Magnetic induction fields and potentials from electrical currents on elliptical cylinders

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Abstract: *Magnetic induction fields and potentials produced by electrical currents along an infinitely long elliptical cylinder and around an infinitely long elliptical-cylindrical solenoid, respectively, are explicitly evaluated. The similarities and differences of the current distributions, magnetic induction fields and potentials of these electromagnetic systems are contrasted with those of the corresponding cylinders with circular cross-sections.*

Key words: *magnetostatics, elliptical cylinders*

Resumo: *Indução magnética de campos e potenciais produzidos por correntes elétricas ao longo de um cilindro elíptico infinito e ao longo de um solenóide cilíndrico infinitamente longo, respectivamente, são calculados explicitamente. As similaridades e as diferenças das distribuições de correntes, campos de indução magnética e potenciais desses sistemas eletromagnéticos são comparados com os dos cilindros correspondentes com secções circulares.*

Palavras-chave: *magnetostática, cilindros elípticos*

1 Introduction

The magnetic induction fields of uniformly distributed electrical currents along an infinitely long circular cylinder and around an infinitely long circular-cylindrical solenoid, respectively, are studied in the introductory courses on electromagnetism (HALLIDAY, 1978; ALONSO, 1967; REITZ, 1967). In the first case, it is established through the application of Ampère's law in its integral form that the field vanishes in the interior of the cylinder; and that in the exterior its lines are circular, and its magnitude is proportional to the electrical current I along the cylinder and inversely proportional to the radial distance ρ to the axis of the cylinder:

$$\vec{B}(\rho < a, \varphi, z) = 0, \quad \vec{B}(\rho > a, \varphi, z) = \frac{2I}{c\rho}\hat{\varphi} \tag{1}$$

where a is the radius of the cylinder and (ρ, φ, z) are circular cylindrical coordinates. In the case of the solenoid, the field vanishes outside and is uniform inside; Ampère's law in its boundary condition form leads to

$$\vec{B}(\rho < a, \varphi, z) = \frac{4\pi}{c} nI\hat{k}, \quad \vec{B}(\rho > a, \varphi, z) = 0 \tag{2}$$

where I is the current in the solenoid and n is the number of turns per centimeter in its winding.

On the other hand, the corresponding magnetic vector potentials are studied at a more advanced level (PURCELL, 1965; LORRAINE, 1990; PANOFSKY, 1964). Here some of their possible explicit forms are given respectively:

$$\vec{A}(\rho < a, \varphi, z) = -\frac{2I}{c}\ln a\,\hat{k}, \quad \vec{A}(\rho > a, \varphi, z) = -\frac{2I}{c}\ln\rho\,\hat{k} \tag{3}$$

and

$$\vec{A}(\rho < a, \varphi, z) = \frac{2\pi}{c} nI\rho\hat{\varphi}, \quad \vec{A}(\rho > a, \varphi, z) = \frac{2\pi}{c} nI\frac{a^2}{\rho}\hat{\varphi} \tag{4}$$

The reader can check their continuity at the boundary $\rho = a$, and that the curl of each one gives the correct values of the fields in Eqs. (1) and (2).

For the sake of completeness, we also identify the current distributions in the respective cylinders:

$$\vec{J}(\rho, \varphi, z) = \frac{I}{2\pi a}\delta(\rho - a)\hat{k} \tag{5}$$

and

$$\vec{J}(\rho, \varphi, z) = nI\delta(\rho - a)\hat{\varphi} \tag{6}$$

where the Dirac-δ function ensures that the current is restricted to the surface of the cylinder, and the factors preceding it correspond to the current per unit length. The reader can also check that the curl of the magnetic induction fields in Eqs. (1) and (2) are zero, in agreement with Ampère's law in its differential equation form for vanishing currents inside and outside the cylinders.

