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Graph coloring and flows

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Graph Coloring and Flows

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ABSTRACT

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Part 1: The Fulkerson Conjecture states that every cubic bridgeless graph has six perfect matchings such that every edge of the graph is contained in exactly two of these perfect matchings. In this paper, we verify the conjecture for some families of snarks (Goldberg snarks, flower snarks) by using a technical lemma.

Part 2: A star coloring of an undirected graph G is a proper vertex coloring of G such that any path of length 3 in G is not bi-colored. The star chromatic number of a family of graphs $\mathcal{G}$, denoted by $\chi_s(\mathcal{G})$, is the minimum number of colors that are necessary to star color any graph belonging to $\mathcal{G}$. Let $\mathcal{F}_\Delta$ be the family of all graphs with maximum degree at most Δ. It was proved by G. Fertin, A. Raspaud and B. Reed (JGT 2004) that $\chi_s(\mathcal{F}_\Delta) \geq 2\Delta$ where $1 \leq \Delta \leq 3$. In this paper, this result is further generalized for every positive integer Δ. That is, $\chi_s(\mathcal{F}_\Delta) \geq 2\Delta$ for every $\Delta \in Z^+$. It was proved by M. Albertson, G. Chappell, H. Kierstead, A. Kundgen, R. Ramamurthi (EJC 2004) that $\chi_s(\mathcal{F}_\Delta) \leq \Delta(\Delta - 1) + 2$. In this paper, a simplified proof is given and this result is further improved for non Δ-regular graph to $\chi_s(\mathcal{F}_\Delta^{ng}) \leq \Delta(\Delta - 1) + 1$ where $\mathcal{F}_\Delta^{ng}$ is the family of non-regular graphs with maximum degree Δ.

Part 3: There are two famous conjectures about integer flows, the 5-flow conjecture raised by Tutte and the orientable 5-cycle double cover by Archdeacon [1] and Jaeger [15]. It is known that the orientable 5-cycle double cover conjecture implies the 5-flow conjecture. But the converse is not known to hold. In this paper, we try to use the reductions of incomplete integer flows to lead us in a direction to attack the problem.

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DEDICATION

To

my parents Linsen Wang and Qinfang Liu , my wife Zheng Zhang

and

my daughter Jennifer Wang

Chapter 1

Fulkerson Coloring of Some Families of Snarks

1.1 Introduction

Edge-3-coloring of cubic graphs has been extensively studied due to its equivalency to the 4-color problem of planar graphs. However, we notice that not all cubic graphs are edge-3-colorable. The following is one of the most famous conjectures in graph theory.

Conjecture 1.1.1 (Fulkerson, [12]) *Every* 2*-connected cubic graph has a collection of six perfect matchings that together cover every edge exactly twice.*

Although the statement of the conjecture is very simple, the solution has eluded many mathematicians over 40 years and remains beyond the horizon. Due to the lack of appropriate techniques, few partial results have ever been achieved and this subject remains as a piece of virgin land in graph theory. In this paper, we would like to introduce some techniques for this problem and verify the conjectures for some families of cubic graphs.

The problem of matching covering is one of the major subjects in graph theory because of its close relation with the problems of cycle cover, integer flow and other problems. Many generalizations and variations of Fulkerson's conjecture have already received extensive attention, and some partial results have been achieved.

An *r-graph* G is an r-regular graph such that $|(X, V(G)\backslash X)_G| \geq r$, for every non-empty vertex subset $X \subseteq V(G)$ of odd order. It was proved by Edmonds that [6] (also see [22]) that, *for a given r-graph G, there is an integer k (a function of G) such that G has a family of perfect matchings which covers each edge precisely k times.* Motivated by this result, Seymour, [22] further conjectured that *every r-graph has a Fulkerson coloring.*

Note that the complement of a perfect matching in a cubic graph is a 2-factor. Fulkerson's conjecture is equivalent to that *every bridgeless cubic graph has a family of six cycles such that every edge is covered precisely four times.* It was proved by Bermond, Jackson and Jaeger, [2] that *every bridgeless graph has a family of seven cycles such that every edge is covered precisely four times;* and proved by Fan [8] that *every bridgeless graph has a family of ten cycles such that every edge is covered precisely six times.*

The relation between Fulkerson coloring and shortest cycle cover problems have been investigated by Fan and Raspaud [7]. In the paper [7], it was proved that *if Fulkerson Conjecture is true, then every bridgeless graph has a family of cycles that covers all edges and has the total length at most* $\frac{22}{15}|E(G)|$. One should notice that the famous cycle double cover conjecture (Szekeres, Seymour [23, 21]) would be verified if one is able to find a cycle cover of every cubic graph with total length at most $\frac{21}{15}|E(G)|$ (Jamshy and Tarsi [16]).

A non-edge-3-colorable, bridgeless, cyclically 4-edge-connected, cubic graph is called a snark. Tutte [25] raised a structural conjecture about snarks that *very snark must contain a subdivision of the Petersen graph.* For Fulkerson coloring, it is sufficient to verify the conjecture for all snarks. In this paper, we verify the conjecture for the families of Goldberg snarks and flower snarks.

1.2 Notations

Most standard terminology and notation can be found in [3] or [28].

Let G be a cubic graph. The graph $2G$ is obtained from G by duplicating every edge to be a pair of parallel edges.

A *circuit* is a connected 2-regular subgraph. A *cycle* is the union of edge-disjoint circuits. An edge is called a *bridge* if it is not contained in any circuit of the graph.

Let $G = (V, E)$ be a graph. The underlying graph, denote by $\overline{G}$, is the graph obtained from G by suppressing all degree-2-vertices. In this paper, it is possible that some graph G may contain a 2-regular component C, and therefore, $\overline{G}$ has a *vertexless loop*.

A vertexless loop is a special case in this paper that is not usually seen in other literatures. For graphs with vertexless loops, we may further extend some popularly used terminology. For example, the degree of a vertex is defined as the same as usual. Therefore, a graph is cubic if the degree of every vertex is 3 while vertexless loops are allowed. An edge-3-coloring of a cubic graph is a mapping $c : E(G) \mapsto \{1, 2, 3\}$ such that every vertex is incident with edges colored with all three colors. Hence, those vertexless loops may be colored with any color.

1.3 A Technical Lemma

In this section, we provide a useful technical lemma.

Lemma 1.3.1 *A cubic graph G admits a Fulkerson Coloring if and only if each $\overline{G \setminus M_i}$ is edge-3-colorable where M_i is a matching of G for $i = 1, 2$ and $M_1 \cup M_2$ forms a cycle in G.*

Remark. The underlying graph $\overline{G \setminus M_i}$ may contain some trivial components (vertexless loops) for $i = 1, 2$. As we discussed in the previous section that every vertexless loop of $\overline{G \setminus M_i}$ corresponds to a 2-regular component C of $G \setminus M_i$. Since $M_1 \bigcup M_2$ is a cycle, every vertex of C must be incident with edges of both M_1 and M_2. Hence, the edges of the circuit C must be alternatively in M_j and $E(G) \setminus \{M_1 \bigcup M_2\}$ for $j \neq i$. It is easy to see that every 2-regular component of $G \setminus M_i$ must be an even length circuit for $i = 1, 2$.

Proof. We only pay attention to non-edge-3-colorable graphs since every edge-3-colorable cubic graph trivially satisfies the conjecture with $M_1 \bigcup M_2 = \emptyset$. Suppose G admits a Fulkerson Coloring $c : E(2G) \mapsto \{a_1, b_1, c_1, a_2, b_2, c_2\}$. Let G_1 be the subgraph of $2G$ induced by edges colored with $\{a_1, b_1, c_1\}$ and G_2 be the subgraph of $2G$ induced by edges colored with $\{a_2, b_2, c_2\}$. If there is no parallel edge in both G_1 and G_2, then G_1 and G_2 are isomorphic to G. Hence G is edge-3-colorable, so we may assume there are parallel edges in G_1 and G_2 and let E_p be the set of edges of G corresponding to parallel edges in either G_1 or G_2.

Let M_j be the set of edges e of G that e corresponds to a parallel edge of G_j for each $j = 1, 2$ and $E_p = M_1 \cup M_2$. Since G_j is cubic, M_j is a matching, and $M_1 \cap M_2 = \emptyset$. Furthermore, each vertex incident with an edge of M_i must be incident with an edge of M_j. Hence, E_p is a set of edge-disjoint even circuits.

Now, we may color each edge $e \in M_j$ with the color $\{a_j, b_j, c_j\} - \{x_j, y_j\}$ where x_j, y_j are colors used for the parallel edges corresponding to e in G_j. So, the resulting coloring is an edge-3-colroing of the underlying graph $\overline{G_j} = \overline{G \setminus M_i}$ for $\{i, j\} = \{1, 2\}$.

For the sufficiency part, suppose that each nontrivial component of $\overline{G \setminus M_1}$ is edge-3-colorable with colors $\{a_1, b_1, c_1\}$, and $\overline{G \setminus M_2}$ is colored with $\{a_2, b_2, c_2\}$, edges of the trivial components (veertexless loops, if exist) of $\overline{G \setminus M_i}$ are colored by a_i for $i = 1, 2$. A coloring $G \setminus M_i$ is obtained by inserting those suppressed degree-2 vertices for $i = 1, 2$ and coloring each edge incident with degree-2 vertex with the same color of the edge before the vertex insertion.

An edge-6-coloring of $2G$ is obtained as follows. For a pair of parallel edges e_1, e_2 with end vertices u, v in $2G$.

Case 1: If there is a corresponding edge in both $G \setminus M_1$ and $G \setminus M_2$ with end vertices u, v, then e_i is colored the same color of the edge with end vertices u, v in $G \setminus M_i$ for $i = 1, 2$.

Case 2: If there is only one corresponding edge with end vertices u, v, in one of $G \setminus M_1$ and $G \setminus M_2$ (not both), then e_1, e_2 are colored with colors of $\{a_i, b_i, c_i\} \setminus \{x_i\}$ where x_i is the color of the corresponding edge in $G \setminus M_i$.

This completes the proof of the lemma. ■

1.4 Goldberg Snarks

Goldberg [13] constructed an infinite family of snarks, G_3, G_5, $G_7, \cdots$, which can be used to give infinitely many counter-examples to the critical graph conjecture [5]. Small examples G_3 and G_5 are illustrated in Figure 1.1.

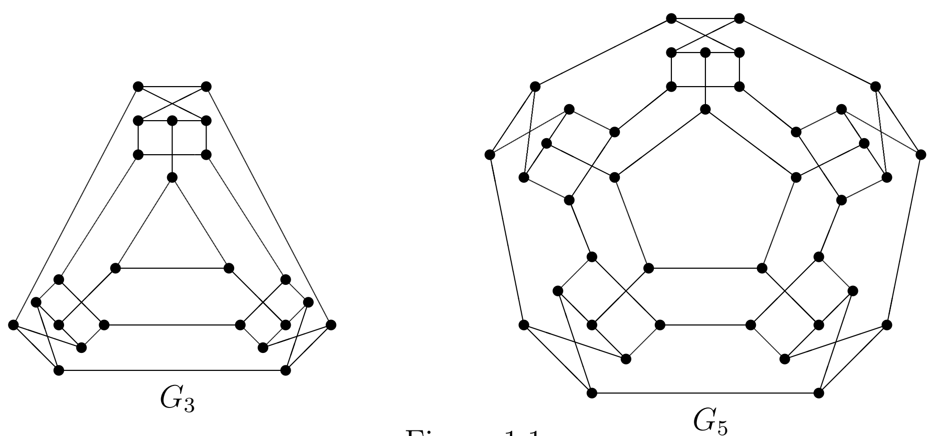


Figure 1.1

For every odd $k \geq 3$, the Goldberg snark G_k can be viewed as: $V(G_k)$=$\{{v_j}^t : 1 \leq t \leq k,\ 1 \leq j \leq 8\}$, the superscript t is under modulo k. The subgraph B_t induced by $\{v_1^t, v_2^t, \cdots, v_8^t\}$ is a basic block. The Goldburg snark is constructed by joining each basic block B_t with B_{t-1} and B_{t+1} (mod(k)). The internal adjacency relation of B_t and the inter-block adjacency relation between B_t, B_{t-1} and B_{t+1} are illustrated in Figure 1 .2.

