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ABSTRACT

A study on Dicycles and Eulerian Subdigraphs

Khalid A. Alsatami

A dicycle cover of a digraph D is a collection $\mathcal{C}$ of dicycles of D such that $\bigcup_{C \in \mathcal{C}} A(C) = A(D)$. If D is obtained from a simple undirected graph G by assigning an orientation to the edges of G, then D is an oriented graph. A digraph D is eulerian if D is connected and for any $v \in V(D)$, $d_D^+(v) = d_D^-(v)$. A digraph D is supereulerian if D contains a spanning eulerian subdigraph. A digraph D is a closed ditrail if it is eulerian. This dissertation focuses on a study of dicycle cover and supereulerian digraphs from the following aspects.

1. Dicycle cover of Hamiltonian oriented graphs.
A dicycle cover of a digraph D is a family $\mathcal{F}$ of dicycles of D such that each arc of D lies in at least one dicycle in $\mathcal{F}$. We investigate the problem of determining the upper bounds for the minimum number of dicycles which cover all arcs in a strong digraph. Best possible upper bounds of dicycle covers are obtained in a number of classes of digraphs, including strong tournaments, Hamiltonian oriented graphs, Hamiltonian oriented complete bipartite graphs, and families of possibly non-hamiltonian digraphs obtained from these digraphs via a sequence of 2-sum operations.

2. Supereulerian digraphs with given local structures .
Catlin in 1988 indicated that there exist graph families $\mathcal{F}$ such that if every edge e in a graph G lies in a subgraph H_e of G isomorphic to a member in $\mathcal{F}$, then G is supereulerian. In particular, if every edge of a connected graph G lies in a 3-cycle, then G is supereulerian. The purpose of this research is to investigate how Catlin's theorem can be extended to digraphs. A strong digraph D is supereulerian if D contains a spanning eulerian subdigraph. We show that there exists an infinite family of non-supereulerian strong digraphs each arc of which lies in a directed 3-cycle. We also show that there exist digraph families $\mathcal{H}$ such that a strong digraph D is supereulerian if every arc a of D lies in a subdigraph H_a isomorphic to a member of $\mathcal{H}$. A digraph D is symmetric if $(x,y) \in A(D)$ implies $(y,x) \in A(D)$; and is symmetrically connected if every pair of vertices of D are joined by a symmetric dipath. A digraph D is partially symmetric if the digraph obtained from D by contracting all symmetrically connected components is symmetrically connected. It is known that a partially symmetric digraph may not be symmetrically

connected. We show that symmetrically connected digraphs and partially symmetric digraphs are such families. Sharpness of these results are discussed.

3. On a class of supereulerian digraphs.

The 2-sum of two digraphs D_1 and D_2, denoted $D_1 \oplus_2 D_2$, is the digraph obtained from the disjoint union of D_1 and D_2 by identifying an arc in D_1 with an arc in D_2. A digraph D is **supereulerian** if D contains a spanning eulerian subdigraph. It has been noted that the 2-sum of two supereulerian (or even hamiltonian) digraphs may not be supereulerian. We obtain several sufficient conditions on D_1 and D_2 for $D_1 \oplus_2 D_2$ to be supereulerian. In particular, we show that if D_1 and D_2 are symmetrically connected or partially symmetric, then $D_1 \oplus_2 D_2$ is supereulerian.

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DEDICATION

To

my father <u>Ali</u> , my mother <u>Fatema</u>, my wife <u>Hanan</u>

and

my lovely children <u>Ali, Abdulkreem, Azam and Haitham</u>

Contents

Chapter 1

Preliminaries

1.1 Notation and Terminology

We consider finite and simple graphs and digraphs. A digraph D is simple if D has no loops and if for any pair of distinct vertices $u, v \in V(D)$, there is at most one arc in D oriented from u to v. Usually, we use G to denote a graph and D a digraph. Undefined terms and notations will follow [1] for graphs and [2] for digraphs. In particular, $\kappa(G)$, $\kappa'(G)$ and $\alpha(G)$ denote the connectivity, the edge connectivity and the independence number of a graph G; and $\kappa(D)$ and $\lambda(D)$ denotes the vertex-strong connectivity and the arc-strong connectivity of a digraph D, respectively. In this chapter we will provide most of the terminology and notation used in this dissertation.

A cycle is a 2-regular connected nontrivial graph. A cycle on n vertices is often called an n-cycle. A cycle cover of a graph G is a collection $\mathcal{C}$ of cycles of G such that $E(G) = \bigcup_{C \in \mathcal{C}} E(C)$. Bondy [4] conjectured that if G is a 2-connected simple graph with $n \geq 3$ vertices, then G has a cycle cover $\mathcal{C}$ with $|\mathcal{C}| \leq \frac{2n-3}{3}$. Bondy [4] showed that this conjecture, if proved, would be best possible. Y. X. Luo and R. S. Chen [14] proved that this conjecture holds for 2-connected simple cubic graphs. It has been shown that for plane triangulations, serial-parallel graphs or planar graphs in general, one can have a better bound for the number of cycles used in a cover ([9–12]). Barnette [3] proved that if G is a 3-connected simple planar graph of order n, then the edges of G can be covered by at most $\frac{n+1}{2}$ cycles. Fan [6] settled this conjecture by showing that it holds for all simple 2-connected graphs. The best possible number of cycles needed to cover cubic graphs has been obtained in [13] and [15].

we use the notation (u, v) to denote an arc oriented from u to v in a digraph. A directed path in a digraph D from a vertex u to a vertex v is called a (u, v)-dipath. To emphasize the distinction between graphs and digraphs, a directed cycle or path in a digraph is often referred as a dicycle or dipath. It is natural to consider the number of dicycles needed to cover a digraph. For a digraph D, $V(D)$ and $A(D)$ denote the vertex set and arc set of D, respectively. If $A' \subseteq A(D)$, then $D[A']$ is the subdigraph induced by A'. For an integer $n > 0$, we use K_n^* to denote the complete digraph on n vertices. Hence for every pair of distinct vertices $u, v \in V(K_n^*)$, there is exactly one arc (u, v) in $A(K_n^*)$. Any simple digraph D on n vertices can be viewed as a subdigraph of K_n^*. If W is an arc subset of $A(K_n^*)$, then $D + W$ denotes the digraph $K_n^*[A(D) \cup W]$. If (u, v) is an arc, we say that u dominates v or (v is dominated by u) and denote it by $u \to v$. For disjoint subsets X and Y of $V(D)$, $X \to Y$ means that every vertex of X dominates every vertex of Y. A digraph D is strong if for any distinct $u, v \in V(D)$, D has a (u, v)-dipath. Following [2], if $X, Y \subseteq V(D)$, then define

$$(X, Y)_D = \{(x, y) \in A(D) : x \in X, y \in Y\}.$$

When $Y = V(D) - X$, we define

$$\partial_D^+(X) = (X, V(D) - X)_D \text{ and } \partial_D^-(X) = (V(D) - X, X)_D.$$

If $X \subseteq V(D) \cup A(D)$, then $D[X]$ denotes the subdigraph induced by X. If S is a subdigraph of a digraph D and if $X \subset A(S)$ and $Y \subseteq A(D) - A(S)$, we use $S - X + Y$ to denote $D[(A(S) - X) \cup Y]$. For a vertex $v \in V(D)$, the **out-degree** of v is $d_D^+(v) = |\partial_D^+(\{v\})|$ and the **in-degree** of v in D is $d_D^-(v) = |\partial_D^-(\{v\})|$. Furthermore, we define

$$\partial_D(v) = \partial_D^+(\{v\}) \cup \partial_D^-(\{v\}) \text{ and } d_D(v) = d_D^+(v) + d_D^-(v).$$

Let $N_D^+(v) = \{u \in V(D) : (v, u) \in A(D)\}$ and $N_D^-(v) = \{u \in V(D) : (u, v) \in A(D)\}$ denote the **out neighbors** and **in neighbors** of v in D, respectively. When the digraph D is understood from the context, we often omit the subscript D. Following [1] and [2], we use $\lambda(D)$ and $c(D)$ to denote the arc-strong connectivity of D and the number of components of the underlying graph $G(D)$ of D, respectively. By the definition of $\lambda(D)$ in [2], for any integer $k \geq 0$,

$$\lambda(D) \geq k \text{ if and only if for any nonempty proper subset } X \subset V(D), |\partial_D^+(X)| \geq k. \tag{1.1}$$

A **dicycle cover** of a digraph D is a collection $\mathcal{C}$ of dicycles of D such that $\bigcup_{C \in \mathcal{C}} A(C) = A(D)$. If D is obtained from a simple undirected graph G by assigning an orientation to the edges of G, then D is an **oriented graph**.

If D is a digraph, we often use $G(D)$ to denote the underlying undirected graph of D, the graph obtained from D by erasing all orientation on the arcs of D.

For graphs H and G, by $H \subseteq G$ we mean that H is a subgraph of G. Similarly, for digraphs H and D, by $H \subseteq D$ we mean that H is a subdigraph of D.

For an integer n, we use the notation $[n]$ to denote the set $\{1, 2, \ldots, n\}$. A **walk** in D is an alternating sequence $W = x_1 a_1 x_2 a_2 x_3 \cdots x_{k-1} a_{k-1} x_k$ of vertices x_i and arcs a_j from D such that the tail of a_j is x_j and the head of a_j is x_{j+1} for every $i \in [k]$ and $j \in [k-1]$. A walk W is **closed** if $x_1 = x_k$, and **open** otherwise. The set of vertices $\{x_i : i \in [k]\}$ is denoted by $V(W)$; the set of arcs $\{a_j : j \in [k-1]\}$ is denoted by $A(W)$.

Following the definition in [2] (Page 11), for a subdigraph H of a digraph D, an (x, y)-dipath P is an (H, H)-dipath if $x, y \in V(H)$ and $V(P) \cap V(H) = \{x, y\}$.

A **ditrail** in D is an alternating sequence of vertices and arcs such that all the arcs are distinct. If all the vertices of a ditrail are distinct we call it a **dipath**. An arc $(u, v) \in A(D)$ is **symmetric** in D if $(u, v), (v, u) \in A(D)$. A digraph D is **symmetric** if $|V(D)| = 1$ or if $|A(D)| > 0$ and every arc of D is symmetric. Especially, a symmetric dipath P is a dipath such that every arc of P is symmetric.

A digraph D is **trialable** if there exist $x, y \in V(D)$, such that D has a spanning (x, y) ditrail.

A dipath P is a hamiltonian dipath if $V(P) = V(D)$. A digraph D is hamiltonian if D contains a hamiltonian dicycle. An (x, y)-hamiltonian dipath is a hamiltonian dipath from x to y. A digraph D is hamiltonian-connected if D has an (x, y)-hamiltonian dipath for every choice of distinct vertices $x, y \in V(D)$.

A digraph D is **weakly connected** if $G(D)$, its underlying graph, is connected. Let $\mathcal{H}$ be a family of digraphs. A strong digraph D is **locally** $\mathcal{H}$ if every arc $a \in A(D)$ lies in a subdigraph H_a of D, where $H_a \in \mathcal{H}$. For convenience, we also call a locally $\{H\}$ digraph D as a **locally** H digraph. A digraph D is **quasi-transitive** if, for every triple of distinct vertices $x, y, z \in V(D)$, with $(x, y), (y, z) \in A(D)$, there is at least one arc between x and z. A digraph D is semicomplete if there is an arc between every pair of vertices in D. A digraph $D = (V, A)$ is semicomplete multipartite if there is a partition $V_1, V_2, \ldots, V_c$ of V into independent sets so that every vertex in V_i shares an arc with every vertex in V_j for $1 \leq i < j \leq c$.

Motivated by the Chinese Postman Problem, Boesch, Suffel, and Tindell [30] in 1977 proposed the supereulerian problem, which seeks to characterize graphs that have spanning Eulerian subgraphs, and they [30] indicated that this problem would be very difficult. Pulleyblank [26] later in 1979 proved that determining whether a graph is supereulerian, even within planar

graphs, is NP-complete. Since then, there have been lots of researches on this topic. Catlin [20] in 1992 presented the first survey on supereulerian graphs. Later Chen et al [21] gave an update in 1995, specifically on the reduction method associated with the supereulerian problem. A recent survey on supereulerian graphs is given in [24].

It is a natural to consider the supereulerian problem in digraphs. A digraph D is **eulerian** if D is connected and for any $v \in V(D)$, $d_D^+(v) = d_D^-(v)$. A digraph D is **supereulerian** if D contains a spanning eulerian subdigraph. A digraph D is a closed ditrail if it is eulerian. The main problems are to investigate the number of dicycles needed to cover a Hamiltonian oriented graph and to determine supereulerian digraphs.

1.2   Previous Work On Supereulerian Digraphs

Several efforts in supereulerian digraphs have been made. However, contrary to the case of undirected graphs, not much work has been done yet for supereulerian digraphs. The earlier studies were done by Gutin ([31, 32]). In [32], G.Gutin stated the following definitions:

Definition 1.2.1. *Given a digraph D and two positive integers $f(x), g(x)$ for every $x \in V(D)$, a subgraph H of D is called a $(g, f)-$factor if $g(x) \leq d_H^+(x) = d_H^-(x) \leq f(x)$ for every $x \in V(D)$. If $f(x) = g(x) = 1$ for every x, then a connected $(g, f)-$factor is a hamiltonian cycle.*

Definition 1.2.2. *An extension of a digraph D is a new digraph H obtained from D by replacing every vertex $x \in V(D)$ with a set of independent vertices S_x such that, for every pair of distinct $x, y \in V(D)$, an arc (u, v), where $u \in S_x$, $y \in S_y$, is in H if and only if (x, y) is in D. An extension of a locally in-semicomplete digraph is called an extended locally $in - semicomplete$ digraph.*

Then he proved the following theorem:

Theorem 1.2.3. *(Gutin [32]) Let D be a semicomplete bipartite digraph or an extended locally $in - semicomplete$ digraph. Then D has a connected $(g, f)-$factor if and only if D is strongly connected and contains a $(g, f)-$factor.*

Using definition1.2.1 Gutin added that: Clearly, a digraph H is supereulerian if and only if H has a connected $(1, d^*)$-factor, where $d^* = \min\{d^+(x), d^-(x)\}$ for every $x \in V(H)$. As a result he presented the following corollary:

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Corollary 1.2.4. *(Gutin [32]) Let D be a semicomplete bipartite digraph or an extended locally in − semicomplete digraph. D is* sup *ereulerian if and only if D is strongly connected and contains a (1,d*)-factor.*

A well known theorem of Chvátal and Erdös [22] states that if $|V(G)| \geq 3$ and if $\kappa(G) \geq \alpha(G)$, then G is hamiltonian. Thomassen [25] gave an infinite family of non hamiltonian (but supereulerian) digraphs such that $\kappa(D) = \alpha(D) = 2$, showing that the Chvátal-Erdös Theorem does not extend to digraphs. This motivates Bang-Jensen and Thommassé in [17] to make the following conjecture.

Conjecture 1.2.5. *Let D be a digraph. If $\lambda(D) \geq \alpha(D)$, then D is supereulerian.*

Moreover in [17] Bang-Jensen and Alessandro Maddaloni analyzed a number of sufficient conditions for a digraph to be supereulerian starting with the following definition:

Definition 1.2.6. *Given a digraph D, a dicycle factor, eulerian factor of D is a collection of vertex-disjoint, arc-disjoint, respectively dicycles spanning $V(D)$.*

Then as an effort towards conjecture 1.2.5 they proved the following theorem:

Theorem 1.2.7. *(J. Bang-Jensen and A. Maddaloni [17]) Let D be a digraph. If $\lambda(D) \geq \alpha(D)$ then D has an eulerian factor.*

In [17], Conjecture 1.2.5 has been verified in several families of digraphs. Gutin in [31] proved the following theorem:

Theorem 1.2.8. *(Gutin [31]) An extended semicomplete digraph is hamiltonian if and only if it is strong and has a cycle factor.*

Using a similar approach to Gutin's proof of Theorem 1.2.8, Bang-Jensen and Alessandro Maddaloni in [17] proved the following supereulerian version of it.

Theorem 1.2.9. *(J. Bang-Jensen and A. Maddaloni [17]) Let D be a semicomplete multipartite digraph. D is supereulerian if and only if it is strong and has an eulerian factor.*

Combining Theorem 1.2.9 and Theorem 1.2.7, Bang-Jensen and Alessandro Maddaloni verify Conjecture 1.2.5 for semicomplete multipartite digraphs.

Theorem 1.2.10. *(J. Bang-Jensen and A. Maddaloni [17]) Let D be a semicomplete multipartite digraph. If $\lambda(D) \geq \alpha(D)$, then D is supereulerian.*

In addition, Bang-Jensen and Alessandro Maddaloni proved that:

Theorem 1.2.11. *(J. Bang-Jensen and A. Maddaloni [17]) Let D be a quasi transitive digraph. If $\lambda(D) \geq \alpha(D)$, then D is supereulerian.*

Also, Bang-Jensen and Alessandro Maddaloni proved the following:

Theorem 1.2.12. *(J. Bang-Jensen and A. Maddaloni [17]) A strong digraph such that $d(x) + d(y) \geq 2n - 3$ for all pairs of non-adjacent vertices x, y is supereulerian.*

Corollary 1.2.13. *(J. Bang-Jensen and A. Maddaloni [17]) A strong digraph such that $d^+(x) + d^-(y) \geq n - 1$ for all ordered pairs (x, y) of non-adjacent vertices is supereulerian.*

Finally, Bang-Jensen and Alessandro Maddaloni proved that the condition in congecture 1.2.5 is not necesary by the following example: consider a directed cycle on 4 vertices C_4: it is eulerian, and hence supereulerian, but $\lambda(C_4) = 1$ and $\alpha(C_4) = 2$. Moreover, they provided infinite families of digraphs with $\lambda(D) = \alpha(D) - 1$ that are not supereulerian. Hence, if true, Congecture 1.2.5 would be best possible.

In [34], Yanmei Hong, Hong-Jian Lai and Qinghai Liu after producing the following definition:

Definition 1.2.14. *Let D be a strong digraph and $U \subsetneq V(D)$. Then in $D[U]$, we can find some ditrails $P_1, \ldots, P_t$ such that $\cup_{i=1}^{t} V(P_i) = U$ and $A(P_i) \cap A(P_j) = \emptyset$ for any $i \neq j$. Let $\tau(U)$ be minimum value of such t. Then $c(G(D[U])) \leq \tau(U) \leq |U|$, where $c(G(D[U]))$ is the number of component of the underlying graph of $D[U]$. For any $A \subseteq V(D) - U$, denote $B := V(D) - U - A$ and let*

$$
\begin{aligned}
h(U, A) : &= \min\{|\partial_D^+(A)|, |\partial_D^-(A)|\} + \min\{|(U, B)_D|, |(B, U)_D|\} - \tau(U), \quad and \quad (1.2) \\
h(U) : &= \min\{h(U, A) : A \cap U = \emptyset\}.
\end{aligned}
$$

Yanmei Hong, Hong-Jian Lai and Qinghai Liu gave the following necessary condition for the existence of a spanning eulerian subdigraph:

Proposition 1.2.15. *(Proposition (2.1) of Hong et al [34]) If D has a spanning eulerian sub-digraph, then for any $U \subset V(D)$, $h(U) \geq 0$.*

Moreover, Hong et al gave the following sufficient degree condition for the existence of a spanning eulerian subdigraph:

Theorem 1.2.16. *(Hong et al [34]) Let D be a strong digraph of order n and minimum out-degree $\delta^+ \geq 4$ and minimum in-degree $\delta^- \geq 4$. If $\delta^+ + \delta^- > n - 4$, then D has a spanning eulerian subdigraph.*

1.3 Main Results

The main results in the dissertation are summarized as follows.