In this article, we study the magnetic induction fields, magnetic vector potentials and current distributions for infinitely long elliptical cylinders with longitudinal and solenoidal currents, respectively. The specific situations of a vanishing field inside the cylinder and a non-vanishing field outside for the first case, and of a non-vanishing field inside and a vanishing field outside for the second case are analyzed. From the experience of the circular cylinders, certain questions can be formulated for the elliptical cylinders in parallel to Eqs. (1)–(6):

(1) Are the magnetic lines elliptical and what is the magnitude of the magnetic induction field?

(2) Is the magnetic field uniform?

(3) What is the magnitude of the magnetic vector potential inside and outside?

(4) Are the lines of the magnetic vector potential elliptical and what is its magnitude inside and outside?

(5) and (6) Are the current distributions uniform and what are their values?

The readers can think about the answers to each of these questions.

In the following section, both situations are formulated using elliptical cylindrical coordinates and the answers to questions (1), (3) and (5) for the longitudinal case, and to (2), (4) and (6) for the solenoidal case are systematically and successively constructed through the application of the laws of magnetostatics. In the final section, we discuss those answers and compare them with their counterparts for the circular cylinder, underlining some points of didactic interest.

2 Magnetostatics in elliptical cylindrical coordinates

The elliptical cylindrical coordinates (u, v, z) are defined through their connections with the cartesian coordinates (ARFKEN, 1970)

$$x = f \cosh u \cos v, \quad y = f \sinh u \sin v, z = z \tag{7}$$

The points $(x = f, y = 0, z = 0)$and $(x = -f, y = 0, z = 0)$ are identified as foci separated by the focal distance $2f$. The locus defined by a fixed value of $u = u_0$, between zero and infinity, and $z = 0$, corresponds to an ellipse with a major semiaxis $f \cosh u_0$ along the x-axis and a minor semiaxis $f \sinh u_0$ along the y-axis; therefore, u is called the elliptical coordinate. The locus defined by a fixed value of $v = v_0$, between zero and 2π, and $z = 0$, corresponds to a hyperbola with a real semiaxis $f \cos v_0$ along the x-axis and an imaginary semiaxis $f \sin v_0$ along the y-axis; therefore, v is called the hyperbolic coordinate. The respective ellipses and hyperbolas can be displaced vertically as z takes on different values $-\infty \leq z_0 \leq \infty$, generating elliptical and hyperbolic cylinders, respectively.

From Eq. (7), the differential displacement can be evaluated, and the scale factors and unit vectors of the elliptical cylindrical coordinates can be identified:

$$d\vec{r} = \hat{i} dx + \hat{j} dy + \hat{k} dz = \hat{u} h_u du + \hat{v} h_u dv + \hat{k} dz \tag{8}$$

where

$$h_u(u,v) = h_v(u,v) = f\sqrt{cosh^2 u - \cos^2 v} \tag{9}$$

and

$$\hat{u} = \frac{\hat{i}\sinh u \cos v + \hat{j}\cosh u \sin v}{\sqrt{\cosh^2 u - \cos^2 v}}, \quad \hat{v} = \frac{-\hat{i}\cosh u \sin v + \hat{j}\sinh u \cos v}{\sqrt{\cosh^2 u - \cos^2 v}} \tag{10}$$

The reader can check the orthogonality of the vectors in Eq. (10), which is associated with the orthogonality of the confocal ellipses and hyperbolas discussed in the previous paragraph. These coordinates are the natural ones to describe the geometry of the elliptical cylinder $u = u_0$, and also the directions of the longitudinal $\hat{k}$, and solenoidal $\hat{v}$, currents.

The magnetic induction field can be constructed and analyzed on the basis of Ampere's law, in its circulation integral form

$$\oint_c \vec{B} \cdot d\vec{l} = \frac{4\pi}{c} I \tag{11}$$

or its boundary condition form

$$\hat{n} \times \left(\vec{B}_2 - \vec{B}_1\right) = \frac{4\pi}{c}\vec{K} \tag{12}$$

Since the currents are restricted to the surface of the cylinder, the Poisson equation satisfied by the magnetic vector potential

$$\nabla^2 \vec{A} = -\frac{4\pi}{c}\vec{J} \tag{13}$$

reduces to the Laplace equation

$$\nabla^2 \vec{A} = 0 \tag{14}$$

inside and outside the cylinder. Therefore, our method of solution consists in selecting the appropriate inner and outer vector harmonic fields solutions of Eqs. (14), and matching them at the elliptical cylinder boundary, so that the magnetic vector potential is continuous.

