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On the s-Hamiltonian index of a graph

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ABSTRACT

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Yehong Shao

In modeling communication networks by graphs, the problem of designing s-fault-tolerant networks becomes the search for s-hamiltonian graphs. This thesis is a study of the s-hamiltonian index of a graph G.

A path P of G is called an **arc** in G if all the internal vertices of P are divalent vertices of G. We define $l(G) = \max\{m : G$ has an arc of length m that is not both of length 2 and in a $K_3\}$. We show that if a connected graph G is not a path, a cycle or $K_{1,3}$, then for a given s, we give the best known bound of the s-hamiltonian index of the graph.

Acknowledgements

I sincerely express my appreciation to my thesis advisor, Dr. Elaine M. Eschen, for her guidance, advice, and continual encouragement. It is a pleasure to work under her supervision.

I would also like to thank my other committee members: Dr. Hong-Jian Lai, and Dr. Frances L. Van Scoy, for their help during my studies.

DEDICATION

To

my father Jinfu , my mother Luolan

and

my sister Honglian

Contents

Chapter 1

Introduction

1.1 Background

Multiprocessors are widely used in recent years. Fault-tolerant multiprocessors are particularly useful in massive parallel systems. How to communicate among the processes running in parallel on the multiple processors becomes an important problem. The network that connects multiprocessors is called an interconnection network.

To design a reliable interconnection network, one expects that the network is fault-tolerant. There are two types of failures in a multiprocessor system, processor failure and link failure. It is important that the computer system still works when one or more processors fail.

We use a graph as a theoretic model to represent an underlying interconnection network. Terminology and notations not defined here can be found in [1]. Let $G = (V, E)$ be a graph. The vertex set $V(G)$ represents the set of processors and the edge set $E(G)$ represents the set of links between processors. A processor failure corresponds to the deletion of a vertex from a graph.

The topology of a network is the way the nodes and links are connected. Figure 1.1

shows two different topologies.

(a) a mesh topology (b) a ring topology

Figure 1.1: Example networks

In many network designs, it is desirable to maintain a fixed network topology. Reliability consideration expects that the network possesses the capability that after a small number of processor failures, the network can reconfigure to keep the same network topology.

A graph G is **hamiltonian** if there exists a cycle containing all the vertices of G.

One of the common network topologies uses a ring connection joining all the processors (see Figure 1.1(b)), which corresponds to a hamiltonian cycle of the graph G modeling such a network.

A network is a k**-fault-tolerant network** for a ring if for any k-processor-failure, the resulting network contains a ring including all of the non-faulty processors. This motivates the following definition.

A graph G is called s**-hamiltonian**, if the removal of any k vertices, $0 \leq k \leq s \leq |V(G)|$, results in a hamiltonian graph.

1.2 Definitions

We consider finite simple connected graphs only. For a graph G and a vertex $v \in V(G)$, define

$$N_G(v) = \{u \in V(G) : \ u \text{ is adjacent to } v \text{ in } G\}$$

and

$$E_G(v) = \{e \in E(G) : \ e \text{ is incident with } v \text{ in } G\}.$$

The **line graph** of a graph G, denoted by $L(G)$ or $L^1(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G have a common vertex. Notice that the vertex set in a line graph $L(G)$ corresponding to each $E_G(v)$ in G induces a complete graph. Denote each complete graph in $L(G)$ which corresponds to $E_G(v)$ in G by K_v. Then $\{E(K_v) : v \in V(G)\}$ is an edge partition of $L(G)$ and we say this is a **complete partition** of $L(G)$. For an integer $m \geq 1$, we define $L^m(G) = L(L^{m-1}(G))$ with $L^0(G) = G$.

In 1973, Chartrand [2] introduced the hamiltonian index of a connected graph G that is not a path to be the minimum number of applications of the line graph operator so that the resulting graph is hamiltonian. He showed that the hamiltonian index exists as a finite number. In 1983, Clark and Wormald [3] extended this idea of Chartrand and introduced the hamiltonian-like indices. Here we define the s-hamiltonian index.

Let $s \geq 0$ be an integer. The **s-hamiltonian index**, $h_s(G)$, of a connected graph G is the least nonnegative integer m such that $L^m(G)$ is s-hamiltonian. Note that when $s = 0$, a 0-hamiltonian graph is a hamiltonian graph and $h_0(G) = h(G)$ is the Hamilton index of a graph G.

Chapter 2

s-Hamilton Index

2.1 Introduction

In this chapter we prove our main result, stated in Theorem 2.1.1. This provides the best known bound for the s-hamiltonian index of a graph.

