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Graph Coloring Problems and Group Connectivity

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Dissertation submitted to the
Eberly College of Arts and Sciences
at West Virginia University
in partial fulfillment of the requirements
for the degree of

Doctor of Philosophy
in
Mathematics

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ABSTRACT

Graph Coloring Problems and Group Connectivity

Miaomiao Han

This dissertation focuses on group connectivity, modulo orientation, neighbor sum distinguishing total coloring and star edge coloring of graphs from the following aspects.

1. Group connectivity.

Let A be an abelian group and let $i_A(G)$ be the smallest positive integer m such that $L^m(G)$ is A -connected. A path P of G is a normal divalent path if all internal vertices of P are of degree 2 in G and if $|E(P)| = 2$, then P is not in a 3-cycle of G . Let $l(G) = \max\{m : G \text{ has a normal divalent path of length } m\}$. We obtain the following result. (i) If $|A| \geq 4$, then $i_A(G) \leq l(G)$. (ii) If $|A| \geq 4$, then $i_A(G) \leq |V(G)| - \Delta(G)$. (iii) Suppose that $|A| \geq 4$ and $d = \text{diam}(G)$. If $d \leq |A| - 1$, then $i_A(G) \leq d$; and if $d \geq |A|$, then $i_A(G) \leq 2d - |A| + 1$. (iv) $i_{\mathbb{Z}_3}(G) \leq l(G) + 2$. All those bounds are best possible.

2. Modulo orientation.

A mod $(2p + 1)$ -orientation D is an orientation of G such that $d_D^+(v) \equiv d_D^-(v) \pmod{2p + 1}$ for any vertex $v \in V(G)$. We prove that for any integer $t \geq 2$, there exists a finite family $\mathcal{F} = \mathcal{F}(p, t)$ of graphs that do not have a mod $(2p + 1)$ -orientation, such that every graph G with independence number at most t either admits a mod $(2p + 1)$ -orientation or is contractible to a member in \mathcal{F} . In particular, the graph family $\mathcal{F}(p, 2)$ is determined, and our results imply that every 8-edge-connected graph G with independence number at most two admits a mod 5-orientation.

3. Neighbor sum distinguishing total coloring.

A proper total k -coloring ϕ of a graph G is a mapping from $V(G) \cup E(G)$ to $\{1, 2, \dots, k\}$ such that no adjacent or incident elements in $V(G) \cup E(G)$ receive the same color. Let $m_\phi(v)$ denote the sum of the colors on the edges incident with the vertex v and the color on v . A proper total k -coloring of G is called neighbor sum distinguishing if $m_\phi(u) \neq m_\phi(v)$ for each edge $uv \in E(G)$. Let $\chi_\Sigma^t(G)$ be the neighbor sum distinguishing total chromatic number of a graph G . Pilśniak and Woźniak conjectured that for any graph G , $\chi_\Sigma^t(G) \leq \Delta(G) + 3$. We show that if G is a graph with treewidth $\ell \geq 3$ and $\Delta(G) \geq 2\ell + 3$, then $\chi_\Sigma^t(G) \leq \Delta(G) + \ell - 1$. This upper bound confirms the conjecture for graphs with treewidth 3 and 4. Furthermore, when $\ell = 3$ and $\Delta \geq 9$, we show that $\Delta(G) + 1 \leq \chi_\Sigma^t(G) \leq \Delta(G) + 2$ and characterize graphs with equalities.

4. Star edge coloring.

A star edge coloring of a graph is a proper edge coloring such that every connected 2-colored subgraph is a path with at most 3 edges. Let $ch'_{st}(G)$ be the list star chromatic index of G : the minimum s such that for every s -list assignment L for the edges, G has a star edge coloring from L . By introducing a stronger coloring, we show with a very concise proof that the upper bound of the star chromatic index of trees also holds for list star chromatic index of trees, i.e. $ch'_{st}(T) \leq \lfloor \frac{3\Delta}{2} \rfloor$ for any tree T with maximum degree Δ . And then by applying some orientation technique we present two upper bounds for list star chromatic index of k -degenerate graphs.

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DEDICATION

To

my father Tongxin Han , my mother Chuanmei Li, my husband Jiaao Li

and

my lovely son Victor Yihan Li

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Chapter 1

Preliminaries

1.1 Notation and Terminology

We follow notations of Bondy and Murty [6] for graphs. We use \mathbb{Z} to denote the set of all integers and \mathbb{N} to denote the set of all natural numbers. For an $m \in \mathbb{Z}$ with $m > 1$, we use \mathbb{Z}_m to denote the set of all integers modulo m as well as the cyclic group of order m . Graphs considered may have multiple edges but no loops. Following [6], for a graph G , $\kappa(G)$, $\kappa'(G)$, $\delta(G)$ and $\Delta(G)$ denote the connectivity, the edge-connectivity, the minimum degree and the maximum degree of G , respectively.

Let A denote an (additive) abelian group with identity 0, and $A^* = A - \{0\}$. Assume that G has an orientation $D(G)$. If an edge $e \in E(G)$ is oriented from a vertex u to a vertex v , then let **tail**(e) = u and **head**(e) = v . For a vertex $v \in V(G)$, define

$$E_D^+(v) = \{e \in E(G) | v = \text{tail}(e)\}, \text{ and } E_D^-(v) = \{e \in E(G) | v = \text{head}(e)\},$$

and let $d^-(v) = |E_D^-(v)|$ and $d^+(v) = |E_D^+(v)|$.

For each integer $k \geq 1$, let $[k] = \{1, 2, \dots, k\}$ and denote $V_k(G)$, $V_{\leq k}(G)$, $V_{\geq k+1}(G)$ the set of vertices with degree k , at most k , at least $k+1$ in G respectively. Denote $E_G(u)$ the set of edges incident with the vertex u in G and $d_G(u) = |E_G(u)|$ the degree of u in G .

1.2 Group Connectivity

Following Jaeger et al [31], we define $F(G, A) = \{f | f : E(G) \rightarrow A\}$ and $F^*(G, A) = \{f | f : E(G) \rightarrow A^*\}$. For a function $f : E(G) \rightarrow A$, define $\partial f : V(G) \rightarrow A$ by

$$\partial f(v) = \sum_{e \in E_D^+(v)} f(e) - \sum_{e \in E_D^-(v)} f(e),$$

where “ \sum ” refers to the addition in A .

A mapping $b : V(G) \rightarrow A$ is an A -valued **zero sum function** on G if $\sum_{v \in V(G)} b(v) = 0$. The set of all A -valued zero sum functions on G is denoted by $Z(G, A)$. A function $f \in F(G, A)$ is an A -**flow** of G if $\partial f(v) = 0$ for every vertex $v \in V(G)$. An A -flow f is a **nowhere-zero A -flow** (abbreviated as A -NZF) if $f \in F^*(G, A)$. If f is a Z -NZF satisfying for all $e \in E(G)$, $|f(e)| < k$, then f is called a **nowhere-zero k -flow** (k -NZF). Tutte [68] (see also Brylawski [9], Arrowsmith and Jaeger [2]) indicated that a graph G has a nowhere-zero k -flow if and only if G has a nowhere-zero \mathbb{Z}_k -flow.

The nowhere-zero flow problem was first introduced by Tutte [67] in his way to attach the 4-color-conjecture. The following fascinating conjectures of Tutte and Jaeger on nowhere zero flows remain open as of today.

Conjecture 1.2.1. (*Tutte [67], [30]*)

- (i) Every graph G with $\kappa'(G) \geq 2$ has a nowhere-zero 5-flow.
- (ii) Every graph G with $\kappa'(G) \geq 2$ without a subgraph contractible to the Peterson graph admits a nowhere-zero 4-flow.
- (iii) Every graph G with $\kappa'(G) \geq 4$ admits a nowhere-zero 3-flow.
- (iv) There exists an integer $k \geq 4$ such that every k -edge-connected graph has a nowhere-zero 3-flow.

For a mapping $b \in Z(G, A)$, a function $f \in F^*(G, A)$ is a **nowhere-zero** (A, b) -**flow** (abbreviated as (A, b) -NZF) if $\partial f = b$. A graph G is **A -connected** if for any $b \in Z(G, A)$, G has an (A, b) -NZF. Let $\langle A \rangle$ be the family of graphs that are A -connected. The **group connectivity number** of a graph G is defined as

$$\Lambda_g(G) = \min\{k \mid G \in \langle A \rangle \text{ for every abelian group } A \text{ with } |A| \geq k\}.$$

The concept of group connectivity was first introduced by Jaeger, Linial, Payan and Tarsi in [31] as a nonhomogeneous form of the nowhere-zero flow problem. They left with several fascinating conjectures in this area, which remain open as of today.

Conjecture 1.2.2. (*Jaeger et al. [31]*)

- (i) If $\kappa'(G) \geq 3$, then $\Lambda_g(G) \leq 5$.
- (ii) If $\kappa'(G) \geq 5$, then $\Lambda_g(G) \leq 3$.
- (iii) There exists an integer $k \geq 5$ such that if $\kappa'(G) \leq k$, then $\Lambda_g(G) \leq 3$.

Many efforts towards these conjectures have been made, as surveyed in [30]. Seymour [62] proves that every 2-edge-connected graph has a nowhere zero 6-flow. Jaeger et al improve this result by showing that if G is a 3-edge-connected graph, then $\Lambda_g(G) \leq 6$. More recently, a break through on \mathbb{Z}_3 -connectivity has been made by Thomassen and by Lov  ze et al.

Theorem 1.2.3. (Thomassen [65]) *If $\kappa'(G) \geq 8$, then G is \mathbb{Z}_3 -connected.*

This lower bound in Theorem 1.2.3 has recently been improved.

Theorem 1.2.4. (Lovasz, Thomassen, Wu and Zhang [54], Wu [69]) *If $\kappa'(G) \geq 6$, then G is \mathbb{Z}_3 -connected.*

The concept of modulo orientation is motivated by the integer flow of graphs introduced by Tutte [66]. If a graph G has an orientation D such that $d_D^+(v) - d_D^-(v) \equiv 0 \pmod{k}$ for every vertex $v \in V(G)$, then we say that G admits a **modulo k -orientation**, or a mod k -orientation for short. Let \mathcal{M}_k denote the family of all graphs admitting a mod k -orientation. As a connected graph G has a modulo $2p$ -orientation if and only if G is Eulerian.

Jaeger [30] observed that, in a graph G , the existence of a mod $(2p+1)$ -orientation is equivalent to the existence of an integer flow (D, f) with $|f(e)| \in \{p, p+1\}$ for each $e \in E(G)$, which is called a *circular $(2 + \frac{1}{p})$ -flow*. In particular, it is well-known that a graph admits a nowhere-zero 3-flow if and only if it admits a mod 3-orientation (see [30, 66, 71]). Tutte's 3-flow conjecture (see [6]) can be stated as follows.

Conjecture 1.2.1. (Tutte) *Every 4-edge-connected graph admits a mod 3-orientation.*

In addition, as observed by Jaeger [30], Tutte's famous 5-flow conjecture [67], which asserts that *every bridgeless graph admits a nowhere-zero 5-flow*, is implied by the following conjecture.

Conjecture 1.2.2. (Jaeger, [30]) *Every 9-edge-connected graph admits a mod 5-orientation.*

Motivated by the group connectivity property defined by Jaeger et al. [31], the concept of strongly \mathbb{Z}_{2p+1} -connectedness was introduced in [43] (see also [40]), serving as contractible configurations for mod $(2p+1)$ -orientations.

Definition 1. *A graph G is **strongly \mathbb{Z}_{2p+1} -connected** if, for every $b \in Z(G, \mathbb{Z}_{2p+1})$, there is an orientation D such that $d_D^+(v) - d_D^-(v) \equiv b(v) \pmod{2p+1}$ for every vertex $v \in V(G)$.*

Conjecture 1.2.2 is further strengthened to the following conjecture.

Conjecture 1.2.3. (Lai [40]) *Every 9-edge-connected graph is strongly \mathbb{Z}_5 -connected.*

Let $\langle \mathcal{SZ}_{2p+1} \rangle$ denote the family of all strongly \mathbb{Z}_{2p+1} -connected graphs. Liang et al. [52] proved that the graph family $\langle \mathcal{SZ}_{2p+1} \rangle$ consists of exactly all *mod $(2p+1)$ -orientation contractible configurations*, that is, all those graphs G such that for every supergraph Γ containing G as a subgraph, Γ/G has a mod $(2p+1)$ -orientation if and only if Γ has a mod $(2p+1)$ -orientation.

A subgraph H of G is called a **maximal $\langle \mathcal{SZ}_{2p+1} \rangle$ -subgraph** of G if $H \in \langle \mathcal{SZ}_{2p+1} \rangle$ and for any subgraph L of G containing H as a proper subgraph, $L \notin \langle \mathcal{SZ}_{2p+1} \rangle$. Since $K_1 \in \langle \mathcal{SZ}_{2p+1} \rangle$ by definition, every vertex of a graph G lies in a maximal $\langle \mathcal{SZ}_{2p+1} \rangle$ -subgraph of G . Let H_1, H_2, \dots, H_c denote the collection of all maximal $\langle \mathcal{SZ}_{2p+1} \rangle$ -subgraph of G . Then

$G' = G/(\cup_{i=1}^c E(H_i))$ is the $\langle \mathcal{SZ}_{2p+1} \rangle$ -**reduction** of G , and we also say G is $\langle \mathcal{SZ}_{2p+1} \rangle$ -reduced to G' . A graph G is $\langle \mathcal{SZ}_{2p+1} \rangle$ -**reduced** if G does not have any nontrivial subgraph in $\langle \mathcal{SZ}_{2p+1} \rangle$. By definition, the $\langle \mathcal{SZ}_{2p+1} \rangle$ -reduction of a graph is always $\langle \mathcal{SZ}_{2p+1} \rangle$ -reduced. Since contraction may bring in new parallel edges, even when G is a simple graph, its $\langle \mathcal{SZ}_{2p+1} \rangle$ -reduction may have multiple edges.

1.3 Graph Coloring

Let $G = (V(G), E(G))$ be a simple graph. Let $[k] = \{1, 2, \dots, k\}$, and $\phi: V(G) \cup E(G) \rightarrow [k]$ be a total k -weighting and $m_\phi(v) = \phi(v) + \sum_{e \in E(v)} \phi(e)$ where $E(x)$ is the set of edges incident with x for a vertex x . We call ϕ a **neighbor sum distinguishing total k -weighting** (abbreviated as NSD total k -weighting) if $m_\phi(u) \neq m_\phi(v)$ for each $uv \in E(G)$.

The NSD total weighting problem was introduced by Przybyło and Woźniak [59] as a variation of a similar problem for edge weighting introduced by Karoński, Łuczak and Thomason [34]. Przybyło and Woźniak [59] proposed the well-known 1-2 Conjecture that every graph has an NSD total 2-weighting. Karoński, Łuczak and Thomason [34] proposed the well-known 1-2-3 Conjecture that every graph without isolated edges has an NSD edge 3-weighting. Towards those conjectures, Kalkowski, Karoński and Pfender [33] showed that every connected graph with at least three vertices has an NSD edge 5-weighting. Kalkowski [32] showed that every graph has an NSD total 3-weighting.

A total k -weighting ϕ of a graph G is called a proper total k -coloring if $\phi(x) \neq \phi(y)$ for each pair of adjacent or incident elements $x, y \in V(G) \cup E(G)$. A proper total k -coloring ϕ of G is called an **NSD total k -coloring** if $m_\phi(u) \neq m_\phi(v)$ for each edge $uv \in E(G)$. The smallest number k in such a coloring of G is called the NSD total chromatic number, denoted by $\chi_\Sigma^t(G)$. We use $\Delta(G)$ (abbreviated as Δ) to denote the maximum degree of G . Note that $\chi_\Sigma^t(G) \geq \Delta + 1$ and if G has two adjacent vertices with degree Δ , then $\chi_\Sigma^t(G) \geq \Delta + 2$.

The concept of NSD total k -coloring was introduced by Piłśniak and Woźniak [58]. They studied the NSD total k -coloring for complete graphs, bipartite graphs, subcubic graphs and proposed the following conjecture.

Conjecture 1.3.1. (Piłśniak and Woźniak [58]) *For a graph G with maximum degree Δ , $\chi_\Sigma^t(G) \leq \Delta + 3$.*

A star coloring of a graph is a proper vertex coloring such that the union of any two color classes induces a star forest. This notion was first introduced by Grünbaum [21] in 1973 and did not attract more attention until 2001 in the paper by Fertin, Raspaud and Reed [19]. Just like relation between concepts of traditional edge and vertex colorings, a star coloring of a line graph is a star edge coloring of the original graph.

A **star edge coloring** of a graph G is a proper edge coloring such that every connected bicolored subgraph is a path of length at most 3 (the length of a path is the number of edges). The notion of the star edge coloring is intermediate between acyclic edge coloring, when every bicolored subgraph is acyclic, and strong edge coloring when every bicolored connected subgraph has at most two edges.

The star chromatic index of G , denoted by $\chi'_{st}(G)$, is the smallest integer k such that G is star k -edge-colorable. It seems very difficult to determine the star chromatic index of graphs even for complete graphs and subcubic graphs. Lei, Shi, and Song [47] showed that it is NP-complete to determine whether a subcubic multigraph is star 3-edge-colorable. Dvořák, Mohar, and Šámal [16] presented the following upper and lower bounds for complete graphs:

$$2n(1 + o(1)) \leq \chi'_{st}(K_n) \leq n \frac{2^{2\sqrt{2}(1+o(1))\sqrt{\log n}}}{(\log n)^{\frac{1}{4}}}.$$

Dvořák, Mohar, and Šámal [16] also studied star edge coloring of subcubic graphs and proved the following.

Theorem 1.3.1 ([16]). *If G is a subcubic graph, then $\chi'_{st}(G) \leq 7$.*

They made the following conjecture.

Conjecture 1.3.2 ([16]). *If G is a subcubic graph, then $\chi'_{st}(G) \leq 6$.*

Chapter 2

Index problem of group connectivity

2.1 Introduction

The **line graph** $L(G)$ of a graph G is defined as the graph whose vertices are the edges of G and where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are incident to a common vertex. We define $L^0(G) = G$ and for integers $k \geq 0$, define recursively $L^{k+1}(G) = L(L^k(G))$. Each $L^k(G)$ is called **the k th iterated line graph** of G , or just an iterated line graph of G . For an integer $n > 0$, let P_n and C_n denote the path on n vertices and the cycle of order n , called an n -path and an n -cycle, respectively. By the definition of line graphs, if $G \in \{K_{1,3}\} \cup \{P_n, C_n | n \in \mathbb{N}\}$, then the iterated line graph of G is either stable as a cycle, or diminishing when k becomes bigger. Therefore, throughout this chapter, we always assume that G is a connected graph that is not in $\{K_{1,3}\} \cup \{P_n, C_n | n \in \mathbb{N}\}$.

The **hamiltonian index** $i_h(G)$ of G is the smallest positive integer k such that $L^k(G)$ is hamiltonian. The concept of hamiltonian index was first introduced by Chartrand and Wall [10], who showed that (Theorem A of [10]) if a connected graph G is not a path, then $i_h(G)$ exists as a finite number. Clark and Wormald [12] considered other indices related to hamiltonicity of the iterated line graphs. More generally, the following is proposed in [45].

Definition 2.1.1. *For a graphical property \mathcal{P} and a connected nonempty simple graph G which is not in $\{K_{1,3}\} \cup \{P_n, C_n | n \in \mathbb{N}\}$, define the \mathcal{P} -index of G , denoted $\mathcal{P}(G)$, as*

$$\mathcal{P}(G) = \begin{cases} \min\{k | L^k(G) \text{ has property } \mathcal{P}\} & \text{if at least one such integer } k \text{ exists} \\ \infty & \text{otherwise} \end{cases}$$

The index problem has been investigated by many, including [10], [12], [17], [41], [46], [61], [75], among others. The purpose of this chapter is to investigate the indices for group connectivity of graphs.

The goal of this chapter is to show that if $G \notin \{K_{1,3}\} \cup \{P_n, C_n | n \in \mathbb{N}\}$, then for any A , there exists a finite integer $m \in \mathbb{N}$ such that $L^m(G) \in \langle A \rangle$. The smallest such m is denoted by

$i_A(G)$, called **the A -connected index** of G . We shall to determine best possible upper bounds for the indices of A -connectedness of graphs, for all abelian groups A . In Section 2, we display the tools we will use in the arguments. Best possible upper bounds of group connectivity are studied in the last section.

2.2 Triangular and triangulated connected indices

Throughout this section, G denotes a connected graph that is not in $\{K_{1,3}\} \cup \{P_n, C_n | n \in \mathbb{N}\}$. For each $i \in \mathbb{N}$, let $D_i(G)$ denote the set of all vertices of degree i in G , and $D_{\leq i}(G) = \cup_{1 \leq j \leq i} D_j(G)$. A graph G is **triangular** if every edge of G lies in a 3-cycle of G .

As our arguments will be back and forth between G and $L(G)$, for each edge $e \in E(G)$, we will often use, in the proof arguments throughout the rest of this chapter, v_e to denote the vertex in $L(G)$ corresponding to $e \in E(G)$. Likewise, if $u \in V(L(G))$, then we often use $e(u)$ to denote the edge in G corresponding to u in $L(G)$.

Proposition 2.2.1. *The following are equivalent.*

- (i) $L(G)$ is triangular;
- (ii) For any $v \in D_1(G)$, $N_G(v) \subseteq D_{\geq 3}(G)$; for any $v \in D_2(G)$, there exists an $K_3 \subseteq G$ such that $v \in V(K_3)$.

Proof. Suppose that (i) holds, or $L(G)$ is triangular. We argue by contradiction to prove (ii). Assume first that for some $v_1 \in D_1(G)$, the only vertex w in $N_G(v_1)$ has degree at most 2. Since G is not a path, we have $w \in D_2(G)$. Thus the vertex in $L(G)$ corresponding to the edge $v_1w \in E(G)$ is a vertex of degree 1, contrary to the assumption that $L(G)$ is triangular. Thus every vertex in $D_1(G)$ must be adjacent to a vertex in $D_{\geq 3}(G)$. Next, we assume that G has a vertex $v_2 \in D_2(G)$ with $N_G(v_2) = \{w_1, w_2\}$. If $w_1w_2 \notin E(G)$, then by the definition of line graphs, the edge in $L(G)$ joining the vertices w_1v_2 and v_2w_2 in $L(G)$ is not in a 3-cycle, contrary to the assumption that $L(G)$ is triangular. This proves (ii).

Conversely, assume that G satisfies (ii). Let $e_1, e_2 \in E(G)$ be an arbitrary pair of adjacent vertices in $L(G)$. Then $L(G)$ has an edge f linking e_1 and e_2 . Then for some $v \in V(G)$, both e_1 and e_2 are incident with v . If $d_G(v) = k \geq 3$, then by the definition of line graphs, edges incident with v are vertices in $L(G)$ which induce a complete subgraph on $k \geq 3$ vertices. As $k \geq 3$, f lies in a 3-cycle of $L(G)$. Therefore, we assume that $d_G(v) = 2$. By (ii), v lies in a 3-cycle of G . Since e_1 and e_2 are the only edges incident with v , the 3-cycle in G containing v must also contain e_1 and e_2 . By the definition of line graphs, the edges of this 3-cycle is also a 3-cycle in $L(G)$, and so f lies in a 3-cycle in this case also. This proves that $L(G)$ must be triangular, and so (i) holds. \square

For any graph Γ , and for distinct edges $e, e' \in E(\Gamma)$, an (e, e') -**path** of Γ is a path P whose initial edge is e and whose terminal edge is e' . The edges in $E(P) - \{e, e'\}$ are called the internal edges of P . By the definition of connectedness, a graph Γ is connected if and only if for any pair of distinct edges $e, e' \in E(\Gamma)$, Γ has an (e, e') -path.