The Laplace equation for the infinitely long cylinder geometry does not have the axial dependence on z. The corresponding equation is bidimensional involving the Laplacian operator constructed with the use of Eq. (9):

$$\frac{1}{f^2([\cosh]^2 u - \cos^2 v)}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) f(u,v) = 0 \tag{15}$$

The first factor in Eq. (15) drops out, and the resulting equation admits separable solutions of the form

$$f(u,v) = U(u)V(v) \tag{16}$$

where each factor satisfies the respective ordinary differential equation

$$\frac{d^2U}{du^2} = AU \tag{17}$$

and

$$\frac{d^2V}{dv^2} = -AV \tag{18}$$

A being the separation constant. The solutions of Eqs. (17) and (18) are hyperbolic sine and cosine functions of $\sqrt{A}u$ and circular sine and cosine functions of $\sqrt{A}v$, respectively. The uniqueness of the function $V(v)$ for $v \to v + 2\pi$ requires that $A = m^2$ with $m = 1, 2, 3, \ldots$. In particular for $A = 0$, the solution of Eq. (17) is linear in u and the only admissible solution of Eq. (18) is a constant. Therefore, the most general solution of Eq. (15) is a linear combination of the separable solutions of the form of Eq. (16) constructed from these solutions of Eqs. (17) and (18):

$$f(u, v) = a_0 u + b_0 + \sum_{m=1}^{\infty} [a_m \sinh(mu) + b_m \cosh(mu)]\,[c_m \sin(mv) + d_m \cos(mv)] \tag{19}$$

2.1 Elliptical cylinder with longitudinal electrical current

Let us consider an electrical current along the longitudinal generatrices of the elliptical cylinder $u = u_0$, with the current density distribution

$$\vec{J}(u, v, z) = \frac{I}{2\pi h_v(u_0, v)} \frac{\delta(u - u_0)}{h_u(u_0, v)} \hat{k} \tag{20}$$

The reader may compare this expression with its counterpart of Eq. (5) for the circular cylinder with special attention to the following similarities and differences. The presence of the Dirac δ- function ensures that the current is restricted to the surface of the cylinder. The presence of the scale factors in the denominators of Eq. (20) is connected with the area element

$$d\vec{a}_z = h_u(u, v)\,du\,h_v(u, v)\,dv\,\hat{k} \tag{21}$$

which ensures that the integrated current in any cross section of the cylinder is

$$\int \vec{J}(u, v, z) \cdot d\vec{a}_z = I \tag{22}$$

It may be instructive for the reader to recognize that in Eq. (5) the scale factors are $h_\varphi = \rho = a$ and $h_\rho = 1$, and the area element is $d\rho\,\rho\,d\varphi$.

The first fraction in Eq. (20) gives the current per unit length

$$\vec{K}(u_0, v, z) = \frac{I\hat{k}}{2\pi f \sqrt{\cosh^2 u_0 - \cos^2 v}} \tag{23}$$

showing that the current is not uniformly distributed, being largest at the ends of the major axis $v = 0, \pi$ and smallest at the ends of the minor axis $v = \pi/2, 3\pi/2$, following the changes of curvature around the ellipse. Thus, Eq. (23) answers question (5).