A nontrivial path P of G is called an **arc** in G if all the internal vertices of P are divalent vertices of G. We define $l(G) = \max\{m : G$ has an arc of length m that is not both of length 2 and in a $K_3\}$. Note that $l(G) \geq 1$.

Theorem 2.1.1 *Let G be a simple connected graph that is not a path, a cycle, or $K_{1,3}$ with $l(G) = l$. Then $h_s(G) \leq l + s + 1$.*

In the case that $s = 0$ Theorem 2.1.1 yields the theorem below, extends a former result by Lai.

Theorem 2.1.2 *(Lai, [5]) Let G be a simple connected graph that is not a path, a cycle, or $K_{1,3}$ with $l(G) = l$. Then $h(G) \leq l + 1$.*

2.2 Proof of the Main Theorem

Let $O(G)$ denote the set of all vertices in G with odd degree. A graph G is **Eulerian** if both $O(G) = \emptyset$ and G is connected. A spanning closed trail of G is called a **spanning Eulerian subgraph** of G. A subgraph H of G is **dominating** if $G - V(H)$ is edgeless. If a closed trail C of G satisfies $E(G - V(C)) = \emptyset$, then C is called a **dominating Eulerian subgraph**.

Theorem 2.2.1 reveals the relationship between a dominating Eulerian subgraph in H and a hamiltonian cycle in $L(H)$.

Theorem 2.2.1 *(Harary and Nash-Williams, [4]) Let H be a graph with $|E(H)| \geq 3$. The line graph $L(H)$ of a graph H is hamiltonian if and only if H has a dominating Eulerian subgraph.*

An edge cut X of G is **essential** if each side of $G - X$ has an edge.

For a graph G and a subset $X \subseteq E(G)$, the **contraction** G/X is the graph obtained from G by identifying the two end vertices of each edge in X and then deleting the edges in X. Note that loops and/or multiple edges may result from a contraction.

Lemma 2.2.2 *Let G be a connected graph and H an edge subset of G.*

(i) if H is an edge set consisting of loops of G and G/H has a spanning Eulerian subgraph, then G has a spanning Eulerian subgraph;

(ii) if H is a pair of parallel edges or the edge set of a C_3 and G/H has a spanning Eulerian subgraph, then G has a spanning Eulerian subgraph.

Proof (i) Let T be a spanning Eulerian subgraph of G/H. Then T or $T + H$ is a spanning Eulerian subgraph of G.

(ii) Let T be a spanning Eulerian subgraph of G/H.

Case 1 $H = \{e_1, e_2\}$ is an edge set of parallel edges of G (see Case 1 in Figure 2.1). Let v_1, v_2 be the two endpoints vertices of the edges in H and v_H the vertex in G/H onto which H is contracted. Let T' be the graph obtained from $T - v_H$ by adding vertices v_1 and v_2 with $N_{T'}(v_1) = N_G(v_1) \cap N_T(v_H)$ and $N_{T'}(v_2) = N_G(v_2) \cap N_T(v_H)$. Since $d_T(v_H)$ is even, $d_{T'}(v_1) + d_{T'}(v_2)$ is even. If $d_{T'}(v_1)$ and $d_{T'}(v_2)$ are both even, then $T' + \{e_1, e_2\}$ is a spanning Eulerian subgraph of G; if $d_{T'}(v_1)$ and $d_{T'}(v_2)$ are both odd, then $T' + e_1$ is a spanning Eulerian subgraph of G.

Case 2 $H = \{e_1, e_2, e_3\}$ is the edge set of a C_3 in G. Let v_1, v_2, v_3 be the three endpoints of the edges in H (see Case 2 in Figure 2.1) and v_H the vertex in G/H onto which H is contracted. Let T' be defined as in Case 7. Since $d_T(v_H)$ is even, $d_{T'}(v_1)+d_{T'}(v_2)+d_{T'}(v_3)$ is even. If $d_{T'}(v_1)$, $d_{T'}(v_2)$ and $d_{T'}(v_3)$ are all even, then $T + H$ is a spanning Eulerian subgraph of G; if two of them are odd and we assume without loss of generality that $d_{T'}(v_1)$ and $d_{T'}(v_2)$ are odd, then $T + \{e_1, e_2\}$ is a spanning Eulerian subgraph of G. □

