(z) \ge 4$ if $d(z) = 11$. By Lemma 2.1.16 (taking $k \le 1$, $d = 6$ and $s = d(z)$), x is adjacent to at least $6 - k - 1 \ge 4$ 10^+-vertices, each of which is adjacent to at least nine 9^+-vertices. Since $\frac{2.37}{2} < 1.37$, we have $M'(x) \ge -2.63 + \frac{d(z)-8.63}{d(z)-6} + \frac{0.37}{3} + \frac{2.37}{2} \times 4 > 2$. This proves (VII-1).

To prove (VII-2), by (VII-1) we assume that x has at most two 9-neighbors, each 9-neighbor of x has only one 8^--neighbor, each 10-neighbor of x has at most two 8^--neighbors, and each 11-neighbor has at most three 7^--neighbors.

Thus we have $M'(x) \ge -2.63 + 0.37 d_9(x) + \frac{1.37}{2} d_{10}(x) + \frac{2.37}{5} d_{11}(x) \ge 0.006$.

If no neighbors of x is adjacent to an 8-vertex, then each 11-neighbor of x is adjacent to at most three 8^--neighbors. Thus $M'(x) \ge -2.63 + 0.37 d_9(x) + \frac{1.37}{2} d_{10}(x) + \frac{2.37}{3} d_{11}(x) \ge -2.63 + \min\{2 \times 0.37 + 1 \times \frac{1.37}{2} + 3 \times \frac{2.37}{3}, 4 \times \frac{1.37}{2} + 2 \times \frac{2.37}{3}\} = 1.165$.

(VIII) Let x be a 7-vertex with seven 8^+-neighbors. Then $M'(x) > 1.5$ if x has a 9-neighbor which is adjacent to two 8^--vertices distinct from x or if x has two 8-neighbors; otherwise $M'(x) \geq -1.63 + \frac{0.37}{2}d_9(x) + \frac{1.37}{5}d_{10}(x) + \frac{2.37}{6}d_{11}(x) \geq 0.295$. In particular, if $d_{11}(x) \geq 5$, then $M'(x) \geq 0.53$.

If x has two 8-neighbors, then by Lemma 2.1.16 (taking $k \leq 1$, $d = 8$ and $s = 7$), x is adjacent to four 10^+-vertices, each of which is adjacent to at least eight 10^+-vertices. Since $d_{11}(x) \geq 11 - 8 + 1 = 4$, at least three of such neighbors are 11-vertices. Hence we have $M'(x) \geq -1.63 + \min\{\frac{2.37}{3} \times 4, \frac{2.37}{3} \times 3 + \frac{1.37}{2} + \frac{2.37}{6}\} = 1.53$.

If x has a 9-neighbor y which is adjacent to two 8^--vertices distinct from x, then by Lemma 2.1.16 (taking $k \leq 1, d = 7, s = 9$), we have that x is adjacent to five 10^+-vertices, each of which is adjacent to at least eight 10^+-vertices. Since $d_{11}(x) \geq 3$, at least two of such neighbors are 11-vertices. Thus $M'(x) \geq -1.63 + \frac{0.37}{4} + \frac{1.37}{2} \times 3 + \frac{2.37}{3} \times 2 > 1.5$.

Now we assume that each 9-neighbor of x is adjacent to at most one 8^--vertex distinct from x and x has at most one 8-neighbor. Thus

$$M'(x) \geq -1.63 + \frac{0.37}{2}d_9(x) + \frac{1.37}{5}d_{10}(x) + \frac{2.37}{6}d_{11}(x).$$

If $d_{11}(x) \geq 5$, then since x has at most one 8-neighbor, we have

$$M'(x) \geq -1.63 + \frac{0.37}{2} \times 1 + \frac{2.37}{6} \times 5 = 0.53.$$

If $d_{11}(x) = 4$, then since x has at most one 8-neighbor, we have

$$M'(x) \geq -1.63 + \frac{0.37}{2} \times 2 + \frac{2.37}{6} \times 4 = 0.32 > 0.295.$$

If $d_{11}(x) = 3$, then x has no 8-neighbor. Thus

$$M'(x) \geq -1.63 + \frac{0.37}{2} \times 4 + \frac{2.37}{6} \times 3 = 0.295.$$

If $d_{11}(x) = 2$, then $d_{10}(x) = 5$ and thus

$$M'(x) \geq -1.63 + \frac{1.37}{5} \times 5 + \frac{2.37}{6} \times 2 = 0.53.$$

(IX) Let x be an 8-vertex with eight 7^+-neighbors. Then $M'(x) \geq 2.53$ if x has three 8^--neighbors; otherwise $M'(x) \geq -0.63 + \frac{0.37}{5}d_9(x) + \frac{1.37}{6}d_{10}(x) + \frac{2.37}{7}d_{11}(x) \geq 0.75$. In particular, if x has a 7-neighbor, then $M'(x) > 1.06$

Note that by Lemma 2.1.7, each 9^+-neighbor y of x is adjacent to at least four 11-vertices and thus sends at least to $\frac{d(y)-8.63}{d(y)-4}$ to x. Therefore, $M'(x) \geq -0.63+\frac{0.37}{5}d_9(x)+\frac{1.37}{6}d_{10}(x)+\frac{2.37}{7}d_{11}(x)$. Note $\frac{0.37}{5} < \frac{1.37}{6} < \frac{2.37}{7}$.

If x has a 7-neighbor, then $d_{11}(x) \geq 5$ and thus we have that $M'(x) \geq -0.63 + \frac{2.37}{7} \times 5 > 1.06$.

Assume x is adjacent to three 8^--vertices. By Lemma 2.1.16 (taking $k \leq 1$, $d \leq 8$ and $s = 8$), x is adjacent to four 10^+-vertices, each of which is adjacent to at least eight 10^+-vertices. In this case $d_{11}(x) \geq 4$. Hence three of such 10^+-neighbors are 11-vertices and we have $M'(x) \geq -0.63 + \min\{\frac{2.37}{3} \times 4, \frac{2.37}{3} \times 3 + \frac{1.37}{2} + \frac{2.37}{7}\} = 2.53$.

If x has an 8-neighbor and has at most two 8^--neighbors, then $d_{11}(x) \geq 4$. Thus $M'(x) \geq -0.63 + \frac{0.37}{5} \times 2 + \frac{2.37}{7} \times 4 \geq 0.87$.

If x has no 8^-neighbor, then either $d_{11}(x) = 2$ and $d_{10}(x) = 6$ or $d_{11}(x) \geq 3$. Thus $M'(x) \geq -0.63 + \min\{\frac{0.37}{5} \times 5 + \frac{2.37}{7} \times 3, \frac{1.37}{6} \times 6 + \frac{2.37}{7} \times 2\} \geq 0.75$.

2.2.2 Main Theorem

Theorem 2.2.1. *Let Σ be a surface of characteristic $\chi(\Sigma) \in \{-6, -7\}$. Then $\Delta(\Sigma) = 10$.*

Proof. First we show that there is a class two graph of maximum degree 10 can be embedded in a surface of characteristic $\chi(\Sigma) \in \{-6, -7\}$.

Consider a surface Σ of characteristic -6. By [14], K_{10} can be embedded on the surface Σ and such an embedding has twenty seven 3-faces, one 4-face and one 5-face. Here we use the same labels $\{0, 1, 2, 3, 4, 5, 6, x, y, z\}$ for vertices of K_{10} as in [14] and a 3-face of the embedding has a facial walk 054, the 4-face has a facial walk $yx05$, and the 5-face has a facial walk $401xz$. We construct G from K_{10} by deleting the edge 04, which creates a 6-face with facial walk $54zx10$, and then adding a vertex v and edges $v5, v4, vz, vx, v1, v0$ in the 6-face of the embedding and adding the edge $0y$ in the 4-face. Then G has maximum degree 10 with 11 vertices, 51 edges and thus is overfull. Hence G is class two and clearly G is embedded in a surface Σ of characteristic -6.

Now we consider a surface Σ of characteristic -7. Since K_{11} with one edge missing can be embedded on Σ and since K_{11} with one edge missing is a class two graph, we have a class two graph of maximum degree 10 that is embedded in a surface Σ of characteristic -7.

Hence, if our theorem is not true, then there is a class two graph H of maximum degree 11 embedded in Σ. Let G be a 11-critical graph that is obtained from H by deleting vertices and edges. By [30], G that is embedded in Σ cannot be embedded in any surface of characteristic greater than $\chi(\Sigma)$.

By Euler formula $|V(G)| + |F(G)| - |E(G)| = \chi(\Sigma)$, we have the following two equations:

$$\sum_{x \in V(G)} (d(x) - 8.63) + \sum_{f \in F(G)} 2(d(f) - 3) = -6\chi(\Sigma) - 2.63|V|, \tag{2.1}$$

$$n_7 + 2n_8 + 3n_9 + 4n_{10} + 5n_{11} = -6\chi(\Sigma) + 4n_2 + 3n_3 + 2n_4 + n_5 - \sum_{f \in F(G)} 2(d(f) - 3). \quad (2.2)$$

By (I-IX), we have

$$0 \le 0.11n_4 + 0.75n_5 + 0.006n_6 + 0.295n_7 + 0.75n_8 \le \sum_{x \in V(G)} M'(x). \quad (2.3)$$

By (I-IX) and (2.1), we have

$$0 \le \sum_{x \in V(G)} M'(x) \le \sum_{x \in V(G) \cup F(G)} M'(x) \le \sum_{x \in V(G) \cup F(G)} M(x) \le -6\chi(\Sigma) - 2.63n. \quad (2.4)$$

As a direct application of Proposition 2.1.2, we obtain the following:

Claim 2.2.1. *$n_{11} \ge 13 - \delta$, where δ is the minimum degree of G.*

Claim 2.2.2. *$|V(G)| = 15$, $\chi(\Sigma) = -7$, and hence $\sum_{x \in V(G)} M'(x) \le \sum_{x \in V(G) \cup F(G)} M'(x) \le 42 - 2.63|V(G)| = 2.55$.*

Proof. By (4), it is clear that $|V(G)| \le 15$. Hence we only need to show that $|V(G)| \ge 15$. Suppose $|V(G)| \le 14$. Then by Lemma 2.1.14, $|V(G)| \in \{11, 13\}$ and the average degree of G, $d_{avg}(G) \ge \frac{2(5 \times 11 + 1)}{11} = \frac{112}{11}$ if $|V(G)| = 11$ and $d_{avg}(G) \ge \frac{2(6 \times 11 + 1)}{13} = \frac{134}{13}$ if $|V(G)| = 13$. On the other hand, by Euler formula, we have $d_{avg}(G) \le 6 + \frac{42}{|V(G)|}$. Then $d_{avg}(G) \le 6 + \frac{42}{11} < \frac{112}{11}$ if $|V(G)| = 11$; and $d_{avg}(G) \le 6 + \frac{42}{13} < \frac{134}{13}$ if $|V(G)| = 13$, a contradiction. Therefore, $|V(G)| \ge 15$ and thus $|V(G)| = 15$. By (4), no 11-critical graph can be embedded in Σ of characteristic -6. Thus $\chi(\Sigma) = -7$, and we have $0 \le \sum_{x \in V(G)} M'(x) \le 42 - 2.63|V(G)| = 2.55$. $\square$

Claim 2.2.3. *$\delta \ge 4$.*

Proof. Suppose to the contrary $\delta \le 3$. Let x be a 3^--vertex.

If $d(x) = 3$, then by Lemma 2.1.9, there are four 11-vertices z in G with $d_{\le 8}(z) = 0$.

If $d(x) = 2$, then by Lemma 2.1.11, every vertex with distance 2 from x is an 11-vertex which has eleven 11-neighbors. Note that there are at least $11 - 2 = 9$ vertices with distance 2 from x.

In either case, there are two 11-vertices z in G with $d_{\le 8}(z) = 0$, say z_1, z_2. By (I), we have $\sum_{x \in V(G)} M'(x) \ge M'(z_1) + M'(z_2) = 2.37 + 2.37 > 2.55$, a contradiction to Claim 2.2.2. $\square$

Claim 2.2.4. *For each edge $xy \in E(G)$, $d(x) + d(y) \ge \Delta + 3 = 14$. Further, if $5 \le d(x) \le d(y)$, then $d(x) + d(y) \ge \Delta + 4 = 15$.*

Proof. Let $xy \in E(G)$ with $d(x) \le d(y)$. If $d(x) + d(y) = \Delta + 2 = 13$, then by Claim 2.2.2 and (II), $d(y) \ge 9$. Further, by Claim 2.2.3 and (II), $d(x) \ge 4$ and there are at least five 11-vertices z with $M'(z) = 2.37$, a contradiction to Claim 2.2.2.

If $d(x)+d(y) = 14$ and $5 \le d(x) \le d(y)$, by Claim 2.2.2 and (V), we have $d(x) = 5$, $d(y) = 9$, $n_9 \ge 1$, and $n_{10} + n_{11} \ge 10$. By Claim 2.2.1, $n_{11} \ge 8$.

Therefore by (2.2) and Claim 2.2.3, $51 \le 42 + n_5 + 2n_4$. Thus $n_4 + n_5 \ge 5$ and $n > 15$, a contradiction to Claim 2.2.2. This proves the claim. □

Claim 2.2.5. $\delta \le 6$.

Proof. Suppose to the contrary $\delta \ge 7$. Then by (2.2) we have

$$15 + n_8 + 2n_9 + 3n_{10} + 4n_{11} = 30 - n_7 + n_9 + 2n_{10} + 3n_{11} = n_7 + 2n_8 + 3n_9 + 4n_{10} + 5n_{11} \le 42.$$

Thus $n_{11} \le 6$. If $n_7 = 0$, then we have $45 - n_8 + n_{10} + 2n_{11} \le 42$, which implies $n_8 \ne 0$. By Claim 2.2.1, $n_{11} \ge 5$ and we have $30 + 3 \times 5 \le 30 + 3n_{11} \le 42$, a contradiction.

Thus $n_7 \ne 0$ and by Claim 2.2.1, $n_{11} \ge 13-7 = 6$. Hence $n_{11} = 6$ and $39+n_8+2n_9+3n_{10} \le 42$. Thus $n_9 + n_{10} \le 1$ and $n_8 + n_9 + n_{10} \le 3$. Since $n = 15$, $n_7 \ge 6$.

Let x be a 7-vertex. By Claim 2.2.4, x has no 7-neighbors.

Assume $n_8 \ne 0$, and let z be an 8-vertex. By (IX), $M'(z) \ge 0.75$. Clearly, x is either adjacent to at least two 8-vertices or at most one 8-vertex. In the later case, we have $d_{11}(x) \ge 5$ due to $n_9 + n_{10} \le 1$. Hence in any case, by (VIII), we always have $M'(x) \ge 0.53$, which implies that $3.93 = 0.75 + 6 \times 0.53 \le \sum_{v \in V} M'(v) \le 2.55$, a contradiction.

Now we assume $n_8 = 0$. Since $n_9 + n_{10} \le 1$, $d_{11}(x) \ge 6$. By (VIII), $M'(x) \ge 0.53$. Hence, $3.18 = 6 \times 0.53 \le \sum_{x \in V} M'(x) \le 2.55$, a contradiction. Hence our claim is true. □

Claim 2.2.6. *G is a triangulation.*

Proof. Since $n_2 = 0$ by Claim 2.2.3, by our discharging rules, we have

$$\sum_{x \in V(G) \cup F(G)} M'(x) = \sum_{x \in V(G)} M'(x) + \sum_{x \in F(G)} M(x).$$

Suppose that G is not a triangulation. Then by Claim 2.2.2, all faces of G are 3-faces except one that is a 4-face and thus we have $\sum_{x \in V(G)} M'(x) \le 0.55$. Hence by (IV), (VI) and (IX), we have $n_4 \le 5$ and $n_5 = n_8 = 0$. Since $n_8 = 0$, $n_6 = 0$ by (VII). Thus by Claims 2.2.3 and 2.2.5 $\delta = 4$ and by Claim 2.2.1 $n_{11} \ge 13-4 = 9$. By (IV) and (VIII), we have $0.11n_4 + 0.295n_7 \le 0.55$ and thus $n_7 \le 1$. Assume $n_7 = 1$ and let x be the 7-vertex. Since $n_5 = n_6 = n_8 = 0$, we have that each 10^+-vertex is adjacent to at most three 7^--vertices. Note by Lemma 2.1.7 either $d_{11}(x) \ge 3$ or $d_{10}(x) = 5$ and $d_{11}(x) = 2$. Hence by (R1), it is easy to check that $M'(x) > 0.55$, a contradiction. So, $n_7 = 0$.

Thus by (2.2), $3n_9 + 4n_{10} + 5n_{11} \leq 40 + 2n_4$. Since $n_{11} \geq 9$, we have $n_4 \geq 3$. By (IV), $M'(x) \geq 0.11$ for each 4-vertex x and thus $0.11n_4 \geq 0.33$. By Claim 2.2.4, no 4-vertex is adjacent to any 9-vertices. Hence if there is a 9-vertex $z \in G$, we have $M'(z) = 0.37$. Thus if $n_9 \neq 0$, then $\sum_{x \in V(G)} M'(x) \geq 0.11n_4 + 0.37n_9 > 0.55$, a contradiction. Therefore $n_9 = 0$. Since the degree sum is even and the only vertices of odd degree have degree 11, it follows that n_{11} is even. Thus $n_{11} \geq 10$. Since $50 + 4n_{10} \leq 4n_{10} + 5n_{11} \leq 40 + 2n_4$, $n_4 = 5$ and $n_{11} = 10$. Let x be a 4-vertex. Then x is adjacent to four 11-vertices. By (IV), each neighbor of x is adjacent to at most two 8^--vertices including x. Since G has only one 4-face, there is at least one neighbor of x, say u, that is not incident with the 4-face. By Lemma 2.1.13, u is not adjacent to any other 8^--vertices. Hence we have $M'(x) \geq -4.63 + 2.37 + 3 \times \frac{2.37}{2} = 1.295$, a contradiction. □

Claim 2.2.7. $n_4 = 0$.

Proof. Suppose to the contrary $n_4 \neq 0$. Then by Claim 2.2.3, $\delta = 4$. Let x be a 4-vertex. Since G is a triangulation, by Lemma 2.1.13, each neighbor of x is adjacent to only one 8^--vertex, which is x. Hence by (R1), we have $M'(x) \geq -4.63 + 2 \times 1.37 + 2 \times 2.37 = 2.85 > 2.55$, a contradiction to Claim 2.2.2. □

Claim 2.2.8. $n_5 + n_6 \geq 4$.

Proof. Suppose to the contrary $n_5 + n_6 \leq 3$. By Claim 2.2.5, $n_5 + n_6 \geq 1$. Denote $a = n_5 + n_6$, By (2.2), we have

$$(\delta - 6)a + n - a + n_8 + 2n_9 + 3n_{10} + 4n_{11} \leq 42.$$

By Claim 2.2.1, $n_{11} \geq 13 - \delta$ and by Claim 2.3.3, $\delta \leq 6$. If $n_{11} \geq 14 - \delta$, then from the above equation we have

$$44 \leq -\delta + 50 \leq 3(\delta - 6) + n - 3 + 4(14 - \delta) \leq 42,$$

a contradiction. Thus $n_{11} = 13 - \delta$.

If $\delta = 5$, then $n_{11} = 8$ and we have $41 = -6 + 15 + 32 \leq -2a + n + n_8 + 2n_9 + 3n_{10} + 4n_{11} \leq 42$. From the above inequalities, $n_8 + 2n_9 + 3n_{10} \leq 2a - 5 \leq 1$. Thus $a = 3, n_9 = n_{10} = 0$ and $n_8 \leq 1$. Furthermore, since $-n_5 + (15 - 3) + n_8 + 4 \times 8 \leq 42$, we have $n_8 \leq n_5 - 2$ and thus $n_5 \geq 2$. By (VI), if x is a 5-vertex, then $M'(x) \geq 1.295$. Hence we have $2.59 \leq 1.295n_5 \leq \sum_{x \in V} M'(x) \leq 2.55$, a contradiction. This implies $\delta = 6$. Thus $n_{11} = 13 - 6 = 7$ and

$$n - a + n_8 + 2n_9 + 3n_{10} + 4n_{11} = 15 - a + n_8 + 2n_9 + 3n_{10} + 28 \leq 42.$$

From the above inequalities, we obtain $n_8 + 2n_9 + 3n_{10} \leq a - 1 \leq 2$. Thus $n_{10} = 0, n_8 + 2n_9 \leq 2$ and $n_7 \geq 15 - 7 - 2 - a = 6 - a \geq 3$.

Let x be a 7-vertex. By Claim 2.2.4, x has no 7-neighbors. Since $n_8 + n_9 + n_{10} \leq 2$, $d_{11}(x) \geq 5$. Hence by (VIII), $M'(x) \geq 0.53$. From $0.53n_7 \leq \sum_{v \in V} M'(v) \leq 2.55$, it follows that

$n_7 \leq 4$. Thus, $n_6 = a \geq 2$. If $n_8 = 0$, then by (VII), $M'(u) \geq 1.165$ for each 6-vertex u and we have $\sum_{x \in V(G)} M'(x) > 2.55$, a contradiction. Hence $n_8 \neq 0$. Then $n_9 = 0$ since $n_8 + 2n_9 \leq 2$. Since the degree sum must be even and n_{11} is odd, n_7 must be odd. Thus $n_7 = 3$. Hence from $3 = n_7 \geq 6 - a \geq 3$, we have $n_6 = 3$. Since $n = 15, n_6 = 3, n_7 = 3, n_{11} = 7$, we have $n_8 = 2$. By (IX), for each 8-vertex v, $M'(v) \geq 0.75$. Therefore $3.09 \leq 3 \times 0.53 + 2 \times 0.75 \leq \sum_{v \in V} M'(v) \leq 2.55$. This contradiction completes the proof of the claim. $\square$

Claim 2.2.9. *$n_5 = 0$ and thus $\delta = 6$ and $n_6 \geq 4$.*

Proof. Suppose to the contrary $n_5 \neq 0$. Then $n_{11} \geq 13 - 5 = 8$ by Claim 2.2.1 and by (2.2), we have

$$n_7 + 2n_8 + 3n_9 + 4n_{10} + 5n_{11} = 42 + n_5.$$

By (VI) and Claim 2.2.4, $M'(x) \geq 1.295$ for each 5-vertex x. Since $1.295n_5 \leq 2.55$, $n_5 = 1$. By Claim 2.3.3, $n_6 \geq 3$ and furthermore we have

$$n_7 + 2n_8 + 3n_9 + 4n_{10} + 5n_{11} \leq 43.$$

This implies $n_{11} \leq 8$. Thus $n_{11} = 8$ and $n_7 + 2n_8 + 3n_9 + 4n_{10} \leq 3$. The above inequality implies $n_8 + n_9 \leq 1$ and $n_{10} = 0$. Using an argument similar to the one in Claim 2.3.3, one can show $n_8 \neq 0$ and $M'(x) \geq 0.53$ for each 7-vertex x. Since $n_8 \neq 0$, we have $n_8 = 1$, $n_9 = n_{10} = 0$, $n_7 \leq 1$. Since $n_5 = 1$ and $n_5 + n_7 + n_{11}$ must be even, $n_5 = n_7 = 1$ and $n_6 = 4$. Therefore, we have $2.575 \leq 1.295 + 0.53 + 0.75 \leq \sum_{x \in V} M'(x) \leq 2.55$. This contradiction completes the proof of the claim. $\square$

Claim 2.2.10. $n_8 = 1$

Proof. By Claim 2.2.9, $n_6 \geq 4$. If $n_8 = 0$, by (VI), every 6-vertex x has $M'(x) \geq 1.165$, and thus we have $\sum_{x \in V} M'(x) \geq 4 \times 1.165 = 4.66 > 2.55$, a contradiction. Hence $n_8 \geq 1$. Suppose to the contrary $n_8 \geq 2$.