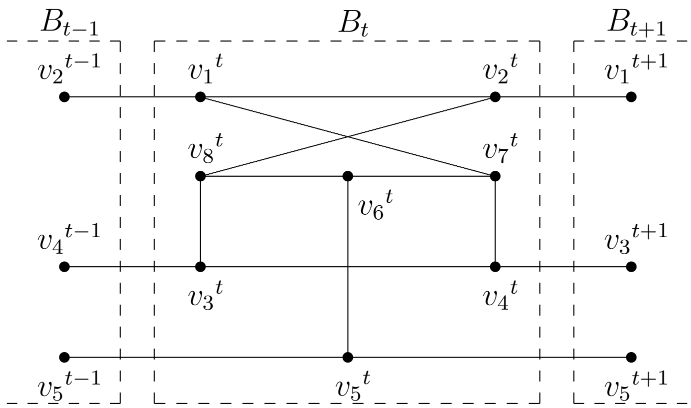


Figure 1.2

Theorem 1.4.1 *The Goldberg snark graph G_k admits a Fulkerson coloring.*

Proof. By Lemma 1.3.1, it is sufficient to show that G_k has a pair of disjoint matching M_1, M_2 such that $M_1 \cup M_2$ is an even subgraph C of G_k, and for $i = 1, 2$, each nontrivial component of $\overline{G_k \setminus M_i}$ is edge-3-colorable.

Let $C = C_1 \cup C_2$, where C_1 is the circuit $v_1^1 v_2^1 v_1^2 v_2^2 \cdots v_1^k v_2^k$ of length $2k$ and C_2 is the circuit $v_3^1 v_4^1 v_3^2 v_8^2 v_6^2 v_7^2 v_4^2 \cdots v_3^k v_8^k v_6^k v_7^k v_4^k$ of length $5k - 3$. Since k is odd, C is an even subgraph of G_k. (See figure 1.3)

Let M_1, M_2 be the two perfect matchings of C as follows:

$M_1 = \{v_2^1 v_1^2, v_2^2 v_1^3, \cdots, v_2^k v_1^1\} \cup \{v_3^1 v_4^1, v_3^2 v_8^2, v_6^2 v_7^2, v_4^2 v_3^3, \cdots, v_8^k v_6^k, v_7^k v_4^k\}$,

$M_2 = \{v_1^1 v_2^1, v_1^2 v_2^2, \cdots, v_1^k v_2^k\} \cup \{v_4^1 v_3^2, v_8^2 v_6^2, v_7^2 v_4^2, \cdots, v_3^k v_8^k, v_6^k v_7^k, v_4^k v_3^1\}$.

Thus $C = M_1 \cup M_2$.

Note that the edges of M_1 and M_2 are selected differently in B_1 or in B_i ($i = 2, 3, \cdots, k$). In order to distinguish the difference, in Figure 3, blocks $B_2, \cdots, B_k$ are lined up in the top row while B_1 is placed in the lower row.

For the suppressed graph $\overline{G_k \setminus M_1}$, there is a Hamilton circuit $v_5^1 v_6^1 v_8^1 v_7^1 v_5^2 v_5^3 \cdots v_5^k$. Thus $\overline{G_k \setminus M_1}$ is edge-3-colorable. (See figure 1.4)

For the suppressed graph $\overline{G_k \setminus M_2}$, it is the union of a cubic component and $\frac{k-1}{2}$ trivial components (vertexless loops). The only cubic component has a Hamilton circuit $v_5^1 v_6^1 v_7^1 v_8^1 v_5^2 v_5^3 \cdots v_5^k$. Thus $\overline{G_k \setminus M_2}$ is also edge-3-colorable. (See figure 1.5)

■

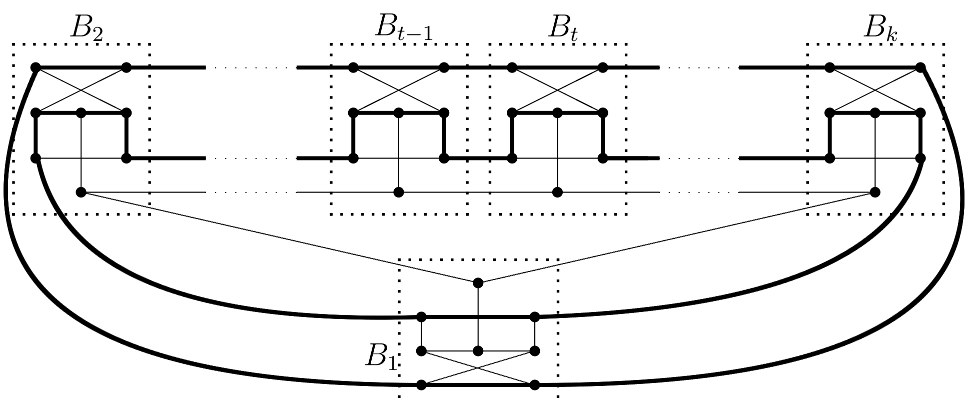


Figure 1.3: $C = M_1 \cup M_2$

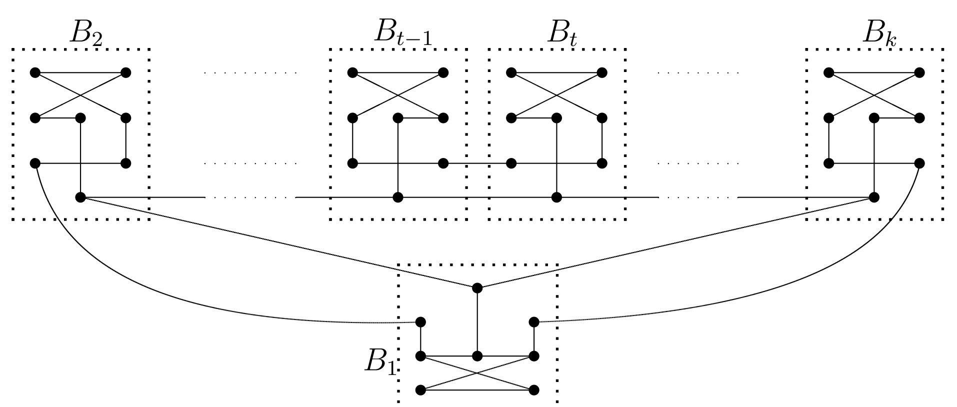


Figure 1.4: $G_k \setminus M_1$

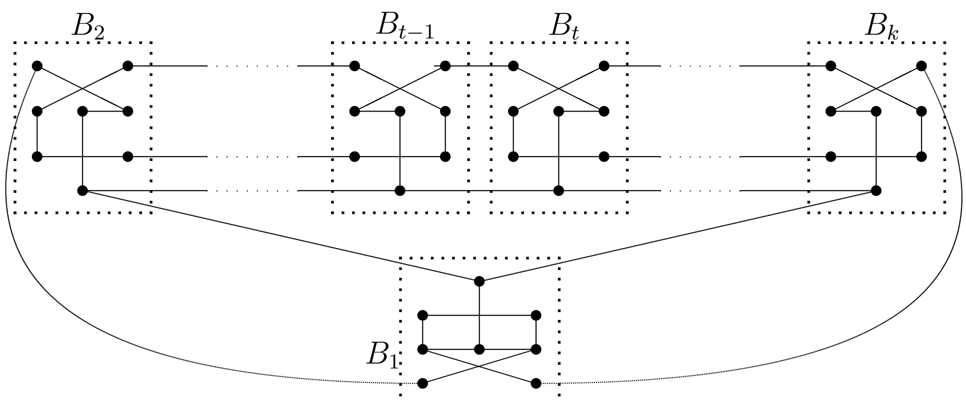


Figure 1.5: $G_k \setminus M_2$

1.5 The Flower Snark

Definition 1.5.1 *For an odd integer $k \geq 3$, the flower snark J_k is constructed as following [14]: the vertex set of J_k consists of $4k$ vertices $v_1, v_2, \cdots, v_k$ and $u_1^1, u_1^2, u_1^3, u_2^1, u_2^2, u_3^3, \cdots, u_k^1, u_k^2, u_k^3$. The graph is comprised of a circuit $u_1^1u_2^1\cdots u_k^1$ of length k and a circuit $u_1^2u_2^2\cdots u_k^2u_1^3u_2^3\cdots u_k^3$ of length $2k$, and in addition, each vertex v_i $(i = 1, 2, \cdots, k)$ is adjacent to u_i^1, u_i^2 and u_i^3. (See Figure 1.6)*

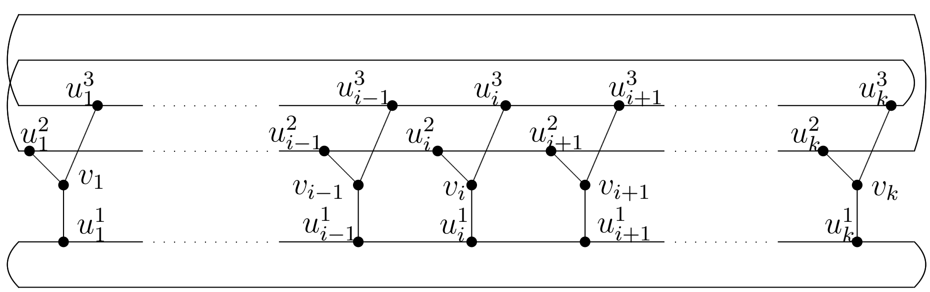
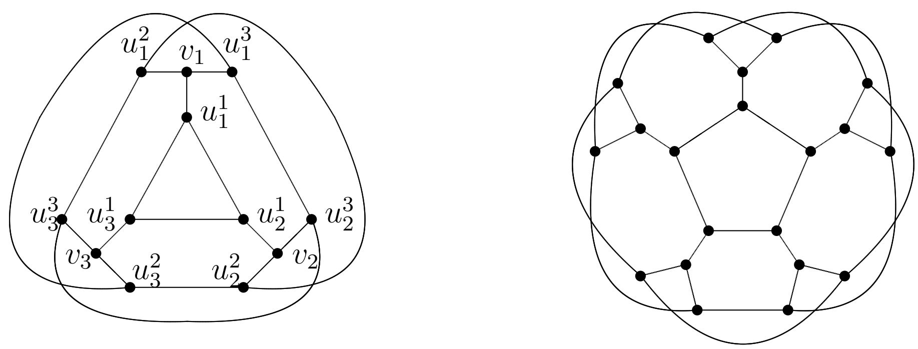


Figure 1.6

The first flower snark J_3 can be obtained from the Peterson graph, with the center vertex replaced by a triangle. In Figure 1.7, we illustrate the first two flower snarks J_3

and J_5.



J_3 J_5

Figure 1.7

Theorem 1.5.2 *The flower snark graph J_k admits a Fulkerson coloring.*

Proof.

By the definition of flower snark, for the odd number k, the vertex set of J_k consists of $4k$ vertices v_1, v_2, $\cdots$, v_k and u_1^1, u_1^2, u_1^3, u_2^1, u_2^2, u_3^3, $\cdots$, u_k^1, u_k^2, u_k^3. The graph is comprised of a circuit $C' = u_1^1 u_2^1 \cdots u_k^1$ of length k and a circuit $C'' = u_1^2 u_2^2 \cdots u_k^2 u_1^3 u_2^3 \cdots u_k^3$ of length $2k$, and in addition, each vertex v_i $(i = 1, 2, \cdots, k)$ is adjacent to u_i^1, u_i^2 and u_i^3.