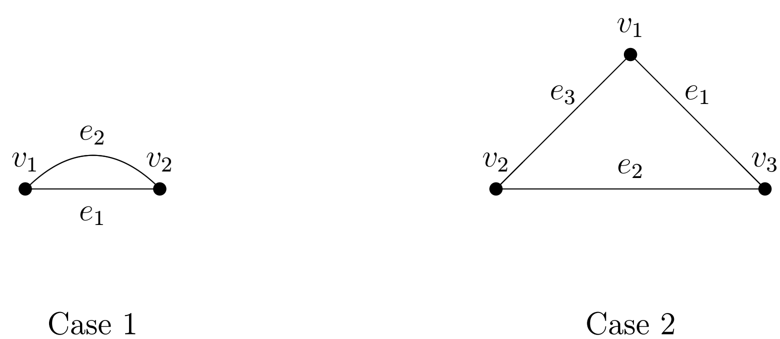

Figure 2.1: The edge set H of Lemma 2.2.2

Lemma 2.2.3 *Let G be a connected graph without essential edges cuts of size 1 and G_1 the graph obtained by contracting all the triangles and multiple edges repeatedly from G. If $|V(G_1)| \leq 4$, then G has a dominating Eulerian subgraph.*

Proof By the assumptions, G_1 is simple, connected and has no 3-cycles.

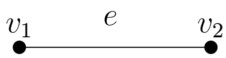

Figure 2.2.1

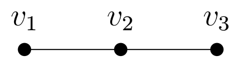

Figure 2.2.2

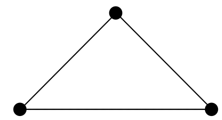

Figure 2.2.3

Figure 2.2: G_1

Case 1 $|V(G_1)| = 1$. By Lemma 2.2.2, G has a spanning Eulerian subgraph.

Case 2 $|V(G_1)| = 2$ (See Figure 2.2.1). Since G has no essential edge cuts of size 1, one of the vertices is a vertex of G. We assume that it is v_1. We delete v_1 and the resulting graph is K_1. By Lemma 2.2.2, $G - v_1$ has a spanning Eulerian subgraph T, and so T is a dominating Eulerian subgraph of G.

Case 3 $|V(G_1)| = 3$. G_1 could be the graph in Figure 2.2.2 or 2.2.3 since G_1 is connected and simple. Since G_1 has no 3-cycles, G_1 must be the graph in Figure 2.2.2. Then since G has no essential edge cuts of size 1, $v_1, v_3 \in V(G)$. And using an argument similar to that in Case 2, we conclude that G has a dominating Eulerian subgraph.

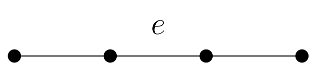

Figure 2.3.1

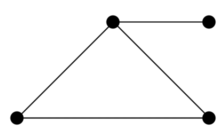

Figure 2.3.2

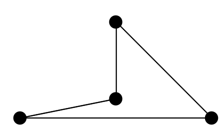

Figure 2.3.3

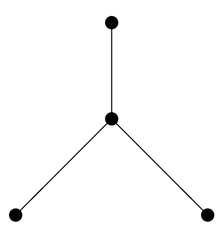

Figure 2.3.4

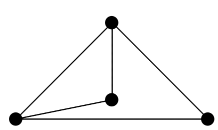

Figure 2.3.5

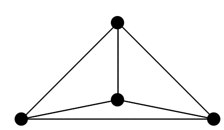

Figure 2.3.6

Figure 2.3: G_1

Case 4 $|V(G_1)| = 4$ (see Figure 2.3).

In Figure 2.3.1, G has an essential cut edge e, a contradiction; In Figure 2.3.3, by Lemma 2.2.2, G has a spanning Eulerian subgraph. Since G_1 has no 3-cycles, G_1 cannot be the graph illustrated in Figure 2.3.2, 2.3.5, 2.3.6. For Figure 2.3.4, we can use an argument similar to that used in Case 2 and 3 to conclude that G has a dominating Eulerian subgraph. □

A graph G is k-**triangular** if each edge of G is in at least k triangles and G is **triangular** if it is 1-triangular.

Lemma 2.2.4 *Let G be a simple connected graph that is not a path, a cycle or $K_{1,3}$, with $l(G) = l$. Then each of the following holds:*

(i). For an integer m, $l \geq m \geq 0$,

$$l(L^m(G)) = \begin{cases} l - m & : \text{if } 0 \leq m < l \\ 1 & : \text{if } m \geq l \end{cases}$$

(ii). For integer $s \geq 0$,

$$\delta(L^{l+s}(G)) \geq \begin{cases} 2 & : \text{if } s = 0 \text{ or } s = 1 \\ 2^{s-2} + 2 & : \text{if } s \geq 2 \end{cases}$$

(iii). $L^l(G)$, $L^{l+1}(G)$ and $L^{l+2}(G)$ are triangular and $L^{l+s}(G)$ is 2^{s-3}-triangular when $s \geq 3$;

(iv). For an integer $s \geq 0$, $\kappa(L^{l+s}(G)) \geq s + 1$.