For any e, e' in a graph G , define $e \sim e'$ if and only if $e = e'$ or there exists a sequence C_1, C_2, \dots, C_k of cycles of length at most 3, such that $e \in E(C_1)$ and $e' \in E(C_k)$ and for any $1 \leq i \leq k-1$, $E(C_i) \cap E(C_{i+1}) \neq \emptyset$. Such a sequence of 3-cycles is called a **triangular sequence** connecting e and e' . It is routine to verify that \sim is an equivalence relation on $E(G)$. Each equivalence class induces a subgraph which is called a **triangularly connected component** of G . If $E(G)$ is a triangularly connected component, then G is **triangularly connected**.

Proposition 2.2.2. *Let G be a connected graph not in $\{K_{1,3}\} \cup \{P_n, C_n | n \in \mathbb{N}\}$ with $|E(G)| \geq 3$. The following are equivalent.*

- (i) $L(G)$ is triangularly connected.
- (ii) For any pair of distinct edges $e, e' \in E(G)$, G has an (e, e') -path P such that every internal edge of P lies in a 3-cycle of G .

Proof. Assume that (ii) holds. Let H_1, H_2, \dots, H_c be the triangularly connected components of $L(G)$. Since G is connected, $L(G)$ is also connected. We may assume that $V(H_1) \cap V(H_2)$ contains a vertex v_e , corresponding to an edge $e \in E(G)$. By definition of v_e , there exists a vertex $v_{e_1} \in V(H_1)$ and a vertex $v_{e_2} \in V(H_2)$ such that e is incident with e_1 and e_2 in G . Therefore, we assume that for $i \in \{1, 2\}$, G has vertices v_1, v_2 such that e_i, e are incident with v_i . Since v_{e_1} and v_{e_2} are not in the same triangularly connected component of $L(G)$, $v_1 \neq v_2$. Thus e_1 and e_2 are distinct edges in G . By (ii), G has an (e_1, e_2) -path P such that every internal edge of P lies in a 3-cycle of G . Thus by the definition of $L(G)$, for the two edges ee_1 and ee_2 , $L(G)$ has a triangular sequence connecting ee_1 and ee_2 . It follows that ee_1 and ee_2 are in the same triangularly connected component, whence $H_1 = H_2$, contrary to the fact that $H_1 \neq H_2$. This contradiction justifies that (ii) implies (i) of Lemma 2.2.2.

Conversely, assume that (i) holds. Let e, e' be distinct edges in G . If e and e' are adjacent in G , then the path in $G[\{e, e'\}]$ is a path satisfying (ii). Thus we assume that e and e' are not adjacent in G . Since G is connected, there exist edges $e_1, e_2 \in E(G)$ such that e, e_1 are adjacent in G , and e' and e_2 are adjacent in G . Thus ee_1 and $e'e_2$ are edges in $L(G)$. Since $L(G)$ is triangularly connected, there exists a triangular sequence C_1, C_2, \dots, C_k connecting the two edges ee_1 and $e'e_2$ in $L(G)$. Among all such sequences, choose one such that k is minimized. Let $v_e, v_{e'}$ denote the vertices in $L(G)$ corresponding to the edges e and e' in G , respectively. Let P' be a $(v_e, v_{e'})$ -path in $L(G)$ with $V(P') \subset \cup_{i=0}^k V(C_i)$. As $V(P') \subseteq E(G)$, we define $P = G[V(P')]$. Since k is minimized, there is no 3-cycle in P , and so P is a path. Let xy be any

internal edge of P . By the definition of P' , we have $v_{xy} \in V(C_i)$, for some $1 \leq i \leq k$. Let uv_{xy} be the common edge of C_{i-1} and C_i in $L(G)$. Then we may assume that $V(C_i) = \{u, v_{xy}, v\}$ and, $V(C_{i-1}) = \{u, v_{xy}, w\}$, for some vertices $v, w \in V(G)$. If $G[e(u) \cup e(v) \cup xy] = C_3$, then xy lies in this 3-cycle in G . Thus we may assume that $G[e(u) \cup e(v) \cup xy] = K_{1,3}$.

Since w is adjacent to u and v_{xy} , and $G[\{e(u), e(w), xy\}] = C_3$, if xy is not in any 3-cycle in G , then we may assume that x is a common vertex in $e(u)$, $e(v)$ and $e(w)$, and so $G[\{e(u), e(v), xy, e(w)\}] = K_{1,4}$, contrary to the assumption that k is minimized. This proves (ii). \square

Corollary 2.2.3. *Each of the following holds.*

(i) *If G is triangular, then $L(G)$ is triangularly connected.*

(ii) *If a graph G is triangularly connected, then $L(G)$ is also triangularly connected.*

Proof. (i) Let e, e' be any pair of distinct edges in G , and $e = u_1u_2$, $e' = v_1v_2$. Since G is connected, there is (u_1, v_1) -path P in G . Since G is triangular, every edge of P is in a 3-cycle C_3 . By Proposition 2.2.2, $L(G)$ is triangularly connected.

(ii) Since triangularly connected graph G is also a triangular graph, by (i), it follows that $L(G)$ is also triangularly connected. \square

Given a connected graph G , a path P of G is a **divalent path** of G if every internal vertex of P has degree 2 in G . By this definition, if an edge is incident with two vertices neither of which is of degree 2, then this edge e induces a divalent path of G . We call P a normal divalent path of G , if all internal vertices of P are of degree 2 in G and if $|E(P)| = 2$, then P is not in a 3-cycle of G . Let $\mathcal{P}(G)$ denote the set of all normal divalent path of G , and define,

$$l(G) = \max\{m \mid G \text{ has a normal divalent path of length } m\}.$$

As in the literature, many studies have used $l(G)$ as an invariant to investigate the hamiltonian index as well as other hamiltonian related indices, see [10], [12], [17], [41], [75], among others. We present the following.

Proposition 2.2.4. *Let G be a connected graph with at least 3 edges not in $\{K_{1,3}\} \cup \{P_n, C_n \mid n \in \mathbb{N}\}$, and let $l = l(G)$. Each of the following holds.*

(i) *(Lemma 3.2 [74]) $L^l(G)$ is triangular.*

(ii) *$L^{l+1}(G)$ is triangularly connected.*

Proof. It suffices to prove (ii). By (i), $L^l(G)$ is triangular. Then, by Corollary 2.2.3 (i), $L^{l+1}(G)$ is triangularly connected. \square

2.3 Group connectivity indices

Throughout this section, we always assume that A is a finite abelian group with at least 3 elements and G is a connected graph not in $\{K_{1,3}\} \cup \{P_n, C_n | n \in \mathbb{N}\}$. Define the A -connected index of G as

$$i_A(G) = \min\{m \in \mathbb{N} \cup \{\infty\} \mid L^m(G) \text{ is } A\text{-connected}\}.$$

We shall show that for any abelian group A , if G is not in $\{K_{1,3}\} \cup \{P_n, C_n | n \in \mathbb{N}\}$ then $i_A(G)$ exists as a finite number. We will determine best possible upper bounds for these indices. The following will be used in our arguments.

Lemma 2.3.1. *Let A be an abelian group with $|A| \geq 3$ and let T be a connected spanning subgraph of a graph G . Each of the following holds.*

- (i) *(Lemma 3.3 of [38]) If G is a cycle of length $n \geq 2$, then G is A -connected if and only if $|A| \geq n + 1$.*
- (ii) *(Lemma 2.1 of [39]) If for each edge $e \in E(T)$, G has an A -connected subgraph H_e with $e \in E(H_e)$, then G is A -connected.*

For a subset $X \subset E(G)$, the contraction G/X is the graph obtained from G by identifying the two ends of each edge in X and then deleting all loops generated by this process. Note that even if G is simple, G/X may have multiple edges. For simplicity, we write G/e for $G/\{e\}$, where $e \in E(G)$. If H is a subgraph of G , then G/H denotes $G/E(H)$. If $v \in V(G/H)$ is obtained by contracting a connected subgraph H of G , then H is called **the preimage** of v , and v is called the **image** of H .

Proposition 2.3.2. *(Proposition 3.2 of [38])*

- (i) *If $H \in \langle A \rangle$ and if $e \in E(H)$, then $H/e \in \langle A \rangle$.*
- (ii) *If $H \in \langle A \rangle$, then $G/H \in \langle A \rangle$ if and only if $G \in \langle A \rangle$.*

Let H be an induced subgraph of G . We define $I_1(H)$ to be $L(G)[E(H)]$, the subgraph of $L(G)$ induced by $E(H)$. Let $I_1 : H \rightarrow L(G)[E(H)]$ be a mapping from the set of all induced subgraph H of G to be the set of all induced subgraphs of $L(G)$. We define $I_1^-(I_1(H)) = H$. Inductively, if I_k and I_k^- are defined, then $I_k(H)$ is an induced subgraph of $L^k(G)$, and so $I_{k+1}(H) = I_1(I_k(H))$ is an induced subgraph of $L^{k+1}(G)$, and define $I_{k+1}^-(H) = I_1^-(I_k^-(H))$. We adopt the notation $I_{k+1}^-(e)$ if $I_{k+1}(H)$ is a path induced by an edge e . Let G be a graph. Define $E' = E'(G) = \{e \in E(G) | e \text{ is in a cycle of } G \text{ of length at most } 3\}$ and $E''(G) = E(G) - E'(G)$. Also define

$$P(G) = \{P | P \text{ is a divalent path in } G \text{ with } |E(P)| = l\}.$$

Lemma 2.3.3. *(Lemma 12 of [46]) Let $d > 0$ be an integer and let $e \in E''(L^{d-1}(G))$. Then $I_{d-1}^-(e)$ is a divalent path in G with length at least d .*

Let $G^* = G - E(P(G))$, and let $G_1^*, G_2^*, \dots, G_t^*$ be the components of G^* , where $t \geq 1$. Let G' be the graph obtained from G by contracting every $G_i^* \in G$ into a vertex, for any $1 \leq i \leq t$ and replace every $P \in P(G)$ with one edge. By definition, if $G \in \langle A \rangle$, then $\kappa'(G) \geq 2$.

Theorem 2.3.4. *Let G be a connected graph with $l = l(G)$, and A be an abelian group with $|A| \geq 4$. Each of the following holds:*

- (i) *If $l = 1$, then $L(G) \in \langle A \rangle$.*
- (ii) *If $l > 1$, then $i_A(G) \leq l$, and the equality holds if and only if $G' \notin \langle A \rangle$.*

Proof. (i). Assume that $l = 1$. By the definition of divalent paths, $l = 1$ if and only if one of the following holds:

- (A) $\delta(G) \geq 3$, or
- (B) $\delta(G) \leq 2$ and every vertex of degree 2 is contained in a triangle.

For every edge $e_1 e_2 \in E(L(G))$, there exists a vertex $v \in V(G)$ such that e_1, e_2 are both incident with v in G . If (A) holds, then the edge $e_1 e_2$ in $L(G)$ lies in a complete subgraph of order $d_G(v) \geq \delta(G) \geq 3$. It follows by Lemma 2.3.1 that $L(G) \in \langle A \rangle$. If (B) holds, then G has a 3-cycle containing both e_1 and e_2 , hence $L(G)$ has a 3-cycle containing the edge $e_1 e_2$. Again by Lemma 2.3.1 $L(G) \in \langle A \rangle$. This proves (i).

(ii). Suppose that $l \geq 2$. By Proposition 2.2.4, every edge e of $L^l(G)$ is in a 3-cycle. By Lemma 2.3.1(i), $K_3 \in \langle A \rangle$, and so by Lemma 2.3.1(ii), $L^l(G) \in \langle A \rangle$. This implies that $i_A(G) \leq l$; and that $i_A(G) = l$ if and only if $L^{l-1}(G) \notin \langle A \rangle$.

By the definition of $P(G)$, we have, for any $1 \leq i \leq t$, $i_A(G_i^*) \leq l(G_i^*) \leq l - 1$. Thus $I_{l-1}(G_i^*) \in \langle A \rangle$. By the definition of line graphs, every divalent path of length l in a graph G will become a divalent path of length $l - 1$ in $L(G)$. It follows that if $P \in P(G)$, then $I_{l-1}(P) \cong K_2$. By Proposition 2.3.2 (ii), $L^{l-1}(G) \notin \langle A \rangle$ if and only if $G' \notin \langle A \rangle$. \square

Let $\Delta \geq 3$ be an integer and $G(\Delta)$ be the graph obtained from $K_{1,\Delta}$ and $P_{n-\Delta}$ by identifying a vertex in $D_1(K_{1,\Delta})$ and a vertex in $D_1(P_{n-\Delta})$. We observe that Δ is the maximum degree of $G(\Delta)$.

Theorem 2.3.5. *Let G be a connected simple graph on $n > 3$ vertices, $\Delta = \Delta(G)$ and A be an abelian group with $|A| \geq 4$. Each of the following holds.*

- (i) $i_A(G) \leq n - \Delta$.
- (ii) *Equality in (i) holds if and only if $G = G(\Delta)$.*

Proof. (i) Note that since G is not a cycle nor a path, we have $\Delta \geq 3$. By the definition of line graphs, $L(G)$ contains a K_Δ as a subgraph. Since $\Delta \geq 3$, by Lemma 2.3.1, $K_\Delta \in \langle A \rangle$. By Proposition 2.3.2 (ii), $L(G) \in \langle A \rangle$ if and only if $L(G)/K_\Delta \in \langle A \rangle$. Let $w \in V(L(G)/K_\Delta)$ be the vertex in $L(G)/K_\Delta$ onto which K_Δ is contracted. By Theorem 2.3.4, $i_A(L(G)) \leq l(L(G)/K_\Delta)$.

If $l(G) = 1$, then by Theorem 2.3.4(i), we have $i_A(G) \leq 1 \leq n - \Delta$. hence we may assume that $l(G) \geq 2$. As every divalent path of length l in G will become a divalent path of length

$l - 1$ in $L(G)$. To prove $i_A(G) \leq n - \Delta$, it suffices to prove $l(L(G)/K_\Delta) \leq n - \Delta - 1$. Let P be any divalent path in $L(G)/K_\Delta$ with $|E(P)| = l(L(G)/K_\Delta)$.

Case 1. $w \in V(P)$.

Suppose that $d_{L(G)/K_\Delta}(w) = 1$, or that $d_{L(G)/K_\Delta}(w) \geq 3$. Let $P' = P - \{w\}$. Then $I_1^-(P')$ is also a divalent path in G , and so

$$l(L(G)/K_\Delta) = |E(P)| \leq |E(I_1^-(P'))| \leq l \leq n - \Delta - 1.$$

Thus we assume that $d_{L(G)/K_\Delta}(v) = 2$. Let P^1 and P^2 be the two component of $P - \{w\}$. Then $I_1^-(P^1)$ and $I_1^-(P^2)$ are divalent paths in G . It follows that

$$l(L(G)/K_\Delta) = |E(P)| \leq |E(P^1)| + |E(P^2)| + 2 \leq |E(I_1^-(P^1))| + |E(I_1^-(P^2))| \leq n - \Delta - 1.$$

Case 2. $w \notin V(P)$.

Fix a vertex $v_0 \in D_\Delta(G)$. Then $I_1^-(P)$ is also a divalent path in G with $V(I_1^-(P)) \cap N_G(v_0) = \emptyset$. Hence $l(L(G)/K_\Delta) = |E(P)| \leq |E(I_1^-(P))| - 1 \leq n - \Delta - 1$.

Since $i_A(G) - 1 = i_A(L(G))$, Combining Cases 1 and 2, we have proved that $i_A(G) \leq n - \Delta$, and so (i) must hold.

(ii) If $G = G(\Delta)$, then $L^{n-\Delta-1}(G)$ has one cut edge, and so $L^{n-\Delta-1}(G) \notin \langle A \rangle$. Thus $i_A(G) = n - \Delta$. Conversely, assume that $i_A(G) = n - \Delta$. By Theorem 2.3.4, $l(G) \geq i_A(G) = n - \Delta$. Thus G must have a divalent path of length at least $n - \Delta$. Since $\Delta \geq 3$, we conclude that $G = G(\Delta)$. \square

The distance of two vertices $u, v \in V(G)$, denoted $dist_G(u, v)$, is the length of a shortest path from u to v of G . The diameter of G , denoted by $diam(G)$, is defined as

$$diam(G) = \max\{dist_G(u, v) \mid u, v \in V(G)\}.$$

Let G_0 be a graph obtained from a cycle C_{2d} by identifying a pendant edge, and for any finite abelian group A with $|A| \geq 4$, define

$$\mathcal{F}_A = \{G : G \text{ has a subgraph } H \text{ such that } G/H \text{ is a cycle of length at least } d + |A|\}.$$

Theorem 2.3.6. *Let G be a connected graph with $d = diam(G) \geq 2$ and A be an abelian group with $|A| \geq 4$.*

- (i) *If $d \leq 3$, then $i_A(G) \leq d$.*
- (ii) *If $d \geq 4$, then $i_A(G) \leq d$ if and only if $G \notin \mathcal{F}_A$.*
- (iii) *If $d \leq |A| - 1$, then $i_A(G) \leq d - 1$.*
- (iv) *If $d \geq |A|$, then $i_A(G) \leq 2d - |A| + 1$.*

Proof. Let $l = l(G)$. If $d \geq l$, by Theorem 2.3.4, then $i_A(G) \leq l \leq d$. Thus we may assume that $l \geq d + 1$. Fix a divalent path $P_0 \in P(G)$. Let u and v denote the two end vertices of P_0 .

If $u \neq v$, then there exists a (u, v) -path P' in G with $|E(P')| = d' \leq d$. Since $l > d$, and since P is a divalent path, we have $V(P_0) \cap V(P') = \{u, v\}$ and $l \leq 2d$. If $u = v$, then P_0 is a cycle. Since $G \neq C_n$, we also have $l \leq 2d$.

(i). $d \leq 3$. Then $l \leq 2d \leq d + 3$. For any divalent path $Q \in P(G)$, we observe that $I_d(Q)$ is a divalent path with length at most 3 in $L^d(G)$. We claim that $L^d(G)$ is triangular. If not, there exists one edge $e \in E(L^d(G))$ such that $e \in E''(L^d(G))$. By Lemma 2.3.3, $I_{d-1}^-(e)$ is a divalent path Q' in G with length at least d . Take a midpoint w of P_0 and a midpoint z of Q' . Then $\text{dist}_G(w, z) \geq l/2 + d/2 \geq (2d+1)/2 > d$, contrary to the assumption that $d = \text{diam}(G)$. Hence $L^d(G)$ must be triangular. By Lemma 2.3.1, we conclude that $i_A(G) \leq d$. Thus (i) must hold.

(ii). $d \geq 4$. Suppose that G has no subgraph H such that G/H is a cycle of length at least $d + |A|$. We claim that $l \leq d + |A| - 1$. If not, then there exists a divalent path $P \in P(G)$ with $|E(P)| \geq d + |A|$. Let P^o denote the set of all internal vertices of P . If $G - P^o$ is connected, then $G/(G - P^o)$ is a cycle of length $|E(P)| \geq d + |A|$, contrary to the assumption. Hence every edge in $E(P)$ is a cut edge of G . Since G is not a path, at least one end of P has degree at least 3 in G . It follows that $d \geq l$, contrary to the assumption that $l \geq d + 1$. Thus we must have $l \leq d + |A| - 1$. It follows that $l(L^d(G)) \leq |A| - 1$. If there exists an edge $e \in E(L^d(G))$ which is not in a cycle of length at most $|A| - 1$ in $L^d(G)$, then as $|A| \geq 4$, we note that $e \in E''(L^d(G))$. By Lemma 2.3.3, $I_{d-1}^-(e)$ is a divalent path Q in G with length at least d . Take a midpoint w of P_0 and a midpoint z of Q . Then $\text{dist}_G(w, z) \geq l/2 + d/2 \geq (2d+1)/2 > d$, contrary to the fact that $d = \text{diam}(G)$. Hence every edge of $L^d(G)$ lies in a cycle of length at most $|A| - 1$. By Lemma 2.3.1, $i_A(G) \leq d$.

Conversely, assume that $d \geq 4$ and $i_A(G) \leq d$. By contradiction, suppose that there exist H such that G/H is a cycle of length at least $d + |A|$. Thus $E(G/H)$ induces a divalent path Q in G , and $Q' = I_d(Q)$ is a divalent path with length at least $|A|$ in $L^d(G)$. Let $(Q')^o$ denote the set of all internal vertices of Q' . It follows that $C' = L^d(G)/(L^d(G) - (Q')^o)$ is a cycle of length at least $|A|$. By Lemma 2.3.1 (i), $C' \notin \langle A \rangle$. Since $L^d(G) \in \langle A \rangle$, by Proposition 2.3.2 (ii), $C' = L^d(G)/(L^d(G) - (Q')^o) \in \langle A \rangle$. This contradiction justifies that $G \notin \mathcal{F}_A$.

(iii) We claim that $\kappa'(L^{d-1}(G)) \geq 2$. If e is a cut edge of $L^{d-1}(G)$, then by Lemma 2.3.3, $I_{d-1}^-(e)$ is a divalent path P of length at least d in G such that every edge of P is a cut edge of G . Let P be a (u, v) -path of G . Since G is not a path, we may assume that $d_G(u) \geq 3$, and so $N_G(u) - V(P)$ has vertex w . It follows that $d \geq \text{dist}_G(w, v) \geq |E(P)| + 1 \geq d + 1$, a contradiction. This proves our claim. Now suppose that $L^{d-1}(G)$ has an induced cycle C of length $|E(C)| \geq |A| \geq 4$. For each edge $e \in E(C)$, by Lemma 2.3.3, $I_{d-1}^-(e)$ is a divalent path of length at least $d \geq 2$ in G . Hence G has a pair of vertices whose distance in G is least $d + 1$, contrary to the fact that $d = \text{diam}(G)$. Hence we conclude that every induced cycle of G must have length at most $|A| - 1$. Since $\kappa'(L^{d-1}(G)) \geq 2$, it follows that every edge of $L^{d-1}(G)$ lies in a cycle of length at most $|A| - 1$. By Lemma 2.3.1, $L^{d-1}(G)$ is A -connected.

(iv) Now assume that $d \geq |A|$. By Theorem 2.3.4, if $l(G) \leq d$, then $i_A(G) \leq d < 2d - |A| + 1$. Hence we may assume that $l(G) \geq d+1$. Note that for any divalent path $P \in P(G)$, $I_{2d-|A|+1}(P)$ is a divalent path with length at most $|A| - 1$ in $L^{2d-|A|+1}(G)$. If there exists an edge $e \in E(L^{2d-|A|+1}(G))$ which is not in a cycle of length at most $|A| - 1$ in $L^{2d-|A|+1}(G)$, then as $|A| \geq 4$, we have $e \in E''(L^{2d-|A|+1}(G))$. By Lemma 2.3.3, $I_{2d-|A|+1}^-(e)$ is a divalent path Q in G with length at least $2d - |A| + 2$. Take the midpoint w of P and a midpoint z of Q . We observe that $d \geq \text{dist}_G(w, z) \geq l/2 + (2d - |A| + 2)/2 \geq d + 1 + (l - |A|)/2 \geq d + 1$, a contradiction. Thus every edge in $L^{2d-|A|+1}(G)$ is in a cycle of length at most $|A| - 1$. By Lemma 2.3.1, $i_A(G) \leq 2d - |A| + 1$. \square

A **wheel** W_n is the graph obtained from C_n by adding one vertex and joining it to each vertex of C_n . A **fan** F_n is the graph obtained from P_n by adding one vertex and joining it to each vertex of P_n . As examples, $K_4 \cong W_3$ and $K_3 \cong F_2$. Let G_1, G_2 be two disjoint graphs. As in [18], $G_1 \oplus_2 G_2$, called the **parallel connection** of G_1 and G_2 , is defined to be the graph obtained from $G_1 \cup G_2$ by identifying exactly one edge. Let \mathcal{WF} be the family of graphs that satisfy the following conditions:

- (i) $K_3, W_{2n+1} \in \mathcal{WF}$;
- (ii) If $G_1, G_2 \in \mathcal{WF}$, then $G_1 \oplus_2 G_2 \in \mathcal{WF}$.