By Lemma 1.3.1, it is sufficient to show that J_k has a pair of disjoint matching M_1, M_2 such that $M_1 \cup M_2$ is an even subgraph of J_k, and for $i = 1, 2$, $\overline{J_k \setminus M_i}$ contains a Hamilton circuit for $j = 1, 2$.

Let C be the circuit $u_1^2 u_2^2 \cdots u_k^2 u_1^3 u_2^3 \cdots u_k^3$ of length $2k$. Let M_1, M_2 be the two perfect matchings of C as follows: $M_1 = \{u_1^2 u_2^2, u_3^2 u_4^2, \cdots, u_k^2 u_1^3, u_2^3 u_3^3, u_4^3 u_5^3 \cdots, u_{k-1}^3 u_k^3\}$, $M_2 = \{u_2^2 u_3^2, u_4^2 u_5^2, \cdots, u_{k-1}^2 u_k^2, u_1^3 u_2^3, u_3^3 u_4^3, \cdots, u_{k-2}^3 u_{k-1}^3, u_k^3 u_1^2\}$. Thus $C = M_1 \cup M_2$.

Let H be the subgraph of J_k induced by $\{v_i, u_i^2, u_i^3 | i = 1, 2, \cdots, k\}$ (Figure 8.a). Here, $\overline{H}$ has a Hamilton circuit $C = u_1^2 u_2^2 \cdots u_k^2 u_1^3 u_2^3 \cdots u_k^3$ and chords $\{u_i^2 v_i u_i^3 | i = 1, 2, \cdots, k\}$ (where v_i is a degree 2 vertex in H). (See Figure 1.8.a)

M_1, M_2 are perfect matching decomposition of the circuit C. Since k is odd, $H \setminus M_j$ is a circuit in which $v_1, v_2, \cdots, v_k$ are in this order for $j = 1, 2$. Hence, $G \setminus M_j$ is constructed by joining circuit $H \setminus M_j$ and $C' = u_1^1 u_2^1 \cdots u_k^1$ by edges $\{v_i u_i^1 | i = 1, 2, \cdots, k\}$. It is easy to see that $\overline{G \setminus M_j}$ is a planar prism, and therefore contains a Hamilton circuit and furthermore is edge-3-colorable. (See Figure 1.8.b) ■

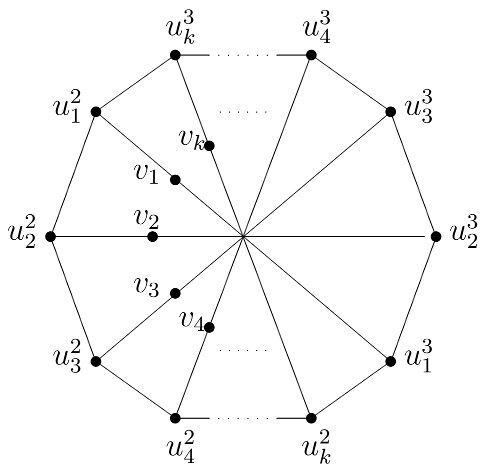


Figure 1.8.a: $\overline{H}$

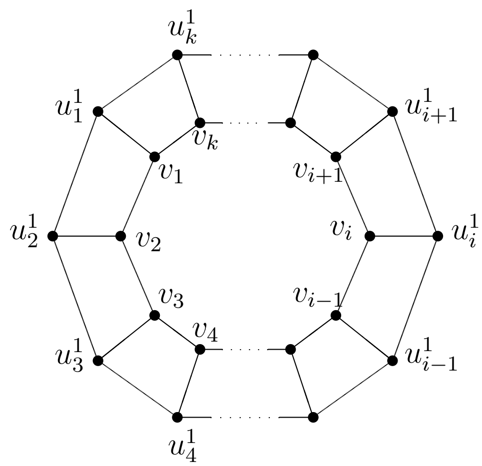
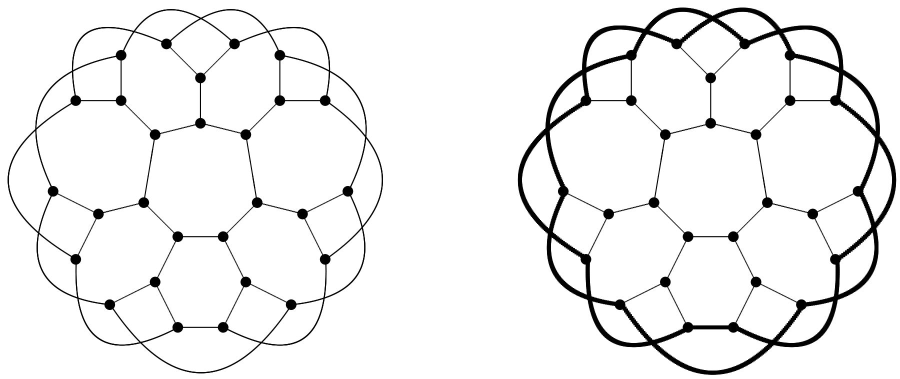


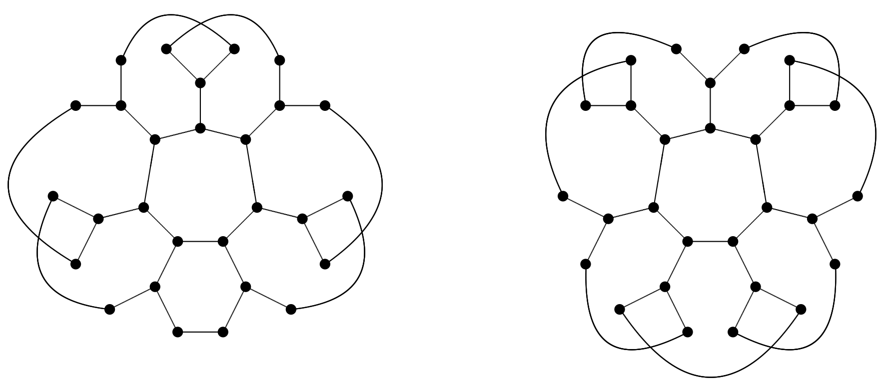
Figure 1.8.b: $\overline{G \setminus M_j}$

In Figures 1.9 and 1.10, a different drawing of flower snarks (J_7) is illustrated. Together with matchings M_1 and M_2, this traditional drawing may help some readers in a different view for the structure of these matchings, and Hamilton circuits in the proof of Theorem 1.5.2.



J_7 $\qquad$ $C = M_1 \cup M_2$

Figure 1.9



$J_7 \setminus M_1$ $\qquad$ $J_7 \setminus M_2$

Figure 1.10

Chapter 2

Star Coloring of Graphs Related to Maximum Degree

2.1 Introduction

All graphs considered here are undirected. In this paper, the term *coloring* refers to *vertex coloring* of graphs. A proper coloring of a graph G is a labelling of the vertices of G such that no two neighbors in G are assigned the same label.

In 1973, Grünbaum [10] considered proper colorings with the additional constraint that the subgraph induced by every pair of color classes is acyclic (contains no cycles). He called such colorings acyclic colorings. Among other problems, he suggested requiring that the union of any two color classes induce a star forest, i.e., a proper coloring avoiding 2-colored paths with four vertices. We call such a coloring a star coloring. Star colorings have recently been investigated by G. Fertin, A. Raspaud, B. Reed [11], and Nešetřil and P. Ossona de Mendez [18].

Definition 2.1.1 *(Star Coloring). A star coloring of a graph G is a proper coloring of G such that no path of length 3 in G is bi-colored.*

We define the star chromatic number of a graph G, denoted by $\chi_s G$, is the minimum number of colors that are necessary to star color G. By extension, the star chromatic number of a family of graphs $\mathcal{G}$, denoted by $\chi_s(\mathcal{G})$, is the minimum number of colors that are necessary to star color any graph belonging to $\mathcal{G}$.

Let $\mathcal{F}_\Delta$ denote the family of graphs with maximum degree Δ. For each $\Delta = 1, 2, 3$, it was proved by G. Fertin, A. Raspaud and B. Reed [9] that $\chi_s(\mathcal{F}_\Delta) \geq 2\Delta$. Here we further generalize this result to $\mathcal{F}_\Delta$ for every $\Delta \in Z^+$.

In [20], it is proved by M. Albertson, G. Chappell, H. Kierstead, A. Kundgen, R. Ramamurthi that $\chi_s(\mathcal{F}_\Delta) \leq \Delta(\Delta-1)+2$. In this paper, we provide another approach to the upper bound and obtained some improvement to this bound for non-regular graphs. Let $\mathcal{F}_\Delta^{ng}$ be the family of non-regular graphs with maximum degree Δ. Together with the lower bound of $\chi_s(\mathcal{F}_\Delta)$, we have the following main theorem.

Theorem 2.1.2 *Let $\mathcal{F}_\Delta$ denote the family of graphs with maximum degree Δ. We have that*

$$2\Delta \leq \chi_s(\mathcal{F}_\Delta) \leq \Delta(\Delta-1)+2;$$

and

$$\chi_s(\mathcal{F}_\Delta^{ng}) \leq \Delta(\Delta-1)+1$$

2.2 Lower Bounds of $\chi_s(\mathcal{F}_\Delta)$

2.2.1 Construction of a family of graphs

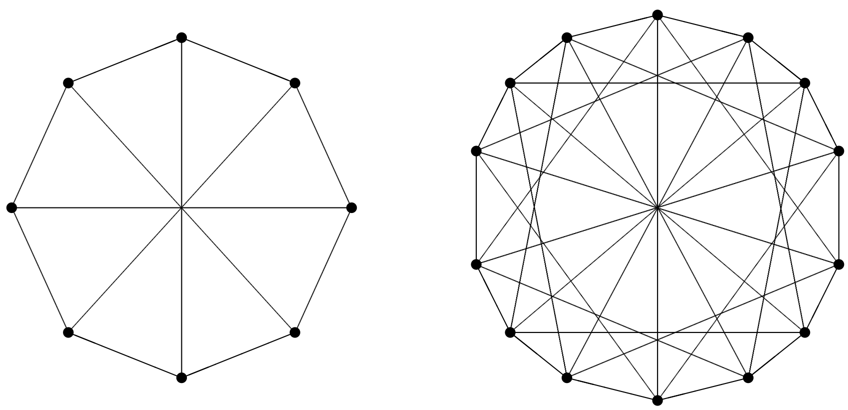
In order to prove that $\chi_s(\mathcal{F}_\Delta) \geq 2\Delta$, a family of graphs are to be constructed to meet the bound.

Definition 2.2.1 *Suppose that N is a group and S is a generating set. The Cayley graph $\Gamma = \Gamma(N, S)$ is a graph constructed as follows.*

Each element n of N is assigned a vertex (that is the vertex set $V(\Gamma)$ of Γ is identified with N). For any $n \in N, s \in S$, the vertices corresponding to the elements n and ns are joined by an edge. Thus the edge set $E(\Gamma)$ consists of pairs of the form (n, ns), where $n \in N$ and $s \in S$.

We construct the graph with maximum degree Δ, denoted by G_Δ, in the following ways: Let $N = Z_{3\Delta-1}$ be the finite cyclic group of order $3\Delta - 1$ and the generating set S (in the additive notation) consists of Δ elements: $\{1, 4, 7, \cdots, 3t + 1, \cdots, 3(\Delta - 1) + 1\}$.

It is easy to see by the construction that G_Δ is a Δ regular graph, and we illustrate G_3, G_5 in Figure 2.1.



G_3 G_5

Figure 2.1

Here are some structural lemma about G_Δ: Let $\{v_0, v_1, \cdots, v_{3\Delta-2}\}$ denote the vertices of G_Δ around the outer circuit (the Hamilton circuit of G_k) in this order.

