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Applications of the Scaled Laplace Transform in some Financial and Risk Models

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Abstract

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In this work, we propose several approximations for the evaluation of some risk measures and option prices based on the inversion of the scaled version of the Laplace transform which was suggested by Mnatsakanov and Sarkisian (2013). The classical risk model is considered for the evaluation of probability of ultimate ruin. Approximations of the inverse function of the ruin probability is proposed and its natural extension to the computation of Value at Risk, a benchmark risk measure for insurance and finance sectors, is proposed. The recovery of the distributions of bivariate models and bivariate aggregate claims amount on insurance policies is suggested. The proposed method is also applied to the Black-Scholes model for the estimation of option prices. Simulation studies and results are presented to demonstrate the performance of the proposed method.

In loving memory of my dear mother (the quintessence of motherhood).

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Chapter 1

Introduction

Individuals and companies often experience risky situations that require decision making in the presence of uncertainties. Among many concerns of financial market participants, one of the most important is the determination of the exact values of financial securities, commodities, and other items of interest. To decide upon a new product, insurance analysts usually consider what is the probability of profitability of the product. Are there adjustments that can be made to the price structure in order to increase the profitability of the product while maintaining a reasonable degree of security and competitiveness? How much capital is necessary to ensure profitability? How much premium should be charged for a policy? When exact solutions to these concerns are not readily available, numerical techniques become essential.

In many cases, the choice of numerical methods tends toward methods of high accuracy while paying little to no attention to how the financial provisions of the contract, instrument, or product may affect the reliability of the numerical solution, see Tagliani et al. [49]. As a consequence of discontinuities that characterize several financial options and risk measures, numerical techniques which are capable of circumventing such issues need to be employed. Methods of Integral Transforms are viable ways of managing issues with discontinuities. The Laplace transform, the Mellin transform, and the Fourier transform, to name a few, are examples of widely applied integral transforms in the literature. In this work, we consider a scaled version of the Laplace transform which was suggested in Mnatsakanov and Sarkisian [40].

The pertinent issue is the inversion of the Laplace transform. The recovery of a function from its Laplace transform is widely known to be a severely ill-posed inverse problem, see Tikhonov [51]. Hence, the need for regularization, which is very helpful in situations involving the inversion of the Laplace transform. Methods of inverting Laplace transforms can be broadly grouped into two classes namely: those that use complex values of the Laplace transform (e.g. the Bromwich Inversion formula, the Talbot method, etc.) and those that use only real values of the Laplace transform (e.g. the Post-Widder method, the Maximum Entropy method, etc.), see Cohen [8]. We considered the Moment recovered (MR) inversion method suggested by Mnatsaknov [39]; this method only uses real values of the Laplace transform. The scaled version of the MR Laplace transform inversion enables the application of the method in cases when the support of the target function F is the positive half of the real line, i.e., $supp\{F\} = \mathbb{R}^+ = [0, \infty)$. The approximation of the probability of ruin and the option pricing problem are examples of such cases. For properties of the moment recovered inversion method and the rate of approximation, see [40] and the references therein.

1.1 Scaled Laplace Transform Inversion

Assume that F is an aboslutely continuous distribution with $\mathbb{R}^+ = [0, \infty)$ as its support. Let f be the probability density function of F with respect to the Lebesgue measure on $\mathbb{R}^+$.

Definition 1. Suppose that F is the distribution function of a random variable X. Assume also that we have a given sequence of exponential moments, $\mu(F) = \{\mu_t(F), t \in \mathbb{N}_\alpha\}$. The scaled Laplace transform of F is defined by

$$\mu_t(F) := \mathcal{L}_{F,b}(t) = \int_{\mathbb{R}^+} e^{-ctx} dF(x), \quad \text{for} \quad t \in \mathbb{N}_\alpha = \{0, 1 \ldots, \alpha\}, \alpha = 1, 2, \ldots \tag{1.1}$$

where $c = \ln b$, for some $b > 1$.

To approximate the cumulative distribution function F and probability density function f, consider the scaled moment-recovered (MR) Laplace transform inversion introduced by Mnatsakanov [40]:

$$F_{\alpha,b}(x) := (\mathcal{L}_{\alpha,b}^{-1}\mu(F))(x) = 1 - \sum_{k=0}^{[\alpha b^{-x}]} \sum_{j=k}^{\alpha} \binom{\alpha}{j} \binom{j}{k} (-1)^{j-k} \mu_j(F)$$

and

$$f_{\alpha,b}(x) = \frac{[\alpha b^{-x}]\ln(b)\Gamma(\alpha+2)}{\alpha\Gamma([\alpha b^{-x}]+1)} \sum_{j=0}^{\alpha-[\alpha b^{-x}]} \frac{(-1)^j \mu_{j+[\alpha b^{-x}]}(F)}{j!(\alpha-[\alpha b^{-x}]-j)!}, \quad x \in \mathbb{R}^+ \tag{1.2}$$

respectively. Here, $\alpha \in \mathbb{N}$ is an integer-valued parameter. The equations above suggest that to evaluate the approximations $F_{\alpha,b}$ and $f_{\alpha,b}$, only the knowledge of the scaled Laplace transforms evaluated at the finite arithmetic progression $\mathbb{N}_{\alpha,b} = \{j\ln b, j = 0, 1, \ldots, \alpha\}$ is required. From [39, 40], we have that $F_{\alpha,b}$ and $f_{\alpha,b}$ converge uniformly to F and f respectively as $\alpha \to \infty$.

1.2 Collective Risk Model

Consider the classical risk model with risk reserve process $\{R_t\}_{t\geq 0}$. A risk reserve process, as defined in broad terms, is a model for the time evolution of the reserves of an insurance company. Let $R_0 = u > 0$ be the initial reserve at time $t = 0$. The company receives income from premiums at a constant rate, say p, per unit time. Claims are paid according to a compound process $S(t) = \sum_{k=1}^{N(t)} X_k$, where $\{N(t), t \geq 0\}$, the total number of claims, is a Poisson process with intensity $\lambda > 0$ and the individual claims, $X_1, X_2, \ldots$, are independent and identically distributed nonnegative random variables, independent of $N(t)$. At time t, the reserve of the company is $R_t = u + pt - S(t)$ and the time to ruin is $\tau(u) = \inf\{t \geq 0 : R_t < 0\}$.

The classical risk model has the property that there exists a constant ρ such that

$$\frac{1}{t}\sum_{k=1}^{N(t)} X_k \to_{a.s} \rho, \; t \to \infty$$

where ρ is defined as the average claim amount per unit time.

Another basic quantity is the *safety loading* (or the *security loading*) η defined as the relative amount by which the premium rate p must exceed ρ,

$$\eta = \frac{p-\rho}{\rho}.$$

To avoid certain ruin, an insurance company must ensure that $\eta > 0$ *always*.

The probability of ultimate ruin $\psi(u)$ is the probability that the reserve ever drops below zero,

$$\psi(u) = \mathbb{P}\left(\inf_{t\geq 0} R_t < 0 | R_0 = u\right) = \mathbb{P}(\tau(u) < \infty). \tag{1.3}$$

The probability that ruin occurs before time T is

$$\psi(u,T) = \mathbb{P}\left(\inf_{0\leq t\leq T} R_t < 0 | R_0 = u\right) = \mathbb{P}(\tau(u) < T). \tag{1.4}$$

The functions $\psi(u)$ and $\psi(u,T)$ are referred to as ruin probabilities with inifinite horizon and finite horizon, respectively.

The study of ruin probabilities, often referred to as *collective risk theory* or simply *risk theory*, orginated largely from Sweden during the first half of the 20th century. Main general ideas were established by Lundberg, see [35]. The first substantial mathematical results were presented by Lundberg [34] and Crámer [9]. Another early important work was also due to Täcklind [48]. The Cramér-Lundberg model is considered as the classical risk model.

Among several textbooks and journal articles on ruin probabilities, some main ones include Bühlmann [7], De Vylder [53], Gerber [20], Grandell [23, 24, 25], Dickson [13]. For more references, see Asmussen and Albrecher [4]. More recently, Gyzl et al. [26] applied the method of maximum entropy for the Laplace transform inversion to estimate the ultimate ruin probability. Albrecher et al. [1] applied the Trefethen-Weideman-Schmelzer (TWS) Laplace transform inversion method. The authors assumed that the claims size distribution represents a completely monotone function.

Mnatsakanov et al. [41] applied the MR scaled Laplace transform inversion method to obtain an approximation of $\psi(u)$. The authors applied the Laplace transform of $\psi(u)$ provided by Pollaczeck-Khinchine formula. In this work, we considered a modified and smoothed version of the approximation presented in [41]. We show that the proposed construction performs considerably better than the former.

1.3 Value-at-Risk

In 1995 the chairman of J.P. Morgan & Co. demanded a 4:15pm report each day on the potential earnings at risk overnight due to global price movements. The result was the concept of *Value at Risk* (VaR). It quantifies how much an economic agent can expect to lose in one day, week, year, ..., with a given probability. It is an estimate of the *worst* possible monetary loss from a financial investment over a future time-period, e.g., one-day, one-week, one-month, etc. For a given time horizon, say t, and confidence level $p \in (0, 1)$, the VaR of a portfolio is the loss in *market value* over the time horizon t that is exceeded with probability $1 - p$. Let X be the loss on an investment, and let $F_X(x)$ be the cumulative distribution function of X. Then the VaR of X is the level p-quantile; that is

$$\mathrm{VaR}_p(X) = \inf\{x \in \mathbb{R} : \mathbb{P}(X > x) \leq 1 - p\} = \inf\{x \in \mathbb{R}^+ : F_X(x) \geq p\}.$$

VaR is the main statistical technique used by banks for modeling financial risk, see Sollis [47]. It is considered as the benchmark risk measure in the financial world. In Dempster et al. [12], several contributors discussed the applications of VaR in the evaluation of different types of financial risk - market (due to price changes), credit (due to counterparty default), liquidity (the risk of unexpectedly large and negative cash flow over a short period due to market imbalance), operational (risk of fraud, trading errors, legal and regulatory risk, and so on), etc.

In [15], Duffie and Pan analyzed market risk using the VaR. Luciano and Kast [33] explored possible applications of VaR to insurance. Consequent upon the recession of 2008, Sollis [47] presented a critical overview of the use of the VaR as a benchmark of risk measurement by banks.

The VaR for a given time horizon does not, in many cases, reflect the possible adverse financial loss in between or beyond the specified time interval. Hence, the need for risk measures that give a more robust reflection of the risk inherent in doing business in a random environment. The ruin probability (or its inverse function) can be interpreted as the continuous alternative to the VaR, see Trufin et al [52].

Consider the continuous-time risk model $\{R_t\}_{t\geq 0}$. Instead of fixing the initial capital u, we fix the safety loading η and ask for the amount of the initial capital needed to bound the ruin probability by an acceptable level, ϵ. We then define a ruin-consistent VaR risk measure as

$$\rho_\epsilon[X] = \inf\{v \geq 0 | \psi(v) \leq \epsilon\} = \psi^{-1}(\epsilon).$$

Hence, $\rho_\epsilon[X]$, which is the equivalent to the inverse function of the ruin probablity, is the smallest amount of capital needed such that the ultimate ruin probability ψ for a risk process with individual claim size distributed as X is at most equal to some specified probability level ϵ.

In this work, we apply the MR inversion method to derive $\psi^{-1}_{\alpha,b}$ as an approximant of ψ^{-1}. We also approximate VaR using the aforementioned approximant of the inverse function of the ruin probability.

1.4 Bivariate Claims Amount Distribution

Most insurance claims involve simultaneous, correlated, multiple lines of business. Parts of a single claim are handled by different claim managers. This makes the study of the joint distribution of claims (not just the marginal distributions) necessary. A bivariate collective model is defined by

$$\begin{pmatrix} S_1 \\ S_2 \end{pmatrix} = \sum_{j=1}^{N} \begin{pmatrix} U_{1j} \\ U_{2j} \end{pmatrix} \tag{1.5}$$

where (S_1, S_2) represents the aggregate claims amount of two non-life insurance porfolios over a specified time period. Here, N is the total number of claims and $\{(U_{1j}, U_{2j})\}$, $j = 1, \ldots, N$, is a sequence of independent and identically distributed couples of non-negative random variables that model claims incurred on the two porfolios. Assume that $\{(U_{1j}, U_{2j})\}$ is independent from N. We are interested in the bivariate distributions of (S_1, S_2) that allow dependence between U_{1j} and U_{2j}.

Cummins and Wiltbank [11] noted that most general insurance claims cases involve a multivariate accident process where each accident can lead to a multivariate claims frequency and severity process. The complexity of the model they presented motivated actuaries to seek simpler models to price property and casualty insurance products. The pioneering work of Hesselager [27] involes models with multivariate counts and univariate claim sizes to model correlated aggregrate claims.

Ambagaspitiya [3] considered two classes of multivariate aggregate claims. The first class arises in cases where multiple claims result from one accident. The second class is the one with multivariate claim counts and unvariate claim sizes. In [2], the aforementioned author considered a book of business, defined as the union of disjoint classes of business, each of which has an aggregate distribution. The classes of business are assumed to be correlated. The author presented formulas to compute the aggragate distribution for the whole book when the claims distribution takes a certain form, e.g., when the claim counts distribution is multivariate, it was shown that the aggregrate distribution of the whole book is compound Poisson. Yuen et al. [55] derived explicit expressions for the ultimate survival probabilities for a risk model with two dependent classes of insurance business when the claim sizes are exponentially distributed. The authors also examined the asymptotic property of the ruin pobability for the special risk process with claim size distributions.

Goffard et al. [22], applied a Laplace transform inversion method involving an orthogonal projection of the probability density function with respect to a probability measure that belongs to the Natural Exponential Family of Quadratic Variance Function (NEF-QVF) to compute bivariate probability distributions from their Laplace transfroms. In this work, from the knowledge of the bivariate scaled Laplace transform of claim amounts, we constructed approximations of their Bivariate Probability Density Functions (BPDFs). We also recover the density and survival functions of bivariate aggregate claims amount.

1.5 Option Pricing

An option in finance is a contract between a buyer and a seller which gives the buyer the right but not the obligation to buy or sell an underlying asset at an agreed predetermined price called the strike price (or strike), say K, on a later date called the exercise time (or maturity date), say T. A call option gives the buyer the right to buy the underlying asset while a put option gives the buyer the right to sell the underlying asset. Financial derivatives such as American and European call and put options are referred to as plain vanilla products. Derivative securities which have certain features that make them more complex than the commonly traded plain vanilla products are called exotic options or simply, exotics. These products are traded in the over-the-counter (OTC) derivative market. Exotic options are important aspects of the portfolio of an investment bank because they are usually more profitable than plain vanilla products. Examples of exotic options are the compound option, chooser option, barrier option, binary/digital option, lookback option, constant proportion portfolio insurance (CPPI), cliquet or ratchet option, variance swap, rainbow option, and Bermudan option, see James [29], Hull [28].

Let $V(S,t)$ be the pay-off function. When $t = T$, for a European call option, the pay-off is defined by

$$V(S,T) = (S-K)^+ = \begin{cases} 0, & S \leq K \quad \text{(option is worthless)} \\ S-K, & S > K \quad \text{(option is exercised)} \end{cases}$$

or

$$V(S,T) = \max(S-K,0) = (S-K)^+.$$

Similarly for a put option,

$$V(S,T) = (K-S)^+.$$

From Wilmott [54], it has been shown that $V(S,t)$ is a unique solution to the partial differential equation

$$\begin{cases} \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V(s,t)}{\partial S^2} + rS\frac{\partial V(S,t)}{\partial S} + \frac{\partial V(S,t)}{\partial t} - rV(S,t) = 0, \\ V(S,T) = h(S), \quad S > 0, \\ \lim_{S\to 0} V(S,t) = r_1(t), \\ \lim_{S\to\infty} V(S,t) = r_2(t), \quad t \in [0,T], \end{cases}$$

where $r_1(t)$, $r_2(t)$ are chosen appropraiately to match $h(S)$.

The widely-used formula by Black, Scholes and Merton [5, 37], provides an exact pricing formula for the simplest model in the case of constant coefficients. However, such a technique is not applicable in the general case with time and space dependent coefficients. Hence, numerical methods are required for the evaluation of non-standard options. To evaluate special options such as the American put and call options, Asian options, discretely monitored barrier options, and options with non-standard pay-offs that are characterized by discontinuities which arise at each monitoring date, carefully chosen numerical methods are required to avoid spurious oscillations when low volatility is assumed.

Several numerical methods have been used to solve the Black-Scholes equation. In Seydel [46], Glasserman [21], Tavella and Randall [50], Wilmott et al [54] and the references therein, one can find well-known numerical methods for option pricing. Since financial markets are prone to stochastic flunctuations, stochastic approaches like the Monte Carlo methods, which are based on formulating and simulating stochastic differental equations, provide natural tools for simulating asset prices. Time marching methods like the Crank-Nicholson and Explicit and

Implicit finite-difference methods are used with sutiable spatial discretization schemes. A major drawback of these time-marching schemes is that they usually require as many time steps as spatial meshes to balance errors arising from discretization. Lee and Sheen [32] claimed that in particular, for the estimation of basket options of reasonable size, the usual time marching schemes seem to be too slow in practice since the cost of solving an elliptic system to advance to a next time step is usually computationally expensive.

Some related works in which the Laplace transform method was applied include Fu et al. [17], Geman and Yor [18], Lee and Sheen [32], Mallier and Alobaidi [36], Pelsser [44], Tagliani and Milev [49]. Most of the earlier works in which the Laplace transform method was applied to solve the Black-Scholes equation were used to obtain the analytic solutions of the various options they studied rather than develop an efficient numerical scheme. In Geman and Yor [18], and Pelsser [44], the Laplace transform is applied for the pricing of a double barrier option and in Mallier and Alobaidi [36], the pricing of the American call option is considered. The Mellin transform method which is similar to the Laplace transform was used to obtain the analytic solution of an option price by Cruz et al. [10], Jódar et al. [30] and Panini and Srivastava [43].

Tagliani and Milev [49] suggested a method they called the *mixed* method for a discretely monitored barrier option. The method involves solving the resulting ordinary differential equation (ODE) from computing the Laplace transform of the Black-Scholes equation by a finite-difference scheme and then transforming the solution of the ODE with the well-known Post-Widder inversion formula, see Cohen [8]. The authors showed that the *mixed* method is positivity-preserving, satisfies the discrete maximum principle, is spurious oscillations free, and is convergent to the exact solution.

In this work, we apply the scaled Laplace transform method to solve the Black-Scholes equation. Our approach does not require either change of variables or solving of diffusion equations. The resulting ODE is an Euler equation which has a closed form solution. To get the value of the option, we applied the MR inversion method to invert the solution of the ODE.

1.6 Dissertation Schema

The remiander of the dissertation is organized as follows. In Chapter 2, we present the modified MR approximations of the ruin probability. We compute smoothed approximations using Poisson weights. We consider models with claims following exponential and gamma distributions to make comparison with approaches in Albrecher at al. [1], Gyzl et al. [26], and Mnatsakanov et al. [41]. The performance of the proposed approximations is illustrated graphically and with tables. In Chapter 3, we propose an approximation of the inverse function of the ruin probability which is based on the MR-approximations of the ruin probability. An approximation of the Value at Risk is also computed. In Chapter 4, we apply the bivariate MR inversion method to approximate the distribution of claims in bivariate risk models. We compare the approximations with those in Goffard et al. [22] which were based on the scale $b = \exp(1)$. In Chapter 5, we estimate the values of call and put options via the MR-inversion of the solution of the ordinary differential equation obtained by taking the scaled Laplace transform of the Black-Scholes equation. Finally in Chapter 6, we outline some concluding remarks and future research directions.