Since $\delta = 6$, $n_{11} \geq 13 - 6 = 7$. Note that by (2.2), $5n_{11} \leq 42 - 2n_8 \leq 38$. Hence we have $n_{11} \leq 7$. Thus $n_{11} = 7$. Since $0.75n_8 \leq 2.55$ by (2.3) and Claim 2.2.2, $n_8 \leq 3$.

If $n_8 = 3$, then by (2) and Claim 2.2.6,

$$n_7 + 3n_9 + 4n_{10} = 42 - 2n_8 - 5n_{11} = 42 - 6 - 35 = 1.$$

Thus, $n_7 = 1$. On the other hand, by (2.3) and Claim 2.2.2, $2.569 \leq 0.006 \times 4 + 0.295 + 0.75 \times 3 \leq \sum_{x \in V} M'(x) \leq 2.55$, a contradiction. Therefore $n_8 = 2$ and thus

$$n_7 + 3n_9 + 4n_{10} = 42 - 2n_8 - 5n_{11} = 42 - 4 - 35 = 3.$$

The above equality implies that $n_{10} = 0, n_9 \leq 1$, and $n_7 \leq 3$. Since $n = 15, n_{11} = 7, n_8 = 2$ and $n_6 \geq 4$, $n_7 + n_9 \leq 2$. Further, since the degree sum must be even and n_{11} is odd, we have $n_7 + n_9$ is odd. Thus, $n_7 = 0$ and $n_9 = 1$ since $n_7 + 3n_9 = 3$. Let x be an 8-vertex. Then by Claim 2.2.4, x has no 6-neighbors. Thus, $d_{9^+}(x) \geq 7$ and $d_{11}(x) \geq 6$. By (R1), $M'(x) \geq -0.63 + \frac{2.37}{7} \times 6 > 1.4$. Thus $2.8 = 2 \times 1.4 \leq \sum_{v \in V} M'(v) \leq 2.55$, a contradiction. This completes the proof of the claim. □

Claim 2.2.11. $n_{10} = n_9 = 0$

Proof. Suppose to the contrary $n_9 + n_{10} \neq 0$. Since $\delta = 6$, $n_{11} \geq 7$. Furthermore, since G is a triangulation, from (2), we have $n_7 + 2n_8 + 3n_9 + 4n_{10} + 5n_{11} = 42$ which implies $n_{11} = 7$ and $n_7 + 2n_8 + 3n_9 + 4n_{10} = 7$ since $n_8 = 1$ and $n_{11} \geq 7$. Solving the equality, together with $n = 15, n_{11} = 7, n_8 = 1$ and $n_6 \geq 4$, we obtain the following possible degree sequences for G: $(6^5, 7^1, 8^1, 10^1, 11^7), (6^4, 7^2, 8^1, 9^1, 11^7)$.

Note that in either case, each 11-vertex z is either adjacent to a 6-vertex or is adjacent to at most four 10^--vertices. Hence by Lemma 2.1.7, $d_{11}(z) \geq 6$. Since by Claim 2.2.4, no 7-vertex is adjacent to any 7-vertex and no 8-vertex is adjacent to any 6-vertex, each 7-vertex has at least five 11-neighbors and each 8-vertex has at least five 11-neighbors. Thus by (VIII) $M'(x) \geq 0.53$ for each 7-vertex x and $M'(y) \geq -0.63 + \frac{2.37}{5} d_{11}(y)$ for the 8-vertex y.

If the degree sequence of G is $(6^5, 7^1, 8^1, 10^1, 11^7)$, then the 8-vertex y has at least six 11-neighbors and thus $M'(y) \geq -0.63 + \frac{2.37}{5} \times 6 > 2.2$. Therefore, $\sum_{x \in V(G)} M'(x) \geq 0.53 + 2.2 > 2.55$, a contradiction.

If the degree sequence of G is $(6^4, 7^2, 8^1, 9^1, 11^7)$, then the 8-vertex y has at least five 11-neighbors and thus $M'(y) \geq -0.63 + \frac{2.37}{5} \times 5 = 1.74$. Therefore, $\sum_{x \in V(G)} M'(x) \geq 0.53 \times 2 + 1.74 > 2.55$, a contradiction. Hence our claim is true. □

The final Step.

Since $\delta = 6$, $n_9 = n_{10} = 0$, and G is a triangulation, from (2), we have $n_7 + 2n_8 + 5n_{11} = 42$. Since $n_8 = 1$, $n_7 + 5n_{11} = 40$. Furthermore, since $n = 15, n_6 \geq 4, n_{11} \geq 7$, and $n_8 = 1$, it follows that $n_7 \leq 3$. By $n_7 + 5n_{11} = 40$, $n_{11} \geq 7$ and $n_7 \leq 3$, we have $n_7 = 0$ and $n_{11} = 8$. Thus the degree sequence of G is $(6^6, 8^1, 11^8)$. Note that any 11-vertex y must be adjacent to a 6-vertex. Hence by Lemma 2.1.7, $d_{11}(y) \geq 6$. Let x be the 8-vertex. Then by Claim 2.2.4, $d_{11}(x) = 8$ and by (R1) we have $M'(x) \geq -0.63 + 8 \times \frac{2.37}{5} = 3.162 > 2.55$, a contradiction. This completes the proof of the theorem. □

2.3 Critical graphs on surfaces with small minimum degrees

In this section, we will show that when the minimum degree is small, the maximum degree of a critical graph embedded in a surface of characteristic χ is at most $H(\chi)$ or $H(\chi)+1$. Specifically we prove the following theorems.

Theorem 2.3.1. *Let G be a Δ-critical graph with $\delta \leq \frac{\Delta+1}{2}$ which can be embedded in a surface with Euler characteristic $\chi \leq -8$. If $\delta \leq 7$, then*

$$\Delta \leq H(\chi).$$

Theorem 2.3.2. *Let G be a Δ-critical graph which can be embedded in a surface with Euler characteristic $\chi \leq -8$. If either $6 \leq \delta \leq 10$ with $\delta \leq \Delta - 2$ or $\delta = 11 \leq \frac{\Delta+1}{2}$, then*

$$\Delta \leq H(\chi) + 1.$$

2.3.1 The minimum degree δ with $\delta \leq 5$

Lemma 2.3.3. *Let G be a Δ-critical graph with $\Delta \geq 11$ and $\delta = 4$. Assume that $n_\Delta \leq \Delta - 1$ and $n_{(\Delta-2)^+} \leq \Delta$. Then each neighbor of a 4-vertex x has degree at least $\Delta - 1$ and is adjacent to only one $(\Delta-3)^-$-vertex which is x. Thus $n_{(\Delta-2)^+} = \Delta$.*

Proof. Let x be a 4-vertex. Let $y \in N(x)$. Since $d(y) \geq \Delta - 2 > 4$ and $n_\Delta < \Delta$, by Corollary 2.1.12, we conclude that $d(y) \geq \Delta - 1$.

First we show that each neighbor of x is adjacent to at most two $(\Delta-3)^-$-vertices. If $d(y) = \Delta - 1$, then by Lemma 2.1.7, y is adjacent to at most two $(\Delta-3)^-$-vertices. Hence we assume $d(y) = \Delta$ and y has three $(\Delta-3)^-$-neighbors. By Lemma 2.1.17 with $k = 0$, we have $n_{(\Delta-1)^+} \geq \Delta + 1$, a contradiction.

Second we show that y has only one $(\Delta-3)^-$-neighbor which is x. Suppose to the contrary that y has another $(\Delta-3)^-$-neighbor z. Then $d(x)+d(z) \leq \Delta+1$ and $d(x)+d(y)+d(z) \leq 2\Delta+1$. Thus by Lemma 2.1.7 z is not adjacent to x and by Corollary 2.1.15, $|N(z) \cap N(y)| \leq 5 + d(y) - \Delta - 3 \leq d(y) - \Delta + 2$ and $|N(z) \setminus N[y]| \geq d(z) - d(y) + \Delta - 2$. Since $d(y) \geq \Delta - 1$ and y is adjacent to at most two $(\Delta-3)^-$-vertices, there are at least $d(y) - 2 + 1 = d(y) - 1$ $(\Delta-2)^+$ vertices in $N[y]$. If z has no $(\Delta-3)^-$neighbors, then $n_{(\Delta-2)^+} \geq d(y) - 1 + d(z) - d(y) + \Delta - 2 \geq \Delta + 1$, a contradiction. If z has a $(\Delta-3)^-$neighbor, then by Lemma 2.1.7, z has at least 4 Δ-neighbors and thus there are at least $4 - (d(y) - \Delta + 2) = \Delta - d(y) + 2$ Δ-vertices in $N(z) \setminus N[y]$. Therefore there are at least $\Delta - d(y) + 2 + d(y) - 1 = \Delta + 1$ $(\Delta-2)^+$-vertices, a contradiction again. This completes the proof of the lemma. □

For the following lemma, we define the following types of Δ-vertices: Let V_Δ^0 be the set of Δ-vertices with no 5^--neighbors, and let V_Δ^1 be the set of Δ-vertices with exactly one 3^--neighbor and no other 5^--neighbors. We also define $n_\Delta^0 = |V_\Delta^0|$ and $n_\Delta^1 = |V_\Delta^1|$. Let $n_\Delta^2 = n_\Delta - (n_\Delta^0 + n_\Delta^1)$ and $V_\Delta^2 = V_\Delta \setminus (V_\Delta^0 \cup V_\Delta^1)$.

Lemma 2.3.4. *Let G be a Δ-critical graph with $\Delta \geq 11$. Then* $\sum_{x\in V(G)}(d(x)-6) \geq$

$$\begin{cases} (\Delta-6)n_\Delta^0 + (\Delta-8)(n_\Delta - n_\Delta^0) + (\Delta-7.5)n_{\Delta-1} + \sum\limits_{7\leq d(x)\leq \Delta-2}(d(x)-6) + 0.5n_5 + 1.5n_3; \\ (\Delta-7)n_\Delta + (\Delta-7.5)n_{\Delta-1} + \sum\limits_{7\leq d(x)\leq \Delta-2}(d(x)-6) + 0.5n_5, \quad \text{if } \delta \geq 4; \\ (\Delta-7)n_\Delta + (\Delta-7.5)n_{\Delta-1} + \sum\limits_{7\leq d(x)\leq \Delta-2}(d(x)-6) + 0.5n_5 + \sum\limits_{d(x)=4}(d_\Delta(x) + 0.5d_{\Delta-1}(x) - 2), \\ \qquad\qquad \text{if } \delta \geq 4,\ n_{(\Delta-2)^+} \leq \Delta, \text{ and } n_\Delta \leq \Delta - 1. \end{cases}$$

Proof. For each $x \in V(G)$ define the initial charge $M(x) = d(x) - 6$. Clearly, it suffices to show the inequalities for $\delta \leq 5$.

We obtain a new charge $M'(x)$ for each vertex x by the following rules:

(R1) Each vertex $y \in V_\Delta^1$ sends 2 to its 3^--neighbor.

(R2) Each vertex $y \in V_\Delta^2$ sends $\frac{1}{d_{5^-}(y)}$ to each 5^--neighbor.

(R3) Each $(\Delta-1)$-vertex y sends $\frac{1}{2d_{5^-}(y)}$ to each 5^--neighbor if $d_{5^-}(y) \neq 0$.

By the above rules, clearly, $M'(y) = \Delta - 7$ if $y \in V_\Delta^2$, $M'(y) = \Delta - 6$ if $y \in V_\Delta^0$, $M'(y) = \Delta - 8$ if $y \in V_\Delta^1$, $M'(y) \geq \Delta - 7.5$ if $d(y) = \Delta - 1$, and $M'(y) = M(y) = d(y) - 6$ if $6 \leq d(y) \leq \Delta - 2$.

Let x be a 2-vertex. Then, by Lemma 2.1.11, x has 2 Δ-neighbors belonging to V_Δ^1. Thus it receives 2 from each neighbor and $M'(x) = 2 - 6 + 2 \times 2 = 0$.

Let x be a 3-vertex. Then, by Lemma 2.1.8, x has at least 2 Δ-neighbors with no other 5^--neighbors and so x receives 2 from each of them. Let y be the third neighbor of x. If $d(y) = \Delta - 1$, then by Lemma 2.1.11, y has no other 5^--neighbor and thus sends 0.5 to x. Otherwise, by Lemma 2.1.7, y has at most one other 5^--neighbor and thus sends at least 0.5 to x. Hence $M'(x) \geq 3 - 6 + 2 \times 2 + 0.5 = 1.5$.

Let $x \in V_4 \cup V_5$.

(I) Assume that x is adjacent to a vertex x' with $d(x)+d(x') = \Delta+2$. Then, $d(x') \geq \Delta-3 \geq 8$ and by Lemma 2.1.11, x has $d(x) - 1$ Δ-neighbors, each of which is adjacent to only one 5^--vertex. Thus x receives 1 from each neighbor other than x' and $M'(x) \geq d(x) - 6 + d(x) - 1 = 2d(x) - 7 \geq 1$.

(II) Assume that x is adjacent to a vertex y, such that $d(y) \geq \Delta - 1$ and y has $d(y) + d(x) - \Delta - 2$ neighbors other than x of degree at most $\Delta - 4$. Then, by Lemma 2.1.16-(2) (taking $k = 0$), all remaining neighbors of x have degree Δ and have no $(\Delta-2)^-$-neighbors other than x. Thus, $M'(x) \geq d(x) - 6 + d(x) - 1 \geq 1$.

Case 1. $d(x) = 4$.

By (I) and (II), we may assume that x has four $(\Delta-1)^+$-neighbors, with each $(\Delta-1)$-neighbor adjacent to only one 5^--vertex, and each Δ-neighbor adjacent to at most two 5^--vertices. Thus by (R1) and (R2) each neighbor of x sends at least $\frac{1}{2}$ to x. Therefore $M'(x) \geq -2 + 4 \times \frac{1}{2} = 0$.

In particular, if $n_{(\Delta-2)^+} \leq \Delta$ and $n_\Delta \leq \Delta - 1$, then by Lemma 2.3.3, x has four $(\Delta-1)^+$-neighbors and each has only one 5^--neighbor. Thus each $(\Delta-1)$-neighbor sends $\frac{1}{2}$ to x and each Δ-neighbor sends 1 to x. Therefore we have $M'(x) \geq -2 + 0.5d_{\Delta-1}(x) + d_\Delta(x)$.

Case 2. $d(x) = 5$.

If x has a $(\Delta-2)$-neighbor, then by Lemma 2.1.10, x has three $(\Delta-1)^+$-neighbors z such that z is adjacent to $(\Delta-2)$ 6^+-vertices and thus z sends $\frac{1}{2}$ to x. Hence $M'(x) \geq 5 - 6 + 3 \times \frac{1}{2} = 0.5$

If x has a $(\Delta-1)$-neighbor y with two 5^--neighbors, then by Lemma 2.1.16-(2) (taking $k = 1$), x has three $(\Delta-1)^+$-neighbors with at least $\Delta - 2$ neighbors of degree at least $\Delta - 2$. By (R1) and (R2), each of these neighbors sends at least $\frac{1}{2}$ to x and we have $M'(x) \geq 5-6+3\times\frac{1}{2} = 0.5$.

Finally we assume that each $(\Delta-1)$-neighbor of x is adjacent to only one 5^--vertex and thus sends $\frac{1}{2}$ to x. Note that by (II) each Δ-neighbor is adjacent to at most three 5^--vertices (including x) and thus sends at least $\frac{1}{3}$ to x. Therefore $M'(x) \geq 5 - 6 + 5 \times \frac{1}{3} = \frac{2}{3} > 0.5$. $\square$

Theorem 2.3.5. *Let G be a Δ-critical graph with $\Delta \geq 11$ which can be embedded in a surface with Euler characteristic χ. If $\delta \leq 5$, then $\Delta \leq H(\chi)$.*

Proof. It is sufficient to show the following inequality by Lemma 2.1.19:

$$\Delta^2 - 7\Delta \leq \sum_{x \in V(G)} (d(x) - 6).$$

Suppose to the contrary $\Delta^2 - 7\Delta > \sum_{x\in V(G)}(d(x) - 6)$.

Claim 2.3.1. $\delta \geq 4$.

Proof. Assume that $\delta \in \{2, 3\}$. Let x be a δ-vertex. Then, by Lemmas 2.1.11, 2.1.8, and 2.1.9, x has a Δ-neighbor with no other $(\Delta-2)^-$-neighbors, and there are at least $\Delta - 4 - \lfloor\frac{\Delta-1}{3}\rfloor$ Δ-vertices z with distance 2 from x that have no $(\Delta-3)^-$-neighbors. Thus, $n_{(\Delta-2)^+} \geq \Delta + 1$, $n_{(\Delta-1)^+} \geq \Delta$, and $n^0_\Delta \geq \Delta - 4 - \lfloor\frac{\Delta-1}{3}\rfloor$. Note that the lower bound on n^0_Δ can be improved if $\delta = 2$. Let $s = 1$ if $\delta = 2$ and $s = 0$ if $\delta = 3$. Then, $n^0_\Delta \geq \Delta - 4 - \lfloor\frac{\Delta-1}{3}\rfloor + 4s$. Additionally, $n_3 \geq 1 - s$. From Lemma 2.3.4, we have

$$\begin{aligned}
\sum_{v\in V}(d(v) - 6) &\geq (\Delta-6)n^0_\Delta + (\Delta-8)(n_{(\Delta-2)^+} - n^0_\Delta) + 1.5(1-s) \\
&\geq (\Delta-6)(\Delta-4-\lfloor\frac{\Delta-1}{3}\rfloor + 4s) + (\Delta-8)(5+\lfloor\frac{\Delta-1}{3}\rfloor - 4s) + 1.5(1-s) \\
&\geq \Delta^2 - 7\Delta + \frac{4\Delta}{3} - \frac{83}{6} + 6.5s \\
&> \Delta^2 - 7\Delta.
\end{aligned}$$

This contradiction completes the proof of the claim. □

Claim 2.3.2. *$n_\Delta \le \Delta - 1$ and $n_\Delta + n_{\Delta-1} + n_{\Delta-2} \le \Delta$.*

Proof. Suppose to the contrary $n_\Delta \ge \Delta$. By Lemma 2.3.4, we have

$$\sum_{x \in V(G)} (d(x) - 6) \ge (\Delta - 7)\Delta = \Delta^2 - 7\Delta,$$

a contradiction. Hence $n_\Delta \le \Delta - 1$.

Suppose to the contrary $n_\Delta + n_{\Delta-1} + n_{\Delta-2} \ge \Delta + 1$. Since $\delta \le 5$ and $n_\Delta \ge \Delta - \delta + 2$ by Proposition 2.1.2, we have $n_\Delta \ge \Delta - 3$. By Lemma 2.3.4, we have

$$\sum_{x \in V(G)} (d(x) - 6) \ge (\Delta - 7)(\Delta - 3) + 4(\Delta - 8) = \Delta^2 - 7\Delta + \Delta - 11 \ge \Delta^2 - 7\Delta,$$

a contradiction again. Hence $n_\Delta + n_{\Delta-1} + n_{\Delta-2} \le \Delta$. □

Claim 2.3.3. *The following hold:*

(1) $d(x) + d(y) \ge \Delta + 3$ for any two adjacent vertices x, y with $d(x) < d(y)$.

(2) Let x be a vertex with $d(x) \in \{4, 5\}$. Then each Δ-neighbor of x is adjacent to at most $d(x) - 2$ vertices with degree at most $\Delta - d(x) + 2$.

Proof. (1) If $d(x) + d(y) = \Delta + 2$ and $d(x) < d(y)$, then by Corollary 2.1.12, $n_\Delta \ge \Delta$, a contradiction to Claim 2.3.2.

(2) If y is adjacent to at least $d(y) + d(x) - \Delta - 1$ vertices with degree at most $2\Delta - (d(x) + d(y)) + 2$, then by Lemma 2.1.17 with $k = 0$, $n_{(\Delta-1)^+} \ge \Delta + 1$, a contradiction again. □

Claim 2.3.4. $\delta = 5$.