Lemma 2.2.2 *Let G' be a graph obtained by deletion of three consecutive vertices v_i, v_{i+1}, v_{i+2} for $i = 0, \cdots, \Delta - 1$ from G_Δ, then $G' = G_{\Delta-1}$.*

Proof. By the construction of G_Δ, the union of the neighbors of v_i, v_{i+1}, v_{i+2} is the entire vertex set of G_Δ. So after deleting v_i, v_{i+1}, v_{i+2}, the degree of every other vertex is decreased by 1 which resulted a $\Delta - 1$ regular graph. One can easily check the edge connection satisfies the construction conditions. ■

The following lemma is proved in [9].

Lemma 2.2.3 $\chi_s(G_k) = 2k$ *for* $k = 1, 2, 3$.

2.2.2 Proof of the lower bound

In this section we show that $\chi_s(G_\Delta) = 2\Delta$ and thus the lower bound for the star chromatic number of the family of graphs with maximum degree Δ should be grater than or equal to 2Δ.

We say a vertex v is **uniquely colored** if graph G has a star coloring such that the color assigned to v is not assigned to any other vertices of G.

First we want to show that $\chi_s(G_\Delta) \geq 2\Delta$. The proof is by the way of contradiction. Suppose $\Delta = k$ is the smallest number such that $\chi_s(G_k) \leq 2k - 1$ with $|V(G_k)| = 3k - 1$.

Claim 2.2.4 $\chi_s(G_k) \geq 2k - 2$

By Lemma 2.2.2, deleting any three consecutive vertices of G_k will result a G_{k-1}, and by the minimality of k we have $\chi_s(G_{k-1}) \geq 2k - 2$. Thus we need at least $2k - 2$ colors to star color G_k.

Claim 2.2.5 $\chi_s(G_k) \geq 2k - 1$

If $\chi_s(G_k) = 2k-2$, by Lemma 2.2.2, after deleting any three consecutive vertices of G_k, we need at least $2k-2$ colors to star color the resulting graph G_{k-1}, thus the vertices we deleted are not uniquely colored in G_k. Also the three consecutive vertices are arbitrarily chosen, which means all the vertices in G_k are not uniquely colored (All the colors are used at least twice in G_k) . Thus the number of colors to star color G_k is at most $\lfloor \frac{3k-1}{2} \rfloor$, which is smaller than $2k-2$ except $k = 1, 2, 3$ and together with Lemma 2.2.3, we have a contradiction to $\chi_s(G_k) = 2k-2$.

So we have $\chi_s(G_k) = 2k-1$. Let $c_1, c_2, \cdots, c_{2k-1}$ denote $2k-1$ colors of the star coloring of G_k, and let $|c_i|$ be the number of vertices colored by color c_i.

Claim 2.2.6 *$|c_i| = 1$ for $i = 1, \cdots, k-1$, $|c_i| = 2$ for $i = k, \cdots, 2k-1$.*

Let t denote the number of colors such that $|c_i| = 1$.

Firstly, $t \geq k-1$. Otherwise if $t \leq k-2$, then every one of the remaining colors is used at least twice in the star coloring of G_k which should color at least $2(2k-1-t)$ vertices. And together with the t uniquely colored vertices, we have at least $4k-2-t \geq 3k$ vertices which contradict to G_k has $3k-1$ vertices.

Secondly, $t \leq k-1$. Otherwise if $t \geq k$, then we consider about the outer circuit of G_k. If any two uniquely colored vertices v_1, v_2 are with distance 2 on the outer circuit, we can delete the 2-path with end vertices v_1, v_2 and by Lemma 2.2.2, we have a G_{k-1} star colored by $2k-3$ colors which contradicts the minimality of k. Thus, any two uniquely colored vertices must be with distance at least 3. So it requires at least $3k$ vertices of G_k if $t \geq k$, hence a contradiction.

Then we have $|c_i| = 1$ for $i = 1, \cdots, k-1$, and consider about the remaining k colors which are used at least twice to color the remaining $2k$ vertices of G_k, it is easy to get $|c_i| = 2$ for $i = k, \cdots, 2k-1$.

Claim 2.2.7 *Let v_{i-1}, v_i, $v_{i+1} \in V(G_k)$, if v_i is a uniquely colored vertex then $c(v_{i-1}) \neq c(v_{i+1})$.*

Otherwise, by Lemma 2.2.2, $G_{k-1} = G_k - \{v_{i-1}v_iv_{i+1}\}$ is $2k-1$ colored, which contradicts that $\chi_s(G_{k-1}) = 2k-2$.

In summary, we have $2k-1$ colors to star color G_k, in which $k-1$ colors are used once and the rest k colors each is used exactly twice. Also by the analysis in Claim 2.2.6, we notice that any two uniquely colored vertices should be with distance at least 3 which requires at least $3k-3$ vertices, consider the total number of vertices is $3k-1$ which leaves us only three possible ways to assign the colors along the outer circuit of G_k which is illustrated in Figure 2.

In Figure 2.2, a solid circle indicates a uniquely colored vertex and all other vertices are assigned with colors which are used twice in the star coloring of G_k.

In Type I color assignment, the distance between the uniquely colored vertices v_0 and v_4 (also v_4 and v_8) is 4. In Type II color assignment, the distance between the uniquely colored vertices v_3 and v_8 is 5. In Type III color assignment, the distance between the uniquely colored vertices v_0 and v_4 (also v_s and v_{s+4} where $4 < s < 3k-5$) is 4. (When $s = 4, 3k-5$, it is Type I color assignment.)

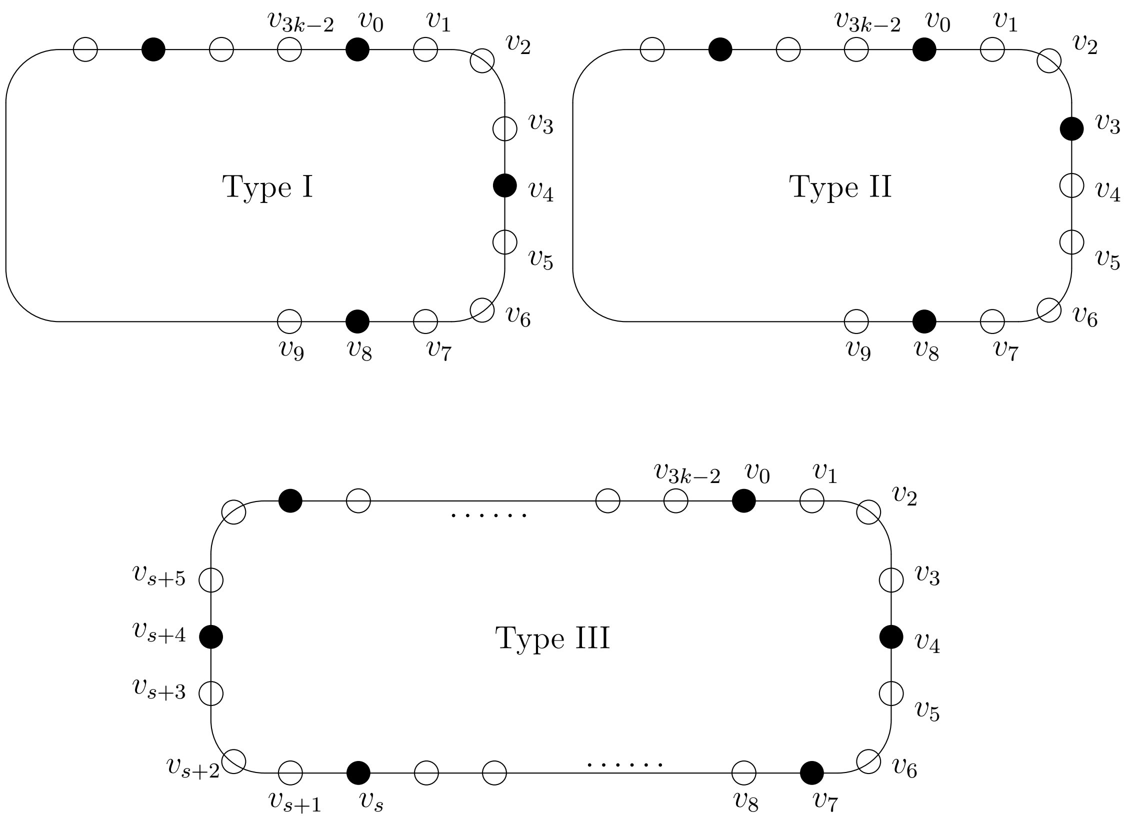


Figure 2.2

Now, we partition all the vertices into $k-1$ parts:

$\{v_1, v_2, v_3, v_4, v_5\}, \{v_6, v_7, v_8\}, \cdots, \{v_{s-1}, v_s, v_{s+1}\}, \cdots, \{v_{3k-3}, v_{3k-2}, v_0\}$. Notice that except the first part contains 5 vertices, every other part contains 3 vertices each and among each part there is only one uniquely colored vertex.

Case 1: Type I color assignment.

By the structure of G_k, there is an edge connecting v_2 and v_6, suppose v_2 is colored by color c_1 and v_6 is colored by c_2. Since v_6 is not a uniquely colored vertex, there must be another vertex colored by c_2, it may lie in the part of 5 vertices or any other 3 vertices part.

Subcase 1.1

Assume the other c_2 colored vertex lies in a 3 vertex part $\{v_{p-1}, v_p, v_{p+1}\}$. In this part, v_{p+1} is a uniquely colored vertex, and v_p is adjacent to v_6 by the structure of G_k, thus v_{p-1} is colored by c_2. And also by the structure of G_k, v_{p-1} is adjacent to v_2 as illustrated in Figure 2.3.

Since v_2 is not a uniquely colored vertex, there must be another vertex colored c_1. If the other vertex colored c_1 lies in the 5 vertices part, it must be v_5, but it will result a bi-colored 4-path $\{v_5v_6v_2v_{p-1}\}$. Thus the other vertex colored c_1 must be in a 3 vertices part $\{v_{p'-1}, v_{p'}, v_{p'+1}\}$. In this part, $v_{p'+1}$ is a uniquely colored vertex, and $v_{p'-1}$ is adjacent to v_2 by the structure of G_k, thus $v_{p'}$ is colored by c_1. But this will give us a bi-colored 4-path $\{v_{p-1}v_2v_6v_{p'}\}$ since v_6 is adjacent to $v_{p'}$ by the structure of G_k.

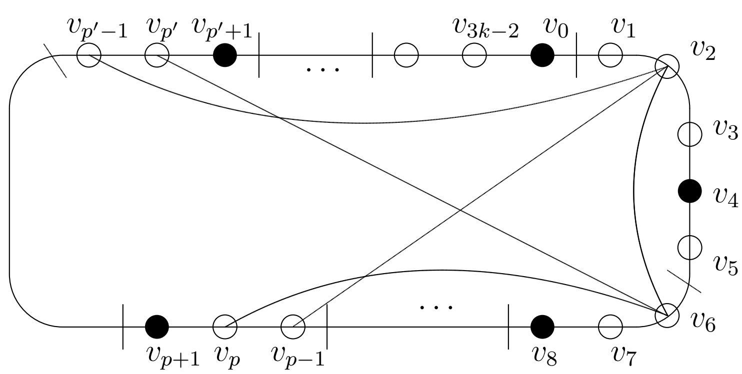


Figure 2.3

Subcase 1.2

Thus, the other c_2 colored vertex should lie in the 5 vertices part which could be v_1 or v_3. Then we can use the same argument to find a bi-colored 4-path which is a contradiction to the star coloring of G_k.

Case 2: Type II color assignment.

Following Case 1, let v_2 be colored c_1, v_6 be colored c_2. By the same argument of Case 1, the following subcases are clearly done:

(1) For each $i = 1, 2$, the other c_i colored vertex lies in a 3 vertex part.

(2) v_5 is colored c_1.

There are only two subcases left:

I. If the other c_2 colored vertex lies in a 3 vertices part and the other c_1 colored vertex lies in the 5 vertices part. The other c_1 colored vertex may be v_4 since in Type II color assignment v_3 is the uniquely colored vertex. But by Claim 2.2.7, the color of v_4 is not c_1, hence a contradiction.