Theorem 2.3.7. (Theorem 1.4 of [18]) *Let G be a triangularly connected graph with $|V(G)| \geq 3$. Then G is not \mathbb{Z}_3 -connected if and only if $G \in \mathcal{WF}$.*

Beineke [3] and Robertson [60] showed that any line graph cannot have an induced subgraph isomorphic to W_5 or $K_{1,3}$. As for $n \geq 3$, any induced W_{2n+1} contains an induced $K_{1,3}$, Beineke and Robertson in fact proved the following.

Theorem 2.3.8. (Beineke [3] and Robertson [60], see also page 74 of [27]) *If a connected graph G is a line graph, then G has no induced subgraph isomorphic to W_{2n+1} for $n \geq 2$.*

Lemma 2.3.9. *If G is triangularly connected, then $L(G) \in \langle \mathbb{Z}_3 \rangle$.*

Proof. By Corollary 2.2.3(ii), $L(G)$ is also triangularly connected. By Theorem 2.3.7, to prove $L(G) \in \langle \mathbb{Z}_3 \rangle$, it suffices to prove that $L(G) \notin \mathcal{WF}$. By contradiction, we assume that $L(G) \in \mathcal{WF}$. By the definition of \mathcal{WF} , either $L(G) = G_1 \oplus_2 K_3$, or $L(G) = G_1 \oplus_2 W_{2n+1}$. By Theorem 2.3.8, we must have $n = 1$.

Case 1. $L(G) = G_1 \oplus_2 K_3$.

Let $V(K_3) = \{v_1, v_2, v_3\}$ in $L(G)$, where $d_{L(G)}(v_2) = 2$. Then $G[\{e(v_1), e(v_2), e(v_3)\}] \in \{K_3, K_{1,3}\}$ in G . Since G is triangularly connected, we must have $G[\{e(v_1), e(v_2), e(v_3)\}] = K_3$. Let u_1, u_2, u_3 denote the vertices of this K_3 in G such that $e(v_1) = u_1u_2$, $e(v_2) = u_2u_3$ and $e(v_3) = u_3u_1$. Since $G \neq K_3$, we may assume that $G - \{u_1, u_2, u_3\}$ has a vertex u_4 such that $u_1u_4 \in E(G)$. Since G is triangularly connected, there must be a 3-cycle sequence connecting

u_1u_4 and u_2u_3 . It follows that there must be a vertex $u_5 \in V(G) - \{u_1, u_2, u_3\}$ such that u_5u_2 or $u_5u_3 \in E(G)$. It follows that $d_{L(G)}(v_2) \geq 3$, contrary to the fact that $d_{L(G)}(v_2) = 2$. This contradiction indicates that Case 1 cannot occur.

Case 2. $L(G) = G_1 \oplus_2 W_{2n+1}$, where $n = 1$.

If $n = 1$, then $W_3 = K_4$ is a subgraph of $L(G)$. Let $V(W_3) = \{e_1, e_2, e_3, e_4\} \subset E(G)$, by the definition of line graphs, $G[\{e_1, e_2, e_3, e_4\}] \cong K_{1,4}$. Since G is triangular, we may assume that for some $e \in E(G)$, $G[e_1, e_2, e]$ is a 3-cycle. It follows that in $L(G)$, e as a vertex is adjacent to both vertices e_1 and e_2 , contrary to the fact that $L(G) = G_1 \oplus_2 W_3$. This contradiction indicates that Case 2 cannot occur as well.

It follows that $L(G) \notin \mathcal{WF}$, and so by Theorem 2.3.7, $L(G) \in \langle \mathbb{Z}_3 \rangle$. \square

Example 1. We consider two examples, which are useful in our discussions below.

(i) A tree T is a $(3,1)$ -tree if every vertex in T has degree equaling 3 or 1. Let T_n denote a $(3,1)$ -tree on $n \geq 4$ vertices. Then $l(T_n) = 1$. Direct computation indicates that $L^2(T_n)$ can be obtained from K_3 and K_4 via parallel connections. Hence $L^2(T_n) \in \mathcal{WF}$. It follows by the theorem below that $L^3(T_n) \in \mathbb{Z}_3$. This shows that $i_{\mathbb{Z}_3}(T_n) = l(T_n) + 2$.

(ii) Let $d \geq 3$ and $l \geq 1$ be integers. Define $J(d, l)$ to be the graph obtained from $K_{1,d}$ by replacing one edge of $K_{1,d}$ by a path of length l . Thus $J(d, 1) = K_{1,d}$. Since any $J(d, l)$ has $n = d + l$ vertices with $d = \Delta(J(d, l))$ being the maximum degree, if $G(\Delta)$ has n vertices, then $G(\Delta) = J(\Delta, n - \Delta)$. Direct computation yields that $L^2(J(3, 2)) = K_4 - e$ and $L^3(J(3, 2)) = W_4$. Therefore, $i_{\mathbb{Z}_3}(J(3, 2)) = 3$.

Lemma 2.3.10. Each of the following holds.

- (i) Let $k > 0$ be an integer. If H is a subgraph of G such that $H \notin \{K_{1,3}\} \cup \{P_n, C_n | n \in \mathbb{N}\}$, then $L^k(H)$ is a subgraph of $L^k(G)$.
- (ii) (Lemma 2.4 of [18]) Let G be a triangularly-connected graph. Then G is \mathbb{Z}_3 -connected if and only if G has a nontrivial \mathbb{Z}_3 -connected subgraph.
- (iii) Let G be a connected graph with a vertex v of $d_G(v) = 1$. If $G - v$ is triangular-connected, then $L(G)$ is \mathbb{Z}_3 -connected.
- (iv) If $l(G) \geq 2$, then $i_{\mathbb{Z}_3}(G) \leq l(G) + 1$.

Proof. (i). By the definition of a line graph, $L(H)$ is a subgraph of $L(G)$. As $H \notin \{K_{1,3}\} \cup \{P_n, C_n | n \in \mathbb{N}\}$, we note that $L(H) \notin \{K_{1,3}\} \cup \{P_n, C_n | n \in \mathbb{N}\}$, and so Lemma 2.3.10 (i) follows from induction.

(iii). Let $H = G - v$. Since H is triangular-connected, both $\delta(H) \geq 2$ and, by Lemma 2.3.9, $L(H)$ is \mathbb{Z}_3 -connected. Let e denote the only edge incident with v in G . Then by the definition of line graphs, $L(G) - e = L(G - v) = L(H)$. Since $\delta(H) \geq 2$, the vertex e is adjacent to at least 2 vertices in $L(H)$. It follows that $L(G)/L(H)$ is spanned by a 2-cycle, which, by Lemma

2.3.1(i), is \mathbb{Z}_3 -connected. Since $L(H)$ is \mathbb{Z}_3 -connected, it follows by Proposition 2.3.2(ii) that $L(G)$ is \mathbb{Z}_3 -connected. This justifies Lemma 2.3.10 (iii).

(iv). By Proposition 2.2.4, $L^{l+1}(G)$ is triangularly-connected. By (ii), it suffices to show that $L^{l+1}(G)$ contains a nontrivial subgraph H such that H is \mathbb{Z}_3 -connected. Since $l(G) = l$, there exists a maximal divalent path P of G with $|E(P)| = l(G) \geq 2$. Since G is not a path, we may assume that P has an end vertex u with $d_G(u) = d \geq 3$. Thus G contains a subgraph $J(3, l)$ with $l \geq 2$. We shall show that Let $H = L^{l+1}(J(3, l))$. By Lemma 2.3.10 (i), H is a subgraph of $L^{l+1}(G)$.

To show that H is \mathbb{Z}_3 -connected, we argue by induction on $k \geq 2$ to show that $L^{k+1}(J(3, k))$ is triangularly-connected and \mathbb{Z}_3 -connected. If $k = 2$, then by Example 1(ii), $L^3(J(3, 2))$ is triangularly-connected and \mathbb{Z}_3 -connected. Assume that $k \geq 3$, and that $L^k(J(3, k-1))$ is triangularly-connected and \mathbb{Z}_3 -connected. By direct computation, $L^k(J(3, k))$ has a unique vertex v of degree 1 such that $L^k(J(3, k)) - v = L^k(J(3, k-1))$. By Lemma 2.3.10 (iii), we conclude that $L^{k+1}(J(3, k))$ is triangularly-connected and \mathbb{Z}_3 -connected. Hence H is \mathbb{Z}_3 -connected. As H is a subgraph of $L^{l+1}(G)$, and as $L^{l+1}(G)$ is triangularly-connected, it follows by Lemma 2.3.10 (ii) that $L^{l+1}(G)$ is \mathbb{Z}_3 -connected. \square

Theorem 2.3.11. *Let $A = \mathbb{Z}_3$ denote the cyclic group of order 3. For an integer $d > 0$, define*

$$\mathcal{F}_d = \{G : G \text{ has a subgraph } H \text{ such that } G/H \text{ is a cycle of length at least } d+5\}.$$

If G is a connected graph with $\text{diam}(G) = d$ and $l = l(G)$, then each of the following holds.

- (i) $i_A(G) \leq l + 2$, and the equality holds if and only if G is a $(3, 1)$ -tree.*
- (ii) If $d \leq 3$, then $i_A(G) \leq d + 2$.*
- (iii) If $d \geq 4$, then $i_A(G) \leq d + 2$ if and only if $G \notin \mathcal{F}_d$.*

Proof. (i) By Proposition 2.2.4, $L^{l+1}(G)$ is triangularly connected. By Lemma 2.3.9, we have $i_A(G) \leq l + 2$. By Lemma 2.3.10 (iv), $i_A(G) = l + 2$ if and only if $l(G) = 1$ and $L^{l+1}(G) \notin \langle \mathbb{Z}_3 \rangle$. This happens, by Theorem 2.3.7, if and only if $L^2(G) \in \mathcal{WF}$. By Theorem 2.3.8, $L^2(G) \in \mathcal{WF}$ if and only if $L^2(G)$ can be built via parallel-connected from K_3 and K_4 . By Example 1(i), if G is a $(3, 1)$ -tree, then $i_{\mathbb{Z}_3}(G) = 3$. Conversely, since $L(G)$ is triangular, if $L^2(G)$ can be built via parallel-connected from K_3 and K_4 , then direct computation indicates that G must be a $(3, 1)$ -tree. This proves (i).

If $d \geq l$, then by Proposition 2.2.4 and Lemma 2.3.9, $i_A(G) \leq d + 2$. Hence we assume that $d < l$. Pick any divalent path $P \in P(G)$. Then $|E(P)| = l \geq d + 1$. Let u and v denote the two end vertices of P . Since $l \geq d + 1$, there exists a (u, v) -path P' in G with $|E(P')| = d' \leq d$ such that $V(P) \cap V(P') = \{u, v\}$. Note that $u = v$ is possible. Since G is not a cycle, we always have $d + 1 \leq l \leq 2d$.

(ii). Assume that $d \leq 3$. Then $l \leq 2d \leq d + 3$. For any divalent path $L \in P(G)$, $I_d(L)$ is a divalent path with length at most 3, and so $L^d(G)$ is triangular. By Corollary 2.2.3 and Lemma 2.3.9, $i_A(G) \leq d + 2$. This proves (ii).

(iii). Assume that $d \geq 4$. Fix a divalent path $P \in P(G)$, and let P^o denote the internal vertices of P . Since $d < l$, edges in P cannot be cut edges of G , and so $H_P = G - P^o$ is connected. Hence G/H_P is a cycle of length l . It follows that if G has no subgraph H such that G/H is a cycle of length at least $d + 5$, then $l \leq d + 4$. We claim that $L^d(G)$ is triangular. If not, then there exists an edge $e \in E''(L^d(G))$. By Lemma 2.3.3, $I_{d-1}^-(e)$ is a divalent path Q in G with length at least d . Take the midpoint w of P and the midpoint z of Q . Direct computation yields then $\text{dist}_G(w, z) \geq l/2 + d/2 \geq (2d + 1)/2 > d$, a contradiction. By Corollary 2.2.3 (i), and Lemma 2.3.9, $i_A(G) \leq d + 2$.

Conversely, assume that $i_A(G) \leq d + 2$. By contradiction, we assume further that G contains a subgraph H such that G/H is a cycle of length at least $d + 5$. Thus $P_0 = G[E(G/H)]$ is a divalent path in G ; and $C' = I_{d+2}(P_0)$ is a divalent path with length at least 4 in $L^{d+2}(G)$. By Lemma 2.3.1 (i), $C' \notin \langle A \rangle$. On the other hand, since $L^{d+2}(G) \in \langle A \rangle$, by Proposition 2.3.2 (ii), $C' = L^{d+2}(G)/L^{d+2}(H) \in \langle A \rangle$. Thus a contradiction is obtained. This completes the proof of (iii). \square

Chapter 3

Independence number and modulo orientation

3.1 Introduction

In this chapter, we investigate mod $(2p + 1)$ -orientations of graphs with bounded independence numbers. It is known that the complete graph K_{4p} does not admit a mod $(2p + 1)$ -orientation. Since the modulo orientation property is preserved under contraction, it is straightforward to construct an infinite family of graphs of independence number two without mod $(2p + 1)$ -orientation by replacing a vertex of K_{4p} with a large complete graph. On the other hand, all those graphs have the behavior that each of them is contractible to K_{4p} . So we may expect to characterize mod $(2p + 1)$ -orientation in the family of graphs with bounded independence number by excluding a list of graphs such that every graph in the family admits a mod $(2p + 1)$ -orientation if and only if it is not contractible to one of the graphs on the list, such as in the Kuratowski's theorem for planar graphs and characterization of graphs embedded on surface by excluding minors.

For any integer $t > 0$, define $\mathcal{F}(t)$ and $\mathcal{G}(t)$ to be graph families such that

$$\mathcal{F}(t) = \{G : G \text{ is } \langle S\mathbb{Z}_{2p+1} \rangle\text{-reduced with } 2 \leq |V(G)| \leq 6pt - 2p \text{ and } \alpha(G) \leq t\} \text{ and}$$

$$\mathcal{G}(t) = \mathcal{F}(t) \setminus \mathcal{M}_{2p+1}.$$

Theorem 3.1.1. *Let $t > 0$ be an integer. Each of the following holds.*

- (i) *A graph G with $\alpha(G) \leq t$ is strongly \mathbb{Z}_{2p+1} -connected if and only if the $\langle S\mathbb{Z}_{2p+1} \rangle$ -reduction of G is not in $\mathcal{F}(t)$.*
- (ii) *A graph G with $\alpha(G) \leq t$ admits a modulo $(2p + 1)$ -orientation if and only if the $\langle S\mathbb{Z}_{2p+1} \rangle$ -reduction of G is not in $\mathcal{G}(t)$.*

More descriptions concerning the graph families $\mathcal{F}(t)$ and $\mathcal{G}(t)$ will be presented below when $t = 2$. In particular, Theorem 3.1.2 below confirms that simple graphs with independence number 2 and large order admit mod $(2p + 1)$ -orientations under edge-connectivity $4p$.

Let K_n denote a complete graph with $V(K_n) = \{v_1, \dots, v_n\}$. For nonnegative integers s_1, s_2, \dots, s_{n-1} , let $K_n(s_1, s_2, \dots, s_{n-1})$ be the graph obtained from K_n by replacing the edge $v_n v_i$ by s_i parallel edges joining v_n and v_i , for each $i \in [n-1]$, and define

$$\begin{aligned}\mathcal{K}(2p+1) &= \{K_n(s_1, s_2, \dots, s_{n-1}) : 2 \leq n \leq 4p+1 \text{ and } 0 \leq s_i \leq 2p-1, \forall i \in [n-1]\}, \\ \mathcal{K}_1(2p+1) &= \mathcal{K}(2p+1) \setminus \mathcal{M}_{2p+1} \text{ and } \mathcal{K}_2(2p+1) = \mathcal{K}(2p+1) \setminus \langle \mathcal{SZ}_{2p+1} \rangle.\end{aligned}\quad (3.1)$$

Theorem 3.1.2. *Let G be a simple graph of order at least $10p+1$ with $\alpha(G) \leq 2$. Each of the following holds.*

- (i) *G admits a mod $(2p+1)$ -orientation if and only if the $\langle \mathcal{SZ}_{2p+1} \rangle$ -reduction of G is not in $\mathcal{K}_1(2p+1)$.*
- (ii) *G is strongly \mathbb{Z}_{2p+1} -connected if and only if the $\langle \mathcal{SZ}_{2p+1} \rangle$ -reduction of G is not in $\mathcal{K}_2(2p+1)$.*
- (iii) *If $\kappa'(G) \geq 2p$ and $\delta(G) \geq 4p$, then G is strongly \mathbb{Z}_{2p+1} -connected (and therefore, admits a mod $(2p+1)$ -orientation).*

As mod 5-orientation of graphs with multiple edges is related to 5-flow conjecture (see [30, 44]), we also show the corresponding Theorem 3.1.3 below for all graphs with independence number two in the mod 5-orientation case. Note that this verifies Conjecture 1.2.2 for all graphs with order at least 21 and independence number at most two.

Let $\mathcal{K}^*(5)$ be the family of graphs such that $H \in \mathcal{K}^*(5)$ if and only if $H \notin \mathcal{M}_5$, H is $\langle \mathcal{SZ}_5 \rangle$ -reduced, and H contains a subgraph isomorphic to $K_{|V(H)|-1}$ with $2 \leq |V(H)| \leq 9$ and $\kappa'(H) \leq 7$.

Theorem 3.1.3. *Let G be a graph of order at least 21 with $\alpha(G) \leq 2$. Each of the following holds.*

- (i) *G admits a mod 5-orientation if and only if the $\langle \mathcal{SZ}_{2p+1} \rangle$ -reduction of G is not in $\mathcal{K}^*(5)$.*
- (ii) *G admits a mod 5-orientation provided it is 8-edge-connected.*

Luo et al. [56] characterized mod 3-orientations of graphs with independence number at most 2, and thus verifies Tutte's 3-flow conjecture for graphs with independence number at most 2. In a consequence paper [50], Li, Luo and Wang adopt a similar idea as in this chapter and develop some new reduction method to obtain analogous results for mod 3-orientations. The results in paper [50] further confirm Tutte's 3-flow conjecture for graphs with independence number at most 4.

This chapter is organized as follows: In Section 2, we introduce some tools and give the proofs of Theorems 3.1.1 and 3.1.2. The proof of Theorem 3.1.3 is presented in Section 3.

3.2 Reductions on Mod $(2p + 1)$ -orientations

We first display the needed tools in our proofs of the main results. Lemma 3.2.1 is a brief summary of certain basic properties from [40, 43, 51].

Lemma 3.2.1. ([40], [43] and [51]) *Let G be a graph and let $m, p > 0$ be integers. Each of the following holds.*

- (i) *If $G \in \langle \mathcal{SZ}_{2p+1} \rangle$ and $e \in E(G)$, then $G/e \in \langle \mathcal{SZ}_{2p+1} \rangle$.*
- (ii) *If $H \subseteq G$, and if both $H \in \langle \mathcal{SZ}_{2p+1} \rangle$ and $G/H \in \langle \mathcal{SZ}_{2p+1} \rangle$, then $G \in \langle \mathcal{SZ}_{2p+1} \rangle$.*
- (iii) *Let mK_2 denote the loopless graph with two vertices and m parallel edges. Then $mK_2 \in \langle \mathcal{SZ}_{2p+1} \rangle$ if and only if $m \geq 2p$.*
- (iv) *The complete graph $K_n \in \langle \mathcal{SZ}_{2p+1} \rangle$ if and only if $n = 1$ or $n \geq 4p + 1$.*
- (v) *$G \in \mathcal{M}_{2p+1}$ if and only if its $\langle \mathcal{SZ}_{2p+1} \rangle$ -reduction $G' \in \mathcal{M}_{2p+1}$.*
- (vi) *$G \in \langle \mathcal{SZ}_{2p+1} \rangle$ if and only if its $\langle \mathcal{SZ}_{2p+1} \rangle$ -reduction $G' = K_1$.*

Let G be a graph and $b \in Z(G, \mathbb{Z}_{2p+1})$ be a boundary function. Define an integer valued mapping $\tau : 2^{V(G)} \mapsto \{0, \pm 1, \dots, \pm(2p + 1)\}$ as follows: for each vertex $x \in V(G)$,

$$\tau(x) \equiv \begin{cases} d(x) & (\text{mod } 2); \\ b(x) & (\text{mod } 2p + 1). \end{cases} \quad (3.2)$$

For a vertex set $A \subset V(G)$, let $b(A) \equiv \sum_{v \in A} b(v) \pmod{2p + 1}$, $d(A) = |[A, V(G) - A]_G|$ and define $\tau(A)$ to be

$$\tau(A) \equiv \begin{cases} d(A) & (\text{mod } 2); \\ b(A) & (\text{mod } 2p + 1). \end{cases} \quad (3.3)$$

Theorem 3.2.2. (Lovász, Thomassen, Wu and Zhang, Theorem 3.1 of [54]) *Let G be a graph and $b \in Z(G, \mathbb{Z}_{2p+1})$. Let z_0 be a vertex of $V(G)$, and let D_{z_0} be a pre-orientation of $E(z_0)$. Assume that*

- (i) $|V(G)| \geq 3$,
- (ii) $d(z_0) \leq 4p + |\tau(z_0)|$, and the edges incident with z_0 are pre-directed such that $d^+(z_0) - d^-(z_0) \equiv b(z_0) \pmod{2p + 1}$.
- (iii) $d(A) \geq 4p + |\tau(A)|$ for each nonempty $A \subseteq V(G) \setminus \{z_0\}$ with $|V(G) \setminus A| \geq 2$.

Then D_{z_0} can be extended to an orientation D of the entire graph G such that, for each vertex $x \in V(G)$,

$$d_D^+(x) - d_D^-(x) \equiv b(x) \pmod{2p + 1}.$$

Theorem 3.2.2 implies that every $6p$ -edge-connected graph is strongly \mathbb{Z}_{2p+1} -connected. We would further explore more properties concerning $\langle \mathcal{SZ}_{2p+1} \rangle$ -reduced graphs below by utilizing Theorem 3.2.2.

Proof of Theorem 3.1.1

Recall that $G \in \mathcal{F}(t)$ if and only if G is $\langle \mathcal{SZ}_{2p+1} \rangle$ -reduced with $2 \leq |V(G)| \leq 6pt - 2p$ and $\alpha(G) \leq t$. By Lemma 3.2.1(iii), every graph in $\mathcal{F}(t)$ has edge multiplicity at most $2p - 1$, and so $\mathcal{F}(t)$ contains finitely many graphs. Note that, by Lemma 3.2.1(v), Theorem 3.1.1(ii) follows from Theorem 3.1.1(i). We will show a variation of Theorem 3.1.1(i), as stated in Theorem 3.2.3 below.