Proof. Suppose to the contrary $\delta \le 4$. Then $\delta = 4$ by Claim 2.3.1. Let x be a 4-vertex. By Claim 2.3.2 and Lemma 2.3.3, we have $n_\Delta + n_{\Delta-1} + n_{\Delta-2} = \Delta$ and $n_\Delta \le \Delta - 1$. Note $n_\Delta \ge \Delta - 4 + 2 = \Delta - 2$ and $d_{\Delta-1}(x) + d_\Delta(x) = 4$. Thus

$$(\Delta-7)(\Delta-2) + (\Delta-7.5)d_{\Delta-1}(x) + (\Delta-8)(2-d_{\Delta-1}(x)) + 0.5d_{\Delta-1}(x) + d_\Delta(x) - 2 = \Delta^2 - 7\Delta.$$

By Lemma 2.3.4,

$$\sum_{x \in V(G)} (d(x) - 6) \ge \Delta^2 - 7\Delta.$$

This contradiction implies $\delta = 5$. □

Claim 2.3.5. *Let x be a 5-vertex and y be a Δ-neighbor of x. Then y has at most three $(\Delta-3)^-$-neighbors and has only one $(\Delta-4)^-$-neighbor which is x. Thus $n_\Delta \ge \Delta - 3$, $n_{(\Delta-2)^+} \ge \Delta - 2$ and $n_{(\Delta-3)^+} \ge \Delta$.*

Proof. By Claim 2.3.3, y has at most three $(\Delta-3)^-$-neighbors. Note that there are at least $\Delta-3$ Δ-vertices and an additional $(\Delta-2)^+$-vertex in $N[y]$.

Now we show that y has only one $(\Delta-4)^-$-neighbor. Suppose to the contrary that y has a $(\Delta-4)^-$-neighbor $z \neq x$. Then z is not adjacent to x and $d(x)+d(y)+d(z) \leq 2\Delta+1$. By Lemma 2.1.15, $|N(z)\cap N(y)| \leq 2$ and $|N(z)\setminus N[y]| \geq d(z)-3$.

If z has a $(\Delta-5)^-$-neighbor, then by Lemma 2.1.7, z is adjacent to at least six Δ-vertices. Since $|N(z)\cap N(y)| \leq 2$ and $y \in N(z)$, there are at least three Δ-vertices not in $N[y]$. Thus $n_\Delta \geq \Delta$, a contradiction to Claim 2.3.2. Hence every neighbor of z has degree at least $\Delta-4$.

Assume that z has a $(\Delta-4)$-neighbor z'. If $d(z) \leq 6$, then by Claim 2.3.3-(1), $d(z') \geq \Delta+3-6 > \Delta-4$, a contradiction. Thus $d(z) \geq 7$ and by Lemma 2.1.7, z is adjacent to at least five Δ-vertices. Since $|N(z)\cap N(y)| \leq 2$ and $y \in N(z)$, there are at least two Δ-vertices not in $N[y]$. Thus $n_\Delta \geq \Delta-1$, $n_{(\Delta-2)^+} \geq \Delta$, and $n_{(\Delta-4)^+} \geq \Delta+1$. By Lemma 2.3.4,

$$\sum_{v\in V(G)} (d(v)-6) \geq (\Delta-7)(\Delta-1)+(\Delta-8)+(\Delta-10) = \Delta^2-7\Delta+\Delta-11 \geq \Delta^2-7\Delta.$$

This contradiction implies that z has no $(\Delta-4)^-$-neighbor. Denote $t=\max\{\Delta-3, \Delta+3-d(z)\}$. Then each neighbor of z has degree at least t. If $d(z) \geq 6$, then $t=\Delta-3$. Otherwise, $d(z)=5$ and $t=\Delta-2$. Since $\Delta \geq 11$, $(d(z)-3)(t-6) \geq \min\{3(\Delta-9), 2(\Delta-8)\} = 2(\Delta-8)$.

If $n_5 \geq 6$, by Lemma 2.3.4,

$$\sum_{v\in V(G)} (d(v)-6) \geq (\Delta-7)(\Delta-3)+(\Delta-8)+2(\Delta-8)+0.5n_5 \geq \Delta^2-7\Delta.$$

If $n_5 \leq 5$, since $\Delta \geq 11$, by Euler's formula,

$$\sum_{v\in V(G)} (d(v)-6) \geq (\Delta-6)(\Delta-3)+(\Delta-8)+2(\Delta-8)-n_5 = \Delta^2-7\Delta+\Delta-11 \geq \Delta^2-7\Delta.$$

In either case, we obtain a contradiction. This completes the proof of the claim. □

Claim 2.3.6. $n_5 \geq 3$.

Proof. Suppose to the contrary $n_5 \leq 2$. Since $\Delta \geq 11$, by Claim 2.3.5, $\sum_{x\in V(G)}(d(x)-6) \geq (\Delta-6)(\Delta-3)+(\Delta-8)+2(\Delta-9)-n_5 \geq \Delta^2-7\Delta+\Delta-8-2 > \Delta^2-7\Delta$, a contradiction. □

Claim 2.3.7. *Let y be a $(\Delta-1)$-vertex adjacent to a 5-vertex x. Then y has exactly one $(\Delta-3)^-$-neighbor which is x.*

Proof. Assume first that y has two $(\Delta-3)^-$-neighbors in addition to x. Then, by Lemma 2.1.17 with $k=0$, since $d(y) < \Delta-k$, $n_\Delta \geq \Delta$, a contradiction to Claim 2.3.2. Now, assume that y has one $(\Delta-3)^-$-neighbor other than x. By Lemma 2.1.17 with $k=1$, since $d(y) \geq \Delta-k$, we have $n_{(\Delta-1)^+} \geq \Delta$. Together with $n_5 \geq 3$ and $n_\Delta \geq \Delta-3$, by Lemma 2.3.4, we have $\sum_{v\in V}(d(v)-6) \geq (\Delta-3)(\Delta-7)+3(\Delta-7.5)+3\times 0.5 = \Delta^2-7\Delta$, again a contradiction. □

Claim 2.3.8.

$$(\Delta-7)n_\Delta+(\Delta-7.5)n_{\Delta-1}+(\Delta-8)n_{\Delta-2}+\sum_{7\le d(x)\le \Delta-3}(d(x)-6)+2n_5\le \sum_{u\in V}(d(u)-6).$$

Proof. It suffices to show that in Lemma 2.3.4, $M'(x)\ge 2$ for each 5-vertex x. By Claim 2.3.5 and (R2) in the proof of Lemma 2.3.4, each Δ-neighbor sends 1 to x. By Claim 2.3.7 and (R3) in the proof of Lemma 2.3.4, each $(\Delta-1)$-neighbor sends 0.5 to x.

If x has three Δ-neighbors, then $M'(x)\ge -1+3=2$. If x has only two Δ-neighbors, then by Lemma 2.1.7, the remaining three neighbors of x are all $(\Delta-1)$-vertices, and so $M'(x)\ge -1+2+3\times 0.5>2$. □

The final step. By Claim 2.3.5, $n_\Delta\ge\Delta-3$, $n_{(\Delta-2)^+}\ge\Delta-2$ and $n_{(\Delta-3)^+}\ge\Delta$. And by Claim 2.3.6, $n_5\ge 3$. Thus by Claim 2.3.8,

$$\sum_{x\in V}(d(x)-6)\ge(\Delta-7)(\Delta-3)+(\Delta-8)+2(\Delta-9)+2n_5\ge\Delta^2-7\Delta-5+2n_5>\Delta^2-7\Delta.$$

This contradiction completes the proof of the theorem. □

2.3.2 The minimum degree δ with $6\le\delta\le 7$

Theorem 2.3.6. *Let G be a Δ-critical graph with minimum degree $6\le\delta\le 7$ embedded in a surface with Euler characteristic $\chi\le -8$. If $\delta\le\frac{\Delta+1}{2}$, then $\Delta\le H(\chi)$.*

Proof. By Lemma 2.1.19, it suffices to show $\Delta^2-7\Delta\le -6\chi$. Suppose to the contrary $\Delta^2-7\Delta>-6\chi$. Then by Lemma 2.1.19, $\sum_{v\in V(G)}(d(v)-6)\le -6\chi<\Delta^2-7\Delta$. Since $\chi\le -8$, we have $\Delta\ge 12$. Let $xy\in E$ with $d(x)=\delta$ and $d(y)=\Delta$. Then, by Lemma 2.1.26, y has at most $\delta-4$ $(\Delta-\delta+2)^-$-neighbors (not including x).

Claim 2.3.9. $n_{(\Delta-3)^+}\le\Delta-1$ *and* $n_{(\Delta-2)^+}\le\Delta-2$.

Proof. If $n_{(\Delta-3)^+}\ge\Delta$, then by Lemma 2.1.19,

$$\sum_{v\in V(G)}(d(v)-6)\ge(\Delta+4)(\delta-6)+(\Delta-\delta+2)(\Delta-\delta)+(\delta-2)(\Delta-3-\delta)=\Delta^2-7\Delta+\Delta+\delta-18\ge\Delta^2-7\Delta.$$

If $n_{(\Delta-2)^+}\ge\Delta-1$, then by Lemma 2.1.19 again,

$$\sum_{v\in V(G)}(d(v)-6)\ge(\Delta+4)(\delta-6)+(\Delta-\delta+2)(\Delta-\delta)+(\delta-3)(\Delta-2-\delta)=\Delta^2-7\Delta+3\delta-18\ge\Delta^2-7\Delta.$$

In either case we obtain a contradiction and thus prove Claim 2.3.9. □

The following claim follows from Lemma 2.1.25 since $\Delta>H(\chi)$.

Claim 2.3.10. *$d(u) + d(v) \geq \Delta + 4$ for any two adjacent vertices u and v.*

Claim 2.3.11. *For each path xuz, $d(x) + d(u) + d(z) \geq 2\Delta + 2$.*

Proof. Suppose to the contrary there is a path xuz such that $d(x)+d(u)+d(z) \leq 2\Delta+1$. Then $d(z) \leq 2\Delta - (d(x) + d(u)) + 1$.

First, we show that $d(u) \geq \Delta + 6 - \delta$. Suppose to the contrary that $d(u) \leq \Delta + 5 - \delta$. Then we have $d(x) + d(u) \leq \Delta + 5$ and $d(u) < \Delta$. Since $d(z) \leq 2\Delta - (d(x) + d(u)) + 1$, u has at most two $(2\Delta - (d(u) + d(x)) + 2)^+$-neighbors besides the $\Delta - \delta + 1$ Δ-neighbors guaranteed by Lemma 2.1.7, a contradiction to Lemma 2.1.26-(ii) (with $k \leq 2$). Thus $d(u) \geq \Delta + 6 - \delta$.

Since $d(u) \geq \Delta + 6 - \delta$, we have that $d(z) \leq 2\Delta - (\Delta + 6) + 1 = \Delta - 5$. Further, since $d(x) + d(z) \leq \delta + \Delta - 5 \leq \Delta + 2$, by Claim 2.3.10, z is not adjacent to x. We may then apply Lemma 2.1.15. Denote $s = d(u)$ and $t = d(z)$. Then by Lemma 2.1.15, $|N(z) \setminus N[u]| \geq t - \delta - s + \Delta + 2$. Note each neighbor of z has degree at least $\Delta + 4 - t$ by Claim 2.3.10.

Next, we show that $\delta \geq 7$. Assume that $\delta = 6$. Then, $d(u) = \Delta$. By Lemma 2.1.26 (with $k \leq 1$), u has at most 3 $(\Delta - 4)^-$-neighbors (including x). We then have $|N[u] \cap V_\Delta| \geq \Delta - 4$ and $|N[u] \cap V_{(\Delta-3)^+}| \geq \Delta - 2$. Thus from $N[u]$ we have a contribution of at least $(\Delta - 4)(\Delta - 6) + 2(\Delta - 9)$ to $\sum_{v \in V(G)}(d(v) - 6)$. From $N(z) \setminus N[u]$, we have a contribution of at least $(t - 6) + (t - 4)(\Delta + 4 - t - 6)$; since $6 \leq t \leq \Delta - 5$, this is minimized when $t = 6$. Thus, we have

$$\sum_{v \in V(G)} (d(v) - 6) \geq \Delta^2 - 8\Delta + 6 + 2(\Delta - 8) = \Delta^2 - 7\Delta + \Delta - 10 > \Delta^2 - 7\Delta.$$

Now we assume that $d(x) = \delta = 7$, which implies $\Delta \geq 13$. Assume $d(u) = \Delta - 1$. Then $d(z) \leq \Delta - 5$. By applying Lemma 2.1.26 with $k \leq 2$, one can conclude that u has exactly two $(\Delta - 4)^-$-neighbors, x and z. Hence $|N[u] \cap V_\Delta| \geq \Delta - 6$ and $|N[u] \cap V_{(\Delta-3)^+}| \geq \Delta - 2$. Thus from $N[u]$ we have a contribution of $(\Delta - 6)(\Delta - 7) + 4(\Delta - 10)$ to $\sum_{v \in V}(d(v) - 6)$. From $N(z) \setminus N[u]$, we have a contribution of at least $(t - 7) + (t - 4)(\Delta + 4 - t - 7)$; this is minimized when either $t = 7$ or $t = \Delta - 5$ for a contribution of at least $3(\Delta - 10)$. Thus,

$$\sum_{v \in V}(d(v) - 6) \geq (\Delta + 4)(7 - 6) + \Delta^2 - 9\Delta + 2 + 3(\Delta - 10) = \Delta^2 - 7\Delta + 2\Delta - 24 > \Delta^2 - 7\Delta.$$

This contradiction implies $d(u) = \Delta$ and thus $d(z) \leq \Delta - 6$. By Lemma 2.1.26 with $k \leq 1$, u has at most 4 $(\Delta - 5)^-$-neighbors (including x). Thus $N[u]$ contributes at least $(\Delta - 5)(\Delta - 7) + 2(\Delta - 11)$ to $\sum_{v \in V}(d(v) - 6)$. We then have $\sum_{v \in V}(d(v) - 6) \geq (\Delta + 4)(7 - 6) + (\Delta - 5)(\Delta - 7) + 2(\Delta - 11) = \Delta^2 - 9\Delta + 17$ by considering $n \geq \Delta + 4$ and $N[u]$. This implies that the vertex z and the at least $t - 5$ vertices in $N(z) \setminus N[u]$ can contribute at most $2\Delta - 18$ to the sum.

First, assume that z has a Δ-neighbor outside of $N[u]$. Thus $N[z] \setminus N[u]$ contributes to the sum at least $(t-7)+(\Delta-7)+(t-6)(\Delta+4-t-7)$; this is minimized when $t=7$ and thus we have a contribution of at least $2\Delta-17$. This contradiction implies that $N(z) \cap V_\Delta \subseteq N[u]$. Thus, $d_\Delta(z) \leq 5$, and by Lemma 2.1.7, we have $d(z') \geq \Delta-4$ for each z' adjacent to z. Then $N[z] \setminus N[u]$ contributes at least $(t-7)+(t-5)(\Delta-4-7)$. If $t \geq 9$, we then have a contribution of at least $2+4(\Delta-11) > 2\Delta-17$. Thus, $t \in \{7,8\}$.

Assume $t=8$. If every neighbor of z in $N(z) \setminus N[u]$ has degree at least $\Delta-3$, then the contribution of $N[z] \setminus N[u]$ is at least $(8-7)+3(\Delta-3-7) > 2\Delta-17$. Hence there is a neighbor z' of z with degree at most $\Delta-4$. By Lemma 2.1.26 (with $k \leq 1$), we have all other neighbors of z have degree at least $\Delta-2$, which implies that $N[z] \setminus N[u]$ contributes at least $(8-7)+2(\Delta-9)+1(\Delta-11) > 2\Delta-17$, a contradiction.

Now we assume $t=7$. Then z has at least 2 neighbors outside of $N[u]$, both of which have degree at least $\Delta+4-7=\Delta-3$. Note that by symmetric reasoning, x has at least 2 neighbors outside of $N[u]$. If $|(N(x) \cup N(z)) \setminus N[u]| \geq 3$, then these vertices contribute at least $3(\Delta-3-7) \geq 2\Delta-17$ to the sum, a contradiction. Thus $N(x) \setminus N[u] = N(z) \setminus N[u]$. Hence if $w \in N(x) \setminus N[u]$, then we have a path xwz with $d(x)+d(w)+d(z) \leq 2\Delta+1$, which implies that $d(w) \geq \Delta+6-\delta \geq \Delta-1$. Therefore these vertices contribute at least $2(\Delta-1-7) > 2\Delta-17$ to the sum. This contradiction completes the proof of the claim. $\square$

Claim 2.3.12. *(1)* $\delta = 7$ *and thus* $\Delta \geq 13$.

(2) $n \geq 8n_7$.

Proof. (1) Suppose to the contrary $\delta=6$. By Lemma 2.1.26-(i) (with $k \geq 2$), y has additional two $(\Delta-3)^+$-neighbors in addition to the $(\Delta-5)$ Δ-neighbors guaranteed by Lemma 2.1.7. Thus together with Claim 2.3.11, y has $(\Delta-3)$ $(\Delta-3)^+$-neighbors and $\Delta-1$ $(\Delta-4)^+$-neighbors. So including y, $n_{(\Delta-3)^+} \geq \Delta-2$ and $n_{(\Delta-4)^+} \geq \Delta$. Hence

$$\Delta^2-7\Delta > -6\chi \geq \sum_{u \in V}(d(u)-6) \geq (\Delta-6)(\Delta-4)+2(\Delta-9)+2(\Delta-10) = \Delta^2-7\Delta+\Delta-14,$$

and we have $12 \leq \Delta \leq 13$. This implies $\Delta^2-7\Delta$ is always a multiple of 6, thus $\Delta^2-7\Delta \geq -6\chi+6 \geq \Delta^2-7\Delta+\Delta-14+6 > \Delta^2-7\Delta$, a contradiction. Therefore $\delta=7$ and $\Delta \geq 2\delta-1=13$.

(2) Since $\Delta \geq 13$, by Claim 2.3.11, no two 7-vertices have a common neighbor and so $n \geq 8n_7$. $\square$

The final step. By Claim 2.3.10, each Δ-neighbor of x has $\Delta-1$ $(\Delta-5)^+$-neighbors. By Lemmas 2.1.7 and 2.1.26 (with $k \leq 1$), among these vertices there are $\Delta-6$ Δ-vertices and additional two $(\Delta-4)^+$-vertices. Thus we have

$$\sum_{v \in V}(d(v)-6) \geq n+(\Delta-5)(\Delta-7)+2(\Delta-11)+3(\Delta-12) = \Delta^2-7\Delta+n-23.$$

$$\begin{aligned}\sum_{v\in V}(d(v)-6) &= 2n+\sum_{v\in V}(d(v)-8)\\ &\geq 2(\Delta+4)+(\Delta-5)(\Delta-8)+2(\Delta-12)+3(\Delta-13)-2\\ &\geq \Delta^2-7\Delta+\Delta-17.\end{aligned}$$

This implies $\Delta \leq 16$. If $\Delta \in \{13, 15, 16\}$, $\Delta^2 - 7\Delta$ is a multiple of 6 and thus

$$\Delta^2-7\Delta \geq -6\chi+6 \geq \Delta^2-7\Delta+\Delta-17+6 > \Delta^2-7\Delta.$$

This contradiction implies $\Delta = 14$. By Claim 2.3.11, no 7-vertex shares a common neighbor with an 8-vertex. Thus if $n_8 \neq 0$, then $n \geq 9 + 8n_7$.

If $n_8 \neq 0$, then $n_7 = 1$ since $n \leq 22$. By Claim 2.3.10, each neighbor of an 8^--vertex is a $(\Delta - 4)^+$-vertex. Thus $n_{(\Delta-4)^+} \geq 15 = \Delta + 1$. Therefore

$$\sum_{v\in V}(d(v)-6) \geq 2(\Delta+4)+(\Delta-5)(\Delta-8)+6(\Delta-12)-1 = 101 > 98 = \Delta^2-7\Delta.$$

If $n_8 = 0$, then $n_7 \leq 2$. Thus we have

$$\sum_{v\in V}(d(v)-6) \geq 3(\Delta+4)+(\Delta-5)(\Delta-9)+2(\Delta-13)+3(\Delta-14)-2 = 99 > 98 = \Delta^2-7\Delta.$$

In either case we obtain a contradiction.

This contradiction completes the proof of the theorem. □

2.3.3 The minimum degree δ with $8 \leq \delta \leq 11$

Theorem 2.3.7. *Let G be a Δ-critical graph with $6 \leq \delta \leq 10$ and $\delta \leq \Delta - 2$ which can be embedded in a surface with Euler characteristic χ. Then, $\Delta \leq H(\chi) + 1$.*

Proof. By Lemma 2.1.19 it suffices to show $\sum_{u\in V}(d(u) - 6) \geq \Delta^2 - 9\Delta + 8$. Let x be a δ-vertex and y be a Δ-neighbor of x. By Lemma 2.1.26, we may assume that y has at least 4 $(\Delta - \delta + 2)^+$-neighbors in addition to $\Delta - \delta + 1$ Δ-neighbors. Then

$$\begin{aligned}\sum_{u\in V}(d(u)-6) &\geq (\Delta+4)(\delta-6)+(\Delta-\delta+2)(\Delta-\delta)+4(\Delta-\delta+2-\delta)\\ &= \Delta^2-\delta\Delta+\delta^2-6\delta-16.\end{aligned}$$

If $\delta \leq 9$, we then have $\sum_{u\in V}(d(u) - 6) \geq \Delta^2 - 9\Delta + 8$. Thus we may assume that $\delta = 10$. By Lemma 2.1.26-(iii), we may further assume that y has at least four $(\Delta - 7)^+$-neighbors in addition to $(\Delta - 9)$ Δ-neighbors. Hence in $N[y]$, there are $(\Delta - 8)$ Δ-vertices with additional four $(\Delta - 7)^+$-vertices. Thus

$$\sum_{u\in V}(d(u)-6) \geq \Delta^2-\delta\Delta+\delta^2-6\delta-16+4 = \Delta^2-9\Delta+8+20-\Delta.$$

The above inequality implies that we may assume $\Delta \geq 21$ and thus $10 \leq \frac{\Delta+1}{2}$. By Lemma 2.1.26-(iii) again ($k \geq 5$), there is one more additional $(\Delta-7)^+$-vertex and we have

$$\sum_{u \in V}(d(u)-6) \geq \Delta^2 - 9\Delta + 8 + 20 - \Delta + \Delta - 7 - 10 > \Delta^2 - 9\Delta + 8.$$

This completes the proof of the theorem. □

Theorem 2.3.8. *Let G be a Δ-critical graph with minimum degree $\delta = 11 \leq \frac{\Delta+1}{2}$ which can be embedded in a surface with Euler characteristic χ. Then, $\Delta \leq H(\chi) + 1$.*

Proof. By Lemma 2.1.19 it suffices to show $-6\chi \geq \Delta^2 - 9\Delta + 8$. Suppose to the contrary $-6\chi < \Delta^2 - 9\Delta + 8$. By Lemma 2.1.19, $\sum_{v \in V}(d(v)-6) \leq -6\chi < \Delta^2 - 9\Delta + 8$.