II. If the other c_2 colored vertex lies in the 5 vertices part, except the cases discussed in subcase 1.2, it could be v_4. Then v_5 must be colored by c_3 which is different as c_2, otherwise $\{v_2v_6v_5v_4\}$ is a bi-colored 4-path. And again, v_5 is not a uniquely colored vertex, there must be another vertex colored c_3. The other c_3 colored vertex can not be in the 5 vertices part since v_1 is adjacent to v_5. Thus we can find the other c_3 colored vertex $v_{p''}$ and result in a bi-colored 4-path $\{v_{p''}v_6v_5v_4\}$ as illustrated in Figure 2.4.

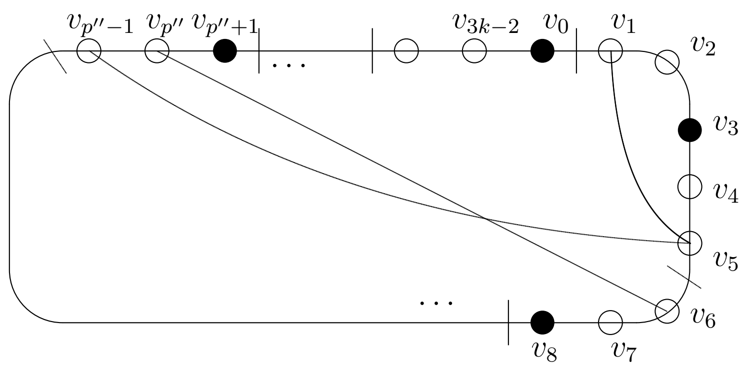


Figure 2.4

Case 3: Type III color assignment.

In Type III color assignment (See Figure 2), there are 3 vertices between uniquely colored vertices v_0 and v_4, and 3 vertices between uniquely colored vertices v_s and v_{s+4} where $4 < s < 3k - 5$. (When $s = 4, 3k - 5$, it is Type I color assignment.)

There is an edge connecting v_2 and v_{s+2}, both v_2 and v_{s+2} are not uniquely colored vertices. Suppose v_2 is colored by color c_4, and v_{s+2} is colored by color c_5. Since v_{s+2} is not a uniquely colored vertex, there must be another vertex colored by color c_5. The vertex could be in the 5 vertices part or a 3 vertices part.

Subcase 3.1 If the other c_5 colored vertex v_t is in a 3 vertices part where $t \leq s$.

Let v_t be in the 3 vertices part $\{v_{q-1}, v_q, v_{q+1}\}$, where v_q is a uniquely colored vertex and v_{p+1} is adjacent to v_{s+2}. Thus v_{q-1} is colored by color c_5. Note that v_2 is not a uniquely colored vertex. So there is another vertex which is colored by color c_4. There are three possible locations for the other c_4 colored vertex: v_5, $v_{q'+1}$ where $q' \leq s$ and $v_{q''}$ where $q'' \geq s$. But all these will result a bi-colored 4-path in G_k as illustrated in Figure 2.5, hence a contradiction to the star coloring.

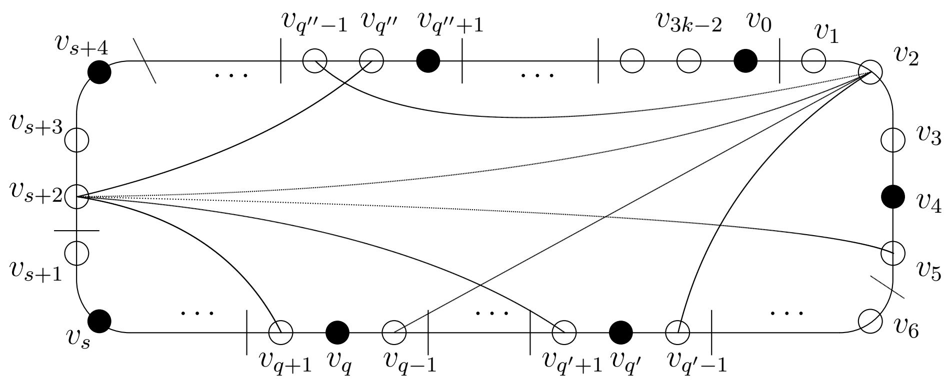


Figure 2.5

Subcase 3.2 If the other c_5 colored vertex v_t is in a 3 vertices part where $t \geq s$. We can use similar argument in subcase 3.1 to reach the contradiction. We omit the details and refer to the Figure 2.6.

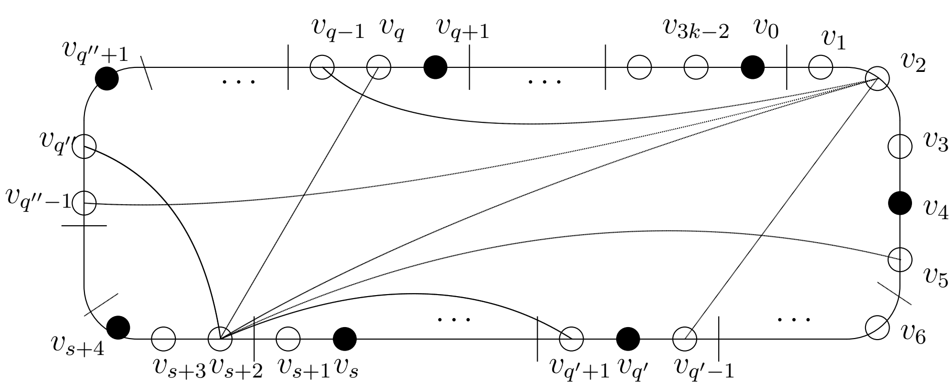


Figure 2.6

Subcase 3.3 If the other c_5 colored vertex v_t is in the 5 vertices part. Then the possible locations for the vertex are v_1 and v_3 (notice that v_5 is adjacent to v_{s+2}). We can use similar argument in subcase 3.1 to reach the contradiction. We omit the details and refer the readers to the Figure 2.7.

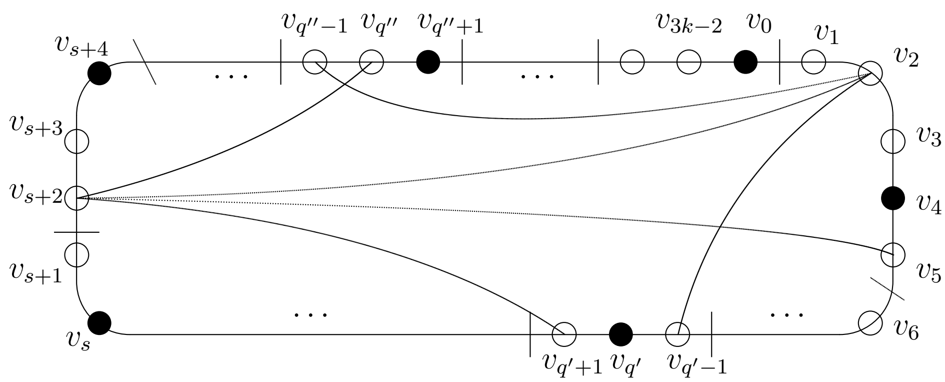


Figure 2.7

Thus, we reach the contradiction of $\chi_s(G_k) \leq 2k-1$, hence $\chi_s(G_\Delta) \geq 2\Delta$. Next we want to show $\chi_s(G_\Delta) = 2\Delta$. We proceed the proof by induction.

By Lemma 2.2.3, when $\Delta = 1, 2, 3$, the equation holds. Assume the equation holds for all $\Delta < m$. When $\Delta = m$, by Lemma 2.2.2, delete any consecutive 3 vertices $\{v_j, v_{j+1}, v_{j+2}\}$ we will have G_{m-1} which can be star colored by $2m-2$ colors. Then we can color the vertices v_j, v_{j+2} by a new color and the vertex v_{j+1} by another new color, together with the coloring of G_{m-1}, we get the star coloring of G_m using $2m$ colors.

2.3 Proof of the upper bound

In [20], it is proved that $\chi_s(G) \leq \Delta(\Delta-1)+2$ using the result from acyclic orientation. Here in this section we give another approach of the upper bound which may have further improvement of the upper bound.

Theorem 2.3.1 *For graphs with maximum degree Δ,*

$$\chi_S(G) \leq \Delta(\Delta-1)+2, \tag{2.1}$$

and furthermore,

$$\chi_S(G) \leq \Delta(\Delta-1)+1, \tag{2.2}$$

if G is not Δ-regular.

Proof. Since G is connected, the vertex set of G can be listed as a sequence $\{x_1, \cdots, x_n\}$ such that $N(x_i) \cap \{x_{i+1}, \cdots, x_n\} \neq \emptyset$ for every $i \in \{1, \cdots, n-1\}$, and $d(x_n) = \delta$ (the minimum degree of G).

I. We claim that we only need at most $\Delta(\Delta-1)+1$ colors to color every proper subsequence $x_1, \cdots, x_\mu$ for each $\mu < n$.

For each $\mu \in \{1, \cdots, n-1\}$, assume that $\{x_1, \cdots, x_{\mu-1}\}$ is already colored. Let H be the subgraph of G induced by the vertices $\{x_1, \cdots, x_\mu\}$. Note that

$$|N_H(x_\mu) \cap \{x_1, \cdots, x_{\mu-1}\}| \leq d(x_\mu) - 1 \tag{2.3}$$

and

$$|N_H^2(x_\mu) \cap \{x_1, \cdots, x_{\mu-1}\}| \leq (\Delta-1)(d(x_\mu)-1). \tag{2.4}$$

By inequalities 2.3 and 2.4, in the subsequence $\{x_1, \cdots, x_{\mu-1}\}$, which was already colored, $N_H(x_\mu) \cup N_H^2(x_\mu)$ uses at most $\Delta(d(x_\mu)-1)$ colors. Hence, x_μ can be colored with a color not used in $\{x_1, \cdots, x_{\mu-1}\} \cap [N_H(x_\mu) \cup N_H^2(x_\mu)]$.

II. By I. $\{x_1, \cdots, x_{n-1}\}$ uses at most $\Delta(\Delta-1)+1$ colors. The last vertex x_n of the sequence can be colored with a new color if we have to.

Assume that $d(x_n) < \Delta$. By the same argument as I., in the subsequence $\{x_1, \cdots, x_{n-1}\}$, which is already colored, $N(x_n) \cup N^2(x_n)$ uses at most $d(x_n)(\Delta-1) \leq (\Delta-1)^2$ colors. Hence, $\Delta(\Delta-1)+1$ colors is enough for the entire graph. ■

Thus, Theorem 2.1.2 is a quick result from Theorem 2.3.1 and the analysis in Section 2.

Chapter 3

Incomplete positive integer flows – missing patterns and flow reductions

3.1 Introduction

3.1.1 Integer Flows

The concept of integer flow was introduced by Tutte as a refinement and a generalization of the face coloring problem of planar graphs. The following are some definition about basic integer flow concepts.

Definition 3.1.1 *Let G be a graph and D be an orientation of G. For a vertex $v \in V(G)$, let $E^+(v)$(or $E^-(v)$) be the set of all arcs of $D(G)$ with their tails (or heads, respectively) at the vertex v. Let f be weights on the arcs.*

Definition 3.1.2 *A flow of a graph G is an ordered pair (D, f) such that $\sum_{e\in E^+(v)} f(e) = \sum_{e\in E^-(v)} f(e)$ every vertex $v \in V(G)$.*

Definition 3.1.3 *A k-flow is a flow (D, f) such that $f(e)$ is an integer and $|f(e)| < k$*

for each $e \in E(G)$. A k-flow is nowhere-zero if the weight of every edge is not zero.

The following are the most famous conjectures in the theory of integer flows proposed by Tutte.