Theorem 3.2.3. *For any graph G with $\alpha(G) \leq t$, G is strongly \mathbb{Z}_{2p+1} -connected if and only if the $\langle \mathcal{SZ}_{2p+1} \rangle$ -reduction of G is not in $\mathcal{F}(t)$.*

Proof. By Lemma 3.2.1(vi), a graph G is strongly \mathbb{Z}_{2p+1} -connected if and only if its $\langle \mathcal{SZ}_{2p+1} \rangle$ -reduction is K_1 , which is not in $\mathcal{F}(t)$ by definition. So it remains to show that

$$\text{if the } \langle \mathcal{SZ}_{2p+1} \rangle\text{-reduction of } G \text{ is not in } \mathcal{F}(t), \text{ then } G \in \langle \mathcal{SZ}_{2p+1} \rangle. \quad (3.4)$$

We shall prove (3.4) by induction on t . When $t = 1$, (3.4) follows from Lemma 3.2.1(iv). Assume that $t \geq 2$ and (3.4) holds for smaller values of t .

Let Γ be a counterexample to (3.4) such that $|V(\Gamma)|$ is minimal. Then Γ' , the $\langle \mathcal{SZ}_{2p+1} \rangle$ -reduction of Γ , satisfies $|V(\Gamma')| \geq 6pt - 2p + 1$ by the definition of $\mathcal{F}(t)$. Hence Γ' itself is a counterexample to (3.4), and so $|V(\Gamma')| = |V(\Gamma)|$ by the minimality of $|V(\Gamma)|$. Therefore, $\Gamma = \Gamma'$ is a $\langle \mathcal{SZ}_{2p+1} \rangle$ -reduced graph.

Claim A. $\delta(\Gamma) \geq 6p$.

Suppose that Γ has minimal degree at most $6p - 1$ and let $z \in V(\Gamma)$ be a vertex with $d_\Gamma(z) = \delta(\Gamma) \leq 6p - 1$. Denote $H = \Gamma - (N_\Gamma(z) \cup \{z\})$. Then $\alpha(H) \leq \alpha(\Gamma) - 1 \leq t - 1$. As H is $\langle \mathcal{SZ}_{2p+1} \rangle$ -reduced, we have $|V(H)| \leq 6p(t - 1) - 2p$ by (3.4) with induction hypothesis on $t - 1$. It follows that $6pt - 2p + 1 \leq |V(\Gamma)| = |V(H)| + |N_\Gamma(z) \cup \{z\}| \leq 6p(t - 1) - 2p + 6p = 6pt - 2p$. This contradiction justifies Claim A.

Now assume $\delta(\Gamma) \geq 6p$. By Theorem 3.2.2, $\kappa'(\Gamma) < 6p$, and so Γ must have an edge cut of size less than $6p$. For a vertex subset $W \subset V(\Gamma)$, let $W^c = V(\Gamma) - W$. Among all edge-cuts $[W, W^c]$ of size at most $6p - 1$ in Γ , choose one with $|W|$ minimized. As $\delta(\Gamma) \geq 6p$, we have $|W| \geq 2$. Let $G_1 = \Gamma / \Gamma[W^c]$ and z_0 be the vertex in G_1 onto which W^c is contracted. Thus $d_{G_1}(z_0) = |[W, W^c]| \leq 6p - 1$.

Arbitrarily add a set Z of $6p + 1 - d_{G_1}(z_0)$ new edges between z_0 and W in G_1 to form a new graph G . Note that $\Gamma[W] = G_1[W] = G[W] = G - z_0$. We will apply Theorem 3.2.2 to show the following Claim B, leading a contradiction to the fact that Γ is a $\langle \mathcal{SZ}_{2p+1} \rangle$ -reduced graph.

Claim B. $\Gamma[W] = G - z_0$ is strongly \mathbb{Z}_{2p+1} -connected.

Let D_{z_0} be a fixed orientation of $E_G(z_0)$ such that

$$4p + 1 \text{ edges are oriented out of } z_0 \text{ and the rest } 2p \text{ edges are oriented into } z_0. \quad (3.5)$$

We also use D_{z_0} to denote the digraph induced by the oriented edges of D_{z_0} . Define $b_1(v) = d_{D_{z_0}}^+(v) - d_{D_{z_0}}^-(v)$ for each vertex $v \in N_G(z_0) \cup \{z_0\}$.

For any $b' \in Z(G - z_0, \mathbb{Z}_{2p+1})$, we are to show that there exists an orientation D' of $G - z_0$ such that $d_{D'}^+(v) - d_{D'}^-(v) \equiv b'(v) \pmod{2p+1}$ for any vertex $v \in V(G - z_0)$. Define a mapping $b : V(G) \rightarrow \mathbb{Z}_{2p+1}$ as follows. For any $x \in V(G)$,

$$b(x) \equiv \begin{cases} b'(x) + b_1(x) \pmod{2p+1} & \text{if } x \in N_G(z_0); \\ b_1(z_0) \pmod{2p+1} & \text{if } x = z_0; \\ b'(x) \pmod{2p+1} & \text{otherwise.} \end{cases}$$

We are going to show Theorem 3.2.2 is applicable to this graph G .

As $b_1(z_0) + \sum_{v \in N_G(z_0)} b_1(v) = 0$ and $b' \in Z(G - z_0, \mathbb{Z}_{2p+1})$, we have $\sum_{x \in V(G)} b(x) = b_1(z_0) + \sum_{v \in N_G(z_0)} b_1(v) + \sum_{v \in V(G - z_0)} b'(v) \equiv 0 \pmod{2p+1}$, and so $b \in Z(G, \mathbb{Z}_{2p+1})$. Since $|W| \geq 2$, $|V(G)| \geq 3$. By (3.5), both $d(z_0) = 6p + 1$ and $b(z_0) = d_{D_{z_0}}^+(z_0) - d_{D_{z_0}}^-(z_0) \equiv 0 \pmod{2p+1}$. This, together with (3.2), implies $|\tau(z_0)| = 2p + 1$, and so Theorem 3.2.2 (i) and (ii) are satisfied.

By (3.3) and by the minimality of W , for any $A \subset W$ with $|A| < |W|$, we have $d(A) \geq 6p$, or $d(A) - 4p \geq 2p$. As $d(A) \equiv \tau(A) \pmod{2}$ and $|\tau(A)| \leq 2p + 1$, it follows by a parity argument that $d(A) \geq 4p + |\tau(A)|$. Thus Theorem 3.2.2 (iii) holds, and hence it holds also for the graph G .

By Theorem 3.2.2, there exists an orientation D of G such that $d_D^+(x) - d_D^-(x) \equiv b(x) \pmod{2p+1}$ for each vertex $x \in V(G)$. Let D' be the restriction of D on $G - z_0$. By the definition of b , we have $d_{D'}^+(v) - d_{D'}^-(v) \equiv b'(v) \pmod{2p+1}$ for each vertex $v \in V(G - z_0)$. It follows by definition that $\Gamma[W] = G - z_0$ is strongly \mathbb{Z}_{2p+1} -connected, and thus Claim B holds.

Since $|W| \geq 2$, Claim B is contrary to the assumption that Γ is $\langle \mathcal{S}\mathbb{Z}_{2p+1} \rangle$ -reduced. This proves Theorem 3.2.3. \square

Theorem 3.2.3 immediately leads the following corollary, which reveals that there are finitely many $\langle \mathcal{S}\mathbb{Z}_{2p+1} \rangle$ -reduced graph in the family of graphs with independence number at most t .

Corollary 3.2.4. *Every $\langle \mathcal{S}\mathbb{Z}_{2p+1} \rangle$ -reduced graph G with $\alpha(G) \leq t$ has order at most $6pt - 2p$.*

Proof of Theorem 3.1.2

We need one more lemma before presenting the proof of Theorem 3.1.2. For a graph G , let $\xi(G)$ be the number of nontrivial maximal $\langle \mathcal{S}\mathbb{Z}_{2p+1} \rangle$ -subgraphs of G .

Lemma 3.2.5. *If G is a simple graph with $\alpha(G) \leq 2$, then $\xi(G) \leq 2$. Furthermore, $\xi(G) = 2$ if and only if $V(G)$ consists of vertex sets of exactly two maximal $\langle \mathcal{SZ}_{2p+1} \rangle$ -subgraphs.*

Proof. Assume that $c = \xi(G) \geq 2$ and let H_1, H_2, \dots, H_c be the nontrivial maximal $\langle \mathcal{SZ}_{2p+1} \rangle$ -subgraphs of G . By Lemma 3.2.1(iv), every strongly \mathbb{Z}_{2p+1} -connected simple graph other than K_1 has order at least $4p + 1$, and so $|V(H_i)| \geq 4p + 1$ for each $1 \leq i \leq c$.

By contradiction, we assume that $c \geq 3$, and so there exists a vertex $v \in V(G) \setminus (V(H_1) \cup V(H_2))$. By Lemma 3.2.1(ii)(iii), both $|[v, V(H_1)]_G| \leq 2p - 1$ and $|[v, V(H_2)]_G| \leq 2p - 1$. Since $|V(H_1)| \geq 4p + 1$, there exists $u_1 \in V(H_1)$ such that $u_1v \notin E(G)$ and $|[u_1, V(H_2)]_G| = 0$. Similarly, there exists $u_2 \in V(H_2)$ such that $u_2v \notin E(G)$ and $|[u_2, V(H_1)]_G| = 0$. Then it follows that $\{u_1, u_2, v\}$ is an independent set of size 3, contradicting to $\alpha(G) \leq 2$. This proves that we must have $\xi(G) \leq 2$, and when $\xi(G) = 2$, $V(G) = V(H_1) \cup V(H_2)$. \square

Proof of Theorem 3.1.2 Since $\langle \mathcal{SZ}_{2p+1} \rangle \subseteq \mathcal{M}_{2p+1}$, we have $\mathcal{K}_1(2p+1) = \mathcal{K}_2(2p+1) \setminus \mathcal{M}_{2p+1}$ by (3.1). Thus by Lemma 3.2.1(v), Theorem 3.1.2(i) follows from Theorem 3.1.2(ii), and so it suffices to show Theorem 3.1.2(ii). Let G be a graph satisfying the hypotheses of Theorem 3.1.2, and let H_1, H_2, \dots, H_c denote the collection of all maximal strongly \mathbb{Z}_{2p+1} -connected subgraphs of G , where $|V(H_1)| \geq |V(H_2)| \geq \dots \geq |V(H_c)|$ and $c \geq 2$, and $G' = G/(H_1 \cup \dots \cup H_c)$ is the $\langle \mathcal{SZ}_{2p+1} \rangle$ -reduction of G .

Proof of (ii). We prove that if G is not strongly \mathbb{Z}_{2p+1} -connected, then G' is in $\mathcal{K}_2(2p+1)$.

If G is not connected, then as $\alpha(G) \leq 2$, G must be a disjoint union of two complete graphs, where the larger one has order at least $5p + 1$. By (3.1) and Lemma 3.2.1(iv), the $\langle \mathcal{SZ}_{2p+1} \rangle$ -reduction of G is a member in $\mathcal{K}_2(2p+1)$ with $s_1 = \dots = s_{n-1} = 0$. Hence we assume that G is connected and not strongly \mathbb{Z}_{2p+1} -connected. By Lemma 3.2.1(iv) and Corollary 3.2.4, $|V(H_1)| \geq 4p + 1$. By Lemma 3.2.5, either $|V(H_2)| > 1$ and $V(G) = V(H_1) \cup V(H_2)$ or $|V(H_2)| = 1$. If $V(G) = V(H_1) \cup V(H_2)$, let $m = |[V(H_2), V(H_1)]_G|$. If $m \geq 2p$, then as $G/(H_1 \cup H_2)$ is an $mK_2 \in \langle \mathcal{SZ}_{2p+1} \rangle$, it follows by Lemma 3.2.1(ii) that $G \in \langle \mathcal{SZ}_{2p+1} \rangle$, contrary to the assumption that G is not strongly \mathbb{Z}_{2p+1} -connected. Hence $m \leq 2p - 1$, and so $G' = mK_2 \in \mathcal{K}_2(2p+1)$.

Assume that $|V(H_2)| = 1$. Then H_1 is the only non-trivial maximal strongly \mathbb{Z}_{2p+1} -connected subgraphs of G . Let $V' = V(G) \setminus V(H_1)$. We claim that $G[V']$ is a complete graph. Suppose to the contrary that there exist vertices $v_1, v_2 \in V'$ such that $v_1v_2 \notin E(G[V'])$. By Lemma 3.2.1(ii)(iii), $|[v_1, V(H_1)]_G| \leq 2p - 1$ and $|[v_2, V(H_1)]_G| \leq 2p - 1$. Thus there exists $u \in V(H_1)$ such that $uv_1 \notin E(G)$ and $uv_2 \notin E(G)$ by $|V(H_1)| \geq 4p + 1$. It follows that $\{u, v_1, v_2\}$ is an independent set, contrary to the assumption of $\alpha(G) \leq 2$. Therefore, $G[V']$ is a complete graph. By Lemma 3.2.1(iv), we have $|V'| \leq 4p$. Thus the $\langle \mathcal{SZ}_{2p+1} \rangle$ -reduction of G is in $\mathcal{K}_2(2p+1)$. This proves (ii).

Proof of (iii). If $\kappa'(G) \geq 2p$ and $\delta(G) \geq 4p$, we show that the $\langle \mathcal{SZ}_{2p+1} \rangle$ -reduction G' is not in $\mathcal{K}_2(2p+1)$, and so $G \in \langle \mathcal{SZ}_{2p+1} \rangle$ follows from (ii). By Lemma 3.2.5, if G has two nontrivial maximal strongly \mathbb{Z}_{2p+1} -connected subgraphs H_1 and H_2 , then $V(G) = V(H_1) \cup V(H_2)$, and so $G/(H_1 \cup H_2)$ is a mK_2 , where $m = |[V(H_2), V(H_1)]_G|$. If $m \leq 2p-1$, then $G' = mK_2 \in \mathcal{K}_2(2p+1)$, contrary to the assumption that $\kappa'(G') \geq \kappa'(G) \geq 2p$. Thus $m \geq 2p$ and so by Lemma 3.2.1(ii) that $G \in \langle \mathcal{SZ}_{2p+1} \rangle$. Hence we assume that G does not have two nontrivial maximal strongly \mathbb{Z}_{2p+1} -connected subgraphs. By Corollary 3.2.4 and Lemma 3.2.5, G has exactly one nontrivial maximal strongly \mathbb{Z}_{2p+1} -connected subgraph H_1 . Moreover, $G - V(H_1)$ is a complete graph as showed above in the proof of (ii). Let u^* be the vertex in G' onto which H_1 is contracted. Since $\delta(G) \geq 4p$, for any vertex $v \in V(G' - u^*)$, we have $|[u^*, v]_{G'}| \geq 4p+1 - |V'|$, and so G' contains a spanning subgraph isomorphic to $K_{4p+1}/K_{4p+1-|V'|}$. By Lemma 3.2.1(i)(iv), $K_{4p+1}/K_{4p+1-|V'|} \in \langle \mathcal{SZ}_{2p+1} \rangle$, and so $G' \in \langle \mathcal{SZ}_{2p+1} \rangle$. This contradicts that G' is $\langle \mathcal{SZ}_{2p+1} \rangle$ -reduced, unless $|V(G')| = 1$. Therefore, $G \in \langle \mathcal{SZ}_{2p+1} \rangle$ by Lemma 3.2.1(vi). \square

3.3 On Mod 5-orientations

The odd-edge-connectivity of a graph is defined as the size of a smallest edge-cut of odd size. A $6p$ -edge-connected graph must be odd- $(6p+1)$ -edge-connected, but not vice versa. Tutte's 3-Flow Conjecture was originally proposed for odd-5-edge-connected graphs (see [6]). Lovász, Thomassen, Wu and Zhang [54] proved the following result for mod $(2p+1)$ -orientations concerning odd-edge-connectivity, which strengthens their theorem on modulo orientations.

Theorem 3.3.1. (Lovász et al. [54]) *Every odd- $(6p+1)$ -edge-connected graph admits a mod $(2p+1)$ -orientation.*

The main result of this section is Theorem 3.3.2 below. For the class of graphs with independence number at most 2, Theorem 3.3.2 improves Theorem 3.3.1 for $p = 2$ and verifies Conjecture 1.2.2 for those values.

Theorem 3.3.2. *Every odd-9-edge-connected graph G of order at least 21 and with $\alpha(G) \leq 2$ has a mod 5-orientation.*

We need a few more tools for the proof of Theorem 3.3.2.

Theorem 3.3.3. (Hakimi [28]) *Let G be a graph and $\ell : V(G) \mapsto \mathbb{Z}$ be a function such that $\sum_{v \in V(G)} \ell(v) = 0$ and $\ell(v) \equiv d_G(v) \pmod{2}$, $\forall v \in V(G)$. Then the following are equivalent.*

- (i) *G has an orientation D such that $d_D^+(v) - d_D^-(v) = \ell(v)$, $\forall v \in V(G)$.*
- (ii) *$|\sum_{v \in S} \ell(v)| \leq |\partial_G(S)|$, $\forall S \subset V(G)$.*

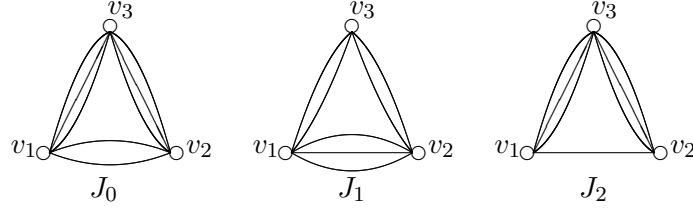


Figure 3.1: Graphs in Lemma 3.3.5, where $J_0 \in \langle SZ_5 \rangle$ and $J_1, J_2 \notin \langle SZ_5 \rangle$.

Let u_1v and u_2v be two distinct edges in G . We define $G_{[v, u_1u_2]}$ to be the graph obtained from G by deleting the edges u_1v, u_2v and adding a new edge u_1u_2 , which is called the *lifting operation* (see [54, 65]). The following lemma of Zhang [73] shows that the odd-edge-connectivity is preserved under certain lifting operation.

Lemma 3.3.4. (Zhang [73]) *Let G be a graph with odd edge-connectivity k . Assume there is a vertex $v \in V(G)$ with $d(v) \neq k$ and $d(v) \neq 2$. Then there exists a pair of edges u_1v, u_2v in $E(v)$ such that $G_{[v, u_1u_2]}$, the graph obtained from G by lifting u_1v, u_2v , remains odd edge-connectivity k .*

Lemma 3.3.5. *Let J_0, J_1 and J_2 be the graphs depicted in Figure 3.1. Each of the following holds.*

- (i) J_0 is strongly \mathbb{Z}_5 -connected.
- (ii) If G' is a $\langle SZ_5 \rangle$ -reduced graph on 3 vertices, then $|E(G')| \leq 7$, where $|E(G')| = 7$ if and only if G' is isomorphic to either J_1 or J_2 .

Proof of (i). Let $b \in Z(J_0, \mathbb{Z}_5)$. If $b(v_1) \neq 0$, lift two edges v_1v_2, v_1v_3 to obtain the graph $G_{[v_1, v_2v_3]}$. Since $|[v_1, \{v_2, v_3\}]_{G_{[v_1, v_2v_3]}}| = 3$ and $b(v_1) \neq 0$, we can modify the boundary $b(v_1)$ with the three edges in $[v_1, \{v_2, v_3\}]_{G_{[v_1, v_2v_3]}}$. Specifically, orient 1, 3, 0, 2 edges towards v_1 when $b(v_1) = 1, 2, 3, 4$, respectively. As $|[v_2, v_3]_{G_{[v_1, v_2v_3]}}| = 4$ and by Lemma 3.2.1(iii), we can also modify the boundaries $b(v_2), b(v_3)$ with those four edges. By symmetry, we assume $b(v_1) = b(v_2) = 0$, then $b(v_3) = 0$ since $b \in Z(J_0, \mathbb{Z}_5)$. Orient all the edges in $E(v_1)$ towards v_1 and orient all the edges in $E(v_2)$ from v_2 to obtain an orientation D of J_0 . Then D is a mod 5-orientation of G , which agrees with the boundary $b(v_1) = b(v_2) = b(v_3) = 0$. Therefore, (i) must hold.

Proof of (ii). Set $b_1(v_1) = b_1(v_2) = 3$ and $b_1(v_3) = 4$. Then $b_1 \in Z(J_1, \mathbb{Z}_5)$. It is routine to check that there is no orientation agreeing with the boundary b_1 in J_1 . Set $b_2(v_1) = b_2(v_2) = 4$ and $b_2(v_3) = 2$. Then $b_2 \in Z(J_2, \mathbb{Z}_5)$. It is easy to see that there is no orientation agreeing with the boundary b_2 in J_2 . Notice that J_1 and J_2 are the only two nonisomorphic graphs on 3 vertices

and 7 edges with edge multiplicity at most 3. Now, Lemma 3.3.5 follows by Lemma 3.2.1(ii) and the fact that $J_0 \in \langle \mathcal{SZ}_5 \rangle$, $J_1, J_2 \notin \langle \mathcal{SZ}_5 \rangle$. \square

Lemma 3.3.6. *Let G be an odd-9-edge-connected graph of order $n \geq 2$. If G contains a subgraph isomorphic to K_{n-1} , then G admits a mod 5-orientation.*

Proof. It is straightforward to verify the statement when $n = 2$ and $n \geq 10$ by Lemma 3.2.1(iii)(iv). Let G be a counterexample with $|V(G)| + |E(G)|$ minimized. The minimality of G implies that G is $\langle \mathcal{SZ}_5 \rangle$ -reduced. Let x be a vertex of G such that $G - x$ contains a subgraph isomorphic to K_{n-1} whose vertex set is denoted by $\{y_1, \dots, y_{n-1}\}$. We may further assume $|[x, y_i]_G| \geq |[x, y_{i+1}]_G|, \forall i \in [n-2]$. If G contains an even degree vertex, say v , then, by Lemma 3.3.4, there exist $\frac{d_G(v)}{2}$ pairs of edges incident with v such that lifting them results a graph, which contains a subgraph isomorphic to K_{n-2} , is still odd-9-edge-connected and has a mod 5-orientation, a contradiction. This implies every vertex has an odd degree, $\delta(G) \geq 9$ and n is even. Moreover, again by Lemma 3.3.4 and the minimality of $|V(G)| + |E(G)|$, we have $d_G(x) = 9$.

If $n = 4$, then $|E(G)| \geq 18$. Since $|[u, v]_G| \leq 3$ for any $u, v \in V(G)$ by Lemma 3.2.1(iii), we have $|E(G)| = 18$, and this, in addition, implies that G is isomorphic to $3K_4$. By Lemma 3.3.5, $3K_3 \in \langle \mathcal{SZ}_5 \rangle$, and so $G \cong 3K_4$ is not $\langle \mathcal{SZ}_5 \rangle$ -reduced, contrary to the assumption that G is $\langle \mathcal{SZ}_5 \rangle$ -reduced. Hence we assume that $n > 4$.

As every vertex of G has an odd degree, we must have $n \geq 6$. The following observations, stated as Claims 1 and 2, follow from Theorem 3.3.3 and Lemma 3.3.5.

Claim 1. *Let $\ell : V(G) \mapsto \{5, -5\}$ be a function such that $\sum_{v \in V(G)} \ell(v) = 0$. Then*

$$\text{there exists } S \subset V(G) \text{ such that } \left| \sum_{v \in S} \ell(v) \right| > |\partial_G(S)|. \quad (3.6)$$

In fact, if (3.6) fails, then by Theorem 3.3.3, G has a mod 5-orientation, contrary to the assumption that G is a counterexample. As $n \leq 9$, by the symmetry between S and $V(G) - S$, we may assume that there exists $S \subset V(G)$ satisfying (3.6) with $|S| \leq 4$ for any given ℓ .