Let x be a δ-vertex and y be a Δ-neighbor of x. By Lemma 2.1.26, we may assume that y has at least five $(\Delta-8)^+$-neighbors in addition to $(\Delta-10)$ Δ-neighbors guaranteed by Lemma 2.1.7.

Assume that y has a neighbor $z \neq x$ with $d(z) \leq \Delta - 10$. Then, x is not adjacent to z and $d(x) + d(y) + d(z) \leq 2\Delta + 1$. Let $d(z) = t$. Then by Lemma 2.1.15, $|N(z) \setminus N[y]| \geq t - 9$. By Lemma 2.1.25, every neighbor of z has degree at least $\Delta + 6 - t$. Thus we have

$$\sum_{u \in V}(d(u)-6) \geq (\Delta+4)(11-6) + (\Delta-9)(\Delta-11) + 5(\Delta-19) + (t-11) + (t-9)(\Delta+6-t-11).$$

This sum is minimized when $t = 11$ and thus we have $\sum_{u \in V}(d(u)-6) \geq \Delta^2 - 9\Delta + 8 + \Delta - 16 > \Delta^2 - 9\Delta + 8$.

Now we assume that every neighbor of y other than x has degree at least $\Delta - 9$. Then, $n_\Delta \geq \Delta - 9, n_{(\Delta-8)^+} \geq \Delta - 4$, $n_{(\Delta-9)^+} \geq \Delta$ and we have

$$\sum_{u \in V}(d(u)-6) \geq 5(\Delta+4) + (\Delta-9)(\Delta-11) + 5(\Delta-19) + 4(\Delta-20) = \Delta^2 - 9\Delta + 8 + 3\Delta - 64.$$

Thus $\Delta \leq 21$. Since $\Delta \geq 21$, we have $\Delta = 21$. Since $\Delta^2 - 9\Delta + 8 = 260 > -6\chi$, we have $\Delta^2 - 9\Delta + 8 \geq -6\chi + 2$. Therefore,

$$\Delta^2 - 9\Delta + 8 \geq \sum_{u \in V}(d(u)-6) + 2 \geq \Delta^2 - 9\Delta + 8 + 3\Delta - 62 = \Delta^2 - 9\Delta + 8 + 1.$$

This contradiction completes the proof of the theorem. □

2.4 Surfaces with Euler characteristic $\chi \in \{-53, \ldots, -8\}$

In this section, as an application of the results in Sections 2.3.1, 2.3.2 and 2.3.3, we will show the following theorem.

Theorem 2.4.1. *Let Σ be a surface with characteristic $\chi(\Sigma) \in \{-53, \ldots, -8\}$. Then*

$$\Delta(\Sigma) \leq \begin{cases} H(\chi(\Sigma)), & \text{if } \chi(\Sigma) \in \{-22, -21, -20, -18, -17, -15, \ldots, -8\} \\ H(\chi(\Sigma)) + 1, & \text{if } \chi(\Sigma) \in \{-53, \cdots, -23, -19, -16\}. \end{cases}$$

Proof. Denote $\Gamma = H(\chi(\Sigma))$ if $\chi(\Sigma) \in \{-22, -21, -20, -18, -17, -15, \ldots, -8\}$ and $\Gamma = H(\chi(\Sigma)) + 1$ if $\chi(\Sigma) \in \{-53, \ldots, 23, -19, -16\}$ and $\chi = \chi(\Sigma)$. Then $\Gamma \geq 11$. Suppose to the contrary that our theorem is not true. Then there is a class two graph B of maximum degree $\Delta > \Gamma$ embedded in a surface Σ of characteristic $\chi(\Sigma)$. Let G be a Δ-critical graph with $\Delta = \Gamma + 1$ obtained from B.

Then $\Delta \geq 12$ and thus by Theorems 2.3.5 and 2.3.6, $\delta \geq 7$. Define:

- $\alpha = (\Delta + 4)(\delta - 6) + (\Delta - \delta + 2)(\Delta - \delta) + 6\chi = \Delta^2 - (4 + \delta)\Delta + \delta^2 + 2\delta - 24 + 6\chi$;
- $\beta = (n - (\Delta + 4))(\delta - 6) + (n_\Delta - (\Delta - \delta + 2))(\Delta - \delta) + \sum_{d(x) \neq \Delta}(d(x) - \delta)$;
- $\eta = \lceil \frac{\Delta^2 - 6\Delta + 3\delta - 22 + 6\chi}{\Delta - \delta} \rceil$;
- $\tau = \lceil \frac{\Delta + 2 + \eta}{2} \rceil$.

Claim 2.4.1. *(1)* $\alpha + \beta \leq 0$.

(2) $d(u) + d(v) \geq \Delta + 2 + \eta$ *for any* $uv \in E(G)$ *if* $\eta \leq \Delta - 4$ *and* $\max\{d(u), d(v)\} < \Delta$.

(3) $n_{\tau^+} \geq \lceil \frac{\Delta+5}{2} \rceil$ *if* $\eta \leq \Delta - 4$ *and* $\beta \geq (n - (\Delta + 4))(\delta - 6) + (n_{\tau^+} - (\Delta - \delta + 2))(\tau - \delta)$.

(1) and (2) follow from Lemma 2.1.19 and Corollary 2.1.24.

For (3), by (2) the set of $(\tau - 1)^-$-vertices is independent and by Lemma 2.1.18, $n_{(\tau-1)^-} < \frac{n}{2}$. Thus $n_{\tau^+} > \frac{n}{2} \geq \frac{\Delta+4}{2}$. Thus $n_{\tau^+} \geq \lceil \frac{\Delta+5}{2} \rceil$. The second part follows from the definition of β.

In the rest of the proof, let xy be an edge in G with $d(x) = \delta$ and $d(y) = \Delta$.

Case 1. $\chi \in \{-9, -8\}$. Then $\Gamma = H(\chi) = 11$ and $\Delta = 12$.

Since $\delta \geq 7$ and $\Delta + 4 = 16$, we have

$$0 \geq \alpha = (\delta - 6) \times 16 + (12 - \delta + 2)(12 - \delta) + 6\chi = (\delta - 5)^2 + 47 + 6\chi.$$

The above inequality implies $\delta = 7$, $\chi = -9$, and $\alpha = -3$. Thus $\beta \leq 3$, $\eta = 4$, and $\tau = 9$. By Claim 2.4.1-(3), we have $n_{9^+} \geq 9$ and thus $\beta \geq 2(9 - 7) = 4$, a contradiction to $\beta \leq 3$.

Hence $\Delta \leq 11$ and thus $\Delta(\Sigma) \leq 11 = H(\chi)$ when $\chi(\Sigma) \in \{-9, -8\}$.

Note that $\Gamma \geq 12$ when $|\chi| \geq 10$. Thus $\Delta \geq 13$. By Theorem 2.3.1, in the rest of the proof, we further assume $\delta \geq 8$.

Case 2. $\chi \in \{-12, -11, -10\}$. Then $\Gamma = H(\chi) = 12$ and $\Delta = 13$.

Thus $\alpha = (\delta - 5.5)^2 + 62.75 + 6\chi \le 0$. Then, $\chi = -12$ and $\delta = 8$. Hence we have $\alpha = -3$, $\beta \le 3$, $\eta = 5$, and $\tau = 10$. By Claim 2.4.1-(3), we have $n_{10^+} \ge 9$ and $\beta \ge 2(10-8) = 4$, a contradiction to $\beta \le 3$.

Hence $\Delta \le 12$ and thus $\Delta(\Sigma) \le 12 = H(\chi)$ when $\chi(\Sigma) \in \{-12, -11, -10\}$.

Case 3. $\chi \in \{-15, -14, -13\}$. Then $\Gamma = H(\chi) = 13$ and $\Delta = 14$.

Thus $\alpha = (\delta - 6)^2 + 80 + 6\chi \le 0$ and we have $\chi \le -14$ and $\delta \le 9$.

We first show that $\delta = 8$ and $\chi = -15$. Otherwise, either $\chi = -14$ or $\delta = 9$. In either case $\eta = \lceil \frac{90+3\delta+6\chi}{14-\delta} \rceil \ge 5$. So $\tau \ge 11$. By Claim 2.4.1-(3), $n_{11^+} \ge 10$ and $\beta \ge (10 - (14-\delta+2))(11-\delta) = (\delta-6)(11-\delta)$. Thus,

$$0 \ge \alpha + \beta \ge (\delta-6)^2 + 80 + 6\chi + (\delta-6)(11-\delta) = 5\delta + 50 + 6\chi \ge 5.$$

This contradiction implies $\chi = -15$ and $\delta = 8$. Thus $\alpha = -6$ and $\beta \le 6$.

Then $\eta = 4$, $\tau = 10$, and $n_{10^+} \ge 10$. Let $t \in \{10, 11\}$. Then $t \ge 20-t$ and by Claim 2.4.1-(2), each t-vertex has t $(20-t)^+$-neighbors. Note $\Delta - \delta + 2 = 8$. If $n_t \ne 0$, then $n_{(20-t)^+} \ge t+1$. If $t = 11$, then $n_{9^+} \ge 12$, $n_{11} \ge 1$ and $n_{10^+} \ge 10$. Thus by Claim 2.4.1-(3), $\beta \ge (11-8) + (10-1-8)(10-8) + 2(9-8) = 7 > 6$, a contradiction. Hence $t = 10$ and $n_{10^+} \ge 11$ and $\beta \ge (11-8)(10-8) = 6$. Thus $\beta = 6$. This implies $n_{10} = 3$, $n_{14} = 8$, $n_8 = 7$, and $n = 18$. Since $\eta = 4$, no 10^--vertex is adjacent to an 8-vertex. Hence each 10^--vertex is adjacent to eight 14-vertices. Thus every 14-vertex is adjacent to all 10^--vertices including all 8-vertices. By Lemma 2.1.7, each 14-vertex is adjacent to at least $14-8+1 = 7$ 14-vertices, which implies that every 14-vertex has at least $10+7 = 17$ neighbors, a contradiction. Thus $n_t = 0$ for each $t \in \{10, 11\}$ and $n_{12^+} \ge 10$. By Claim 2.4.1-(3), $\beta \ge (10-8)(12-8) = 8 > 6$, a contradiction.

Hence $\Delta \le 13$ and thus $\Delta(\Sigma) \le 13 = H(\chi)$ when $\chi(\Sigma) \in \{-15, -14, -13\}$.

Case 4. $\chi \in \{-18, -17, -16\}$. Then $\Gamma = 14$ and $\Delta = 15$.

Thus $\alpha = \delta^2 - 13\delta + 141 + 6\chi \le 0$. Since $\delta \ge 8$, we have $\delta \le 9$ and $\eta \ge \lceil \frac{113+3\delta+6\chi}{15-\delta} \rceil \ge 5$. By Claim 2.4.1, $n_{11^+} \ge 10$ and $\beta \ge (10 - (17-\delta))(11-\delta) = -\delta^2 + 18\delta - 77$. Thus

$$0 \ge \alpha + \beta \ge 5\delta + 64 + 6\chi.$$

This implies $\delta = 8$ and $\chi = -18$. Hence $\alpha = -7$ and $\beta \le 7$. Further, $\eta = 5$ and $\tau = 11$. Note $8 \le \frac{\Delta+1}{2}$ and $\Delta - \delta + 3 = 10$. By Lemma 2.1.26-(ii) and (iii), the Δ-neighbor y of an 8-vertex x has at least three 10^+ neighbors in addition to $(\Delta - 7)$ Δ-neighbors guaranteed by Lemma 2.1.7. Thus $n_{10^+} \ge 3 + 9 = 12$. Note $n_{11^+} \ge 10$. Therefore we have $\alpha = -7$ and $\beta \ge (11-8) + 2(10-8) = 7$ and thus $\beta = 7$. This implies $n = 19$, $n_{15} = 9$, $n_{10} = 2$, $n_{11} = 1$, and $n_8 = 7$. Since $\eta \ge 5$ and $\Delta + 2 + \eta \ge 22$, no 11^--vertex is adjacent to a 10^--vertex. Thus by Lemma 2.1.18, $11 = |V_{11^-}| < \frac{19}{2}$, a contradiction.

Hence $\Delta \le 14$ and thus $\Delta(\Sigma) \le 14$ when $\chi(\Sigma) \in \{-18, -17, -16\}$.

Case 5. $\chi = \chi(\Sigma) \in \{-22, -21, -20, -19\}$. Then $\Gamma = 15$ and $\Delta = 16$.

Thus $\alpha = (\delta - 7)^2 + 119 + 6\chi \leq 0$. Therefore $\delta \leq 10$.

We first consider the case when $\delta = 8$. Then $\alpha = 120 + 6\chi \geq -12$, $8 \leq \frac{\Delta+1}{2}$, and $\eta = \lceil \frac{138+3\delta+6\chi}{16-\delta} \rceil = \lceil \frac{162+6\chi}{8} \rceil \geq 4$. Thus $\beta \leq 12$ and by Lemma 2.1.26-(ii) and (iii), y has at least three 11^+-neighbors in addition to 9 Δ-neighbors guaranteed by Lemma 2.1.7.

If y has no 9^--neighbors other than x, then $\beta \geq 3(11-8) + 3(10-8) = 15$, a contradiction.

If y has a 9^--neighbor other than x, say z, then $d(x) + d(y) + d(z) \leq 33 = 2\Delta + 1$. By Lemma 2.1.15, $|N(z) \setminus N[y]| \geq d(z) - 8 + 2$. Since $\eta \geq 4$, the degree of each neighbor of z is at least $22 - d(z)$. Thus $\beta \geq 3(11-8) + (d(z)-8+2)(22-d(z)-8) = 9 + (d(z)-6)(14-d(z)) \geq 21$ since $d(z) \in \{8, 9\}$, a contradiction.

Now we consider the case when $\delta \geq 9$. Then $\eta = \lceil \frac{138+3\delta+6\chi}{16-\delta} \rceil \geq 5$ and thus $\tau \geq 12$ and $n_{12^+} \geq 11$. Moreover $\beta \geq (11-(18-\delta))(12-\delta) = -\delta^2 + 19\delta - 84$ and thus $\alpha + \beta \geq 5\delta + 84 + 6\chi$. Since $\alpha + \beta \leq 0$, we have $\delta = 9$ and $\chi = -22$. This implies $\alpha = -9$ and $\beta \leq 9$.

If $n_{12} = n_{13} = 0$, then $n_{14^+} \geq 11$ and so $\beta \geq (11-9)(14-9) = 10$, a contradiction. So $n_t \neq 0$ for some $t \in \{12, 13\}$. We choose t to be the smaller integer such that $n_t \neq 0$. Then $t \geq 12 > 23 - t$ and by Claim 2.4.1-(2), each t-vertex has t $(23-t)^+$-neighbors. Since $\Delta - \delta + 2 = 9$ and $n_{12^+} \geq 11$, we have $n_{16} \geq 9, n_{t^+} \geq 11$, and $n_{(23-t)^+} \geq t + 1$. Therefore $\beta \geq (t+1-11)(23-t-9) + 2(t-9) \geq 10 > 9$, a contradiction.

Hence $\Delta \leq 15$ and thus $\Delta(\Sigma) \leq 15$ when $\chi(\Sigma) \in \{-22, -21, -20, -19\}$.

Case 6. $\chi \in \{-52, \ldots, -23\}$. Then $\Gamma = H(\chi) + 1 \geq 16$. More specifically,

- If $\chi \in \{-23, \ldots, -20\}$, then $\Delta = 17$.
- If $\chi \in \{-28, \ldots, -24\}$, then $\Delta = 18$.
- If $\chi \in \{-32, \ldots, -29\}$, then $\Delta = 19$.
- If $\chi \in \{-37, \ldots, -33\}$, then $\Delta = 20$.
- If $\chi \in \{-43, \ldots, -38\}$, then $\Delta = 21$.
- If $\chi \in \{-48, \ldots, -44\}$, then $\Delta = 22$.
- If $\chi \in \{-53, \ldots, -49\}$, then $\Delta = 23$.

By Lemmas 2.3.7 and 2.3.8, $\delta \geq 11$ and if $\delta = 11$, then $\Delta \leq 20$.

Subcase 6.1: $\chi \in \{-37, \ldots, -23\}$.

First, assume $\delta = 11$. Then, $\alpha = \Delta^2 - 15\Delta + 119 + 6\chi \leq 0$. Since $-37 \leq \chi \leq -23$, it is easy to check that $\Delta = 20$ and $\chi = -37$. Thus, $\alpha = -3$ and $\beta \leq 3$. Further, $n_{20} \geq 11, \eta = 8, \tau = 15$. By Claim 2.4.1-(3), $n_{\tau^+} \geq 13$. Then, $\beta \geq (13-11)(15-11) = 8$, a contradiction.

Thus $\delta \geq 12$. Since $\delta \geq 12 > \frac{\Delta-2}{2}$, α is increasing with respect to δ and we have $\alpha \geq \Delta^2 - 16\Delta + 144 + 6\chi$.

If $\chi \in \{-23, \ldots, -20\}$ and $\Delta = 17$, then $\alpha \geq \Delta^2 - 16\Delta + 144 + 6\chi \geq 17^2 - 16 \times 17 + 144 + 6 \times (-23) = 23 > 0$.

If $\chi \in \{-28, \dots, -24\}$ and $\Delta = 18$, then $\alpha \geq \Delta^2 - 16\Delta + 144 + 6\chi \geq 18^2 - 16 \times 18 + 144 + 6 \times (-28) = 12 > 0$.

If $\chi \in \{-32, \dots, -29\}$ and $\Delta = 19$, then $\alpha \geq \Delta^2 - 16\Delta + 144 + 6\chi \geq 19^2 - 16 \times 19 + 144 + 6 \times (-32) = 9 > 0$.

If $\chi \in \{-37, \dots, -33\}$ and $\Delta = 20$, then $\alpha \geq \Delta^2 - 16\Delta + 144 + 6\chi \geq 20^2 - 16 \times 20 + 144 + 6 \times (-37) = 2 > 0$.

In each case above, we obtain $\alpha > 0$, a contradiction. Therefore when $\chi \in \{-37, \dots, -23\}$, $\Delta \leq H(\chi) + 1$.

In the remaining cases, $\Delta \geq 21$ and $\delta \geq 12$ as noted before.

Subcase 6.2: $\chi \in \{-43, \dots, -38\}$. Then $\Delta = 21$ and $\alpha = \delta^2 - 19\delta + 333 + 6\chi$.

Thus we have $\eta = \lceil \frac{21^2 - 6\times 21 + 3\delta - 22 + 6\chi}{21-\delta} \rceil \geq \lceil \frac{35+3\delta}{21-\delta} \rceil \geq \lceil \frac{71}{9} \rceil = 8$ and $\tau = \lceil \frac{\Delta+2+\eta}{2} \rceil \geq 16$. By Claim 2.4.1-(3), $n_{16^+} \geq 13$.

If $n_{16} \neq 0$, then each neighbor of a 16-vertex has degree at least 15. Thus $n_{15^+} \geq 17$. By Lemma 2.1.7, $n_{21} \geq 23 - \delta$ and by Claim 2.4.1-(3), $n_{16^+} \geq 13$. Hence $\beta \geq 4(15-\delta) + (13 - (21 - \delta + 2))(16 - \delta) = -\delta^2 + 22\delta - 100$. Therefore $\alpha + \beta \geq 3\delta + 233 + 6\chi \geq 3 \times 12 + 233 + 6 \times (-43) > 0$, a contradiction. Hence $n_{16} = 0$ and $n_{16^+} = n_{17^+} \geq 13$. Thus $\beta \geq (13 - (21 - \delta + 2))(17 - \delta) = -\delta^2 + 27\delta - 170$. Therefore $\alpha + \beta \geq 8\delta + 163 + 6\chi \geq 8 \times 12 + 163 + 6 \times (-43) > 0$, a contradiction again.

Therefore when $\chi \in \{-43, \dots, -38\}$, $\Delta \leq H(\chi) + 1$.

Subcase 6.3: $\chi \in \{-47, \dots, -44\}$. Then $\Delta = 22$ and $\alpha = \delta^2 - 20\delta + 372 + 6\chi$.

Thus we have $\eta = \lceil \frac{22^2 - 6\times 22 + 3\delta - 22 - 282}{22-\delta} \rceil \geq \lceil \frac{48+3\delta}{22-\delta} \rceil \geq \lceil \frac{48+36}{22-12} \rceil = 9$ and $\tau \geq 17$. By Claim 2.4.1-(3), $n_{17^+} \geq 14$ and $\beta \geq (17-\delta)(14 - (22 - \delta + 2)) = -\delta^2 + 27\delta - 170$. Hence $\alpha + \beta \geq 7\delta + 202 + 6\chi \geq 4 > 0$, a contradiction.

Therefore when $\chi \in \{-47, \dots, -44\}$, $\Delta \leq H(\chi) + 1$.

Subcase 6.4: $\chi(\Sigma) \in \{-53, \dots, -48\}$. Then $\Delta = 23$ and $\alpha = \delta^2 - 21\delta + 413 + 6\chi$.

Thus we have $\eta \geq \lceil \frac{23^2 - 6\times 23 + 3\delta - 22 - 318}{23-\delta} \rceil = \lceil \frac{51+3\delta}{23-\delta} \rceil \geq \lceil \frac{87}{11} \rceil = 8$ and $\tau = \lceil \frac{\Delta+2+\eta}{2} \rceil \geq 17$. By Claim 2.4.1-(3), $n_{17^+} \geq 14$.

We first consider the case when $n_t \neq 0$ for some $t \in \{17, 18, 19, 20\}$. Let $t \in \{17, 18, 19, 20\}$ be the smallest integer such that $n_t \neq 0$. Since $t > 33 - t$, $n_{(33-t)^+} \geq t + 1$. Note $n_{t^+} \geq 14$. Hence we have $\beta \geq (t + 1 - 14)(33 - t - \delta) + (14 - (23 - \delta + 2)(t - \delta) = -\delta^2 + 24\delta + t(35 - t) - 429$. Thus $0 \geq \alpha + \beta \geq 3\delta + 413 + 6\chi - 429 + t(t - 35) \geq 36 - 16 - 318 + t(35 - t)$. Since $17 \leq t \leq 20$, $t(35 - t) \geq 300$. Hence $\alpha + \beta \geq 2 > 0$, a contradiction.