1. A dicycle cover of a digraph D is a family $\mathcal{F}$ of dicycles of D such that each arc of D lies in at least one dicycle in $\mathcal{F}$. We investigate the problem of determining the upper bounds for the minimum number of dicycles which cover all arcs in a strong digraph. Best possible upper bounds of dicycle covers are obtained in a number of classes of digraphs, including strong tournaments, Hamiltonian oriented graphs, Hamiltonian oriented complete bipartite graphs, and families of possibly non-hamiltonian digraphs obtained from these digraphs via a sequence of 2-sum operations.

2. Catlin in 1988 indicated that there exist graph families $\mathcal{F}$ such that if every edge e in a graph G lies in a subgraph H_e of G isomorphic to a member in $\mathcal{F}$, then G is supereulerian. In particular, if every edge of a connected graph G lies in a 3-cycle, then G is supereulerian. The purpose of this research is to investigate how Catlin's theorem can be extended to digraphs. A strong digraph D is supereulerian if D contains a spanning eulerian subdigraph. We show that there exists an infinite family of non-supereulerian strong digraphs each arc of which lies in a directed 3-cycle. We also show that there exist digraph families $\mathcal{H}$ such that a strong digraph D is supereulerian if every arc a of D lies in a subdigraph H_a isomorphic to a member of $\mathcal{H}$. A digraph D is symmetric if $(x, y) \in A(D)$ implies $(y, x) \in A(D)$; and is symmetrically connected if every pair of vertices of D are joined by a symmetric dipath. A digraph D is partially symmetric if the digraph obtained from D by contracting all symmetrically connected components is symmetrically connected. It is known that a partially symmetric digraph may not be symmetrically connected. We show that symmetrically connected digraphs and partially symmetric digraphs are such families. Sharpness of these results are discussed.

3. The 2-sum of two digraphs D_1 and D_2, denoted $D_1 \oplus_2 D_2$, is the digraph obtained from the disjoint union of D_1 and D_2 by identifying an arc in D_1 with an arc in D_2. A digraph D is **supereulerian** if D contains a spanning eulerian subdigraph. It has been noted that the 2-sum of two supereulerian (or even hamiltonian) digraphs may not be supereulerian.

We obtain several sufficient conditions on D_1 and D_2 for $D_1 \oplus_2 D_2$ to be supereulerian. In particular, we show that if D_1 and D_2 are symmetrically connected or partially symmetric, then $D_1 \oplus_2 D_2$ is supereulerian.

Chapter 2

Dicycle cover of Hamiltonian oriented graphs

2.1 The problem

We consider finite loopless graphs and digraphs, and undefined notations and terms will follow [1] for graphs and [2] for digraphs. In particular, a cycle is a 2-regular connected nontrivial graph. A cycle cover of a graph G is a collection $\mathcal{C}$ of cycles of G such that $E(G) = \bigcup_{C \in \mathcal{C}} E(C)$. Bondy [4] conjectured that if G is a 2-connected simple graph with $n \geq 3$ vertices, then G has a cycle cover $\mathcal{C}$ with $|\mathcal{C}| \leq \frac{2n-3}{3}$. Bondy [4] showed that this conjecture, if proved, would be best possible. Y. X. Luo and R. S. Chen [14] proved that this conjecture holds for 2-connected simple cubic graphs. It has been shown that for plane triangulations, serial-parallel graphs or planar graphs in general, one can have a better bound for the number of cycles used in a cover ([9–12]). Barnette [3] proved that if G is a 3-connected simple planar graph of order n, then the edges of G can be covered by at most $\frac{n+1}{2}$ cycles. Fan [6] settled this conjecture by showing that it holds for all simple 2-connected graphs. The best possible number of cycles needed to cover cubic graphs has been obtained in [13] and [15].

A directed path in a digraph D from a vertex u to a vertex v is called a (u, v)-dipath. To emphasize the distinction between graphs and digraphs, a directed cycle or path in a digraph is often referred as a dicycle or dipath. It is natural to consider the number of dicycles needed to cover a digraph. Following [2], for a digraph D, $V(D)$ and $A(D)$ denote the vertex set and arc set of D, respectively. If $A' \subseteq A(D)$, then $D[A']$ is the subdigraph induced by A'. Let K_n^*

denote the complete digraph on n vertices. Any simple digraph D on n vertices can be viewed as a subdigraph of K_n^*. If W is an arc subset of $A(K_n^*)$, then $D + W$ denotes the digraph $K_n^*[A(D) \cup W]$.

A digraph D is strong if for any distinct $u, v \in V(D)$, D has a (u, v)-dipath. As in [2], $\lambda(D)$ denotes the arc-strong-connectivity of D. Thus a digraph D is strong if and only if $\lambda(D) \geq 1$. We use (u, v) denotes an arc with tail u and head v. For $X, Y \subseteq V(D)$, we define

$$(X, Y)_D = \{(x, y) \in A(D) : x \in X \text{ and } y \in Y\}; \text{ and } \partial_D^+(X) = (X, V(D) - X)_D.$$

Let

$$d_D^+(X) = |\partial_D^+(X)|, \text{ and } d_D^-(X) = |\partial_D^-(X)|.$$

When $X = \{v\}$, we write $d_D^+(v) = |\partial_D^+\{v\}|$ and $d_D^-(v) = |\partial_D^-\{v\}|$. Let $N_D^+(v) = \{u \in V(D) - v : (v, u) \in A(D)\}$ and $N_D^-(v) = \{u \in V(D) - v : (u, v) \in A(D)\}$ denote the **out-neighbourhood** and **in-neighbourhood** of v in D, respectively. We call the vertices in $N_D^+(v)$, $N_D^-(v)$ the **out-neighbours**, **in-neighbours** of v. Thus for a digraph D, $\lambda(D) \geq 1$ if and only if for any proper non empty subset $\emptyset \neq X \subset V(D)$, $|\partial_D^+(X)| \geq 1$.

A **dicycle cover** of a digraph D is a collection $\mathcal{C}$ of dicycles of D such that $\bigcup_{C \in \mathcal{C}} A(C) = A(D)$. If D is obtained from a simple undirected graph G by assigning an orientation to the edges of G, then D is an **oriented graph**. The main purpose is to investigate the number of dicycles needed to cover a Hamiltonian oriented graph. We prove the following.

Theorem 2.1.1. *Let D be an oriented graph on n vertices and m arcs. If D has a Hamiltonian dicycle, then D has a dicycle cover $\mathcal{C}$ with $|\mathcal{C}| \leq m - n + 1$. This bound is best possible.*

In the next section, we will first show that every Hamiltonian oriented graph with n vertices and m arcs can be covered by at most $m - n + 1$ dicycles. Then we show that for every Hamiltonian graph G with n vertices and m edges, there exists an orientation $D = D(G)$ of G such that any dicycle cover of D must have at least $m - n + 1$ dicycles.

2.2 Proof of the Main Result

In this section, all graphs are assumed to be simple. We start with an observation, stated as a Lemma below. A digraph D is **weakly connected** if the underlying graph of D is connected.

Lemma 2.2.1. *A weakly connected digraph D has a dicycle cover if and only if $\lambda(D) \geq 1$.*

Proof. Suppose D has a dicycle cover $\mathcal{C}$. If D is not strong, then there exists a proper nonempty subset $\emptyset \neq X \subset V(D)$ such that $|\partial_D^+(X)| = 0$. Since D is weakly connected, D contains an arc $(u, v) \in (V(D) - X), X)_D$. Since $\mathcal{C}$ is a dicycle cover of D, there exists a dicycle $C \in \mathcal{C}$ with $(u, v) \in A(C)$. Since $(u, v) \in (V(D) - X), X)_D$, we conclude that $\emptyset \neq A(C) \cap (X, V(D) - X))_D \subseteq \partial_D^+(X)$, contrary to the assumption $|\partial_D^+(X)| = 0$. This proves that D must be strong.

Conversely, assume that D is strong. For any arc $a = (u, v) \in A(D)$, since D is strong, there must be a directed (v, u)-path P in D. It follows that $C_a = P + a$ is a dicycle of D containing a, and so $\{C_a : a \in A(D)\}$ is a dicycle cover of D. $\square$

Let C be a dicycle and let $a = (u, v)$ be an arc not in $A(C)$ but with $u, v \in V(C)$. Then $C + a$ contains a unique dicycle C_a containing a. In the following, we call C_a the fundamental dicycle of a with respect to C.

Lemma 2.2.2. *Let D be an oriented graph on n vertices and m arcs. If D has a Hamiltonian dicycle, then D has a dicycle cover $\mathcal{C}$ with $|\mathcal{C}| \leq m - n + 1$.*

Proof. Let C_0 denote the directed Hamiltonian cycle of D. For each $a \in A(D) - A(C)$, let C_a denote the fundamental dicycle of a with respect to C. Then $\mathcal{C} = \{C_0\} \cup \{C_a : a \in A(D) - A(C)\}$ is a dicycle cover of D with $|\mathcal{C}| \leq m - n + 1$. $\square$

In the following, we present some concepts and tools needed to construct such an example of Hamiltonian oriented graphs to show the inequality of Theorem 2.1.1 is best possible.

Definition 2.2.3. *Let D be a directed graph. A topological order of a digraph D is an order of vertices of $V(D) = \{v_1, v_2, ..., v_n\}$ such that for each edge $(v_i, v_j) \in E(D)$ we have $i < j$.*

Proposition 2.2.4. *A directed graph D has a topological order if and only if D is acyclic.*

The undirected graph say G will be acyclic digraph, by assigning an orientation to the edges of graph G based on Proposition 2.2.4, as the following:

- **A cyclic Algorithm:**

– **Input**: A undirected simple graph G.

– **Output**: A cyclic orientation of simple graph G.

* **Step(1):** Order the vertices of $V(G) = \{v_1, v_2, ..., v_n\}$ in the clockwise direction.
* **Step(2):** Oriented all edges of $E(v_1)$ as out edges.
* **Step(3):** Oriented all edges of $E(v_2)$ as out edges.
* **Step(4):** After oriented $E(v_1) \cup E(v_2) \cup ... \cup E(v_k)$ oriented the edges of $E(v_{k+1}) - (\bigcup_{i=1}^{k} E(v_i))$ as out edges.

Let G be a Hamiltonian graph, We need to construct an orientation D for a graph G so that $A(D(G)) - A(C)$ be a cyclic digraph. The structured way to construct such an orientation D by using the following approaches :

- **Approach (1):** Oriented a Hamilton cycle C.

- **Approach (2):** Oriented all edges of $E(G) - E(C)$ by using acyclic algorithm.

According to the approach (1), there are two ways to assign a directions to the edges of a Hamilton cycle C. Let define such of these directions as as the following:

- $D(C) :=$ Oriented the edges in the a Hamilton cycle C in the anticlockwise direction.

- $D'(C) :=$ Oriented the edges in the a Hamilton cycle C in the clockwise direction

In the following table show the difference number of dicycle cover for the Hamilton oriented graph $D(G)$ according to the orientation of a Hamilton cycle C.

n	$dcc(D(G))$	$D'(C)$	$D(C)$
5	6	5	7
6	10	8	11
7	15	11	15
8	21	15	21
9	28	19	28
10	36	25	38
11	45	29	45
12	55	35	55
13	66	42	68
14	78	48	78
15	91	55	91
16	105	63	105

To prove Theorem 2.1.1 is best possible, we need to construct, for each integer $n \geq 4$, a Hamiltonian oriented graph on n vertices and m arcs D such that any dicycle cover $\mathcal{C}$ of D must have at least $m - n + 1$ dicycles in $\mathcal{C}$.

We present a summary of our structure as Definition below. Let G be a Hamiltonian simple graph. We present a construction of such an orientation $D = D(G)$. Since G is Hamiltonian, we may assume that $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $C = v_1 v_2, \ldots, v_n v_1$ is a Hamiltonian cycle of G.

Definition 2.2.5. *Define an orientation $D = D(G)$ as follows.*
(i) Orient the edges in the Hamiltonian cycle $C = v_1 v_2, \ldots, v_n v_1$ as follows.

$$(v_{i+1}, v_i) \in A(D), \ i = 1, 2, \ldots, n - 1, (v_1, v_n) \in A(D).$$

(ii) For each $i = 2, 3, \ldots, n - 2$, and for each $j = i + 2, i + 3, \ldots n$, assign directions to edges of G not in $E(C)$ as follows.

$$(v_i, v_j) \in A(D), \ if \ v_i v_j \in E(G) - E(C) \ and \ i + 1 < j \leq n.$$

and

$$(v_1, v_j) \in A(D), \ if \ v_1 v_j \in E(G) - E(C) \ and \ i + 1 < j \leq n - 1.$$
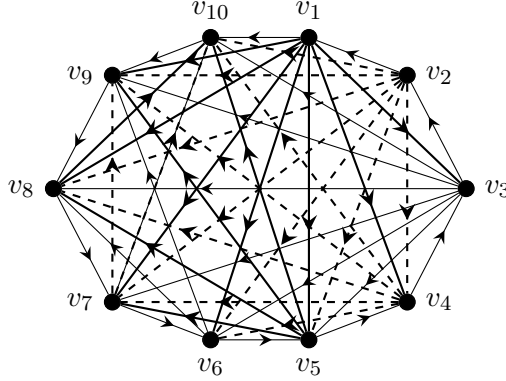
Figure 1: An orientation of a complete graph $D(K_{10})$.

We make the following observations stated in the Lemma below.

Lemma 2.2.6. *Each of the following holds for the digraph D.*

(i) The dicycle $C_0 = v_1 v_n v_{n-1} \ldots v_3 v_2 v_1$ is a Hamiltonian dicycle of D.

(ii) The digraph $D - A(C_0)$ is acyclic.

(iii) $N_D^+(v_n) = \{v_{n-1}\}$; $N_D^-(v_1) = \{v_2\}$; $N_D^-(v_2) = \{v_3\}$.

(iv) The dicycle C_0 is the only dicycle of D containing the arc (v_1, v_n).

(v) The dicycle C_0 is the unique Hamiltonian dicycle of D.

(vi) If C'' is a dicycle of D, then C'' contains at most one arc in $A(D) - A(C_0)$.

Proof. (i). Follows immediately from Definition 2.2.5 (i).

(ii). By Definition 2.2.5, the labels of the vertices $V(D) = \{v_1, v_2, \ldots, v_n\}$ satisfies $(v_i, v_j) \in A(D) - A(C_0)$ only if $i < j$. It follows (for example, Section 2.1 of [2]) that $D - A(C_0)$ is acyclic, and so (ii) holds.

(iii). This follows immediately from Definition 2.2.5.

(iv). Let C' be a dicycle of D with $(v_1, v_n) \in A(C')$. Since $(v_1, v_n) \in A(C') \cap A(C_0)$, we choose the largest label $i \leq n$, such that $(v_1, v_n), (v_n, v_{n-1}), \ldots, (v_{i+1}, v_i) \in A(C') \cap A(C_0)$. Since $C' \neq C_0$, we have $i \geq 3$. Since C' is a dicycle, there must be a vertex $v_j \in V(D)$ such that $(v_i, v_j) \in A(C')$. By the choice of i, we must have $(v_i, v_j) \notin A(C_0)$, and so $(v_i, v_j) \in A(D) - A(C_0)$. By Definition 2.2.5 (ii), we have $i + 2 \leq j \leq n$, contrary to the fact that C' is a dicycle of D containing (v_1, v_n). This proves (iv).

(v). Let C' be a Hamiltonian dicycle of D. Since $V(C') = V(D)$, we have $v_n \in V(C')$. We claim that $(v_1, v_n) \in A(C')$. If $(v_1, v_n) \notin A(C')$, then there exists a $v_i \in V(C)$ ($i \in \{v_2, v_3, \ldots, v_{n-1}\}$) such that $(v_i, v_n) \in A(C')$. Hence, $(v_i, v_n), (v_n, v_{n-1}), \ldots, (v_{i+2}, v_{i+1}) \in A(C')$. By Definition 2.2.5 (i) and (ii), $N^+(v_{i+1}) \subset \{v_{i+2}, v_{i+3}, \ldots, v_n\}$, contrary to the fact that C' is a Hamiltonian

dicycle of D. Thus, $(v_1, v_n) \in A(C')$. It follows by Lemma 2.2.6 (iv) that we must have $C' = C_0$,

(vi). By contradiction, we assume that D has a dicycle C'' which contains two arcs $a_1, a_2 \in A(D) - A(C_0)$. Since $V(D) = \{v_1, v_2, \ldots, v_n\}$, we assume that $a_1 = (v_i, v_{i'})$ and $a_2 = (v_j, v_{j'})$. Without lost of generality and by Lemma 2.2.5, we further assume that $1 \leq i < j < n$. Let $i \geq t \geq 1$ be the smallest integer such that $v_t \in V(C'')$. Since C'' is a dicycle of D, there must be a $v_s \in V(C'')$ such that $(v_s, v_t) \in A(C'')$. By Definition 2.2.5, either $(v_s, v_t) \in A(C_0)$ and $s = t + 1 < j$ or $(v_s, v_t) \in A(D) - A(C_0)$ and $1 < s + 1 < t$. By the choice of t, we can only have $s = t + 1$ and $(v_{t+1}, v_t) \in A(C'') \cap A(C_0)$. Choose the largest integer h with $t + 1 \leq h < j$ such that $(v_{t+1}, v_t), (v_{t+2}, v_{t+1}), \ldots, (v_h, v_{h-1}) \in A(C'') \cap A(C_0)$. Since C'' is a dicycle, there must be a v_k with $1 \leq k \leq n$ such that $(v_k, v_h) \in A(C'')$. By the maximality of h and by Definition 2.2.5 (i), we conclude that $(v_k, v_h) \notin A(C_0)$. By Definition 2.2.5 (ii), $1 \leq k \leq h - 2$. By the minimality of t, we must have $t \leq k \leq h - 2$. It follows by $j > h$ that C'' cannot contain $a_2 = (v_j, v_{j'})$, contrary to the assumption. This contradiction justifies (vi). $\square$

To complete the proof of Theorem 2.1.1, we present the next Lemma.

Lemma 2.2.7. *Let G be a Hamiltonian simple graph. There exists an orientation $D = D(G)$ such that every dicycle cover of D must have at least $m - n + 1$ dicycles.*

Proof. Let G be a Hamiltonian graph and let $D = D(G)$ be the orientation of G given in Definition 2.2.5. For notational convenience, we adopt the notations in Definition 2.2.5 and denote $V(D) = \{v_1, v_2, \ldots, v_n\}$. Thus by Lemma 2.2.6 (v), $C_0 = v_1 v_n v_{n-1} \ldots v_2 v_1$ is the unique Hamiltonian dicycle of D.