Claim 2. *Let S be a vertex subset of G . Each of the following holds.*

- (i) $|\partial_G(S)| \geq \begin{cases} 9 & \text{if } |S| = 1, \\ 12 & \text{if } |S| = 2. \end{cases}$
- (ii) If $|S| = 3$, then $|\partial_G(S)| \geq 13$. Moreover, if $|\partial_G(S)| = 13$, then $d_G(s) = 9, \forall s \in S$, and $G[S] \in \{J_1, J_2\}$. (See Figure 3.1).
- (iii) If $n = 8$ and $|S| = 4$, then $|\partial_G(S)| \geq 12$ since G contains K_{n-1} .

When $n = 6$, denote $X = \{x, y_4, y_5\}$ and $Y = \{y_1, y_2, y_3\}$. As $d_G(x) = 9$, we have $|[x, y_5]_G| \leq 1$ and $|[x, y_4]_G| \leq 2$. These, together with $|[y_4, y_5]_G| \leq 3$, imply that

$$\begin{aligned} |[X, Y]_G| &= d_G(x) + d_G(y_4) + d_G(y_5) - 2(|[x, y_4]| + |[x, y_5]| + |[y_4, y_5]|) \\ &\geq 21 - 2(2 + 1 + 3) = 15. \end{aligned} \quad (3.7)$$

Set $\ell(x) = \ell(y_4) = \ell(y_5) = 5$ and $\ell(y_1) = \ell(y_2) = \ell(y_3) = -5$. We will obtain a contradiction by showing that ℓ violates Claim 1. Choose an $S \subset V(G)$ satisfying (3.6) with $|S|$ minimized. Then $|S| \leq 3$. By Claim 2(i), $|S| \neq 1, 2$, and so $|S| = 3$. Thus $|\sum_{v \in S} \ell(v)| \in \{5, 15\}$. By Claim 2, $|\sum_{v \in S} \ell(v)| = 15$ implying $S \in \{X, Y\}$, contrary to (3.7).

Therefore, we assume $n = 8$ in the following. Since $d_G(x) = 9$ and $|[x, y_i]_G| \geq |[x, y_{i+1}]_G|, \forall i \in [7]$, we have

$$|[x, y_7]_G| \leq |[x, y_6]_G| \leq |[x, y_5]_G| \leq 1, \quad (3.8)$$

and

$$|[x, \{y_5, y_6, y_7\}]| \leq |[x, \{y_4, y_6, y_7\}]| \leq 3. \quad (3.9)$$

Let $X_1 = \{x, y_5, y_6, y_7\}$, $Y_1 = \{y_1, y_2, y_3, y_4\}$, $X_2 = \{x, y_4, y_6, y_7\}$, and $Y_2 = \{y_1, y_2, y_3, y_5\}$. Define two functions ℓ_1 and ℓ_2 to be as follows.

$$\ell_1(v) = \begin{cases} 5, & \text{if } v \in X_1; \\ -5, & \text{if } v \in Y_1. \end{cases} \quad \text{and} \quad \ell_2(v) = \begin{cases} 5, & \text{if } v \in X_2; \\ -5, & \text{if } v \in Y_2. \end{cases}$$

We are to show that either ℓ_1 or ℓ_2 violates Claim 1, leading to a contradiction.

For $i = 1, 2$, choose $S_i \subset V(G)$ satisfying (3.6) with $|S_i|$ minimized. By Claim 2(i), we have $3 \leq |S_i| \leq 4$.

Claim 3. *If $|S_i| = 3$, then $|\partial_G(S_i)| = 13$ and $S_i = X_i \setminus \{x\}$.*

As $|S_i| = 3$, $|\sum_{v \in S_i} \ell_i(v)| \in \{5, 15\}$. By (3.6) and Claim 2(ii), we must have $15 = |\sum_{v \in S_i} \ell_i(v)| > |\partial_G(S_i)| = 13$. Thus $S_i \subset X_i$ or $S_i \subset Y_i$. Moreover, $G[S_i]$ is isomorphic to J_1 or J_2 as $|\partial_G(S_i)| = 13$ and by Claim 2(ii).

If $x \in S_i$, then by Claim 2(ii), $S_i \subset X_i$ and $|[x, S_i \setminus \{x\}]| \geq 4$ as $G[S_i]$ is isomorphic to J_1 or J_2 , contradicting to (3.8). If $S_i \subset Y_i$, then we have $13 = |\partial_G(S_i)| = |[x, S_i]_G| + |[S_i, V(G) \setminus (S_i \cup \{x\})]_G| \geq |[x, S_i]_G| + 12$. Thus $|[x, S_i]_G| \leq 1$, and so $|[x, \{y_4, y_5, y_6, y_7\}]_G| = 0$. Denote $\{y_t\} = Y \setminus S_i$. Then $|[x, y_t]_G| \geq 9 - |[x, S_i]_G| - |[x, \{y_4, y_5, y_6, y_7\}]_G| \geq 8$. So, by Lemma 3.2.1(iii), G is not $\langle S\mathbb{Z}_5 \rangle$ -reduced, a contradiction to the assumption on G . Therefore, we conclude that $S_i = X_i \setminus \{x\}$ if $|S_i| = 3$.

Claim 4. *If $|S_i| = 3$, then $|S_{3-i}| \notin \{3, 4\}$.*

Assume $|S_1| = |S_2| = 3$ first. We claim that there exists $s \in S_1 \cup S_2 = \{y_4, y_5, y_6, y_7\}$ such that $d_{G[S_1 \cup S_2]}(s) \geq 7$. If one of $G[S_1]$, $G[S_2]$ is isomorphic to J_2 , it is routine to verify that the vertex s corresponding to v_3 in J_2 has degree at least 7 in $G[S_1 \cup S_2]$. Otherwise, we have $G[S_1] \cong G[S_2] \cong J_1$ by Claim 2(ii), and so one of the vertices y_6, y_7 has degree at least 7 in $G[S_1 \cup S_2]$. Since $d_{G[S_1 \cup S_2]}(s) \geq 7$, it follows by $|[s, \{y_1, y_2, y_3\}]| \geq 3$ that $d_G(s) \geq 10$, contradicting to $d_G(s) = 9$ by Claim 2(ii).

We assume $|S_i| = 3$ and $|S_{3-i}| = 4$. By Claim 3, we have $y_{6-i} \in S_i \subset X_i$, and it follows by Claim 2(ii) and Claim 3 that

$$|[y_{6-i}, \{y_6, y_7\}]| \geq 4. \quad (3.10)$$

Since $|S_{3-i}| = 4$ and by Claim 2(iii), we have $20 > |\partial_G(S_{3-i})| = |[X_{3-i}, Y_{3-i}]|$ from (3.6). However, it follows from (3.9), (3.10) and $y_{6-i} \in X_i$ that

$$\begin{aligned} |[X_{3-i}, Y_{3-i}]_G| &= d_G(x) - |[x, \{y_{3+i}, y_6, y_7\}]_G| + |[\{y_{3+i}, y_6, y_7\}, Y_{3-i}]_G| \\ &\geq 9 - 3 + 10 + |[y_{6-i}, \{y_6, y_7\}]_G| \\ &\geq 20 = \left| \sum_{v \in S_{3-i}} \ell_{3-i}(v) \right|, \end{aligned}$$

a contradiction to (3.6). Hence Claim 4 holds.

The final step. By Claim 4, we may assume that $|S_1| = |S_2| = 4$. Thus, for $i \in \{1, 2\}$, $20 = |\sum_{v \in S_i} \ell_i(v)| > |\partial_G(S_i)| = |[X_i, Y_i]|$ by (3.6) and Claim 2(iii). Then $|\partial_G(S_i)| = |[X_i, Y_i]| \leq 18$, since $|X_i|$ is even. However, it follows from (3.8) and (3.9) that

$$\begin{aligned} 36 &\geq |[X_1, Y_1]_G| + |[X_2, Y_2]_G| \\ &= 2d_G(x) - |[x, \{y_4, y_6, y_7\}]_G| - |[x, \{y_5, y_6, y_7\}]_G| \\ &\quad + 2|[\{y_6, y_7\}, \{y_1, y_2, y_3\}]_G| + (d_G(y_4) - |[x, y_4]_G|) + (d_G(y_5) - |[x, y_5]_G|) \\ &\geq 18 - 3 - 3 + 12 + 6 + 8 = 38, \end{aligned}$$

a contradiction. The proof is completed. \square

Proof of Theorem 3.3.2. Let G be an odd-9-edge-connected graph with $\alpha(G) \leq 2$ and G' be the $\langle \mathcal{SZ}_5 \rangle$ -reduction of G . We shall show that $|V(G')| \leq 9$ and G' contains a subgraph isomorphic to $K_{|V(G')|-1}$. Then G' admits a mod 5-orientation by Lemma 3.3.6, and so Theorem 3.3.2 follows from Lemma 3.2.1(v).

Denote G_1 to be the underline simple graph of G . Since $|V(G_1)| \geq 21$, G_1 is not $\langle \mathcal{SZ}_5 \rangle$ -reduced by Corollary 3.2.4, and hence $\xi(G_1) \neq 0$. By Lemma 3.2.5, we have $1 \leq \xi(G_1) \leq 2$.

If $\xi(G_1) = 2$, again by Lemma 3.2.5, G'_1 , the $\langle \mathcal{SZ}_5 \rangle$ -reduction of G_1 , is a graph with at most two vertices, so does G' . Notice that $|V(G')| \leq |V(G'_1)|$. Assume $\xi(G_1) = 1$ and let H_1 be the corresponding nontrivial maximal $\langle \mathcal{SZ}_5 \rangle$ -subgraphs of G_1 . Clearly, $|V(H_1)| \geq 9$ by Lemma 3.2.1(iv). Let H be a nontrivial maximal $\langle \mathcal{SZ}_5 \rangle$ -subgraphs of G with $|V(H)|$ maximized. As $G[V(H_1)] \in \langle \mathcal{SZ}_5 \rangle$, we have $|V(H)| \geq |V(H_1)| \geq 9$. We claim that $\alpha(G - V(H)) = 1$. In fact, suppose that u, v are two non-adjacent vertices in $G - V(H)$. Then, by Lemma 3.2.1(ii)(iii), we have $|[u, V(H)]| \leq 3$ and $|[v, V(H)]| \leq 3$. Since $|V(H)| \geq 9$, there exists $w \in V(H)$ such that $\{w, u, v\}$ forms an independent set of size 3, a contradiction to $\alpha(G) \leq 2$. Hence $\alpha(G - V(H)) = 1$. Now, by Lemma 3.2.1(iv), the $\langle \mathcal{SZ}_5 \rangle$ -reduction of $G - V(H)$ has size at most 8 and independence number 1. Hence G' has order at most 9 and contains a subgraph isomorphic to $K_{|V(G')|-1}$. Therefore, Theorem 3.3.2 follows from Lemma 3.2.1(v) and Lemma 3.3.6. \square

Note that Theorem 3.1.3 follows from Theorems 3.3.2 and 3.1.1.

Chapter 4

Neighbor sum distinguishing total coloring

4.1 Introduction

For a vertex v of a graph G , we use $d(v) = d_G(v)$ and $N(v) = N_G(v)$ to denote the degree and the neighbors, respectively, of v in G . Let $V_{\leq \ell} = \{v \in V(G) : d(v) \leq \ell\}$. For $V_1, V_2 \subseteq V(G)$, let $E(V_1, V_2) = \{v_1v_2 \in E(G) : v_1 \in V_1, v_2 \in V_2\}$. For a total coloring ϕ of G , let $C_\phi(x)$ denote the set of colors of edges incident with x under coloring ϕ , and $C_\phi[x] = C_\phi(x) \cup \{\phi(x)\}$.

A tree decomposition (T, \mathcal{V}) of a graph G consists of a tree T and a collection $\mathcal{V} = \{V_t : t \in V(T)\}$ of bags $V_t \subseteq V(G)$ such that

- (1) $V(G) = \cup_{t \in V(T)} V_t$;
- (2) for each $vw \in E(G)$, there exists a $t \in V(T)$ such that V_t contains both v and w ;
- (3) for each $v \in V(G)$, the subgraph induced by $\{t \in V(T) \mid v \in V_t\}$ is a subtree of T .

The width of a tree decomposition (T, \mathcal{V}) is $\max_{t \in V(T)} |V_t| - 1$. The treewidth $\text{tw}(G)$ of G is the minimum width over all tree decompositions of G .

Given a tree decomposition (T, \mathcal{V}) of G , where T is rooted in some vertex $r \in V(T)$, we define the height $h(t)$ of a vertex $t \in V(T)$ to be the distance from r to t . For $v \in V(G)$, we define t_v as the unique vertex of minimum height in T for $v \in V_{t_v}$. In particular, if $v \in V_r$, then $t_v = r$. First, we display the following useful structure of graphs with treewidth ℓ and $\Delta(G) \geq \ell + 1$.

Lemma 4.1.1. *Let G be a graph with treewidth ℓ and $\Delta(G) \geq \ell + 1$. Then there are two non-empty disjoint subsets $W, U \subseteq V(G)$ and a vertex $x \notin W \cup U$ satisfying the following (see Figure 4.1):*

- (1) $N(W) \subseteq U \cup \{x\} \cup V_{\leq \ell}$;

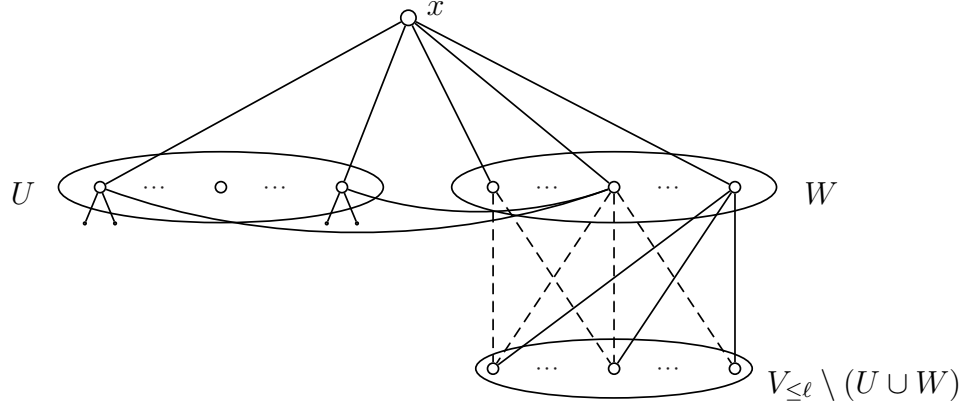


Figure 4.1: Structure of the graph G in Lemma 4.1.1

- (2) $d(x) \geq \ell + 1$ and $d(w) \leq \ell$ for each $w \in W$;
- (3) $W \subseteq N(x) \subseteq W \cup U$;
- (4) $|U| \leq \ell$.

Proof. Let $X = \{v \in V(G) : d(v) \geq \ell + 1\}$. Since $\Delta(G) \geq \ell + 1$, $X \neq \emptyset$.

Fix a width ℓ tree decomposition (T, \mathcal{V}) of G and root the associated tree T in an arbitrary vertex $r \in V(T)$. Note that we may choose a tree decomposition (T, \mathcal{V}) such that each bag V_t contains at least two vertices. Let $x \in X$ with $h(t_x) = \max_{v \in X} h(t_v)$. Define T' as the subtree of T rooted at t_x , the subgraph of T induced by all vertices $t \in V(T)$ where the path from t to the root r contains t_x .

Define $U = V_{t_x} \setminus \{x\}$ and $W = N(x) \setminus U$. Then $W \subseteq N(x) \subseteq W \cup U$ and by the definitions of treewidth and U , we have $|U| \leq \ell$. As $|N(x)| \geq \ell + 1$ and $1 \leq |U| \leq \ell$, we have that U and W are non-empty sets, and so (3) and (4) follow.

Let $Y = \cup_{t \in V(T')} V_t$. By the definition of t_x , x does not appear in any bag V_t of a vertex $t \in T \setminus T'$. Thus $N(x) \subseteq Y$. Furthermore, we have $X \cap Y \subseteq U \cup \{x\}$; otherwise, if there is a vertex $v \in (X \cap Y) \setminus (U \cup \{x\})$, then $h(t_v) > h(t_x)$, a contradiction to the choice of x .

Since $X \cap Y \subseteq U \cup \{x\}$ and $N(x) \subseteq Y$, we have $W = N(x) \setminus U \subseteq Y \setminus U \subseteq Y \cap V_{\leq \ell}$, which implies that $d(w) \leq \ell$ for each $w \in W$. This proves (2).

Finally, we will show (1): $N(W) \subseteq U \cup \{x\} \cup V_{\leq \ell}$. Let $v \in N(w) \setminus (U \cup \{x\})$ for a vertex $w \in W$. If $d(v) \geq \ell + 1$, then there is a bag V_t of a vertex $t \in T \setminus T'$ that contains both v and w , and thus $w \in V_{t_x} = U \cup \{x\}$, a contradiction. Therefore $d(v) \leq \ell$ and $N(W) \subseteq U \cup \{x\} \cup V_{\leq \ell}$.

We study the NSD total coloring of graphs with bounded treewidth and present an upper bound on the NSD total chromatic number.

Theorem 4.1.2. *Let $\ell \geq 3$ be an integer and G be a graph with treewidth at most ℓ . Then*

$$\chi_{\Sigma}^t(G) \leq \max\{\Delta + \ell - 1, 3\ell + 2\}.$$

Furthermore, we prove the following stronger result for graphs with treewidth 3.

Theorem 4.1.3. *Let G be a graph with treewidth at most 3. Then $\chi_{\Sigma}^t(G) \leq \max\{\Delta + 1, 10\}$ if G contains no two adjacent Δ -vertices.*

4.2 Preliminaries

Let $t \geq 2$ be an integer and S_1, S_2, \dots, S_t be t finite sets of integers. Define

$$\sum_{i=1}^t S_i = S_1 + S_2 + \dots + S_t = \{a_1 + a_2 + \dots + a_t : a_i \in S_i, a_i \neq a_j, \text{ for all } i \neq j\}.$$

The following theorem is a corollary of Combinatorial Nullstellensatz.

Theorem 4.2.1. (Alon [1]) *Let $t \geq 2$ be an integer and S_1, S_2, \dots, S_t be t finite sets of integers, where $|S_i| = s_i$ and $s_1 \geq s_2 \geq \dots \geq s_t$. Define s'_1, s'_2, \dots, s'_t by*

$$s'_1 = s_1, \text{ and } s'_i = \min\{s'_{i-1} - 1, s_i\}, \text{ for } 2 \leq i \leq t.$$

If $s'_t > 0$, then

$$|S_1 + S_2 + \dots + S_t| \geq \sum_{i=1}^t s'_i - \frac{1}{2}t(t+1) + 1.$$

By the above theorem, we have the following corollaries.

Corollary 4.2.2. *If $|S_i| \geq p \geq t$ for each $i = 1, \dots, t$, then*

$$|S_1 + S_2 + \dots + S_t| \geq (p - t)t + 1.$$

Corollary 4.2.3. *If $|S_i| \geq t - 1$ for each $i \in [t]$ and $|S_p| \geq t + 1$, $|S_q| \geq t$ for two distinct integers $p, q \in [t]$, then*

$$|S_1 + S_2 + \dots + S_t| \geq t + 1.$$

For a subset $A \subseteq V(G)$, a mapping $\phi: (V(G) \setminus A) \cup E(G) \rightarrow [k]$ is called an A -partial NSD total k -coloring of G if it is a proper total k -coloring of G except that only the vertices in A are not colored and $m_{\phi}(u) \neq m_{\phi}(v)$ for each edge $uv \in E(G)$ with $\{u, v\} \subseteq V(G) \setminus A$. Note that in an A -partial NSD total k -coloring, only the two adjacent vertices not in A are sum distinguished.

Lemma 4.2.4. *Let ℓ and k be two positive integers with $k \geq 3\ell + 1$. If $B \subseteq V_{\leq \ell}$, then each B -partial NSD total k -coloring of G can be extended to an NSD total k -coloring of G .*

Proof. It suffices to consider the case when $B = \{v\}$. Let ϕ be a B -partial NSD total k -coloring of G , and let

$$S(v) = \{\phi(u), \phi(uv), m_\phi(u) - m_\phi(v) : u \in N_G(v)\}.$$

Since $d(v) \leq \ell$, we have $|S(v)| \leq 3\ell$. Since $k \geq 3\ell + 1$, there is a color available for v and ϕ can be extended to an NSD total k -coloring of G .

4.3 Proofs of Theorems 4.1.2 and 4.1.3

In this section, we first prove some structural properties of smallest counterexamples to both Theorems 4.1.2 and 4.1.3 and then complete their proofs in separate subsections.

Suppose to the contrary that G is a counterexample to Theorem 4.1.2 or Theorem 4.1.3 with $|E(G)|$ minimum. Let $k = \max\{\Delta(G) + \ell - 1, 3\ell + 2\}$ or $k = \max\{\Delta(G) + 1, 10\}$, respectively. Then $k \geq 3\ell + 1$. In order to obtain a contradiction, by Lemma 4.2.4, it is sufficient to show that G has a B -partial NSD total k -coloring for some $B \subseteq V_{\leq \ell}$.

For any proper subgraph H of G , $tw(H) \leq tw(G)$ (see [5]), so, by minimality of G , H has an NSD total k -coloring ψ . Thus G is connected and for any $B \subseteq V(H) \cap V_{\leq \ell}$, we obtain a B -partial NSD total k -coloring of H from ψ by uncoloring all vertices in B .

Claim 1. *G does not contain an edge uv with $d(u) \leq \ell$ and $d(v) \leq \ell + 1$. Thus $V_{\leq \ell}$ is independent.*

Proof. Suppose to the contrary that there is an edge uv such that $d(u) \leq \ell$ and $d(v) \leq \ell + 1$. Let $B = V_{\leq \ell} \setminus \{v\}$ and ϕ be a B -partial NSD total coloring of $G - \{uv\}$. Let

$$S(u, v) = \{\phi(ux), \phi(vy), \phi(v), m_\phi(y) - m_\phi(v) : x \in N_G(u) \setminus \{v\}, y \in N_G(v) \setminus \{u\}\}.$$

Then $|S(u, v)| \leq d_G(u) - 1 + 2(d_G(v) - 1) + 1 \leq 3\ell < k$. Thus there is a color available for the edge uv and we can extend ϕ to a B -partial NSD total k -coloring of G , a contradiction.

By Lemma 4.1.1, there are two non-empty disjoint subsets $W, U \subseteq V(G)$ and a dedicated vertex $x \notin W \cup U$ satisfying (1)–(4) of Lemma 4.1.1 (see Figure 4.1). Furthermore, $N(W) \subseteq U \cup \{x\}$ by (1) of Lemma 4.1.1 and Claim 1. Thus by (3) and (4) of Lemma 4.1.1 and Claim 1, $\ell + 2 \leq d(x) \leq |W| + |U| \leq |W| + \ell$, so $|W| \geq d(x) - \ell \geq 2$.

Let W' be a nonempty subset of W . Let ϕ be a proper total k -coloring of $G' = G - \{xw : w \in W'\}$ except that the vertices in W are not colored. We call ϕ a *W -partial almost NSD total k -coloring of G'* if $m_\phi(u) \neq m_\phi(v)$ for any $uv \in E(G')$ and $\{u, v\} \subset V(G) \setminus (W \cup \{x\})$. Note that a W -partial NSD total k -coloring of G' is also a W -partial almost NSD total k -coloring of G' .