Now we assume $n_t = 0$ for each $t \in \{17, 18, 19, 20\}$. Then $n_{21^+} \geq 14$. Thus $\beta \geq (14 - (25 - \delta))(21 - \delta) = (\delta - 11)(21 - \delta)$. So, $0 \geq \alpha + \beta \geq 11\delta + 182 + 6\chi \geq 11\delta + 182 - 318$. This implies $\delta = 12$ and so $\alpha \geq -13$ and $\beta \leq 13$.

Since $\delta = 12 \leq \frac{\Delta+1}{2}$, by Lemma 2.1.26-(iii), for each Δ-vertex with a 12-neighbor, there are 5 14^+-neighbors in addition to the $23-12+1 = 12$ Δ-neighbors guaranteed by Lemma 2.1.7. Thus we have $n_{23} \geq 13, n_{21^+} \geq 14$, and $n_{14^+} \geq 18$. Hence $\beta \geq (14-13)(21-12)+(18-14)(14-12) = 17 > 13$.

This contradiction completes the proof of Subcase 6.4 and thus of the theorem. $\square$

By [19], since the class two graph $K_{13} - K_3$ can be embedded in an orientable surface of characteristic -12, we have the following corollary.

Corollary 2.4.2. *Let Σ be an orientable surface. If $\chi(\Sigma) = -12$, then $\Delta(\Sigma) = H(\chi(\Sigma)) = 12$.*

Chapter 3

Star Edge Coloring

3.1 Introduction

As with standard proper edge coloring, star edge coloring originates from a proper vertex coloring problem.

Definition 3.1.1 (Star (vertex) coloring, Grünbaum (1973), [15])**.** *Let $f : V(G) \to [k]$ be a proper vertex coloring of a graph G. If $G[V_i \cup V_j]$ is a star forest for each $1 \leq i < j \leq k$, then f is a k star coloring of G. Equivalently, a star coloring is a proper coloring in which there are no bicolored paths on more than 3 vertices or bicolored cycles. The star chromatic number $\chi_{st}(G)$ is the minimum k such that G admits a k star coloring.*

One may obtain an edge coloring problem from a vertex coloring problem in the usual fashion, by considering a vertex coloring of the line graph of G, $L(G)$. The star edge coloring problem was introduced by Liu and Deng [26] in 2008:

Definition 3.1.2 (Star edge coloring)**.** *Let $f : E(G) \to [k]$ be a proper edge coloring of a graph G. If f is a star (vertex) coloring of $L(G)$, then f is a star edge coloring of G. Equivalently, a star edge coloring is a proper edge coloring in which there are no bicolored 4-cycles or bicolored paths on 4 edges. The star chromatic index $\chi'_{st}(G)$ is the minimum k such that G admits a k star edge coloring.*

Note that the star edge chromatic index can be bounded by acyclic chromatic index $\chi'_a(G)$ (in which every bicolored connected subgraph is acyclic) and the strong chromatic index $\chi'_s(G)$ (in which every bicolored connected subgraph has at most two edges). A greedy algorithm for strong edge coloring then gives an upper bound for the star chromatic index.

Proposition 3.1.1. $\chi'_a(G) \leq \chi'_{st}(G) \leq \chi'_s(G) \leq 2\Delta(\Delta - 1) + 1.$

Liu and Deng [26] found a general bound for graphs with sufficiently large maximum degree.

Theorem 3.1.1. *Let G be a graph with maximum degree $\Delta \geq 7$. Then, $\chi'_{st}(G) \leq \lceil 16(\Delta-1)^{\frac{3}{2}} \rceil$.*

Dvořák, Mohar, and Šámal [11] give the following bounds for complete graphs, which imply a near-linear upper bound for any graph.

Theorem 3.1.2. *For the complete graph K_n:*

$$2n(1+o(1)) \leq \chi'_{st}(K_n) \leq n\frac{2^{2\sqrt{2}(1+o(1))\sqrt{\log n}}}{(\log n)^{\frac{1}{4}}}$$

From this bound, for any graph G, $\chi'_{st}(G) \leq \Delta \cdot 2^{O(1)\sqrt{\log \Delta}}$

It is also known that the star chromatic index is difficult to compute, even for subcubic graphs: Lei, Shi, and Song [25] proved that it is NP-complete to determine whether $\chi'_{st}(G) \leq 3$ for an arbitrary graph G.

Much is known about star chromatic index for subcubic graphs. In the same paper in which they studied complete graphs, Dvořák, Mohar, and Šámal showed that if $\Delta(G) \leq 3$, then $\chi'_{st}(G) \leq 7$. They further proposed the following conjecture:

Conjecture 3.1.1. *If G is a subcubic graph, then $\chi'_{st}(G) \leq 6$.*

Wang, Wang, and Wang in [44] and [45] prove bounds for graphs G under different planarity conditions and when $\Delta(G) \leq 4$.

Theorem 3.1.3.

Let G be a planar graph with girth g.
(i) $\chi'_{st}(G) \leq 2.75\Delta + 18$
(ii) If G is K_4-minor free, then $\chi'_{st}(G) \leq 2.25\Delta + 6$
(iii) If G has no 4-cycles, then $\chi'_{st}(G) \leq \lfloor 1.5\Delta \rfloor + 18$
(iv) If $g \geq 5$, then $\chi'_{st}(G) \leq \lfloor 1.5\Delta \rfloor + 13$
(v) If $g \geq 8$, then $\chi'_{st}(G) \leq \lfloor 1.5\Delta \rfloor + 3$
(vi) If G is outerplanar, then $\chi'_{st}(G) \leq \lfloor 1.5\Delta \rfloor + 5$
Let G be a graph with $\Delta(G) \leq 4$.
(i) $\chi'_{st}(G) \leq 14$
(ii) If G is bipartite, then $\chi'_{st}(G) \leq 13$

The star edge coloring problem extends naturally to a list version:

Definition 3.1.3. *Let L be an assignment to lists of $E(G)$. Then, f is a list star edge coloring of G if f is a star edge coloring of G such that $f(e) \in L(e)$ for each $e \in E(G)$. The star edge-choosability of G, $ch'_{st}(G)$, is the minimum k such that for any assignment to lists L of $E(G)$ with $|L(e)| = k$ for each $e \in E(G)$, G admits a star edge coloring f with $f(e) \in L(e)$ for each $e \in E(G)$.*

Lužar, Mockovčiaková, and Sotá [32] proved that $ch'_{st}(G) \leq 7$ for a subcubic graph G.

Bezegova et al. [3] and Deng et al. [10] independently proved that for each tree T with maximum degree Δ, its star chromatic index $\chi'_{st}(T) \leq \lfloor \frac{3\Delta}{2} \rfloor$. This result was extended to list star chromatic index by Han et al. in [16].

Theorem 3.1.4. *For any tree T with maximum degree Δ,*

$$\chi'_{st}(T) \leq ch'_{st}(T) \leq \lfloor \frac{3\Delta}{2} \rfloor.$$

Furthermore, this bound is sharp.

Han et al. in [16] additionally studied k-degenerate graphs, giving the following theorem.

Theorem 3.1.5. *Let G be k-degenerate with $k \geq 2$. Then,*
(i) $ch'_{st}(G) \leq \frac{5k-1}{2}\Delta(G) - \frac{k(k+3)}{2}$
(ii) $ch'_{st}(G) \leq 2k\Delta(G) + k^2 - 4k + 2$

The maximum average degree of a graph G, denoted $mad(G)$ is the maximum of average degrees of subgraphs of G:

$$mad(G) = \max\{\frac{2|E(H)|}{|V(H)|} : H \subseteq G\}$$

If $mad(G)$ is small, we say that G is sparse. Note that the condition that $mad(G) < 2$ is equivalent to the condition that G is acyclic. In some sense, the closer maximum average degree is to 2 for a connected graph G containing some cycle, the closer G is to being a tree.

Kerdjoudj, Kostochka, and Raspaud [21], Kerdjoudj and Kostochka [22], and Kerdjoudj, Pradeep, and Raspaud [23] studied the list star edge coloring of graphs with small maximum average degree.

Theorem 3.1.6 ([21, 22])**.** *Let G be a subcubic graph. Then each of the following holds.*
(i) If $mad(G) < \frac{7}{3}$, *then* $ch'_{st}(G) \leq 5$.
(ii) If $mad(G) < \frac{5}{2}$, *then* $ch'_{st}(G) \leq 6$.
(iii) If $mad(G) < \frac{30}{11}$, *then* $ch'_{st}(G) \leq 7$.

Theorem 3.1.7 ([22, 23])**.**
(i) If $mad(G) < \frac{7}{3}$, *then* $ch'_{st}(G) \leq 2\Delta(G) - 1$.
(ii) If $mad(G) < \frac{5}{2}$, *then* $ch'_{st}(G) \leq 2\Delta(G)$.
(iii) If $mad(G) < \frac{8}{3}$, *then* $ch'_{st}(G) \leq 2\Delta(G) + 1$.
(iv) If $mad(G) < \frac{14}{5}$, *then* $ch'_{st}(G) \leq 2\Delta(G) + 2$.
(v) If $mad(G) < 3$, *then* $ch'_{st}(G) \leq 2\Delta(G) + 3$.

Wang et al., in [45], also give the following lemma:

Lemma 3.1.8. *Consider C_n.*

(i) If $n \neq 5$, then $ch'_{st}(C_n) = \chi'_{st}(C_n) = 3$.

(ii) If $n = 5$, then $ch'_{st}(C_n) = \chi'_{st}(C_n) = 4$.

Here, I determine structural properties of graphs G with $mad(G) < \frac{8}{3}$ by use of the discharging method. I then use these substructures to star edge color G, giving bounds on the same order as the (sharp) bounds for star edge chromatic index for trees. A note on methodology: in the results on the usual edge coloring problem, we used the tool of alternating paths (Kempe chains) in partially colored graphs. To understand why this tool cannot be extended to the star edge coloring problem in a straightforward way, consider the path P_7 on six edges. If the edges of this path are colored blue-red-blue-green-red-green, then this is a star edge coloring. The blue-red-blue subpath is a maximal component of the graph induced by the red and blue color classes. Switching colors along this path results in a new coloring of P_7: red-blue-red-green-red-green. While this is indeed a proper edge coloring of P_7, it contains a bicolored path on 4 edges and so is not a star edge coloring.

The following are the results that I prove here. The first concerns the non-list version of the star edge coloring problem, while the second two concern the list version. I will give reasons at the end of this section for my choice in the presentation of these theorems.

Theorem 3.1.9. *Let G be a graph with $mad(G) \leq 2.4$. Then, $\chi'_{st}(G) \leq \lfloor \frac{3\Delta}{2} \rfloor + 2$.*

Theorem 3.1.10. *Let G be a graph with maximum degree Δ and $mad(G) < \frac{15}{7}$. Then, $ch'_{st}(G) \leq \lfloor \frac{3\Delta}{2} \rfloor + 1$.*

Theorem 3.1.11. *Let $\varepsilon > 0$ be a real number and $d = 2\lceil \frac{8-3\varepsilon}{9\varepsilon} \rceil$. Let G be a graph with maximum average degree $mad(G) < \frac{8}{3} - \varepsilon$. Then*

$$ch'_{st}(G) \leq \max\{\frac{3}{2}\Delta + \frac{d}{2} + 2, \Delta + 2d + 1\}.$$

For a planar graph G with girth g, it is well known that the maximum average degree $mad(G) < \frac{2g}{g-2}$. Thus we have the following corollaries:

Corollary 3.1.12. *If G is a planar graph with girth $g \geq 12$, then $\chi'_{st}(G) \leq \lfloor \frac{3\Delta}{2} \rfloor + 2$.*

Corollary 3.1.13. *If G is a planar graph with girth $g \geq 30$, then $ch'_{st}(G) \leq \lfloor \frac{3\Delta}{2} \rfloor + 1$.*

Corollary 3.1.14. *Let G be a planar graph with maximum degree Δ and girth g.*

- *If $g \geq 9$, then $ch'_{st}(G) \leq max\{\frac{3\Delta}{2} + 11, \Delta + 37\}$.*
- *If $g \geq 16$, then $ch'_{st}(G) \leq \frac{3\Delta}{2} + 4$.*

A further note on my results: In the proof of the non-list bounds on planar graphs with large girth, Wang et al. employ an edge partition technique. They assign colors $\{1, \cdots, s\}$ to one subgraph and $\{s+1, \cdots, t\}$ to a second subgraph, then combine the two colorings. As such, their technique does not apply to the general case of low maximum average degree and does extend easily to the list version of the problem, which is the primary concern of my results. Further, I give structural properties of sparse graphs which may be of interest in other problems.

3.2 Overview of the Proofs and Notation

Although there are some key differences between the proofs of each of the three above theorems, they follow the same overall structure. I begin by assuming that G is a minimal counterexample with respect to $|E(G)|$. I first find reducible "tree-like" structures using arguments analagous to those used by Bezegova et al. [3] and Deng et al. [10] to prove the bounds for trees. I then find reducible configurations in a minimum degree 2 subgraph. I finally use the the discharging method to show that these reducible configurations must exist, providing a contradiction.

I begin by defining the primary "non-tree-like" graph in use:

Definition 3.2.1. *Let G be any graph. From G one can obtain a (possibly empty) graph by iteratively deleting vertices of degree 1 until no such vertices remain. I denote this graph G'. Equivalently, G' is the maximal subgraph of G with minimum degree 2.*

Next, I will make use of the "tree-like" structures the extend from G':

Definition 3.2.2. *Let $x \in V(G')$. Define T_x as the maximal tree rooted at x such that $V(T_x) \cap V(G') = \{x\}$. Note that it is possible that T_x consists of an isolated vertex (x); in this case, we say that T_x is trivial.*

I also use paths in G' where the internal vertices all have degree 2:

Definition 3.2.3. *A t-**thread** of G' is a path $x_0x_1 \cdots x_tx_{t+1}$ where $d_{G'}(x_i) = 2$ for $1 \leq i \leq t$.*

For $y \in V(G')$, denote $d_2(y)$ as the number of 2-neighbors of y in G', $d_{k^-}(y)$ as the number of k^--neighbors of y in G', and $d_{k^+}(y)$ as the number of k^+-neighbors in G'.

3.3 Some Claims for Minimal Counterexamples

The proof of Theorem 3.1.11 requires a stronger form of list star coloring, and so the full proof is postponed. However, the proof uses the same key ideas of the proofs of Theorems 3.1.9 and 3.1.10. I give here some structural results for the list version of the problem for minimal counterexamples.

Let G be a minimal counterexample to Theorem 3.1.9 or 3.1.10 with respect to $|E(G)|$. Let $k = \lfloor \frac{3\Delta(G)}{2} \rfloor + c$ where $c = 2$ if $mad(G) < 2.4$ and $c = 1$ if $mad(G) < \frac{15}{7}$.

I give below a slightly modified proof of the bound for trees. I include the proof here because similar ideas are used in the "tree-like" structures in the subsequent lemma.

Lemma 3.3.1. *Let T be a tree. Then, $ch'_{st}(T) \leq \frac{3\Delta(T)}{2}$.*

Proof. Let T be a tree rooted at a vertex x. Let L be an assignment to lists of $E(T)$ with $|L(e)| = \lfloor \frac{3\Delta(T)}{2} \rfloor$ for each $e \in E(T)$.

Without loss of generality, assume for any vertex $v \in V(T)$, either $d(v) = \Delta(T)$ or $d(v) = 1$; if $1 < d(v) < \Delta(T)$, simply add leaf neighbors to v until $d(v) = \Delta(T)$. We use the terminology that a vertex v is in *Level* i if the distance between x and v is i. So x is in Level 0 and the neighbors of x are in Level 1. An edge is called in *Layer* i if its two end vertices are in Levels $i-1$ and i, respectively. Denote $Lay(i)$ to be the set of all edges in Layer i.

We proceed algorithmically by layer to build an L-star-edge-coloring f. Firstly, $Lay(1)$ may be (properly) colored greedily, since each edge has more than $\Delta(T)$ colors available. Let $\{x_1, \cdots, x_{\Delta(T)}\}$ be the vertices in Level 1. Let $N(x_i) = \{x, x_{i,1}, \cdots, x_{i,\Delta(T)-1}\}$. For each $x_{i,j}$, let $L'(x_i x_{i,j}) = L(x_i x_{i,j}) \setminus \{f(xx_s) : i - \lfloor \frac{\Delta(T)}{2} \rfloor \leq s \leq i\}$ (indices taken modulo $\Delta(T)$). Then, $|L'(e)| \geq \Delta(T)$ for each $e \in Lay(1)$ and so each edge can be assigned a color $f(x_i x_{i,j})$ distinct from its incident edges. Let $x_{i,j}$ be a vertex in Level 2 (and so $x_{i,j}$ has parent x_i). Let $N(x_{i,j}) = \{x_i, x_{i,j,1}, \cdots x_{i,j,\Delta(T)-1}\}$. Define, for each k, $L'(x_{i,j} x_{i,j,k}) = L(x_{i,j} x_{i,j,k}) \setminus \{f(xx_i), f(x_i x_{i,s} : j - \lfloor \frac{\Delta(T)}{2} \rfloor \leq s \leq j\}$. Then, $|L'(x_{i,j} x_{i,j,k})| \geq \Delta - 1$ for each $x_{i,j} x_{i,j,k} \in Lay(2)$ and so each edge can be assigned a color $f(x_{i,j} x_{i,j,k})$ distinct from its incident edges. To determine $f(e)$ for an edge in $Lay(t)$ where every edge in a layer of lower index has already been colored, we follow the same algorithm as for $Lay(2)$.

It remains to show that f is an L-star-edge coloring of T. Clearly, f is a proper L-edge coloring of T. Since T is acyclic, there cannot be a bicolored 4-cycle. Assume that $uvwyz$ is a bicolored path on 4 edges in T. Without loss of generality, assume that, of vertices $\{u, v, w, y, z\}$, u is in the furthest level from x. If $uvwy$ is a subpath of the (unique) (x, u)-path, then by the algorithm, $f(uv) \neq f(wy)$, contradicting the assumption that the path is bicolored. Thus, u and z are in the same level of T, say l. We can follow the indexing scheme above, assuming that $w = x_{i_1, i_2, \cdots, i_{l-2}}$, $v = x_{i_1, i_2, \cdots, i_{l-2}, j}$, $u = x_{i_1, \cdots, i_{l-2}, j, s}$, $y = x_{i_1, \cdots, i_{l-2}, k}$, and $z = x_{i_1, \cdots, i_{l-2}, t}$. Assume that $j < k$. Then, since $f(yz) = f(wv)$ and the algorithm forbids for yz colors used at $x_{i_1, \cdots, i_{l-2}, c}$ with $k - \lfloor \frac{\Delta(T)}{2} \rfloor \leq c \leq k$, we have that $k - j \geq \lfloor \frac{\Delta(T)}{2} \rfloor$. Symmetrically, $j - k \geq \lfloor \frac{\Delta(T)}{2} \rfloor$. Since w has only $\Delta(T) - 1$ neighbors in Level $l - 1$, this is a contradiction.

□

The following lemmas give structural results for the "tree-like" structures of G:

Claim 3.3.1. *Let $x \in V(G')$ and $y \in V(T_x) \setminus \{x\}$. Then, the distance $dist(x,y) \leq 2$.*

Proof. Let y be a vertex in T_x at farthest distance from x. Assume that $dist(x,y) \geq 3$. Let H be the graph obtained by deleting all vertices in T_x at distance $dist(x,y)$ from x. By minimality of G, we have an L-star-edge-coloring f of H. We may simply extend f to G by use of the tree coloring algorithm above: This will guarantee that no bicolored path on 4 edges exists in T_x. Further, since this algorithm guarantees that no bicolored path on 3 edges extends to 3 different levels of a tree, we guarantee that no bicolored path on 4 vertices extends from T_x into G'. □

Let $d = 5$ if $mad(G) < 2.4$ and $d = 2$ if $mad(G) < \frac{15}{7}$. Note that we then have $\lfloor \frac{3\Delta(G)}{2} \rfloor + \lfloor \frac{d}{2} \rfloor$ colors available for each $e \in E(G)$.

Claim 3.3.2. *Let $x \in V(G')$ with $d_{G'}(x) \leq d$. Then, T_x is a star rooted at x.*

Proof. Assume to the contrary that T_x is not a star. By Lemma 3.3.1, we know that vertices in T_x are at distance at most 2 from x. Assume that $d_{G'}(x) = k \leq d$ and, without loss of generality, $d_{T_x}(x) = \Delta - k$ and further that for $x_i \in N_{T_x}(x), d_G(x_i) = d_{T_x}(x_i) = \Delta$.