Chapter 2

Ruin Probability

The probability of ultimate ruin under the classical model is defined by

$$\psi(u) = \mathbb{P}\left\{ u + pt - \sum_{k=1}^{N(t)} X_k < 0, \text{ for some } t \geq 0 \right\}, \ u \geq 0. \tag{2.1}$$

From actuarial literature, when the distribution F of the claims sizes X_k, $k = 1, 2, \ldots, N(t)$, does not follow the exponential model, the evaluation of the ruin probability is a difficult problem. Recall, the constant premium $p > 0$, the claims sizes X_k, $k = 1, 2, \ldots, N(t)$, are independent of the total number of claims $\{N(t), t \geq 0\}$ which is a Poisson process with intensity $\lambda > 0$, and $u \in \mathbb{R}^+$ is the initial reserve.

2.1 Moment Recovered Approximation

In this section, we recall the moment recoverd approximation of the ruin probability computed by Mnatsakanov et al. [41]. The Laplace transform of the ruin probability according to the Pollaczek-Khinchine formula, has the form

$$\mathcal{L}_\psi(s\ln b) = \frac{1}{s\ln b} - \frac{1-\rho}{s\ln b - \lambda_p\{1-\mathcal{L}_f(s\ln b)\}}, \tag{2.2}$$

where $\mathcal{L}_g(s\ln b) = \int_{\mathbb{R}^+} e^{-(s\ln b)x} g(x)dx$ is the scaled Laplace transform of some function g defined on $\mathbb{R}^+ = [0,\infty)$, $\rho = \lambda_p E(X) < 1$, and $\lambda_p = \lambda/p$.

Applying (2.2), the MR-approximation of $\psi(u)$ is derived as

$$\begin{aligned}\psi_{\alpha,b}(u) &= (\mathcal{L}^{-1}_{\alpha,\psi}\mathcal{L}_\psi)(u)\\ &= \frac{[\alpha b^{-u}]\ln(b)\Gamma(\alpha+2)}{\alpha\Gamma([\alpha b^{-u}]+1)} \sum_{m=0}^{\alpha-[\alpha b^{-u}]} \frac{(-1)^m \mathcal{L}_\psi((m+[\alpha b^{-u}])\ln b)}{m!(\alpha-[\alpha b^{-u}]-m)!}.\end{aligned} \tag{2.3}$$

Let us denote the the sup-norm of a function $f:\mathbb{R}^+\to\mathbb{R}$ by $||f||$. For some $b>1$, assume that

$$M_k = \sup_{x\in\mathbb{R}^+} |\psi'(x)b^{kx}|,\ \ k=1,2 \tag{2.4}$$

and

$$M_3 = \sup_{x\in\mathbb{R}^+} |\psi''(x)b^{2x}| < \infty. \tag{2.5}$$

Theorem 1. *Assume that ψ is continuous on $\mathbb{R}^+$ and that $\phi(x) = b^{-x}$ for some $b>1$. Then $\psi_{\alpha,b}$ converges uniformly to ψ. Additionally,*

(i) $\psi_{\alpha,b}(x)-\psi(x)=\frac{1}{\alpha+1}\left[-\frac{b^x\psi'(x)}{\ln b}([\alpha\phi(x)]-\alpha\phi(x)-\phi(x))+\right.$
$$\left.\frac{1}{2}\left(\frac{b^{2x}\psi'(x)}{\ln b}+\frac{b^{2x}\psi''(x)}{\ln^2 b}\right)\phi(x)(1-\phi(x))\right]+o\left(\frac{1}{\alpha}\right)$$

and

(ii) $||\psi_{\alpha,b}-\psi||\leq\frac{1}{\alpha+1}\left\{\frac{2M_1}{\ln b}+\frac{M_2}{2\ln b}+\frac{M_3}{2\ln^2 b}\right\}+o\left(\frac{1}{\alpha}\right)$

as $\alpha\to\infty$.

Proof. From (2.3), let

$$\begin{aligned}\psi_{\alpha,b}(x)&=\frac{\alpha}{\alpha+1}(\mathcal{L}^{-1}_{\alpha,\psi}\mathcal{L}_\psi)(x)=\frac{\ln b\;\Gamma(\alpha+1)}{\Gamma([\alpha\phi(x)])}\times\\&\sum_{m=0}^{\alpha-[\alpha\phi(x)]}\frac{(-1)^m}{m!(\alpha-[\alpha\phi(x)]-m)!}\int_0^\infty e^{-((m+[\alpha\phi(x)])\ln b)}\psi(y)\mathrm{d}y\\&=\frac{\ln b\;\Gamma(\alpha+1)}{\Gamma([\alpha\phi(x)])\Gamma(\alpha-[\alpha\phi(x)]+1)}\int_0^\infty(b^{-y})^{[\alpha\phi(x)]}(1-b^{-y})^{\alpha-[\alpha\phi(x)]}\psi(y)\mathrm{d}y.\end{aligned}\tag{2.6}$$

Let $\xi=b^{-y}$ in (2.6). Then,

$$\psi_{\alpha,b}(x)=\int_0^1\beta(\xi;[\alpha\phi(x)],\alpha-[\alpha\phi(x)]+1)\psi(-\log_b\xi)\mathrm{d}\xi,\tag{2.7}$$

where $\beta(\cdot,[\alpha\phi(x)],\alpha-[\alpha\phi(x)]+1)=\beta_{\alpha,x}(\cdot)$ is the beta density function. Setting $g(\xi)=\psi(-\log_b\xi)$, (2.7) becomes

$$\psi_{\alpha,b}(x)=\int_0^1 g(\xi)\beta_{\alpha,x}(\xi)d\xi.\tag{2.8}$$

From a Lemma due to Feller [16], since $g(\xi)$ is continuous and bounded, the integration in (2.8) converges uniformly to $g(\xi)|_{\xi=\phi(x)}$ in every finite interval in which the variance of the beta distribution described in (2.7) converges to zero uniformly. Consequently,

$$\int_0^1 g(\xi)\beta_{\alpha,x}(\xi)d\xi \approx g(\phi(x)) = \psi\left(\log_b \frac{1}{\phi(x)}\right) = \psi(x).$$

Next we show the rate of approximation of $\psi_{\alpha,b}$. First note that

$$\psi_{\alpha,b}(x) - \psi(x) = \int_0^1 \beta_{\alpha,x}(\xi)g(\xi)\mathrm{d}\xi - \int_0^1 \psi(-\log_b b^{-x})\mathrm{d}\xi. \tag{2.9}$$

Setting $g(b^{-x}) = \psi(-\log_b b^{-x})$, (2.9) becomes

$$\psi_{\alpha,b}(x) - \psi(x) = \int_0^1 \beta_{\alpha,x}(\xi)\left[g(\xi) - g(b^{-x})\right]\mathrm{d}\xi. \tag{2.10}$$

From Taylor's series expansion of $g(\xi)$ around b^{-x}, (2.10) becomes

$$\psi_{\alpha,b}(x) - \psi(x) = \int_0^1 \beta_{\alpha,x}(\xi)\left[g'(b^{-x})(\xi - b^{-x}) + \frac{1}{2}g''(\eta)(\xi - b^{-x})^2\right]\mathrm{d}\xi \tag{2.11}$$

for some η between ξ and b^{-x}.

Note that

$$g'(\xi) = -\frac{\psi'(-\log_b \xi)}{\xi \ln b} \quad \text{and} \quad g''(\xi) = \frac{\psi'(-\log_b(\xi))}{\xi^2 \ln b} + \frac{\psi''(-\log_b \xi)}{\xi^2 \ln^2 b}.$$

Applying (2.4) and (2.5), we get that

$$\sup_{\xi\in[0,1]} |g'(\xi)| \leq \frac{M_1}{\ln b} \tag{2.12}$$

and

$$\sup_{\xi\in[0,1]} |g''(\xi)| = \frac{M_2}{\ln b} + \frac{M_3}{\ln^2 b}. \tag{2.13}$$

Also, the mean and variance of the beta distribution $\beta_{\alpha,x}(\xi)$ are such that

$$\theta_\alpha = \frac{[\alpha\phi(x)]}{\alpha+1}, \tag{2.14}$$

$$\sigma_\alpha^2 = \frac{[\alpha\phi(x)](\alpha - [\alpha\phi(x)] + 1)}{(\alpha+1)^2(\alpha+2)} < \frac{1}{\alpha+1}, \tag{2.15}$$

and

$$|\theta_\alpha - b^{-x}| \leq \frac{2}{\alpha+1}. \tag{2.16}$$

Using (2.12)-(2.16), (2.11) becomes

$$\begin{aligned}\psi_{\alpha,b}(x) - \psi(x) &= g'(b^{-x})(\theta_\alpha - b^{-x}) + \frac{1}{2}g''(b^{-x})(\sigma_\alpha^2 - (\theta_\alpha - b^{-x})^2)\\ &= \frac{1}{\alpha+1}\Bigg[-\frac{b^x\psi'(x)}{\ln b}([\alpha\phi(x)] - \alpha\phi(x) - \phi(x)) +\\ &\quad \frac{1}{2}\left(\frac{b^{2x}\psi'(x)}{\ln b} + \frac{b^{2x}\psi''(x)}{\ln^2 b}\right)\phi(x)(1-\phi(x))\Bigg] + o\left(\frac{1}{\alpha}\right).\end{aligned} \tag{2.17}$$

Thus

$$||\psi_{\alpha,b} - \psi|| \leq \frac{1}{\alpha+1}\left(\frac{2M_1}{\ln b} + \frac{M_2}{2\ln b} + \frac{M_3}{2\ln^2 b}\right) + o\left(\frac{1}{\alpha}\right), \tag{2.18}$$

as $\alpha \to \infty$. ■

It is worth noting that in practice, the distribution of the claim sizes $X_1, \ldots, X_n$ might be unknown. Hence, the Laplace transform can be estimated by using the sample mean $\overline{X}$, $\widehat{\rho} = \lambda_p \overline{X}$, and the empirical scaled Laplace transform

$$\widehat{\mathcal{L}}_f(s\ln b) = \frac{1}{n}\sum_{i=1}^{n} e^{-(s\ln b)X_i}.$$

Applying $\widehat{\mathcal{L}}$ instead of $\mathcal{L}$ on the right hand side of (2.3), ψ is estimated by

$$\begin{aligned}\widehat{\psi}_{\alpha,b}(u) &= (\mathcal{L}^{-1}_{\alpha,\psi}\widehat{\mathcal{L}}_\psi)(u)\\ &= \frac{[\alpha b^{-u}]\ln b\Gamma(\alpha+2)}{\alpha\Gamma([\alpha b^{-u}]+1)}\sum_{m=0}^{\alpha-[\alpha b^{-u}]}\frac{(-1)^m\widehat{\mathcal{L}}_\psi((m+[\alpha b^{-u}])\ln b)}{m!(\alpha-[\alpha b^{-u}]-m)!},\end{aligned} \tag{2.19}$$

where

$$\widehat{\mathcal{L}}_\psi(s\ln b) = \frac{1}{s\ln b} - \frac{1-\widehat{\rho}}{s\ln b - \lambda_p(1-\widehat{\mathcal{L}}_f(s\ln b))}. \tag{2.20}$$

2.2 Smooth Approximations

Let $\psi_{\alpha,b}$ be as defined in (2.3). With Poisson probabilities $P_\alpha(k,x) = \frac{(\alpha x)^k}{k!}e^{-\alpha x}$, $k = 0, 1, \ldots$, smooth approximations of $\psi(u)$ can be constructed from $\psi_{\alpha,b}$ by using the definition

$$\psi_{\alpha,P}(x) := (\psi_{\alpha,b} \bullet P_\alpha)(x) = \sum_{k=0}^{\infty}\psi_{\alpha,b}\left(\frac{k}{\alpha}\right)P_\alpha\left(k,x\right). \tag{2.21}$$

From simulation studies in Section 2.4, we see that $\psi_{\alpha,P}$ converges to ψ for each $b > 1$.

2.3 Modified MR-approximations

With the goal of constructing an approximant that converges faster and has smaller absolute approximation errors, we consider a modified version of the MR-approximation. This version is in some way a linear combination of the MR-approximation with different integer moments. Let $\widetilde{\alpha} = 2\alpha$. The modified MR-approximation of $\psi(u)$ is defined as

$$\widetilde{\psi}_{\alpha,b}(x) := 2\psi_{\widetilde{\alpha},b}(x) - \psi_{\alpha,b}(x). \tag{2.22}$$

Corollary 1. *Assume that ψ is continuous. Then for each $b > 1$, $\widetilde{\psi}_{\alpha,b}$ converges uniformly to ψ and*

$$\begin{aligned}\widetilde{\psi}_{\alpha,b}(x) - \psi(x) = \frac{1}{(2\alpha+1)(\alpha+1)}\Bigg[&-\frac{\psi'(x)b^x}{\ln b}(2\alpha+3)([\alpha\phi(x)] - \alpha\phi(x) - \phi(x)) \\ &+ \left(\frac{\psi'(x)b^{2x}}{\ln b} + \frac{\psi''(x)b^{2x}}{\ln^2 b}\right)\phi(x)(1-\phi(x))\Bigg] + O\left(\frac{1}{\alpha^2}\right),\end{aligned}$$

as $\alpha \to \infty$.

Proof. Note that using (2.22) we have that

$$\widetilde{\psi}_{\alpha,b} - \psi = 2(\psi_{\widetilde{\alpha},b} - \psi) - (\psi_{\alpha,b} - \psi). \tag{2.23}$$

Convergence of $\widetilde{\psi}_{\alpha,b}$ follows from Theorem 1. Combining (2.23) and Theorem 1(i), we have that

$$
\begin{aligned}
\widetilde{\psi}_{\alpha,b}(x) - \psi(x) &= \frac{2}{2\alpha+1}\left[-\frac{b^x\psi(x)}{\ln b}([2\alpha\phi(x)] - 2\alpha\phi(x) - \phi(x)) + \right.\\
&\quad \left.\frac{1}{2}\left(\frac{b^{2x}\psi'(x)}{\ln b} + \frac{b^{2x}\psi''(x)}{\ln^2 b}\right)\phi(x)(1-\phi(x))\right] + 2\ O\left(\frac{1}{(2\alpha+1)^2}\right)\\
&\quad - \frac{1}{\alpha+1}\left[-\frac{b^x\psi(x)}{\ln b}([\alpha\phi(x)] - \alpha\phi(x) - \phi(x)) + \right.\\
&\quad \left.\frac{1}{2}\left(\frac{b^{2x}\psi'(x)}{\ln b} + \frac{b^{2x}\psi''(x)}{\ln^2 b}\right)\phi(x)(1-\phi(x))\right] - O\left(\frac{1}{(\alpha+1)^2}\right)\\
&= \frac{1}{(2\alpha+1)(\alpha+1)}\left[-\frac{\psi'(x)b^x}{\ln b}(2\alpha+3)([\alpha\phi(x)] - \alpha\phi(x) - \phi(x))\right.\\
&\quad \left. + \left(\frac{\psi'(x)b^{2x}}{\ln^b} + \frac{\psi''(x)b^{2x}}{\ln^2 b}\right)\phi(x)(1-\phi(x))\right] + O\left(\frac{1}{\alpha^2}\right), \qquad (2.24)
\end{aligned}
$$

as $\alpha \to \infty$. ■

2.3.1 *Estimation and Smooth approximations*

Applying the Poisson probabilities defined above, smooth approximations of the modifield MR-approximations can be constructed as follows:

$$
\widetilde{\psi}_{\alpha,P}(x) := (\widetilde{\psi}_{\alpha,b} \bullet P_\alpha)(x) = \sum_{k=0}^{\infty} \widetilde{\psi}_{\alpha,b}\left(\frac{k}{\alpha}\right) P_\alpha(k,x). \qquad (2.25)
$$

From simulation studies in Section 2.4, we see that $\widetilde{\psi}_{\alpha,P}$ converges to ψ.

When the distribution of the claims sizes is unknown, $\psi(u)$ is estimated by

$$
\widehat{\widetilde{\psi}}_{\alpha,b}(u) = 2\hat{\psi}_{\widetilde{\alpha},b}(u) - \hat{\psi}_{\alpha,b}(u), \qquad (2.26)
$$

where $\hat{\psi}_{\alpha,b}(u)$ is defined by (2.19).

2.4 Numerical Implementation

Obtaining an explicit expression for the probability of ultimate ruin for a surplus process with initial reserve u is generally a difficult problem. Hence, the need for a numerical approximation. To illustrate the performance of the proposed smoothed and modified MR-approximations of ψ, we conducted simulation studies with five examples. Among the examples considered, four have explicit expressions for the probability of ruin. For the other example, we compare our approximations with an existing method that has been used to approximate the probability of ruin. In examples 1 and 2, we consider models with claims sizes distribution specified as Exponential with different rates $\beta = 1, 2$, arrival intensity $\lambda = 1$, and constant premium $p = 5$. For the models in examples 3 and 4, we assume that the claims sizes follow the Gamma distribution with two different pairs of (shape $= a$, scale $= \beta$) $\in \{(2,\ 1),\ (2.5,\ 0.4)\}$. In example 5, we consider a model with claims amounts exponentially distributed with rate, $\beta = 1/5$, Poisson arrival intensity $\lambda = 50$, and constant premium rate $p = 300$. All evaluations of $\psi_{\alpha,b}(u)$ and $\widetilde{\psi}_{\alpha,b}(u)$ were conducted at $u \in \{\ln(\alpha/(\alpha - j + 1))/\ln b, 1 \le j \le \alpha\}$.

