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Edge coloring of simple graphs and edge -face coloring of simple plane graphs

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by

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Abstract

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Let G be a simple graph embedded in the surface Σ of Euler characteristic $\chi(\Sigma) \geq 0$. Denote by $\chi_e(G)$, Δ, and g the edge chromatic number, the maximum degree and the girth of the graph G, respectively. We prove that $\chi_e(G) = \Delta$ if $\Delta \geq 5$ and $g \geq 4$, or $\Delta \geq 4$ and $g \geq 5$, or $\Delta \geq 3$ and $g \geq 9$. In addition, if $\chi(\Sigma) > 0$, then $\chi_e(G) = \Delta$ if $\Delta \geq 3$ and $g \geq 8$.

Let G be a simple graph with the average degree $\overline{d}$ and the maximum degree Δ. It is proved that G is not critical if $\overline{d} \leq 6$ and $\Delta \geq 8$, or $\overline{d} \leq \frac{20}{3}$ and $\Delta \geq 9$. This result generalizes earlier results of Vizing[18], Mel'nikov[11], Hind and Zhao[6], and Yan and Zhao[21]. It also improves a result by Fiorini[5] on the number of edges of critical graphs for $8 \leq \Delta \leq 12$.

Given a simple plane graph G, an edge-face k-coloring of G is a function $\phi : E(G) \cup F(G) \mapsto \{1, \cdots, k\}$ such that, for any two adjacent elements a, $b \in E(G) \cup F(G)$, $\phi(a) \neq \phi(b)$. Denote $\chi_e(G), \chi_{ef}(G), \Delta(G)$ the edge chromatic number, the edge-face chromatic number and the maximum degree of G, respectively. We prove that $\chi_{ef}(G) = \chi_e(G) = \Delta(G)$ for any 2-connected simple plane graph G with $\Delta(G) \geq 24$.

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Chapter 1

Introduction

1.1 Notations and Definitions

Let $G = (V, E)$ be a finite and simple graph where V is the vertex set of G and E is the edge set of G. We denote $\delta(G)$, $g(G)$, $\Delta(G)$ the minimum degree, the girth, and the maximum degree of G, respectively. A k-vertex (or $(\geq k)$-vertex, $(\leq k)$-vertex, respectively) is a vertex of degree k (or $\geq k$, $\leq k$, respectively). The *girth* of a graph is the length of the shortest cycle in the graph. Denote V_k the set of $k-$vertices in G.

A graph is *k-edge colorable* if its edges can be colored with k colors in such a way that adjacent edges receive different colors. The *edge chromatic number*, denoted by $\chi_e(G)$, of a graph G is the smallest integer k such that G is k-edge colorable. A simple graph G is *class one* if it is Δ-edge colorable, where Δ is the maximum degree of G. Otherwise, Vizing's Theorem [17] guarantees that it is $(\Delta + 1)$-edge colorable, in which case it is said to be *class two*.

A *critical graph* G is a connected graph such that G is class two and $G - e$ is class one for any edge e of G.

Let $L(e)$ be a color set assigned to an edge $e \in E(G)$. G is *L-edge colorable* if each edge e can be colored with a color from $L(e)$ such that adjacent edges receive different colors. In particular, if for each edge $e \in E(G)$ and any $L(e)$ with $|L(e)| = k$, G is L-edge colorable, we say that G is k-edge choosable. The *edge-list chromatic number* of G, denoted by $\chi_{elist}(G)$, is the smallest integer k such that G is k-edge choosable.

If G is a graph embedded in a surface, we denote $F(G)$ the face set of G.

For a plane graph G and $f \in F(G)$, let $B(f)$ be the *boundary* of the face f. An *element* of G is a member of $E(G) \cup F(G)$. Any two elements are *adjacent* if they are either adjacent to or incident with each other in the traditional sense. An *edge-face k-coloring* of the plane graph G is a function $\phi : E(G) \cup F(G) \mapsto \{1, \cdots, k\}$ such that for any two adjacent elements $a,\ b \in E(G) \cup F(G)$, $\phi(a) \neq \phi(b)$. Denote $\chi_{ef}(G)$ the *edge-face chromatic number* of G, i.e., the smallest integer k such that G has an edge-face k-coloring.

1.2 Adjacency Lemmas

Lemma 1.2.1 *(Vizing's Adjacent Lemma [18]) If H is a critical graph with the maximum degree Δ, that is, $\chi_e(H) = \Delta + 1$ and $\chi_e(H - e) = \Delta$ for every edge $e \in E(H)$, and if u and v are adjacent vertices of H, where the degree of v is d, then,*

(i) if $d < \Delta$, then u is adjacent to at least $\Delta - d + 1$ vertices of degree Δ, and,

(ii) if $d = \Delta$, then u is adjacent to at least two vertices of degree Δ.

From Vizing's Adjacent Lemma, we can easily get the following corollary.

Corollary 1.2.2 *Let H be a critical graph with maximum degree Δ. Then*

(1) every vertex is adjacent to at most one 2-vertex and at least two Δ-vertices;

(2) the sum of the degree of any two adjacent vertices is at least $\Delta + 2$;

(3) if a vertex is adjacent to a 2-vertex, then the rest of its neighbors are Δ-vertices.

Let $x \in V(G)$. Denote $N(x)$ the set of vertices adjacent to x. For $V' \subseteq V(G)$, denote $N(V') = \cup_{x \in V'} N(x)$.

Lemma 1.2.3 *(Limin Zhang [25]) Let G be a critical graph with the maximum degree Δ, $xy \in E(G)$ and $d(x) + d(y) = \Delta + 2$. The following hold:*

(1) every vertex of $N(\{x, y\}) \setminus \{x, y\}$ is a Δ-vertex;

(2) every vertex of $N(N(\{x, y\})) \setminus \{x, y\}$ is of degree at least $\Delta - 1$;

(3) if $d(x),\ d(y) < \Delta$, then every vertex of $N(N(\{x, y\})) \setminus \{x, y\}$ is a Δ-vertex.