Let $\mathcal{C}$ be a dicycle cover of D. By Lemma 2.2.6 (iv), we must have $C_0 \in \mathcal{C}$. For each arc $a \in A(D) - A(C_0)$, since $\mathcal{C}$ is a dicycle cover of D, there must be a dicycle $C(a) \in \mathcal{C}$ such that $a \in A(C(a))$. By Lemma 2.2.6 (vi), $A(C(a)) \cap A(D) - A(C_0) = \{a\}$. It follows that if $a, a' \in A(D) - A(C_0)$, then $a \neq a'$ implies $C(a) \neq C(a')$ in $\mathcal{C}$. Thus we have $\{C(a) \mid a \in A(D) - A(C_0)\} \subseteq \mathcal{C}$. Hence

$$|\mathcal{C}| \geq |\{C(a) \ : \ a \in A(D) - A(C_0)\} \cup \{C_0\}| = m - n + 1.$$

This proves the Lemma. $\square$

By Lemmas 2.2.2 and 2.2.7, Theorem 2.1.1 follows.

We are about to show that Theorem 2.1.1 can be applied to obtain dicycle cover bounds for certain families of oriented graphs.

2.2.1   Tournament

Definition 2.2.8. *A tournament is an orientation of a complete graph K_n. An alternate definition for a tournament is for any two distinct vertices $u, v \in V(K_n)$ one of these arcs belong to $E(K_n)$ either $(u, v) \in E(K_n)$ or $(v, u) \in E(K_n)$ not all.*

Let T_n denote a tournament of order n. Then T_n is an oriented graph.

Lemma 2.2.9. *If T_n is k-strong, then every length $(k-1)$ dipath can be extended to a Hamiltonian dicycle.*

Proof. For $k = 1$, Camion [5, 7] proved that every strong tournament is Hamiltonian.
For $k = 2$, we have a counterexample T_6 show that T_6 is a 2- strong tournament , but there exist an arc $a \in A(T_6)$ lies between the two vertices v_4 and v_6, can not be extended to a Hamilton dicycle. See Figure 2 .
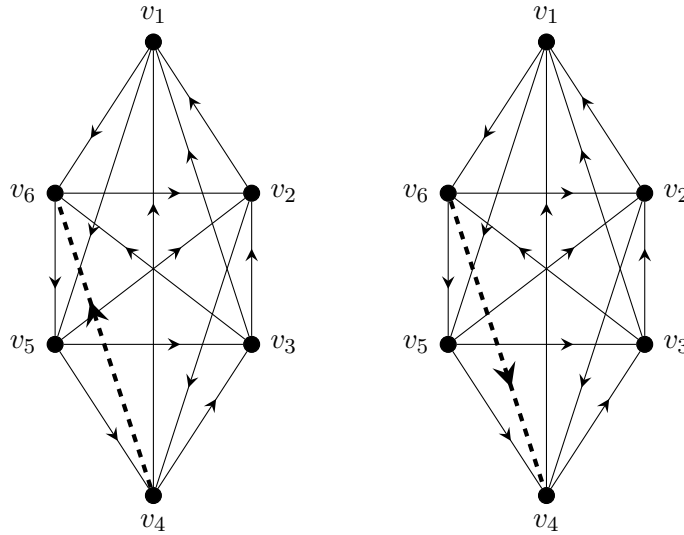

Figure 2: An arc $a \in A(T_6)$ can not extend to a Hamiltonian dicycle.

For $k = 3$, we have a counterexample T_7 show that T_7 is a 3- strong tournament , but there exist a dipath $P = \{(x, y), (y, z)\}$ with length 2, can not be extended to a Hamilton dicycle. See Figure 3.
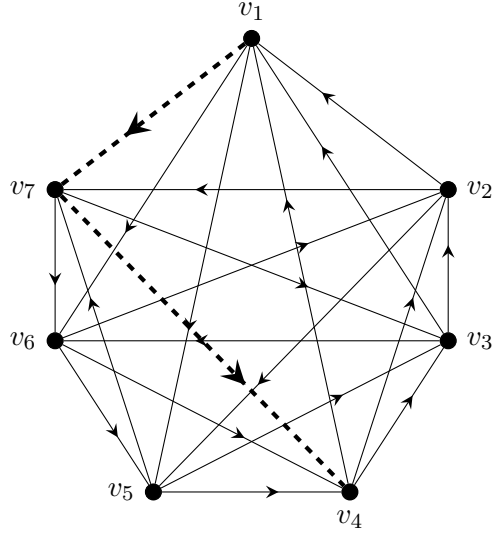
Figure 3: A dipath $P = \{(v_1, v_7), (v_7, v_4\}$ in T_7 can not extend to a Hamiltonian dicycle.

$\square$

Due to Camion Theorem, hence the corollary below follows from Theorem 2.1.1.

Corollary 2.2.10. *Every strong tournament on n vertices has a dicycle cover $\mathcal{C}$ with $|\mathcal{C}| \leq \dfrac{n(n-1)}{2} - n + 1$. This bound is best possible.*

2.2.2 A Complete Bipartite

Definition 2.2.11. *A bipartite graph G with vertex bipartition (A, B) is balanced if $|A| = |B|$.*

Theorem 2.2.12. *A bipartite graph G with vertex bipartition (A, B), has a Hamiltonian cycle if and only if G is balanced.*

Let $K_{m,n}$ be a complete bipartite graph with vertex bipartition (A, B) and $|A| = m, |B| = n$, then $K_{m,n}$ has Hamiltonian cycle if and only if $m = n \geq 2$, that is, $K_{m,n}$ is balanced. Let $K_{n,n}$ denote a balanced complete bipartite graph.

Corollary 2.2.13. *Every Hamiltonian orientation of balanced complete bipartite graph $K_{n,n}$ has a dicycle cover $\mathcal{C}$ with $|\mathcal{C}| \leq (n-1)^2$. This bound is best possible.*

Lemma 2.2.14. *Let $K_{n,n}$ be a Hamiltonian orientation of balanced bipartite on $2n$, then D has a dicycle cover $\mathcal{C}$ with $|\mathcal{C}| \leq (n-1)^2$.*

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Proof. Since a Hamiltonian orientation of balanced bipartite have $m = n^2$ and so by following Lemma 2.2.2 we have $|\mathcal{C}| \leq m - n + 1 \leq n^2 - 2n + 1 = (n-1)^2$. $\hfill\square$

To prove Corollary 2.2.13 is best possible, we have to construct, for $n \geq 3$ a Hamiltonian oriented balanced bipartite graph on $2n$ such that any dicycle cover $\mathcal{C}$ of $K_{n,n}$ must have at least $(n-1)^2$ dicycle cover in $\mathcal{C}$.

The structured way to construct an orientation for a Hamiltonian oriented balanced bipartite graph on $2n$ to be acyclic digraph by using the following steps :

- **Step (1):** Oriented a Hamilton cycle C.

- **Step (2):** Oriented all edges of $E(K_{n,n}) - E(C)$ by using acyclic algorithm.

According to the step (1), in the following table show the difference number of dicycle cover for the Hamiltonian oriented balanced bipartite graph $K_{n,n}$ according to the orientation of a Hamilton cycle C.

n	$dcc(K_{n,n})$	$D'(C)$	$D(C)$
4	9	7	9
5	16	12	16
6	25	17	25
7	36	24	36
8	49	31	49
9	64	40	64

Now we can summarizing these steps into the following Definition. Let $k_{n,n}$ be a Hamiltonian balanced bipartite graph. We construct an orientation $D = D(k_{n,n})$. Since $k_{n,n}$ is Hamiltonian balanced bipartite , we may assume that $A = \{u_1, u_2, \cdots, u_n\}$, $B = \{v_1, v_2, \cdots, v_n\}$ and $C = \{u_1 v_1 u_2 v_2 \cdots u_n v_n u_1\}$ is a Hamiltonian cycle of $k_{n,n}$.

Definition 2.2.15. *We define an orientation $D = D(k_{n,n})$ as following*
(i) Orient the edges in the Hamiltonian dicycle $C = \{u_1 v_1 u_2 v_2 \cdots u_n v_n u_1\}$ as follows:

$$(v_i, u_i) \in A(D) \ for \ i = 1, 2, \cdots n.$$

$$(u_{i+1}, v_i) \in A(D) \ for \ i = 1, 2, \cdots n - 1.$$

and

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$$(u_1, v_n) \in A(D).$$

(ii)We orient to edges of $k_{n,n}$ not in $E(C)$ as follows:

$$(u_j, v_{i+j}) \in A(D) \text{ for } i = 1, 2, \cdots n - j; \text{ and } j = 1, 2, \cdots n - 1.$$

and

$$(v_l, u_{i+(l+1)}) \in A(D) \text{ for } i = 1, 2, \cdots n - (l+1); \text{ and } l = 1, 2, \cdots n - 2.$$
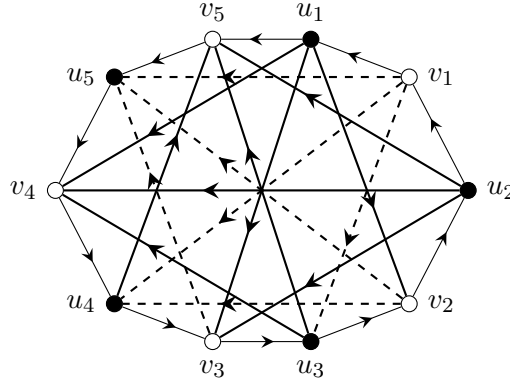

Figure 4: An orientation of a complete bipartite $D(K_{5,5})$.

Remark 2.2.16. *Since the orientation $D = D(k_{n,n})$ of a Hamiltonian of balanced bipartite graph is similar to the orientation of Definition 2.2.5, then lemma 2.2.6 is still hold for $D = D(k_{n,n})$.*

To complete proof of Corollary 2.2.13, we present the following lemma. Thus Corollary 2.2.13 follows from lemmas 2.2.14 and 2.2.17.

Lemma 2.2.17. *Let $k_{n,n}$ be a Hamiltonian of balanced bipartite graph with $n \geq 2$. There exist an orientation $D = D(k_{n,n})$ such that every dicycle cover of D must have at least $(n-1)^2$ dicycles.*

Proof. Let $k_{n,n}$ be a Hamiltonian of balanced bipartite graph and let $D = D(k_{n,n})$ be the orientation of $k_{n,n}$ given in Definition 2.2.15. By following Lemma 2.2.7, a Hamiltonian orientation of balanced bipartite have $m = n^2$ and hence $|\mathcal{C}| \geq m - n + 1 \geq n^2 - 2n + 1 = (n-1)^2$. $\qquad \square$

2.3 Dicycle Covers of 2-Sums of Digraphs

In this section, we shall show that Theorem 2.1.1 can also be applied to certain non-hamiltonian digraphs which can be built via 2-sums. We start with 2-sums of digraphs.

Definition 2.3.1. Let $D_{n_1} = (V(D_{n_1}), A(D_{n_1}))$ and $D_{n_2} = (V(D_{n_2}), A(D_{n_2}))$ be two disjoint digraphs, $a_1 = (v_{12}, v_{11}) \in A(D_{n_1})$ and $a_2 = (v_{22}, v_{21}) \in A(D_{n_2})$. The 2-sum $D_{n_1} \oplus_2 D_{n_2}$ of D_{n_1} and D_{n_2} is obtained from the union of D_{n_1} and D_{n_2} by identifying the arcs a_1 and a_2, that is, $v_{11} = v_{21}$ and $v_{12} = v_{22}$.

Definition 2.3.2. Let $D_{n_1}, D_{n_2}, \ldots, D_{n_s}$ be s disjoint digraphs with $n_1, n_2, \ldots, n_s$ vertices, respectively. Let $D_{n_1} \oplus_2 D_{n_2} \oplus_2 \ldots \oplus_2 D_{n_s}$ denote a sequence of 2-sum of $D_{n_1}, D_{n_2}, \ldots, D_{n_s}$, that is, $(((D_{n_1} \oplus_2 D_{n_2}) \oplus_2 D_{n_3}) \oplus_2 \ldots) \oplus_2 D_{n_s}$.

Theorem 2.3.3. Let $D_{n_1}, D_{n_2}, \ldots, D_{n_s}$ be s disjoint Hamiltonian oriented graphs on $n_1, n_2, \ldots, n_s$ vertices and $m_1, m_2, \ldots, m_s$ arcs, respectively, and let $D = D_{n_1} \oplus_2 D_{n_2} \oplus_2 \ldots \oplus_2 D_{n_s}$. Then D has a dicycle cover $\mathcal{C}$ with $|\mathcal{C}| \leq |A(D)| - |V(D)| + 1$. This bound is best possible.

Proof. By Theorem 2.1.1, D_{n_i} $(i = 1, 2, \ldots s)$ has a dicycle cover $\mathcal{C}_i$ with $|\mathcal{C}_i| \leq m_i - n_i + 1$. Let $\mathcal{C} = \bigcup_{i=1}^{s} \mathcal{C}_i$. Then $|\mathcal{C}| \leq (m_1 - n_1 + 1) + (m_2 - n_2 + 1) + \ldots + (m_s - n_s + 1) = (m_1 + m_2 + \ldots + m_s) - (n_1 + n_2 + \ldots + n_s) + s = (m_1 + m_2 + \ldots + m_s - (s-1)) - (n_1 + n_2 + \ldots + n_s - 2(s-1)) + 1 = |A(D)| - |V(D)| + 1$. By Definition 2.3.2, $\mathcal{C}$ is a dicycle cover of D. Thus, D has a dicycle cover $\mathcal{C}$ with $|\mathcal{C}| \leq |A(D)| - |V(D)| + 1$.

Let G_{n_i} be s disjoint Hamiltonian simple graphs for $i \in \{1, 2, \ldots, s\}$. We may assume that $V(G_{n_i}) = \{v_{i1}, v_{i2}, \ldots, v_{in_i}\}$ and $C_i = v_{i1}v_{i2} \ldots v_{in_i}v_{i1}$ is a Hamiltonian cycle of G_{n_i}, and let

$$D_{n_i} = D(G_{n_i}) \text{ be the orientation of } G_{n_i} \text{ given in Definition 2.2.5.} \qquad (2.1)$$

For notational convenience, we adopt the notations in Definition 2.2.5 and denote $V(D_{n_i}) = \{v_{i1}, v_{i2}, \ldots, v_{in_i}\}$. Thus by Lemma 2.2.6 (v), $C_{i_0} = v_{i1}v_{in_i} \ldots v_{i2}v_{i1}$ is the unique Hamiltonian dicycle of D_{n_i}. Let $a_i = (v_{i2}, v_{i1})$ is an arc of D_{n_i}. We construct the 2-sum digraph $D_{n_1} \oplus_2 D_{n_2} \oplus_2 \ldots \oplus_2 D_{n_s}$ from the union of $D_{n_1}, D_{n_2}, \ldots, D_{n_s}$ by identifying the arcs $a_1, a_2, \ldots, a_s$ such that $v_{11} = v_{21} = \ldots = v_{s1}$, $v_{12} = v_{22} = \ldots = v_{s2}$. We assume that $v_1 := v_{11} = v_{21} = \ldots = v_{s1}$, $v_2 := v_{12} = v_{22} = \ldots = v_{s2}$. (The case when $s = 2$ is depicted in Figure 5).
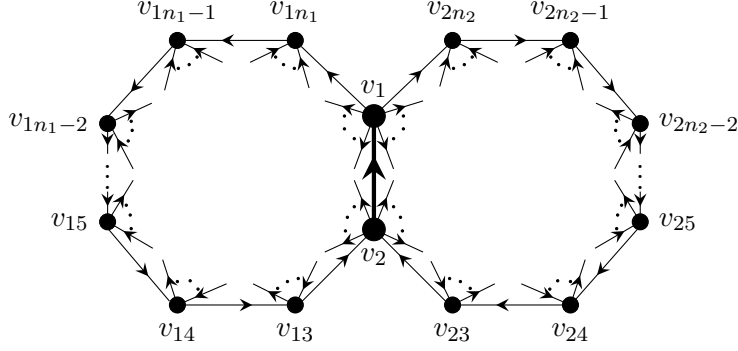
Figure 5. The 2-sum digraph for D_{n_1} and D_{n_2}.

Claim 1. There does not exist a dicycle whose arcs intersect arcs in two or more D_{n_i}'s, $(i = 1, 2, \ldots, s)$.

By Definition 2.3.1, we have $V(D_{n_i}) \cap V(D_{n_j}) = \{v_1, v_2\}(i \neq j)$. Without loss of generality, we consider oriented graphs D_{n_1} and D_{n_2}, suppose that there exists a dicycle C_0 such that

$$\{A(C_0) - (v_2, v_1)\} \cap A(D_{n_1}) \neq \emptyset \ \ and \ \ \{A(C_0) - (v_2, v_1)\} \cap A(D_{n_2}) \neq \emptyset.$$

Thus, there must exist four different arcs

$$\{(v_{1i'}, v_1), (v_1, v_{2i''}), (v_{2j''}, v_2), (v_2, v_{1j'})\} \in A(C_0) \tag{2.2}$$

with $(v_{1i'}, v_1), (v_2, v_{1j'}) \in A(D_{n_1})$ and $(v_1, v_{2i''}), (v_{2j''}, v_2) \in A(D_{n_2})$, as shown in Figure 6; or four different arcs

$$\{(v_{1s'}, v_2), (v_2, v_{2s''}), (v_{2k''}, v_1), (v_1, v_{1k'})\} \in A(C_0) \tag{2.3}$$

with $(v_{1s'}, v_2), (v_1, v_{1k'}) \in A(D_{n_1})$ and $(v_2, v_{2s''}), (v_{2k''}, v_1) \in A(D_{n_2})$, as shown in Figure 7.

By the Definition 2.3.1, Lemma 2.2.6 (iii) and (1), we have $N_D^-(v_1) = \{v_2\}$, and so $v_{1i'} = v_2$ or $v_{2k''} = v_2$, contrary to the assumption that C_0 is a dicycle. This proves Claim 1.

By Claim 1, for every dicycle C in D, all arcs in C (except for the arc (v_2, v_1)) belong to exactly one of oriented graphs $D_{n_i}(i = 1, 2, \ldots, n)$. By Definition 2.2.5 and Lemma 2.2.7, every dicycle cover of oriented graph $D_{n_i}(i = 1, 2, \ldots, n)$ must have at least $m_i - n_i + 1$ dicycles. This completes the proof. $\qquad\square$
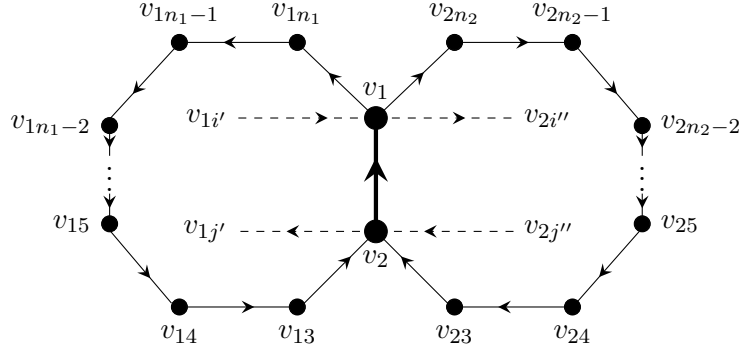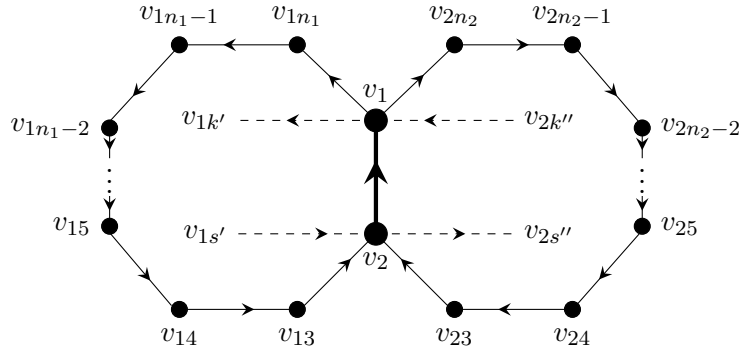
Figure 6

Figure 7

By Corollary 2.2.10 and Theorem 2.3.3, we have the following Corollary.