Concerning the magnetic induction field produced by the above currents, let us assume that it vanishes inside the cylinder, and that its lines are elliptical outside. In other words, our tentative answers to question (1) are

$$\vec{B}(u < u_0, v, z) = 0, \quad \vec{B}(u > u_0, v, z) = B(u > u_0, v, z)\hat{v} \tag{24}$$

Then the remaining task is to test the validity of such answers and to determine the magnitude outside. This task is accomplished next in two steps based on Ampère's law in its boundary condition and circulation integral forms, respectively. In fact, at any point of the elliptical cylinder boundary $u = u_0$, the first form, Eq. (12), becomes

$$\hat{u} \times \vec{B}(u = u_0^+, v, z) = \frac{4\pi}{c}\vec{K}(u_0, v, z) \tag{25}$$

which through the use of Eqs. (24) and (23), and the orthonormality of the unit vectors $\hat{u}, \hat{v}, \hat{k}$, leads to

$$B(u = u_0^+, v, z) = \frac{2I}{c h_v(u_0, v)} \tag{26}$$

The extension of Eq. (26) valid at the boundary $u = u_0$, to any values of $u > u_0$ leads to the more explicit form of Eq. (24)

$$\vec{B}(u > u_0, v, z) = \frac{2I\hat{v}}{c h_v(u, v)} \tag{27}$$

Let us now evaluate the circulation of the magnetic induction field of Eq. (27) around any ellipse with $u = u_1 > u_0$:

$$\oint_0^{2\pi} \vec{B}(u = u_1, v, z) \cdot \hat{v} h_v(u_1, v)\, dv = \frac{4\pi I}{c} \tag{28}$$

consistent with Ampère's law, Eq. (11). The corresponding evaluation around any closed curve surrounding the cylinder $u = u_0$, using the general form of Eq. (8) for the line element, gives the same result as Eq. (28), because the contributions from the displacements along $\hat{u}$ and $\hat{k}$ vanish due to their orthogonality to $\vec{B}$. On the other hand, if the closed curve is in the interior of the cylinder, both sides of Eq. (28) vanish, because the field and the current are identically zero there. In conclusion, we have given affirmative answers to question (1) through Eqs. (24) and (27). They are the natural extensions of Eqs. (1), with the replacements $\hat{\varphi} \rightarrow \hat{u}$ and $h_\varphi \rightarrow h_v$ from the circular to the elliptical geometry.

The longitudinal currents of Eqs. (20) or (23) can be expected to produce a longitudinal magnetic vector potential. Of the harmonic functions of Eq. (19), the ones with $m = 0$ are compatible with the geometry under consideration. Thus, we consider explicitly

$$\vec{A}(u < u_0, v, z) = a_0\hat{k}, \quad \vec{A}(u > u_0, v, z) = b_0 u\hat{k} \tag{29}$$

since we expect a vanishing magnetic induction field inside the cylinder and a non-vanishing one outside, Eq. (23). The evaluation of the curl of the potential outside

$$\nabla \times \vec{A} = \frac{1}{h_u h_v} \begin{vmatrix} h_u \hat{u} & h_v \hat{v} & \hat{k} \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial z} \\ h_u A_u & h_v A_v & A_z \end{vmatrix} = \frac{\hat{v}}{h_u}\left(-\frac{\partial}{\partial u}\right)(b_0 u) = \frac{-b_0 \hat{v}}{h_u(u,v)} \tag{30}$$

leads to the magnetic induction field of Eqs. (27), with the identification of

$$b_0 = -\frac{2I}{c} \tag{31}$$

The continuity of the potential at $u = u_0$ also determines the value of a_0, so that Eq. (29) takes the specific forms:

$$\vec{A}(u < u_0, v, z) = -\frac{2I}{c} u_0 \hat{k}, \quad \vec{A}(u > u_0, v, z) = -\frac{2I}{c} u \hat{k} \tag{32}$$

which give the answer to question (3).

2.2 Elliptical cylindrical solenoid

Let us consider the electrical current along elliptical loops winding around the cylinder $u = u_0$, with n loops per unit length, so that the current density is

$$\vec{J}(u, v, z) = \frac{nI\delta(u - u_0)}{h_u(u, v)} \hat{v} \tag{33}$$

In this case, the area element is

$$d\vec{a}_v = \hat{v}\, dz\, h_u(u, v)\, du \tag{34}$$

and the integration of the current over a length h of the cylinder gives

$$\int \vec{J}(u, v, z) \cdot d\vec{a}_v = nI \int_0^h dz \int_0^\infty \delta(u - u_0) du = nhI \tag{35}$$

where nh is the number of loops in such a length. The corresponding current per unit length is

$$\vec{K}(u_0, v, z) = nI\hat{v} \tag{36}$$

which is uniformly distributed along the cylinder following the uniform winding. Equation (36) is the answer to question (6).