Obtain a graph H by deleting all vertices in T_x at distance 2 from x. Let f be an L-star-edge-coloring of H. To extend f to G while ensuring that no bicolored paths on 4 edges exist starting at a vertex in T_x and ending at a vertex in G', it is necessary to forbid colors $f(xy)$ with $y \in V(G')$. Let $y_1, \cdots, y_k$ be the neighbors of x in G' and let $x_1, \cdots, x_{\Delta-k}$ be the neighbors of x in T_x. Consider x_i for some $1 \leq i \leq \Delta - k$. Let $u_1, \cdots, u_{\Delta-1}$ be the neighbors of x_i other than x. To color edges at leaf neighbors of x_i, we forbid $f(xx_s)$ for $i - \lfloor \frac{\Delta-k}{2} \rfloor \leq s$ (indices taken modulus $\lfloor \frac{\Delta-d}{2} \rfloor$) in addition to $f(xy_t)$ for $1 \leq t \leq k$. Thus, the total number of colors available is $(\lfloor \frac{3\Delta}{2} \rfloor + 2) - (\lfloor \frac{\Delta-k}{2} \rfloor + k) \geq \Delta - 1$. Thus, sufficient colors remain to extend f to edges at leaf neighbors of x_i. Further, this algorithm guarantees a star-edge-coloring, analagous to the above proof for trees. □

Claim 3.3.3. *Let $x \in V(G')$ such that $d_{G'}(x) = 2$ with neighbors y, z in $V(G')$. If $d_{G'}(y) = 2$ and $d_{G'}(z) \leq d$, then T_x is trivial .*

Proof. By Lemma 3.3.2, we know that T_x, T_y, and T_z are stars rooted at x, y, and z, respectively. Assume that x has some neighbors in T_x. Let x' be a leaf neighbor of x, $H = G - x'$, and f be an L-star-edge-coloring of H. In extending f to G, consider the possibilities for a path on 4 edges containing xx'. Necessarily, x' is one of the path's endpoints. Clearly, no path on 4 edges exists between x' and a leaf neighbor of y or z. Thus, such a path must contain at least 2 edges in G'. Thus, to ensure a star edge coloring of G, it suffices to forbid colors used at edges of y and z inside G', as well as the colors used at other leaf neighbors of x. Thus, the total number of forbidden colors for xx' is at most $2 + d + \Delta - 3 = \Delta + d - 1$. Thus xx' has at least $\lfloor \frac{3\Delta}{2} \rfloor + \lfloor \frac{d}{2} \rfloor - (\Delta + d - 1) \geq \frac{\Delta - d}{2} + 1 \geq 1$ color available. This contradiction completes the proof of the lemma. □

Claim 3.3.4. $\Delta(G') \geq 3$

Proof. Assume that $\Delta(G') = 2$, i.e., that G' consists of disjoint cycles. Let $x \in V(G')$. By Lemmas 3.3.2 and 3.3.3, T_x is trivial. By Lemma 3.1.8, $ch'_{st}(G) \leq 4 = \lfloor \frac{3\Delta}{2} \rfloor + 1$. $\square$

3.4 Proof of Theorem 3.1.9

Now, assume that G is a minimal counterexample (w.r.t $|E(G)|$) to Theorem 3.1.9, i.e., $\chi'_{st}(G) > k = \frac{3\Delta}{2} + 2$ but every proper subgraph H has $\chi'_{st}(H) \leq \frac{3\Delta}{2} + 2$.

To guarantee that some configurations are reducible in G, in addition to the lemmas above, I also need a lemma to switch colors at leaf edges. This is an easy observation in the non-list version of the problem, but poses difficulties in the list version that are remedied by applying a stronger version of star-edge-coloring.

Observation 3.4.1. *Let H be a graph with an s-star-edge-coloring f. Let u be a vertex with distinct leaf neighbors v, w. Assume that $f(uv) = i$ and $f(uw) = j$. Let f' be obtained by switching the colors used at uv and uw: specifically, $f'(uv) = j, f'(uw) = i$, and $f'(e) = f(e)$ for every edge $e \in E(G) \setminus \{uv, uw\}$. Then f' is also an s-star-edge-coloring of H.*

Claim 3.4.1. *G' contains no path $uxyv$ with $d_{G'}(u) \leq 3$, $d_{G'}(x) = d_{G'}(y) = 2$, and $d_{G'}(v) \leq 5$.*

Proof. Assume such a path exists. By Lemma 3.3.2, T_u, T_x, T_y, and T_z are stars rooted at u, x, y, and z, respectively. By Lemma 3.3.3, T_x and T_y are trivial.

Let $H = G - xy$ and let f be a k-star-edge-coloring of H.

If $f(ux) = f(yv)$ and T_u contains a leaf neighbor u' of u, then, by Observation 3.4.1, we obtain a new star edge coloring f' of H by switching $f(ux)$ and $f(uu')$; in f', we now have $f'(ux) \neq f'(yv)$. Thus, wlog, we may assume that $f(ux) \neq f(yv)$ or both T_u and T_v are trivial. In this case, whatever color is assigned to xy, it is not possible to have a bicolored path on 4 edges that contains an edge in T_u or T_v. Thus the only colors that must be forbidden are colors used at edges of u and v inside G'. There are at most $3 + \Delta$ such colors, and so a color remains available for xy, a contradiction. $\square$

Lemma 3.4.1. *Let G be a graph with $mad(G) < 2.4$. Then, G' has a t-thread $x_0 x_1 \cdots x_t x_{t+1}$ such that one of the following holds.*

1. $t = 2, d_{G'}(x_0) = 3,$ *and* $d_{G'}(x_{t+1}) \leq 5.$

2. $t = 3$ *and* $\min\{d_{G'}(x_0), d_{G'}(x_{t+1})\} \leq 5.$

3. $t \geq 4$

Proof. Assume to the contrary that G' contains none of these structures. I proceed by the discharging method:

Assign to each vertex $x \in V(G')$ the initial charge $M(x) = d(x) - 2.4$. Then, since $mad(G') \leq mad(G) < 2.4$, it follows that $\sum_{x \in V(G')} M(x) < 0$. For the remainder of this proof, a c-vertex refers to the vertex degree in G'.

Note that if x is a 2-vertex, then $M(x) = -0.4$, and if y is a 3^+-vertex, then $M(y) > 0$.

We obtain for each vertex a new charge $M'(x)$ by the following rule:

R1: Each 3^+-vertex y sends each 2-neighbor $\frac{M(y)}{d_2(y)}$ if $d_2(y) > 0$.

Clearly, by the initial charge and the discharging rule, $M'(y) \geq 0$ for each 3^+-vertex y. It then suffices to show that any t-thread has net positive charge, which would imply that $\sum_{x \in V(G')} M'(x) \geq 0$, a contradiction.

Note that each 3^+-vertex y sends at least $\frac{d(y)-2.4}{d(y)} = 1 - \frac{2.4}{d(y)}$ to any 2-neighbors, and $f(d(y)) = 1 - \frac{2.4}{d(y)}$ is an increasing function on $d(y)$.

First, assume that G' has no t-thread with $t \geq 2$, i.e., each 2-vertex has no 2-neighbors. Let $x \in V_2$. Then, x receives at least $1 - \frac{2.4}{3} = 0.2$ from each 3^+-neighbor, and so $M'(x) \geq -0.4 + 2 \times 0.2 = 0$. This contradiction guarantees that G' has some t-thread with $t \geq 2$.

Assume that G' has no t-thread with $t \geq 3$. Then, let ux_1x_2v be a 2-thread that is guaranteed. Assume that $d_{G'}(u) = 3$; then, $d_{G'}(v) \geq 6$. Then, u sends x_1 at least 0.2, and v sends x_2 at least $1 - \frac{2.4}{6} = 0.6$. Thus, $M'(u) + M'(x_1) + M'(x_2) + M'(v) \geq -0.4 + 0.2 - 0.4 + 0.6 = 0$, a contradiction. Thus, $d_{G'}(u) \geq 4$ (and symmetrically, $d_{G'}(v) \geq 4$). So, u sends x_1 (and v sends x_2) at least $1 - \frac{2.4}{4} = 0.4$. Then, $M'(u) + M'(x_1) + M'(x_2) + M'(v) \geq 2 \times -0.4 + 2 \times 0.4 = 0$. This contradiction implies that if G' has no t-thread with $t \geq 3$, then G' has the 2-thread described above as a subgraph.

Thus, G' has a t-thread with $t \geq 3$. Assume that G has no t-thread with $t \geq 4$. Let $ux_1x_2x_3v$ be a 3-thread in G'. Then, we may assume that $\min\{d_{G'}(u), d_{G'}(v)\} \geq 6$. Then, u sends x_1 (and v sends x_3) at least 0.6. Then, $M'(u) + M'(x_1) + M'(x_2) + M'(x_3) + M'(v) \geq -0.4 + 0.6 - 0.4 - 0.4 + 0.6 = 0$. This contradiction implies that if G' has no t-thread with $t \geq 4$, then G' has the 3-thread described above as a subgraph.

Thus, G' has a t-thread with $t \geq 4$. This contradiction completes the proof of the lemma. $\square$

Lemma 3.4.1 is a direct contradiction to Claim 3.4.1, which proves Theorem 3.1.9.

Note: it is easy to see in the discharging of Lemma 3.4.1 that in order to guarantee that a 3-thread has net positive charge, we need to guarantee that one of the endpoints of the thread has degree at most 5. Hence arises the constant added term of $\lfloor \frac{5}{2} \rfloor = 2$ to the bound for trees.

3.5 Proof of Theorem 3.1.10

Let G be a minimal counterexample (w.r.t $|E(G)|$) to Theorem 3.1.10, i.e., there is some list assignment L for $E(G)$ such that each list has size $\lfloor \frac{3\Delta}{2} \rfloor + 1$, G admits no star L-edge coloring, but every proper subgraph does.

In addition to Claims 3.3.2 and 3.3.3, we also must show that the following configuration is forbidden for G':

Claim 3.5.1. *G' has no path $x_1x_2x_3x_4x_5$ such that $d(x_i) = 2$ for each $i \in \{1, 2, 3, 4, 5\}$.*

Proof. Assume that such a path exists. Let $H = G - x_3x_4$ and let f be an L-star-edge-coloring of H.

By Lemma 3.3.2, we have that T_{x_i} is a star rooted at x_i for each i. By Lemma 3.3.3, T_{x_2}, T_{x_3}, and T_{x_4} are all trivial. Thus, to extend f to G, it suffices to forbid colors used at edges incident to x_2 and x_5. There are at most $\Delta + 2$ such colors. Since $k = \lfloor \frac{3\Delta}{2} \rfloor + 1$, there is at least one color still available for x_3x_4, a contradiction. $\square$

Below is a lemma that shows that the configuration in Lemma 3.5.1 is guaranteed to exist in G' if $mad(G) < \frac{15}{7}$.

Lemma 3.5.1. *If $mad(G) < \frac{15}{7}$, then G' contains a t-thread with $t \geq 5$.*

Proof. For each vertex $x \in V(G')$, define an initial charge $M(x) = d(x) - \frac{15}{7}$. Then, since $mad(G') \leq mad(G) < \frac{15}{7}$, we have $\sum_{x \in V(G')} M(x) < 0$. For the remainder of this proof, a c-vertex refers to the vertex degree in G'.

Note that $M(x) = \frac{-1}{7}$ for each 2-vertex x and $M(y) > 0$ for each 3^+-vertex y.

Obtain a new charge $M'(x)$ for each $x \in V(G')$ by the following rule:

R1: Every 3^+-vertex y sends $\frac{M(x)}{d_2(x)}$ to each 2-neighbor in G', if $d_2(x) \neq 0$.

If y is a 3^+-vertex, then y sends at least $1 - \frac{15}{7d(y)}$ to each 2-neighbor. This is an increasing value as $d(y)$ increases. Thus each 3^+-vertex y sends each 2-neighbor at least $\frac{2}{7}$. And clearly, $M'(y) \geq 0$ for each 3^+-vertex y.

Let $x_0x_1 \cdots x_tx_{t+1}$ be a t-thread with $t \leq 4$. Then, x_0 sends x_1 (and x_{t+1} send x_t) at least $\frac{2}{7}$. Thus, $\sum_{i=0}^{t+1} M'(x_i) \geq 2 \times (\frac{2}{7}) - 4 \times (\frac{1}{7}) = 0$.

Since $\sum_{x \in V(G')} M'(x) = \sum_{x \in V(G')} M(x) < 0$, it follows that G' contains some t-thread with $t \geq 5$. $\square$

Lemma 3.5.1 provides a contradiction that completes the proof of Theorem 3.1.10.

3.6 A stronger form of Star Edge Coloring

Let G be a graph, $Z \subseteq E(G)$, and L be a list assignment of G. A *Z-star-sublist of L* is a list assignment L_0 satisfying the following two conditions:
(a) $L_0(e) \subseteq L(e)$ for each edge e and $|L_0(e)| \in \{1, 3\}$ with $|L_0(e)| = 3$ if and only if $e \in Z$.
(b) Any edge coloring ϕ with $\phi(e) \in L_0(e)$ is a star edge coloring of G.

A vertex $v \in V(G)$ is called a *pre-pendent vertex* if $d(v) \geq 2$ and $d_1(v) \geq d(v) - 1$. In other words, the vertex v has at least $d(v) - 1$ neighbors of degree one. An edge $e = uv$ is called a *twig* of G if $d(u) = 1$ and v is a pre-pendent vertex.

Theorem 3.6.1. *Let T be a tree with maximum degree Δ rooted at x. Let W be a set of twigs in T such that W is a matching and x is not a pre-pendent vertex of a twig in W. Then T has a W-star-sublist of L for any list assignment L satisfying the following:*
(a) $|L(e)| = 1$ for any $e \in Lay(1)$ and $L(e) \neq L(e')$ for any two distinct edges $e, e' \in Lay(1)$;
(b) for each $e \in Lay(2)$, $|L(e)| \geq \Delta + \lfloor \frac{d(x)}{2} \rfloor$ if $e \notin W$ and $|L(e)| \geq \Delta + \lfloor \frac{d(x)}{2} \rfloor + 2$ otherwise;
(c) for each $e \in E - Lay(1) - Lay(2)$, $|L(e)| \geq \Delta + \lfloor \frac{\Delta}{2} \rfloor$ if $e \notin W$ and $|L(e)| \geq \Delta + \lfloor \frac{\Delta}{2} \rfloor + 2$ otherwise.

Note that Theorem 3.1.4 follows from Theorem 3.6.1 by taking $W = \emptyset$.

Proof. If there are two twigs whose pre-pendant vertices are adjacent, then there is an edge $uv \in E(T) - W$ such that each of u and v is a pre-pendant vertex of some twig in W. Thus T is a bistar and the result follows easily. Now we assume that W is an induced matching.

Note that for each edge e in Level 1, $L_0(e) = L(e)$. We proceed algorithmically from lower to higher levels to find L_0 as follows.

From Level 1 to Level 2 (construct $L(e)$ for each $e \in Lay(2)$):

Denote $l = \Delta + \lfloor \frac{d(x)}{2} \rfloor - d(x)$ and $N(x) = \{y_1, \ldots, y_{d(x)}\}$. Suppose that the sublist for each $e \in Lay(2)$ incident with y_i is selected for $i \leq t-1$. Now we select the sublists for edges incident with y_t. Denote $N(y_t) = \{u_0, u_1, \ldots, u_{d(y_t)-1}\}$ where $x = u_0$. In case when y_t is a pre-pendant of a twig in W, $y_t u_{d(y_t)-1}$ is the twig in W.

Step 1: Select $L_0(y_t u_i)$ for $y_t u_i$ for $1 \leq i \leq l$.

Note that $E(x)$ and W are disjoint. Thus $|L_0(e)| = 1$ for each $e \in E(x)$ and for each $1 \leq i \leq l$,

$$|L(y_t u_i) - \bigcup_{e \in E(x)} L_0(e)| \geq |L(y_t u_i)| - d(x) \geq \Delta + \lfloor \frac{d(x)}{2} \rfloor - d(x) = l.$$

Thus for edges $y_t u_1, \ldots, y_t u_l$, one can choose mutually disjoint $L_0(y_t u_1), \ldots, L_0(y_t u_l)$ such that $L_0(y_t u_i) \subseteq L(y_t u_i) - \cup_{e \in E(x)} L_0(x)$ for each $i = 1, \ldots, t$ with size 1 or 3 depending on whether $l = d(y_t) - 1$ and whether $y_t u_{d(y_t)-1} \in W$ or not.

Step 2: Select $L_0(y_tu_i)$ for $l+1 \le i \le d(y_t)-1$.

We select $L_0(y_tu_i)$ one by one for $l+1 \le i \le d(y_t)-1$. Note $y_tx = y_tu_0$ and for each $1 \le j \le d(y_t)-2$, $y_tu_j \notin W$.

We forbid the colors in $L_0(y_tu_j)$ for each $j = 1, \ldots, i-1$ and in $L_0(xy_j)$ for each $s = t, \ldots, t + \lfloor \frac{d(x)}{2} \rfloor$ where the subindex s is taken mod $d(x)$.

Since

$$\begin{aligned} |A_i = L(y_tu_i) - \bigcup_{j=1}^{i-1} L_0(y_tu_j) - \bigcup_{s=t}^{t+\lfloor \frac{d(x)}{2} \rfloor} L_0(xy_s)| &\ge |L(y_tu_i)| - (i-1) - \lfloor \frac{d(x)}{2} \rfloor - 1 \\ &= |L(y_tu_i)| - i - \lfloor \frac{d(x)}{2} \rfloor \\ &\ge 1 \quad (\text{or} \ge 3 \text{ if } y_tu_i \in W), \end{aligned}$$

one can choose $L_0(y_tu_i) \subset A_i$ with size 1 or 3 depending on whether $y_tu_i \in W$.

Assume that $L_0(e)$ is selected for all edges e in Layers up to Layer i. We are going to select $L_0(e)$ for edges e in Layer $i+1$ using the same strategy by modifying l and replacing $\lfloor \frac{d(x)}{2} \rfloor$ with $\lfloor \frac{\Delta}{2} \rfloor$.

From level i to level $i+1$ ($i \ge 2$):

Let u be a vertex in level $i-1$ with a neighbor in level i of degree at least 2. Denote $N(u) = \{w_0, w_1, \ldots, w_{d(u)-1}\}$ where w_0 is the parent of u. Denote T_u to be the subtree rooted at u with $E(T_u) = \{uw_0\} \cup [\cup_{t=1}^{d(u)-1} E(w_t)]$. Then T_u is a rooted tree with two layers, and $E(u) \cap W = \emptyset$.

Suppose that the sublist for each edge $e \in Lay(i)$ incident with w_i is selected for each $i \le t-1$. Now we select the sublist for edges incident with w_t. Denote $N(w_t) = \{u_0, u_1, \ldots, u_{d(w_t)-1}\}$ where $u = u_0$. In case when w_t is a pre-pendant of a twig in W, $w_tu_{d(w_t)-1}$ is the twig in W. Let $l = \lfloor \frac{3\Delta}{2} \rfloor - d(u)$. Then we can follow Steps 1 and 2 in the above to pick $L_0(w_tu_i)$ for $1 \le i \le d(w_t)-1$ (replacing $\lfloor \frac{d(x)}{2} \rfloor$ with $\lfloor \frac{\Delta}{2} \rfloor$), where similar as in Step 1 we have

$$|A_i| \ge |L(w_tu_i)| - (i-1) - \lfloor \frac{\Delta}{2} \rfloor - 1 \ \ge 1 \quad (\text{or} \ge 3 \text{ if } w_tu_i \in W).$$

Note that for each $l+1 \le i \le d(w_t)-1$,

$$i + \lfloor \frac{\Delta}{2} \rfloor \ge l + 1 + \lfloor \frac{\Delta}{2} \rfloor = \lfloor \frac{3\Delta}{2} \rfloor - d(u) + 1 + \lfloor \frac{\Delta}{2} \rfloor \ge \Delta.$$

Therefore by the algorithm, we have the following observation.

Observation: *$L_0(w_tu_i) \cap L_0(uw_k) = \emptyset$ for each $i \in \{1, \ldots, d(u)-1\}$ and $k \in \{t, t+1, \ldots, t + \lfloor \frac{d(u)}{2} \rfloor\}$ (mod $d(u)$). Furthermore, for each $k \in \{1, 2, \ldots, \lfloor \frac{3\Delta}{2} \rfloor - d(w_0)\}$, $L_0(w_0z) \cap L_0(uw_k) = \emptyset$ for each $z \in N_T(w_0)$ by the coloring process of $E_T(u)$.*

Let c be a coloring of $E(T)$ with $c(e) \in L_0(e)$ for each edge e. It remains to verify that c is a star edge coloring of T . Clearly, c is a proper edge coloring. Suppose to the contrary that $vwuyz$ is a path of length 4 in T such that $c(vw) = c(uy)$ and $c(uw) = c(yz)$. Denote $N(u) = \{w_0, w_1, \ldots, w_{d(w)-1}\}$ as above. Then $w = w_i$ and $y = w_j$ for some i, j. WLOG assume $i < j$.

We first assume $i \geq 1$. Since $c(w_i v) = c(uw_j)$, by Observation, we have $j-i \notin \{0, 1, \ldots, \lfloor \frac{d(u)}{2} \rfloor\}$ (mod $d(u)$). Similarly, we also have $i - j \notin \{0, 1, \ldots, \lfloor \frac{d(u)}{2} \rfloor\}$ (mod $d(u)$) as $c(uw_i) = c(w_j z)$. This implies $j - i \notin \{0, 1, \ldots, d(u) - 1\}$ (mod $d(u)$), a contradiction.

Now we assume $i = 0$. Since $c(uw_j) = c(w_0 v)$, by Observation, $j \geq \lfloor \frac{3\Delta}{2} \rfloor - d(w_0) + 1$. Since $c(w_j z) = c(uw_0)$, $0 \notin \{j, j+1, \ldots, j + \lfloor \frac{d(u)}{2} \rfloor\}$ (mod $d(u)$), meaning $j + \lfloor \frac{d(u)}{2} \rfloor \leq d(u) - 1$. Since $d(w_0) \leq \Delta$ and $d(u) \leq \Delta$, we have

$$d(u) \leq \lfloor \frac{d(u)}{2} \rfloor + 1 + \lfloor \frac{d(u)}{2} \rfloor \leq \lfloor \frac{3\Delta}{2} \rfloor - d(w_0) + 1 + \lfloor \frac{d(u)}{2} \rfloor \leq j + \lfloor \frac{d(u)}{2} \rfloor \leq d(u) - 1.$$

This is a contradiction again and thus completes the proof of the theorem. □

The following structural result of graphs with small maximum average degree is needed in the proof of Theorem 3.1.11 and its proof will be completed in the next section.

Lemma 3.6.2. *Let $\varepsilon > 0$ be a real number and $d = 2\lceil \frac{8-3\varepsilon}{9\varepsilon} \rceil$. Let G be a graph with $mad(G) < \frac{8}{3} - \varepsilon$ and minimum degree $\delta(G) \geq 2$. Then, G contains one of the following t-threads $ux_1x_2 \cdots x_t v$ (see Figure 5.).*

(C1) $t \geq 4$;

(C2) $t = 3$ *and* $d(u) \leq d$;

(C3) $t = 2$, $d(u) = 3$, *and* $d_{(d+1)^+}(u) = 0$;

(C4) $t = 2$, $d(u) \leq d$, $d_2(u) = d(u)$, *and* $d(v) \leq d$;

(C5) $t = 2$, $d(u) \leq \frac{d}{2}$, $d_2(u) \geq d(u) - 1$, $d_{(d+1)^+}(u) = 0$, *and* $d(v) \leq d$;

(C6) $t = 1$, $d(u) = 3$, $d_{(d+1)^+}(u) = 0$, $d(v) \leq d$, *and* $d_2(v) = d(v)$;

(C7) $t = 1$, $d(u) = 3$, $d_{(d+1)^+}(u) = 0$, $d(v) \leq \frac{d}{2}$, $d_2(v) \geq d(v) - 1$, *and* $d_{(d+1)^+}(v) = 0$.