Proof. (i). **Case 1** $l(G) = 1$.

By the definition of an arc, $l(G) = 1$ if and only if one of the following holds

(A) $\delta(G) \geq 3$;

(B) $\delta(G) \leq 2$ and every vertex of degree 2 is contained in a triangle.

If (A) holds, then $\delta(L^m(G)) \geq 3$ and so $l(L^m(G)) = 1$ for any $m \geq 0$. Hence, we assume that (B) holds. By way of contradiction we suppose that $l(L(G)) \geq 2$. Let $P_0 = v_0v_1 \cdots v_l$ be an arc of length of $l(L(G)) \geq 2$. So $d_{L(G)}(v_1) = d_{L(G)}(v_2) = \cdots = d_{L(G)}(v_{1-1}) = 2$. By the definition of an arc, $v_0v_1v_2$ is an induced path of length 2 in $L(G)$, i.e., $v_0v_1, v_1v_2 \in E(L(G))$, but $v_0v_2 \notin E(G)$. Assume that $e_{v_0}, e_{v_1}, e_{v_2}$ are edges in G corresponding to v_0, v_1, v_2 in $L(G)$, respectively. So we have a path $e_{v_0}, e_{v_1}, e_{v_2}$ of length 3 in G whose internal vertices are of degree 2. Hence, $l(G) \geq 3$, contrary to $l(G) = 1$.

Case 2 $l(G) \geq 2$.

Let $P_0 = v_0v_1 \cdots v_l$ be an arc of length of $l(G) \geq 2$. By the definition of an arc, $d_G(v_i) = 2$ for $i = 1, 2, \cdots, l-1$. Let $u_1, u_2, \cdots, u_l$ be the vertices in $L(G)$ corresponding to the edges $v_0v_1, v_1v_2, \cdots, v_{l-1}v_l$ in G respectively. Then $u_1u_2 \cdots u_l$ is an arc in $L(G)$ with length $l-1$. So $l(L(G)) = l-1$. Inductively, we have $l(L^m(G)) = l-m$ when $0 \leq m < l$.

In particular, $l(L^m(G)) = 1$ when $m \geq l$ by Case 1 since $l(L^{l-1}(G)) = 1$.

(ii). First we prove that $\delta(L^l(G)) \geq 2$. We assume by way of contradiction that there exists a vertex v of degree 1 in $L^l(G)$. So the edges corresponding to v and its only adjacent vertex induce a path of length 2 with an internal vertex of degree 2 in $L^{l-1}(G)$, which is an arc of length 2, contrary to the fact that $l(L^{l-1}(G)) = 1$. Hence, $\delta(L^l(G)) \geq 2$. For any graph G that is neither a path nor a cycle, the sequence $\delta(L^i(G)), i = i, 2, \cdots$, is nondecreasing. So we also have $\delta(L^{l+1}(G)) \geq 2$.

Next we prove that $\delta(L^{l+2}(G)) \geq 3$.