The magnetic induction field produced by the infinitely long solenoid with the uniformly distributed current of Eq. (36) is expected to be axial and uniform inside and vanishing outside:

$$\vec{B}(u < u_0, v, z) = B\hat{k}, \quad \vec{B}(u > u_0, v, z) = 0 \tag{37}$$

Its magnitude inside follows from the application of Eq. (12) for the field and current described by Eqs. (37) and (36), respectively

$$-\hat{u}\times\hat{k}B = \frac{4\pi}{c}nI\hat{v} \tag{38}$$

Therefore, the fields inside and outside the elliptical cylinder of Eqs. (37) are the same as those of its circular counterpart of Eq. (12), and question (2) is answered.

In order to construct the magnetic vector potential inside and outside the elliptical cylindrical solenoid and find the answers to question (4), we start out by examining some of the possibilities leading to the uniform magnetic induction field of Eq. (37). Two independent possibilities in cartesian coordinates are

$$\vec{A}_1 = -\hat{i}By, \quad \vec{A}_2 = \hat{j}Bx \tag{39}$$

as the reader can verify immediately by taking their respective curls. Another possibility is the half sum of $\vec{A}_1$ and $\vec{A}_2$,

$$\vec{A} = \frac{B\rho}{2}(-\hat{i}\sin\varphi + \hat{j}\cos\varphi) \tag{40}$$

where the unit vector in the parenthesis is identified as $\hat{\varphi}$, and coincides with Eq. (4) for the circular geometry. This suggests considering the general linear combination

$$\vec{A} = p\vec{A}_1 + q\vec{A}_2 = Bf(-\hat{i}p\sinh u\sin v + \hat{j}q\cosh u\cos v), \quad p+q=1 \tag{41}$$

where the cartesian coordinates have been written in terms of elliptical coordinates using Eqs. (7). By making reference to Eq. (19), it is recognized that it is the harmonic functions with $m=1$ that are involved.

From Eqs. (10) the cartesian unit vectors are given as linear combinations of the elliptical unit vectors

$$\hat{i} = \frac{\hat{u}\sinh u\cos v - \hat{v}\cosh u\sin v}{\sqrt{\cosh^2 u - \cos^2 v}}, \quad \hat{j} = \frac{\hat{u}\cosh u\sin v + \hat{v}\sinh u\cos v}{\sqrt{\cosh^2 u - \cos^2 v}} \tag{42}$$

Then Eq. (41) can be rewritten completely in terms of elliptical coordinates and unit vectors

$$\begin{aligned} \vec{A}(u<u_0, v, z) &= \frac{fB}{\sqrt{\cosh^2 u - \cos^2 v}}\Big\{\hat{u}\left[-p\sinh^2 u + q\cosh^2 u\right]\sin v\cos v \\ &\qquad + \hat{v}\sinh u\cosh u\left[p\sin^2 v + q\cos^2 v\right]\Big\} \\ &= \frac{Bf^2}{4h_u(u,v)}\Big\{\hat{u}[(p+q) + (-p+q)\cosh 2u]\sin 2v \\ &\qquad + \hat{v}\sinh 2u\,[(p+q) + (-p+q)\cos 2v]\Big\} \end{aligned} \tag{43}$$

The reader can check that the curl of the terms with the coefficient $(p+q)$ gives the constant magnetic field, while the curl of the terms with the coefficient $(-p+q)$

vanishes. Also the divergences of the respective combinations vanish. Notice also that the terms with the coefficient $(p+q)$ are not harmonic and the terms with the $(-p+q)$ coefficient are harmonic, with $m = 2$ in Eq. (19).