Conjecture 3.1.4 (5-flow conjecture, [26]) *Every bridgeless graph admits a nowhere-zero 5-flow.*

Conjecture 3.1.5 (4-flow conjecture, [27]) *Every bridgeless graph containing no subdivision of the Petersen graph admits a nowhere-zero 4-flow.*

3.1.2 Sum of Flows

The following is a theorem about representing a positive k-flow as a sum of $k-1$ positive 2-flows raised by Little, Younger and Tutte.

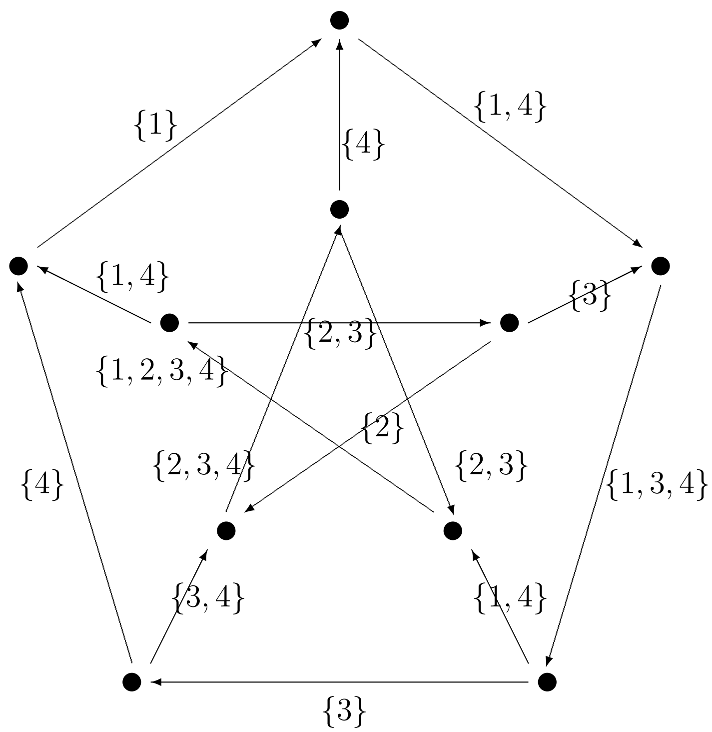
Theorem 3.1.6 (Little, Younger and Tutte, [19]) *For each nonnegative k-flow (D, f) of a graph G, G has $k-1$ nonnegative 2-flows (D, f_μ) $(\mu = 1, \cdots, k-1)$ such that $f = \sum_{\mu=1}^{k-1} f_\mu$.*

Due to this theorem, for a positive 5-flow (D, f) of a graph G, we can write $f = \sum_{i=1}^{4} f_i$ where each (D, f_i) is a non-negative 2-flow.

3.2 Incomplete 5-flow, missing patterns of a positive 5-flow

By the coverage of these 4 non-negative 2-flows, each edge of G can be classified as 15 different *types*: each type is a non-empty element of the power set $\mathcal{P}(\{1, 2, 3, 4\})$.

One may have an immediate impression that the edges of the Petersen graph P_{10} *must have all* 15 *different types.* Actually, it is *incorrect*. The following figure shows a nowhere-zero 5-flow (D, f) of P_{10} where there are only 13 types of edges - *some types do not appear.*



Definition 3.2.1 *Let* (D, f) *be a positive* 5*-flow of a graph* G *and let* $f = \sum_{\mu=1}^{4} f_\mu$ *be a* LYT *decomposition of the* 5*-flow (that is, each* (D, f_μ) *is a non-negative* 2*-flow). The subset* S *of* $\mathcal{P}(1, 2, 3, 4\})$ *consists of all missing types of edges of* (D, f) *(with respect to the decomposition* $f = \sum_{\mu=1}^{4} f_\mu$*) is called the* missing pattern of G with respect to $(D, f = \sum_{\mu=1}^{4} f_\mu)$.

Definition 3.2.2 *A positive* 5*-flow* (D, f) *of a graph* G *is* incomplete *if* (D, f) *has a* LYT *decomposition* $f = \sum_{\mu=1}^{4} f_\mu$ *such that the missing pattern is not empty.*

The study of incomplete 5-flow and its missing patterns was originally initiated by Tarsi.

Theorem 3.2.3 (Jamshy, Raspaud and Tarsi, [17]) *If a graph G admits a nowhere-zero 5-flow, then it must admit a positive 5-flow which is incomplete.*

Let $S \in \mathcal{P}(\{1,2,3,4\})$. Let

$$E_S = \{e \in E(G) : e \in \bigcap_{i \in S} supp(f_i) - \bigcup_{i \notin S} supp(f_i)\}.$$

Denote $\mathcal{M}(G, f = f_1 + f_2 + f_3 + f_4) = \{S \in \mathcal{P}(\{1,2,3,4\}) : \quad E_S = \emptyset\}$. to be the missing patterns of a positive 5-flow decomposition of graph G.

Definition 3.2.4 *For each $S \in \mathcal{P}(\{1,2,3,4\})$, let $\mathcal{G}(S)$ be the family of all graphs G such that G admits some positive 5-flow (D, f) such that a* superset *of S is the missing pattern of (D, f) with respect to some* LYT *decomposition $(D, f = \sum_{\mu=1}^{4} f_\mu)$.*

It is trivial that

$$S_1 \supseteq S_2 \ \ iff \ \ \mathcal{G}(S_1) \subseteq \mathcal{G}(S_2).$$

With this observation, we have a poset $\mathcal{Q}$ in which each node is a family of graphs with a given missing pattern.

One of the major goals of this paper to further study is:
For the set relation between $\mathcal{G}(S_1)$ and $\mathcal{G}(S_2)$ for a pair of distinct subsets $S_1, S_2 \in \mathcal{P}(\{1,2,3,4\})$, which one of the following is true:
1. One is a subset of another.
2. They are the same set.
3. Neither.

Let $\mathcal{F}_\mu$ be the family of all graphs admitting a nowhere-zero μ-flow.

In this paper, we mainly interested in the interval $\mathcal{Q}[4,5]$ of $\mathcal{Q}$ between $\mathcal{F}_4$ and $\mathcal{F}_5$.

It is obvious that the number of all possible missing patterns of all 5-flow graphs G with respect to a 5-flow decomposition is $\frac{2^{15}}{4!}$. Do we really have this amount of nodes in the poset $\mathcal{Q}[4,5]$? Of course not, some trivial cases ($\mathcal{F}_4$-graphs) can be easily identified, for example,

$$\mathcal{F}_4 = \mathcal{G}(\{1\}) = \mathcal{G}(\{2\}) = \mathcal{G}(\{3\}) = \mathcal{G}(\{4\}) = \mathcal{G}(\{1,2,3,4\}).$$

Problem 3.2.5 *What is the structure of the poset $\mathcal{Q}[4,5]$? How many distinct nodes are there in $\mathcal{Q}[4,5]$?*

Is it possible that the poset $\mathcal{Q}[4,5]$ has only two nodes: $\mathcal{F}_4$ and $\mathcal{F}_5$? Or it has all possible $\frac{2^{10}}{4!}$ nodes? The answers are "no" for both questions.

Many $\mathcal{G}(S)$'s are identical although the subsets S are different. For example, Theorem 3.2.3 already shows that $\mathcal{F}_5$ is identical to some $\mathcal{G}(S)$ for some non-empty $S \in \mathcal{P}(\{1,2,3,4\})$. And Theorem 3.3.1 lists all $S \in \mathcal{P}(\{1,2,3,4\})$ that $\mathcal{G}(S) = \mathcal{F}_4$. With the study of the Petersen graph, we will further to show that there *must be some nodes between $\mathcal{F}_5$ and $\mathcal{F}_4$, but is neither of them.*

Based on the pervious definitions, we can construct a poset using the symbols in *Figure* 3.1. If the pattern E_{ab} is missing, then draw the corresponding line, if the pattern E_{abc} is missing, then draw the corresponding dot. *Figure* 3.2 shows the family of graphs with a positive 5-flow decomposition missing E_{12}, E_{24}, E_{124}, E_{123}.

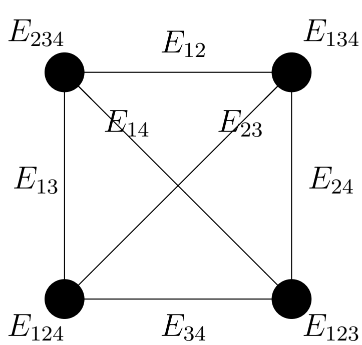


Figure 3.1

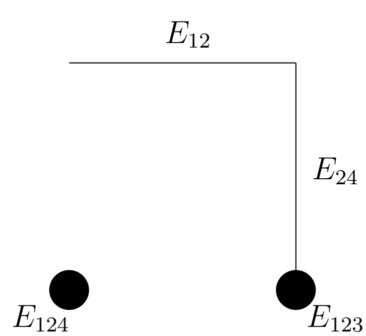


Figure 3.2

3.3 From 5-flow to 4-flow

In this section, we consider about the relation between the missing patterns of a positive 5-flow decomposition and a nowhere zero 4-flow.

Theorem 3.3.1 *Let (D, f) be a positive 5-flow of G with $f = f_1 + f_2 + f_3 + f_4$ where each (D, f_j) is a non-negative 2-flow. Then there is a linear combination*

$$\sum_{j=1}^{4} \alpha_j f_j$$

which is a nowhere-zero modular 4-flow if and only if the missing parts of f is one of the following several cases:
(1) (trivial cases) missing E_{1234} or E_a for any $a \in \{1, 2, 3, 4\}$;
(2) missing E_{ab}, E_{ac}, E_{bc} for any $\{a, b, c\} \subset \{1, 2, 3, 4\}$;
(3) missing E_{ab}, E_{ac}, E_{ad} for any $\{a, b, c, d\} = \{1, 2, 3, 4\}$;
(4) missing $E_{abc}, E_{acd}, E_{abd}$ for any $\{a, b, c, d\} = \{1, 2, 3, 4\}$;
(5) missing E_{ab}, E_{ac}, E_{bcd} for any $\{a, b, c, d\} = \{1, 2, 3, 4\}$;
(6) missing E_{abc}, E_{abd}, E_{cd} for any $\{a, b, c, d\} = \{1, 2, 3, 4\}$.

Proof. "$\Leftarrow$": We are to find a nowhere-zero mod-4-flow (D, f').

(1) is trivial. If $E_{1234} = \emptyset$, then

$$f' = f.$$

If $E_1 = \emptyset$, then

$$f' = f_2 + f_3 + f_4.$$

(2):

$$f' \equiv 2f_a + 2f_b + 2f_c + f_d \ mod(4).$$

(3):

$$f' \equiv 3f_a + f_b + f_c + f_d \ mod(4).$$

(Or, a $Z_2 \times Z_2$-flow: $f' = (0,1)f_1 + (1,0)f_2 + (1,0)f_3 + (1,0)f_4$ is a nowhere-zero $Z_2 \times Z_2$-flow.)

(4):

$$f' \equiv 2f_a + f_b + f_c + f_d \; mod(4).$$

(5):

$$f' \equiv f_a + 3f_b + 3f_c + 2f_d \; mod(4).$$

(6):

$$f' \equiv f_a + f_b + 2f_c + 2f_d \; mod(4).$$

"$\Rightarrow$": Prove by contradiction. Assume the missing pattern is any of above and let $\alpha : \{1,2,3,4\} \mapsto Z_4 - \{0\}$ (simply $\alpha(j) = \alpha_j$) be the coefficients for a nowhere-zero 4-flow (described in Theorem).

Denote $|h|$ be the number of α_j's that $\alpha_j = h \in Z_4 - \{0\}$.

I. Due to some automorphism of Z_4, we may choose a mapping α such that $|1| \geq |3|$.

II. Since the induced flow is nowhere-zero 4-flow,

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \not\equiv 0 \;\; mod(4) \tag{3.1}$$

for otherwise, E_{1234} must be empty.