Corollary 2.3.4. *Let $D_{n_1}, D_{n_2}, \ldots, D_{n_s}$ be s disjoint strong tournaments with $n_1, n_2, \ldots, n_s$ vertices, respectively. Then $D_{n_1} \oplus_2 D_{n_2} \oplus_2 \ldots \oplus_2 D_{n_s}$ has a dicycle cover $\mathcal{C}$ with $|\mathcal{C}| \leq (\frac{n_1(n_1-1)}{2} + \frac{n_2(n_2-1)}{2} + \ldots + \frac{n_s(n_s-1)}{2}) - (n_1 + n_2 + \ldots + n_s) + s$. This bound is best possible.*

Let G_n be a Hamiltonian graph with n vertices and m arcs, let $D_n^i (i$ is an integer $)$ denote a Hamiltonian orientation of G_n. For a positive integer s, let $H(G_n, s)$ denote the family of all 2-sum generated digraphs $D_n^1 \oplus_2 D_n^2 \oplus_2 \ldots \oplus_2 D_n^s$, as well as a member in the family (for notational convenience). By the definition of $H(G_n, s)$, we have $H(G_n, 1) = D_n^1$, $H(G_n, s) = H(G_n, s-1) \oplus_2 D_n^s$. The conclusions of next corollaries follow from Theorem 2.1.1. The sharpness of these corollaries can be demonstrated using similar constructions displayed in Lemma 2.2.7 and Corollary 2.2.13.

Corollary 2.3.5. *Let $m, n \geq 3$ be integer, G_n be a Hamiltonian graph with n vertices and m edges, and K_n be a complete graph on $n \geq 3$ vertices.*

(i) Any member in $H(G_n, s)$ has a dicycle cover $\mathcal{C}$ with $|\mathcal{C}| \leq s(m - n + 1)$. This bound is best possible.

(ii) In particular, any $H(K_n, s)$ has a dicycle cover $\mathcal{C}$ with $|\mathcal{C}| \leq s(\frac{n(n-1)}{2} - n + 1)$. This bound is best possible.

Corollary 2.3.6. *Let $m, n \geq 3$ be integer, B_n be a Hamiltonian bipartite graph with $2n$ vertices and m edges, and $K_{n,n}$ be a complete bipartite graph.*

(i) Any $H(B_n, s)$ has a dicycle cover $\mathcal{C}$ with $|\mathcal{C}| \leq s(m - 2n + 1)$. This bound is best possible.

(ii) In particular, any $H(K_{n,n}, s)$ has a dicycle cover $\mathcal{C}$ with $|\mathcal{C}| \leq s(n - 1)^2$. This bound is best possible.

Chapter 3

Supereulerian Digraphs with given local structures

3.1 Introduction

We consider finite graphs and digraphs. Undefined terms and notations will follow [1] for graphs and [2] for digraphs. As in [2], (u, v) represents an arc oriented from a vertex u to a vertex v. A digraph D is **simple** if D has no loops and if for any pair of distinct vertices $u, v \in V(D)$, there is at most one arc in D oriented from u to v. For an integer $n > 0$, we use K_n^* to denote the complete digraph on n vertices. Hence for every pair of distinct vertices $u, v \in V(K_n^*)$, there is exactly one arc (u, v) in $A(K_n^*)$. A cycle on n vertices is often called an n-cycle. For a digraph D, the underlying graph of D, denoted by $G(D)$, is obtained from D by erasing the orientations of all arcs of D. A **ditrail** in D is an alternating sequence of vertices and arcs such that all the arcs are distinct. If all the vertices of a ditrail are distinct we call it a **dipath**. An arc $(u, v) \in A(D)$ is **symmetric** in D if $(u, v), (v, u) \in A(D)$. A digraph D is **symmetric** if $|V(D)| = 1$ or if $|A(D)| > 0$ and every arc of D is symmetric. Especially, a symmetric dipath P is a dipath such that every arc of P is symmetric. Following [2], if $X, Y \subseteq V(D)$, then define

$$(X, Y)_D = \{(x, y) \in A(D) : x \in X, y \in Y\}.$$

When $Y = V(D) - X$, we define

$$\partial_D^+(X) = (X, V(D) - X)_D \text{ and } \partial_D^-(X) = (V(D) - X, X)_D.$$

If $X \subseteq V(D) \cup A(D)$, then $D[X]$ denotes the subdigraph induced by X. If S is a subdigraph of a digraph D and if $X \subset A(S)$ and $Y \subseteq A(D) - A(S)$, we use $S - X + Y$ to denote $D[(A(S) - X) \cup Y]$.

For a vertex $v \in V(D)$, the **out-degree** of v is $d_D^+(v) = |\partial_D^+(\{v\})|$ and the **in-degree** of v in D is $d_D^-(v) = |\partial_D^-(\{v\})|$. Furthermore, we define

$$\partial_D(v) = \partial_D^+(\{v\}) \cup \partial_D^-(\{v\}) \text{ and } d_D(v) = d_D^+(v) + d_D^-(v).$$

Let $N_D^+(v) = \{u \in V(D) : (v,u) \in A(D)\}$ and $N_D^-(v) = \{u \in V(D) : (u,v) \in A(D)\}$ denote the **out neighbors** and **in neighbors** of v in D, respectively. When the digraph D is understood from the context, we often omit the subscript D. Following [1] and [2], we use $\lambda(D)$ and $c(D)$ to denote the arc-strong connectivity of D and the number of components of the underlying graph $G(D)$ of D, respectively. By the definition of $\lambda(D)$ in [2], for any integer $k \geq 0$,

$$\lambda(D) \geq k \text{ if and only if for any nonempty proper subset } X \subset V(D), |\partial_D^+(X)| \geq k. \tag{3.1}$$

Boesch, Suffel, and Tindell [30] in 1977 proposed the supereulerian problem, which seeks to characterize graphs that have spanning eulerian subgraphs, and they indicated such that problem would be very difficult. Pulleyblank [26] later in 1979 proved that determining whether a graph is supereulerian, even within planar graphs, is NP-complete. Since then, there have been lots of researches on this topic. Catlin [20] in 1992 presented the first survey on supereulerian graphs. Later Chen et al [21] gave an update in 1995, specifically on the reduction method associated with the supereulerian problem. A recent survey on supereulerian graphs is given in [24].

It is natural to consider the supereulerian problem in digraphs. A strong digraph D is **eulerian** if for each vertex $v \in V(D)$, we have $d_D^+(v) = d_D^-(v)$. A strong digraph D is **supereulerian** if D contains a spanning eulerian subdigraph. One of the central problems is to characterize supereulerian digraphs.

Theorem 3.1.1. *(Jaeger [23] and Catlin [19]) Every 4-edge-connected graph is supereulerian.*

Theorem 3.1.2. *(Catlin, Corollary 1 of [19]) there exist graph families $\mathcal{F}$ such that if every edge of a connected graph G lies in a subgraph of G isomorphic to a member in $\mathcal{F}$, then G is supereulerian. In particular, if every edge of G lies in a 3-cycle of G, then G is supereulerian.*

The purpose of this paper is to see if the above-mentioned results of Jaeger and Catlin can be extended to digraphs. Firstly, we shall show that both Theorem 3.1.1 and Theorem 3.1.2 cannot be directly extended to digraphs. To be more precise, we in the next section will show that for any integer $k > 0$, there exists infinitely many non-supereulerian digraphs D with $\lambda(D) \geq k$. To show that Theorem 3.1.2 cannot be extended to digraphs, we need more concepts. A digraph D is **weakly connected** if $G(D)$, its underlying graph, is connected. Let $\mathcal{H}$ be a

family of digraphs. A strong digraph D is **locally $\mathcal{H}$** if every arc $a \in A(D)$ lies in a subdigraph H_a of D, where $H_a \in \mathcal{H}$. For convenience, we also call a locally $\{H\}$ digraph D as a **locally H** digraph. Let C_3 denote a directed 3-cycle. Unlike graphs, we in Section 2 will also show that a strong and locally C_3 digraph may not be supereulerian. Thus what local structures will assure supereulerian property will be the objective of this research. We will introduce the graph families of symmetrically connected digraphs and partially symmetric digraphs, in subsequent sections below, and prove the following results.

Theorem 3.1.3. *Each of the following holds.*
(i) Every symmetrically connected digraph is supereulerian.
(ii) Every partially symmetric digraph is supereulerian.

Moreover, in the Sections 3 and 4, we will show that every weakly connected locally symmetrically connected digraph is supereulerian and every weakly connected locally partially symmetric digraph is supereulerian. The sharpness of these results are also discussed.

3.2   Examples of nonsupereulerian strong digraphs

As Catlin in [19] indicated that

> every connected graph in which every edge lies in a 3-cycle is supereulerian,   (3.2)

it is natural to see if every strong digraph in which every arc lies in a directed 3-cycle is supereulerian. In this section, we shall present, for any integer $k > 0$, an infinite family of $\mathcal{D}$ such that every digraph in $\mathcal{D}$ is locally $\{C_3\}$ with $\lambda(D) \geq k$ but nonsupereulerian.

We need the following necessary condition for a digraph to be supereulerian. Let D be a digraph and $U \subset V(D)$. We call a collection of ditrails $P_1, P_2, \cdots, P_t$ of the induced subdigraph $D[U]$ a **cover** of U if $\cup_{i=1}^{t} V(P_i) = U$ and $A(P_i) \cap A(P_j) = \emptyset$, whenever $i \neq j$. The minimum value of such t is denoted by $\tau(U)$. For any subset $A \subseteq V(D) - U$, define $B =: V(D) - U - A$. Let

$$h(U, A) =: \min\{|\partial_D^+(A)|, |\partial_D^-(A)|\} + \min\{|(U, B)_D|, |(B, U)_D|\} - \tau(U).$$

Then we have the following proposition.

Proposition 3.2.1. *(Hong, Lai and Liu, Proposition 2.1 of [34]) If D has a spanning eulerian subdigraph, then for any $U \subset V(D)$, and for any subset $A \subseteq V(D) - U$, we have $h(U, A) \geq 0$.*

The authors in [17] and [34] have independently presented infinite families of non supereulerian digraphs with arbitrarily high arc-strong connectivity. In those digraphs shown in [17] and [34], there exist some arcs which are not lying in a directed 3-cycle. In this section, we will construct an infinite family of non supereulerian digraphs with arbitrarily high arc-strong connectivity such that every arc of each of these digraphs lies in a directed 3-cycle.
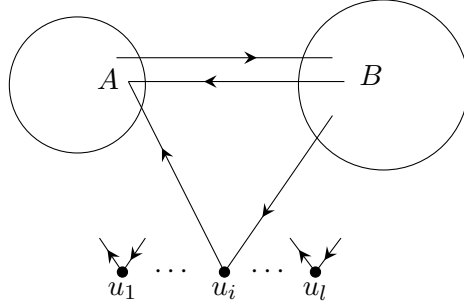

Figure 1. The digraph $D = D(\alpha, \beta, k, \ell)$.

Example 3.2.2. *Let $\alpha, \beta, k > 0$ be integers with $\alpha, \beta \geq k + 1$, and let A and B be two disjoint set of vertices with $|A| = \alpha$ and $|B| = \beta$. Let $\ell \geq \alpha\beta + 1$ be an integer, and U be a set of vertices disjoint from $A \cup B$ with $|U| = \ell$. We construct a digraph $D = D(\alpha, \beta, k, \ell)$ such that $V(D) = A \cup B \cup U$ and the arcs of D are given as required in (D1) and (D2) below. (See Figure 1).*
(D1) $D[A \cup B] \cong K^*_{\alpha+\beta}$ is a complete digraph.
(D2) For every vertex $u \in U$, and for every $v \in A$, $(u, v) \in A(D)$ and for every $w \in B$, $(w, u) \in A(D)$. Thus for any $u \in U$, we have $N_D^+(u) = A$ and $N_D^-(u) = B$. No two vertices in U are adjacent.

A digraph D is **quasitransitive** if, for every triple of distinct vertices $x, y, z \in V(D)$, with $(x, y), (y, z) \in A(D)$, there is at least one arc between x and z. Thus the digraphs in Example 3.2.2 are quasitransitive. Using canonical decompositions of quasitransitive digraphs by Bang-Jensen and Huang in [16], Bang-Jensen and Maddaloni characterized supereulerian quasitransitive digraphs and further showed that there exists a polynomial algorithm to determine if a quasitransitive digraph is supereulerian in [17].

Proposition 3.2.3. *Let $D = D(\alpha, \beta, k, \ell)$ for some given parameters α, β, k and ℓ as defined in Example 3.2.2. Then each of the following holds.*
(i) $\lambda(D) > k$.
(ii) Every arc of D lies in a directed 3-cycle.
(iii) D is not supereulerian.

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Proof. (i) We use (3.1) to show (i). Let $\emptyset \neq X \subset V(D)$ be a proper nonempty subset. Let $n_X = |X \cap (A \cup B)|$. If $0 < n_X < \alpha + \beta$, then by Example 3.2.2(D1), $\partial_D^+(X) \geq |(X \cap U, A - X)_D \cup (X \cap (A \cup B), (A \cup B) - X)_D| \geq n_X(\alpha + \beta - n_X) \geq \alpha + \beta - 1 \geq 2k + 1 > k$. Hence we assume that either $A \cup B \subseteq X$ or $X \cap (A \cup B) = \emptyset$. If $X \cap (A \cup B) = \emptyset$, then $X \subseteq U$, and so by Example 3.2.2(D2), $|\partial_D^+(X)| \geq |(X, A)_D| \geq |A| > k$. Thus we may assume that $A \cup B \subseteq X$. Then $|\partial_D^+(X)| = |(B, U - X)_D| \geq |B| > k$. Hence by (1.1), $\lambda(D) > k$.

(ii) Let $a = (u, v)$ be an arc in D. If $u, v \in A \cup B$, then by Example 3.2.2(D1), a is in a $K_{\alpha+\beta}^*$ and so a lies in a directed 3-cycle of D. Since U is an independent set of D, by Example 3.2.2(D2), we may assume that $u \in B$ and $v \in U$, whence for any $w \in A$, $wuvw$ is a directed 3-cycle; if $u \in U$ and $v \in A$, then for any $w' \in B$, $w'uvw'$ is a directed 3-cycle. This justifies (ii).

(iii) We apply Proposition 3.2.1. By (D1), $D[A \cup B] \cong K_{\alpha+\beta}^*$, and so $|\partial_D^+(A)| = \alpha\beta$. By (D2), $|(U, B)_D| = 0$ and so $\tau(U) = |U| > \alpha\beta$. It follows that

$$h(U, A) = |\partial_D^+(A)| + |(U, B)_D| - \tau(U) = \alpha\beta - |U| < 0.$$

It follows from Proposition 3.2.1 that D is not supereulerian. $\square$

Thus Example 3.2.2 indicates that there exists an infinite family of non supereulerian digraphs with arbitrarily high arc-strong connectivity such that every arc of each of these digraphs lies in a directed 3-cycle. Hence both Theorem 3.1.1 and Theorem 3.1.2 cannot be directly extended to digraphs.

3.3 Locally symmetrically connected supereulerian digraphs

In this section, we will introduce symmetrically connected digraphs and show that every locally symmetrically connected digraph is supereulerian. We will also show that this result is best possible in some sense.

Definition 3.3.1. *Let D be a digraph such that either $D = K_1$ or $A(D) \neq \emptyset$. If for any $u, v \in V(D)$, D contains a symmetric dipath from u to v, then D is called a **symmetrically connected** digraph. Let $\mathcal{SC}$ be the family of all symmetrically connected digraphs.*

Theorem 3.3.2. *Every symmetrically connected digraph is supereulerian.*

Proof. By contradiction, we assume that D is not supereulerian. By Definition 3.3.1 and (3.1), D is strong. Thus D contains a nontrivial eulerian subdigraph. Choose S to be an eulerian

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subdigraph of D such that

$$|V(S)| \text{ is maximized among all eulerian subdigraph of } D. \qquad (3.3)$$

Since D is not supereulerian, we have $|V(S)| < |V(D)|$. Pick a vertex $u \in V(D) - V(S)$ and $v \in V(S)$. As $u, v \in V(D)$, and D is symmetrically connected, D contains a symmetric dipath $P = v_0 v_1 \cdots v_m$ with $v_0 = u$ and $v_m = v$. Since $u \in V(D) - V(S)$ and $v \in V(S)$, there exists a smallest integer $t > 0$ such that $v_t \in V(S)$. Since P is symmetric, the arcs $(v_1, v_0), (v_2, v_1), \cdots, (v_t, v_{t-1}) \in A(D)$. It follows that $D[A(S) \cup \{(v_0, v_1), (v_1, v_2), \cdots, (v_{t-1}, v_t), (v_1, v_0), (v_2, v_1), \cdots, (v_t, v_{t-1})\}]$ is also an eulerian subdigraph of D, contrary to (3.3). $\qquad \square$

The **symmetric difference** between two digraphs D_1 and D_2, written as $D_1 \triangle D_2$, is the induced digraph on the arc set $(A(D_1) \cup A(D_2)) - (A(D_1) \cap A(D_2))$. Thus $D_1 \triangle D_2$ has no isolated vertices and contains arcs that are either in D_1 or in D_2 but not in both.

Corollary 3.3.3. *Every weakly connected locally $\mathcal{SC}$ digraph is supereulerian.*

Proof. Let D be a weakly connected and locally symmetrically connected digraph. We will prove that D is symmetrically connected. For any $u_1, u_k \in V(D)$, we shall follow Definition 3.3.1 to show there exists a symmetric dipath from u_1 to u_k. Since D is weakly connected then there exists an undirected path P in $G(D)$ lying between u_1 and u_k. If P has only one arc and since $\{(u_1, u_k), (u_k, u_1)\} \cap A(D) \neq \emptyset$, by Definition 3.3.1, $\{u_1, u_k\}$ are endpoints of a symmetrically connected dipath P. Then D contains a symmetric dipath from u_1 to u_k and we are done. If not, then for any continuous three vertices $\{u_{j-1}, u_j, u_{j+1}\} \in V(P)$ where $1 \leq j - 1$ and $j + 1 \leq k$. Since $\{(u_{j-1}, u_j), (u_j, u_{j-1})\} \cap A(D) \neq \emptyset$ and $\{(u_j, u_{j+1}), (u_{j+1}, u_j)\} \cap A(D) \neq \emptyset$, by Definition 3.3.1, $\{u_{j-1}, u_j\}$ are endpoints of a symmetrically connected dipath P_1 and $\{u_j, u_{j+1}\}$ are endpoints of a symmetrically connected dipath P_2. Since $u_j \in V(P_1) \cap V(P_2)$, implies that $V(P_1) \cap V(P_2) \neq \emptyset$. Then there exists a symmetrically connected dipath $P_3 \subseteq (P_1 \triangle P_2)$ with endpoints u_{j-1} and u_{j+1}. Then there exists a symmetrically connected subdigraph lies between u_{j-1} and u_{j+1}. This shows that the locally symmetrically connected digraph D is transitive for the arcs of any undirected path. Hence, by transitivity, (u_1, u_k) lies in a symmetrically connected subdigraph. This shows that D contains a symmetric dipath from u_1 to u_k implies that D is symmetrically connected. By Theorem 3.3.2, D is supereulerian. $\qquad \square$

The sharpness of Corollary 3.3.3 will be justified in the proposition below.