Example 1. Assume that $X \sim \text{Exp}(\text{rate} = \beta)$ with density function, $f(x) = \beta e^{-\beta x}$, $x > 0$. The scaled Laplace transform of f is

$$\mathcal{L}_f(s \ln b) = \frac{\beta}{s \ln b + \beta}. \tag{2.27}$$

From (2.2) and (2.27), the scaled Laplace transform of the probability of ultimate ruin for a model with claim sizes dristributed as X is

$$\mathcal{L}_{\psi}(s\ln b) = \frac{1}{s\ln b} - \frac{(1-\rho)(\beta+s\ln b)}{\lambda_p s\ln b}. \tag{2.28}$$

We substitute (2.28) into (2.3) to approximate the ruin probability by

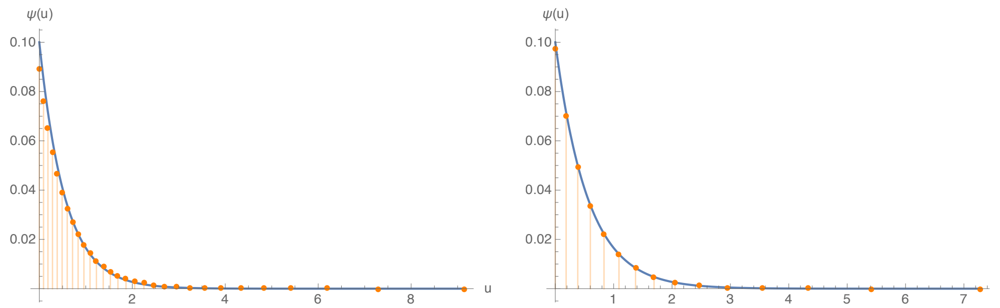
$$\psi_{\alpha,b}(u) = \frac{[\alpha b^{-u}]\ln(b)\Gamma(\alpha+2)}{\alpha\Gamma([\alpha b^{-u}]+1)} \sum_{m=0}^{\alpha-[\alpha b^{-u}]} \frac{(-1)^m}{m!(\alpha-[\alpha b^{-u}]-m)!}\times \\ \left(\frac{1}{(m+[\alpha b^{-u}])\ln b} - \frac{(1-\rho)(\beta+(m+[\alpha b{-}u])\ln b)}{\lambda_p(m+[\alpha b^{-u}])\ln b}\right).$$

The probability of ultimate ruin for a surplus Poission process with claims sizes distribution specified by the exponential distribution has the form

$$\psi(u) = \frac{1}{1+\eta}e^{-\frac{\beta\eta}{1+\eta}u}, \tag{2.29}$$

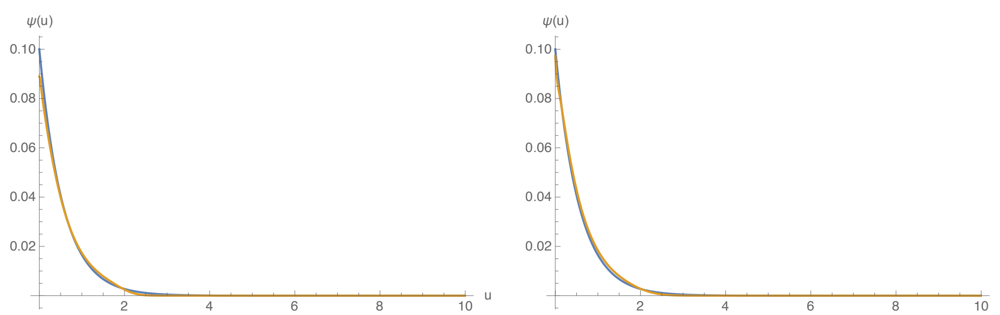
where η is the safety (security) loading, see Boland [6].

Consider the case with $\beta = 2$, $\lambda = 1$, $p = 5$, $\lambda_p = 0.2$, and $\eta = (p-\rho)/\rho = 9$. In Figure 2.1, we display $\widetilde{\psi}_{\alpha,b}$ and $\psi_{\alpha,b}$ along with the theoretical ruin probability $\psi(u)$. It is clearly seen that the approximations are close to the true values. We see also that the $\widetilde{\psi}_{\alpha,b}$ outperforms $\psi_{\alpha,b}$ even with $\alpha = 15$. To illustrate the smooth versions of the approximations, we truncated the infinite series in (2.21) and (2.25) at $M = 60$ and $M = 30$ respectively. Figure 2.2 displays the smoothed MR-approximations and the smoothed version of modified approximations.



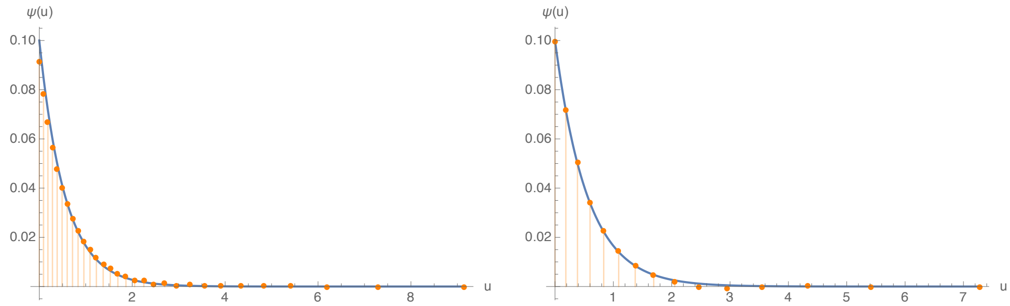
(a) (b)

Figure 2.1: Approximations of $\psi(u)$ when $X \sim \text{Exp}(\text{rate} = 2)$ by (a) $\psi_{\alpha,b}$ (dots), $\alpha = 30$ (b) $\widetilde{\psi}_{\alpha,b}$ (dots), $\alpha = 15$. In both plots, $b = 1.45$.



(a) (b)

Figure 2.2: Smooth approximations of $\psi(u)$ when $X \sim \text{Exp}(\text{rate} = 2)$ by (a) $\psi_{\alpha,P}(u)$ (yellow), $\alpha = 30$ (b) $\widetilde{\psi}_{\alpha,P}(u)$(yellow), $\alpha = 15$. In both plots, $b = 1.45$.



(a) (b)

Figure 2.3: Estimations of $\psi(u)$ when $X \sim \text{Exp}(\text{rate} = 2)$ by (a) $\widehat{\psi}_{\alpha,b}$ (dots) $\alpha = 30$ (b) $\widehat{\widehat{\psi}}_{\alpha,b}$ (dots) $\alpha = 15$. In both plots, $n = 1000$.

When the claims sizes distribution is not known, we employ empirical estimations of the scaled Laplace transform and its inversion defined in (2.19) and (2.20) to estimate $\psi(u)$. In Figure 2.3, we compare estimates $\widehat{\psi}_{\alpha,b}(u)$ and $\widehat{\widehat{\psi}}_{\alpha,b}(u)$ with the true ruin probability $\psi(u)$. Again, the estimates are close to the true values and the modified MR-estimates perform better.

To demonstrate the accuracy of $\widetilde{\psi}_{\alpha,b}$ as α increases, Table 2.1 gives several values of $10^4 \times ||\tilde{\psi}_{\alpha,b} - \psi||$ when $\alpha \in \{60, 90, 120, 150, 200, 400\}$, $\widetilde{\alpha} = 2\alpha$ and $1.35 \leq b \leq 1.50$. We observed that for this case, the pair $(\alpha = 400, b = 1.50)$ is the best among the considered parameters with normalized sup-norm error 0.00469297. Table 2.2, presents several values of the normalized sup-norm error $10^4 \times ||\psi_{\alpha,b} - \psi||$. Comparing values in Tables 2.1 and 2.2, we see that $\widetilde{\psi}_{\alpha,b}$ performs better than $\psi_{\alpha,b}$ in the normalized sup-norm error. Figure 2.4 and the values of the normalized errors in Table 2.3 further demonstrate that $\widetilde{\psi}_{\alpha,b}$ performs better than $\psi_{\alpha,b}$.

Example 2. Now assume that the claims sizes X_k's have Exp(1) distribution.

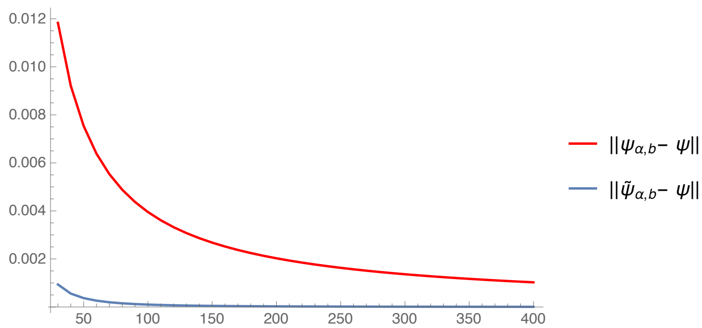


Figure 2.4: $X \sim \text{Exp}(\text{rate} = 2)$. Error approximations in sup-norm for $\alpha = 10k, 3 \leq k \leq 40$ and $1.35 \leq b \leq 1.50$.

Table 2.1: Records of $10^4 \times ||\widetilde{\psi}_{\alpha,b} - \psi||$ with $X \sim \text{Exp}(\text{rate} = 2)$ and $\lambda_p = 0.2$.

$\alpha \setminus b$	1.35	1.40	1.41	1.415	1.4175	1.42	1.425	1.43	1.45	1.50
60	3.60491	2.84059	2.71788	2.65958	2.63114	2.60318	2.54861	2.49578	2.30052	1.90408
90	1.67888	1.31663	1.25872	1.23123	1.21783	1.20465	1.17895	1.15409	1.06231	0.876574
120	0.967149	0.7566	0.723016	0.707079	0.699313	0.691678	0.676789	0.662389	0.609267	0.501956
150	0.627987	0.490532	0.468636	0.458249	0.453188	0.448213	0.438513	0.429132	0.394542	0.32474
200	0.358428	0.279545	0.266996	0.261045	0.258146	0.255297	0.249741	0.24437	0.224572	0.184662
400	0.0916074	0.0712789	0.0680519	0.0665222	0.0657772	0.0650449	0.0636177	0.0622382	0.0571567	0.0469297

In Figure 2.5, we present plots showing the $\widetilde{\psi}_{\alpha,b}(u)$ and $\psi_{\alpha,b}(u)$ along with $\psi(u)$. It is not clearly evident from the plots that $\widetilde{\psi}_{\alpha,b}$ performs better than $\psi_{\alpha,b}$. To better examine the performance of both approximations, we computed the normalized sup-norm errors. Comparing values of the normalized sup-norm errors in Tables 2.4, 2.5, and 2.6, we see that $\tilde{\psi}_{\alpha,b}(u)$ outperforms $\psi_{\alpha,b}(u)$ in terms of the sup-norm of errors. Furthermore, in Figure 2.6, we see that $\widetilde{\psi}_{\alpha,b}$ converges faster to $\psi(u)$ than $\psi_{\alpha,b}$. In Figure 2.7, we present smooth versions of both approximations. We truncated the infinite series in (2.21) and (2.22) at $M = 700$ and $M = 150$ respectively. The choices of M were dictated by the behavior of the approximations at the tails. How to determine the optimal truncation value is open for future research.

If the distribution of the claims sizes $X_1, X_2, \ldots, X_n$ is unknown, we apply

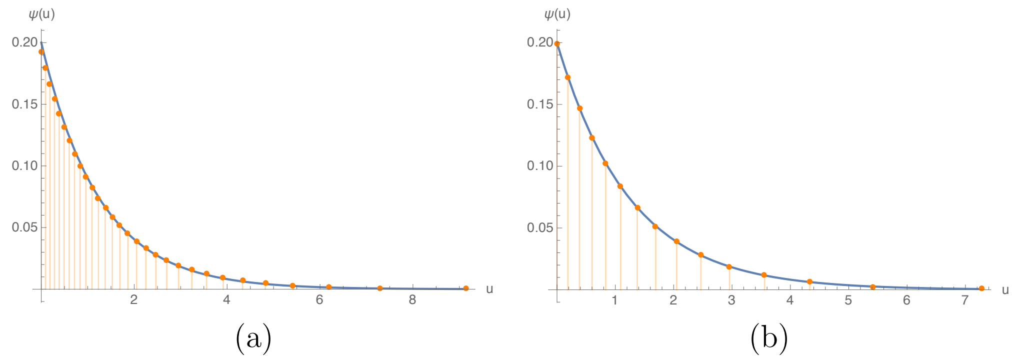


Figure 2.5: Approximations of $\psi(u)$ when $X \sim \text{Exp}(1)$ by (a) $\psi_{\alpha,b}$ (dots) $\alpha = 30$. (b) $\widetilde{\psi}_{\alpha,b}$ (dots), $\alpha = 15$. In both plots, $b = 1.45$.

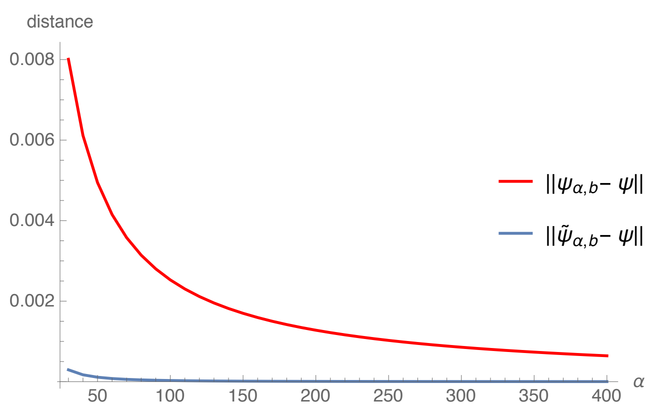


Figure 2.6: $X \sim \text{Exp}(1)$. Approximation errors in sup-norm for $\alpha = 10k, 3 \leq k \leq 40$ and $1.35 \leq b \leq 1.50$.

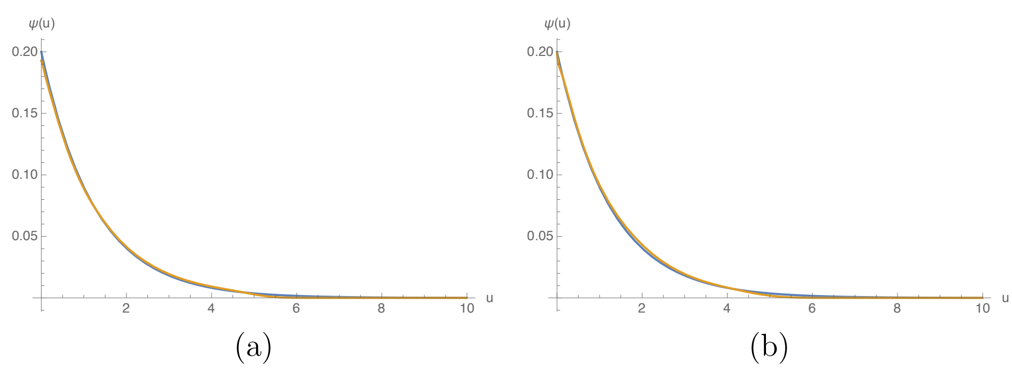


Figure 2.7: Smooth approximations of $\psi(u)$ when $X \sim \text{Exp}(1)$ by (a) $\psi_{\alpha,P}(u)$ (yellow), $\alpha = 30$ (b) $\widetilde{\psi}_{\alpha,P}(u)$ (yellow), $\alpha = 15$. In both plots, $b = 1.45$.

Table 2.2: Records of $10^4 \times ||\psi_{\alpha,b} - \psi||$ with $X \sim \text{Exp}(\text{rate} = 2)$ and $\lambda_p = 0.2$.

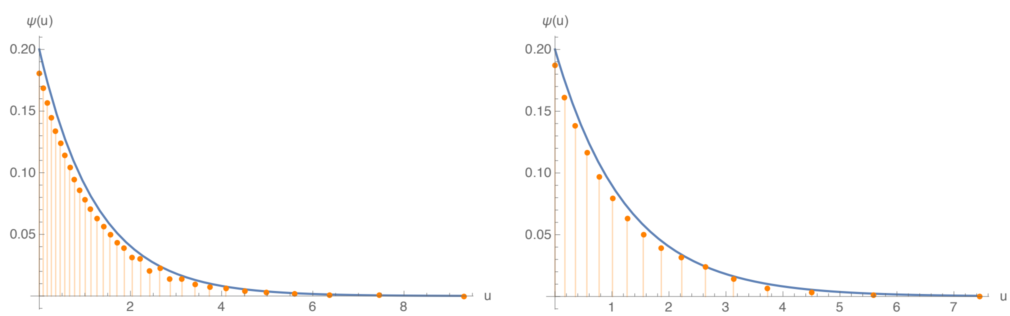
$\alpha \setminus b$	1.35	1.40	1.41	1.415	1.4175	1.42	1.425	1.43	1.45	1.50
60	75.7283	66.5593	64.9737	64.2074	63.8306	63.458	62.7249	62.0076	59.2864	53.3734
90	52.0627	45.6176	44.5071	43.9709	43.7073	43.4467	42.9342	42.4329	40.5337	36.4186
120	39.6666	34.6999	33.8458	33.4335	33.2309	33.0306	32.6368	32.2517	30.7935	27.6387
150	32.0383	27.9989	27.3051	26.9702	26.8057	26.6431	26.3233	26.0107	24.8275	22.2699
200	24.2619	21.1816	20.653	20.3981	20.2728	20.149	19.9056	19.6677	18.7674	16.8233
400	12.3102	10.7305	10.46	10.3296	10.2655	10.2021	10.0777	9.95601	9.49598	8.50399

Table 2.3: Records of $10^4 \times ||\widetilde{\psi}_{\alpha,b} - \psi||$ with $X \sim \text{Exp}(\text{rate} = 2)$ and $\lambda_p = 0.2$.

$\alpha \setminus b$	1.35	1.40	1.41	1.415	1.4175	1.42	1.425	1.43	1.45	1.50
30	12.6177	10.0727	9.65937	9.46245	9.36632	9.27168	9.08679	8.90755	8.24256	6.88001
45	6.12314	4.84685	4.64112	4.54327	4.49554	4.44858	4.3569	4.26811	3.93948	3.27016
50	5.05032	3.9905	3.81993	3.73884	3.69929	3.66038	3.58443	3.51089	3.23884	2.68545
75	2.37244	1.86415	1.78275	1.7441	1.72525	1.70672	1.67056	1.63558	1.50635	1.24452
100	1.37287	1.07559	1.02811	1.00557	0.994591	0.983791	0.962728	0.942353	0.867157	0.715099

(2.19) and (2.20) to estimate $\psi(u)$. Figure 2.8 presents estimates of the ruin probability.

Example 3. Consider a surplus model with distribution of claims sizes on an insurance policy specified by Gamma(shape $= a$, scale $= \beta$). The scaled Laplace transform of the claims sizes distribution is



(a) (b)

Figure 2.8: Estimations of $\psi(u)$ when $X \sim \text{Exp}(\text{rate} = 1/2)$ by (a) $\widehat{\psi}_{\alpha,b}$ (dots), $\alpha = 32$ (b) $\widehat{\widehat{\psi}}_{\alpha,b}$ (dots), $\alpha = 16$. In both plots, $b = 1.45$ and $n = 1000$.

Table 2.4: Records of $10^4 \times ||\widetilde{\psi}_{\alpha,b} - \psi||$ with $X \sim \exp(\text{rate} = 1)$ and $\lambda_p = 0.2$.

$\alpha \setminus b$	1.35	1.40	1.41	1.415	1.4175	1.42	1.425	1.43	1.45	1.50
60	1.15532	0.858157	0.811302	0.789129	0.778339	0.76774	0.7471	0.727174	0.653993	0.507963
90	0.52466	0.388829	0.367456	0.357347	0.352428	0.347597	0.338192	0.329114	0.295797	0.22942
120	0.298347	0.220853	0.208672	0.202911	0.200109	0.197357	0.191999	0.186829	0.167859	0.130097
150	0.192198	0.142176	0.134318	0.130603	0.128796	0.127021	0.123566	0.120232	0.108002	0.0836688
200	0.108825	0.0804454	0.0759902	0.0738839	0.0728594	0.0718534	0.0698951	0.0680056	0.0610757	0.0472942
400	0.0274773	0.0202901	0.0191629	0.0186302	0.018371	0.0181166	0.0176214	0.0171436	0.0153919	0.0119108

Table 2.5: Records of $10^4 \times ||\psi_{\alpha,b} - \psi||$ with $X \sim \text{Exp}(\text{rate} = 1)$ and $\lambda_p = 0.2$.