Denote $d = d(x)$. For each $w \in W'$, let

$$S_{w, \phi} = [k] \setminus (C_\phi[x] \cup C_\phi(w)).$$

Then for each $w \in W'$,

$$\begin{aligned}
|S_{w,\phi}| &= k - |C_\phi[x]| - |C_\phi(w)| + |C_\phi[x] \cap C_\phi(w)| \\
&= k - (d - |W'| + 1) - (d(w) - 1) + |C_\phi[x] \cap C_\phi(w)| \\
&= k - \ell - d + |W'| + (\ell - d(w)) + |C_\phi[x] \cap C_\phi(w)|.
\end{aligned} \tag{4.1}$$

Claim 2. For each $W' \subseteq W$ and each W -partial almost NSD total k -coloring ϕ of $G - \{xw : w \in W'\}$, we have

$$|\sum_{w \in W'} S_{w,\phi}| \leq \ell.$$

Proof. Suppose to the contrary that $|\sum_{w \in W'} S_{w,\phi}| \geq \ell + 1$ for some $W' \subseteq W$ and some W -partial almost NSD total k -coloring ϕ of $G - \{xw : w \in W'\}$. Let $A = \{m_\phi(u) - m_\phi(x) : u \in N(x) \cap U\}$. By (4) of Lemma 4.1.1, $|A| \leq |U| \leq \ell$. Since $|\sum_{w \in W'} S_{w,\phi}| \geq \ell + 1$, $\sum_{w \in W'} S_{w,\phi} \setminus A \neq \emptyset$. Thus we can pick one color $\alpha_w \in S_{w,\phi}$ for each edge xw such that $\alpha_w \neq \alpha_u$ for any two distinct vertices w, u in W' and $\sum_{w \in W'} \alpha_w \notin A$. So we can extend ϕ to a W -partial NSD total k -coloring of G , a contradiction.

Claim 3. $d(x) \geq k - \ell$.

Proof. Suppose to the contrary that $d = d(x) \leq k - \ell - 1$.

Denote $t = |W|$. Then $t \geq 2$. Let ϕ be a W -partial almost NSD total k -coloring of $G - \{xw : w \in W\}$. Since $d(w) \leq \ell$ for each $w \in W$, by Eq.(4.1), $|S_{w,\phi}| \geq k - d - \ell + t \geq t + 1$. By Corollary 4.2.2, we have that

$$|\sum_{w \in W} S_{w,\phi}| \geq (k - d - \ell + t - t)t + 1 = (k - d - \ell)t + 1.$$

By Claim 2, $(k - d - \ell)t + 1 \leq \ell$. Thus $t \leq \ell - 1$ and $d \leq t + \ell \leq 2\ell - 1$, implying $k - d - \ell \geq k - 3\ell + 1 \geq 2$. On the other hand,

$$d \geq k - \ell - \frac{\ell - 1}{t} \geq (2\ell + 1) - \frac{\ell - 1}{2} = \frac{3\ell + 3}{2}.$$

Thus $t \geq d - \ell \geq \frac{\ell + 3}{2}$. Therefore, $(k - d - \ell)t + 1 \geq 2t + 1 \geq \ell + 4$, a contradiction to Claim 2. This completes the proof of the claim.

Claim 4. Let $\epsilon = 1$ or 2 . If $|W| \geq \ell + \epsilon$, then there is a triple (w_1, W', ϕ) satisfying

- (1) $w_1 \in W' \subseteq W$ and $d(w_1)$ is the maximum among all vertices in W ;
- (2) $|W'| \geq \ell$;
- (3) ϕ is a W -partial almost NSD total k -coloring of $G - \{xw : w \in W'\}$ such that
 - (i) $\phi(x) \notin C_\phi(w_1)$ and $\{\phi(xw) \mid w \in W \setminus W'\} \subseteq C_\phi(w_1)$;
 - (ii) $(\ell - d(w_1)) + |C_\phi(w_1) \cap C_\phi[x]| \geq \epsilon$.

Proof. Denote $t = |W|$. Let $w_1 \in W$ such that $d(w_1)$ is the maximum among all vertices in W . Let ψ be a W -partial almost NSD total k -coloring of $G - \{xw : w \in W\}$. We can choose ψ such that $\psi(x) \notin C_\psi(w_1)$. Otherwise, we can recolor the vertex x with a color not in $C_\psi(x) \cup C_\psi(w_1) \cup \{\psi(u) : u \in U\}$ since $|C_\psi(x) \cup C_\psi(w_1) \cup \{\psi(u) : u \in U\}| \leq d(w_1) - 1 + 2|U| \leq 3\ell - 1$.

The claim is trivial if $(\ell - d(w_1)) + |C_\psi(w_1) \cap C_\psi[x]| \geq \epsilon$ by simply taking the triple (w_1, W, ψ) .

Now we assume that $(\ell - d(w_1)) + |C_\psi(w_1) \cap C_\psi[x]| \leq \epsilon - 1$. We observe that there are at most $|U| \leq \ell$ edges in $E(W, U)$ colored with a same color since $N(W) \subseteq \{x\} \cup U$.

If $(\ell - d(w_1)) + |C_\psi(w_1) \cap C_\psi[x]| = \epsilon - 1$, then let $a \in C_\psi(w_1) \setminus C_\psi[x]$. Since $|W| \geq \ell + \epsilon \geq \ell + 1$, there must be a vertex, say w_t , in W such that $a \notin C_\psi(w_t)$. Then let $W' = W \setminus \{w_t\}$ and ϕ be the coloring obtained from ψ by coloring the edge xw_t with a . Since $|C_\phi(w_1) \cap C_\phi[x]| = |C_\psi(w_1) \cap C_\psi[x]| + 1$, it is easy to see that (w_1, W', ϕ) is the desired triple.

If $(\ell - d(w_1)) + |C_\psi(w_1) \cap C_\psi[x]| < \epsilon - 1$, then $\epsilon = 2$, $d(w_1) = \ell$, and $|C_\psi(w_1) \cap C_\psi[x]| = 0$. Thus $C_\psi(w_1) \setminus C_\psi[x] = C_\psi(w_1)$. Since $|C_\psi(w_1)| = d(w_1) - 1 \geq 2$, let $a, b \in C_\psi(w_1) \setminus C_\psi[x]$ be two different colors. Since $|W| \geq \ell + 2$, there must be two distinct vertices, say w_{t-1}, w_t in $W \setminus \{w_1\}$ such that $a \notin C_\psi(w_t)$ and $b \notin C_\psi(w_{t-1})$. Let $W' = W \setminus \{w_{t-1}, w_t\}$ and ϕ be the coloring obtained from ψ by coloring the edge xw_t, xw_{t-1} with a and b , respectively. It is easy to check that (w_1, W', ϕ) is a triple satisfying (1), (2) and (3).

Claim 5. *If $d(x) \leq k - \ell + 1$, then $|W| \leq \ell + 1$.*

Proof. Suppose to the contrary that $|W| \geq \ell + 2$. We first prove the following fact. No triple (w_1, W', ϕ) satisfies

- (a) all requirements in Claim 4 with $\epsilon = 2$ except that $\phi(x)$ may or may not belong to $C_\phi(w_1)$;
- (b) there is a vertex $w_2 \in W' \setminus \{w_1\}$ such that $(\ell - d(w_2)) + |C_\phi(w_2) \cap C_\phi[x]| \geq 1$.

Suppose to the contrary that the fact is false. Then $p = |W'| \geq \ell$. Since $d = d(x) \leq k - \ell + 1$, by Eq. (4.1), we have the following:

- $|S_{w, \phi}| \geq k - \ell - d + |W'| + (\ell - d(w)) + |C_\phi[x] \cap C_\phi(w)| \geq p - 1$ for each $w \in W'$,
- $|S_{w_1, \phi}| \geq -1 + p + 2 = p + 1$,
- $|S_{w_2, \phi}| \geq -1 + p + 1 = p$.

Thus by Corollary 4.2.3, $|\sum_{w \in W'} S_{w, \phi}| \geq p + 1 \geq \ell + 1$, a contradiction to Claim 2.

Let (w_1, W', ϕ) be a triple satisfying Claim 4 with $\epsilon = 2$. By above Fact, we have $d(w) = \ell$ (and thus $d(w_1) = \ell$ by (1) of Claim 4) and $C_\phi[x] \cap C_\phi(w) = \emptyset$ for each $w \in W' \setminus \{w_1\}$. Let $A = \cup_{w \in W' \setminus \{w_1\}} C_\phi(w)$ and $B = \{\phi(u) \mid u \in N(x) \cap U\}$.

If $A \setminus B \neq \emptyset$, then we can pick a color $a \in A \setminus B$. Since $C_\phi[x] \cap C_\phi(w) = \emptyset$ for each $w \in W' \setminus \{w_1\}$, we have $a \notin C_\phi[x]$. Recolor the vertex x with a to obtain a new W -partial

almost NSD total k -coloring ϕ_1 . Since $\phi(x) \notin C_\phi(w_1)$ by (3)(i) of Claim 4 with $\epsilon = 2$, it is easy to check that $|C_{\phi_1}(w_1) \cap C_{\phi_1}[x]| \geq |C_\phi(w_1) \cap C_\phi[x]|$ and there exists $w_2 \in W' \setminus \{w_1\}$ such that $|C_{\phi_1}(w_2) \cap C_{\phi_1}[x]| \geq 1$. This implies that (w_1, W', ϕ_1) is a triple satisfying the requirements (a) and (b) in above Fact, a contradiction.

Thus $A \setminus B = \emptyset$ and $A \subseteq B$. Note that $|W' \setminus \{w_1\}| \geq \ell - 1 \geq 2$ by (2) of Claim 4. Let $w_2, w_3 \in W' \setminus \{w_1\}$. Then $d(w_2) = d(w_3) = \ell$, and so

$$\begin{aligned} |N(x) \cap U| &\geq |B| \geq |A| \geq |C_\phi(w_2)| \geq \ell - 1, \text{ and} \\ |C_\phi(w_2) \cap C_\phi(w_3)| &= |C_\phi(w_2)| + |C_\phi(w_3)| - |C_\phi(w_2) \cup C_\phi(w_3)| \\ &\geq 2(\ell - 1) - |A| \geq 2\ell - 2 - |B| \geq 2\ell - 2 - |U| \geq \ell - 2 \geq 1. \end{aligned}$$

Let $a \in C_\phi(w_2) \cap C_\phi(w_3)$ with $\phi(w_2u_2) = \phi(w_3u_3) = a$, where $u_2, u_3 \in U$. Since $|N(x) \cap U| \geq \ell - 1$ and $|U| \leq \ell$, at least one of u_2 and u_3 , say u_3 , is in $N(x) \cap U$. Let $\phi(xu_3) = b$. Then $b \notin C_\phi(w_3)$. Swap colors of xu_3 and w_3u_3 to obtain a new W -partial almost NSD total k -coloring ϕ_1 . Clearly, $|C_{\phi_1}(w_2) \cap C_{\phi_1}[x]| \geq 1$. By above Fact, $|C_{\phi_1}(w_1) \cap C_{\phi_1}[x]| \leq 1$, which implies that $a \notin C_\phi(w_1)$, $b \in C_\phi(w_1)$, and

$$|W \setminus W'| = |\{\phi(xw) \mid w \in W \setminus W'\}| \leq |C_{\phi_1}(w_1) \cap C_{\phi_1}[x]| \leq 1.$$

Since $|W| \geq \ell + 2$, we have $|W'| \geq \ell + 1$.

If there is a vertex $w_4 \in W' \setminus \{w_1, w_2, w_3\}$ such that $b \notin C_\phi(w_4)$, then color xw_4 with b to obtain a new W -partial almost NSD total k -coloring ϕ_2 from ϕ_1 . It is not difficult to check that $(w_1, W' \setminus \{w_4\}, \phi_2)$ is a triple satisfying the requirements in above Fact, a contradiction.

Thus $b \in C_\phi(w)$ for each $w \in W' \setminus \{w_1, w_2, w_3\}$. Note that $b \in C_\phi(w_1)$ and $\phi(xu_3) = b$. There are at least $|W'| - 2 + 1 \geq \ell$ edges in $E(W' \cup \{x\}, U)$ colored with b , meaning that there are at least ℓ vertices in U adjacent to an edge colored with b . Since $b \in A \subseteq B$, there is a vertex $u_4 \in N(x) \cap U$ such that $\phi(u_4) = b$, so u_4 is not incident with an edge colored with b . Therefore $|U| \geq \ell + 1$, a contradiction to the fact that $|U| \leq \ell$. The proof of the claim is completed.

Proof of Theorem 4.1.2 In this subsection, let G be a counterexample to Theorem 4.1.2 with $|E(G)|$ minimum and $k = \max\{\Delta + \ell - 1, 3\ell + 2\}$. By Claim 3, we have $d \geq k - \ell$. Thus $|W| \geq k - \ell - \ell \geq \ell + 2$. By Claim 5, we have $d > k - \ell + 1 \geq \Delta$. This contradiction completes the proof of Theorem 4.1.2.

Proof of Theorem 4.1.3 In this subsection, let G be a counterexample to Theorem 4.1.3 with $|E(G)|$ minimum and $k = \max\{\Delta + 1, 10\}$. Note $\ell = 3$, $k - \ell \geq \Delta - 2$ and $d = d(x)$.

By Claim 3, we have $d \geq k - \ell$. If $d = k - \ell + 1$, then $|W| \geq k - 2 - 3 \geq 5 = \ell + 2$, a contradiction to Claim 5. Thus $d \neq k - \ell + 1$. Since $k - \ell \geq \Delta - 2$, we have $d = k - \ell + 2$ or $d = k - \ell$.

We first assume $d = k - \ell + 2$. Then $d \geq \Delta$ and $d \geq 10 - 3 + 2 = 9$. Thus $d = \Delta \geq 9$,

$|W| \geq d - \ell \geq 6$ and $k = \Delta + 1$. Let $w \in W$ and choose a W -partial almost NSD total k -coloring ϕ of $G - \{xw\}$ such that $|C_\phi[x] \cap C_\phi(w)|$ is as large as possible.

We first show $C_\phi(w) \subseteq C_\phi[x]$. Suppose to the contrary $C_\phi(w) \not\subseteq C_\phi[x]$. Let $a \in C_\phi(w) \setminus C_\phi[x]$. Recall that there are at most $|U| \leq \ell$ edges in $E(W, U)$ colored with a same color since $N(W) \subseteq \{x\} \cup U$. Since $|U| \leq 3$ and $|W| \geq 6$, there is a vertex $w' \in W$ such that $a \notin C_\phi(w')$ and $\phi(xw') \notin C_\phi(w)$. We recolor the edge xw' with a to obtain a new W -partial almost NSD total k -coloring ϕ_1 of $G - \{xw\}$ satisfying $C_{\phi_1}(w) = C_\phi(w)$ and $|C_{\phi_1}[x] \cap C_{\phi_1}(w)| = |C_\phi[x] \cap C_\phi(w)| + 1$, a contradiction to the choice of ϕ . Thus $C_\phi(w) \subseteq C_\phi[x]$, so $|C_\phi[x] \cap C_\phi(w)| = d(w) - 1$.

By Eq. (4.1), we have

$$|S_{w,\phi}| \geq 2 - d(w) + |C_\phi[x] \cap C_\phi(w)| = 1.$$

We color the edge xw with a color in $S_{w,\phi}$ to obtain a new coloring ϕ_2 . Since $d = \Delta$ and no two Δ -vertices are adjacent, the degree of each vertex in U is at most $\Delta - 1$. Since $k = \Delta + 1$, $C_{\phi_2}[x] = [k]$. Thus $m_{\phi_2}(x) > m_{\phi_2}(u)$ for each $u \in U$. Therefore ϕ_2 is a $\{w\}$ -partial NSD total k -coloring of G , a contradiction.

Now we assume $d = k - \ell$. Then $k - \ell - d = 0$ and $|W| \geq d - \ell = k - \ell - \ell \geq 4$.

Let (w_1, W', ϕ) be a triple described in Claim 4 with $\epsilon = 1$. Denote $p = |W'|$. Then $p \geq 3 = \ell$ and $\ell - d(w_1) + |C_\phi(w_1) \cap C_\phi[x]| \geq 1$. Since $k - \ell - d = 0$, by Eq. (1),

$$|S_{w_1,\phi}| \geq |W'| + 1 = p + 1$$

and for each $w \in W' \setminus \{w_1\}$,

$$|S_{w,\phi}| \geq |W'| = p.$$

Therefore by Corollary 4.2.3, $|\sum_{w \in W'} S_{w,\phi}| \geq p + 1 \geq \ell + 1$, a contradiction to Claim 2. This contradiction completes the proof of Theorem 4.1.3.

Chapter 5

Star edge coloring

5.1 Introduction

Recall that a star edge coloring of a graph G is a proper edge coloring such that every connected bicolored subgraph is a path of length at most 3 (the length of a path is the number of edges). The star chromatic index of G , denoted by $\chi'_{st}(G)$, is the smallest integer k such that G is star k -edge-colorable. A natural generalization of star edge coloring is the list star edge coloring and it was pointed out in [16]: It would be interesting to understand the list version of star edge-coloring.

For a given list assignment L which assigns to each edge e a finite set $L(e)$, a graph is said to be L -star-edge-colorable if G has a star edge coloring c such that $c(e) \in L(e)$ for each edge e . L is called an edge k -list if each $L(e)$ is a set of size k . A graph G is star k -edge-choosable if for any edge k -list L there is a star edge coloring c such that $c(e) \in L(e)$ for every edge e . The list star chromatic index of a graph G , denoted by $ch'_{st}(G)$, is the minimum k such that G is star k -edge-choosable.

Liu and Deng [53] showed that $\chi'_{st}(G) \leq \lceil 16(\Delta - 1)^{\frac{3}{2}} \rceil$ when $\Delta \geq 7$. Dvořák, Mohar, and Šámal [16] presented a near-linear upper bound for $\chi'_{st}(G)$.

Theorem 5.1.1 ([16]). *For any graph G with maximum degree Δ , $\chi'_{st}(G) \leq \Delta \cdot 2^{O(1)\sqrt{\log \Delta}}$.*

Bezegová et al. [4] and Deng et al. [13] independently proved the following bound for trees.

Theorem 5.1.2 ([4], [13]). *Let T be a tree with maximum degree Δ . Then*

$$\chi'_{st}(T) \leq \lfloor \frac{3\Delta}{2} \rfloor,$$

and the bound is tight.

It seems very difficult to determine the star chromatic index of graphs even for complete graphs and subcubic graphs. Lei, Shi, and Song [47] showed that it is NP-complete to deter-

mine whether a subcubic multigraph is star 3-edge-colorable. Dvořák, Mohar, and Šámal [16] presented the following upper and lower bounds for complete graphs:

$$2n(1 + o(1)) \leq \chi'_{st}(K_n) \leq n \frac{2^{2\sqrt{2}(1+o(1))\sqrt{\log n}}}{(\log n)^{\frac{1}{4}}}.$$

Dvořák, Mohar, and Šámal [16] also studied star edge coloring of subcubic graphs and proved the following.

Problem 1 ([16]). *Is it true that $ch'_{st}(G) \leq 7$ for every subcubic graph G ? (Perhaps even ≤ 6).*

Problem 2 ([16]). *Is it true that $ch'_{st}(G) = \chi'_{st}(G)$ for every graph G ?*

In an attempt to solve Problem 1, Kerdjoudj and Kostochka [35] proved the following results on list version for subcubic graphs.

Theorem 5.1.3 ([35]). *Let G be a subcubic graph. Then each of the following holds.*

- (i) $ch'_{st}(G) \leq 8$.
- (ii) If $mad(G) < \frac{7}{3}$, then $ch'_{st}(G) \leq 5$.
- (iii) If $mad(G) < \frac{5}{2}$, then $ch'_{st}(G) \leq 6$.

As far as we know, Theorem 5.1.3 is the only published result on the list star edge coloring. In this chapter, we attempt to study the list star edge coloring of general graphs and present a couple of upper bounds on the list star chromatic index in terms of degeneracy.

By introducing the notion of a slightly stronger edge coloring (than star edge coloring). We first give a concise proof for the list star chromatic index of trees, and thus extend the star chromatic index of trees to the list star chromatic index. Then by modifying the ideas of the proof for trees and introducing some orientation technique, we present some upper bounds of list star chromatic index of k -degenerate graphs for general $k \geq 2$. Our method is new and we believe that it will be useful in the study of star edge coloring. Specifically we prove the following two theorems.

Theorem 5.1.4. *For every tree T with maximum degree Δ ,*

$$ch'_{st}(T) \leq \lfloor \frac{3\Delta}{2} \rfloor,$$

and this bound is tight.

Theorem 5.1.5. *Let $k \geq 2$ be an integer. For every k -degenerate graph G with maximum degree Δ , we have the following two upper bounds:*

- (a) $ch'_{st}(G) \leq \frac{5k-1}{2}\Delta - \frac{k(k+3)}{2}$. *The bound is tight for C_5 as $ch'_{st}(C_5) = 4$.*
- (b) $ch'_{st}(G) \leq 2k\Delta + k^2 - 4k + 2$.

Remark. Theorem 5.1.4 implies that if $\chi'_{st}(T) = \lfloor \frac{3\Delta}{2} \rfloor$, then $\chi'_{st}(T) = ch'_{st}(T)$. In particular, it is proved in [4] and [13] that if T is a tree which has a Δ -vertex whose neighbors are all Δ -vertices, then $\chi'_{st}(T) = \lfloor \frac{3\Delta}{2} \rfloor$ and thus $\chi'_{st}(T) = ch'_{st}(T) = \lfloor \frac{3\Delta}{2} \rfloor$ by Theorem 5.1.4. This responds to Problem 2 for some trees.

By comparing those two bounds together with an upper bound of a stronger coloring which we call list $\frac{1}{2}$ -strong edge coloring to be introduced in section 2, we have the following corollary.

Corollary 5.1.6. *Let $k \geq 2$ be an integer. For every k -degenerate graph G with maximum degree Δ ,*

$$ch'_{st}(G) \leq \begin{cases} 2k\Delta + k^2 - 4k + 2, & \text{if } k \leq \frac{\Delta}{3}; \\ \frac{5k-1}{2}\Delta - \frac{k(k+3)}{2}, & \text{if } k \geq \frac{\Delta}{3}. \end{cases}$$

5.2 Star edge coloring and $\frac{1}{2}$ -strong edge coloring

Our main idea of coloring is to find a partition of each $E(v)$ into two parts such that the colors used by the edges in one part can be repeated by some edges with distance two from them. This will help estimate the number of forbidden colors. We first apply this idea on trees and then generalize it to general graphs.

List star edge coloring and list $\frac{1}{2}$ -strong edge coloring on trees

In this subsection we will prove Theorem 5.1.4. Let G be a planar graph embedded on the plane. For each pair of adjacent edges $u_1v, u_2v \in E(v)$, define the distance from u_1v to u_2v at v to be

$$d_v(u_1v, u_2v) = 1 + |\{uv \in E(v) : u_1v, uv, u_2v \text{ are located in the clockwise order}\}|.$$

It is obvious that $d_v(u_1v, u_2v) + d_v(u_2v, u_1v) = d_G(v)$.

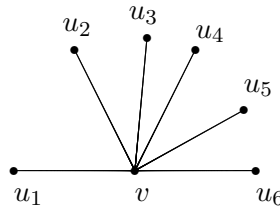


Figure 5.1: An example on definition of distance.

Example: In Figure 5.1, $d_v(u_1v, u_2v) = 1$, $d_v(u_2v, u_1v) = 5$, $d_v(u_3v, u_5v) = 2$, $d_v(u_6v, u_3v) = 3$.