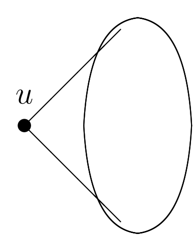

Figure 2.4.1: $L^{l+2}(G)$

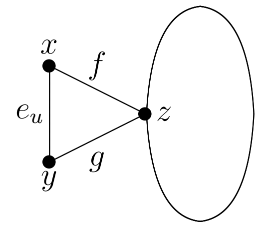

Figure 2.4.2: $L^{l+1}(G)$

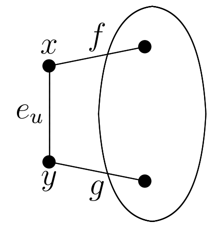

Figure 2.4.3: $L^{l+1}(G)$

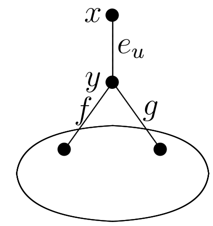

Figure 2.4.4: $L^{l+1}(G)$

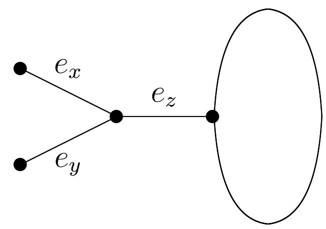

Figure 2.4.5: $L^{l}(G)$

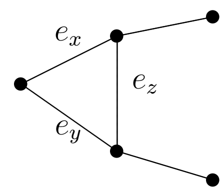

Figure 2.4.6: $L^{l}(G)$

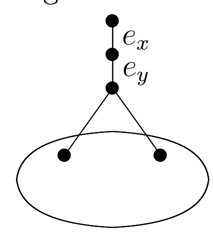

Figure 2.4.7: $L^{l}(G)$

Assume that there exists a vertex u in $L^{l+2}(G)$ of degree 2 (see Figure 2.4.1), then the corresponding edge $e_u = xy$ of u in $L^{l+1}(G)$ is incident with 2 edges. That means e_u is either a pendent edge (see Figure 2.4.4) or an edge with both ends x, y of degree 2 (see Figure 2.4.2 and 2.4.3). The graph $L^{l+1}(G)$ in Figure 2.4.4 corresponds to $L^{l}(G)$ in Figure 2.4.7, contrary to (i) that $l(L^{l})(G) = 1$. So we can assume that the end vertices of e_u both have degree 2. Then $E_{L^{l+1}(G)}(x) = \{e_u, f\}$ and $E_{L^{l+1}(G)}(y) = \{e_u, g\}$. And it is easy to see that $\{f, e_u, g\}$ forms an arc of length 3 if f and g have no common vertices (see Figure 2.4.3), contrary to the fact that $L^{l+1}(G)$ has no arcs of length greater than 1 by (i). So $\{f, e_u, g\}$ forms a triangle in $L^{l+1}(G)$ (see Figure 2.4.2). Assume that e_x, e_y, e_z are edges in $L^{l}(G)$ corresponding to the vertices x, y, z in $L^{l+1}(G)$. The graph induced by $\{x, y, z\}$ corresponds to a $K_{1,3}$ (see Figure 2.4.5) or C_3 (see Figure 2.4.6). First consider the case of Figure 2.4.6. Since G is not a C_3 nor a $K_{1,3}$, $L^{l}(G)$ can not be a C_3. So e_z is adjacent to at least one more edge other than e_x, e_y, which is contrary to the fact that e_x and e_y can only be adjacent to e_z. Since $l \geq 1$, $L^{l}(G)$ is a line graph and so it is claw free, which excludes the graph in Figure 2.4.5.

So we have $\delta(L^{l+2}(G)) \geq 3$.

Define $a_1 = 3$, $a_2 = 4$. Since $\delta(L^{l+2}(G)) \geq 3 = a_1$, every edge in $L^{l+2}(G)$ is adjacent to at least $4 = 2(3-1) = a_2$ edges and so $\delta(L^{l+3}(G)) \geq 4 = a_2$. Inductively, suppose that every edge in $L^{l+s-1}(G)$ is adjacent to at least $a_{s-1} = 2a_{s-2} - 2$ edges. Then $\delta(L^{l+s}(G)) \geq 2^{s-2} + 2$, as desired.

(iii). First we prove that $L^l(G)$ is triangular; that is, for any $e = xy \in E(L^l(G))$, e lies in at least one triangle. If e is a pendent edge in $L^l(G)$ (see Figure 2.5.2), then $\delta(L^l(G)) = 1 < 2$, contrary to (ii). And since $l(L^l(G)) = 1$, if one of $d_{L^l(G)}(x)$ and $d_{L^l(G)}(y)$ is 2, then e must lie in a triangle. So we can assume that $d_{L^l(G)}(x) \geq 3$ and $d_{L^l(G)}(y) \geq 3$ (see Figure 2.5.1). Since $xy \in E(L^l(G))$, the corresponding edges e_x and e_y in $L^{l-1}(G)$ share a common vertex v in $L^{l-1}(G)$. So the corresponding $L^{l-1}(G)$ of $L^l(G)$ is a graph in Figure 2.5.3 or Figure 2.5.4 by $l(L^{l-1}(G)) = 1$. Then e lies in a triangle in $L^l(G)$.