$\alpha \setminus b$	1.35	1.40	1.41	1.415	1.4175	1.42	1.425	1.43	1.45	1.50
60	53.1626	44.17	42.6245	41.8786	41.5121	41.1497	40.4374	39.7409	37.104	31.4021
90	35.9515	29.8256	28.7747	28.2676	28.0185	27.7723	27.2883	26.8152	25.025	21.1593
120	27.159	22.5141	21.7179	21.3339	21.1452	20.9587	20.5922	20.234	18.879	15.955
150	21.822	18.0815	17.4407	17.1317	16.9798	16.8298	16.5349	16.2467	15.1566	12.8055
200	16.4383	13.6143	13.1307	12.8975	12.783	12.6698	12.4473	12.2299	11.4078	9.63537
400	8.27355	6.84735	6.60335	6.48571	6.42793	6.37083	6.25861	6.14897	5.73445	4.84133

$$\mathcal{L}_f(s \ln b) = \left(\frac{1}{\beta s \ln b + 1} \right)^a. \tag{2.30}$$

From (2.27) and (2.30), the scaled Laplace transform of the ruin probability is

$$\mathcal{L}_\psi(s \ln b) = \frac{1}{s \ln b} - \frac{(1-\rho)(\beta s \ln b + 1)^a}{\lambda_p \left((\beta s \ln b + 1)^a - 1 \right)}. \tag{2.31}$$

By substituting (2.31) into (2.2), we derive the approximation of the ruin probability as

$$\psi_{\alpha,b}(u) = \frac{[\alpha b^{-u}] \ln(b) \Gamma(\alpha+2)}{\alpha \Gamma([\alpha b^{-u}]+1)} \sum_{m=0}^{\alpha - [\alpha b^{-u}]} \frac{(-1)^m}{m!(\alpha - [\alpha b^{-u}] - m)!} \times \left(\frac{1}{(m + [\alpha b^{-u}]) \ln b} - \frac{(1-\rho)(\beta(m + [\alpha b^{-u}]) \ln b + 1)^a}{\lambda_p((\beta(m + [\alpha b^{-u}]) \ln b + 1)^a - 1)} \right).$$

Consider the case with claims sizes $X \sim \text{Gamma}(2,1)$ and the total number of claims N, a Poisson process with arrival intensity $\lambda = 1$. This example is

Table 2.6: Records of $10^4 \times ||\widetilde{\psi}_{\alpha,b} - \psi||$ with $X \sim \text{Exp}(\text{rate} = 1)$ and $\lambda_p = 0.2$.

$\alpha \setminus b$	1.35	1.40	1.41	1.415	1.4175	1.42	1.425	1.43	1.45	1.50
30	4.33842	3.24357	3.06991	2.98762	2.94756	2.90818	2.83146	2.75734	2.48459	1.93781
45	2.01061	1.49678	1.41559	1.37716	1.35845	1.34007	1.30427	1.26971	1.14267	0.88877
50	1.64248	1.22165	1.15521	1.12377	1.10846	1.09343	1.06414	1.03587	0.931985	0.724494
75	0.749002	0.555598	0.525141	0.510731	0.50372	0.496834	0.483425	0.470482	0.422967	0.328241
100	0.426823	0.316177	0.298774	0.290543	0.286538	0.282605	0.274949	0.267559	0.240441	0.186432

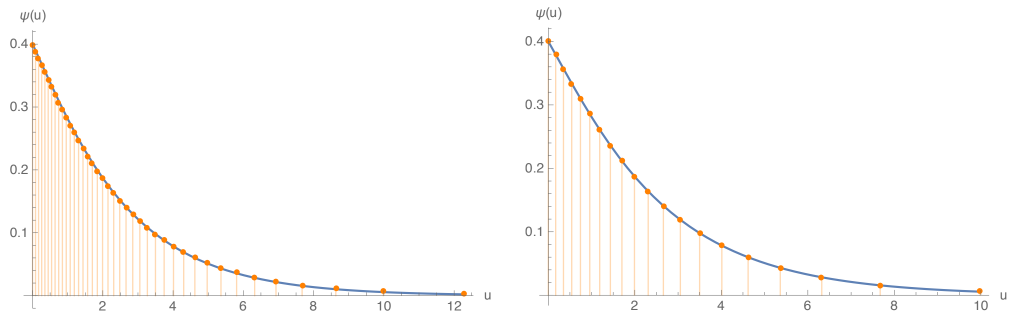
adapted from Gyzl et al. [26]. The authors derived the exact form of the ruin probability as

$$\psi(u) = 0.461862e^{-0.441742u} - 0.061862e^{-1.358257u} \text{ for } u > 0. \tag{2.32}$$

Figure 2.9 illustrates approximations of $\psi(u)$ by $\widetilde{\psi}_{\alpha,b}$ and $\psi_{\alpha,b}$. Figure 2.10 shows the smoothed approximations $\psi_{\alpha,P}$ and $\widetilde{\psi}_{\alpha,P}$. Tables 2.7 and 2.8 present normalized errors in sup-norm for various values of α and b. From Figure 2.11, we again see that $\widetilde{\psi}_{\alpha,b}$ has smaller sup-norm errors than $\psi_{\alpha,b}$ as α increases. In Table 2.9, several values of $\widetilde{\psi}_{\alpha,b}$ and $\psi_{\alpha,b}$ are presented for $\alpha = 5000$, $b = 1.415$. Additionally, values corresponding to ψ_{FT} obtained by applying the Fixed Talbot Laplace transform inversion algorithm and thoeretical values of the ruin probability are also presented. We see that the values of the modified MR-approximations are comparable with the values of FT algorithm.

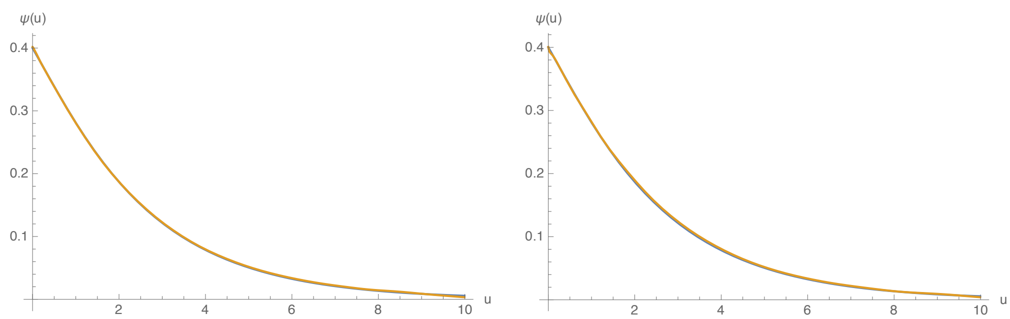
When the claims size distribution is unknown, we again apply the the emprical Laplace transform to estimate the probability of ruin. Figure 2.12 shows estimated values using $\widehat{\psi}_{\alpha,b}(u)$ and $\widehat{\widetilde{\psi}}_{\alpha,b}(u)$.

Example 4. Assume now that the claims sizes distribution is specified by Gamma$(2.5, 0.4)$. This example is adapted from Mnatsakanov et al. [41]. We



(a) (b)

Figure 2.9: Approximations of $\psi(u)$ when $X \sim \text{Gamma}(2,1)$ by (a) $\psi_{\alpha,b}$ (dots), $\alpha = 40$ (b) $\tilde{\psi}_{\alpha,b}$ (dots), $\alpha = 20$. In both plots, $b = 1.35$.



(a) (b)

Figure 2.10: Smooth approximations of $\psi(u)$ when $X \sim \text{Gamma}(2,1)$ by (a) $\psi_{\alpha,P}(u)$ (yellow), $\alpha = 40$, $M = 400$ (b) $\tilde{\psi}_{\alpha,P}(u)$ (yellow), $\alpha = 20$, $M = 200$.

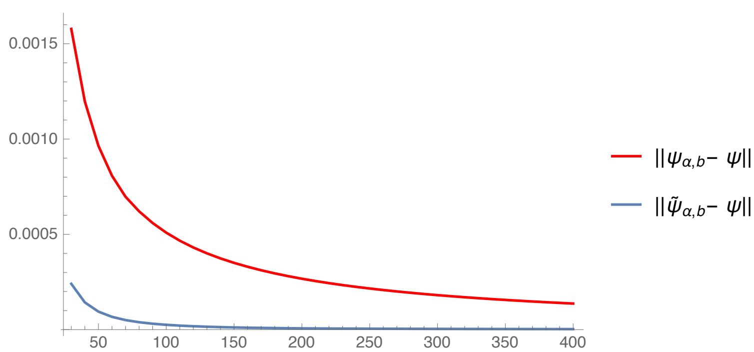


Figure 2.11: $X \sim \text{Gamma}(2,1)$. Approximation errors in sup-norm for $\alpha = 10k, 3 \leq k \leq 40,\ 1.35 \leq b \leq 1.50$.

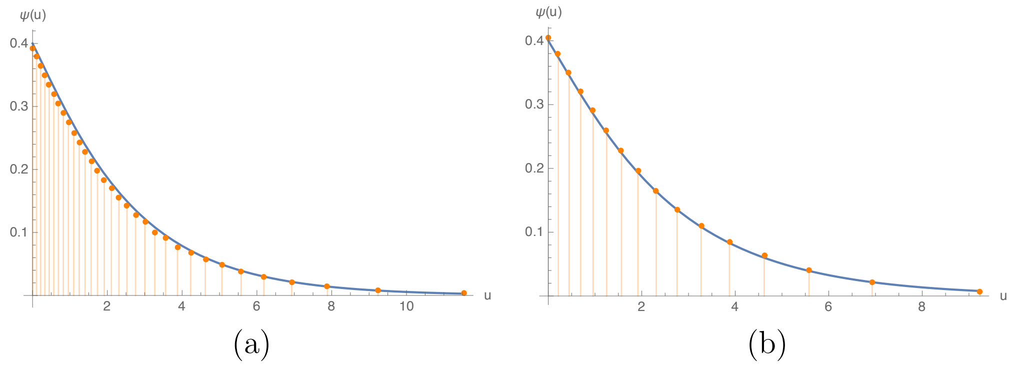


Figure 2.12: Estimations of $\psi(u)$ when $X \sim \text{Gamma}(2,1)$ by (a) $\widehat{\psi}_{\alpha,b}$, $\alpha = 40$ (b) $\widehat{\widehat{\psi}}_{\alpha,b}$ (dots) of $\psi(u)$, $\alpha = 20$. In both plots, $b = 1.35$ and $n = 1000$.

Table 2.7: Records of $10^4 \times ||\widetilde{\psi}_{\alpha,b} - \psi||$ with $X \sim \text{Gamma}(2,1)$ and $\lambda_p = 0.2$.

$\alpha \setminus b$	1.35	1.40	1.41	1.415	1.4175	1.42	1.425	1.43	1.45	1.50
30	2.77275	2.49112	2.44116	2.4169	2.40494	2.3931	2.36977	2.34688	2.25958	2.06786
45	1.35393	1.19594	1.16873	1.15559	1.14914	1.14276	1.13021	1.11796	1.07158	0.971673
60	0.798906	0.699685	0.682807	0.674683	0.670697	0.666759	0.659028	0.651483	0.623045	0.562313
75	0.526332	0.458597	0.447158	0.441659	0.438963	0.436302	0.431079	0.425987	0.406834	0.366135
100	0.30483	0.264232	0.257423	0.254154	0.252553	0.250972	0.247874	0.244855	0.233523	0.20956
200	0.0796509	0.0685068	0.0666558	0.0657692	0.0653352	0.0654537	0.0663055	0.0669547	0.0673546	0.0537883

compare the values of $\widetilde{\psi}_{\alpha,b}$ with $\psi_{\alpha,b}$ and ψ_{FT}. In Table 2.10, $\alpha = 4000$ and $b = 1.14795$, while in Table 2.11, $\alpha = 1000$ and $b = 1.1485$. In both tables, the arrival intensity $\lambda = 1$ and the premium $p = 1.1$. We see from the tables that using large values of α, approximations of ψ by $\widetilde{\psi}_{\alpha,b}$ are closer to ψ_{FT} than $\psi_{\alpha,b}$.

Example 5. Consider the case when claim sizes $X \sim \text{Exp}(\text{rate} = 1/5)$, $\lambda = 50$, $p = 300$, and safety loading $\eta = 0.2$. Figure 2.13 presents the visualizations of the MR-approximations along with the true ruin probability. The high probability of ruin reflected in the plots is due to low security loading. We clearly see that $\tilde{\psi}_{\alpha,b}(u)$ outperforms $\psi_{\alpha,b}(u)$. In Figure 2.14 we present smooth approximations of $\psi(u)$. To implement the infinite series in (2.21) and (2.25),

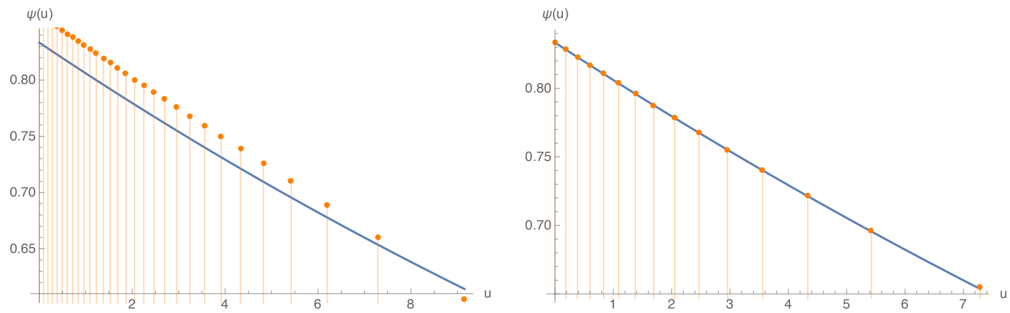
Table 2.8: Records of $10^4 \times ||\psi_{\alpha,b} - \psi||$ with $X \sim \text{Gamma}(2, 1)$ and $\lambda_p = 0.2$.

$\alpha \setminus b$	1.35	1.40	1.41	1.415	1.4175	1.42	1.425	1.43	1.45	1.50
60	15.398	9.64228	8.72093	8.291	8.07721	8.24543	8.82845	9.39798	11.5503	16.1823
90	10.3284	6.47838	5.86249	5.57266	5.59349	5.78961	6.17496	6.5514	7.97399	11.0357
120	7.77521	4.87904	4.41573	4.19713	4.3097	4.45609	4.74374	5.02473	6.08666	8.3723
150	6.2329	3.91266	3.54197	3.38572	3.50388	3.62064	3.85007	4.07419	4.92122	6.7444
200	4.68474	2.94233	2.66394	2.5823	2.67064	2.75794	2.92948	3.09705	3.73036	5.09364
400	2.35087	1.47804	1.33861	1.32403	1.36799	1.41142	1.49677	1.58015	1.89528	2.57371

Table 2.9: Values of ψ, $\widetilde{\psi}_{\alpha,b}$, $\psi_{\alpha,b}$, and ψ_{FT} computed at several values of $u_j = \ln(\alpha/(\alpha - j + 1))/\ln b$. $X \sim \text{Gamma}(2,\ 1)$, $\alpha = 5000$, $b = 1.425$, $\lambda = 1$, and $p = 5$.

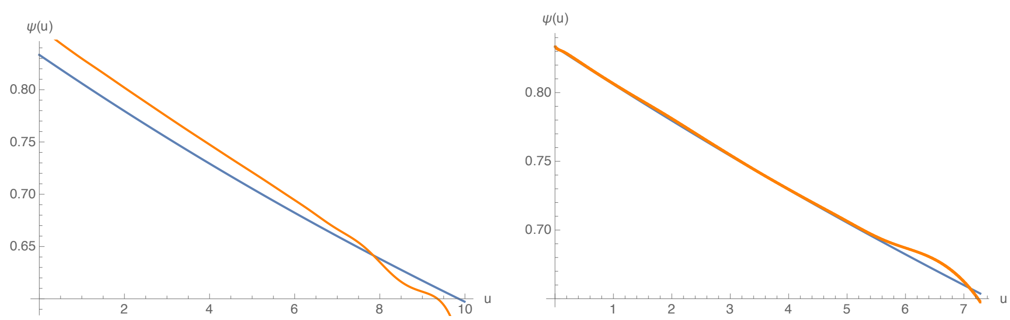
j	500	600	700	800	900	1000
$\psi(u_j)$	0.362835	0.354857	0.346728	0.338460	0.330063	0.321547
$\widetilde{\psi}_{\alpha,b}(u_j)$	0.362835	0.354857	0.346728	0.338460	0.330062	0.321546
$\psi_{\alpha,b}(u_j)$	0.362832	0.354853	0.346723	0.338453	0.330055	0.321538
$\psi_{FT}(u_j)$	0.362834	0.354856	0.346728	0.338460	0.330062	0.321546
j	2000	2500	3000	3500	4000	4500
$\psi(u_j)$	0.232093	0.186355	0.141464	0.098557	0.058914	0.024346
$\widetilde{\psi}_{\alpha,b}(u_j)$	0.232093	0.186355	0.141464	0.098557	0.058914	0.024346
$\psi_{\alpha,b}(u_j)$	0.232084	0.186349	0.141461	0.098559	0.058919	0.024352
$\psi_{FT}(u_j)$	0.232092	0.186355	0.141463	0.098556	0.058913	0.024345

we truncated the series at $M = 300$ and $M = 137$, respectively. We see from the plots that there are big drop-offs at large values of u. This is an edge effect due to fragment issues between the truncation point and the number of integer moments α. In Figure 2.15 we display estimated values for situations when the distribution of the claims sizes is unknown.



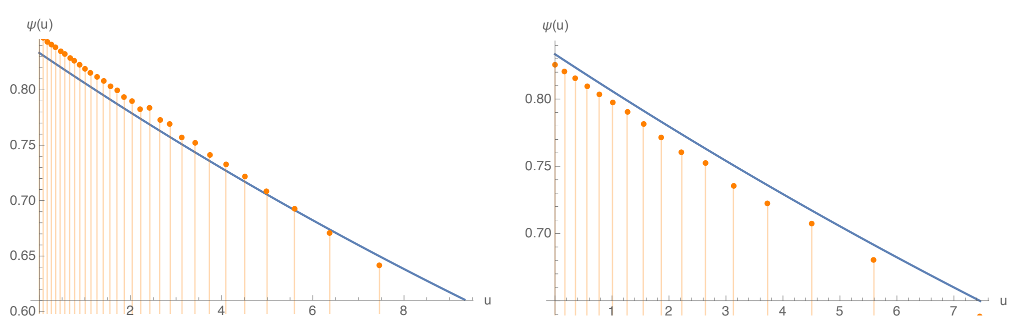
(a) (b)

Figure 2.13: Approximations of $\psi(u)$ when $X \sim \text{Exp}(1/5)$ by (a) $\psi_{\alpha,b}$, $\alpha = 30$ (b) $\tilde{\psi}_{\alpha,b}$ (dots), $\alpha = 15$. In both plots, $b = 1.45$.



(a) (b)

Figure 2.14: Smooth approximations of $\psi(u)$ when $X \sim \text{Exp}(1/5)$ by (a) $\psi_{\alpha,P}(u)$ (orange), $\alpha = 30$ (b) $\tilde{\psi}_{\alpha,P}(u)$ (orange), $\alpha = 15$. In both plots, $b = 1.45$.



(a) (b)

Figure 2.15: Estimations of $\psi(u)$ when $X \sim \text{Exp}(1/5)$ by (a) $\psi_{\alpha,b}$ (dots), $\alpha = 32$ (b) $\tilde{\psi}_{\alpha,b}$ (dots), $\alpha = 16$. In both plots, $b = 1.45$ and $n = 1000$.