Hence, by (3.1),

$$|j| < 4 \;\; \forall j = 1, \cdots, 4. \tag{3.2}$$

III. Claim that $|1| \geq 1$. For otherwise, $|3| \leq |1| = 0$, and therefore, $|2| = 4$, this contradicts (3.1).

IV. If $|2| = 3$, say $\alpha_j = 2$ for $j = 1,2,3$, then $E_{ab} = \emptyset$ for each pair of $a, b \in \{1,2,3\}$, which is the excluded case (2). So, $|2| \leq 2$.

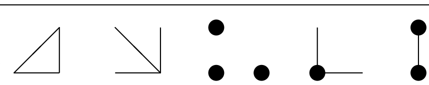
V. If $|2| = 2$, say $\alpha_j = 2$ for $j = 1, 2$, then $E_{12} = \emptyset$. By III., let $\alpha_3 = 1$. Thus, by (3.1), $\alpha_4 \neq 3$. Hence, $\alpha_4 = 1$ and therefore, $E_{134} = E_{234} = \emptyset$, which is the excluded case (6). So, $|2| \leq 1$.

VI. So, by (3.2), $|2| = 1$, say $\alpha_1 = 2$. By I and III., $|1| \geq 2$, let $\alpha_j = 1$ for $j = 2, 3$. If $\alpha_4 = 1$, then Hence, $E_{1ab} = \emptyset$ for each $a, b \in \{2, 3, 4\}$, which is the excluded case (2). If $\alpha_4 = 3$, then Hence, $E_{123} = \emptyset$ and $E_{c4} = \emptyset$ for each $c \in \{2, 3\}$ which is the excluded case (5).

This complete the proof for group Z_4.

There is another group $Z_2 \times Z_2$ of order 4. The proof for $Z_2 \times Z_2$ is the same as above.
■

Therefore, we can use the poset symbol system to list the equivalent families of the family of graphs admit a nowhere zero 4-flow.

4-flow $\Longleftrightarrow$ E_{abcd}, E_a 

3.4 Orientable 5-cycle double cover

Definition 3.4.1 (orientable cycle double cover)
(1) Let $F = C_1, \ldots, C_r$ be a cycle double cover of a graph G. The set F is an orientable cycle double cover if there is an orientation D_μ on $E(C_\mu)$, for each $\mu = 1, \ldots, r$, such that $D_\mu(C_\mu)$ is a directed cycle, and for each edge e contained in two cycles C_α and C_β, the directions of D_α on $E(C_\alpha)$ and D_β on $E(C_\beta)$ are opposite on e.
(2) An orientable k-cycle double cover F is an orientable cycle double cover consisting of k members.

The equivalency of orientable 4-cycle double cover and $\mathcal{F}_4$ was found by Tutte in [24].

Proposition 3.4.2 (Tutte [24]) *G admits a nowhere-zero 4-flow iff G has an orientable 4-cycle double cover.*

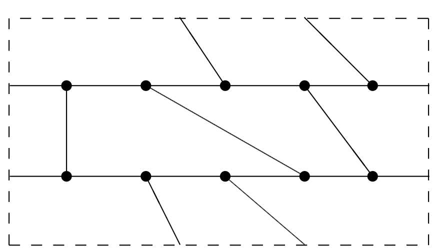
A few conjectures stronger than Tutte's 5-flow conjecture have been proposed. One of them is the orientable 5-cycle double cover conjecture raised by Archdeacon and Jaeger.

Conjecture 3.4.3 (Archdeacon [1] and Jaeger [15]) *Every bridgeless graph has an orientable cycle double cover consisting of at most five cycles.*

It is known that the orientable 5-cycle double cover conjecture implies the 5-flow conjecture. It is remain unknown if the 5-flow conjecture implies the orientable 5-cycle double cover conjecture. And the reductions of incomplete integer flows may lead us a direction to attack the problem.

The relation between orientable 5-cycle double cover and $\mathcal{F}_5$ is unknown. Let $\mathcal{G}(O5CDC)$ be the family of all graphs admitting an orientable 5-cycle double cover. Since the Petersen graph has an orientable 5-cycle double cover (See Figure: Petersen embedded in a turos), we have the following relation:

$$\mathcal{F}_4 \subset \mathcal{G}(O5CDC) \subseteq \mathcal{F}_5.$$



Some mathematicians believes that $\mathcal{G}(O5CDC)$ is proper subset of $\mathcal{F}_5$. That is, $\mathcal{G}(O5CDC)$ is a node of the poset $\mathcal{Q}[4,5]$ distinct from the upper bound $\mathcal{F}_5$ and the lower bound $\mathcal{F}_4$.

The relation that $\mathcal{G}(O5CDC) \subseteq \mathcal{F}_5$ can be seen as follows: (add some detailed discussion, or copy some theorem from my book).

In this subsection, the missing patterns of graphs in $\mathcal{G}(O5CDC)$ are identified.

Theorem 3.4.4 *A graph G has an orientable 5-cycle double cover if and only if G admits a positive 5-flow with decomposition $f = f_1 + \cdots + f_4$ such that $E_{13} = E_{14} = E_{23} = E_{124} = E_{123} = \emptyset$.*

Proof. "$\Rightarrow$": Let $\{C_1, \cdots, C_5\}$ be an orientable cycle double cover of G each of which has an orientation D_j.

For $H' = C_1 \cup C_2$ let $(D', f_1 + f_2)$ be a positive 3-flow of H': D' is the same as D_1 on C_1 and the same as D_2 on $C_2 - C_1$, and f_1 has the support C_1, f_2 has the support $C_1 \Delta C_2$.

For $H'' = C_3 \cup C_4$ let $(D'', f_3 + f_4)$ be a positive 3-flow of H'': D'' is opposite to D_3 on C_3 and opposite to D_4 on $C_4 - C_3$, and f_3 has the support C_3, f_4 has the support $C_3 \Delta C_4$.

Then, the missing patterns are $E_{13} = E_{14} = E_{23} = E_{134} = E_{123} = \emptyset$.

"$\Leftarrow$": Reverse the above operation, then we can get an orientable 5-cycle double cover. ■

Proposition 3.4.5 *Let (D, f) be a positive 5-flow of G. Then there is a decomposition of f: $f = f_1 + f_2 + f_3 + f_4$ where each (D, f_j) is a non-negative 2-flow and $E_{ab} = E_{cd} = \emptyset$ for some $\{a, b, c, d\} = \{1, 2, 3, 4\}$. That is,*

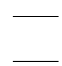
$$\mathcal{G}(\emptyset) = \mathcal{G}(\{ab\}, \{cd\}).$$

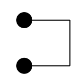
Proposition 3.4.6 *Let (D, f) be a positive 4-flow of G. Then there is a decomposition of f: $f = f_1 + f_2 + f_3$ where each (D, f_j) is a non-negative 2-flow and $E_{ab} = \emptyset$ for some $\{a, b\} \subset \{1, 2, 3\}$.*

Proposition 3.4.6 is the best possible. Motivation: $|E(K_4)| = 6$ but $f_1 + f_2 + f_3$ has 7 patterns.

Proof. Similar to Theorem 3.4.4 by using orientable 4-cycle double cover. ■

Therefore, we have two more families using the poset symbol system.

5-flow $\Longleftrightarrow$ 

Orientable 5CDC $\Longleftrightarrow$ 

3.5 Peterson graph and missing patterns

Problem 3.5.1 *Is there a node $\mathcal{G}(S)$ in the poset $\mathcal{Q}[4,5]$ other than $\mathcal{F}_4$ that $P_{10} \notin \mathcal{G}(S)$?*

Although some mathematician believes that it is possible that $\mathcal{G}(S)$ is a such node, but no proof yet. Our discussion in Subsection 3.4 may unveil some evidence to support this, but it is still far away from finding such a node $\mathcal{G}(S)$.

With the study of the Petersen graph, we found a such node: a node $\mathcal{G}(S)$ between $\mathcal{F}_5$ and $\mathcal{F}_4$, but is neither of them.

In this section, we prove that Positive 5-flow on Peterson graph can not miss pattern $E_{12}, E_{23}, E_{34}, E_{41}$ at the same time, that is:

Theorem 3.5.2 $P_{10} \notin \mathcal{G}(E_{12}, E_{23}, E_{34}, E_{41})$

Here is the proof:

Claim 3.5.3 *(i)*$E_i \neq \emptyset$ *for* $\forall i \in 1, 2, 3, 4$

(ii)$E_{13}, E_{24} \neq \emptyset$

(iii)$E_{abc} \neq \emptyset$ *for* $\forall a, b, c \in 1, 2, 3, 4$

(iv)$E_{1234} \neq \emptyset$

Proof. (i)Missing any one of $E_{f=1}$ will lead to 4-flow. (By Theorem 3.3.1(1))

(iii)Missing any one of $E_{f=3}$ with $E_{12}, E_{23}, E_{34}, E_{41}$ missing will lead to 4-flow. (By Theorem 3.3.1(5))

(ii) and (iv) are trivial. ■

Given a positive 5-flow on P_{10}, $C = E_{f=odd}$ is a cycle, where $E_{f=odd}$ is the union of E_s for every $|s| = odd$. By claim 3.5.3, none of them is empty set, hence $|C| \geq 8$.

Now by the way of contradiction, suppose that $P_{10} \in \mathcal{G}(E_{12}, E_{23}, E_{34}, E_{41})$.

Claim 3.5.4 *In* $C = E_{f=odd} \in P_{10}$*:*

(i) $E_{f=3}$ *edges are not adjacent to each other.*

(ii) If two $E_{f=3}$ *edges are connected by an* $E_{f=1}$ *edge, and they share the same direction, then they must have the same pattern.*

(iii) If two $E_{f=1}$ *edges are connected by an* $E_{f=3}$ *edge, and they share the same direction, then they must have the same pattern.*

Proof. (i) If two $E_{f=3}$ edges are adjacent to each other at vertex v, then the value of flow on the third edge of v must be either 6 or 0 which is no longer a positive 5-flow.

(ii) There are two cases depending on the direction of the $E_{f=1}$ in middle. Without lose of generality, let $e_1 \in E_{123}$ as in $Figure$ 3.3, and it is easy to verify the claim.

(iii) The similar argument in (ii) will work here. ■

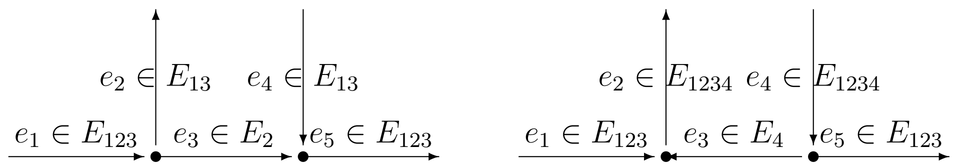


Figure 3.3

Claim 3.5.5 $|C| \neq 8$.