Since $(p+q)$ is one, Eq. (43) indicates that both components in the $\hat{u}$ and $\hat{v}$ directions have to be present. Consequently, the answer to the first part of question (4) is negative, *i. e.* , the lines of the magnetic vector potential are not elliptical in general. Additionally, the value of the coefficient $(-p+q)$ is still to be determined. This will be done by matching the magnetic vector potential of Eq. (43) at the boundary $u = u_0$ with the outside potential which is constructed next.

Since the magnetic induction field outside the solenoid vanishes, Eq. (37), the corresponding magnetic vector potential must be constructed with harmonic functions from Eq. (19), apart from the unit vectors and scale factor that it must have in common with Eq. (43). The boundary condition that the magnetic vector potential vanishes as $u \to \infty$ selects the negative exponential of mu in Eq. (19). Let us recall that in 2.1 the $m = 0$ harmonic composition of the magnetic potential of Eq. (32) leads to the magnetic induction field of Eq. (27), which shares the $m = 0$ harmonicity in the elliptical vector basis, and also has vanishing curl and divergence. These considerations are made in order to justify the inclusion of $m = 0$ and 2 harmonic functions from Eq. (19) in the magnetic vector potential outside the solenoid:

$$\vec{A}(u > u_0, v, z) = \frac{Bf^2}{4h_u(u,v)}\left[\hat{u} s e^{-2u} \sin 2v + \hat{v}\left(t - s e^{-2u} \cos 2v\right)\right] \tag{44}$$

The reader can check that the terms with the coefficient s are harmonic functions with $m = 2$ with vanishing curl and divergence. The term with the coefficient t is the $m = 0$ harmonic anticipated in the previous paragraph.

The matching of the magnetic vector potential from Eqs. (43) and (44) at the elliptical boundary $u = u_0$ by equating the coefficients of the linearly independent terms in $\hat{u}\sin 2v$, $\hat{v}$, and $\hat{v}\cos 2v$ give three equations

$$\begin{aligned} 1 + (-p+q)\cosh 2u_0 &= s e^{-2u_0} \\ \sinh 2u_0 &= t \\ (-p+q)\sinh 2u_0 &= -s e^{-2u_0} \end{aligned} \tag{45}$$

which determine the three coefficients

$$(-p+q) = -e^{-2u_0}, \quad s = t = \sinh 2u_0 \tag{46}$$

In conclusion, the magnetic vector potential inside and outside has the explicit forms of Eqs. (43) and (44)

$$\begin{aligned} \vec{A}(u < u_0, v, z) &= \frac{Bf^2}{4h_u(u,v)}\left[\hat{u}\left(1 - e^{-2u_0}\cosh 2u\right)\sin 2v\right. \\ &\quad \left. + \hat{v}\sinh 2u\left(1 - e^{-2u_0}\cos 2v\right)\right] \end{aligned} \tag{47}$$

$$\begin{aligned}\vec{A}(u>u_0,v,z) &= \frac{Bf^2}{4h_u(u,v)}\Big\{\hat{u}\sinh 2u_0 e^{-2u}\sin 2v \\ &\qquad + \hat{v}\sinh 2u\left[1-e^{-2u}\cos 2v\right]\Big\} \qquad (48)\end{aligned}$$

These equations complete the answer to question (4) and reiterate that the lines of the magnetic vector potential are not elliptical. Also, from Eqs. (41) and (46), the individual values of the coefficients

$$p = e^{-u_0}\cosh u_0 \quad \text{and} \quad q = e^{-u_0}\sinh u_0 \qquad (49)$$

are immediately obtained. Then the magnetic vector potential has the alternative forms of Eq. (47)

$$\vec{A}(u<u_0,v,z) = \frac{Bf}{2}e^{-u_0}\left(-\hat{i}\cosh u_0 \sinh u \sin v + \hat{j}\sinh u_0 \cosh u \cos v\right) \qquad (50)$$

and of Eq. (48)

$$\vec{A}(u>u_0,v,z) = \frac{Bf}{2}\sinh 2u_0\left(-\hat{i}e^{-u}\sin v + \hat{j}e^{-u}\cos v\right) \qquad (51)$$

in terms of the cartesian unit vectors. The reader can check that they match at $u = u_0$ and involve harmonic functions with $m = 1$.