For an edge coloring c and each vertex x , denote $c(x) = \{c(xu) : xu \in E(G)\}$.

Definition 2. Let G be a plane graph and $0 \leq r \leq 1$ be a rational number. An r -strong edge coloring of G is an edge coloring $c : E(G) \mapsto [k]$ such that

- (i) $c(e_1) \neq c(e_2)$ for any two adjacent edges e_1, e_2 ;
- (ii) for any edge $xy \in E(G)$, if $d_x(vx, yx) \leq rd_G(x)$, then $c(vx) \notin c(y)$; if $d_y(uy, xy) \leq rd_G(y)$, then $c(uy) \notin c(x)$.

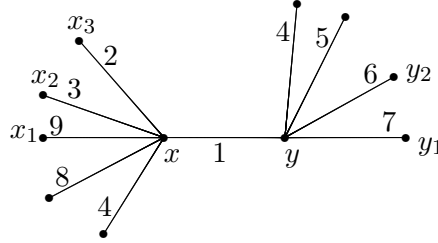


Figure 5.2: A $\frac{1}{2}$ -strong 9-edge-coloring: $c(x_1x), c(x_2x), c(x_3x) \notin c(y)$ and $c(y_1y), c(y_2y) \notin c(x)$.

A 0-strong edge coloring is a proper edge coloring, and a 1-strong edge coloring is a strong edge coloring. In this chapter, we focus on $\frac{1}{2}$ -strong edge coloring of graphs. We first show that a $\frac{1}{2}$ -strong edge coloring is always a star edge coloring and then show that every tree T with maximum degree Δ has a list $\frac{1}{2}$ -strong edge coloring as long as $|L(e)| \geq \lfloor \frac{3\Delta}{2} \rfloor$ for each edge e .

Lemma 5.2.1. Let G be a plane graph and c be a proper edge coloring of G . If c is a $\frac{1}{2}$ -strong edge coloring, then c is a star edge coloring of G .

Proof. Suppose to the contrary that c is not a star edge coloring. Let $P = xyzuv$ be a bicolored path (or cycle) where $c(xy) = c(zu)$ and $c(yz) = c(uv)$. By the definition of $\frac{1}{2}$ -strong edge coloring, we have $c(tz) \neq c(xy)$ for any $tz \in E(z)$ with $d_z(tz, yz) \leq \frac{1}{2}d_G(z)$. Thus $d_z(uz, yz) \geq \lfloor \frac{d_G(z)}{2} \rfloor + 1$ since $c(uz) = c(xy) \in c(y)$. For the same reason, we have $d_z(yz, uz) \geq \lfloor \frac{d_G(z)}{2} \rfloor + 1$. This implies

$$d_G(z) = d_z(yz, uz) + d_z(uz, yz) \geq \lfloor \frac{d_G(z)}{2} \rfloor + 1 + \lfloor \frac{d_G(z)}{2} \rfloor + 1 \geq d_G(z) + 1,$$

a contradiction.

Now we are ready to prove our result on trees (Theorem 5.1.4). By Lemma 5.2.1, Theorem 5.1.4 follows directly from the theorem below.

Theorem 5.2.2. Let T be a tree with maximum degree Δ embedded on the plane and L be a list assignment with $|L(e)| \geq \lfloor \frac{3\Delta}{2} \rfloor$ for each $e \in E(G)$. Then there exists a $\frac{1}{2}$ -strong edge coloring c such that $c(e) \in L(e)$ for every $e \in E(G)$.

Proof. We prove the theorem by induction on $|V(T)|$. The theorem is obvious if $|V(T)| = 2$. We assume $|V(T)| \geq 3$. Let x be a vertex in T such that x is adjacent to at least $d_T(x) - 1$ leaves. Denote $t = d_T(x) - 1$ and let x_1x, \dots, x_tx, yx be the edges in $E_T(x)$ in counterclockwise where x_1, x_2, \dots, x_t are leaves. Let $T' = T - \{x_1, \dots, x_t\}$. By induction hypothesis, T' has a $\frac{1}{2}$ -strong edge coloring c' such that $c'(e) \in L(e)$ for every $e \in E(T')$. We shall extend c' to be a $\frac{1}{2}$ -strong edge coloring c of T .

Denote $s = \lfloor \frac{d_T(x)}{2} \rfloor$. For every $1 \leq i \leq s$, we have

$$|L(x_ix) \setminus c'(y)| \geq \lfloor \frac{3\Delta}{2} \rfloor - \Delta \geq s.$$

Thus we can first color the edges x_1x, \dots, x_sx properly by coloring each x_ix with a color from $L(x_ix) \setminus c'(y)$ for every $1 \leq i \leq s$.

Denote $l = \lfloor \frac{d_T(y)}{2} \rfloor$. Let y_1, \dots, y_l be all the neighbors of y with $d_y(y_jy, xy) \leq l$ ($j \in [l]$) and denote $L_0 = \{c(x_ix) : i \in [s]\} \cup \{c(y_jy) : j \in [l]\} \cup \{c(xy)\}$. By the definition of $\frac{1}{2}$ -strong edge coloring, L_0 is the set of all forbidden colors for xx_j for each $s+1 \leq j \leq t$.

Then for each $s+1 \leq j \leq t$, we have

$$|L(xx_j) \setminus L_0| \geq \lfloor \frac{3\Delta}{2} \rfloor - \lfloor \frac{\Delta}{2} \rfloor - 1 - \lfloor \frac{\Delta}{2} \rfloor = \Delta - 1 - \lfloor \frac{\Delta}{2} \rfloor \geq t - s.$$

Thus we can color the edges $x_{s+1}x, x_{s+2}x, \dots, x_tx$ properly by coloring xx_j with a color from $L(xx_j) \setminus L_0$ for each $s+1 \leq j \leq t$.

Finally, we show this coloring is a $\frac{1}{2}$ -strong edge coloring of T . It suffices to verify the edge xy satisfying condition (ii) of Definition 2. Let $v \in \{x_1, \dots, x_s\}$ and $u \in \{y_1, \dots, y_l\}$. If $d_x(vx, xy) \leq \lfloor \frac{1}{2}d_T(x) \rfloor = s$, we have $c(vx) \notin c(y)$; and if $d_y(uy, xy) \leq \lfloor \frac{1}{2}d_T(y) \rfloor = l$, we have $c(uy) \notin c(x)$. Therefore, the resulting coloring c is a $\frac{1}{2}$ -strong edge coloring of T . The proof is completed.

A generalization of $\frac{1}{2}$ -strong edge coloring

Note that in the definition of $\frac{1}{2}$ -strong edge coloring of a plane graph G , we only use the clockwise order of $E(v)$ for each vertex v , but not any planar structures. So the idea of $\frac{1}{2}$ -strong edge coloring can be generalized to arbitrary graphs as long as we have a cyclic ordering of edges in $E(v)$ for each vertex v .

Definition 3. Let G be a graph and let $\sigma(v)$ be a cyclic ordering of the edges in $E(v)$ for each vertex v . σ is called a local ordering of $E(G)$. The distance from edge uv to wv at v with respect to σ , denoted by $d_{\sigma,v}(uv, wv)$, is their distance in $\sigma(v)$.

One may consider $\sigma(v)$ as a directed cycle with vertex set $E(v)$ and the distance from uv to wv is the length of the directed path from uv to wv in the directed cycle. Thus $d_{\sigma,v}(uv, wv) +$

$d_{\sigma,v}(wv, uv) = d(v)$. Denote

$$F_{\sigma,v}(uv) = \{wv \in E(v) : d_v(uv, wv) \leq \lfloor \frac{d(v)}{2} \rfloor\}.$$

Let G be a graph and σ be a local ordering of $E(G)$. A proper edge coloring c is a $\frac{1}{2}$ -strong edge coloring with respect to σ provided that for each edge $uv \in E(G)$, $c(uv) \notin c(w)$ if $wv \in F_{\sigma,v}(uv)$ (or equivalently $d_v(uv, wv) \leq \lfloor \frac{d(v)}{2} \rfloor$).

For convenience, the local ordering σ will be mentioned explicitly only when needed. If σ is understood from the context, we simply use $d_v(uv, wv)$ and $F_v(uv)$ to denote $d_{\sigma,v}(uv, wv)$ and $F_{\sigma,v}(uv)$, respectively. Note that $|F_v(uv)| = \lfloor \frac{d(v)}{2} \rfloor$.

Similar to Lemma 5.2.1, a $\frac{1}{2}$ -strong edge coloring c of G with respect to σ is a star edge coloring.

Lemma 5.2.3. *Let G be a graph. For any local ordering, every $\frac{1}{2}$ -strong edge coloring of G is a star edge coloring.*

Proof. Suppose to the contrary that $P = xyzuv$ is a bicolored path (or cycle) of length four in a $\frac{1}{2}$ -strong edge coloring c of G . Since $c(yz) \in c(u)$, we have $d_z(yz, uz) > \lfloor \frac{d(z)}{2} \rfloor$. Since $c(zu) \in c(y)$, $d_z(uz, yz) > \lfloor \frac{d(z)}{2} \rfloor$. Thus $d_z(uz, yz) + d_z(yz, uz) \geq 2(\lfloor \frac{d(z)}{2} \rfloor + 1) > d(z)$, a contradiction to the fact $d_z(uz, yz) + d_z(yz, uz) = d_G(z)$.

We show a general upper bound on the list $\frac{1}{2}$ -strong edge coloring chromatic index of graphs, which provides an upper bound for list star edge coloring as well by Lemma 5.2.3.

For two positive integers Δ and k , denote

$$\ell = \begin{cases} \frac{3}{4}\Delta^2 + (k-1)\Delta, & \text{if } k \leq \lfloor \frac{\Delta}{2} \rfloor \text{ and } \Delta \text{ is even;} \\ \frac{3}{4}\Delta^2 + \frac{2k-3}{2}\Delta + \frac{3}{4}, & \text{if } k \leq \lfloor \frac{\Delta}{2} \rfloor \text{ and } \Delta \text{ is odd;} \\ \Delta^2 + \frac{k-4}{2}\Delta + 2k-1, & \text{if } k \geq \lfloor \frac{\Delta}{2} \rfloor + 1 \text{ and } \Delta \text{ is even;} \\ \Delta^2 + \frac{k-5}{2}\Delta + \frac{3k+3}{2}, & \text{if } k \geq \lfloor \frac{\Delta}{2} \rfloor + 1 \text{ and } \Delta \text{ is odd.} \end{cases}$$

Theorem 5.2.4. *Let G be a k -degenerate graph with maximum degree $\Delta \geq 3$. Then, for any local ordering and for any list assignment L with $|L(e)| \geq \ell$ for each edge $e \in E(G)$, there exists a $\frac{1}{2}$ -strong edge coloring c such that $c(e) \in L(e)$ for every $e \in E(G)$.*

Proof. Let σ be a local ordering of $E(G)$. Let G be a counterexample with $|E(G - V_1)|$ minimized. By Theorem 5.1.2, G is not a tree and $G - V_1$ is connected. Let v be a vertex such that $d_{G-V_1}(v)$ is the minimum in $G - V_1$. Denote $E_G(v) = \{x_1v, \dots, x_tv, y_1v, \dots, y_sv\}$, where $d_G(x_i) \geq 2$ and $d_G(y_j) = 1$ for each $1 \leq i \leq t$ and each $1 \leq j \leq s$. Construct a new graph G' from $G - v$ by adding new degree one vertex x'_i connecting x_i for each $1 \leq i \leq t$ where the edge x'_ix_i plays the same role as vx_i in the ordering $\sigma(x_i)$. Since v is adjacent to at least one vertex of degree large than one in G , we have $|E(G' - V_1(G'))| < |E(G - V_1)|$. By the minimality of

G , there exists a $\frac{1}{2}$ -strong edge coloring c' such that $c'(e) \in L(e)$ for every $e \in E(G')$. Uncolor the edges $x'_i x_i$'s and we still use c' to denote the new coloring. Then the coloring c' restricted to $G - v$ is a partial $\frac{1}{2}$ -strong edge coloring of G , and we shall extend c' to a $\frac{1}{2}$ -strong edge coloring c of G by coloring the edges in $E(v)$ appropriately.

We color the edges $x_i v$ in $\{x_1 v, x_2 v, \dots, x_t v\}$ with $|F_v(x_i v) \cap \{x_1 v, x_2 v, \dots, x_t v\}| = \lfloor \frac{\Delta}{2} \rfloor$ first, and then color the remaining edges in $\{x_1 v, x_2 v, \dots, x_t v\}$, and finally we color the edges $y_1 v, \dots, y_s v$.

In the following, we estimate the maximum number of forbidden colors in order to color the edges in $E(v)$. Let $uv \in E(v)$ where $u \in \{x_1, \dots, x_t\}$. Suppose we pick a color α to color uv .

We first consider the forbidden colors on u 's side. By the definition of $\frac{1}{2}$ -strong edge coloring, we have

(i) for each edge $uw \in F_u(vu)$, $\alpha \notin c'(w)$. Since $|c'(w)| = d_G(w) \leq \Delta$ and there are $|F_u(vu)|$ such edges, the total number of forbidden colors from those edges is at most $|F_u(vu)|\Delta = \lfloor \frac{d(u)}{2} \rfloor \Delta$;

(ii) for each edge $zu \notin F_u(vu)$ and for any $z'z \in E(G)$ with $uz \in F_z(z'z)$, $c'(z'z)$ does not appear in $c'(u)$. Since $c'(z'z) \notin c'(u)$, we have $\alpha \neq c'(z'z)$ and thus including $c'(zu)$, there are at most $\lfloor \frac{\Delta}{2} \rfloor + 1$ forbidden colors in $c'(z)$. Since uv is not colored yet, there are $(d(u) - 1 - \lfloor \frac{d(u)}{2} \rfloor)$ such edges zu . Therefore the total number of forbidden colors from those edges is at most $(d(u) - 1 - \lfloor \frac{d(u)}{2} \rfloor)(\lfloor \frac{\Delta}{2} \rfloor + 1)$.

So the number of forbidden colors on u 's side is at most

$$\lfloor \frac{d(u)}{2} \rfloor \Delta + (d(u) - 1 - \lfloor \frac{d(u)}{2} \rfloor)(\lfloor \frac{\Delta}{2} \rfloor + 1) \leq \lfloor \frac{\Delta}{2} \rfloor \Delta + (\Delta - 1 - \lfloor \frac{\Delta}{2} \rfloor)(\lfloor \frac{\Delta}{2} \rfloor + 1).$$

Now we consider the forbidden colors on v 's side. Note that $y_1 v, \dots, y_s v$ are not colored yet. It is clear that the number of forbidden colors on v 's side is at most $(t - 1)\Delta \leq (k - 1)\Delta$. However we can have better estimation when $t \geq \lfloor \frac{\Delta}{2} \rfloor + 1$.

Denote $A = F_v(uv) \cap \{x_1 v, x_2 v, \dots, x_t v\}$ and $a = |A|$. Let h be the number of colored edges in $F_v(uv)$, and let $u'v$ be the colored edge in A with $d_v(uv, u'v)$ maximized.

Similar to (i) and (ii) we have the following:

(iii) For each edge $wv \in A$, $\alpha \notin c'(w)$ and thus there are $d_G(w) \leq \Delta$ or $d_G(w) - 1 \leq \Delta - 1$ (depending on whether wv is already colored or not) forbidden colors at w .

(iv) For each edge $wv \in \{x_1 v, x_2 v, \dots, x_t v\} - F_v(uv)$, similar to (ii) there are at most $\lfloor \frac{\Delta}{2} \rfloor + 1$ forbidden colors. Note there are at most $(t - 1 - a)$ such edges.

If $a = |A| \leq \lfloor \frac{\Delta}{2} \rfloor - 1$, by (iii) and (iv) the total number of forbidden colors caused by the v 's

side is at most

$$\begin{aligned}
& a\Delta + (t-1-a)(\lfloor \frac{\Delta}{2} \rfloor + 1) \\
\leq & (\lfloor \frac{\Delta}{2} \rfloor - 1)\Delta + (t - \lfloor \frac{\Delta}{2} \rfloor)(\lfloor \frac{\Delta}{2} \rfloor + 1) \\
\leq & \lfloor \frac{\Delta}{2} \rfloor(\Delta - \lfloor \frac{\Delta}{2} \rfloor - 1) - \Delta + k(\lfloor \frac{\Delta}{2} \rfloor + 1) \\
\leq & \lfloor \frac{\Delta}{2} \rfloor(\Delta - \lfloor \frac{\Delta}{2} \rfloor - 2) + k(\lfloor \frac{\Delta}{2} \rfloor + 2) - \Delta - 1 \quad (\text{by } \lfloor \frac{\Delta}{2} \rfloor + 1 \leq k) \\
\leq & \lfloor \frac{\Delta}{2} \rfloor(\Delta - \lfloor \frac{\Delta}{2} \rfloor - 4) + k(\lfloor \frac{\Delta}{2} \rfloor + 2) - 1 \quad (\text{by } 2\lfloor \frac{\Delta}{2} \rfloor \leq \Delta).
\end{aligned}$$

Now assume $|A| = \lfloor \frac{\Delta}{2} \rfloor$. Then $F_v(uv) \subseteq \{x_1v, x_2v, \dots, x_tv\}$ and $\lfloor \frac{d_G(v)}{2} \rfloor = \lfloor \frac{\Delta}{2} \rfloor$. Since $u'v$ is already colored, by the coloring algorithm, $|F_v(u'v) \cap \{x_1v, x_2v, \dots, x_tv\}| = |A| = \lfloor \frac{\Delta}{2} \rfloor$. Thus $F_v(u'v) \subseteq \{x_1v, x_2v, \dots, x_tv\}$ and $|F_v(u'v)| = \lfloor \frac{\Delta}{2} \rfloor$. Note that $h \leq d_v(uv, u'v)$. Since the colored edges in $F_v(uv)$ do not belong to $F_v(u'v)$ if $h \neq 0$, we have $h + \lfloor \frac{\Delta}{2} \rfloor \leq d_v(uv, u'v) + |F_v(u'v)| \leq t$. Thus

$$h \leq t - \lfloor \frac{\Delta}{2} \rfloor. \quad (5.1)$$

By (iii) and (iv), the total number of forbidden colors on v 's side is at most

$$\begin{aligned}
& h\Delta + (\lfloor \frac{\Delta}{2} \rfloor - h)(\Delta - 1) + (t-1-\lfloor \frac{\Delta}{2} \rfloor)(\lfloor \frac{\Delta}{2} \rfloor + 1) \\
= & \lfloor \frac{\Delta}{2} \rfloor(\Delta - 1) + h + (t-1-\lfloor \frac{\Delta}{2} \rfloor)(\lfloor \frac{\Delta}{2} \rfloor + 1) \\
\leq & \lfloor \frac{\Delta}{2} \rfloor(\Delta - 1) + t - \lfloor \frac{\Delta}{2} \rfloor + (t-1-\lfloor \frac{\Delta}{2} \rfloor)(\lfloor \frac{\Delta}{2} \rfloor + 1) \quad (\text{by Inequality (5.1)}) \\
= & \lfloor \frac{\Delta}{2} \rfloor(\Delta - 2) + t(\lfloor \frac{\Delta}{2} \rfloor + 2) - (1 + \lfloor \frac{\Delta}{2} \rfloor)^2 \\
\leq & \lfloor \frac{\Delta}{2} \rfloor(\Delta - 2) + k(\lfloor \frac{\Delta}{2} \rfloor + 2) - (1 + \lfloor \frac{\Delta}{2} \rfloor)^2 \quad (\text{by } t \leq k) \\
= & \lfloor \frac{\Delta}{2} \rfloor(\Delta - \lfloor \frac{\Delta}{2} \rfloor - 4) + k(\lfloor \frac{\Delta}{2} \rfloor + 2) - 1.
\end{aligned}$$

Therefore, if $k \geq \lfloor \frac{\Delta}{2} \rfloor + 1$, then the total number of forbidden colors for uv is at most

$$\lfloor \frac{\Delta}{2} \rfloor\Delta + (\Delta - 1 - \lfloor \frac{\Delta}{2} \rfloor)(\lfloor \frac{\Delta}{2} \rfloor + 1) + \lfloor \frac{\Delta}{2} \rfloor(\Delta - \lfloor \frac{\Delta}{2} \rfloor - 4) + k(\lfloor \frac{\Delta}{2} \rfloor + 2) - 1 \leq \ell - 1.$$

If $k \leq \lfloor \frac{\Delta}{2} \rfloor$, the total number of forbidden colors for uv is at most

$$\lfloor \frac{\Delta}{2} \rfloor\Delta + (\Delta - 1 - \lfloor \frac{\Delta}{2} \rfloor)(\lfloor \frac{\Delta}{2} \rfloor + 1) + (k-1)\Delta \leq \ell - 1.$$

Finally, when we color y_jv ($j \in [s]$), the total number of forbidden colors is at most

$$\lfloor \frac{\Delta}{2} \rfloor\Delta + (\Delta - 1 - \lfloor \frac{\Delta}{2} \rfloor)(\lfloor \frac{\Delta}{2} \rfloor + 1) \leq \ell - 1.$$

Therefore, we can complete the coloring process to obtain a $\frac{1}{2}$ -strong edge coloring c of G , a contradiction. This completes the proof of the theorem.

Note that Theorem 5.2.4 also provides a general upper bound $\frac{3}{2}\Delta^2 - 1$ (and $\frac{3}{2}\Delta^2 - \Delta + \frac{3}{2}$ when Δ is odd) for $\frac{1}{2}$ -strong edge coloring of graphs with maximum degree Δ . In addition, Corollary 5.1.6 follows from Theorem 5.1.5 and Theorem 5.2.4 with a straightforward calculation.

5.3 List star edge coloring of k -degenerate graphs-two more upper bounds

In this section, we modify the idea of the proof of trees by introducing a special orientation of a graph G to handle star edge coloring and present two more upper bounds.

Definition 4. Let G be a graph on n vertices with maximum degree Δ , and $p, q \leq \Delta$ be two positive integers. A well-ordered (p, q) -star orientation (\mathcal{V}, D) of G is a vertex enumeration $\mathcal{V} = (v_1, v_2, \dots, v_n)$ together with the orientation D such that, for each $i \in [n]$,

- (a) $d_D^+(v_i) = |E_D^+(v_i)| \leq p$;
- (b) for any $uv_i \in E_D^-(v_i)$, $|E_{G_i}(u)| \leq q$, where G_i is the subgraph of G induced by $\cup_{j=1}^i E_D^-(v_j)$.

We also need to modify the definition of local ordering of G (see Definition 3) for digraphs.

Definition 5. Let G be a graph and D be an orientation of G . Let $\sigma(v)$ be a cyclic ordering of the edges in $E_D^-(v)$ for each vertex v . σ is called a local ordering of D . The distance from edge uv to wv at v with respect to σ , denoted by $d_{\sigma,v}(uv, wv)$, is their distance in $\sigma(v)$.

Theorem 5.3.1. Let G be a graph with maximum degree Δ and let $p, q \leq \Delta$ be two positive integers. Assume that G has a well-ordered (p, q) -star orientation (\mathcal{V}, D) . Then

$$ch'_{st}(G) \leq \begin{cases} \frac{3q+2p-1}{2}\Delta - \frac{p(q+1)}{2}, & \text{if } \Delta \leq p+2; \\ \frac{3q+2p-1}{2}\Delta - \frac{p(q+3)}{2}, & \text{if } \Delta \geq p+3. \end{cases}$$

Proof. Let σ be a local ordering of D . We will define a coloring of G recursively by coloring G_1, G_2 until G_n such that the coloring of G_i is indeed a star edge coloring of G_i for each $i \in [n]$. For a given edge $uv \in E_D^-(v)$, denote

$$F_v(uv) = \{wv \in E_D^-(v) : d_v(uv, wv) \leq \lfloor \frac{d_D^-(v)}{2} \rfloor\} \text{ and } g_v(uv) = wv \text{ where } d_v(uv, wv) = 1.$$

First, we color G_1 with a proper edge coloring. Note that $E(G_1)$ induces a star (possibly empty).