Proof. Due to Lemma 3.5.4, $E_{f=1}$ and $E_{f=3}$ edges must be alternating and each pattern appears only once on C, directions of two $E_{f=3}$ edges connected by an $E_{f=1}$ edge must be opposite and directions of two $E_{f=1}$ edges connected by an $E_{f=3}$ edge must be opposite. Then by counting patterns on edges as in *Figure* 3.4, at least two of $E_{f=1}$ edges share the same pattern which is a contradiction to each pattern appears only once on C. ■

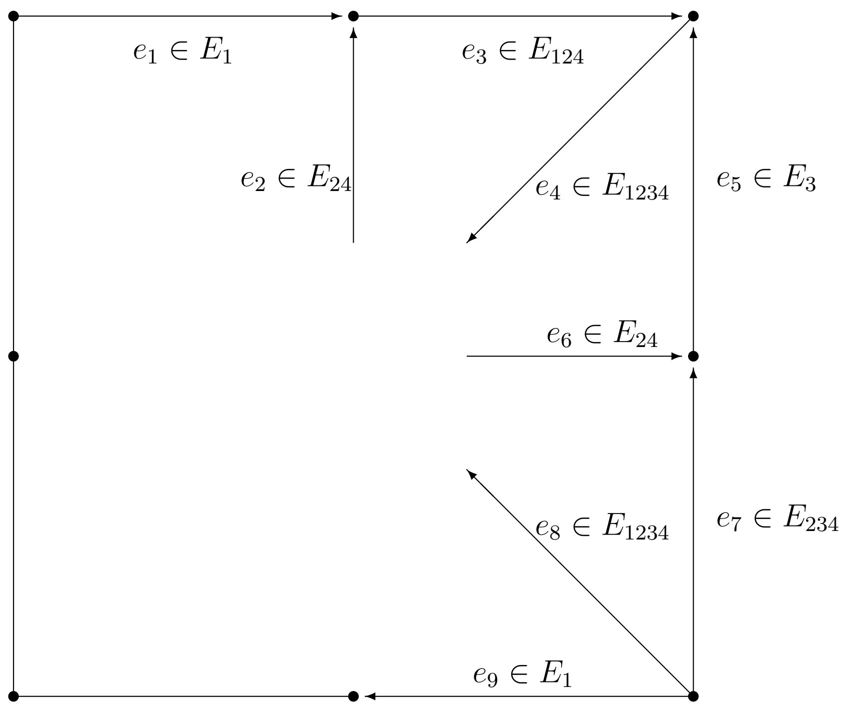


Figure 3.4

Claim 3.5.6 $|C| \neq 9$.

Proof. The remaining vertex v_{10} adjacent to the vertices on C by v_1, v_4, v_7 as in *Figure* 3.5. Denote $\overline{v_i v_j}$ be the edge connecting vertices v_i and v_j. By Claim 3.5.3, among the 9 edges on C, four of them are $E_{f=3}$ edges, and the rest five edges are $E_{f=1}$ edges.

By symmetry of the graph, we can assume that $\overline{v_1 v_2}$, $\overline{v_3 v_4}$ and $\overline{v_5 v_6}$ are $E_{f=3}$ edges. By Claim 3.5.4, $\overline{v_1 v_2}$ and $\overline{v_5 v_6}$ must share the same direction, but that will lead to a contradiction by simply considering the direction of $\overline{v_2 v_6}$. ■

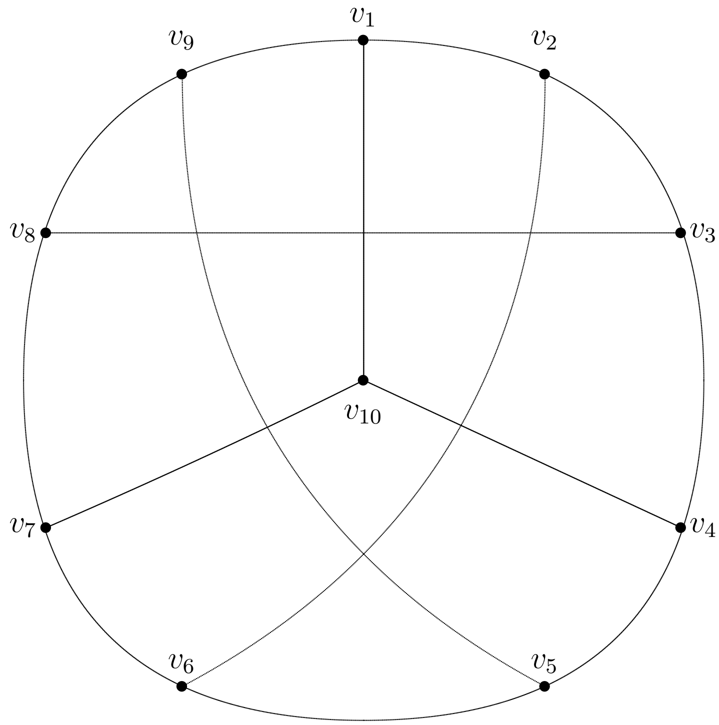


Figure 3.5

Claim 3.5.7 $|C| \neq 10$.

Proof. $|C|$ can be viewed as two 5-cycle joined by 5 edges (as in *Figure* 3.6). Due to Claim 3.5.4, there are only 4 types of 5-cycle and each one is the inverse direction of another which can be joined by 5 edges. We may assume *Type* 3 is the inverse direction of *Type* 1 and *Type* 4 is the inverse direction of *Type* 2.

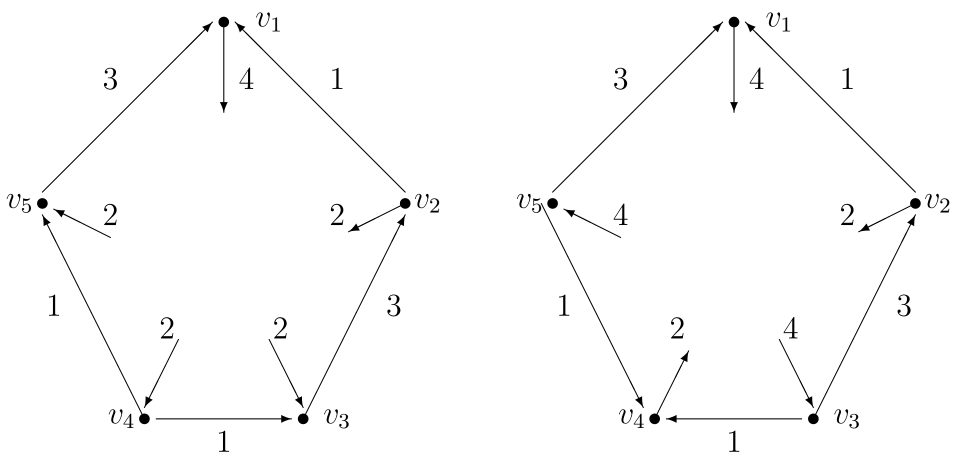
Since *Type* 1 and *Type* 2 do not share the same flow patterns on half edges, so the only two cases are *Type* 1 joins *Type* 3 and *Type* 2 joins *Type* 4.

Now consider the case *Type* 1 joins *Type* 3: notice the direction of the half edges on v_1 and v_4 are out, so these two half edges should be connected with the corresponding v_1

and v_4 in *Type* 3 which will be a 4-cycle in the Peterson Graph, a contradiction.

For the case *Type* 2 joins *Type* 4: notice the only two half edges with flow pattern in $E_{f=2}$ should be connected with the corresponding half edges in *Type* 4 and that still contradicts the structure of Peterson Graph.

■



Type 1 *Type* 2

Figure 3.6

Then Theorem 3.5.2 is a quick result from Claim 3.5.5,Claim 3.5.6,Claim 3.5.7.

3.6 Poset of missing patterns

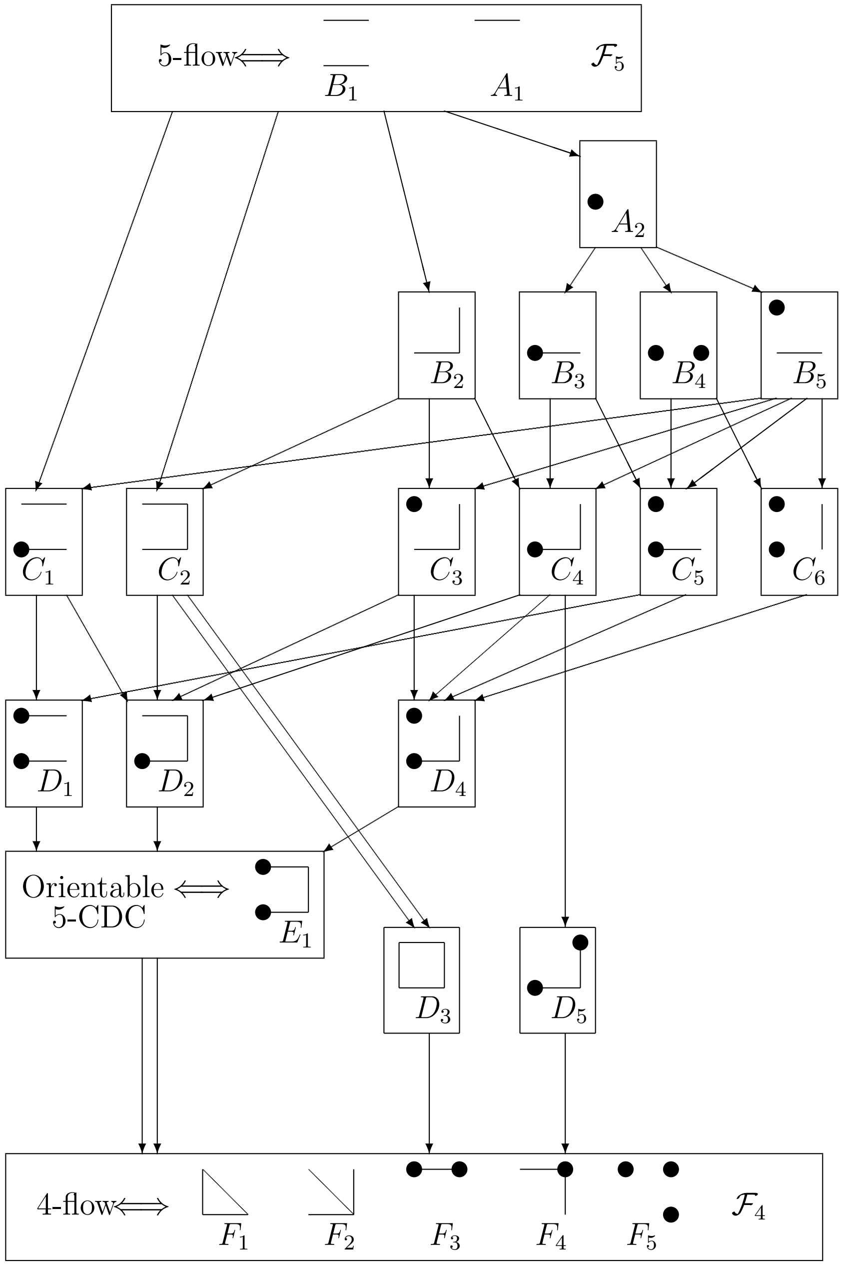
To make a conclusion of the results in this paper, we use a poset system of missing patterns to illustrate the relation between 5-flow conjecture and the 4-flow conjecture. See Figure 3.7.

Every box indicate a family of graphs with a positive 5-flow decomposition missing the corresponding patterns but can not be reduced to a 4-flow due to the Theorem 3.3.1. The top box $\mathcal{F}_5$ containing B_1 and A_1 is the family of graphs satisfy the equivalent statement of the 5-flow conjecture by Theorem 3.4.5. Box E_1 is the family of graphs satisfy the equivalent statement of the orientable 5-cycle double cover conjecture by Theorem 3.4.4.

The bottom box $\mathcal{F}_4$ containing F_1, F_2, F_3, F_4 and F_5 satisfy the equivalent statement of the 4-flow conjecture by Theorem 3.3.1.

The single arrows in the figure indicate "contains" relation between any two boxes. Any single arrow can be an "equal" relation, and if we can prove the arrows in between $\mathcal{F}_5$ and E_1 are all "equal" relation then we can show the equivalence of 5-flow conjecture and orientable 5-cycle double cover conjecture.

The double arrows in the figure indicate "proper contains" relation between any two boxes. For instance the double arrow between C_2 and D_3, which is a result from Theorem 3.5.2 such that the Petersen graph is in C_2 but not in D_3. The same explanation for the double arrow between E_1 and $\mathcal{F}_4$, the Petersen graph is in E_1 but not in $\mathcal{F}_4$.



5-flow⟺
B_1
A_1
$\mathcal{F}_5$
A_2
B_2
B_3
B_4
B_5
C_1
C_2
C_3
C_4
C_5
C_6
D_1
D_2
D_4
Orientable ⟺
5-CDC
E_1
D_3
D_5
4-flow⟺
F_1
F_2
F_3
F_4
F_5
$\mathcal{F}_4$

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