Proposition 3.3.4. *Let H be a strong digraph with $|V(H)| = n > 1$. If there exists a vertex $v \in V(H)$ such that $d(v) \leq n - 1$ and v is not incident with any symmetric arcs, then there exists an infinite family $\mathcal{F}(H)$ of strong, locally H, non-supereulerian digraphs.*

Proof. Let H be such a given digraph. Then H contains a vertex v such that v is not incident with any symmetric arcs. We have $N_H^+(v) \cap N_H^-(v) = \emptyset$.

Let $\alpha \geq |N_H^+(v)|$ and $\beta \geq |N_H^-(v)|$ be integers such that $\alpha + \beta \geq n$, and let $k = 1$, and $\ell \geq \alpha\beta + 1$ be integers. Define $D = D(\alpha, \beta, k, \ell)$ as in Example 3.2.2. We have the following observations, which justify the proposition.

(A) D is strong and nonsupereulerian. This observation follows from Proposition 3.2.3 with $k = 1$.

(B) For any arc $a \in A(D)$, there exists a subdigraph H_a of D such that H_a is isomorphic to H.

Let $a = (x, y)$ be an arc in $A(D)$. If $a \in A(D[A \cup B])$, then by (D1), $D[A \cup B] \cong K^*_{\alpha+\beta}$. It follows by the assumption that $\alpha + \beta \geq n = |V(H)|$ that $D[A \cup B]$ has a subdigraph isomorphic to H which contains a. Hence by (D2), we may assume that $(x, y) \in (A \cup B, U)_D$ or $(x, y) \in (U, A \cup B)_D$. In either case, let $A' \subseteq A$ and $B' \subseteq B$ be subsets with $|A'| = |N_H^+(v)|$ and $|B'| = |N_H^-(v)|$, respectively. Then $D[A' \cup B' \cup \{x, y\}]$ contains a subdigraph isomorphic to H and contains (x, y). This proves (B) and justifies that D is locally H. $\qquad\square$

3.4 Partially symmetric supereulerian digraphs

We investigate a different kind of local structural condition which warrants a digraph to be supereulerian. For any digraph D, define a relation on $V(D)$ such that $u \sim v$ if and only if $u = v$ or D has a symmetrically connected subdigraph H with $u, v \in V(H)$. If H_1 and H_2 are two subdigraphs of D, then we define $H_1 \cup H_2$ to be the subdigraph of D with vertex set $V(H_1) \cup V(H_2)$ and arc set $A(H_1) \cup A(H_2)$.

Lemma 3.4.1. *If H_1 and H_2 are two symmetrically connected subdigraphs of D such that $V(H_1) \cap V(H_2) \neq \emptyset$, then $H_1 \cup H_2$ is also a symmetrically connected subdigraph of D.*

Proof. It suffices to show that for any $u, v \in V(H_1 \cup H_2)$, $H_1 \cup H_2$ contains a symmetric dipath from u to v. If $u, v \in V(H_1)$, then since H_1 is symmetrically connected, by Definition 3.3.1, H_1 contains a symmetric dipath from u to v. Hence $H_1 \cup H_2$ has a symmetric dipath from u to v. Similarly, If $u, v \in V(H_2)$, then $H_1 \cup H_2$ also has a symmetric dipath from u to v.

Thus we assume that $u \in V(H_1) \backslash V(H_2)$ and $v \in V(H_2) \backslash V(H_1)$. Since $V(H_1) \cap V(H_2) \neq \emptyset$, we can take a vertex $r \in V(H_1) \cap V(H_2)$. Since H_1 and H_2 are symmetrically connected, H_1 contains a (u, r)-symmetric dipath P_1 and H_2 contains a (r, v)-symmetric dipath P_2. It follows that $D[A(P_1) \cup A(P_2)]$ contains a symmetric dipath from u to v. $\qquad\square$

By Lemma 3.4.1, the relation $\sim$ is an equivalence relation on $V(D)$. Each equivalence class induces a maximal symmetrically connected subdigraph of D. We have the following observation.

Observation 3.4.2. *Let D be a digraph. Each of the following holds.*
(i) D has a unique collection of maximal symmetrically connected subdigraphs.
(ii) If H_1 and H_2 are two maximal symmetrically connected subdigraphs, then either $H_1 = H_2$, or $V(H_1) \cap V(H_2) = \emptyset$.

Definition 3.4.3. *Let $c \geq 2$ be an integer and let D be a weakly connected digraph and let $\{H_1, H_2, \cdots, H_c\}$ be the set of maximal symmetrically connected subdigraphs of D.*
(i) If for any proper nonempty subset $\mathcal{J} \subset \{H_1, H_2, \cdots, H_c\}$, there exist an $H_i \in \mathcal{J}$ and a vertex $v \in V(H_i)$, and an $H_j \notin \mathcal{J}$ such that

$$N_D^+(v) \cap V(H_j) \neq \emptyset \text{ and } N_D^-(v) \cap V(H_j) \neq \emptyset,$$

*then D is **partially symmetric**.*
(ii) Let $\mathcal{PS}$ denote the family of all partially symmetric digraphs.

For a digraph D and let $\{H_1, H_2, \cdots, H_c\}$ be the set of all symmetrically connected components of D. Define D' to be the digraph obtained from D by contracting all symmetrically connected components. By Definition 3.4.3, D is partially symmetric if and only if D' is symmetrically connected. In fact, if D' is symmetrically connected, then Definition 3.4.3 (i) holds. Conversely, let M be a symmetrically connected component of D'. If $M \neq D'$, then $\mathcal{J} = V(M)$ is a subset of all symmetrically connected components of D, and so by Definition 3.4.3(i), there exists a vertex $H_i \in V(M)$ and a $H_j \in V(D') - V(M)$ such that both $(H_i, H_j), (H_j, H_i) \in A(D')$, contrary to the assumption that M is a component.

Let K_s^* and K_t^* be two complete digraphs of order $s \geq 2$ and $t \geq 2$, respectively, such that u_1, u_2 are two distinct vertices of K_s^* and v_1, v_2 are two distinct vertices of K_t^*. Let D be the digraph obtained from the disjoint union of K_s^* and K_t^* by adding the arcs (u_1, v_1) and (v_2, u_2). Then by the remark above, D is partially symmetric but not symmetrically connected.

Example 3.4.4. *Let J_1, J_2 be digraphs with $V(J_1) = V(J_2) = \{v_1, v_2, v_3, v_4\}$ and $A(J_1) = \{(v_1, v_2), (v_2, v_1),$*

$(v_1, v_4), (v_2, v_3), (v_3, v_4), (v_4, v_3), (v_3, v_1), (v_4, v_2)\}$, and $A(J_2) = \{(v_1, v_2), (v_2, v_1), (v_1, v_3), (v_1, v_4), (v_3, v_4),$
$(v_4, v_3), (v_3, v_2), (v_4, v_2)\}$. See Figure 2. Then we have these observations.

(i) For $i \in \{1, 2\}$, J_i does not have a symmetric (v_2, v_3)-dipath, and so J_i is not symmetrically connected.

(ii) For $i \in \{1, 2\}$, the maximal symmetrically connected subdigraph of J_i are $H_1 = D[\{v_1, v_2\}]$ and $H_2 = D[\{v_3, v_4\}]$.

(iii) By Definition 3.4.3(i), J_1 is partially symmetric but J_2 is not partially symmetric.

(iv) For any arc $a \in A(K_4^*) - A(J_2)$, $J_2 + a$ is symmetrically connected.
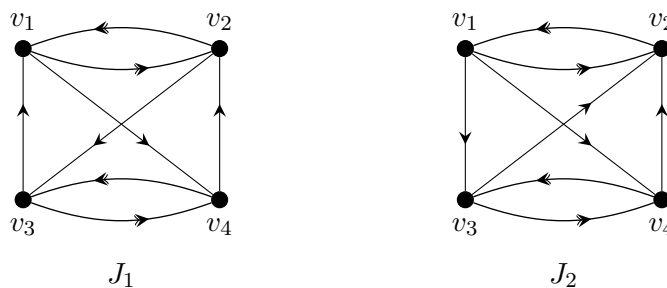

Figure 2: J_1 is partially symmetric digraph but J_2 is not partially symmetric digraph.

Lemma 3.4.5. *Let D be a partially symmetric digraph. Then each of the following holds.*
(i) For every $1 \le i \le c$, we have that $|V(H_i)| \ge 2$.
(ii) D is strong.
(iii) $|V(D)| \ge 4$.
(iv) D does not have symmetric arcs connecting a vertex in H_i and a vertex in H_j for any distinct $i, j \in \{1, 2, \cdots, c\}$.

Proof. Let $\{H_1, H_2, \cdots, H_c\}$ be the set of maximal symmetrically connected subdigraphs of D with $c \ge 2$.

(i) By contradiction, we assume that for some maximal symmetrically connected subdigraph H_i of D, $V(H_i) = \{u\}$. Let $\mathcal{J} = \{H_1, H_2, \cdots, H_c\} - \{H_i\}$. Since D is partially symmetric, there must be an $H_j \in \mathcal{J}$ and a vertex $v \in V(H_j)$ such that $(v, u), (u, v) \in A(D)$. It follows by the maximality of H_j that $u \in V(H_j)$, contrary to Observation 3.4.2 (ii). Hence Lemma 3.4.5(i) must hold.

(ii) Let X be a nonempty proper subset of $V(D)$. By (3.1), it suffices to show that $|\partial_D^+(X)| \ge 1$. If for some i, both $X \cap V(H_i) \ne \emptyset$ and $V(H_i) - X \ne \emptyset$, then by Definition 3.3.1, $|\partial_D^+(X)| \ge$

$|\partial_{H_i}^+(X \cap V(H_i))| \geq 1$. Hence we assume that no such H_i exists. Since $V(D) = \cup_{i=1}^c V(H_i)$, we may assume that for some integer m with $1 \leq m < c$, $X = \cup_{j=1}^m V(H_j)$. Since D is partially symmetric, by Definition 3.4.3, there exist a vertex $x \in V(H_h)$ for some $1 \leq h \leq m$ and a maximal symmetrically connected subdigraph H_r with $m+1 \leq r \leq c$ such that $N_D^+(x) \cap V(H_r) \neq \emptyset$ and $N_D^-(x) \cap V(H_r) \neq \emptyset$. This implies that $|\partial_D^+(X)| \geq 1$, and so D must be strong.

(iii) By Definition 3.4.3 we have that $c \geq 2$, and by (i) each maximal symmetrically connected subdigraph H_i with $|V(H_i)| \geq 2$, this shows that $|V(D)| \geq 4$.

(iv) If for some $i, j \in \{1, 2, \cdots, c\}$, D has symmetric arcs connecting a vertex in H_i and a vertex in H_j, then $H_i \cup H_j$ is symmetrically connected. By Definition 3.4.3, contrary to H_i is maximal symmetrically connected. Hence Lemma 3.4.5(iv) must hold.

$\square$

Theorem 3.4.6. *Every partially symmetric digraph is supereulerian.*

Proof. We argue by contradiction and assume that D is partially symmetric and

$$D \text{ is not supereulerian.} \tag{3.4}$$

Let $\{H_1, H_2, \cdots, H_c\}$ be the set of all maximal symmetrically connected subdigraphs of D. Since D is partially symmetric digraph, by Lemma 3.4.5(ii), D is strong and so D contains a nontrivial eulerian subdigraph. Choose an eulerian subdigraph S of D such that

$$|V(S)| \text{ is maximized among all eulerian subdigraphs of } D. \tag{3.5}$$

Since D is not supereulerian, $V(D) - V(S) \neq \emptyset$. Since D is strong, there exists an arc $(u, v) \in \partial_D^+(V(S))$.

If for some i with $1 \leq i \leq c$, $u, v \in V(H_i)$. Since H_i is symmetrically connected, by Definition 3.3.1, H_i has a (v, u)-dipath $P = v_1 v_2 \cdots v_k$ with $v = v_1$ and $u = v_k$ such that P is symmetric. Since $v_1 = v \notin V(S)$ and $v_k = u \in V(S)$, there exists a smallest index $i_0 > 1$ such that $v_{i_0} \in V(S)$. It follows that $D[V(S) \cup \{v_{i_0-1}, v_{i_0}\}]$ is eulerian with one more vertex than S, contrary to (3.5).

Therefore, there does not exist such H_i, consequently, for each maximal symmetrically connected subdigraph H_i of D, either $V(H_i) \cap V(S) = \emptyset$ or $V(H_i) \subseteq V(S)$. Without loss of generality, we assume that for some t with $1 \leq t \leq c$, $H_1, H_2, \cdots, H_t$ are contained in S and $H_{t+1} \cdots H_c$ are disjoint from $V(S)$.

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Since D is partially symmetric digraphs, by Definition3.4.3(i), there exist a vertex $x \in V(H_k)$ for some k with $1 \le k \le t$ and for some j with $t + 1 \le j \le c$ such that $N_D^+(x) \cap V(H_j) \ne \emptyset$ and $N_D^-(x) \cap V(H_j) \ne \emptyset$. Suppose that $x' \in N_D^+(x) \cap V(H_j)$ and $x'' \in N_D^-(x) \cap V(H_j)$. Since H_j is strong, H_j has a (x', x'')-dipath $x_1 x_2 \cdots x_q$ with $x' = x_1$ and $x'' = x_q$. Since $V(H_j) \cap V(S) = \emptyset$, it follows that $C = D[A(P) \cup \{(x, x'), (x'', x)\}]$ is a dicycle of $D - A(S)$. Thus $S' = D[A(S) \cup A(C)]$ is also an eulerian subdigraph of D with $|V(S')| \ge |V(S)| + 1$, contrary to (3.5). $\square$

Corollary 3.4.7. *Every weakly connected locally $\mathcal{PS}$ digraph is supereulerian.*

Proof. Let D be a weakly connected and locally partially symmetric digraph, $\{H_1, H_2, \cdots, H_c\}$ be the set of maximal symmetrically connected subdigraphs of D with $c \ge 2$. We shall verify Definition 3.4.3(i) to prove that D is partially symmetric.

Let $\mathcal{J} = \{H_{i_1}, H_{i_2}, \cdots, H_{i_m}\}$ and let $\mathcal{J}' = \{H_{i_{m+1}}, H_{i_{m+2}}, \cdots, H_{i_{m+s}}\}$ with $m + s = c$. Let $X = \cup_{j=1}^m V(H_{i_j})$ and $Y = \cup_{k=1}^s V(H_{i_{m+k}})$. Since D is weakly connected then there exists an arc $a \in (X, Y)_D \cup (Y, X)_D$. By Lemma 3.4.5(iv), we may assume that

D does not have any symmetric dipath connecting a vertex in X and a vertex in Y. (3.6)

Since D is locally partially symmetric, by Definition 3.4.3(ii), D contains a partially symmetric subdigraph Q of D with $a \in A(Q)$. Without loss of generality, let $a = (u, v) \in (X, Y)_D$. Since $a \in A(Q)$, we have $u \in V(Q) \cap V(X)$ and $v \in V(Q) \cap V(Y)$. Let $\{Q_1, Q_2, \cdots, Q_d\}$ be the set of maximal symmetrically connected subdigraphs of Q with $d \ge 2$. By (3.6) and by Definition 3.4.3(i), we may assume that for some index l with $1 \le l \le d - 1$, we have $\cup_{r=1}^l V(Q_{i_r}) \subseteq X$ and $\cup_{r=l+1}^d V(Q_{i_r}) \subseteq Y$. Since Q is partially symmetric, by Definition 3.4.3(i), for some h with $1 \le h \le l$, there exists an $x' \in V(Q_{i_h})$, and for some k with $l + 1 \le k \le d$, we have $N_Q^+(x') \cap V(Q_{i_k}) \ne \emptyset$ and $N_Q^-(x') \cap V(Q_{i_k}) \ne \emptyset$. Since $\{H_1, H_2, \cdots, H_c\}$ is the set of maximal symmetrically connected subdigraphs of D, there must be an s' with $1 \le s' \le m$ such that $x' \in V(Q_{i_h}) \subseteq V(H_{i_{s'}})$; and there must be an s'' with $m + 1 \le s'' \le m + s$ such that $V(Q_{i_k}) \subseteq V(H_{i_{s''}})$. Therefore, by Definition 3.4.3(i), D must be partially symmetric. By Theorem 3.4.6, D is supereulerian. $\square$

Observe that in Example 3.4.4 (iv), J_2 is "nearly symmetrically connected". The next examples 3.4.8 and 3.4.10, presents an infinite family of non supereulerian digraphs, such that for each digraph D in the family, every arc of D lies in a subdigraph isomorphic to J_2. This, in some sense, indicates that Corollary 3.4.7 is best possible.

Example 3.4.8. *Let $\alpha, \beta, k > 0$ be integers with $\alpha, \beta \geq k+1$, and let A and B be two disjoint set of vertices with $|A| = \alpha$ and $|B| = \beta$. Let $\ell \geq \alpha\beta + 1$ be an integer, and U be a set of vertices disjoint from $A \cup B$ with $|U| = 2\ell$. We construct a digraph $J = J(\alpha, \beta, k, \ell)$ such that $V(J) = A \cup B \cup U$ and the arcs of J are given as required in (J1) and (J2) below.*
(J1) $D[A \cup B] \cong K^*_{\alpha+\beta}$ is a complete digraph.
(J2) Write $U = \{u_1, u'_1, u_2, u'_2, \cdots, u_\ell, u'_\ell\}$ such that for each $1 \leq i \leq \ell$, $J[\{u_i, u'_i\}] \cong K^*_2$, $N^+_J(u_i) = N^+_J(u'_i) = A$ and $N^-_J(u_i) = N^-_J(u'_i) = B$. (See Figure 3)
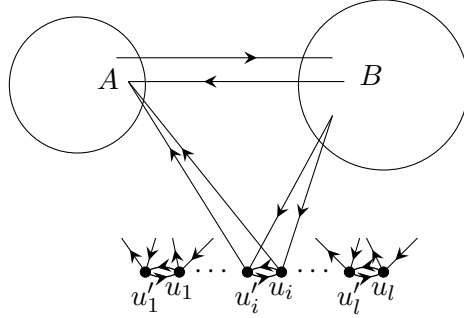

Figure 3. The digraph $D = D(\alpha, \beta, k, \ell)$.