Table 2.10: Values of ultimate ruin probabilities $\widetilde{\psi}_{\alpha,b}$, $\psi_{\alpha,b}$ and ψ_{FT} computed at several values of $u_j = \ln(\alpha/(\alpha - j + 1))/\ln b$. $X \sim \text{Gamma}(2.5,\ 0.4)$, $\alpha = 4000$, $b = 1.14795$, $\lambda = 1$ and $p = 1.1$.

j	500	600	700	800	900	1000
$\psi_{\alpha,b}(u_j)$	0.811541	0.789651	0.767593	0.745439	0.723223	0.700961
$\widetilde{\psi}_{\alpha,b}(u_j)$	0.811535	0.789644	0.767587	0.745432	0.723216	0.700954
$\psi_{FT}(u_j)$	0.811533	0.789643	0.767586	0.745431	0.723215	0.700954
j	1500	2000	2500	3000	3500	4000
$\psi_{\alpha,b}(u_j)$	0.589078	0.476154	0.361914	0.245881	0.127030	0.000331
$\widetilde{\psi}_{\alpha,b}(u_j)$	0.589073	0.476151	0.361914	0.245882	0.127033	0.000337
$\psi_{FT}(u_j)$	0.589071	0.476150	0.361913	0.245882	0.127031	0.000337

Table 2.11: Values of ultimate ruin probabilities $\widetilde{\psi}_{\alpha,b}$, $\psi_{\alpha,b}$ and ψ_{FT} computed at several values of $u_j = \ln(\alpha/(\alpha - j + 1))/\ln b$. $X \sim \text{Gamma}(2.5,\ 0.4)$, $\alpha = 1000$, $b = 1.1485$, $\lambda = 1$ and $p = 1.1$.

j	25	50	100	200	300	400
$\psi_{\alpha,b}(u_j)$	0.893671	0.875107	0.834075	0.746673	0.657782	0.568228
$\widetilde{\psi}_{\alpha,b}(u_j)$	0.893506	0.875027	0.834045	0.746644	0.657756	0.568208
$\psi_{FT}(u_j)$	0.893504	0.875025	0.834044	0.746643	0.657754	0.568208
j	500	600	700	800	900	1000
$\psi_{\alpha,b}(u_j)$	0.477934	0.386743	0.294416	0.200537	0.104223	0.001269
$\widetilde{\psi}_{\alpha,b}(u_j)$	0.477921	0.386738	0.294417	0.200546	0.104240	0.001293
$\psi_{FT}(u_j)$	0.47792	0.386737	0.294416	0.200546	0.104238	0.001293

Chapter 3

Inverse Function of Ruin Probability and Value at Risk

The need to guard against loss motivates the active research area of risk management. Regulators often require that insurance and financial institutions have available capital that is at least equal to some appropriate VaR of the one-year risk. This capital helps protect against solvency issues. For instance, in insurance, such capital is used to mitigate against the risk that the premiums and initial reserve, in addition to income from investment, becomes insufficient to cover future claims. Questions about the appropriateness of measuring risk over a fixed time horizon motivated the suggestion of a ruin consistent mesaure by Trufin et al. [52]. In this chapter, we present moment recovered approximations of the inverse function of the ruin probability. We also show their application in the compuation of the average VaR.

Recall

$$\rho_\epsilon[X] = \inf\{u \geq 0 | \psi(u) \leq \epsilon\} = \psi^{-1}(\epsilon),$$

where u is the initial reserve of an insurance company and ϵ is a specified confidence level.

3.1 Moment Recovered Approximation

In the preceding chapter, we defined the MR approximation of the ultimate ruin probability $\psi(u)$ in (2.3). To approximate the inverse function $\psi^{-1}(x)$ of the ruin probability from (2.3), we define the MR-approximation as

$$\psi^{-1}_{\alpha,b}(x) = \int_0^\infty \bar{B}_\alpha(\psi_{\alpha,b}(u), x)du, \tag{3.1}$$

where $\bar{B}_\alpha(t,x) = 1 - B_\alpha(t,x)$ and

$$B_\alpha(t,x) = \sum_{k=0}^{[\alpha x]} \binom{\alpha}{k} t^k (1-t)^{\alpha-k} \to \left\{ \begin{array}{ll} 1, & t < x \\ 0, & t > x \end{array} \right. \quad \text{as } \alpha \to \infty. \tag{3.2}$$

Note that (3.1) can be expressed as

$$\psi^{-1}_{\alpha,b}(x) = \sum_{l=1}^{\alpha} \int_{u_{l-1}}^{u_l} \bar{B}_\alpha(\psi_{\alpha,b}(u), x)\ dx = \sum_{l=1}^{\alpha} \bar{B}_\alpha(\psi_{\alpha,b}(u_{l-1}), x)\Delta u_l, \tag{3.3}$$

where $u_l = [\ln\alpha - \ln(\alpha - j + 1)]/\ln b$ for $1 \leq l \leq \alpha$, $\Delta u_l = u_{l-1} - u_l$, and

$$\psi_{\alpha,b}(u_l) = \frac{(\alpha - l + 1)\ln(b)\Gamma(\alpha+2)}{\alpha\Gamma(\alpha - l + 2)} \sum_{j=0}^{l-1} \frac{(-1)^j \mathcal{L}_\psi((j + \alpha - l + 1)\ln(b))}{j!(l-1-j)!}. \tag{3.4}$$

Theorem 2. *Assume that ψ^{-1} is continuous on $(0, 1)$. Then for each fixed $b > 1$, $\psi^{-1}_{\alpha,b}$ converges uniformly to ψ^{-1} as $\alpha \to \infty$.*

Proof. Let $1 - \psi(u) = G(u)$ be the survival function. We check that $\psi^{-1}(t) = G^{-1}(1 - t)$. Let $G^{-1}(x)$ be the inverse function of $G(u)$ such that $G^{-1}(x) := \inf\{t \geq 0 : G(t) \geq x\}$. First, we approximate G by $G_{\alpha,b}(u)$. Then, we approximate G^{-1} by

$$G^{-1}_{\alpha,b}(x) = \int_0^{\infty} B_\alpha(G_{\alpha,b}(u), x)\mathrm{d}u. \tag{3.5}$$

Now

$$G^{-1}_{\alpha,b}(x) - G^{-1}(x) = \underbrace{\int_0^{\infty} B_\alpha(G_{\alpha,b}(u), x)\mathrm{d}u - \int_0^{\infty} B_\alpha(G(u), x)\mathrm{d}x}_{(a)} + \underbrace{\int_0^{\infty} B_\alpha(G(u), x)\mathrm{d}u - G^{-1}(x)}_{(b)}. \tag{3.6}$$

Let $\int_0^{\alpha} B_\alpha(G(u), x)\mathrm{d}x = G^{-1}_\alpha(x)$. Then expression (3.6)(a) becomes

$$G^{-1}_{\alpha,b}(x) - G^{-1}_\alpha(x) = \int_0^{\infty} \left[B_\alpha(G_{\alpha,b}(u), x) - B_\alpha(G(u), x)\right] \mathrm{d}x. \tag{3.7}$$

Taking the Taylor's series expansion of $B_\alpha(G_{\alpha,b}(u), x)$ around $G(u)$, we get

$$\begin{aligned} G^{-1}_{\alpha,b}(x) - G^{-1}_\alpha(x) = &-\int_0^{\infty} \beta(G(u), c, d-1)(G_{\alpha,b}(u) - G(u))\mathrm{d}u \\ &+ \frac{1}{2}\int_0^{\infty} \Delta\beta(\tilde{G}(u), x)(G_{\alpha,b}(u) - G(u))^2\mathrm{d}u, \end{aligned} \tag{3.8}$$

where $\beta(\cdot, c, d)$ is the density function of the Beta distribution with $c = [\alpha x]$,

$d = \alpha - [\alpha x] + 1$, $\Delta\beta(\cdot, x) = \beta(\cdot, c, d-2) - \beta(\cdot, c-1, d-1)$, and $\tilde{G}(u)$ between $G_{\alpha,b}(u)$ and $G(u)$, see Mnatsakanov [38].

Let $\tau = G(u)$ in (3.8). We get that $G(0) = 1 - \rho$, where $\rho = \frac{\lambda}{p}E(X)$. From Mnatsakanov et al. [42], we have that $||G_{\alpha,b}(u) - G(u)|| < C/(\alpha+1)$, for postive constant C that can be identified by the constants in (2.4). Likewise

$$\int_{1-\rho}^{1} \beta(\tau, c, d-1) Q'(\tau) \mathrm{d}\tau \leq ||Q'||, \tag{3.9}$$

and

$$\int_{1-\rho}^{1} |\Delta\beta(\tilde{G}(u), x)| \leq 2||\tilde{Q}'||, \tag{3.10}$$

where Q' is the derivative of G^{-1}. Finally,

$$||G_{\alpha,b}^{-1} - G_{\alpha}^{-1}|| \leq \frac{C}{\alpha+1}(||Q'|| + ||\tilde{Q}'||) + o\left(\frac{1}{\alpha}\right). \tag{3.11}$$

Assume $supp\{G\} = [0, T]$, $0 < T < \infty$. We have that $B_\alpha(G(u), x) \to \mathbb{I}(G(u) < x)$ and $|B_\alpha(G(u), x)| \leq 1$. By Lebesgue's dominated convergence theorem, see Royden [45], we have that

$$G_{\alpha}^{-1}(x) = \int_0^T B_\alpha(G(u), x)\mathrm{d}x \to \int_0^T \mathbb{I}(u < G^{-1}(x))\mathrm{d}u. \tag{3.12}$$

Hence, for each $x \in supp\{G\}$ with $(G^{-1}(x) < T)$, we have that

$$G_{\alpha}^{-1}(x) \to G^{-1}(x), \tag{3.13}$$

uniformly as $\alpha \to \infty$. Combining (3.11) and (3.13), completes the proof. ∎

In cases when the distribution of the claims sizes $X_1, X_2, \cdots, X_n$ is not known, we can consider the estimator of ψ^{-1} as

$$\hat{\psi}_{\alpha,b}(x) = \int_0^\infty \bar{B}_\alpha(\hat{\psi}_{\alpha,b}(u), x)dx = \sum_{l=1}^{\alpha} \bar{B}_\alpha(\hat{\psi}_{\alpha,b}(u_{l-1}), x)\Delta u_l. \tag{3.14}$$

3.2 Smooth Approximation

From the Weierstrass approximation theorem, we know that every function defined on a compact interval can be uniformly approximated by a polynomial function. The inverse function of the ruin probability is defined on the interval $[0, 1]$, hence it can be approximated by polynomials. To achieve smooth approximations of $\psi^{-1}(x)$, we apply the Bernstein polynomials in conjunction with the MR-approximations.

Definition 2. The Bernstein polynomial of order n for a function f defined on $[0, 1]$ is defined by

$$B_n(f,\ x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}. \tag{3.15}$$

From (3.3) and (3.15), we construct polynomial approximations of $\psi^{-1}(x)$ as follows; let

$$b_\alpha(k, x) = \binom{\alpha}{k} x^k (1-x)^{\alpha-k} \text{ for } k = 0, 1, \cdots, \alpha.$$

Then,

$$\psi^{-1}_{\alpha,B}(x) := (\psi^{-1}_{\alpha,b} \bullet b_\alpha)(x) = \sum_{k=0}^{\alpha} \psi^{-1}_{\alpha,b}\left(\frac{k}{\alpha}\right) b_\alpha(k,x) \tag{3.16}$$

is the smoothed MR-approximation of $\psi^{-1}(u)$.

Corollary 2. *Assume that ψ^{-1} is continuous. For each fixed $b > 1$, $\psi^{-1}_{\alpha,B}$ converges uniformly to ψ^{-1} as $\alpha \to \infty$.*

Proof. We see that

$$\begin{aligned}
\psi^{-1}_{\alpha,B}(x) - \psi^{-1}(x) &= \sum_{k=0}^{\alpha} \psi^{-1}_{\alpha,b}\left(\frac{k}{\alpha}\right) b_\alpha(k,x) - \sum_{k=0}^{\alpha} \psi^{-1}\left(\frac{k}{\alpha}\right) b_\alpha(k,x) \\
&\quad + \sum_{k=0}^{\alpha} \psi^{-1}\left(\frac{k}{\alpha}\right) b_\alpha(k,x) - \psi^{-1}(x) \\
&= \sum_{k=0}^{\alpha} \left[\psi^{-1}_{\alpha,b}\left(\frac{k}{\alpha}\right) - \psi^{-1}\left(\frac{k}{\alpha}\right)\right] b_\alpha(k,x) + \sum_{k=0}^{\alpha} \psi^{-1}\left(\frac{k}{\alpha}\right) b_\alpha(k,x) - \psi^{-1}(x) \\
&= \sum_{k=0}^{\alpha} \left[\psi^{-1}_{\alpha,b}\left(\frac{k}{\alpha}\right) - \psi^{-1}\left(\frac{k}{\alpha}\right)\right] b_\alpha(k,x) + B_\alpha(\psi^{-1},x) - \psi^{-1}(x). \qquad (3.17)
\end{aligned}$$

Combining Theorem 2 and Definition 2, we see that

$$\begin{gathered}
||\psi^{-1}_{\alpha,B} - \psi^{-1}|| \le \sup_{x\in(0,1)} |\psi^{-1}_{\alpha,b}(x) - \psi^{-1}(x)| \sum_{k=0}^{\alpha} b_\alpha(k,x) + \\
\sup_{x\in(0,1)} |B_\alpha(\psi^{-1},x) - \psi^{-1}(x)| \to 0
\end{gathered}$$

as $\alpha \to \infty$. $\blacksquare$

3.3 Average Value at Risk

As mentioned above, the probability of ruin can be interpreted as the continuous alternative to the risk measure VaR. Hence a natural extension of the inverse function $\psi^{-1} : [0,\ 1] \to \mathbb{R}^{+}$ is the estimation of the Tail-VaR (or average VaR). Trufin et al. [52] defined the Tail-VaR as

$$\mathrm{TVaR}(\epsilon) := \frac{1}{\epsilon}\int_0^{\epsilon} \psi^{-1}(x)dx, \text{ for } \epsilon \in [0,\ 1].$$

From the approximant $\psi_{\alpha,b}^{-1}$ of ψ^{-1}, we construct the MR-approximation of $\mathrm{TVaR}(\epsilon)$ as

$$\mathrm{TVaR}_{\alpha,b}(\epsilon) := \frac{1}{\epsilon}\int_0^{\epsilon} \psi_{\alpha,b}^{-1}(x)dx = \frac{1}{\alpha\epsilon}\sum_{l=1}^{[\alpha\epsilon]} \psi_{\alpha,b}^{-1}\left(\frac{l-1}{\alpha}\right). \tag{3.18}$$

3.4 Numerical Implementation

We present three examples to demonstrate the performance of $\psi_{\alpha,b}^{-1}$. In examples 1 and 2, we considered again the models with claims sizes ditribution specified by the exponential distribution with rates $\delta = 2$ and $\delta = 0.5$. In both examples the arrival intensity $\lambda = 1$. For the third example, we consider a model with claims sizes distribution specified as Exp(1/5). For all the models, we approximate the Tail-VaR.

Example 6. Assume that the claims sizes distributions is specified by Exponential(rate $= \delta$). For a surplus Poisson process, the inverse function of the probability of ultimate ruin has the form

$$\psi^{-1}(x) = -\frac{1+\eta}{\delta\eta}\ln(x(1+\eta)), \tag{3.19}$$

where η is the safety (or security) loading.

We insert the scaled Laplace transforms from (2.27) and (2.28) into (3.4) to approximate inverse function of the ulmitate ruin probability $\psi^{-1}(x)$ by

$$\psi^{-1}_{\alpha,b}(x) = \sum_{l=1}^{\alpha} 1 - B(\psi_{\alpha,b}(l-1), x)\frac{\ln\left(\frac{\alpha-l+2}{\alpha-l+1}\right)}{\ln b}, \tag{3.20}$$

where

$$\psi_{\alpha,b}(l-1) = \frac{(\alpha-l+1)\ln(b)\Gamma(\alpha+2)}{\alpha\Gamma(\alpha-l+2)}\sum_{j=0}^{l-1}\frac{(-1)^j}{j!(l-1-j)!}\times \left(\frac{1}{(j+\alpha-l+1)\ln b} - \frac{(1-\rho)(\beta+(j+\alpha-l+1)\ln b)}{\lambda_p(j+\alpha-l+1)\ln b}\right). \tag{3.21}$$

Now, consider the case when $X \sim \text{Exp}(2)$, $\eta = 9$, and constant premium $p = 5$. In Figure 3.1, we present the MR-approximation of the inverse ruin function alongside the exact values. The approximations required large values of integer moments, α. To demonstrate the performance of the approximation as α increases, we present, in Tables 3.1 and 3.2, several values of $\psi^{-1}_{\alpha,b}(x)$ and $\psi^{-1}(x)$ at $\alpha = 2000$ and $\alpha = 5000$ with $b = 1.05$. We observe that indeed as α increases, the approximations gets closer to the exact value.

In Table 3.3, we present approximations of the Tail-VaR of the claims sizes along

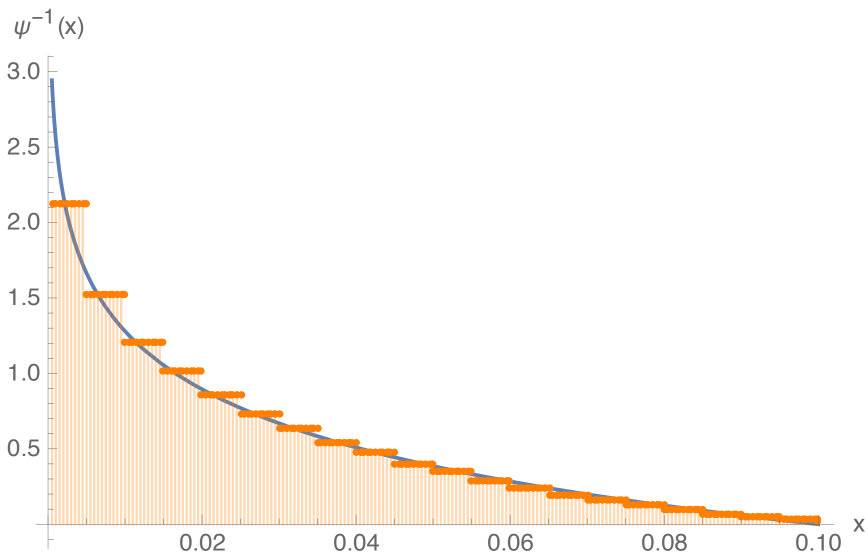


Figure 3.1: Approximation of ψ^{-1}, $X \sim \text{Exp}(2)$, $\psi^{-1}_{\alpha,b}$ (orange), $\alpha = 200$, $b = 1.05$.

Table 3.1: Values of $\psi^{-1}_{\alpha,b}$ and ψ^{-1} $x_j = j/10\alpha$. $X \sim \text{Exp}(\text{rate} = 2)$, $\alpha = 2000$, $b = 1.05$

j	25	50	100	500	700
$\psi^{-1}(x_j)$	2.43446	2.04938	1.66430	0.770164	0.583235
$\psi^{-1}_{\alpha,b}(x_j)$	2.44992	2.01025	1.64795	0.767088	0.579976
j	900	1000	1500	1800	2000
$\psi^{-1}(x_j)$	0.443615	0.385082	0.159823	0.0585336	0
$\psi^{-1}_{\alpha,b}(x_j)$	0.439927	0.381154	0.154730	0.0543829	0.0121134

with the exact values. We see that approximations by $\text{TVAR}_{\alpha,b}$ are comparable to the true values.