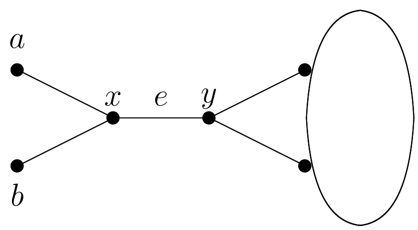

Figure 2.5.1: $L^l(G)$

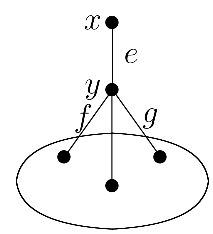

Figure 2.5.2: $L^l(G)$

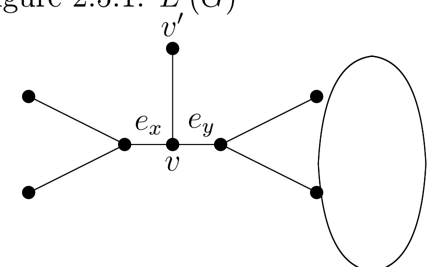

Figure 2.5.3: $L^{l-1}(G)$

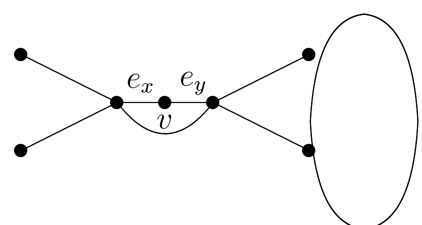

Figure 2.5.4: $L^{l-1}(G)$

Figure 2.5

Let $xy \in E(L^{l+1}))$. Then e_x and e_y share a common vertex in $L^l(G)$. Since e_x lies in a triangle, there exists an edge e_a that is incident with both e_x and e_y. Hence, $L^{l+1}(G)$ is triangular. Similarly, $L^{l+2}(G)$ is triangular. And so $L^{l+1}(G)$ and $L^{l+2}(G)$ are both triangular.

If $s \geq 3$, by (ii), $\delta(L^{l+s-1}(G)) \geq 2^{s-3} + 2$, so the incident edges of each vertex form a complete graph with order at least $2^{s-3}+2$ in $L^{l+s}(G)$. Then each edge of $L^{l+s}(G)$ lies in at least 2^{s-3} triangles, that is, $L^{l+s}(G)$ is 2^{s-3}-triangular since $L^{l+s}(G)$ is the edge-disjoint union of complete graphs.

(iv) Since G is connected, $L^l(G)$ is connected, i.e., $\kappa(L^l(G)) \geq 1$. So we now assume $s \geq 1$.

Notice that, for an integer $k \geq 0$, a non-complete line graph $L(H)$ has no vertex cut of size less than k if and only if H has no essential edge cut of size less than k. Next we prove that $\kappa(L^{l+s}(G)) \geq s+1$ by showing that $L^{l+s-1}(G)$ has no essential edge cut of size less than $s+1$. Suppose that $L^{l+s-1}(G)$ has an essential edge cut X of size less than $s+1$ and $L^{l+s-1}(G) - X$ has two nontrivial components C_1 and C_2. Since $L^{l+k}(G)$ is connected and triangular, $|X| \geq 2$.

Since $|X| \geq 2$, $L^l(G)$ has no essential edge cuts of size 1, and so $\kappa(L^{l+1}(G)) \geq 2$.

If $k = 2$ and $|X| = 2$, then there must exist a vertex v in C_1 or C_2 such that $X \subseteq E_{L^{l+1}(G)}(v)$. Since X is essential, v is incident with one more edge except X. Then v is a cut vertex of $L^{l+1}(G)$, a contradiction. So $|X| \geq 3$, that is, $\kappa(L^{l+2}(G)) \geq 3$.

Next we prove that $\kappa(L^{l+s}(G)) \geq s+1$ when $s \geq 3$ by induction. Assume that $\kappa(L^{l+s-1}(G)) \geq s$ and we consider the graph $L^{l+s-1}(G)$. Since each edge lies in at least one triangle, at least two edges of X are incident with the same vertex. So we can assume without loss of generality that X has at most $s-1$ vertices in C_1 and denote the set of these vertices by Y. Since $|X| \leq s$, at least one vertex y of Y is incident with exactly one edge of X. Since $\delta(L^{l+s-1}(G)) \geq 2^{s-3}+2 \geq s$ when $s \geq 3$, $d_{L^{l+s-1}(G)}(v) \leq s-2+1 = s-1$, a contradiction. Hence $E(C_1 - Y) \neq \emptyset$. So Y is a $(s-1)$-cut of $L^{l+s-1}(G)$, contrary to the induction hypothesis.