Proposition 3.4.9. *Let $J = J(\alpha, \beta, k, \ell)$ for some given parameters α, β, k and ℓ as defined in Example 3.4.8. Then each of the following holds.*
(i) $\lambda(J) > k$.
(ii) Every arc of J lies in a subdigraph isomorphic to J_2.
(iii) D is not supereulerian.

Proof. (i) The proof (i) similar to that of Proposition 3.2.3(i), and will be omitted.

(ii) Let $a = (u, v)$ be an arc in D. If $u, v \in A \cup B$, then by Example 3.4.8(D2), a is in $K^*_{\alpha+\beta}$ and so as a lies in a subdigraph isomorphic to J_2. Thus we may assume that either $a \in A(D[\{u_i, u'_i\}])$ for some i, or $a \in (B, U)_D \cup (U, A)_D$. In any case, by Example 3.4.8(J2), for any $w \in A$ and $w' \in B$, and for any i with $1 \leq i \leq \ell$, $J[\{w, w', u_i, u'_i\}] \cong J_2$. Hence (ii) must hold.

(iii) We apply Proposition 3.2.1. By (J1), $J[A \cup B] \cong K^*_{\alpha+\beta}$, and so $|\partial^+_J(A)| = \alpha\beta$. By (J2), $|(U, B)_J| = 0$ and so $\tau(U) = |U| > \alpha\beta$. It follows that

$$h(U, A) = |\partial^+_J(A)| + |(U, B)_J| - \tau(U) = \alpha\beta - |U| < 0.$$

It follows from Proposition 3.2.1 that J is not supereulerian. $\square$

Example 3.4.10. *(1) Let H_1 and H_2 be digraph with $V(H_1) = V(H_2) = \{v_1, v_2, v_3, v_4\}$ and $A(H_1) = \{(v_1, v_2), (v_2, v_1), (v_1, v_4), (v_2, v_3), (v_3, v_1), (v_4, v_3), (v_3, v_4), (v_2, v_4)\}$, and $A(H_2) = \{(v_1, v_3), (v_1, v_4), (v_4, v_3),$*

$(v_3, v_4), (v_3, v_2), (v_2, v_4), (v_2, v_1), (v_1, v_2)\}$. Then we have these observations.

(i) For $i \in \{1, 2\}$, H_i are strong.

(ii) For $i \in \{1, 2\}$, H_i is not partially symmetric digraphs, as depicted in Figure 4.
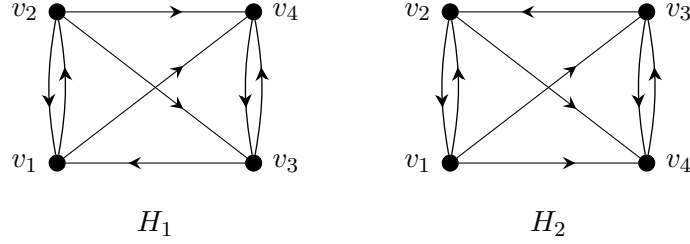

Figure 4: H_1 and H_2.

(2) We will present $H_1 \oplus_2 H_2$ digraph, and then we will use the $H_1 \oplus_2 H_2$ digraph to built an infinite family of non supereulerian digraphs,such that for each digraph D in the family, every arc of D lies in a subdigraph isomorphic to H_i for $i = 1, 2$.

(i) Let $V(H_1 \oplus_2 H_2) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $A(H_1 \oplus_2 H_2) = \{(v_1, v_2), (v_2, v_1), (v_1, v_4), (v_2, v_3),$

$(v_2, v_4), (v_3, v_1),$

$(v_4, v_3), (v_3, v_4), (v_3, v_6), (v_3, v_5), (v_4, v_5), (v_5, v_6), (v_6, v_5), (v_6, v_4)\}$. Then we observe $H_1 \oplus_2 H_2$ may not be supereulerian since $d^+_{H_1 \oplus_2 H_2}(v_4) = 1$ as depicted in Figure 5
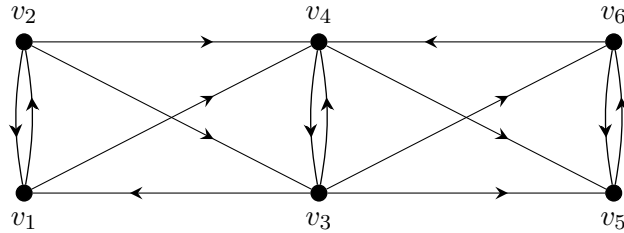

Figure 5: $H_1 \oplus_2 H_2$.

(ii) Using the notation in Figure 5, we can fix the vertex set of one of the two K_2^*'s in $H_1 \oplus_2 H_2$, $H_1 \oplus_2 H_2[\{v_1, v_2\}]$ and $H_1 \oplus_2 H_2[\{v_5, v_6\}]$, and define it as a distinguished pair. Let $\{x, y\}$ be a distinguished pair of $H_1 \oplus_2 H_2$, and denote $H_1 \oplus_2 H_2$ by $D(x, y) := H_1 \oplus_2 H_2(x, y)$.

Let $D_1(x', y')$ and $D_2(x'', y'')$ be the two copies of $D(v_1, v_2)$. Obtain a new digraph $D_1 \oplus_2 D_2$ from D_1 and D_2 by identifying x' and x'' to form a new vertex x, and identifying y' and y'' to form a new vertex y. See figure 6 for an drawing of $D_1 \oplus_2 D_2$ with $x = v_1$ and $y = v_2$.
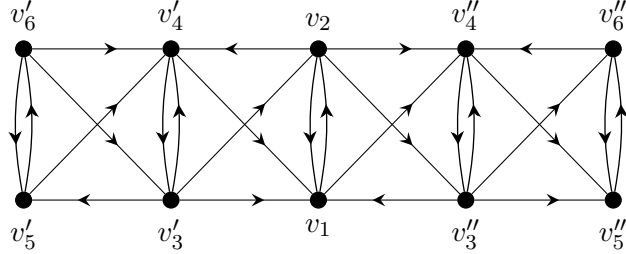

Figure 6: $D_1 \oplus_2 D_2$.

3.5 Locally k-Eulerian Digraphs

Let D be a weakly connected digraph and $k \geq 2$ be an integer. We call D **a locally k-Eulerian** digraph if for any arc $a \in A(D)$, there exists an Eulerian subdigraph H_a contain a in D contains a, such that $|A(H_a)| \leq k$.

Proposition 3.5.1. *Let $\mathcal{E}_k$ be the family of all locally k-Eulerian digraphs and let D be a weakly connected locally $\mathcal{E}_k$ digraph. Each of the following holds.*
(I) D is a strong.
(II) There exist an Example of infinite family of digraphs show that for any integer $k \geq 3$, there exists locally k-Eulerian non supereulerian digraph D with $\lambda(D) \geq k$.
(III)If for any $H \in \mathcal{E}_k$, H is symmetrically connected, then D is supereulerian.

Proof. (I) Assume that D is not strongly connected. Then there exists a proper nonempty subset $\emptyset \neq X \subset V(D)$ with $|\partial_D^+(X)| = 0$. By assumption D is weakly connected, D contains an arc $a = (u, v) \in (V(D) - X), X)_D$. Since D is a locally $\mathcal{E}_k$ digraph, there exist an Eulerian subdigraph H_a contains $a \in A(H_a)$. Since $(u, v) \in (V(D) - X), X)_D$, we conclude that $\emptyset \neq A(H_a) \cap (X, V(D) - X))_D \subseteq \partial_D^+(X)$, contrary to the assumption $|\partial_D^+(X)| = 0$. This proves that D must be strong.

(II)

Example 3.5.2. *Let $h, l \geq 1$ be integers and let A and B be two disjoint set of vertices with*

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$|A| = h$ and $|B| = l$. Let U be a set of vertices disjoint from $A \cup B$ with $|U| = h \times l + 1$. We construct a digraph $E = E(h, l, h \times l + 1)$ such that $V(E) = A \cup B \cup U$ and the arcs of E are given as required in (E1) and (E2) below.

(E1) $D[A \cup B] \cong K_{h+l}^*$ is a complete digraph.

(E2) For every vertex $u \in U$, and for every $v \in A$, $(u, v) \in A(D)$ and for every $w \in B$, $(w, u) \in A(D)$. Thus for any $u \in U$, we have $N_D^+(u) = A$ and $N_D^-(u) = B$. No two vertices in U are adjacent.

Proposition 3.5.3. *Let* $E = E(h, l, h \times l + 1)$ *for some given parameters* h, l *as defined in Example 3.5.2. Then each of the following holds.*

(i) $\lambda(D) \geq min\{h, l\}$.

(ii) For any arc $a \in A(D)$, *there exists a subdigraph* H_a *of* D *contain* a *such that* $|A(H_a)| \leq h + l + 1$.

(iii) D *is not supereulerian.*

Proof. (i) The proof (i) similar to that of Proposition 3.2.3(i), and will be omitted.

(ii) Now let $a = (x, y)$ be an arc in $A(D)$. If $a \in A(D[A \cup B])$, then by (D1), $D[A \cup B] \cong K_{h+l}^*$. By our assumption there exist many of subdigraphs say H_a contain a with each of them $|A(H_a)| \leq h + l$, implies the condition on subdigraphs is hold. Hence by (D2), we may assume that $(x, y) \in (A \cup B, U)_D$ or $(x, y) \in (U, A \cup B)_D$. In either case, let $A' \subseteq A$ and $B' \subseteq B$ be subsets with $|A'| = |N_H^+(v)|$ and $|B'| = |N_H^-(v)|$, respectively. Then $D[A' \cup B' \cup \{x, y\}]$ contain a an Eulerian subdigraph H_a contains a with $|A(H_a)| \leq h + l + 1$, then our condition is hold. This justifies that D is a locally k-Eulerian digraph.

(iii) We apply Proposition 3.2.1. By (E1), $E[A \cup B] \cong K_{h+l}^*$, and so $|\partial_E^+(A)| = h \times l$. By (E2), $|(U, B)_E| = 0$ and so $\tau(U) = |U| = h \times l + 1$. It follows that

$$h(U, A) = |\partial_E^+(A)| + |(U, B)_E| - \tau(U) = h \times l - |U| < 0.$$

It follows from Proposition 3.2.1 that E is not supereulerian. $\square$

(III) If for any $H \in \mathcal{E}_k$, H is symmetrically connected, then by Corollary 3.3.3, D is supereulerian. $\square$

3.6 A k-free trailable Digraphs

Definition 3.6.1. *Let D a digraph and $k \geq 2$ be an integer, then D called a k-free trailable digraph, if for any $X \subseteq V(D)$ with $|X| = k$. Then $D[X]$ the digraph induced by X has spanning closed ditrail. For simplicity of condition, if every k vertices lies in closed ditrail of D .*

We will begin with this section by introducing the following worths theorems in supereulerian digraphs:

Theorem 3.6.2. *(Jensen and Maddalini, Theorem(3.7) [17])*

Let D be a strong simple digraph on n vertices. If $d(x) + d(y) \geq 2n - 3$ for any pair of non adjacent vertices x and y, then D is supereulerian.

Theorem 3.6.3. *(Hong, Lai and Liu, Theorem(3.4) of [34])*

Let D be a strong digraph with $|V(D)| = n$ and $\delta^+(D) \geq 4$ and $\delta^-(D) \geq 4$; If $\delta^+(D) + \delta^-(D) \geq n - 4$ then D is supereulerian.

Theorem 3.6.4. *Let $k \geq 2$ be an integer number. Let D be a nonsingular digraph with*

$$|V(D)| = n \geq \begin{cases} 2k - 3 & \text{if } 2 \leq k \leq 6, \\ k + 3 & \text{if } 6 \leq k \leq 9, \\ 2k - 6 & \text{if } 9 \leq k. \end{cases}$$

or $n = k$. If $\forall X \subseteq V(D)$ with $|X| = k$, $D[X]$ has spanning closed ditrail, then D is supereulerian.

Proof. It is clear that D is strong because for any $x, y \in V(D)$ by our condition there exists a C_k contains x and y, so there exist a dipath from x to y and a dipath from y to x. If $n = k$, then D is a Hamiltonian and we are done. If $n > k$, then let $X_0 \subset V(D)$ with $|X_0| = k$ and $v \in X_0$. Then by our condition v has an out-neighbor y_1 in X_0. Let $\{x_1, x_2, ..., x_{n-k}\} = V(D) - X_0$ and $X_1 = (X_0 \cup \{x_1\}) - y_1$. Then v has a nother out-neighbor y_2 in X_1. Repeating this procedure v has at least $n - k + 1$ out-neighbors in D. By the same procedure v has at least $n - k + 1$ in-neighbors. Therefore, $d(v) \geq 2(n - k + 1)$.

(I) By using Theorem 3.6.2 , we determine when $d(x) + d(y) \geq 2n - 3 \;\forall x, y \in V(D)$

$$\Rightarrow 4(n - k + 1) \geq 2n - 3$$

$$\Rightarrow 4n - 4k + 4 \geq 2n - 3$$

$$\Rightarrow 2n \geq 4k - 7$$

$$\Rightarrow n \geq 2k - \frac{7}{2}$$

$$n \geq 2k - 3$$

It means that:

When $n \geq 2k - 3$, then D is supereulerian.

(II) Since v has at least $n - k + 1$ in-neighbors in D and has at least $n - k + 1$ out-neighbors in D as well. Then

(A) By using the first condition of Theorem 3.6.3 that $\delta^+(D) \geq 4$ and $\delta^-(D) \geq 4$ we have the following:

$$n - k + 1 \geq 4$$

$$\Rightarrow n \geq k + 3.$$

(B) By using the second condition of Theorem 3.6.3:

$$\delta^+(D) + \delta^-(D) \geq n - 4$$

$$\Rightarrow 2(n - k + 1) \geq n - 4$$

$$\Rightarrow n \geq 2k - 6.$$

Combining (A) with (B) implies that when $n \geq max\{2k-6, k+3\}$, then D is supereulerian.

Combining (I) with (II), we get that when $n \geq min\{max\{2k - 5, k + 3\}, 2k - 3\}$, then D is supereulerian. Thus, if

$$n \geq \begin{cases} 2k - 3 & \text{if } 2 \leq k \leq 6, \\ k + 3 & \text{if } 6 \leq k \leq 9, \\ 2k - 6 & \text{if } 9 \leq k, \end{cases}$$

then D is supereulerian. $\qquad\square$

In the following Definition, is a special case of the general case of Definition 3.6.1:

Definition 3.6.5. *Let D a digraph and $k \geq 2$ be an integer, then D called a k-free trailable digraph, if for any $X \subseteq V(D)$ with $|X| = k$. Then $D[X]$ the digraph induced by X has a dicycle C_k with k vertices. For simplicity of condition, if every k vertices lies in closed dicycle C_k of D.*

Theorem 3.6.6. *Let $k \geq 2$ be an integer number. Let D be a nonsingular digraph with*

$$|V(D)| = n \geq \begin{cases} 2k - 3 & \text{if } 2 \leq k \leq 6, \\ k + 3 & \text{if } 6 \leq k \leq 9, \\ 2k - 6 & \text{if } 9 \leq k \leq 13, \\ k + \lceil \frac{k}{2} \rceil & \text{if } 14 \leq k. \end{cases}$$

or $n = k$. If for any $X \subseteq V(D)$ with $|X| = k$, $D[X]$ has a dicycle C_k, then D is supereulerian.

Proof. It is clear that D is strong because for any $x, y \in V(D)$ by our condition there exists a C_k contains x and y, so there exist a dipath from x to y and a dipath from y to x. If $n = k$, then D is a Hamiltonian and we are done. For $n > k$, by Lemma 2.2.14 we have that $d(v) \geq 2(n-k+1)$ for all vertex $v \in V(D)$.

(I) By using Theorem 3.6.2 , we determine when $d(x) + d(y) \geq 2n - 3 \; \forall x, y \in V(D)$. Then we have that $4(n - k + 1) \geq 2n - 3$ implies that $n \geq 2k - 3$. That mean when $n \geq 2k - 3$ then D is supereulerian.

(II) Since every vertex $v \in V(D)$ has at least $n - k + 1$ in-neighbors and at least $n - k + 1$ out-neighbors . We have the following:

(A) By using the first condition of Theorem 3.6.3, we determine when $\delta^+(D) \geq 4$ and $\delta^-(D) \geq 4$, implies that $n - k + 1 \geq 4$. We have that $n \geq k + 3$.

(B) By using the second condition of Theorem 3.6.3, we determine when $\delta^+(D) + \delta^-(D) \geq n - 3$. Then we have that $2(n - k + 1) \geq n - 3$ that give us $n \geq 2k - 5$

Then, by using (A) and (B) together we get that when $n \geq max\{2k - 5, k + 3\}$, then D is supereulerian.

Hence, by combining (I) and (II), we get that when $n \geq min\{max\{2k - 5, k + 3\}, 2k - 3\}$, then D is supereulerian. Thus, if

$$n \geq \begin{cases} 2k - 3 & \text{if } 2 \leq k \leq 6, \\ k + 3 & \text{if } 6 \leq k \leq 9, \\ 2k - 6 & \text{if } 9 \leq k \leq 13, \end{cases}$$

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then D is supereulerian.

Now we need to prove the reminder case when $n \geq k + \lceil \frac{k}{2} \rceil$. Let $X \subset V(D)$ with $|X| = k$. By definition of D let C_k be a dicycle consist of all vertices of X. Let $Y = V(D) - V(C_k) = \lceil \frac{k}{2} \rceil$. Let $\bar{X} \subset V(C_k)$ consist of nonadjacent vertices of C_k. Then we have the following cases:

- **Case(1)** If $14 \leq k$, such that k is an even number.
  By picking $\bar{X} = \lceil \frac{k}{2} \rceil = \frac{k}{2}$ and join them with $Y = \lceil \frac{k}{2} \rceil = \frac{k}{2}$, we get that $\bar{X} \cup Y = \frac{k}{2} + \frac{k}{2} = k$. Since $|\bar{X} \cup Y| = k$ then by definition of D consider a cycle $\bar{C}_k$ consist of $\bar{X} \cup Y$. Let $v \in \bar{X}$, then start at the vertex v and go over all vertices belong to $\bar{C}_k$ until back the same starting vertex, then move to all vertices belong to C_k until back to same vertex. Hence, we prove that D is supereulerian.

- **Case(2)** If $14 \leq k$, such that k is an odd number.
  By picking $\bar{X} = \lceil \frac{k}{2} \rceil = \frac{k}{2}$ and join them with $Y = \lceil \frac{k}{2} \rceil = (\frac{k}{2}) + 1$, we get that $\bar{X} \cup Y = \frac{k}{2} + \lceil \frac{k}{2} \rceil = (\frac{k}{2} + \frac{k}{2}) + 1 = (k-1) + 1 = k$. Since $|\bar{X} \cup Y| = k$ then by definition of D consider a cycle $\bar{C}_k$ consist of $\bar{X} \cup Y$. Let $v \in \bar{X}$, then start at the vertex v and go over all vertices belong to $\bar{C}_k$ until back the same starting vertex, then move to all vertices belong to C_k until back to same vertex. Hence, we prove that D is supereulerian.

We conclude from case(1) and (2) that D is supereulerian.