As it is well known, the magnetic vector potential is not uniquely defined. For a given potential $\vec{A}$, it is always possible to add the gradient of any scalar function χ, so that the new potential $\vec{A} + \nabla\chi$ produces the same magnetic induction field $\vec{B} = \nabla \times \vec{A}$, because the curl of the gradient vanishes. This is the so-called gauge transformation invariance. In Eq. (48), the harmonic vector contributions

$$\frac{\hat{u}e^{-2u}\sin 2v - \hat{v}e^{-2u}\cos 2v}{h_v(u,v)} = \nabla\left(-\tfrac{1}{2}e^{-2u}\sin 2v\right)$$

can be written as a gradient. If we subtract such terms from the magnetic vector potential of Eqs. (47) and (48), we obtain

$$\begin{aligned}\vec{A}(u<u_0,v,z) &= \frac{Bf^2}{4h_u(u,v)}\Big\{\hat{u}\left[1-e^{-2u_0}\cosh 2u - \sinh 2u_0 e^{-2u}\right]\sin 2v \\ &\quad + \hat{v}\left[\sinh 2u(1-e^{-2u_0}\cos 2v) + \sinh 2u_0 e^{-2u}\cos 2v\right]\Big\} \qquad (52)\end{aligned}$$

and

$$\vec{A}(u>u_0,v,z) = \frac{Bf^2}{4h_u(u,v)}\sinh 2u_0 \hat{v} \qquad (53)$$

The magnetic potential inside the solenoid, Eq. (52), still has both $\hat{u}$ and $\hat{v}$ components and its lines are not elliptical, except the one at the boundary $u = u_0$. Both expressions match at the boundary, and Eq. (53) indicates that the lines of the magnetic potential are indeed elliptical. So after all, the first part of question (4) may have a positive answer outside the solenoid. The reader can recognize that the expressions in Eqs. (53) and (27) have the same harmonic vector composition.

In addition, the circulation of the magnetic potential of Eq. (53) around any ellipse with $u = u_1 > u_0$:

$$\int_0^{2\pi} \vec{A}(u = u_1, v, z) \cdot \hat{v} h_v(u_1, v) dv = \frac{B}{4} f^2 \sinh 2u_0 2\pi = B\pi \left(f \cosh u_0 \right)\left(f \sinh u_0 \right) \tag{54}$$

similar to Eq. (28), is identified as the magnetic flux through any elliptical cross-section of the solenoid. The same result is obtained if Eq. (48) is used instead of Eq. (53), and it is also extended to any closed curve surrounding the solenoid. The circulation of Eq. (52) or (47) around any closed curve inside the cylinder gives the magnetic flux through the surface limited by the curve.

3 Discussion

In the introduction, the familiar longitudinal and solenoidal electrical current distributions on infinitely long circular cylinders and their respective magnetic induction fields and vector potentials were recalled and used as references to formulate the corresponding problems for elliptical cylinders, including six specific questions. In the main section, both situations were analyzed systematically, identifying the longitudinal and solenoidal currents of Eqs. (23) and (36), constructing the respective magnetic induction fields of Eqs. (24, 27) and (36, 38) using Ampère's law in its boundary condition form, and constructing the corresponding magnetic vector potentials Eqs. (29–32) and (41–53) as harmonic functions subject to the conditions that they are continuous at the elliptical boundary and their curls reproduce the respective magnetic induction fields. The six questions were answered qualitatively and quantitatively, most of them in the affirmative; only question (4) has the answer that the lines of the magnetic vector potential may be elliptical outside the solenoid, Eq. (53), but not inside Eq. (52).