Example 7. Now assume the the claim sizes distribution follow Exp(0.5). Consider the case when safety loading $\eta = 1.5$, premium $p = 5$, and arrival intensity $\lambda = 1$. Figure 3.2 demonstrates the performance of $\psi^{-1}_{\alpha,b}$. Evaluations were conducted at $x_j \in \{j/2.5\alpha, 1 \leq j \leq \alpha\}$. It is clear from plots that as α increases, for fixed b, the approximations get closer to the true values.

Table 3.2: Values of $\psi^{-1}_{\alpha,b}$ and ψ^{-1} $x_j = j/10\alpha$. $X \sim \text{Exp}(\text{rate} = 2)$, $\alpha = 5000$, $b = 1.05$

j	400	500	600	700	800	900	1000
$\psi^{-1}(x_j)$	1.403183	1.279214	1.177924	1.092285	1.01810	0.952666	0.894132
$\psi^{-1}_{\alpha,b}(x_j)$	1.399710	1.276637	1.175875	1.090563	1.01659	0.951284	0.892834
j	2000	2500	3000	3500	4000	4500	5000
$\psi^{-1}(x_j)$	0.509050	0.385082	0.283792	0.198153	0.123969	0.058534	0
$\psi^{-1}_{\alpha,b}(x_j)$	0.507674	0.383517	0.282034	0.196208	0.121848	0.056340	0.008259

Table 3.3: Values of Tail-VaR. $X \sim \text{Exp}(2)$, $\alpha = 200$, $b = 1.05$

ϵ	$\text{TVaR}_{\alpha,b}(\epsilon)$	$\text{TVaR}(\epsilon)$
0.05	0.951237	0.940637
0.10	0.556362	0.555556
0.20	0.279887	0.170474

Example 8. Consider the case when the claims amounts distribution is specified by $X \sim \text{Exp}(1/5)$, where Poisson arrival intensity $\lambda = 50$, constant premiun rate $p = 300$, and safety loading $\eta = 0.2$. In Figure 3.3, we present approximations of the inverse function of the ruin probability by $\psi^{-1}_{\alpha,b}$. Evaluations were conducted at points $x_j \in \{j/1.2\alpha, j = 1, \ldots, \alpha\}$ for $\alpha = 400$ and $b = 1.05$.

Next, we compute some values of the Tail-VaR (average VaR) of the claims sizes. Recall the VaR is the level p-quantile. In Table 3.4, we present values of $\text{TVaR}_{\alpha,b}$ along with the exact values $\text{TVAR}(\epsilon)$.

Table 3.4: Values of Tail-VaR. $X \sim \text{Exp}(1/5)$, $\alpha = 200$, $b = 1.05$

ϵ	$\text{TVaR}_{\alpha,b}(\epsilon)$	$\text{TVaR}(\epsilon)$
0.80	30.7981	31.2247
0.90	27.4187	27.6912
0.95	25.9756	26.0692
0.99	24.9261	25.9756

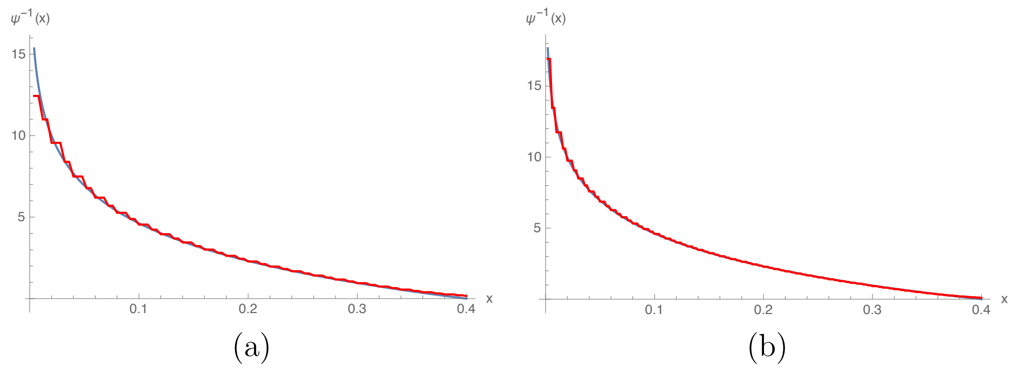


Figure 3.2: Approximation of ψ^{-1} when claims sizes $X \sim \text{Exp}(0.5)$, (a) $\psi^{-1}_{\alpha,b}$ (red line) $\alpha = 100$. (b) $\psi^{-1}_{\alpha,b}$ (red line), $\alpha = 200$. In both plots, $b = 1.05$.

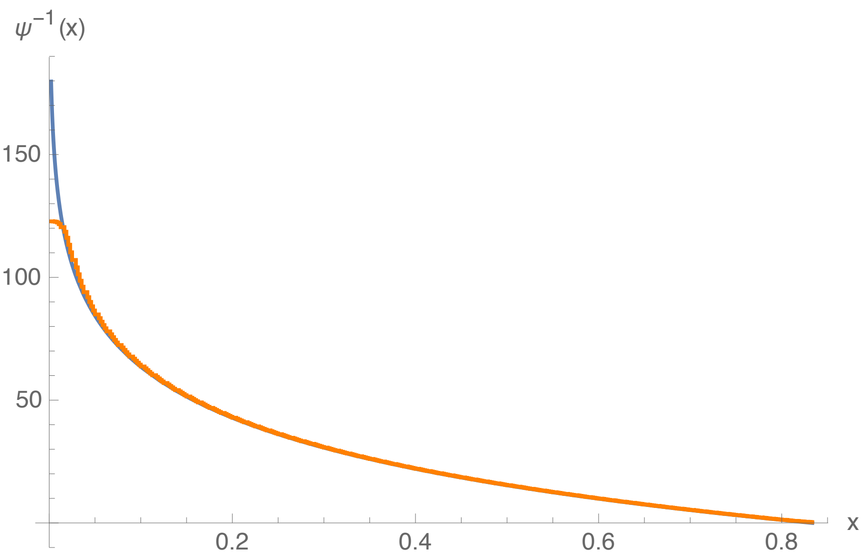


Figure 3.3: Approximation of ψ^{-1}, $X \sim \text{Exp}(1/5)$, $\psi^{-1}_{\alpha,b}$ (orange), $\alpha = 400$, $b = 1.05$.

Chapter 4

Bivariate Distribution

Recall a bivariate collective risk model is defined in (1.4). In this chapter, we present moment recovered approximations of Bivariate Probability Distribution Functions (BPDFs) of random vectors from the knowledge of their bivariate scaled Laplace transforms. The recovery of the BPDF of aggragrate claims amount is also discussed.

4.1 Bivariate Scaled Laplace Transform

Assume that F is an absolutely continuous distribution with $\mathbb{R}_+^2$ as its support. Let f be the probability density function of F with respect to the Lebesgue measure on $\mathbb{R}_+^2$.

Definition 3. Let (X, Y) be a random vector distributed according to F. Assume that we have a given sequence of moments $\mu(F) = \{\mu_{j,m}(F); (j, m) \in$

$\mathbb{N} \times \mathbb{N}\}$. Then, the bivariate scaled Laplace transform is defined as

$$\mu_{j,m}(F) := \mathcal{L}_f^{(2)}(s \ln b, t \ln b) = \int_0^\infty \int_0^\infty e^{-(sx+ty)\ln b} f(x,y) \mathrm{d}x\mathrm{d}y. \tag{4.1}$$

From Mnatsakanov [39], we recover the bivariate density function and bivariate survival function from the scaled Laplace transform inversion by the following approximations:

$$\begin{aligned} f_{a,b}(x,y) := (\mathcal{B}_a^{-1}\mathcal{L}_{f,b}^{(2)})(x,y) &= \frac{[\alpha b^{-x}][\alpha' b^{-y}] \ln^2 b \Gamma(\alpha+2)\Gamma(\alpha'+2)}{\alpha\alpha'\Gamma([\alpha b^{-x}]+1)\Gamma([\alpha' b^{-y}]+1)} \\ &\times \sum_{m=0}^{\alpha-[\alpha b^{-x}]} \sum_{l=0}^{\alpha'-[\alpha' b^{-y}]} \frac{(-1)^{m+l}\mathcal{L}_f^2((m+[\alpha b^{-x}])\ln b, (l+[\alpha' b^{-y}])\ln b)}{m!l!(\alpha-[\alpha b^{-x}]-m)!(\alpha'-[\alpha' b^{-y}]-l)!}, \end{aligned} \tag{4.2}$$

and

$$\bar{F}_{a,b} = \sum_{k=0}^{[\alpha b^{-x}]} \sum_{l=0}^{[\alpha' b^{-y}]} \sum_{j=k}^{\alpha} \sum_{m=l}^{\alpha'} \binom{\alpha}{j}\binom{j}{k}\binom{\alpha'}{m}\binom{m}{l}(-1)^{j+m-k-l}\mathcal{L}_f^2(j\ln b, m\ln b), \tag{4.3}$$

where $a = (\alpha, \alpha')$.

4.2 Smooth Approximations

Using Poisson probabilities, we construct smooth approximations of the bivariate density $f(x, y)$ of (X, Y):

$$P_a(k, x, j, y) = \frac{(\alpha x)^k}{k!} e^{-\alpha x} \frac{(\alpha' y)^j}{j!} e^{-\alpha' y}. \tag{4.4}$$

We define

$$\begin{aligned} f_{a,P}(x, y) &= (f_{a,b} \bullet P_a)(x, y) \\ &= \sum_{k=0}^{\infty} \sum_{k=0}^{\infty} f_{a,b} \left(\frac{k}{\alpha}, \frac{j}{\alpha'} \right) P_a(k, x, j, y), \end{aligned} \tag{4.5}$$

and we construct a modified version of $f_{a,P}$ as

$$\begin{aligned} \widetilde{f}_{a,P}(x, y) &= (\widetilde{f}_{a,b} \bullet P_a)(x, y) \\ &= \sum_{k,j=0}^{\infty} \widetilde{f}_{a,b} \left(\frac{k}{\alpha}, \frac{j}{\alpha'} \right) P_a(k, x, j, y), \end{aligned} \tag{4.6}$$

where $\widetilde{f}_{a,b} = 2 f_{\tilde{a},b} - f_{a,b}$.

4.3 Numerical Implementation

In this section, we demonstrate the performance of the moment recovered approximations in the recovery of the joint BPDF of (S_1, S_2). In many cases, there are no closed forms of the joint density functions of claims sizes distributions.

Hence, to assess the accuracy of our approximations, we apply the MR method to well known bivariate distributions.

Example 9. Let (X, Y) be a couple of independent random variables exponentially distributed with rates β_1 and β_2. The bivariate scaled Laplace transform of (X, Y) is given as

$$\mathcal{L}_f^{(2)}(s \ln b, t \ln b) = \frac{\beta_1}{\beta_1 + s \ln b} \times \frac{\beta_2}{\beta_2 + t \ln b}. \tag{4.7}$$

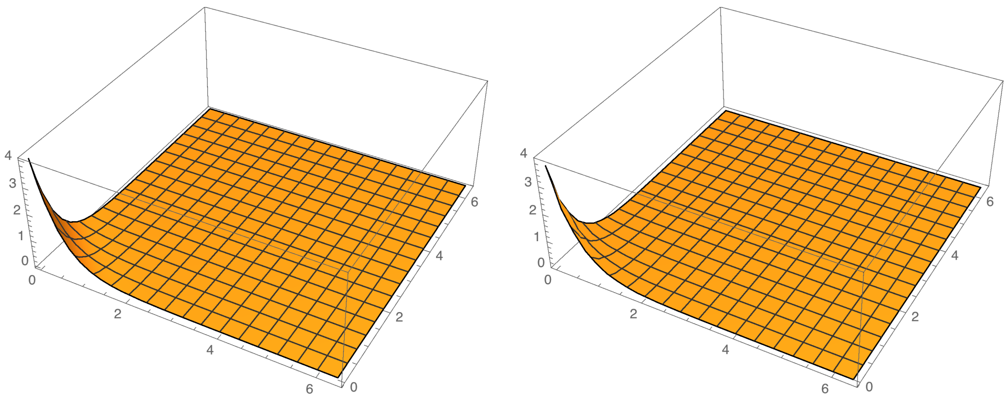
We input the bivariate scaled Laplace transform from (4.7) into (4.2) to approximate the joint density function by

$$\begin{aligned} f_{a,b}(x, y) &= \frac{[\alpha b^{-x}][\alpha' b^{-y}] \ln^2 b \Gamma(\alpha + 2)\Gamma(\alpha' + 2)}{\alpha\alpha'\Gamma([\alpha b^{-x}] + 1)\Gamma([\alpha' b^{-y}] + 1)} \\ &\times \sum_{m=0}^{\alpha-[\alpha b^{-x}]} \sum_{l=0}^{\alpha'-[\alpha' b^{-y}]} \frac{(-1)^{m+l}}{m!l!(\alpha - [\alpha b^{-x}] - m)!(\alpha' - [\alpha' b^{-y}] - l)!} \times \\ &\qquad \frac{\beta_1}{\beta_1 + (m + [\alpha b^{-x}]) \ln b} \times \frac{\beta_2}{\beta_2 + (l + [\alpha' b^{-y}]) \ln b}. \end{aligned} \tag{4.8}$$

Consider the case when $\beta_1 = 2$ and $\beta_2 = 2$. Figure 4.1 shows the MR-approximations of the joint density function. We set $\alpha = \alpha' = 32$, $b = 1.45$. Evaluations were conducted on the grid

$$(x_j, y_j) = \left[\ln\left(\frac{\alpha}{\alpha - i + 1}\right), \ln\left(\frac{\alpha'}{\alpha' - j + 1}\right)\right];\ i = 1 \ldots \text{ and } \alpha,\ j = 1, \ldots, \alpha'.$$

Example 10. Let (X_1, X_2) be a random vector with independent Gamma distributions such that $X_1 \sim \text{Gamma}(a_1, \beta_1)$ and $X_2 \sim \text{Gamma}(a_2, \beta_2)$, respec-



(a) (b)

Figure 4.1: Approximation of $f(x,y)$ when (a) $(X,Y) \sim \beta_1\beta_2 e^{-(\beta_1 x+\beta_2 y)}$ and (b) $f_{\alpha,b}(x,y)$, $\alpha=\alpha'=32$.

tively. The bivariate scaled Laplace of (X_1, X_2) is given by

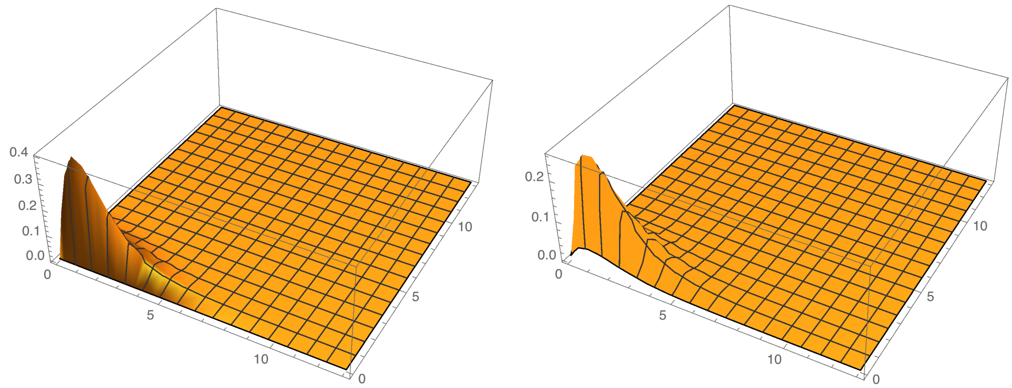
$$\mathcal{L}_f^{(2)}(s\ln b, t\ln b)) = \left(\frac{1}{\beta_1 s\ln b+1}\right)^{a_1}\left(\frac{1}{\beta_2 t\ln b+1}\right)^{a_2}. \tag{4.9}$$

We input (4.9) into (4.2) to approximate the joint density function of (X_1, X_2) by

$$\begin{aligned} f_{a,b}(x,y) = {} & \frac{[\alpha b^{-x}][\alpha' b^{-y}]\ln^2 b\Gamma(\alpha+2)\Gamma(\alpha'+2)}{\alpha\alpha'\Gamma([\alpha b^{-x}]+1)\Gamma([\alpha' b^{-y}]+1)} \\ & \times \sum_{m=0}^{\alpha-[\alpha b^{-x}]}\sum_{l=0}^{\alpha'-[\alpha' b^{-y}]}\frac{(-1)^{m+l}}{m!l!(\alpha-[\alpha b^{-x}]-m)!(\alpha'-[\alpha' b^{-y}]-l)!}\times \\ & \left(\frac{1}{\beta_1(m+[\alpha b^{-x}])\ln b+1}\right)^{a_1}\left(\frac{1}{\beta_2(l+[\alpha b^{-y}])\ln b+1}\right)^{a_2}, \end{aligned} \tag{4.10}$$

where $a=(\alpha,\alpha')$.

Consider the case where $\{a_1=2, \beta_1=1, a_2=2.5, \beta=0.4\}$. We set $\alpha=50, \alpha'=60$ and $b=1.35$. We see from Figure 4.2 that the moment recovered



(a) (b)

Figure 4.2: Approximation of $f(x_1, x_2)$ when (a) $(X_1, X_2) \sim$ Gamma$(2, 1) \times$ Gamma$(2.5, 0.4)$ and (b) $f_{\alpha,b}(x, y)$, $\alpha = 50$, $\alpha' = 60$, $b = 1.35$.

approximations are quite close to the theoretical distribution.