Theorem 3.1.11 is a corollary of the following slightly stronger result.

Theorem 3.6.3. *Let $\varepsilon > 0$ be a real number and $d = 2\lceil \frac{8-3\varepsilon}{9\varepsilon} \rceil$. Let G be a graph with $mad(G) < \frac{8}{3} - \varepsilon$ and maximum degree Δ, and let W be a set of twigs in G which is a matching. Let $k = \max\{\frac{3}{2}\Delta + \frac{d}{2} + 2, \Delta + 2d + 1\}$. Then for any k-list assignment L of $E(G)$, there exists a W-star-sublist of L.*

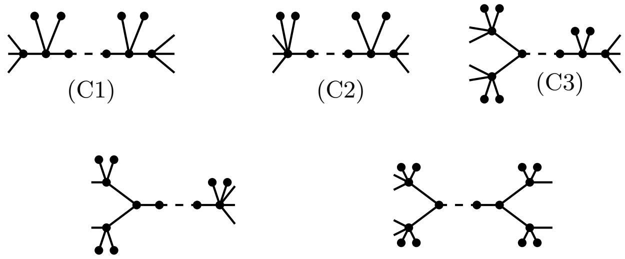

Figure 5.

Proof. Let the pair (G, W) be a counterexample to Theorem 3.6.3 with $|E(G)|$ minimum where W is a set of twigs. Thus G is connected and by Theorem 3.6.1 G is not a tree and so G' is non-empty. Note that no pre-pendent vertex of G is in $V(G')$ by definition and $G - E(G')$ is a forest.

Claim 3.6.1. *If $d_{G'}(x) \le d$, then T_x is a star rooted at x.*

Proof. Let $E_0 = E(T_x) - E_G(x)$ be the set of edges in T_x not incident to x. Then the claim is equivalent to $E_0 = \emptyset$. Suppose to the contrary $E_0 \ne \emptyset$. Denote $G_1 = G - E_0$ and $W_1 = W \cap E(G_1)$. By the minimality of G, for each $e \in E(G_1)$, one can have a desired W_1-star-sublist $L_0^1(e)$ of $L(e)$ in G_1. Since $x \in V(G')$, x is not a pre-pendent vertex in G_1, so $|L_0^1(e)| = 1$ for each $e \in E_{G_1}(x)$. For each edge e in $Lay(2)$ of T_x, let

$$L'(e) = L(e) - \bigcup_{e' \in E(x) - E(T_x)} L_0^1(e').$$

Since $\Delta \ge d_G(x) = d_{G'}(x) + d_{T_x}(x)$ and $d_{G'}(x) \le d$, we have

$$|L'(e)| \ge \frac{3}{2}\Delta + \frac{d}{2} + 2 - d_{G'}(x) \ge \Delta + \frac{\Delta - d_{G'}(x)}{2} + 2 \ge \Delta + \frac{d_{T_x}(x)}{2} + 2.$$

By Theorem 3.6.1, one can obtain a desired sublist L_0^2 of T_x with $L_0^1(e) = L_0^2(e)$ for each $e \in E_{T_x}(x)$. Then we combine L_0^1 and L_0^2 to obtain a W-star-sublist of L of G, a contradiction. □

Claim 3.6.2. *Assume that $x \in V(G)$ such that $d_{G'}(x) \le d$, $d_{G'}(u) \le d$ for every $u \in N_{G'}(x)$, and $\sum_{u \in N_{G'}(x)} d_{G'}(u) \le 2d + 2$. Then $d_G(x) = d_{G'}(x)$. That is, T_x is trivial.*

Proof. Suppose to the contrary that T_x is nontrivial. Since $d_{G'}(x) \le d$, $d_{G'}(u) \le d$ for every $u \in N_{G'}(x)$, by Claim 3.6.1, T_x and T_u are stars rooted at x and u respectively. By the minimality of G, $H = G - E(T_x)$ has a desired W-star-sublist L_0 of L. Denote

$$A(x) = \bigcup_{u \in N_{G'}(x),\ e \in E_{G'}(u)} L_0(e).$$

Then $|A(x)| \leq 2d+2$ since $|L_0(e)| = 1$ for each $e \in E_{G'}(u)$ with $u \in N_{G'}(x)$. For each $u \in N_{G'}(x)$, there is no path with four edges containing both one edge in T_u and one edge in T_x, since T_x and T_u are stars. Thus, to find sublists for edges in T_x, it suffices to exclude the colors in $A(x)$. Additionally, since x is not a pre-pendant vertex, we only need one available color for each edge in $E(T_x)$. For any edge $e \in E(T_x)$, $|L(e) \setminus A(x)| \geq \Delta + 2d + 1 - (2d+2) = \Delta - 1 \geq d_{T_x}(x)$. Thus, L_0 can be extended to be a desired W-star-sublist of L in G, a contradiction. $\square$

Now we are ready to present main reductions by utilizing W-star-sublist argument.

Claim 3.6.3. *There is no path xyz in G' such that $d_{G'}(x) \leq 3$, $d_{G'}(u) \leq d$ for every $u \in N_{G'}(x)$, and $d_{G'}(y) = d_{G'}(z) = 2$. Therefore G' doesn't contain Configurations* **(C1)**, **(C2)**, *or* **(C3)**.

Proof. Suppose to the contrary that there is such a path xyz in G'. By Claim 3.6.1, T_z is a star and T_u is also a star for each $u \in N_{G'}(x)$.

Let x_1 (and x_2) be the other neighbor(s) of x. Since $\sum_{u \in N_{G'}(x)} d_{G'}(u) \leq 2d+2$ and $d_{G'}(x) + d_{G'}(z) \leq 5 < 2d+2$, by Claim 3.6.2, both T_x and T_y are trivial.

Let $H = G - xy$. Note $xy, yz \notin W$ and z is a pre-pendant vertex in H (but not in G). Let $W' = W \cup \{yz\}$. By the minimality of G, H has a W'-star sublist L_0' of L. Since $yz \in W'$, $|L_0'(yz)| = 3$. We first pick a sublist L_0 of L for $G - xy$.

(1) $L_0(e) = L_0'(e)$ for each $e \in E(H) \setminus \{yz\}$.

(2) Pick any color

$$\alpha \in L_0'(yz) - L_0'(xx_1) \cup L_0'(xx_2) \text{ if } x_2 \text{ exists; and } \alpha \in L_0'(yz) - L_0'(xx_1) \text{ otherwise.}$$

Set $L_0(yz) = \{\alpha\}$. Note that this is possible since $xx_1 \notin W'$ (and $xx_2 \notin W'$).

For the edge xy, one can further pick a color

$$\beta \in A = L(xy) - \bigcup_{e \in E_{G'}(u), u \in \{z, x_1, x_2\}} L_0(e)$$

(or $\beta \in A = L(xy) - \bigcup_{e \in E_{G'}(u), u \in \{z, x_1\}} L_0(e)$ if x_2 does not exist), and let $L_0(xy) = \{\beta\}$, since we have $|A| \geq \Delta + 2d + 1 - (2d+2) = \Delta - 1 \geq 1$.

It is easy to see that L_0 is a W-star sublist of L in G. This contradicts the fact that G is a counterexample.

The second part of the claim follows from the fact that Configurations **(C1)**, **(C2)** and **(C3)** all satisfy the conditions of Claim 3.6.3. $\square$

Claim 3.6.4. *There is no path $uxyz$ in G' such that $d_{G'}(u) \leq d$, $d_{G'}(w) \leq d$ for each $w \in N_{G'}(u)$, $\sum_{w \in N_{G'}(u)} d_{G'}(w) \leq 2d+2$, $d_{G'}(x) = d_{G'}(y) = 2$, and $d_{G'}(z) \leq d$. Therefore G' does not contain Configurations* **(C4)** *or* **(C5)**.

Proof. By Claim 3.6.1, T_z is a star rooted at z. By Claim 3.6.2, T_u, T_x, and T_y are trivial. Note $xy \notin W$.

Let $H = G - xy$. By the minimality of G, H has a W-star-sublist L_0 of L. Since T_u is trivial, we have $d_G(u) + d_G(z) \leq d + \Delta$, and so

$$|L(xy) - \bigcup_{e \in E_G(u) \cup E_G(z)} L_0(e)| \geq \Delta + 2d + 1 - (\Delta + d) = d + 1 > 1.$$

Pick a color $\alpha \in L(xy) - \bigcup_{e \in E_G(u) \cup E_G(z)} L_0(e)$ and let $L_0(xy) = \{\alpha\}$. Therefore L_0 is extended to be a W-star-sublist of L in G, a contradiction. □

Claim 3.6.5. *There is no path uxy in G' such that $d_{G'}(y) \leq 3$, $d_{G'}(v) \leq d$ for each $v \in N_{G'}(y)$, $d_{G'}(x) = 2$, $d_{G'}(u) \leq d$, $d_{G'}(w) \leq d$ for each $w \in N_{G'}(u)$, and $\sum_{w \in N_{G'}(u)} d_{G'}(w) \leq 2d + 2$. Therefore G' does not contain Configurations* **(C6)** *or* **(C7)**.

Proof. Suppose to the contrary that there is such a path. We first show $d_{G'}(y) = 3$. Otherwise if $d_{G'}(y) = 2$, let z be the other neighbor distinct from x. Then $uxyz$ is a path forbidden in Claim 3.6.4. This contradiction implies $d_{G'}(y) = 3$. Denote $N_{G'}(y) = \{x, y_1, y_2\}$.

By Claim 3.6.2, T_y, T_x, and T_u are all trivial. By Claim 3.6.1 both T_{y_1} and T_{y_2} are stars.

Let $H = G - xy$. By minimality of G, H has a W-star-sublist L_0 of L. We consider two cases.

Case 1: $L_0(xu) \subset L_0(yy_1) \cup L_0(yy_2)$. WLOG, assume $L_0(xu) = L_0(yy_1)$.

Then $L_0(xu) \cap L_0(yy_2) = \emptyset$. Note that in this case it is allowed to have $L_0(xy) = L_0(wy_2)$ for a leaf edge wy_2 in G. Let

$$\alpha \in A = L(xy) - \bigcup_{w \in N_G(y_1)} L_0(y_1w) - \bigcup_{w \in N_{G'}(y_2)} L_0(y_2w) - \bigcup_{w \in N_{G'}(u)} L_0(uw)$$

and set $L_0(xy) = \{\alpha\}$. This is possible since $|A| \geq \Delta + 2d + 1 - (\Delta + d + d) = 1$.

Case 2: $L_0(xu) \cap [L_0(yy_1) \cup L_0(yy_2)] = \emptyset$.

In this case it is allowed to have $L_0(xy) = L_0(wy_i)$ for a leaf edge wy_i for each $i = 1, 2$. Let

$$\beta \in B = L(xy) - \bigcup_{w \in N_{G'}(y_1)} L_0(y_1w) - \bigcup_{w \in N_{G'}(y_2)} L_0(y_2w) - \bigcup_{w \in N_{G'}(u)} L_0(uw)$$

and set $L_0(xy) = \{\beta\}$, since $|B| \geq \Delta + 2d + 1 - (d + d + d) \geq 1$.

In either case, we can extend L_0 from H to G, a contradiction. □

By Claims 3.6.3, 3.6.4, and 3.6.5, G' does not contain configurations **(C1)-(C7)**, which contradicts Lemma 3.6.2. This contradiction completes the proof of the theorem. □

In this section, we will prove Lemma 3.6.2. For convenience, we copy the lemma in the following.

Lemma 3.6.4. *3.6.2 Let $\varepsilon > 0$ be a real number and $d = 2\lceil \frac{8-3\varepsilon}{9\varepsilon} \rceil$. Let G be a graph with $mad(G) < \frac{8}{3} - \varepsilon$ and minimum degree $\delta(G) \geq 2$. Then, G contains one of the following t-threads $ux_1x_2 \cdots x_tv$.*

(C1) $t \geq 4$;

(C2) $t = 3$ *and* $d(u) \leq d$;

(C3) $t = 2$, $d(u) = 3$, *and* $d_{(d+1)^+}(u) = 0$;

(C4) $t = 2$, $d(u) \leq d$, $d_2(u) = d(u)$, *and* $d(v) \leq d$;

(C5) $t = 2$, $d(u) \leq \frac{d}{2}$, $d_2(u) \geq d(u) - 1$, $d_{(d+1)^+}(u) = 0$, *and* $d(v) \leq d$;

(C6) $t = 1$, $d(u) = 3$, $d_{(d+1)^+}(u) = 0$, $d(v) \leq d$, *and* $d_2(v) = d(v)$;

(C7) $t = 1$, $d(u) = 3$, $d_{(d+1)^+}(u) = 0$, $d(v) \leq \frac{d}{2}$, $d_2(v) \geq d(v) - 1$, *and* $d_{(d+1)^+}(v) = 0$.

Proof. If $\varepsilon \geq \frac{2}{3}$, then $mad(G) < 2$ and so G is acyclic, contradicting $\delta(G) \geq 2$. Thus $0 < \varepsilon < \frac{2}{3}$.

We prove by contradiction and proceed by the discharging method. Suppose to the contrary that there is no t-thread as in **(C1)-(C7)**. For each $x \in V(G)$, define the initial charge $M(x) = d(x) - (\frac{8}{3} - \varepsilon)$. Note that $M(x) = 2 - \frac{8}{3} + \varepsilon = -\frac{2}{3} + \varepsilon$ for each 2-vertex x. For each 3^+-vertex x, $M(x) \geq \frac{1}{3} + \varepsilon > 0$.

Obtain a second charge $M'(x)$ by the following rule:

R1: *Each $(d+1)^+$-vertex y sends $\frac{d(y)-8/3+\varepsilon}{d_{d^-}(y)}$ to each d^--neighbor, if $d_{d^-}(y) \neq 0$.*

Obtain a third charge $M''(x)$ by the following rule:

R2: *Each 3^+-vertex y sends $\frac{M'(y)}{d_2(y)}$ to each 2-neighbor if $d_2(y) \neq 0$.*

It is of interest to consider the amount sent from a 3^+-vertex y to a 2-neighbor, given properties of y. Firstly, we compute charges of several types in the following. Note that the function $\frac{c-8/3+\varepsilon}{c}$ is increasing with respect to c given $\varepsilon < 2/3$.

(A) By **R1**, if y is a $(d+1)^+$-vertex, then y sends each d^--neighbor x at least

$$\frac{d+1-8/3+\varepsilon}{d+1} = 1 - \frac{8-3\varepsilon}{6\lceil (8-3\varepsilon)/(9\varepsilon) \rceil + 3} > 1 - \frac{8-3\varepsilon}{6(8-3\varepsilon)/(9\varepsilon)} = 1 - \frac{3\varepsilon}{2}.$$

Thus y sends x at least $1 - \frac{3\varepsilon}{2}$.

(B) Assume that y is a 3-vertex with a 2-neighbor x.

- If $d_2(y) = 1$, then y sends x exactly $\frac{1}{3} + \varepsilon$.
- Assume that y has a $(d+1)^+$-neighbor z. Then, $d_2(y) \leq 2$ and y receives at least $1 - \frac{3\varepsilon}{2}$ from z by (A). Thus y sends x at least $\frac{1}{2}[(\frac{1}{3} + \varepsilon) + (1 - \frac{3\varepsilon}{2})] \geq \frac{2}{3} - \frac{\varepsilon}{4}$.

(C) Assume that $4 \le d(y) < \frac{d}{2} + 1$. Let x be a 2-neighbor of y.

- If $d_2(y) = d(y)$, then y sends x at least $\frac{4-8/3+\varepsilon}{4} = \frac{1}{3} + \frac{\varepsilon}{4}$.
- If $d_2(y) \le d(y) - 1$, then y sends x at least $\frac{4-8/3+\varepsilon}{3} \ge \frac{4}{9}$.
- If $d_2(y) \le d(y) - 2$, then y sends x at least $\frac{4-8/3+\varepsilon}{2} \ge \frac{2}{3}$.
- Assume that y has a $(d+1)^+$-neighbor z. Then, $d_2(y) \le d(y) - 1$ and y receives at least $1 - \frac{3\varepsilon}{2}$ from z by (A). Thus, y sends x at least $\frac{1}{3}[4 - \frac{8}{3} + \varepsilon + (1 - \frac{3\varepsilon}{2})] \ge \frac{7}{9} - \frac{\varepsilon}{6} \ge \frac{2}{3}$.

(D) Assume $\frac{d}{2} + 1 \le d(y) \le d$.

- If $d_2(y) = d(y)$, then y sends x at least

$$\frac{d/2 + 1 - 8/3 + \varepsilon}{d/2 + 1} = 1 - \frac{8/3 - \varepsilon}{\lceil (8 - 3\varepsilon)/(9\varepsilon) \rceil + 1} \ge 1 - \frac{3\varepsilon(8 - 3\varepsilon)}{8 + 6\varepsilon}.$$

- If $d_2(y) \le d(y) - 1$, then y sends x at least

$$\frac{d/2 + 1 - 8/3 + \varepsilon}{d/2} = 1 - \frac{5/3 - \varepsilon}{\lceil (8 - 3\varepsilon)/(9\varepsilon) \rceil} \ge 1 - \frac{5/3 - \varepsilon}{(8 - 3\varepsilon)/(9\varepsilon)} = 1 - \frac{3\varepsilon(5 - 3\varepsilon)}{8 - 3\varepsilon}.$$

Clearly, $\sum_{x \in V(G)} M''(x) = \sum_{x \in V(G)} M'(x) = \sum_{x \in V(G)} M(x) < 0$ since $mad(G) < \frac{8}{3} - \varepsilon$. As $M''(y) \ge 0$ for each 3^+-vertex y, we have

$$\sum_{x \in V_2(G)} M''(x) < 0. \tag{3.1}$$

In the following, we shall show that each of the t-threads not forbidden in G receives nonnegative charge to yield a contradiction.

- Let $ux_1x_2x_3v$ be a 3-thread. Since **(C2)** is forbidden, we have $\min\{d(u), d(v)\} \ge d + 1$. By (A), u sends x_1 at least $1 - \frac{3\varepsilon}{2}$, and v sends x_3 at least $1 - \frac{3\varepsilon}{2}$ as well. Thus, $M''(x_1) + M''(x_2) + M''(x_3) \ge 3(\varepsilon - \frac{2}{3}) + 2(1 - \frac{3\varepsilon}{2}) = 0$.

- Let ux_1x_2v be a 2-thread. We further divide our discussion according to the value of $d(u)$:

  1. Assume $d(u) = 3$. Since **(C3)** is forbidden, we conclude $d_{(d+1)^+}(u) \ge 1$. Then, u sends x_1 at least $\frac{2}{3} - \frac{\varepsilon}{4}$ by (B).
     (a) Assume $d(v) \ge \frac{d}{2} + 1$. Then, v sends x_2 at least $1 - \frac{3\varepsilon(8-3\varepsilon)}{8+6\varepsilon}$ by (D). Since $\varepsilon < \frac{2}{3}$, we have
     $M''(x_1) + M''(x_2) \ge 2(\varepsilon - \frac{2}{3}) + (\frac{2}{3} - \frac{\varepsilon}{4}) + (1 - \frac{3\varepsilon(8-3\varepsilon)}{8+6\varepsilon}) = \frac{117\varepsilon^2 - 48\varepsilon + 16}{12(4+3\varepsilon)} > 0$.
     (b) Assume $3 \le d(v) < \frac{d}{2} + 1$. Since **(C4)** is forbidden and $d_2(u) < d(u)$, we have $d_2(v) < d(v)$.

Case 1: Assume $d_{(d+1)^+}(v) \geq 1$. Then, v sends x_2 at least $\frac{2}{3} - \frac{\varepsilon}{4}$ by (B) or (C). Hence $M''(x_1) + M''(x_2) \geq 2(\varepsilon - \frac{2}{3}) + 2(\frac{2}{3} - \frac{\varepsilon}{4}) > 0$.

Case 2: Assume $d_{(d+1)^+}(v) = 0$. Since **(C3)** is forbidden, $d(v) \geq 4$, and since **(C5)** is forbidden, $d_2(v) \leq d(v) - 2$. Then, v sends x_2 at least $\frac{2}{3}$ by (C) and so $M''(x_1) + M''(x_2) \geq 2(\varepsilon - \frac{2}{3}) + (\frac{2}{3} - \frac{\varepsilon}{4}) + \frac{2}{3} > 0$.

2. Assume $\min\{d(u), d(v)\} \geq 4$. Then, u sends x_1 at least $\frac{1}{3} + \frac{\varepsilon}{4}$ by (C).

(a) Assume $d(v) \geq d+1$. Then, v sends x_2 at least $1 - \frac{3\varepsilon}{2}$ by (A), so $M''(x_1) + M''(x_2) \geq 2(\varepsilon - \frac{2}{3}) + (\frac{1}{3} + \frac{\varepsilon}{4}) + (1 - \frac{3\varepsilon}{2}) > 0$.

(b) Assume $\max\{d(u), d(v)\} < \frac{d}{2} + 1$. Since **(C4)** is forbidden, we have $d_2(u) < d(u)$ and $d_2(v) < d(v)$. Since **(C5)** is forbidden, for $w \in \{u, v\}$, either $d_{(d+1)^+}(w) \geq 1$ or $d_2(w) \leq d(w) - 2$. If $d_{(d+1)^+}(w) \geq 1$, then w sends its 2-neighbors at least $\frac{2}{3}$ by (C). If $d_2(w) \leq d(w) - 2$, then w sends its 2-neighbors at least $\frac{2}{3}$ by (C). Thus, $M''(x_1) + M''(x_2) \geq 2(\varepsilon - \frac{2}{3}) + 2 \cdot \frac{2}{3} > 0$.

(c) Assume $d \geq d(v) \geq \frac{d}{2} + 1$. Since $d_2(v) < d(v)$, v sends x_2 at least $1 - \frac{3\varepsilon(5-3\varepsilon)}{8-3\varepsilon}$ by (D).