$\square$

Chapter 4

On a class of supereulerian digraphs

4.1   Introduction

We consider finite graphs and digraphs, and undefined terms and notations will follow [1] for graphs and [2] for digraphs. Throughout this paper, the notation (u, v) denotes an arc oriented from u to v. A digraph D is **strict** if it contains no parallel arcs nor loops; and is **symmetric** if for any vertices $u, v \in V(D)$, if $(u, v) \in A(D)$, then $(v, u) \in A(D)$. If two arcs of D have a common vertex, we say that these two arcs are adjacent in D. A directed path in a digraph D from a vertex u to a vertex v is called a (u, v)-dipath. To emphasize the distinction between graphs and digraphs, a directed cycle or path in a digraph is often referred as a dicycle or dipath. A dipath P is a hamiltonian dipath if $V(P) = V(D)$. A digraph D is hamiltonian if D contains a hamiltonian dicycle. An (x, y)-hamiltonian dipath is a hamiltonian dipath from x to y. A digraph D is hamiltonian-connected if D has an (x, y)-hamiltonian dipath for every choice of distinct vertices $x, y \in V(D)$.

As in [2], $\lambda(D)$ denotes the arc-strong-connectivity of D. A digraph D is strong if and only if $\lambda(D) \geq 1$. For $X, Y \subseteq V(D)$, we define

$$(X, Y)_D = \{(x, y) \in A(D) : x \in X \text{ and } y \in Y\}; \text{ and } \partial_D^+(X) = (X, V(D) - X)_D.$$

For a subset $A' \subseteq A(D)$ the subdigraph **arc-induced** by A' is the digraph $D[A'] = (V', A')$, where V' is the set of vertices in V which are incident with at least one arc in A'.

Let

$$d_D^+(X) = |\partial_D^+(X)|, \text{ and } d_D^-(X) = |\partial_D^-(X)|.$$

When $X = \{v\}$, we write $d_D^+(v) = |\partial_D^+\{v\}|$ and $d_D^-(v) = |\partial_D^-\{v\}|$. Let $N_D^+(v) = \{u \in V(D) - v : (v, u) \in A(D)\}$ and $N_D^-(v) = \{u \in V(D) - v : (u, v) \in A(D)\}$ denote the **out-neighbourhood** and **in-neighbourhood** of v in D, respectively. Vertices in $N_D^+(v)$, $N_D^-(v)$ are called the **out-neighbours**, **in-neighbours** of v.

Boesch, Suffel, and Tindell [30] in 1977 proposed the supereulerian problem, which seeks to characterize graphs that have spanning eulerian subgraphs. They indicated that this problem would be very difficult. Pulleyblank [26] later in 1979 proved that determining whether a graph is supereulerian, even within planar graphs, is NP-complete. Catlin [20] in 1992 presented the first survey on supereulerian graphs. Chen et al [21] surveyed the reduction method associated with the supereulerian problem and their applications. An updated survey presenting the more recent developments can be found in [35].

It is natural to consider the supereulerian problem in digraphs. A digraph D is **eulerian** if it contains a closed ditrail W such that $A(W) = A(D)$, or, equivalently, if D is strong and for any $v \in V(D)$, $d_D^+(v) = d_D^-(v)$. A digraph D is **supereulerian** if D contains a closed ditrail W such that $V(W) = V(D)$, or, equivalently, if D contains a spanning eulerian subdigraph. Some recent developments on supereulerian digraphs are given in [17, 27, 31, 32, 34].

A central problem is to determine or characterize supereulerian digraphs. In Section 2, the **2-sum** $D_1 \oplus_2 D_2$ of two digraphs D_1 and D_2 is defined, and some basic properties of 2-sums are discussed. We will observe that a 2-sum of two supereulerian (or even hamiltonian) digraphs may not be supereulerian. Thus it is natural to seek sufficient conditions on D_1 and D_2 for the 2-sum of D_1 and D_2 to be supereulerian. In the last section, we will present several sufficient conditions for supereulerian 2-sums of digraphs. In particular, we show that if D_1 and D_2 are either symmetrically connected or partially symmetric (to be defined in Section 3), then $D_1 \oplus_2 D_2$ is supereulerian.

4.2 The 2-sums of digraphs

The definition and some elementary properties of the 2-sums of digraphs are presented in this section. A digraph is nontrivial if it contains at least one arc. Throughout this section, all digraphs are assumed to be nontrivial.

Definition 4.2.1. *Let D_1 and D_2 be two vertex disjoint digraphs, and let $a_1 = (v_{11}, v_{12}) \in A(D_1)$ and $a_2 = (v_{21}, v_{22}) \in A(D_2)$ be two distinguished arcs. The **2-sum** $D_1 \oplus_{a_1, a_2} D_2$ of D_1 and D_2 with base arcs a_1 and a_2 is obtained from the union of D_1 and $D_2 - a_2$ by identifying v_{11} with*

v_{21} and v_{12} with v_{22}, respectively. When the arcs a_1 and a_2 are not emphasized or is understood from the context, we often use $D_1 \oplus_2 D_2$ for $D_1 \oplus_{a_1,a_2} D_2$.

Lemma 4.2.2. *Let D_1 and D_2 be two vertex disjoint strong digraphs. Then*

$$\lambda(D_1 \oplus_2 D_2) \geq \min\{\lambda(D_1), \lambda(D_2)\}.$$

Proof. Let $k \geq 0$ be an integer such that $\min\{\lambda(D_1), \lambda(D_2)\} = k$, and let $\lambda(D_1 \oplus_2 D_2) = k'$. We shall show that $k' \geq k$. By (1.1), there exists a proper nonempty vertex subset $X \subset V(D_1 \oplus_2 D_2)$ such that $|\partial^+_{D_1 \oplus_2 D_2}(X)| = k'$. Let $S = \partial^+_{D_1 \oplus_2 D_2}(X)$. We argue by contradiction and assume that $k' < k$.

By Definition 4.2.1, we have $v_{11} = v_{21} \in V(D_2)$ and $v_{12} = v_{22} \in V(D_2)$ in $D_1 \oplus_2 D_2$. If $X \cap V(D_1) \neq \emptyset$ and $X \cap V(D_2) = \emptyset$, we obtain that $v_{11} = v_{21} \notin X$ and $v_{12} = v_{22} \notin X$, then $X \subset V(D_1)$ and $S = \partial^+_{D_1}(X)$. It follows by (1.1) that $k' = |S| \geq \lambda(D_1) \geq k$, contrary to the assumption that $k' < k$. Similarly, if $X \cap V(D_1) = \emptyset$ and $X \cap V(D_2) \neq \emptyset$, then $X \subset V(D_2)$ and $S = \partial^+_{D_2}(X)$, hence a contradiction to the assumption that $k' < k$ is obtained from $k' = |S| \geq \lambda(D_2) \geq k$.

Thus we may assume that $X \cap V(D_1) \neq \emptyset$ and $X \cap V(D_2) \neq \emptyset$. Let $X' = X \cap V(D_1)$. Then X' is a proper nonempty subset of $V(D_1)$, and $\partial^+_{D_1}(X') \subseteq S$. It follows by (1.1) that $k' = |S| \geq |\partial^+_{D_1}(X')| \geq \lambda(D_1) \geq k$ contrary to the assumption that $k' < k$. $\qquad\square$

Example 4.2.3. *The converse of Lemma 4.2.2 may not always stand, as indicated by the example below, depicted in Figure 1. Let $V(D_1) = \{v_{11}, v_{12}, v_{13}, v_{14}\}$ and $V(D_2) = \{v_{21}, v_{22}, v_{23}, v_{24}\}$. Let $A(D_1) = \{(v_{11}, v_{12}), (v_{13}, v_{12}), (v_{14}, v_{13}), (v_{11}, v_{14}), (v_{11}, v_{13}), (v_{14}, v_{12})\}$ and $A(D_2) = \{(v_{21}, v_{22}), (v_{22}, v_{23}), (v_{23}, v_{24}), (v_{24}, v_{21}), (v_{23}, v_{21}), (v_{24}, v_{22})\}$. Let $a_1 = (v_{11}, v_{12})$ and $a_2 = (v_{21}, v_{22})$. Then it is routine to verify that $\lambda(D_1 \oplus_{a_1,a_2} D_2) \geq 1$. While D_2 is strong, the digraph D_1 contains a vertex v_{11} with $d^-_{D_1}(v_{11}) = 0$, and so $\lambda(D_1) = 0$.*
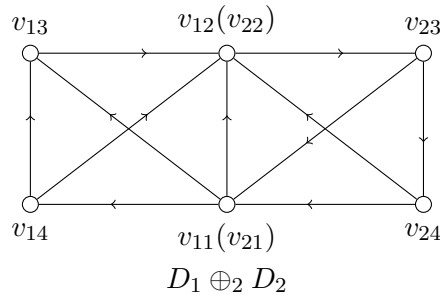

$$D_1 \oplus_2 D_2$$

Figure 1. $\lambda(D_1 \oplus_2 D_2) = 1$ but $\min\{\lambda(D_1), \lambda(D_2)\} = 0$.

Lemma 4.2.4. *A digraph D is not supereulerian if for some integer $m > 0$, $V(D)$ has vertex disjoint subsets $\{B, B_1, ..., B_m\}$ satisfying both of the following:*
(i) $N_D^-(B_i) \subseteq B, \forall i \in \{1, 2, ..., m\}$.
(ii) $|\partial_D^-(B)| \leq m - 1$.

Proof. By contradiction, we assume that both (i) and (ii) hold and D is supereulerian. Let S be a spanning eulerian subdigraph of D, then $B \subset V(S) = V(D)$ and $A(S) \subset A(D)$. Since S is eulerian, for any subset $X \subset V(S)$, it follows that $|\partial_S^+(X)| = |\partial_S^-(X)|$. Thus, by (ii), we conclude that

$$|\partial_D^+(B) \cap A(S)| = |\partial_D^-(B) \cap A(S)| \leq |\partial_D^-(B)| \leq m - 1. \tag{4.1}$$

By (i) and by (4.1), there must be a B_j with $j \in \{1, 2, ..., m\}$ such that $\partial_D^-(B_j) \cap A(S) = \emptyset$, contrary to the assumption that $V(S) = V(D)$. $\square$

Lemma 4.2.4 can be applied to find examples of hamiltonian digraphs whose 2-sum is not supereulerian, as shown in Example 4.2.5 below.

Example 4.2.5. *Let $n_1, n_2 \geq 3$ be integers and C_{n_1} and C_{n_2} be two vertex disjoint dicycles with length n_1 and n_2, respectively. We claim that $C_{n_1} \oplus_2 C_{n_2}$ is not supereulerian. To justify this claim, we denote $V(C_{n_1}) = \{v_{11}, v_{12}, \ldots, v_{1n_1}\}$, and $V(C_{n_2}) = \{v_{21}, v_{22}, \ldots, v_{2n_2}\}$. Without loss of generality, we assume that $a_1 = (v_{11}, v_{12})$ and $a_2 = (v_{21}, v_{22})$, and $C_{n_1} \oplus_2 C_{n_2} = C_{n_1} \oplus_{a_1, a_2} C_{n_2}$. Let B, B_1 and B_2 be subdigraphs of $C_{n_1} \oplus_2 C_{n_2}$ with $V(B) = \{v_{12}\}$, $V(B_1) = \{v_{13}\}$ and $V(B_2) = \{v_{23}\}$, respectively. By Lemma 4.2.4, we conclude that $C_{n_1} \oplus_2 C_{n_2}$ is not supereulerian (See Figure 2).*
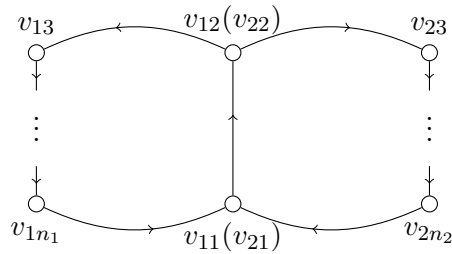

Figure 2. The 2-sum $C_{n_1} \oplus_2 C_{n_2}$ of C_{n_1} and C_{n_2}

4.3 Sufficient conditions for supereulerian 2-sums of digraphs

In this section, we will show several sufficient conditions on D_1 and D_2 to assure that the 2-sum $D_1 \oplus_2 D_2$ is supereulerian.

Proposition 4.3.1. *Let D_1 and D_2 be two vertex disjoint supereulerian digraphs with $a_1 = (v_{11}, v_{12}) \in A(D_1)$ and $a_2 = (v_{21}, v_{22}) \in A(D_2)$, and let $D_1 \oplus_2 D_2$ denote $D_1 \oplus_{a_1, a_2} D_2$. Each of the following holds.*

(i) For some $i \in \{1, 2\}$, if D_i has a spanning eulerian subdigraph S_i such that $a_i \notin A(S_i)$, then $D_1 \oplus_2 D_2$ is supereulerian.

(ii) If for some $i \in \{1, 2\}$, D_i is hamiltonian-connected, then $D_1 \oplus_2 D_2$ is supereulerian.

Proof. (i) Since D_1 and D_2 are supereulerian digraphs, D_1 and D_2 are strongly connected, and so by Lemma 4.2.2, $D_1 \oplus_2 D_2$ is also strongly connected. Without loss of generality, we assume that $i = 1$ and D_1 has a spanning eulerian subdigraph S_1 such that $a_1 \notin A(S_1)$. Since D_2 is supereulerian, we can pick a spanning eulerian subdigraph S_2' in D_2. Then $A(S_1) \cap A(S_2') = \emptyset$ and $V(S_1) \cap V(S_2') \neq \emptyset$. It follows that $D[A(S_1) \cup A(S_2')]$ is a spanning eulerian subdigraph in $D_1 \oplus_2 D_2$.

(ii) Without loss of generality, we assume that $i = 1$ and D_1 is hamiltonian-connected, and so D_1 has a (v_{11}, v_{12})-hamiltonian dipath T_1 and a (v_{12}, v_{11})-hamiltonian dipath T_2. Since D_2 is supereulerian, D_2 contains a spanning eulerian subdigraph S_2'. Define

$$S = \begin{cases} D[A(T_1) \cup A(S_2' - \{(v_{21}, v_{22})\})] & \text{if } (v_{21}, v_{22}) \in A(S_2') \\ D[(A(T_2) \cup \{(v_{11}, v_{12})\}) \cup A(S_2')] & \text{if } (v_{21}, v_{22}) \notin A(S_2') \end{cases} .$$

As in any case, S is strongly connected and every vertex $v \in V(S)$ satisfies $d_S^+(v) = d_S^-(v)$, and so S is eulerian. Since $V(S) = V(T_i) \cup V(S_2') = V(D_1) \cup V(D_2)$, for $i \in \{1, 2\}$, we conclude that S is a spanning eulerian subdigraph of $D_1 \oplus_2 D_2$, and so $D_1 \oplus_2 D_2$ is supereulerian. $\qquad \square$

If the absence of condition (i) in Proposition 4.3.1 is not sufficient to guarantee the existence $D_1 \oplus_2 D_2$ be a supereulerian. We have introduce the following counterexample.

Example 4.3.2. *(1) Let H_1 and H_2 be digraph with $V(H_1) = V(H_2) = \{v_1, v_2, v_3, v_4\}$ and $A(H_1) = \{(v_1, v_2), (v_2, v_1), (v_1, v_4), (v_3, v_2), (v_3, v_1), (v_4, v_3), (v_2, v_4)\}$, and $A(H_2) = \{(v_1, v_3), (v_1, v_4), (v_4, v_3), (v_3, v_4), (v_3, v_2), (v_4, v_2), (v_2, v_1)\}$. Then we have these observations.*

(i) For $i \in \{1, 2\}$, H_i are supereulerians .

(ii) Let H_1 and H_2 have spanning closed ditrails S_1 and S_2 respectively, and $a_1 = (v_4, v_3) \in A(S_1)$

and $a_2 = (v_2, v_1) \in A(S_2)$.

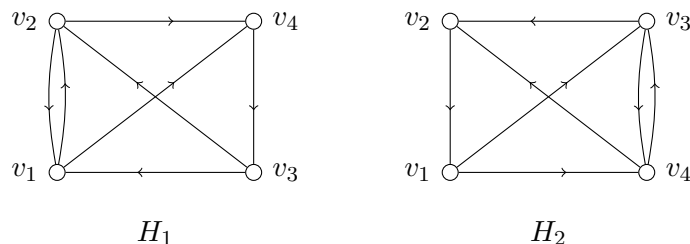*(iii) For $i \in \{1, 2\}$, $a_i \in A(S_i)$ as depicted in Figure 3.*

H_1 H_2

Figure 3: H_1 and H_2.

(2) We will present $H_1 \oplus_2 H_2$ digraph, and then we will use the $H_1 \oplus_2 H_2$ digraph to built an infinite family of digraphs to show that if condition (i) in Proposition 4.3.1 does not hold then any 2 sum with the $H_1 \oplus_2 H_2$ digraph will may not supereulerian.

(i) Let $V(H_1 \oplus_2 H_2) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $A(H_1 \oplus_2 H_2) = \{(v_1, v_2), (v_2, v_1), (v_1, v_4), (v_2, v_3),$ $(v_2, v_4), (v_3, v_1), (v_4, v_3), (v_3, v_6), (v_3, v_5), (v_5, v_4), (v_5, v_6), (v_6, v_5), (v_6, v_4)\}$. Then we observe $H_1 \oplus_2$ H_2 may not be supereulerian since $a = (v_4, v_3) \in A(S_i)$ for $i = \{1, 2\}$ where $v_2 = v_4$ and $v_1 = v_3$, as depicted in Figure 4
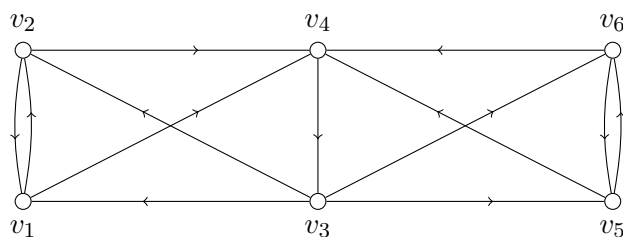

Figure 4: $H_1 \oplus_2 H_2$.

(ii) Using the notation in Figure 3, we can fix the vertex set of one of the two K_2^*'s in $H_1 \oplus_2 H_2$, $H_1 \oplus_2 H_2[\{v_1, v_2\}]$ and $H_1 \oplus_2 H_2[\{v_5, v_6\}]$, and define it as a distinguished pair. Let $\{x, y\}$ be a distinguished pair of $H_1 \oplus_2 H_2$, and denote $H_1 \oplus_2 H_2$ by $D(x, y) := H_1 \oplus_2 H_2(x, y)$. Let $D_1(x', y')$ and $D_2(x'', y'')$ be the two copies of $D(v_1, v_2)$. Obtain a new digraph $D_1 \oplus_2 D_2$ from D_1 and D_2 by identifying x' and x'' to form a new vertex x, and identifying y' and y'' to form a new vertex y. See figure 5 for an drawing of $D_1 \oplus_2 D_2$ with $x = v_1$ and $y = v_2$.
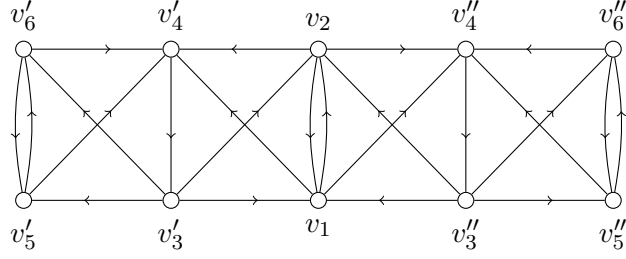
Figure 5: $D_1 \oplus_2 D_2$.