Therefore $L^{l+s}(G)$ has no essential edge cuts of size less than $s+2$ and so $\kappa(L^{l+1+s}(G)) \geq s+2$. □

Lemma 2.2.5 *Let G be a simple connected graph that is neither a path nor a cycle with $l(G) = l$. Then for any $S' \subseteq E(L^{l+s}(G))$ with $|S'| \leq s$, $L^{l+s}(G) - S'$ has a dominating Eulerian subgraph.*

Proof By Lemma 2.2.4(iii), $L^{l+s}(G)$ is $(2^{s-3} \geq 8 \geq s+1)$-triangular when $s \geq 6$. In this case, every edge of $L^{l+s}(G) - S'$ lies in at least one triangle since $|S'| = s$. By Lemma 2.2.3, $L^{l+s}(G) - S'$ has a spanning Eulerian trail since we can contract all the triangles to get a K_1. Next we consider the cases when $s = 0, 1, 2, 3, 4, 5$.

Since $L^{l+s}(G)$ is an edge-disjoint union of complete graphs, we can assume that $\{E_1, E_2, \cdots, E_n\}$ is a complete edge partition of $L^{l+s}(G)$ where E_i is a complete graph for $1 \leq i \leq n$. Consider $L^{l+s}(G) - S'$ and notice that deleting some edges in $L^{l+s}(G)$ may result in some of the edges in $L^{l+s}(G) - S'$ not lying in any triangles. For any complete E_i with order t, if $E_i \cap S' = \emptyset$, then E_i is still triangular; if $E_i \cap S' \neq \emptyset$ and $E_i - S'$ is not triangular, then

$$|E_i \cap S'| \geq t - 2. \tag{2.1}$$

Let G_1 be the graph obtained by contracting all the triangles and multiple edges repeatedly from $L^{l+s}(G) - S'$. From the proof of Lemma 2.2.4 (iv), $L^{l+s}(G)$ has no essential edge cuts of size less than $s+2$. Hence $L^{l+s}(G) - S'$ has no essential edge cuts of size less than 2. By Lemma 2.2.3, it suffices to show that $|V(G_1)| \leq 4$ in each of the cases below, we shall show that $|V(G)| \leq 4$.

Case 1 $s = 0$. Since $L^l(G)$ is triangular, then $G_1 = K_1$.

Case 2 $s = 1$. Let $S' = \{e\}$ and consider $L^{l+1}(G) - S'$. By (1), the only possibility of making some $K_t - S'$ not triangular is that $S' \subseteq K_3$. We can assume that $E_1 = K_3$ and by (1), for any $e \in E_2 \cup E_3 \cdots \cup E_n$, e lies in at least one triangle of $L^{l+1}(G) - S'$. That means the induced graph of $E_2 \cup E_3 \cdots \cup E_n$ is triangular. Since $\kappa(L^{l+1}(G)) \geq 1 + 1 = 2$, $E_1 = K_3$ shares at least two vertices with other E_i's. Thus G_1 has at most 2 vertices left.

Case 3 $s = 2$. Let $S' = \{e_1, e_2\}$ and consider $L^{l+2}(G) - S'$. By (1), the possibilities of making some $K_t - S'$ not triangular are listed below (see Table 1).

Table 1

	e_1	e_2	$\lvert V(G_1)\rvert \leq$	$\lvert E(G_1)\rvert \leq$
Case 2.1	K_4	K_4	4	2
Case 2.2	K_3	K_3	3	1
Case 2.3	K_3	K_3'	5	4

In Table 1, Case 2.1 is the case when e_1, e_2 are contained in the same K_4; Case 2.2 is the case when they are both contained in the same K_3; Case 2.3 is the case when one of them is contained in a K_3 and the other is contained in a different K_3.

Let X be the union of the complete graphs each of which contains some edges of S'. So we have that $X = K_3$ (Case 2.2), $X = K_4$ (Case 2.1) or $X = K_3 \cup K_3'$ (Case 2.3) and for any $e \in G - X$, e lies in at least one triangle of $L^{l+2}(G) - S'$. Since $\kappa(L^{l+2}(G)) \geq 2+1 = 3$, X shares at least three vertices with other E_i's.

If $X = K_3$, then G_1 has at most 1 vertex left; if $X = K_4$, then K_4 has at most two vertices left; if $X = K_3 \cup K_3'$, then G_1 has at most 4 vertices left.

Case 4 $s = 3$. Let $S' = \{e_1, e_2, e_3\}$ and consider $L^{l+3}(G) - S'$. By Lemma 2.2.4(ii), $\delta(L^{l+2}(G)) \geq 3$. So each E_i is an edge-disjoint union of complete graphs with orders at least 3. By (1), the possibilities of making some $K_n - S'$ not triangular are that either S' is contained in some K_5, or at least 2 edges of S' are contained in some K_4, or at least 1 edge of S' is contained in some K_3.