From Eqs. (7) and their geometrical interpretation it follows that in the asymptotic limit $u \to \infty$, the major and minor axis tend to become equal. In other words, the ellipses tend to become circles and the hyperbolic coordinate becomes the inclination of the corresponding asymptotes. Therefore, the lines of the magnetic induction field of Eq. (27) and of the magnetic vector potential of Eq. (53) tend to become circular far enough from the solenoid, being closer to their counterparts of Eqs. (1) and (4), respectively.

The general results for the elliptical cylinders reduce to those of the circular cylinders in the double limit $f \to 0, u \to \infty$, such that

$$f \cosh u \approx f \sinh u \approx \tfrac{1}{2} f e^u \to \rho \quad \text{and} \quad v \to \varphi$$

and also $h_u(u, v) \to \rho$. Then, except for an additive constant term, $u \to ln\rho$, and a constant factor, $e^{mu} \to \rho^m$. The reader can easily verify that Eqs. (27), (32), (52-53), (20) and (33) reduce to Eqs. (1), (3), (4), (5) and (6), respectively.

We complete the discussion of this section with two additional comments on each of the two problems analyzed in section II. For the elliptical cylinder with the longitudinal current, we consider, as a reference problem, the electrostatic situation in which the cylinder is kept at a fixed potential. The corresponding solution involves the electrostatic potential with the lowest harmonic $m = 0$ in Eq. (19):

$$\Phi(u < u_0, v, z) = V_0; \quad \Phi(u > u_0, v, z) = V_0 + V_0(u_0 - u) \tag{55}$$

such that the equipotentials are elliptical cylinders. Then the electric field intensity, evaluated as the negative gradient of the potential, becomes:

$$\vec{E}(u < u_0, v, z) = 0; \quad \vec{E}(u > u_0, v, z) = +\frac{V_0 \hat{u}}{h_u(u, v)} \tag{56}$$

such that the electric lines are hyperbolic. And the surface charge density on the cylinder follows from the application of Gauss's law:

$$\sigma(u = u_0, v, z) = \frac{\vec{E}(u = u_0^+, v, z) \cdot \hat{u}}{4\pi} = \frac{V_0}{4\pi h_u(u_0, v)} \tag{57}$$

Apart from their scalar or vector nature, the common space dependences of the sources Eqs. (57) and (23), the potentials Eqs. (55) and (32), and force fields Eqs. (56) and (24–27), can be recognized. The two situations can be connected with each other upon recognition that observers in an inertial frame of reference moving with a velocity $\vec{v} = -v\hat{k}$, relative to the cylinder at the fixed potential, will observe a linear density current

$$\vec{K}(u = u_0, v, z) = v\sigma = \frac{vV_0 \hat{k}}{4\pi h_u(u_0, v)} \tag{58}$$

a magnetic vector potential

$$\vec{A}(u < u_0, v, z) = -\frac{\vec{v}}{c}\Phi = \frac{vV_0}{c}\hat{k}; \quad \vec{A}(u > u_0, v, z) = \frac{vV_0 \hat{k}}{c}\left(1 + u_0 - u\right) \tag{59}$$

and a magnetic induction field

$$\vec{B}(u > u_0, v, z) = -\frac{\vec{v}}{c} \times \vec{E}(u > u_0, v, z) = \frac{vV_0 \hat{v}}{c h_u(u, v)} \tag{60}$$

which have the same directions and relative magnitudes as their counterparts of Eqs. (23), (32) and (27). They become identical for $vV_0 = 2I$.

On the other hand, our interest in the case of the elliptical cylindrical solenoid has its origin in the search for alternative manifestations of the Aharonov-Bohm effect [AHARANOV, 1959; CHAMBERS, 1960; KOUZNETSOV, 1999). This effect consists in physical changes in regions where the magnetic induction field vanishes but the magnetic vector potential is present, like a change in the electronic interference pattern for beams moving around a cylindrical solenoid, or a change in the eigenenergies and eigenstates of electrons confined to move inside annular cylindrical boxes. Such changes depend on the magnetic flux inside the solenoid as expressed by Eq. (54).

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