Example 11. Assume that a bivariate random variable (X_1, X_2) follows the Downton Bivariate Exponential distribution - $DBVE(\mu_1, \mu_2, \rho)$ introduced by Downton [14]. This distribution has been used to describe the reliability of the behavior of two simple systems consisting of two components in parallel and of two systems in series. The joint density function of (X, Y) is given by

$$f(x, y) = \frac{\mu_1 \mu_2}{1-\rho} \exp\left(-\frac{\mu_1 x + \mu_2 y}{1-\rho}\right) I_0\left(\frac{2\sqrt{\rho \mu_1 \mu_2 x y}}{1-\rho}\right), \quad x, y \in \mathbb{R}_+^2, \tag{4.11}$$

where $I_0(\cdot)$ is the Bessel function of the first kind, $\mu_1,\ \mu_2 \geq 0$, and $0 \leq \rho \leq 1$. The bivariate scaled Laplace transform of (X, Y) is given by

$$\mathcal{L}_f^{(2)}(s \ln b, t \ln b) = \frac{\mu_1 \mu_2}{(\mu_1 + s \ln b)(\mu_2 + t \ln b) - \rho\ s\ t \ln^2 b}. \tag{4.12}$$

To approximate the joint density function of (X, Y) we substitute the bivariate

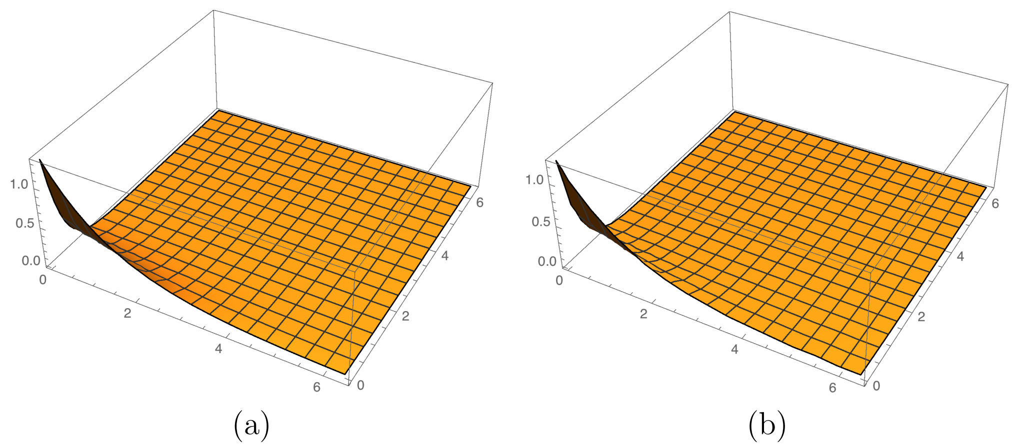


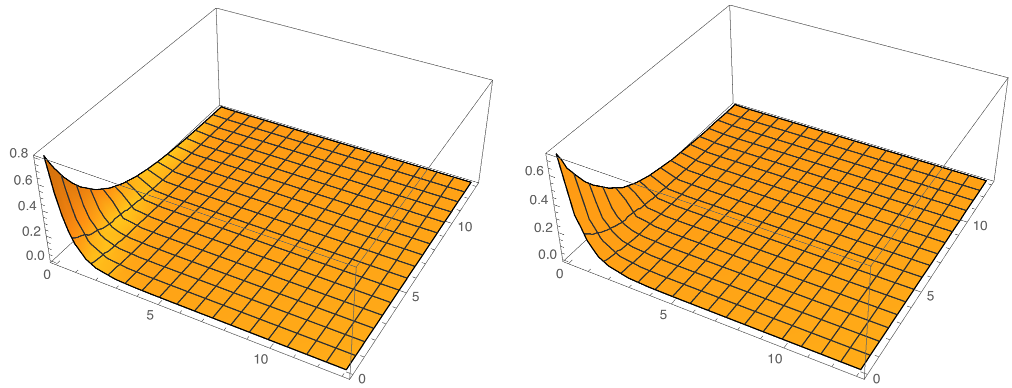
Figure 4.3: Approximation of $f(x,y)$ when (a) $(X,Y) \sim DBVE(1/2,2,1/4)$ and (b) $f_{\alpha,b}(x,y)$, $\alpha=\alpha'=50$, $b=1.45$.

scaled Laplace transform into (4.2) to get

$$f_{a,b}(x,y) = \frac{[\alpha b^{-x}][\alpha' b^{-y}]\ln^2 b\Gamma(\alpha+2)\Gamma(\alpha'+2)}{\alpha\alpha'\Gamma([\alpha b^{-x}]+1)\Gamma([\alpha' b^{-y}]+1)}$$
$$\times \sum_{m=0}^{\alpha-[\alpha b^{-x}]} \sum_{l=0}^{\alpha'-[\alpha' b^{-y}]} \frac{(-1)^{m+l}}{m!l!(\alpha-[\alpha b^{-x}]-m)!(\alpha'-[\alpha' b^{-y}]-l)!} \times$$
$$\frac{\mu_1\mu_2}{(\mu_1+(m+[\alpha b^{-x}])\ln b)(\mu_2+(l+[\alpha' b^{-y}])\ln b)-\rho(m+[\alpha b^{-x}])(l+[\alpha' b^{-y}])\ln^2 b},$$

where $a=(\alpha,\alpha')$.

For illustration, we consider the cases when $\{\mu_1=1/2,\mu_2=2,\rho=1/4\}$ and $\{\mu_1=2,\mu_2=1/2,\rho=-1/4\}$, where the first case is adapted from Goffard et al. [22]. Figures 4.3 and 4.4 shows plots of both cases; the plots on the right are those of the MR-approximations. In this first plot, we set $\alpha=\alpha'=50$ and $b=1.45$. In the second plot, $\alpha=\alpha'=32$. Comparing with the theoretical joint density functions, we see that the approximations are relatively close to the true distributions.



(a) (b)

Figure 4.4: Approximation of $f(x, y)$ when (a) $(X, Y) \sim DBVE(2, 1/2, -1/4)$ and (b) $f_{\alpha,b}(x, y)$, $\alpha = \alpha' = 32$, $b = 1.45$.

4.4 Aggregate Claims Amount Distribution

Assume that the claims amount (U_{1j}, U_{2j}) in the collective model defined in (1.4) are i.i.d. and $DBVE(\mu_1, \mu_2, \rho)$-distributed. Let the total number of claims N be specified by a negative binomial distribution $\mathcal{NB}(r, p)$. Our goal is to approximate the joint density function and joint survival function of the bivariate aggregate claims amount (S_1, S_2). The bivariate scaled Laplace transform is given by

$$\mathcal{L}_g^{(2)}(s \ln b, t \ln b) = \left[\frac{1 - p}{1 - p\mathcal{L}_f^{(2)}(s \ln b, t \ln b)} \right]^r - (1 - p)^r, \tag{4.13}$$

where $\mathcal{L}_f^{(2)}(s \ln b, t \ln b)$ is the scaled Laplace transform of the joint density function of (U_{1j}, U_{2j}). We note that $\mathcal{L}_g^{(2)}(s \ln b, t \ln b)$ exists only if $\mathcal{L}_f^{(2)}(s \ln b, t \ln b) < 1/p$.

We insert the scaled Laplace transform in (4.13) into (4.2) to approximate the joint density and survival functions of (S_1, S_2) by

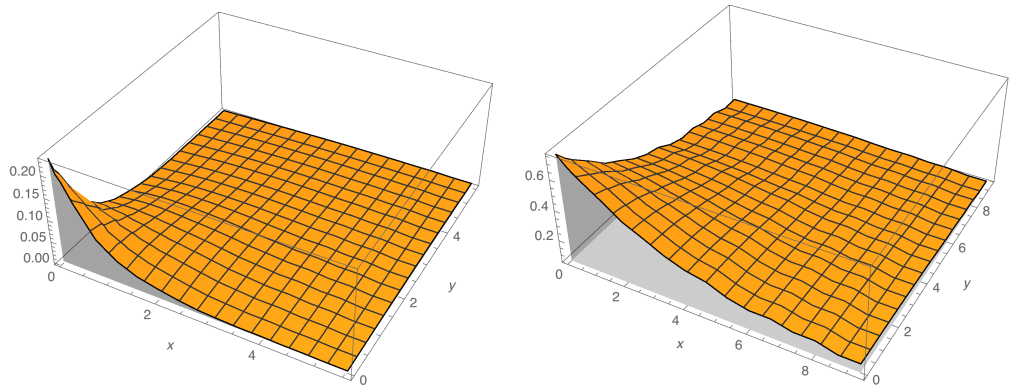
$$g_{a,b}(x,y) = \frac{[\alpha b^{-x}][\alpha' b^{-y}] \ln^2 b \Gamma(\alpha+2)\Gamma(\alpha'+2)}{\alpha\alpha'\Gamma([\alpha b^{-x}]+1)\Gamma([\alpha' b^{-y}]+1)}$$
$$\times \sum_{m=0}^{\alpha-[\alpha b^{-x}]} \sum_{l=0}^{\alpha'-[\alpha' b^{-y}]} \frac{(-1)^{m+l}}{m!l!(\alpha-[\alpha b^{-x}]-m)!(\alpha'-[\alpha' b^{-y}]-l)!} \times$$
$$\left(\left[\frac{1-p}{1-p\mathcal{L}_f^{(2)}((m+[\alpha b^{-x}])\ln b, (l+[\alpha' b^{-y}])\ln b)} \right]^r - (1-p)^r \right), \tag{4.14}$$

and

$$\bar{G}_{a,b} = \sum_{k=0}^{[\alpha b^{-x}]} \sum_{l=0}^{[\alpha' b^{-y}]} \sum_{j=k}^{\alpha} \sum_{m=l}^{\alpha'} \binom{\alpha}{j} \binom{j}{k} \binom{\alpha'}{m} \binom{m}{l} (-1)^{j+m-k-l} \times$$
$$\left(\left[\frac{1-p}{1-p\mathcal{L}_f^{(2)}(j\ln b, m\ln b)} \right]^r - (1-p)^r \right), \tag{4.15}$$

where $\mathcal{L}_f^{(2)}(\cdot,\cdot)$ is as defined in (4.12).

Example 12. Consider the case when the parameters of the Downton distribution are $\{\mu_1 = 1, \mu_2 = 1, \rho = 1/4\}$ and those of the negative binomial distribution are $\{r = 1, p = 3/4\}$. These parameters were chosen to ensure that $\mathcal{L}_f^{(2)}(s\ln b, t\ln b) < 1/p$. The exact form of the survial function is not known. Figure 4.5 provides a visualization of the moment recovered approximations of the joint probability density and the survival function of the aggregate claims amount distribution, where $\alpha = \alpha' = 32$ and $b = 1.45$. The moment recovered approximations are comparable to those obtained by Goffard et. al. [22].



(a) (b)

Figure 4.5: Approximations of the distribution of bivariate aggragate claims amount (a) Joint density function of (S_1, S_2) (b) Joint survival function of (S_1, S_2).

From the simulation studies presented, the moment recovered method seems well-suited for the approximations of the distributions of bivariate distributions.

Chapter 5

Option Pricing

Financial markets have become increasingly more sophisticated in recent years and so, have the offered products. More complex financial options and derivatives have replaced the simple buy/sell trade deals of earlier years, see Lee and Sheen [32]. The determination of the actual value of financial options is a major concern for market participants. While the writer of an option contract might be largely concerned by how much to charge for the contract and what his profit margin would be, the holder of the contract wants to be certain that he is paying a fair price and that he stands to make gains from the exercise of the contract. These and many other concerns have motivated a large volume of research, too numerous to mention, in the area of option pricing. In this chapter, we present applications of the moment recovered approximation method in the estimation of the prices of some well known financial options and derivatives.

5.1 European Call and Put Options

We apply the scaled Laplace transform method to solve the Black-Scholes equation which depends on one stock asset. Applying the MR inversion method, we obtain approximations of the European style put and call options.

5.1.1 Put Option

Let $P(S,\tau)$ be the value of a European put option, where S is the current value of the underlying asset and $\tau = T - t$ is the time left till maturity. Assume that the volatility σ and the risk-free interest rate r depend only on S (i.e. $\sigma = \sigma(S)$ and $r = r(S)$). The price $P(S,\tau)$ satisfies the Black-Scholes equation

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P(s,\tau)}{\partial S^2} + rS\frac{\partial P(S,\tau)}{\partial S} - \frac{\partial P(S,\tau)}{\partial \tau} - rP(S,\tau) = 0, \tag{5.1}$$

$$P(S,0) = \max\{K - S, 0\}, \tag{5.2}$$

$$P(S,\tau) \to Ke^{-r\tau} \text{ as } S \to 0,\ P(S,\tau) \to 0 \text{ as } S \to \infty, \tag{5.3}$$

where (5.2) is the final value condition.

To solve (5.1), we take the scaled Laplace transform of $P(S,\tau)$ and its partial derivatives as follows:

$$\mu_\lambda(P(S,.)) = \mathcal{L}_{P(S,.),b}(\lambda) = \int_0^\infty e^{-\lambda \ln(b)\tau} P(S,\tau)\mathrm{d}\tau, \tag{5.4}$$

and

$$\mu_\lambda\left(\frac{\partial P(S,.)}{\partial \tau}\right) = \mathcal{L}_{\frac{\partial P(S,.)}{\partial \tau},b}(\lambda) = \int_0^\infty e^{-\lambda \ln(b)\tau}\frac{\partial P(S,.)}{\partial \tau}\mathrm{d}\tau. \tag{5.5}$$

Applying integration by parts, we get

$$\mu_\lambda\left(\frac{\partial P(S,.)}{\partial \tau}\right) = -P(S,0) + \lambda \ln(b)\mu_\lambda(P(S,.)), \tag{5.6}$$

$$\begin{aligned}\mu_\lambda\left(\frac{\partial P(S,.)}{\partial S}\right) &= \mathcal{L}_{\frac{\partial P(S,.)}{\partial S},b}(\lambda) = \int_0^\infty e^{-\lambda \ln(b)\tau}\frac{\partial P(S,\tau)}{\partial S}\mathrm{d}\tau \\ &= \frac{\partial}{\partial S}\int_0^\infty e^{-\lambda \ln(b)\tau}P(S,\tau)\mathrm{d}\tau = \frac{\partial}{\partial S}\mu_\lambda(P(S,.)),\end{aligned} \tag{5.7}$$

and

$$\mu_\lambda\left(\frac{\partial^2 P(S,.)}{\partial S^2}\right) = \mathcal{L}_{\frac{\partial^2 P(S,.)}{\partial S^2},b}(\lambda) = \int_0^\infty e^{-\lambda \ln(b)\tau}\frac{\partial^2 P(S,\tau)}{\partial S^2}\mathrm{d}\tau = \frac{\partial^2}{\partial S^2}\mu_\lambda(P(S,.)). \tag{5.8}$$

Transforming the boundary conditions, we get

$$\begin{aligned}\lim_{S\to 0}\mu_\lambda(P(S,.)) &= \lim_{S\to 0}\int_0^\infty e^{-\lambda \ln(b)\tau}P(S,\tau)\mathrm{d}\tau = \int_0^\infty e^{-\lambda \ln(b)\tau}\lim_{S\to 0}P(S,\tau)\mathrm{d}\tau \\ &= \int_0^\infty e^{-\lambda \ln(b)\tau}Ke^{-r\tau}\mathrm{d}\tau = \frac{K}{r+\lambda \ln(b)}.\end{aligned} \tag{5.9}$$

Applying (5.4) - (5.9), (5.1) becomes,

$$\frac{1}{2}\sigma^2 S^2\frac{\partial^2 \mu_\lambda(P(S,.))}{\partial S^2} + rS\frac{\partial \mu_\lambda(P(S,.))}{\partial S} - (r+c\lambda)\mu_\lambda(P(S,.)) = -P(S,0), \tag{5.10}$$

with boundary conditions

$$\mu_\lambda(P(S,.)) \to \frac{K}{r+c\lambda} \text{ as } S \to 0 \text{ and } \mu_\lambda(P(S,.)) \to 0 \text{ as } S \to \infty. \tag{5.11}$$

Solving the homogeneous part of (5.10), we get

$$\frac{1}{2}\sigma^2 S^2 \mu_\lambda'' + rS\mu_\lambda' - (r+\lambda\ln(b))\mu_\lambda = 0, \tag{5.12}$$

$$S^2\mu_\lambda'' + \frac{2r}{\sigma^2}S\mu_\lambda' - \frac{2(r+\lambda\ln(b))}{\sigma^2}\mu_\lambda = 0, \tag{5.13}$$

and observe that (5.12) is an Euler equation. By setting $\mu_\lambda = CS^\gamma$, $\mu_\lambda'' = C\gamma S^{\gamma-1}$ and $\mu_\lambda'' = C\gamma(\gamma-1)S^{\gamma-2}$, for some $\gamma, C \in \mathbb{R}$. Substituting this into (5.13), we get

$$\begin{aligned} S^2(CS^\gamma)'' + \frac{2r}{\sigma^2}S(CS^\gamma)' - \frac{2(r+\lambda\ln(b))}{\sigma^2}CS^\gamma &= 0, \\ \Rightarrow CS^\gamma\left[\gamma(\gamma-1) + \frac{2r}{\sigma^2}\gamma - \frac{2(r+\lambda\ln(b))}{\sigma^2}\right] &= 0. \end{aligned} \tag{5.14}$$

Solving

$$\gamma^2 + \left(\frac{2r}{\sigma^2} - 1\right)\gamma - \frac{2(r+\lambda\ln(b))}{\sigma^2} = 0, \tag{5.15}$$

we get

$$\gamma_1 = \frac{-\left(r-\frac{1}{2}\sigma^2\right) + \sqrt{\left(r-\frac{1}{2}\sigma^2\right)^2 + 2\sigma^2(r+\lambda\ln(b))}}{\sigma^2}, \tag{5.16}$$

and

$$\gamma_2 = \frac{-\left(r-\frac{1}{2}\sigma^2\right) - \sqrt{\left(r-\frac{1}{2}\sigma^2\right)^2 + 2\sigma^2(r+\lambda\ln(b))}}{\sigma^2}. \tag{5.17}$$

Hence, the solution to (5.12) is

$$\mu_\lambda^{(c)}(P(S,\cdot)) = C_1 S^{\gamma_1} + C_2 S^{\gamma_2}, \tag{5.18}$$

where the superscript (c) indicates that the solution is the complementary solution of (5.10), which is also the solution in cases when $S \geq K$.

When $K \geq S$ at maturity T, i.e., $\tau = 0$, $P(S,0) = K - S$. So, the non-homogeneous part of (5.10) corresponds to cases when the strike K is greater than the underlying stock price. Hence, the put option will be exercised.