Suppose that at least 2 edges of S' are contained in some K_4, or at least 1 edge of S' is contained in some K_3. Then since edges of $L^{l+3}(G)$ not in these complete graphs are lying in triangles disjoint from these complete graphs, and since $\kappa(L^{l+3}(G)) \geq 3+1 = 4$, the contraction of all the triangles of $L^{l+s}(G) - S'$ will result in a graph G_1 with at most 4 vertices.

Now suppose that a K_5 contains all 3 edges in S'. Since $\kappa(L^{l+3}(G)) \geq 3+1 = 4$, this K_5 shares at least four vertices with other E_i's. And so G_1 has at most 2 vertices left.

Case 5 $s = 4$. Let $S' = \{e_1, e_2, e_3, e_4\}$ and consider $L^{l+4}(G) - S'$. By Lemma 2.2.4(ii), $\delta(L^{l+3}(G)) \geq 4$. So each E_i is an edge-disjoint union of complete graphs with orders at least 4. By (1), the possibilities of making some $K_n - S'$ not triangular are that either S'

is contained in some K_6, or at least 3 edges of S' are contained in some K_5, or at least 2 edges of S' are contained in some K_4.

Suppose that at least 3 edges of S' are contained in some K_5, or at least 2 edge of S' is contained in some K_4. Then since edges of $L^{l+4}(G)$ not in these complete graphs are lying in triangles disjoint from these complete graphs, and since $\kappa(L^{l+4}(G)) \geq 5$, the contraction of all the triangles of $L^{l+s}(G) - S'$ will result in a graph G_1 with at most 4 vertices.

Now suppose that a K_6 contains all 4 edges in S'. Since $\kappa(L^{l+4}(G)) \geq 4 + 1 = 5$, this K_6 shares at least five vertices with other E_i's. And so G_1 has at most 2 vertices left.

Case 6 $s = 5$. Let $S' = \{e_1, e_2, e_3, e_4, e_5\}$ and consider $L^{l+5}(G) - S'$. By Lemma 2.2.4(ii), $\delta(L^{l+3}(G)) \geq 6$. So each E_i is an edge-disjoint union of complete graphs with orders at least 6. By (1), the possibilities of making some $K_n - S'$ not triangular are that either S' is contained in some K_7 or 4 edges of S' are contained in some K_6. Since $\kappa(L^{l+5}(G)) \geq 5 + 1 = 6$, this concerned K_7 or K_6 shares at least six vertices with other E_i's. And so G_1 has at most 2 vertices left. □

Proof of Theorem 2.1.1 Let S be a vertex set of $L^{l+1+s}(G)$ with $|S| = s$. Let S' be the edge set of $L^{l+s}(G)$ corresponding to S in $L^{l+1+s}(G)$. By Lemma 2.2.4(iv), $\kappa(L^{l+1+s}(G)) \geq s + 2$. Then $\kappa(L^{l+1+s}(G) - S) \geq 2$ and so $L^{l+s}(G) - S'$ has no essential edge cuts of size 1, or equivalently, all the edge cuts of size 1 of $L^{l+s}(G) - S'$ are pendent edges. And since $\kappa'(L^{l+s}(G)) \geq \kappa(L^{l+s}(G)) \geq s + 1$ and $|S'| = s$, $\kappa'(L^{l+s}(G) - S') \geq 1$. So $L^{l+s}(G) - S'$ is connected. By Lemma 2.2.5, $L^{l+s}(G) - S'$ has a dominating Eulerian subgraph. Since deleting an edge in $L^{l+s}(G)$ will not affect the adjacency relationship between any two edges in $E(L^{l+s}(G)) - S'$, the adjacency relationship between any two corresponding vertices in $V(L^{l+s+1}(G)) - S$. So the induced graph of $V(L^{l+s+1}(G)) - S$ in $L^{l+s+1}(G)$ is the line graph of the induced graph of $E(L^{l+s}(G)) - S'$ in $L^{l+s}(G)$. Then by Theorem 2.2.1, $L^{l+1+s}(G) - S$ is hamiltonian . And so $L^{l+1+s}(G)$ is s-hamiltonian. □

From Theorem 2.1.1, we know the s-Hamilton index of a graph G $h_s(G)$ is at most $l + s + 1$. It is natural to ask what kind of graphs have $h_s(G) = l + s + 1$.

Question 1 Characterize the graph G with $h_s(G) = l + s + 1$.

Question 2 If $h_s(G) < l + s + 1$, is the graph $L^{h_s+k}(G)$ is also s-hamiltonian for any integer $k \geq 1$?

Question 3 Is this line graph model an optimal model for s-fault-tolerant network?

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