Theorem 4.3.3. *[36] If a strict digraph on $n \geq 3$ vertices has $(n-1)^2 + 1$ or more arcs, then it is hamiltonian-connected.*

Corollary 4.3.4. *Let D_1 be a strict digraph on $n_1 \geq 3$ vertices and with $|A(D_1)| \geq (n_1-1)^2+1$. If D_2 is a supereulerian digraph, then $D_1 \oplus_2 D_2$ is supereulerian.*

Proof. By Theorem 4.3.3, D_1 is hamiltonian-connected. Then by Proposition 4.3.1(ii), $D_1 \oplus_2 D_2$ is supereulerian. □

Two classes of supereulerain digraphs seem to be of particular interests in studying supereulerian digraph 2-sums. We first present their definitions.

Definition 4.3.5. *Let D be a digraph such that either $D = K_1$ or $A(D) \neq \emptyset$. If for any $u, v \in V(D)$, D contains a symmetric dipath from u to v, then D is called a **symmetrically connected** digraph.*

Given a digraph D, define a relation $\sim$ on $V(D)$ such that $u \sim v$ if and only if $u = v$ or D has a symmetrically connected subdigraph H with $u, v \in V(H)$. By definition, one can routinely verify that $\sim$ is an equivalence relation. Each equivalence class induces a symmetrically connected component of D. Hence D is symmetrically connected if and only if D has only one symmetrically connected component. A symmetrically connected component of D is also called a maximal symmetrically connected subdigraph of D. When D has more than one symmetrically connected components, we have the following definition.

Definition 4.3.6. *Let D be a weakly connected digraph and $\{H_1, H_2, \ldots, H_c\}$ be the set of maximal symmetrically connected subdigraphs of D with $c \geq 2$. If for any proper nonempty subset $\mathcal{J} \subset \{H_1, H_2, \ldots, H_c\}$,*

$$\text{there exist an } H_i \in \mathcal{J}, \text{ a vertex } v \in V(H_i), \text{ and an } H_j \notin \mathcal{J} \text{ such that} \qquad (4.2)$$
$$N_D^+(v) \cap V(H_j) \neq \emptyset \text{ and } N_D^-(v) \cap V(H_j) \neq \emptyset,$$

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then D is **partially symmetric**.

It is known that both symmetrically connected digraphs and partially symmetric digraphs are supereulerian.

Theorem 4.3.7. *([28]) Each of the following holds.*
(i) Every symmetrically connected digraph is supereulerian.
(ii) Every partially symmetric digraph is supereulerian.

A main result of this section is to show that the digraph 2-sums of symmetrically connected or partially symmetric digraphs are supereulerian.

Lemma 4.3.8. *Let D_1 and D_2 be two vertex disjoint digraphs with $a_1 = (v_{11}, v_{12}) \in A(D_1)$ and $a_2 = (v_{21}, v_{22}) \in A(D_2)$, and let $D_1 \oplus_2 D_2$ denote $D_1 \oplus_{a_1, a_2} D_2$. Each of the following holds.*
(i) If D_1 and D_2 are symmetrically connected, then $D_1 \oplus_2 D_2$ is symmetrically connected.
(ii) If D_1 and D_2 are partially symmetric, then $D_1 \oplus_2 D_2$ is partially symmetric.
(iii) If D_1 is symmetric and D_2 is partially symmetric, then $D_1 \oplus_2 D_2$ is partially symmetric.

Proof. (i) For any vertices $x, y \in V(D_1 \oplus_2 D_2)$, we shall show that $D_1 \oplus_2 D_2$ always has a symmetric (x, y)-dipath. If for some $i \in \{1, 2\}$, we have $x, y \in V(D_i)$, then as D_i is symmetrically connected, D_i contains a symmetric (x, y)-dipath P. Since D_i is a subdigraph of $D_1 \oplus_2 D_2$, P is also a symmetric (x, y)-dipath of $D_1 \oplus_2 D_2$. Hence we may assume that $x \in V(D_1)$ and $y \in V(D_2)$. Since D_1 and D_2 are symmetrically connected, D_1 contains a symmetric (x, v_{11})-dipath P_1 and D_2 contains a symmetric (v_{21}, y)-dipath P_2. By Definition 4.2.1, v_{11} and v_{21} represent the same vertex in $D_1 \oplus_2 D_2$, and so $D_1 \oplus_2 D_2[A(P_1) \cup A(P_2)]$ is a symmetric (x, y)-dipath in $D_1 \oplus_2 D_2$.

(ii) Fix $i \in \{1, 2\}$. Since D_i is partially symmetric, for some integer $c_i > 1$, let $\{H'_{i1}, H'_{i2}, \ldots, H'_{ic_i}\}$ be the set of all maximal symmetrically connected subdigraphs of D_i. Without loss of generality, we assume that $v_{11} \in V(H'_{11})$ and $v_{21} \in V(H'_{21})$; and for some s, t with $1 \leq s \leq c_1$ and $1 \leq t \leq c_2$, $v_{12} \in V(H'_{1s})$ and $v_{22} \in V(H'_{2t})$. (We allow the possibility that $s = 1$ and/or $t = 1$). Define, for $1 \leq h \leq c_1$ and $1 \leq j \leq c_2$,

$$H_{1h} = \begin{cases} H'_{1h} & \text{if } h \notin \{1, s\} \\ H'_{11} \cup H'_{21} & \text{if } h = 1 \\ H'_{1s} \cup H'_{2t} & \text{if } h = s \end{cases} \quad \text{and} \quad H_{2j} = \begin{cases} H'_{2j} & \text{if } j \notin \{1, t\} \\ H'_{11} \cup H'_{21} & \text{if } j = 1 \\ H'_{1s} \cup H'_{2t} & \text{if } j = t \end{cases}.$$

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Then $\mathcal{H} = \{H_{11}, H_{12}, \ldots, H_{1c_1}, H_{21}, H_{22}, \ldots, H_{2c_2}\}$ is the set of all maximal symmetrically connected subdigraphs of $D_1 \oplus_2 D_2$. Note that $H_{11} = H_{21}$ and $H_{1s} = H_{2t}$. We shall show by definition that $D_1 \oplus_2 D_2$ is partially symmetric. To do that, let $\mathcal{J}$ be a nonempty proper subset of $\mathcal{H}$. We shall show that (4.2) holds.

Since $\mathcal{H} = \{H_{11}, H_{12}, \ldots, H_{1c_1}, H_{21}, H_{22}, \ldots, H_{2c_2}\}$, we either have $\mathcal{J} \cap \{H_{11}, H_{12}, \ldots, H_{1c_1}\} \neq \emptyset$ or $\mathcal{J} \cap \{H_{21}, H_{22}, \ldots, H_{2c_2}\} \neq \emptyset$. By symmetry, we may assume that $\mathcal{J} \cap \{H_{11}, H_{12}, \ldots, H_{1c_1}\} \neq \emptyset$.

Suppose first that $\{H_{11}, H_{12}, \ldots, H_{1c_1}\} - \mathcal{J} \neq \emptyset$. Let $\mathcal{J}' = \{H'_{1h} | H_{1h} \in \mathcal{J}\}$. Then $\{H'_{11}, H'_{12}, \ldots, H'_{1c_1}\} - \mathcal{J}' \neq \emptyset$. Since D_1 is partially symmetric, there exist an $H'_{1h_0} \in \mathcal{J}'$, a vertex $v \in V(H'_{1h_0})$, and an $H'_{1j_0} \in \{H'_{11}, H'_{12}, \ldots, H'_{1c_1}\} - \mathcal{J}'$ such that

$$N^+_{D_1}(v) \cap V(H'_{1j_0}) \neq \emptyset \text{ and } N^-_{D_1}(v) \cap V(H'_{1j_0}) \neq \emptyset.$$

This implies that the vertex $v \in V(H_{1h_0})$, $H_{1h_0} \in \mathcal{J}$, and $H_{1j_0} \notin \mathcal{J}$ such that

$$N^+_{D_1 \oplus_2 D_2}(v) \cap V(H_{1j_0}) \neq \emptyset \text{ and } N^-_{D_1 \oplus_2 D_2}(v) \cap V(H_{1j_0}) \neq \emptyset.$$

Thus (4.2) holds in this case.

Hence we may assume that $\{H_{11}, H_{12}, \ldots, H_{1c_1}\} \subset \mathcal{J}$. Since $\mathcal{J}$ is a proper subset, we must have $\{H_{21}, H_{22}, \ldots, H_{2c_2}\} - \mathcal{J} \neq \emptyset$. Since $H_{21} = H_{11} \in \mathcal{J}$, we also have $\{H_{21}, H_{22}, \ldots, H_{2c_2}\} \cap \mathcal{J} \neq \emptyset$. With a similar argument, we conclude that (4.2) must also hold in this case.

(iii) Let $H_0 = D_1$ and let $\{H'_1, H'_2, \ldots, H'_c\}$ be the set of all maximal symmetrically connected subdigraphs of D_2 with $v_{21} \in V(H'_1)$ and for some $j \in \{1, 2, \ldots, c\}$, $v_{22} \in V(H'_j)$. (We allow the possibility that $j = 1$). Define

$$H_i = \begin{cases} H'_1 \cup H_0 \cup H'_j & \text{if } i = 1 \text{ or } i = j \\ H'_i & \text{if } i \notin \{1, j\} \end{cases}.$$

Then $\mathcal{H} = \{H_1, H_2, \ldots, H_c\}$ is the set of all maximal symmetrically connected subdigraphs of $D_1 \oplus_2 D_2$. Note that $H_1 = H_j$ with this notation. Let $\mathcal{J}$ be a nonempty proper subset of $\mathcal{H}$. We shall show that (4.2) holds.

Let $J' = \{H'_i | H_i \in \mathcal{J}\}$. Since $\mathcal{J}$ is proper, $\mathcal{J}'$ is a nonempty proper subset of $\{H'_1, H'_2, \ldots, H'_c\}$. Since D_2 is partially symmetric, by Definition 4.3.6, there exist an $H'_{i_0} \in \mathcal{J}'$, a vertex $v \in V(H'_{i_0})$, and an $H'_{j_0} \in \{H'_1, H'_2, \ldots, H'_c\} - \mathcal{J}'$ such that

$$N^+_{D_1}(v) \cap V(H'_{j_0}) \neq \emptyset \text{ and } N^-_{D_1}(v) \cap V(H'_{j_0}) \neq \emptyset.$$

This implies that vertex $v \in V(H_{i_0})$, $H_{i_0} \in \mathcal{J}$ and $H_{j_0} \notin \mathcal{J}$ such that

$$N^+_{D_1 \oplus_2 D_2}(v) \cap V(H_{j_0}) \neq \emptyset \text{ and } N^-_{D_1 \oplus_2 D_2}(v) \cap V(H_{j_0}) \neq \emptyset.$$

Thus (4.2) holds, and so by definition, $D_1 \oplus_2 D_2$ is partially symmetric. □

Theorem 4.3.9. *Let D_1 and D_2 be two digraphs. Each of the following holds.*
(i) If D_1 and D_2 are symmetrically connected, then $D_1 \oplus_2 D_2$ is supereulerian.
(ii) If D_1 and D_2 are partially symmetric, then $D_1 \oplus_2 D_2$ is supereulerian.
(iii) If D_1 is symmetric and D_2 is partially symmetric, then $D_1 \oplus_2 D_2$ is supereulerian.

Proof. This follows from Theorem 4.3.7 and Lemma 4.3.8. □

If the absence of condition (ii) in Theorem 4.3.9 is not sufficient to guarantee that $D_1 \oplus_2 D_2$ will be a supereulerian. We have introduce the following counterexample.

Example 4.3.10. *(1)Let D_1 be a digraph with $V(D_1) = \{v_1, v_2, v_3, v_4\}$ and $A(D_1) = \{(v_1, v_2), (v_2, v_1),$*
$(v_1, v_4), (v_2, v_3), (v_3, v_4), (v_4, v_3), (v_3, v_1), (v_4, v_2)\}$, and let D_2 be a digraph with $V(D_2) = \{v_1, v_2, v_3,$
$v_4, v_5, v_6\}$ and $A(D_2) = \{(v_1, v_2), (v_2, v_1), (v_2, v_4), (v_2, v_3), (v_3, v_4), (v_4, v_3), (v_4, v_1), (v_3, v_1), (v_5, v_4),$
$(v_5, v_6), (v_6, v_5), (v_6, v_4), (v_3, v_6), (v_3, v_5)\}$, as depicted in Figure 6. Then we have these observa-
tions.
(i) For $i \in \{1, 2\}$, D_i does not have a symmetric (v_3, v_2)-dipath, and so D_i is not symmetrically
connected.
(ii) For $i \in \{1, 2\}$, the maximal symmetrically connected subdigraph of D_i are $H_1 = D_i[\{v_1, v_2\}]$
, $H_2 = D_i[\{v_3, v_4\}]$ and $H_3 = D_2[\{v_5, v_6\}]$.
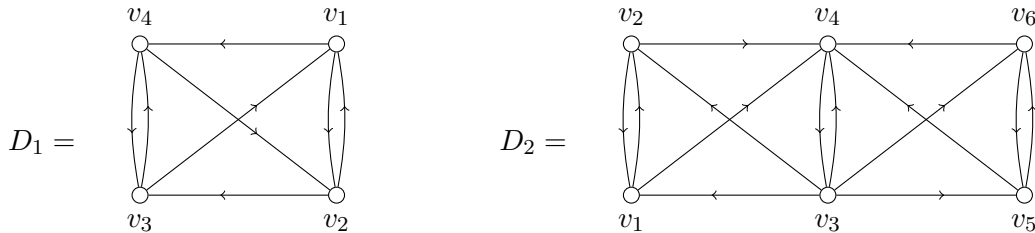*(iii) By Definition 4.3.6(i),D_1 is partially symmetric but D_2 is not partially symmetric.*

Figure 6. The digraphs D_1 and D_2.

(2) We will present $D_1 \oplus_2 D_2$ digraph, and then we will use the $D_1 \oplus_2 D_2$ digraph to built
an infinite family of digraphs to show that the condition (ii) in Therorem 4.3.9 in some sense is

the best possible.

*(i) Let $V(D_1 \oplus_2 D_2) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$ and $A(D_1 \oplus_2 D_2) = \{(v_1, v_2), (v_2, v_1), (v_1, v_4), (v_2, v_3),$
$(v_3, v_4), (v_4, v_3), (v_3, v_1), (v_4, v_2)(v_5, v_6), (v_6, v_5), (v_4, v_5), (v_3, v_5), (v_6, v_4), (v_6, v_3), (v_6, v_7), (v_6, v_8), (v_8, v_5),$
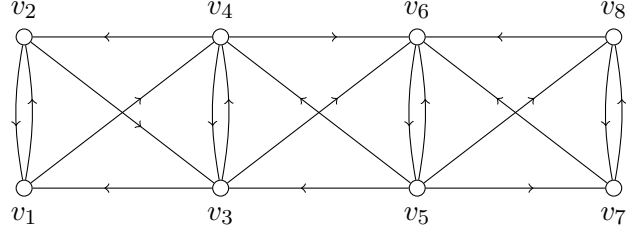$(v_7, v_5), (v_7, v_8), (v_8, v_7)\}$, as depicted in Figure 7*

Figure 7: $D_1 \oplus_2 D_2$ may not be a supereulerian.

(ii) Using the notation in Figure 13, we can fix the vertex set of one of the two K_2^*'s in $D_1 \oplus_2 D_2$, $D_1 \oplus_2 D_2[\{v_1, v_2\}]$, and $D_1 \oplus_2 D_2[\{v_7, v_8\}]$, and define it as a distinguished pair. Let $\{a, b\}$ be a distinguished pair of $D_1 \oplus_2 D_2$, and denote $D_1 \oplus_2 D_2$ by $Y(a, b) := D_1 \oplus_2 D_2(a, b)$. Let $Y_1(a', b')$ and $Y_2(a'', b'')$ be the two copies of $Y(v_1, v_2)$. Obtain a new digraph $Y_1 \oplus_2 Y_2$ from Y_1 and Y_2 by identifying a' and a'' to form a new vertex a, and identifying b' and b'' to form a new vertex b. See figure 8 for an drawing of $Y_1 \oplus_2 Y_2$ with $a = v_1$ and $b = v_2$.
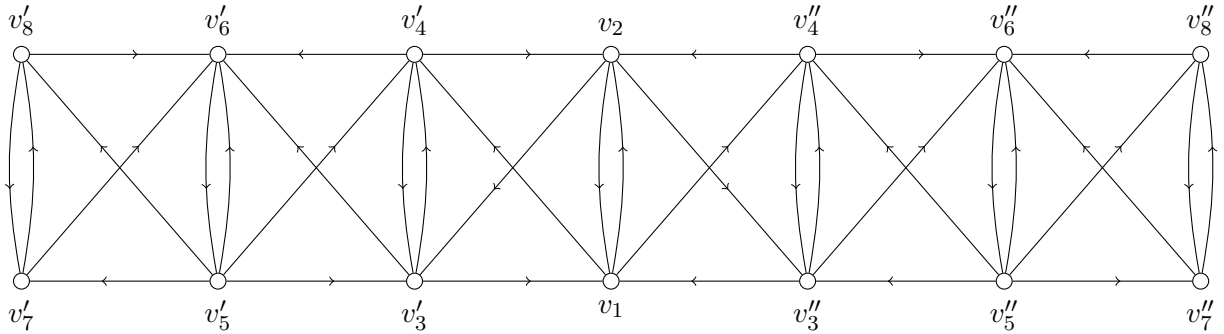

Figure 8: $Y_1 \oplus_2 Y_2$.

It is also natural to consider sufficient conditions on D_1 and D_2 for $D_1 \oplus_2 D_2$ to be hamiltonian.

Theorem 4.3.11. *If D_1 is hamiltonian and D_2 is hamiltonian-connected digraphs, then $D_1 \oplus_2 D_2$ is hamiltonian.*

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Proof. Let $V(D_1) = \{v_{11}, v_{12}, \ldots, v_{1n_1}\}$ with $C = v_{11}v_{12} \ldots v_{1n_1}v_{11}$ be a hamiltonian dicycle of D_1 and $V(D_2) = \{v_{21}, v_{22}, \ldots, v_{2n_2}\}$. Let $a_1 = (v_{11}, v_{12}) \in A(D_1)$ and $a_2 = (v_{21}, v_{22}) \in A(D_2)$, and $D_1 \oplus_2 D_2 = D_1 \oplus_{a_1,a_2} D_2$. Since D_2 is hamiltonian-connected, D_2 contains a (v_{21}, v_{22})-hamiltonian dipath P. Thus $(C - \{a_1\}) \cup P$ is a hamiltonian dicycle in $D_1 \oplus_2 D_2$. $\qquad\square$

Theorem 4.3.12. *[37] If a semicomplete digraph D is 4-strong, then D is hamiltonian-connected.*

By Theorem 4.3.11 and 4.3.12, we have the following corollary.

Corollary 4.3.13. *Let D_1 and D_2 be two 4-strong semicomplete digraphs, then $D_1 \oplus_2 D_2$ is hamiltonian.*

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