Let $\mu_\lambda^{(p)}(P(S,\cdot)) = AS + B$ be a particular solution of (5.10), for some $A, B \in \mathbb{R}$. Then, $\mu_\lambda^{(p)'} = A$ and $\mu_\lambda^{(p)''} = 0$. Substituting these into (5.10) we get

$$\begin{aligned} & rSA - (r + \lambda \ln(b))(AS + B) = -(K - S), \\ & \Rightarrow -(r + \lambda \ln(b))B - \lambda \ln(b) AS = -K + S. \end{aligned} \tag{5.19}$$

Matching coefficients of the lhs and rhs of (5.19), we get

$$A = -\frac{1}{\lambda \ln(b)} \quad \text{and} \quad B = \frac{K}{r + \lambda \ln(b)}.$$

Hence,

$$\mu_\lambda^{(p)}(P(S,\cdot)) = -\frac{1}{\lambda \ln(b)} S + \frac{K}{r + \lambda \ln(b)}. \tag{5.20}$$

Combining (5.18) and (5.20), we get the general solution of (5.10)

$$\mu_\lambda(P(S,\cdot)) = \mathcal{L}_{P(S,\cdot),b}(\lambda) = C_1 S^{\alpha_1} + C_2 S^{\alpha_2} - \frac{1}{\lambda \ln(b)} S + \frac{K}{r + \lambda \ln(b)}. \quad (5.21)$$

Thus,

$$\mu_\lambda(P(S,\cdot)) = \begin{cases} C_1 S^{\gamma_1} + C_2 S^{\gamma_2} - \frac{1}{\lambda \ln(b)} S + \frac{K}{r+\lambda \ln(b)}, & S < K \\ C_1 S^{\gamma_1} + C_2 S^{\gamma_2}, & S \geq K \end{cases}. \quad (5.22)$$

We have that $\gamma_1 \geq 0 \geq \gamma_2$. So in the case when $S < K$, $C_2 = 0$ ensures the boundedness of the derivative $\mu'_\lambda(P(S,\cdot))$. In the case when $S \geq K$, $C_1 = 0$ ensures the value of the option goes to zero when the stock price goes to infinity. Thus the general solution to (5.10) reduces to

$$\mu_\lambda(P(S,\cdot)) = \begin{cases} C_1 S^{\gamma_1} - \frac{1}{\lambda \ln(b)} S + \frac{K}{r+\lambda \ln(b)}, & S < K \\ C_2 S^{\gamma_2}, & S \geq K \end{cases}, \quad (5.23)$$

where (5.23) satisfies the boundary conditions in (5.11). Next we solve for C_1 and C_2:

$$\mu_\lambda(P(S,\cdot))|_{S=K} = C_1K^{\gamma_1} - \frac{1}{\lambda\ln(b)}K + \frac{K}{r+\lambda\ln(b)}, \qquad S < K, \tag{5.24}$$

$$\mu_\lambda(P(S,\cdot))|_{S=K} = C_2K^{\gamma_2}, \qquad S \geq K, \tag{5.25}$$

$$\mu'_\lambda(P(S,\cdot))|_{S=K} = \alpha_1C_1K^{\gamma_1-1} - \frac{1}{\lambda\ln(b)}, \qquad S < K, \tag{5.26}$$

$$\mu'_\lambda(P(S,\cdot))|_{S=K} = \gamma_2C_2K^{\gamma_2-1} \qquad S \geq K. \tag{5.27}$$

Setting (5.24) = (5.25) and (5.26) = (5.27), we get

$$C_1K^{\gamma_1} - \frac{K}{\lambda\ln(b)} + \frac{K}{r+\lambda\ln(b)} = C_2K^{\gamma_2}, \tag{5.28}$$

$$\gamma_1C_2K^{\gamma_1-1} - \frac{1}{\lambda\ln(b)} = \gamma_2C_1K^{\gamma_2-1}. \tag{5.29}$$

Multiply (5.28) by $\frac{\gamma_2}{K}$ and subtract (5.29) to get

$$\gamma_2C_1K^{\gamma_1-1} - \gamma_1C_1K^{\gamma_1-1} + \frac{\gamma_2}{r+\lambda\ln(b)} - \frac{\gamma_2}{\lambda\ln(b)} + \frac{1}{\lambda\ln(b)} = 0. \tag{5.30}$$

Solving for C_1, we get

$$C_1 = \left[\frac{\gamma_2}{r+\lambda\ln(b)} - \frac{\gamma_2-1}{\lambda\ln(b)}\right]\frac{K^{1-\gamma_1}}{\gamma_1-\gamma_2}. \tag{5.31}$$

Substituting in (5.29), we get

$$C_2 = \left[\frac{\gamma_1}{r+\lambda\ln(b)} - \frac{\gamma_1-1}{\lambda\ln(b)}\right]\frac{K^{1-\gamma_2}}{\gamma_1-\gamma_2}. \tag{5.32}$$

Finally, to estimate the value of the put option, we apply the moment recovered (MR) inversion method to (5.23). Thus, the value of the European put option is approximated by

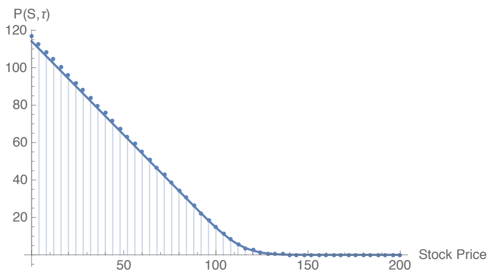
$$P_{\alpha,b}(S,\tau) = \frac{[\alpha b^{-\tau}]\ln(b)\Gamma(\alpha+2)}{\alpha\Gamma([\alpha b^{-\tau}]+1)} \sum_{m=0}^{\alpha-[\alpha b^{-\tau}]} \frac{(-1)^m \mu_{m+[\alpha b^{-\tau}]}(P(S,\cdot))}{m!(\alpha-[\alpha b^{-\tau}]-m)!},$$

where

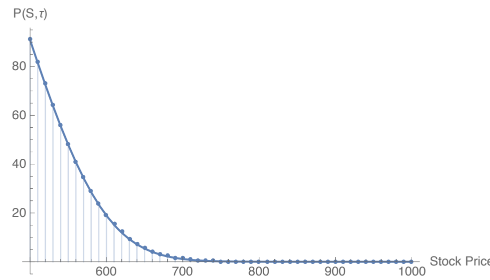
$$\mu_{m+[\alpha b^{-\tau}]}(P(S,\cdot)) = \begin{cases} C_1 S^{\gamma_1} - \frac{1}{\ln(b)(m+[ab^{-\tau}])} S + \frac{K}{r+\ln(b)(m+[ab^{-\tau}])}, & S < K \\ C_2 S^{\gamma_2}, & S \geq K \end{cases}.$$

In C_1 and C_2, $\lambda \equiv m + [\alpha b^{-\tau}]$.

Example 13. We demonstrate the performance of the scaled Laplace method and its inversion on a simulated data set of stock prices. Let $\tau \in (0,1)$, where $\tau = 0$ implies maturity, and recall $\tau = T - t$. Assume interest rate $r = 0.05$ and volatility $\sigma = 0.1$. Figure 5.1 presents visualizations of the MR-approximations of the price of a put option with S as the price of the underlying asset. In the first plot we consider stock prices in the interval $[0, 200]$, with srike, $K = 121$. In the second, prices are in the interval $[500, 1000]$ with strike $K = 620$. We set $\alpha = 32$ and $b = 1.45$ in both plots. We see clearly that the MR-approximations are comparable with the Black-Scholes prices.



(a)



(b)

Figure 5.1: Value of European Put option. MR-approximation (dots), Black-Scholes prices (blue lines). (a) Strike, $K = 121$, $S \in (0, 200)$ (b) Strike $K = 620$, Stock price $S \in (500, 1000)$.

5.1.2 Call Option

Recall, the Black-Scholes equation for a European style call option is

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C(s,\tau)}{\partial S^2} + rS\frac{\partial C(S,\tau)}{\partial S} - \frac{\partial C(S,\tau)}{\partial \tau} - rC(S,\tau) = 0, \tag{5.33}$$

$$C(S,0) = \max\{S - K, 0\}, \tag{5.34}$$

$$C(S,\tau) \to 0 \text{ as } S \to 0,\ C(S,\tau) \to S \text{ as } S \to \infty. \tag{5.35}$$

We take the scaled Laplace transform of (5.33) and solve the resulting ordinary differential equation with boundary conditions $\mu_\lambda C(S,\cdot) \to \frac{S}{c\lambda}$ as $S \to \infty$ and $\mu_\lambda C(S,\cdot) \to 0$ as $S \to 0$. We get the general solution

$$\mu_\lambda(C(S,\cdot)) = \begin{cases} C_1 S^{\gamma_1} + C_2 S^{\gamma_2} + \frac{1}{\lambda \ln(b)} S - \frac{K}{r + \lambda \ln(b)}, & S > K \\ C_1 S^{\gamma_1} + C_2 S^{\gamma_2}, & S \le K \end{cases}, \tag{5.36}$$

where γ_1, γ_2, C_1, and C_2 are as defined in (5.16), (5.17), (5.31), and (5.32), respectively.

We observe that in the case when $S > K$, $C_1 = 0$ ensures the boundedness of the derivative $\mu'_\lambda(C(S,\cdot))$ and in the case when $S \leq K$, $C_2 = 0$ ensures that the option value goes to zero as the stock price goes to zero, see Kumar et al. [31]. Thus, the general solution reduces to

$$\mu_\lambda(C(S,\cdot)) = \begin{cases} C_2 S^{\gamma_2} + \frac{S}{\lambda \ln(b)} - \frac{K}{r+\lambda \ln(b)}, & S > K \\ C_1 S^{\gamma_1}, & S \leq K \end{cases} \tag{5.37}$$

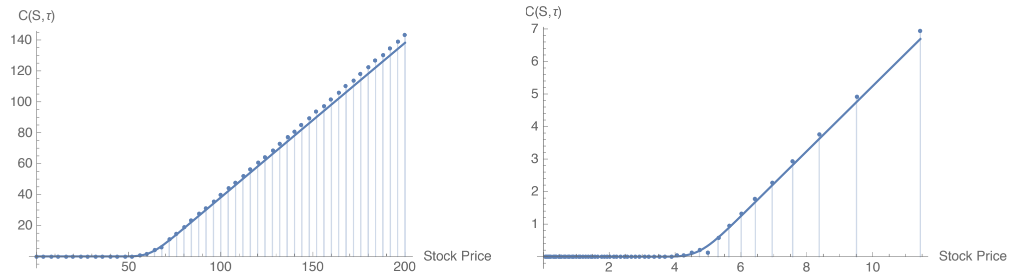
where (5.37) satisfies the boundary conditions.

Applying the MR inversion method, the call option can be approximated by

$$C_{\alpha,b}(S,\tau) = \frac{[\alpha b^{-\tau}]\ln(b)\Gamma(\alpha+2)}{\alpha\Gamma([\alpha b^{-\tau}]+1)} \sum_{m=0}^{\alpha-[\alpha b^{-\tau}]} \frac{(-1)^m \mu_{m+[\alpha b^{-\tau}]}(C(S,\cdot))}{m!(\alpha-[\alpha b^{-\tau}]-m)!}, \tag{5.38}$$

where $\lambda \equiv m + [\alpha b^{-\tau}]$ in γ_1, γ_2, C_1, and C_2.

Example 14. Assume interest rate $r = 0.05$ and volatility $\sigma = 0.1$. Figure 5.2 presents plots of the Black-Scholes prices and MR-approximations of the price of a call option with S as the price of the underlying asset. In the first plot we consider stock prices in the interval $[0, 200]$, with strike $K = 121$. In the second plot we conducted evaluations at stock prices $S \in \{(\ln\alpha - \ln(\alpha-j-1))/\ln b, 1 \leq j \leq \alpha\}$ and strike $K = 5$. We set $\alpha = 32$ and $b = 1.45$ in both plots. We see clearly that the MR-approximations are comparably close to the prices derived by the Black-Scholes formula.



(a) (b)

Figure 5.2: Value of European Call option. MR-approximation (dots), Black-Scholes prices (blue line). (a) Strike, $K = 65$, $S \in (0, 200)$ (b) Strike $K = 4$.

5.2 Exotic Options

Exotic options are generally more profitable than plain vanilla options, hence the enormous research interest in the their pricing techniques. In this section we examine the double-barrier knock-out call option and the Asian call option.

5.2.1 Double-barrier options

A double-barrier option is a type of financial option where the option to exercise depends on the price of the underlying asset crossing or reaching two barriers namely: L (lower barrier) and U (upper barrier). The payoff depends on whether the underlying asset price reaches L or U during the transaction period. The option knocks out (becomes worthless) if either barrier is reached during its lifetime. If neither barrier is reached by maturity T, the option pays the standard Black-Scholes pay-off $\max\{0, S(T)-K\}$, where K, the strike price of the option, satisfies $L < K < U$.

A double-barrier knock-out call option satisfies the following equation:

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 B(s,\tau)}{\partial S^2} + rS\frac{\partial B(S,\tau)}{\partial S} - \frac{\partial B(S,\tau)}{\partial \tau} - rB(S,\tau) = 0, \tag{5.39}$$

$$B(S,0) = \max\{S-K,0\}\mathbf{1}_{[L,U]}(S), \tag{5.40}$$

$$B(S,\tau) \to 0 \text{ as } S \to 0 \text{ or } S \to \infty, \tag{5.41}$$

where $B(S,\tau)$ is the price of a Barrier call option and

$$B(S,0) = \max\{S-K,0\} = \begin{cases} S-K, & S \in [L,U] \ \& \ S > K \\ 0, & S \leq K \ \& \ S \notin [L,U] \end{cases}. \tag{5.42}$$

Taking the scaled Laplace transform of (5.39) and solving the resulting ordinary differential equation, we get the general solution

$$\mu_\lambda(B(S,\cdot)) = \begin{cases} C_2 S^{\gamma_2} + \frac{S}{\lambda\ln(b)} - \frac{K}{r+\lambda\ln(b)}, & S > K \ \& \ S \in [L,U] \\ C_1 S^{\gamma_1}, & S \leq K \ \& \ S \notin [L,U] \end{cases}. \tag{5.43}$$

Applying the MR inversion method, a barrier call option can be approximated by

$$B_{\alpha,b}(S,\tau) = \frac{[\alpha b^{-\tau}]\ln(b)\Gamma(\alpha+2)}{\alpha\Gamma([\alpha b^{-\tau}]+1)} \sum_{m=0}^{\alpha-[\alpha b^{-\tau}]} \frac{(-1)^m \mu_{m+[\alpha b^{-\tau}]}(B(S,\cdot))}{m!(\alpha-[\alpha b^{-\tau}]-m)!}, \tag{5.44}$$

where $\lambda \equiv m + [\alpha b^{-\tau}]$ in γ_1, γ_2, C_1, and C_2.

Example 15. Consider the case when volatility $\sigma = 0.001$, interest rate $r = 0.05$, stock prices $S \in (0, 200)$. Figure 5.3 presents plots of the MR-

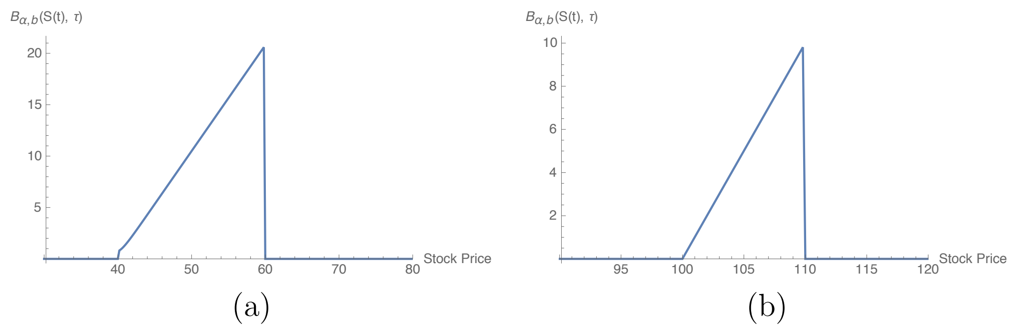


Figure 5.3: Approximations of the value of Double-barrier Knock-out Call options (a) Parameters: $K = 65$, $L = 0$, $U = 60$ (b) Parameters: $K = 100$, $L = 90$, $U = 110$.

approximations of the value of double-barrier knock-out barrier options with S as the price of the underlying asset. In the first plot we set $\alpha = 32$. In the second plot, to demonstrate the performance of the approximations as α increases, we set $\alpha = 10000$. (cf. Tagliani and Milev [49]).

5.2.2 Asian Option

Asian options gives the holder of the contract the right to buy or sell an asset for its average price over a pre-set transaction period. Consider a time interval $[0, T]$, and let $\{A(x), x \geq 0\}$ be the average underlying price. Then

$$A(x) = \frac{1}{x} \int_0^x S(u)du.$$

Hence pay-off at maturity T is

$$(A(T) - K)^+ = \max\{A(T) - K), 0\}.$$

The price V of an Asian option is a function of the underlying asset price S, the average price A, and time to maturity τ.

Geman and Yor [19] established that the Asian option can be valued by the Laplace transform method. They showed that the price $V(t)$ is given as

$$V(t) = \frac{4S(t)}{\sigma^2 T} e^{-r(T-t)} C^{(\nu)}(h, q), \tag{5.45}$$

where r is the constant interest rate,

$$\nu = \frac{2r}{\sigma^2} - 1; \quad h = \frac{\sigma^2}{4}(T-t); \quad q = \frac{\sigma^2}{4S(t)} \left\{ KT - \int_0^t S(u)\mathrm{d}u \right\},$$

and the Laplace transform of $C^{(\nu)}(h, q)$ with respect to h is given as

$$\int_0^\infty e^{-sh} C^{(\nu)}(h, q) = \frac{\int_0^{1/2q} e^{-x} x^{\frac{\mu-\nu}{2}} (1-2qx)^{\frac{\mu+\nu}{2}+1} \mathrm{d}x}{\lambda(\lambda - 2 - 2\nu)\Gamma(\frac{\mu-\nu}{2} - 2)}. \tag{5.46}$$

Applying the MR-inversion method, we approximate $V(t)$ by

$$V_{\alpha,b}(t) = \frac{4S(t)}{\sigma^2 T} e^{-r\tau} \frac{[\alpha b^{-h}] \ln(b) \Gamma(\alpha+2)}{\alpha \Gamma([\alpha b^{-h}] + 1)} \sum_{m=0}^{\alpha - [\alpha b^{-h}]} \frac{(-1)^m \mu_{m+[\alpha b^{-h}]}(C^{(\nu)}(\cdot, q))}{m!(\alpha - [\alpha b^{-h}] - m)!}.$$

Chapter 6

Conclusion and Future Work

At the outset of this work, we sought to present extensions of the research already done by Mnatsakanov and several collaborators in showing the applications of the moment recovered Laplace transform inversion method in the approximations of distribution functions, regression analysis, ruin probability, and many other areas. We set out to show that the method can be conveniently applied to compute several risk measures that are used to mitigate against loss in the finance and insurance sectors. We have shown through several examples that the method is well-suited for solving different types of physical problems that arise in practice.

In the bid to extend the application of the scaled Laplace transform and its inversion in computing the likelihood of the ultimate ruin of an insurance company, we constructed a modified version of the approximations derived by the aforementioned researcher. The modified version has been shown to perform better in computation, convergence, and rate of approximation. While on paper, differences do not seem to be acutely obvious between our method of approximation and already existing methods in the literature, in practice, where the threshold between insolvency and solvency is very small, the accuracy in computation, even to the smallest digit, is an advantage. Comparisons with already established methods of computing the ruin probability showed the moment recovered approximation method and the modified version surely deserve consideration.

Our main contribution is presented in Chapter 3. We presented a method of computing the Value at Risk on aggregated insurance claims by computing the inverse function of the ultimate ruin proability. This method proved to be well-suited to the problem. In comparing with exact solutions, where available, we showed that our approximations were quite close to the true values. A major drawback of the method is the computational time. This is due to the large number of integer moments and double summations required to achieve accurate approximations.

In Chapter 4, we presented extensions of the scaled Laplace transform and its moment recovered inversions to bivariate compound models. From several examples presented, we have shown that the approximations were quite close to the true functions, thereby giving us the confidence to apply the method in cases where exact representations of the distributions of claims specified by such models is not readily available. The scaled Laplace transform has also been shown to perform better in this scenario than the classical transform with $b = \exp(1)$, in cases when the support of the distributions being examined is unbounded. Comparison with existing methods further shows that the method is highly efficient.

We also presented applications of the moment recovered density function for the estimation of vanilla and exotic options. Comparisons with the Black-Scholes formula, showed that the MR method performs very well. The ease of application of the method should encourage use among technical and non-technical participants of financial markets.

Some questions we hope to address in the future include but are not limited to, the approximation of the probability of ruin in finite time, the distribution of time until ruin, valuations of more mathematically interesting stock options such as the American option and other exotic options. From this work, questions left unanswered include how to choose the truncation value to implement the infinite series for the smoothing of the approximations with Poisson probabilities. At the point, the choice is haphazard; we shall seek to formalize this in the future. Additionally, the relationship between the number of integer moments α and the truncation value is a direction currently being explored. In the examples presented, we showed that for a fixed scaling parameter b, the approximation error reduces as α increases. The choice of the global optimal pair of parameters (α, b) is an interesting question that will be addressed in the future.

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