Now we assume that G_{i-1} is already colored with an edge coloring c . We are to extend the coloring c to the edges in $E_D^-(v_i)$ to obtain a star edge coloring of G_i . Denote $\sigma(v_i) = \{u_1v_i, u_2v_i, \dots, u_{d_D^-(v_i)}v_i\}$. Suppose that all the edges $u_1v_i, \dots, u_{j-1}v_i$ are colored and we are to color the edge u_jv_i according to the following rules (see Figure 5.3).

- (i) $c(u_j v_i) \neq c(u_t v_i)$ for any $t \in [d_D^-(v_i)]$ with $t \leq j - 1$;
- (ii) $c(u_j v_i) \notin c(y)$ for any $y u_j \in E_{G_i}(u_j)$ with $y \neq v_i$;
- (iii) $c(u_j v_i) \notin c(z)$ for any $z v_i \in F_{v_i}(u_j v_i)$;
- (iv-a) $c(u_j v_i) \notin c(x)$ for any $v_i x \in E_D^+(v_i)$ with $c(v_i x) \in c(u_j)$ or $d_{G_i}(x) \leq \Delta - 1$;
- (iv-b) for any $v_i x \in E_D^+(v_i)$ with $c(v_i x) \notin c(u_j)$ and $d_{G_i}(x) = \Delta$,

$$c(u_j v_i) \notin \begin{cases} c(x) \setminus c(g_x(v_i x)), & \text{if } \Delta \geq p + 3; \\ c(x), & \text{if } \Delta \leq p + 2. \end{cases}$$

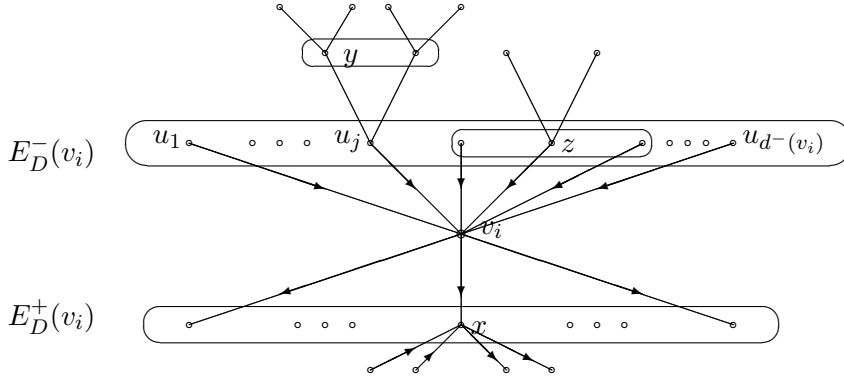


Figure 5.3: Local structure of $E(v_i)$.

Now we estimate the number of forbidden colors for $u_j v_i$.

(a) By (i) $u_j v_i$ and $u_k v_i$ should be colored with different colors for any $k \neq j$, and this requires at most $d_D^-(v_i) - 1$ forbidden colors.

(b) The number of forbidden colors from (ii) is at most $(q - 1)\Delta$, since $|E_{G_{i-1}}(u_j)| = |E_{G_i}(u_j)| - 1 \leq q - 1$.

(c) For each z with $z v_i \in F_{v_i}(u_j v_i)$, $|c(z)| \leq |E_{G_i}(z)| \leq q$ and the color $c(z v_i)$ is already counted as a forbidden color in (a). Thus the number of forbidden colors from (iii) not counted in (a) is at most $(q - 1) \lfloor \frac{d_D^-(v_i)}{2} \rfloor$.

Let $a = |\{v_i x \in E_D^+(v_i) : c(v_i x) \in c(u_j) \text{ or } d_{G_i}(x) \leq \Delta - 1\}|$. Then $0 \leq a \leq d_D^+(v_i)$.

(d) If $c(v_i x) \in c(u_j)$, then $c(v_i x)$ is counted in (b). Thus the number of forbidden colors from (iv-a) not counted in (b) is at most $a(\Delta - 1)$, and the number of forbidden colors from (iv-b) is at most $(d_D^+(v_i) - a)\Delta$ (when $\Delta \leq p + 2$) or $(d_D^+(v_i) - a)(\Delta - 1)$ (when $\Delta \geq p + 3$). Hence the number of forbidden colors from (iv-a) and (iv-b) is at most $d_D^+(v_i)\Delta$ (when $\Delta \leq p + 2$) or $d_D^+(v_i)(\Delta - 1)$ (when $\Delta \geq p + 3$).

Therefore, when $\Delta \geq p + 3$, the total number of forbidden colors for $u_j v_i$ is at most

$$\begin{aligned}
& (q-1)\Delta + (q-1)\lfloor \frac{d_D^-(v_i)}{2} \rfloor + d_D^+(v_i)(\Delta-1) + d_D^-(v_i) - 1 \\
= & (q-1)\Delta + (q-1)\lfloor \frac{d_D^-(v_i)}{2} \rfloor + d_D^+(v_i)(\Delta-2) + (d_D^+(v_i) + d_D^-(v_i)) - 1 \\
\leq & (q-1)\Delta + (q-1)(\frac{\Delta - d_D^+(v_i)}{2}) + d_D^+(v_i)(\Delta-2) + \Delta - 1 \quad (\text{since } d_D^+(v_i) + d_D^-(v_i) \leq \Delta) \\
= & \frac{3q-1}{2}\Delta + \frac{2\Delta-3-q}{2}d_D^+(v_i) - 1 \\
\leq & \frac{3q-1}{2}\Delta + \frac{2\Delta-3-q}{2}p - 1 \quad (\text{since } d_D^+(v_i) \leq p) \\
= & \frac{3q+2p-1}{2}\Delta - \frac{p(q+3)}{2} - 1.
\end{aligned}$$

If $\Delta \leq p + 2$, then similar calculation yields that the number of forbidden colors is at most $\frac{3q+2p-1}{2}\Delta - \frac{p(q+1)}{2} - 1$.

Therefore, $\frac{3q+2p-1}{2}\Delta - \frac{p(q+1)}{2}$ colors (when $\Delta \leq p + 2$) or $\frac{3q+2p-1}{2}\Delta - \frac{p(q+3)}{2}$ colors (when $\Delta \geq p + 3$) are enough to complete the coloring process.

Finally we show that this coloring is indeed a star edge coloring. It suffices to show, in the graph G_i , for each $j \in [d_D^-(v_i)]$, after coloring $u_j v_i$, it does not produce a bicolored path or cycle of length four. Suppose to the contrary that there is a bicolored path or cycle P of length four containing the edge $u_j v_i$. Obviously by (ii), P is not a cycle and v_i is not an endpoint of P . Let $u_j v_i x$ be a subpath in P . Then either $c(u_j v_i) \in c(x)$ or $c(v_i x) \in c(u_j)$.

If $x v_i \in E_D^-(v_i)$, then $x = u_k$ for some $k \in [d_D^-(v_i)]$. By (ii), $c(u_j v_i) \notin c(y)$ for any $y u_j \in E_{G_i}(u_j)$ with $y \neq v_i$, and so u_k is not an endpoint of P . Similarly, u_j is not an endpoint of P . This implies $c(u_j v_i) \in c(u_k)$ and $c(u_k v_i) \in c(u_j)$. By (iii), $u_k v_i \notin F_{v_i}(u_j v_i)$ and $u_j v_i \notin F_{v_i}(u_k v_i)$. Thus $d_{v_i}(u_j v_i, u_k v_i) \geq \lfloor \frac{d_D^-(v_i)}{2} \rfloor + 1$ and $d_{v_i}(u_k v_i, u_j v_i) \geq \lfloor \frac{d_D^-(v_i)}{2} \rfloor + 1$. Therefore we can obtain the following contradiction:

$$d_D^-(v_i) = d_{v_i}(u_j v_i, u_k v_i) + d_{v_i}(u_k v_i, u_j v_i) \geq 2\lfloor \frac{d_D^-(v_i)}{2} \rfloor + 2 \geq d_D^-(v_i) + 1.$$

Now we assume $v_i x \in E_D^+(v_i)$. By (ii) again, x is not an endpoint of P which implies $c(u_j v_i) \in c(x)$. By (iv-a) and (iv-b), $c(v_i x) \notin c(u_j)$ and $d_{G_i}(x) = \Delta \geq p + 3$. Thus u_j is an endpoint of P . Let $P = u_j v_i x x_1 x_2$. By (iv-b), $x_1 \in E_D^-(x)$ and $x x_1 = g_x(v_i x)$ (meaning $d_x(v_i x, x_1 x) = 1$).

Since P is bicolored, we have $c(x x_1) = c(u_j v_i)$, and so $c(u_j v_i) = c(x x_1) = c(g_x(v_i x))$ and $d_{G_{i-1}}(x) = d_{G_i}(x) = \Delta \geq p + 3$ by (iv-a) and (iv-b). Hence $d_D^-(x) \geq \Delta - p \geq 3$. Note $c(v_i x) \in c(x_1)$.

If $x_1 x_2$ is colored before $v_i x$, then $x_1 x \notin F_x(v_i x)$ by (iii). But we have $d_D^-(x) \geq 3$ and $1 = d_x(v_i x, x_1 x) \leq \lfloor \frac{d_D^-(x)}{2} \rfloor$, which implies $x_1 x \in F_x(v_i x)$ by definition, a contradiction.

Now assume that x_1x_2 is colored after v_ix . By (ii), x_1x_2 is oriented from x_2 to x_1 since $c(x_1x_2) \in c(x)$. By (iv-a) and (iv-b), we have $c(x_1x_2) = c(g_x(x_1x))$ which implies $d_x(x_1x, v_ix) = 1$. Thus we obtain the following contradiction:

$$3 \leq d_D^-(x) = d_x(v_ix, x_1x) + d_x(x_1x, v_ix) = 2.$$

Therefore c is a star edge coloring and thus completes the proof of the theorem.

By modifying the coloring algorithm in the proof of Theorem 5.3.1, we also obtain another upper bound for $ch'_{st}(G)$ for any graph G with a well-ordered (p, q) -star orientation.

Theorem 5.3.2. *Let G be a graph with a well-ordered (p, q) -star orientation (\mathcal{V}, D) . Let $\Delta \geq 3$ be the maximum degree of G and let $q \geq 2$. Then*

$$ch'_{st}(G) \leq (p + q)\Delta + q^2 - 3q - p + 2.$$

Proof. We adopt the same notations as in Theorem 5.3.1, but apply a modified coloring rules as below.

We assume that G_{i-1} is already colored with an edge coloring c and we extend the coloring c to the edges in $E_D^-(v_i)$ to obtain a star edge coloring of G_i . Assume that all the edges $u_1v_i, \dots, u_{j-1}v_i$ are colored. We are to color the edge u_jv_i according to the following rules.

- (i) $c(u_jv_i) \neq c(u_tv_i)$ for each $t \leq j - 1$;
- (ii) $c(u_jv_i) \notin c(y)$ for any $yu_j \in E_{G_i}(u_j)$ with $y \neq v_i$;
- (iii) $c(u_jv_i) \notin c(z)$ for any $zv_i \in E_D^-(v_i)$ with $c(zv_i) \in c(u_j)$;
- (iv) $c(u_jv_i) \notin c(x)$ for any $v_ix \in E_D^+(v_i)$.

Denote $b = |c(E_D^-(v_i)) \cap c(u_j)|$. Then $b \leq q - 1$. Similar to Theorem 5.3.1, the total number of forbidden colors for u_jv_i is at most

$$\begin{aligned} & (q - 1)\Delta + (q - 1)b + d_D^+(v_i)\Delta + (d_D^-(v_i) - b - 1) \\ &= (q - 1)\Delta + (q - 2)b + d_D^+(v_i)(\Delta - 1) + (d_D^+(v_i) + d_D^-(v_i)) - 1 \\ &\leq (q - 1)\Delta + (q - 2)(q - 1) + p(\Delta - 1) + \Delta - 1 \\ &= (p + q)\Delta + q^2 - 3q - p + 1. \end{aligned}$$

Since there are $(p + q)\Delta + q^2 - 3q - p + 2$ colors, one can always find a color for u_jv_i .

Now we show that after coloring u_jv_i , the new coloring is a star edge coloring. Suppose to the contrary that P is a bicolored path or cycle of length four containing the edge u_jv_i . By (ii), P is not a cycle and v_i is not an endpoint of P . Let u_jv_ix be a subpath of P . By (ii) again, x is not an endpoint of P . Thus $c(u_jv_i) \in c(x)$, and so $xv_i \in E_D^-(v_i)$ by (iv). Thus by

(iii), $c(xv_i) \notin c(u_j)$, which implies that u_j is an endpoint of P . Denote $P = u_j v_i x_1 x_2$ where $u_j v_i, x_1 x_2 \in E_D^-(v_i)$ and $c(xv_i) = c(x_1 x_2) \in c(x_1)$. Thus x_1 and $x_1 x_2$ both are colored before xv_i . By (ii), $c(xv_i) \notin c(x_1)$, a contradiction. This proves Theorem 5.3.2.

We shall show that every k -degenerate graph admits a well-ordered (k, k) -star orientation, and then apply Theorems 5.3.1 and 5.3.2 to obtain upper bounds of list star edge chromatic index of k -degenerate graphs, which will prove Theorem 5.1.5 (a) and (b).

Lemma 5.3.3. *Every k -degenerate graph admits a well-ordered (k, k) -star orientation.*

Proof. Let G be a k -degenerate graph. We shall find G_n, G_{n-1}, \dots, G_1 and v_n, \dots, v_1 recursively. Define $G_n = G$. We assume G_i is determined and we are to find v_i and G_{i-1} according to the following.

- (A1) If $V_{\geq k+1}(G_i) \neq \emptyset$, choose v_i to be a vertex in $V_{\geq k+1}(G_i)$ whose degree is at most k in the subgraph $G_i[V_{\geq k+1}(G_i)]$ of G_i induced by $V_{\geq k+1}(G_i)$.
- (A2) If $V_{\geq k+1}(G_i) = \emptyset$ and $E(G_i) \neq \emptyset$, choose v_i to be a vertex with maximum degree in G_i .
- (A3) If $V_{\geq k+1}(G_i) = \emptyset$ and $E(G_i) = \emptyset$, let v_i be any vertex in $V(G) \setminus \{v_n, \dots, v_{i+1}\}$.
- (B) For each edge $uv_i \in E(G_i)$ with $|E_{G_i}(u)| \leq k$, orient the edge uv_i from u to v_i .
- (C) Set $G_{i-1} = G_i - \{uv_i \in E(G_i) : |E_{G_i}(u)| \leq k\}$.

Note that, in (A1) such a vertex exists since G is k -degenerate graph and $G_i[V_{\geq k+1}(G_i)]$ is a subgraph of G . We claim that this defines a proper vertex enumeration $\mathcal{V} = (v_1, v_2, \dots, v_n)$. To this end, we show that $v_i \neq v_j$ for any $i \neq j$. Suppose to the contrary that a vertex v is labelled with v_i and v_j for some $i > j$.

We first claim that the degree of v_j in G_j is not zero. Otherwise, $E(G_j) = \emptyset$ by (A2) and (A3), and by (A3) again, v_j is not selected, a contradiction. Thus $v_i \in V_{\geq k+1}(G_i)$ otherwise the degree of v_i is zero in G_t for any $t = i-1, \dots, j$ by (C). By (C), for every w such that $v_i w \in E_{G_i}(v_i) \setminus E_D^-(v_i)$, $|E_{G_{i-1}}(w)| \geq k+1$. Furthermore, by (A1) and (C), we have $|E_{G_{i-1}}(v_i)| \leq k$. Thus according to (A1), w is always chosen before vertex v_j for every w such that $v_i w \in E_D^-(w)$. This implies that v_j has degree zero in G_j , a contradiction. This proves $v_i \neq v_j$ for any $i \neq j$.

Clearly, this defines an orientation D satisfying (a) and (b) in Definition 4. Therefore, (\mathcal{V}, D) is a well-ordered (k, k) -star orientation with $\mathcal{V} = (v_1, v_2, \dots, v_n)$.

Proof of Theorem 5.1.5 (a) and (b): Theorem 5.1.5 (a) with $\Delta \geq k+3$ and Theorem 5.1.5 (b) are implied by Theorems 5.3.1 and 5.3.2 with $p = q = k$, together with Lemma 5.3.3. It remains to show Theorem 5.1.5 (a) when $\Delta \in \{k, k+1, k+2\}$. We may also assume $\Delta \geq 4$ as the case of $\Delta = 2$ is trivial and the case of $\Delta = 3$ follows by Theorem 5.1.3.

We compare the bounds in Theorem 5.2.4 with the desired bound $\frac{5k-1}{2}\Delta - \frac{k(k+3)}{2}$ in all cases. The bounds in Theorem 5.2.4 are better when $\Delta \in \{k, k+1\}$. For the case of $\Delta = k+2$, when

Δ is odd we have $k \geq 3$ and

$$\left(\frac{5k-1}{2}\Delta - \frac{k(k+3)}{2}\right) - \left(\Delta^2 + \frac{k-5}{2}\Delta + \frac{3k+3}{2}\right) = \frac{1}{2}k^2 - k - \frac{3}{2} \geq 0;$$

when Δ is even and $k \geq 4$, we have

$$\left(\frac{5k-1}{2}\Delta - \frac{k(k+3)}{2}\right) - \left(\Delta^2 + \frac{k-4}{2}\Delta + 2k - 1\right) = \frac{1}{2}k^2 - 2k \geq 0.$$

Now it remains to verify the final case that $\Delta = 4$ and $k = 2$. That is, we will show the following statement.

Every 2-degenerate graph G with maximum degree 4 is star 13-edge-choosable.

Let G together with an edge 13-list L be a counterexample to the above statement with $|E(G)|$ minimized.

Let $xy \in E(G)$. By the minimality of G , $G - xy$ has a list star edge coloring c with $c(e) \in L(e)$ for each $e \in E(G) \setminus \{xy\}$. Denote $A(xy) = \bigcup_{w \in N(x) \cup N(y) \setminus \{x,y\}} c(w)$.

(I) For any $xy \in E(G)$, $|A(xy)| \geq 13$ and thus $\delta(G) = 2$.

Otherwise, $L(xy) \setminus A(xy) \neq \emptyset$. Thus one can always pick a color in $L(xy) \setminus A(xy)$ to color xy to extend c to be a list star edge coloring of G , a contradiction.

Let z be a vertex with minimum degree in $G[V_{\geq 3}]$. Then z has a neighbor x_1 of degree 2 in G since G is 2-degenerate and $\delta(G) = 2$.

(II) $d_G(z) = 4$ and z has exactly two neighbors of degree 2.

If $d_G(z) = 3$, then $|A(zx_1)| \leq 4 + 4 + 4 = 12 < 13$, a contradiction to (I).

If z has at least three neighbors of degree 2, then $|A(zx_1)| \leq 4 + 4 + 2 + 2 = 12 < 13$, a contradiction to (I) again.

By (II), let x_1 and x_2 be the two neighbors of z with degree 2 and z_1, z_2 be the other two neighbors of z . Let $x_{ii} \neq z$ be the other neighbor of x_i for each $i = 1, 2$ (see Figure 5.4).

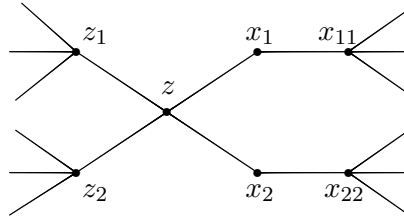


Figure 5.4: A Possible Configuration.

By the minimality of G , let c' be a star edge coloring of $G - x_1 - x_2$. We are to extend c' to a star edge coloring c of G below. Since $|\bigcup_{x \in N(x_{ii}) \setminus \{x_i\}} c'(x)| \leq 12$ and $|L(x_{ii})| = 13$ for each $i = 1, 2$, we first color $x_i x_{ii}$ with a color in $L(x_{ii}) \setminus \bigcup_{x \in N(x_{ii}) \setminus \{x_i\}} c'(x)$. Denote c to be the new coloring of $G - zx_1 - zx_2$ after coloring $x_1 x_{11}$ and $x_2 x_{22}$.

(III) $c(x_ix_{ii}) \notin c(z)$ for each $i = 1, 2$.

Without loss of generality, assume that $c(x_1x_{11}) \in c(z)$. Then $|c(z_1) \cup c(z_2) \cup c(x_{11})| \leq 11$, and we first color zx_1 with a color α such that $\alpha \in L(zx_1) \setminus [c(z_1) \cup c(z_2) \cup c(x_{11})]$ and $\alpha \neq c(x_2x_{22})$. Clearly, this coloring of $G - zx_2$ is a star edge coloring of $G - zx_2$. If $c(x_2x_{22}) \in c(z)$, then $|A(zx_2)| \leq 12$ since $c(x_1x_{11}) \in c(z)$, a contradiction to (I). Thus, $c(x_2x_{22}) \notin c(z)$.

Since $c(x_1x_{11}) \in c(z)$, $|c(z_1) \cup c(z_2) \cup c(x_1) \cup \{c(x_2x_{22})\}| \leq 10$, and so we color zx_2 with a color $\beta \in L(zx_2) \setminus [c(z_1) \cup c(z_2) \cup c(x_1) \cup \{c(x_2x_{22})\}]$.

We verify that this results a star edge coloring. Suppose that P is a bicolored path (or cycle) of length four containing zx_2 . By the coloring of $c(zx_2)$, we have $|P \cap E(t)| \leq 1$ for each $t \in \{z_1, z_2, x_1\}$, and so $|P \cap E(x_{22})| = 2$ and z is an endpoint of P since $c(x_2x_{22}) \notin c(z) \cup \{c(zx_1)\}$. However $c(x_2x_{22}) \notin c(w)$ for each $w \in N(x_{22})$ and $w \neq x_2$. This implies that the length of P is at most three and thus proves (III).

The final step: By (III), we may assume $c(x_ix_{ii}) \notin c(z)$ for each $i = 1, 2$. Since $|c(z_1) \cup c(z_2) \cup \{c(x_1x_{11}), c(x_2x_{22})\}| \leq 10$, one can color the edges zx_1, zx_2 properly such that $c(zx_i) \in L(zx_i) \setminus [c(z_1) \cup c(z_2) \cup \{c(x_1x_{11}), c(x_2x_{22})\}]$ for each $i = 1, 2$.

It remains to check this is a star edge coloring. Suppose that P is a bicolored path or cycle of length four containing zx_1 or zx_2 . Without loss of generality, assume that P contains zx_1 . For each $i = 1, 2$, z_i is not an endpoint of P since $c(x_1x_{11}) \notin c(z)$ and z_i is not contained in P either since $c(zx_1) \notin c(z_i)$ for each $i = 1, 2$.

Since $c(x_1x_{11}) \notin \bigcup_{x \in N(x_{11}) \setminus \{x_1\}} c(x)$, z is not an endpoint of P . Thus P contains x_1zx_2 . Since $|E(P)| = 4$, either $c(zx_1) = c(x_2x_{22})$ or $c(zx_2) = c(x_1x_{11})$. However by the choice of $c(zx_i)$, $c(zx_i) \notin \{c(x_1x_{11}), c(x_2x_{22})\}$ for each $i = 1, 2$. This contradiction proves that c is a star edge coloring of G and thus completes the proof. ■

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