Case 1: $d(u) \leq \frac{d}{2}$. Then, u sends x_1 at least $\frac{2}{3}$ by (C). Thus, $M''(x_1) + M''(x_2) \geq 2(\varepsilon - \frac{2}{3}) + \frac{2}{3} + (1 - \frac{3\varepsilon(5-3\varepsilon)}{8-3\varepsilon}) = \frac{9\varepsilon^2+8}{3(8-3\varepsilon)} > 0$.

Case 2: $d(u) \geq \frac{d}{2} + 1$. Then we have
$M''(x_1) + M''(x_2) \geq 2(\varepsilon - \frac{2}{3}) + 2(1 - \frac{3\varepsilon(5-3\varepsilon)}{8-3\varepsilon}) = \frac{(6\varepsilon-4)^2}{3(8-3\varepsilon)} > 0$.

- Let uxv be a 1-thread.

(1) Assume $\min\{d(u), d(v)\} \geq 4$. Then by (C), $M''(x) \geq (\varepsilon - \frac{2}{3}) + 2(\frac{1}{3} + \frac{\varepsilon}{4}) > 0$.

(2) Assume $d(u) = 3$ and $d(v) \geq d+1$. Then, u sends x at least $\frac{1}{3}(3 - \frac{8}{3} + \varepsilon) = \frac{1}{9} + \frac{\varepsilon}{3}$. Then by (A), $M''(x) \geq (\varepsilon - \frac{2}{3}) + (\frac{1}{9} + \frac{\varepsilon}{3}) + (1 - \frac{3\varepsilon}{2}) = \frac{4}{9} - \frac{\varepsilon}{6} > 0$.

(3) Assume $d(u) = 3$ and $d_{(d+1)^+}(u) \geq 1$. Then, u sends x at least $\frac{2}{3} - \frac{\varepsilon}{4}$ by (B), so $M''(x) \geq (\varepsilon - \frac{2}{3}) + (\frac{2}{3} - \frac{\varepsilon}{4}) > 0$.

(4) Assume $d(u) = 3, d_{(d+1)^+}(u) = 0$, and $d(v) \geq \frac{d}{2} + 1$. Since **(C6)** is forbidden, $d_2(v) < d(v)$. Thus, v sends x at least $1 - \frac{3\varepsilon(5-3\varepsilon)}{8-3\varepsilon}$ by (D), so
$M''(x) \geq (\varepsilon - \frac{2}{3}) + (\frac{1}{9} + \frac{\varepsilon}{3}) + (1 - \frac{15\varepsilon-9\varepsilon^2}{8-3\varepsilon}) = \frac{45\varepsilon^2-51\varepsilon+32}{9(8-3\varepsilon)} > 0$.

(5) Assume $d(u) = 3, d_{(d+1)^+}(u) = 0$, and $d(v) \leq \frac{d}{2}$. Since **(C7)** is forbidden, $d_2(v) \leq d(v) - 2$ or $d_{(d+1)^+}(v) \geq 1$. Thus by (C), $M''(x) \geq (\varepsilon - \frac{2}{3}) + (\frac{1}{9} + \frac{\varepsilon}{3}) + \frac{2}{3} > 0$.

All the t-threads allowed in G are examined in the above arguments. This proves that every t-thread in G receives nonnegative charge, which is a contradiction to Eq. (3.1) that $\sum_{x \in V_2(G)} M''(x) < 0$. □

3.7 Concluding remarks on star edge coloring

All of my bounds are of the form $\lfloor\frac{3\Delta}{2}\rfloor + c$, where c is a constant only dependent on $mad(G)$. It is worth noting that in Theorem 3.6.3, little attention is taken to minimize c. However, in Theorems 3.1.9 and 3.1.10, c is minimized to the greatest extent our methods allow. Greater care could be taken with further specific values of $mad(G)$ to give better bounds. I have chosen to present the values of $mad(G) < 2.4$ and $mad(G) < \frac{15}{7}$ for the following reasons:

$mad(G) < 2.4$ is the threshold to guarantee a 2-thread in G'. If $mad(G) \geq 2.4$, it is possible that there is no 2-thread, and so we would need to consider reducible configurations involving a 2-vertex x with only 3^+-neighbors (in G'). In these configurations, it was necessary to guarantee that neighbors of x not only are low degree but also have neighbors of low degree; the need to consider the 2nd-neighborhood of a 2-vertex is the impetus for the double-discharging method used in Lemma 3.6.2. Further, some of the configurations for a 1-thread also required the ability to choose from one of three possible colors–for the non-list problem; this would have required a strengthening of the simple switch in Lemma 3.4.1.

$mad(G) < \frac{15}{2}$ is the threshold to guarantee a 5-thread in G'. If $mad(G) \geq \frac{15}{7}$, then t-threads with $t \leq 4$ must be considered in reducible configurations. As in the proof of Theorem 3.6.3, this would require a stronger form of list star coloring.

Wang et al. [44] obtain the following upper bounds on the non-list star edge coloring for planar graphs with large girth.

Theorem 3.7.1 ([44]). *Let G be a planar graph with maximum degree Δ and girth g.*
(i) If G has no cycles of length 4, then $\chi'_{st}(G) \leq \lfloor 1.5\Delta\rfloor + 18$.
(ii) If $g \geq 5$, then $\chi'_{st}(G) \leq \lfloor 1.5\Delta\rfloor + 13$.
(iii) If $g \geq 8$, then $\chi'_{st}(G) \leq \lfloor 1.5\Delta\rfloor + 3$.

Their proof in [44] applies a clever edge-partition technique and assigns certain specific colors to certain given part of edges. However their methods cannot be easily extendable to the list version of the problem. We believe that the girth condition in Corollary 3.1.14 are not tight. For planar graphs with girth 4, we have an infinite family of such graphs whose list star chromatic index can not be bounded by $\frac{3\Delta}{2} + c$ (see Figure 6.).

Proposition 3.7.1. *For each integer $\Delta \geq 3$, there exists a planar graph K of girth 4 with maximum degree Δ such that*

$$ch'_{st}(K) \geq \chi'_{st}(K) \geq \frac{13\Delta}{8} - \frac{3}{4}.$$

In view of Theorem 3.7.1, Proposition 3.7.1 and Corollary 3.1.14, we conjecture that, in the list star edge coloring problem, the tree-like upper bound holds for planar graphs with girth at least 5.

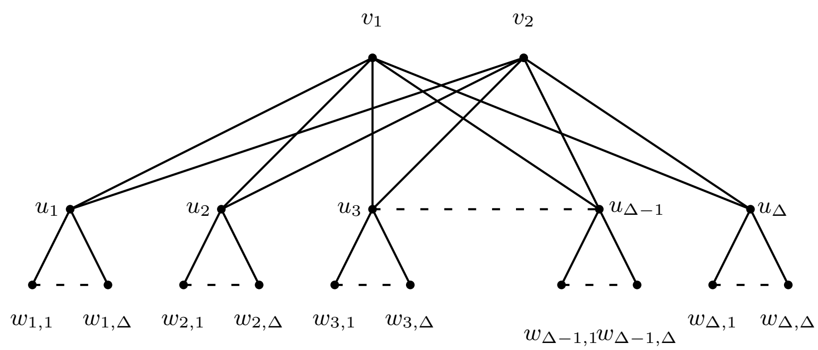

Figure 6.

Conjecture 3.7.1. *There exists a constant $c > 0$ such that for any planar graph G of girth at least 5 with maximum degree Δ, we have*

$$ch'_{st}(G) \leq \frac{3\Delta}{2} + c.$$

Proof of Proposition 3.7.1.

Let K be a graph obtained from the complete bipartite graph $K_{2,\Delta}$ as follows: Let v_1, v_2 be the Δ-vertices of $K_{2,\Delta}$, and let $u_1, \cdots, u_\Delta$ be the 2-vertices of $K_{2,\Delta}$. Obtain K by adding $(\Delta-2)$ leaves to u_i $(1 \leq i \leq \Delta)$, so that u_i is now a Δ-vertex. Let ϕ be a star $(\Delta + k)$-edge-coloring of K. We shall show that $k \geq \frac{5\Delta}{8} - \frac{3}{4}$ below.

We first claim $|c_\phi(v_1) \cap c_\phi(v_2)| \leq \frac{\Delta}{2}$. By contradiction, we may assume, without loss of generality, that $\phi(v_1u_i) \in c_\phi(v_2)$ for each $1 \leq i \leq \lfloor \frac{\Delta}{2} \rfloor + 1$. Then it follows from the Pigon-Hole Principle that there exist $1 \leq i, j \leq \lfloor \frac{\Delta}{2} \rfloor + 1$ such that $\phi(v_2u_i) = \phi(v_1u_j)$. Since $\phi(v_1u_i) \in c_\phi(v_2)$, we denote $\phi(v_1u_i) = \phi(v_2u_\ell)$. Then $u_\ell v_2 u_i v_1 u_j$ is a bicolored path (or cycle) of length four, a contradiction.

Now we assume, wlog, that $\phi(v_1u_i) = i$ for each $i \in [\Delta]$ and $\phi(v_2u_j) = \Delta + j$ for each $j \in [t]$, for some t with $\frac{\Delta}{2} \leq t \leq k$. There are $(\Delta - 2)\Delta$ leaves incident with $u_1, \ldots, u_\Delta$. To determine the number of colors needed to color those leaves, we may view that each u_i is incident with $\Delta + k - 2$ colored pseudo-leaves (that is, u_i sees all colors) and then delete certain colored pseudo-leaves to obtain a proper star-edge coloring of K. For each color pair (i, j) with $1 \leq i < j \leq \Delta$, the pseudo-leaf with color i incident with u_j and the pseudo-leaf with color j incident with u_i, together with v_1u_i, v_1u_j form a bicolored path of length four, and so at least one of the pseudo-leaves should be deleted. There are $\binom{\Delta}{2}$ such colored pairs (i, j), which implies at least $\binom{\Delta}{2}$ pseudo-leaves (with colors in $\{1, 2, \ldots, \Delta\}$) are deleted. Similarly, for each color pair $(\Delta + i, \Delta + j)$ with $1 \leq i < j \leq t$, one of the pseudo-leaves, either the one with color $\Delta + i$ incident with u_j or the one with color $\Delta + j$ incident with u_i, should be deleted. Thus at least $\binom{t}{2}$ pseudo-leaves (with color in $\{\Delta + 1, \ldots, \Delta + t\}$) are deleted. On the other hand, there are

$(\Delta-2)\Delta$ remaining leaves, which implies there are $k\Delta$ pseudo-leaves are deleted. Thus, we have

$$k\Delta \geq \binom{\Delta}{2} + \binom{t}{2}.$$

Since $t \geq \frac{\Delta}{2}$, we conclude that $k \geq \frac{5\Delta}{8} - \frac{3}{4}$. This completes the proof of the proposition.

Chapter 4

Final Remarks

I conclude with thoughts on future problems for research for the edge coloring parameters studied here.

4.1 On chromatic index of embedded graphs

Vizing's Planar Graph Conjecture, that any planar graph with maximum degree at least 6 is Class 1, remains open. This conjecture is considered to be likely as difficult to prove as the Four Color Conjecture. However, the conjecture for graphs embedded on other surfaces seems more tractable.

Problem 1. *Find $\Delta(\Sigma)$ for further values of $\chi(\Sigma)$.*

It is natural to attempt to verify the Embedded Graphs Conjecture for further given values of $\chi(\Sigma)$. Continuing this approach may result in further useful tools such as adjacency lemmas and provide evidence toward the conjecture (or against, should the conjecture prove to be false). Beyond confirming parts of the conjecture, many of the tools such as adjacency lemmas have been developed for specific cases, and these have use in other problems. However, there are some reasons why confirming special cases of $\chi(\Sigma)$ is not a preferable long-term approach: most importantly, there are infinitely many cases that must be checked. Beyond this, as $\chi(\Sigma)$ decreases (becomes more negative), $\Delta(\Sigma)$ increases. As seen in the proof of Theorem 1.2.2, many cases of vertex degree must be considered. As $\Delta(G)$ grows, so too does the number of cases, making the proof increasingly unwieldy. Thus, moving forward, it may be of more interest to look at other tools, such as Tashnikov trees. This is an extension of the brooms used in the proof of Theorem 1.2.3, first conceived by Tashnikov [41]. This tool has so far been difficult to use in its full generality; however, Cao et al. [7] have some nice new lemmas arising from Tashnikov trees for the number of large degree vertices in a critical graph. Lemmas of this sort may allow general improvements to the bounds for $\Delta(\Sigma)$.

Problem 2. *For a graph G embedded on a surface Σ of Euler charactistic $\chi(\Sigma) < 0$, can the bound for $\Delta(\Sigma)$ be improved, for example to $\sqrt{-7\chi(\Sigma)}$?*

4.2 On star edge coloring

Since star edge chromatic index and choosability are relatively new graph parameters, there remain a great number of open problems.

Problem 3. *Can we find sharp or at least improved bounds for $\chi'_{st}(G)$ or $ch'_{st}(G)$ for planar graphs? bipartite graphs? graphs with large girth?*

As shown by Proposition 3.7.1, for a planar graph G with girth at most 4, G may have non-tree-like bounds for edge star choosability. We make the following conjecture for graphs with larger girth:

Conjecture 4.2.1. *Let G be a planar graph of girth at least 5. Then, there exists some constant c such that $\chi'_{st}(G) \leq ch'_{st}(G) \leq \lfloor \frac{3\Delta(G)}{2} \rfloor + c$.*

So far, the methodology for most results on star edge coloring has been rooted in traditional, structural graph theory. Many of the techniques used for star edge coloring are inspired by those used to study the strong edge coloring (in which no path on more than 2 edges is bicolored). Can other tools from probablistic and algebraic graph theory be of use? For example, Molloy and Reed [34] used the probabilistic method to show that the strong chromatic index of G is at most $1.998\Delta(G)^2$ given sufficiently large $\Delta(G)$. A reasonable next step is to see if the ideas can be applied to star edge coloring.

For standard proper edge coloring, a major unresolved conjecture is that $\chi'(G) = ch'(G)$, i.e., chromatic index is equal to edge choosability, for every graph G. This has been proved to be true for some classes of (multi)graphs such as bipartite multigraphs (Galvin, [13]). It is reasonable to ask a similar question for star edge coloring.

Problem 4. *Is $\chi'_{st}(G) = ch'_{st}(G)$ for every graph G?*

This is currently only known to be the case for very few classes of graphs, specifically paths, cycles, stars, and trees that obtain the sharp upper bound. A possible starting point is the following possibly more tractable problem.

Problem 5. *Is $\chi'_{st}(t) = ch'_{st}(T)$ for every tree T?*

Bibliography

[1] K. Appel and W. Haken, Every planar map is four colorable, I. Discharging, *Illinois J. Math.* **21** (1977) 429-490.

[2] K. Appel, W. Haken and J. Koch, Every planar map is four colorable, II. Reducibility, *Illinois J. Math.* **21** (1977) 491-567.

[3] L. Bezegová, B. Lužar, M. Mockovčiaková, R. Soták and R. Škrekovski. Star edge coloring of some classes of graphs, J. Graph Theory **81** (2016) 73-82.

[4] L.W. Beineke and S. Fiorini, On small graphs critical with respect to edge-colourings, *Discrete Math.* **16** (1976) 109-121.

[5] D. Bokal, G. Brinkmann and S. Grünewald, Chromatic-index-critical graphs of orders 13 and 14, *Discrete Math.* **300** (2005) 16-29.

[6] G. Brinkmann and E. Steffen, Chromatic-index-critical graphs of orders 11 and 12, *Europ. J. Combinatorics* **19** (1998) 889-900.

[7] Y. Cao, G. Chen, S. Jiang, H. Liu, and F. Lu, Average degrees of edge-chromatic critical graph, submitted.

[8] G. Chen, X. Chen and Y. Zhao. Hamiltonicity of edge chromatic critical graphs., *Discrete Math.* **340** (2017) 3011-3015.

[9] A. G. Chetwynd and H. P. Yap, Chromatic index critical graphs of order 9, *Discrete Math.* **47** (1983) 23-33.

[10] K. Deng, X. Liu and S. Tian. Star edge coloring of trees (in Chinese), *J. Shandong Univ. Nat. Sci.* **46** (2011) 84-88.

[11] Z. Dvořák , B. Mohar and R. Šámal, Star chromatic index, *J. Graph Theory* **72** (2013) 313-326.

[12] G. Fertin, A. Raspaud and B. Reed, On star coloring of graphs, *Lecture Notes 483 in Comput. Sci. 2204*, Springer, Berlin, (2001) 140-153.

[13] F. Galvin, The list chromatic index of a bipartite multigraph, *J. Combin. Theory Ser. B* **63** (1995) 153-158.

[14] J.L. Gross and T.W. Tucker, Topological graph theory, Dover Publications.

[15] B. Grünbaum, Acyclic colorings of planar graphs, *Israel J. Math.* **14** (1973) 390-408.

[16] M. Han, J. Li, R. Luo and Z. Miao, List star edge coloring of k-degenerate graphs, submitted.

[17] H. Hind and Y. Zhao, Edge colorings of graphs embeddable in a surface of low genus, *Discrete Math.* **190** (1998) 107-114.

[18] I. Holyer, The NP-completeness of edge-coloring, *SIAM J. Comput.* **10** (1981) 718-720.

[19] M. Jungerman, Orientable Triangular Embeddings of $K_{18}-K_3$ and $K_{13}-K_3$, *J. of Combin. Theory, Ser. B* **16** (1974) 293-294.

[20] A.B. Kempe, On the geographical problem of four colors, *Amer. J. Math.* **2** (1879) 193-200.

[21] S. Kerdjoudj, A. Kostochka and A. Raspaud, List star edge coloring of subcubic graphs, *Discuss. Math. Graph Theory* **38** (2018) 1037-1054.

[22] S. Kerdjoudj and A. Raspaud, List star edge coloring of sparse graphs, *Discrete Appl. Math.* **238** (2018) 115-125.

[23] S. Kerdjoudj, K. Pradepp and A. Raspaud. List star chromatic index of sparse graphs, Discrete Math., 341 (2018), 1835-1849.

[24] D. König, Über Graphen und ihre Anwendung auf Determinantentheorie und Mengenlehre (in German), *Math. Ann.* **77** (1916) 453-465.

[25] H. Lei, Y. Shi and Z.-X. Song, Star chromatic index of subcubic multigraph, *J. Graph Theory* **88** (2018) 566-576.

[26] X.S. Liu and K. Deng, An upper bound on the star chromatic index of graphs with $\Delta \geq 7$, *J. Lanzhou Univ. Nat. Sci.* **44** (2008), 98-99.

[27] R. Luo, L.Y. Miao and Y. Zhao, The size of edge chromatic critical graphs with maximum degree 6, *J. Graph Theory* **60** (2009) 149-171.

[28] R. Luo, Z. Miao, and Y.Zhao, Hamiltonian cycles in critical graphs with large maximum degree, *Graphs and Combinatorics* **32** (2016) 2019-2028.

[29] R. Luo and Y. Zhao, Finding the exact bound of the maximum degree of class two graphs embeddable in a surface of characteristic $\epsilon \in \{-1, -2, -3\}$, *J. of Combin. Theory, Ser. B* **98** (2008) 707-720.

[30] R. Luo and Y. Zhao, Finding $\Delta(\Sigma)$ for a surface Σ of characteristic $\chi(\Sigma) = -5$, *J. Graph Theory* **68** (2011) 148-168.

[31] R. Luo, Z.K. Miao and Y. Zhao, Finding $\Delta(\Sigma)$ for a surface Σ of characteristic -4, *J. Graph Theory* **83** (2016) 277-302.

[32] B. Lužar, M. Mockovčiaková, and R. Soták. Note on list star edge-coloring of subcubic graphs, J. Graph Theory, 2018, https://doi.org/10.1002/jgt.22402 .

[33] L.S. Mel'nikov, The chromatic class and location of a graph on a closed surface (in Russian), *Mat. Zametki* **7** (1970) 671-681/*Math. Notes* **7** (1970) 405-411.

[34] M. Molloy and B. Reed, A Bound on the Strong Chromatic Index of a graph, *J. Combin. Theory Ser. B* **69** (1997) 103-109.

[35] N. Robertson, D. Sanders, P. Seymour, and R. Thomas, The four-color theorem, *J. Combin. Theory Ser. B* **70** 1997 2-44.

[36] D. Sanders and Y. Zhao, Planar graphs of maximum degree seven are class I, *J. Combin. Theory Ser. B* **83** (2001) 201-212.

[37] D. Sanders and Y. Zhao, Coloring edges of graphs embedded in a surface of characteristic zero, *J. Combin. Theory Ser. B* **87** (2003) 254-263.

[38] C.E. Shannon, A theorem on coloring the lines of a network, *J. Math. Phys.* **28** (1949) 148-151.

[39] M. Steibitz, D. Scheide, B. Toft, L. Favrholdt, Graph Edge Coloring, Wiley.

[40] P.G. Tait, Note on a theorem in geometry of position, *Trans. Roy. Soc. Edinburgh* **29** (1880) 657-660.

[41] V.A. Tashnikov, On an algorithm for the edge coloring of multigraphs (in Russian), *Diskretn. Anal. Issled. Oper. Ser. 1* **7** (2000) 72-85.

[42] V.G. Vizing, Critical graphs with a given chromatic class (in Russian), *Diskret. Analiz.* **5** (1965) 9-17.

[43] V.G. Vizing, Some unsolved problems in graph theory, *Uspekhi Mat. Nauk* **23** (1968) 117-134, *Russian Math. Surveys* **23** (1968) 125-142.

[44] Y. Wang, W. Wang and Y. Wang, Edge-partition and star chromatic index, *Appl. Math. Comput.* **333** (2018) 480-489.

[45] Y. Wang, Y. Wang and W. Wang, Star edge-coloring of graphs with maximum degree four, *Appl Math. Comput.* **340** (2019) 268-275.

[46] P. Wernicke, Über den kartographischen Vierfarbensatz, *Math. Ann.* **58**, 413-426.

[47] Z. Yan and Y. Zhao, Edge colorings of embedded graphs, *Graphs and Combinatorics* **16** (2000) 245-256.

[48] L. Zhang, Every planar graph with maximum degree 7 is of class 1, *Graphs and Combinatorics* **16** (2000) 467-495.