Chapter 2

Edge Coloring of Embedded Graphs with Large Girth

2.1 Introduction

In 1965, Vizing [18] proposed the following conjecture:

Conjecture 2.1.1 *(Vizing planar graph conjecture) Every planar graph with maximum degree at least 6 is class one.*

And he himself proved a partial result of Conjecture 2.1.1 as follows.

Theorem 2.1.2 *(Vizing [18] 1965) Every simple planar graph with the maximum degree at least 8 is class one.*

In 1970, Mel'nikov [11] generalized Theorem 2.1.2 to projective plane and in 1998, Hind and Zhao further generalized Theorem 2.1.2 to torus and Klein bottle.

For maximum degree $\Delta \leq 5$, there are graphs of class two.

Vizing planar graph conjecture seems to be very difficult since $\Delta = 6$ is so close to the average degree of planar graphs. The case $\Delta = 7$ was recently confirmed independently by Zhang [25] and Sanders and Zhao [16]. The case $\Delta = 6$ still remains open.

In [3], Borodin *et al.* considered the edge-list coloring of embedded graphs with large girth.

Theorem 2.1.3 *(Borodin et. al. [3]) Let G be a simple graph with the maximum degree Δ and the girth g that is embeddable in a surface Σ of characteristic $\chi(\Sigma) \geq 0$. Then $\chi_{elist} = \Delta$ in each of the following cases:*

(1) $\Delta \geq 5$ and $g \geq 5$;

(2) $\Delta \geq 4$ and $g \geq 6$;

(3) $\Delta \geq 3$ and $g \geq 10$.

In [3], they also pointed out that

> *We know of no conditions for $\chi_e(G)$ (edge chromatic number of G) to equal Δ that are weaker than these.*

To respond the above comments, we show in this paper that if the edge-list coloring is replaced by edge coloring in Theorem 2.1.3, the girth requirement can be weakened.

Theorem 2.1.4 *Let G be a simple graph with the maximum degree Δ and the girth g that is embeddable in a surface Σ of characteristic $\chi(\Sigma) \geq 0$. Then $\chi_e = \Delta$ in each of the following cases:*

(1) $\Delta \geq 5$ and $g \geq 4$;

(2) $\Delta \geq 4$ and $g \geq 5$;

(3) $\Delta \geq 3$ and $g \geq 9$, or, $\Delta \geq 3$, $g \geq 8$ and $\chi(\Sigma) > 0$.

The main tools used in the proof of Theorem 2.1.4 are Vizing's Adjacent Lemma, Zhang's Adjacent Lemma, Euler formula and the Euler contribution of faces. We will first introduce these known results.

2.2 Euler contribution

Let G be a graph embedded in a surface Σ with Euler characteristic $\chi(\Sigma)$. Then, the *Euler contribution* of a face f of G is defined as follows:

$$\Phi(f) = 1 - \frac{d(f)}{2} + \sum_{v \in V(f)} \frac{1}{d(v)}$$

where $V(f)$ is the set of vertices on the boundary of the face f.

Theorem 2.2.1 [1] (Lebesgue [10], [13]) *Let G be a connected, loopless, bridgeless graph embedded in a surface Σ with Euler characteristic $\chi(\Sigma)$. Then*

$$\sum_{f\in F(G)} \Phi(f) = \chi(\Sigma). \tag{2.1}$$

A face is said to be *non-negative* (*positive, zero*, respectively) if it has non-negative (positive, zero, respectively) Euler contribution.

With a simple computation, we may characterize the positive faces and zero faces of a simple graph with minimum degree at least 3. Since we only need the structure of non-negative faces of length at least 4 in the proof of our main results, we only list the non-negative faces of length at least 4 in the following two tables.

Corollary 2.2.2 *Let H be a graph embedded in a surface Σ with Euler characteristic $\chi(\Sigma) \geq 0$. Assume that $\delta(H) \geq 3$ and $g(H) \geq 4$ where $\delta(H)$ and $g(H)$ are the minimum degree and the girth of H, respectively. Then, any positive face must be a face listed in Table 1 and any zero face must be a face listed in Table 2.*

$d_H(f)$	*degree sequence around the face*
5	3, 3, 3, 3, ≤ 5
4	3, 3, 3, $\leq \Delta$
4	3, 3, 4, ≤ 11
4	3, 3, 5, ≤ 7
4	3, 4, 4, ≤ 5

Table 1, Positive Faces

$d_H(f)$	*degree sequence around the face*
6	3, 3, 3, 3, 3, 3
5	3, 3, 3, 3, 6
5	3, 3, 3, 4, 4
4	4, 4, 4, 4
4	3, 3, 4, 12
4	3, 3, 6, 6
4	3, 4, 4, 6

Table 2, Zero Faces

2.3 Proof of Theorem 2.1.4

2.3.1 Proof of (1) of Theorem 2.1.4

Let G be a counterexample to (1) of Theorem 2.1.4 with $|E(G)|$ as small as possible. Then $\delta(G) \geq 2$ and $\Delta \geq 5$.

From Euler formula $|V(G)| + |F(G)| - |E(G)| = \chi(\Sigma)$, we have

[1] The theorem presented here is a slightly revised version of Lebesgue Theorem. The original Theorem was for plane graphs only.

$$\sum_{x \in V(G) \cup F(G)} (d(x) - 4) = -4\chi_(\Sigma) \leq 0. \tag{2.2}$$

We call $c(x) = d(x) - 4$ the initial charge of x for each $x \in V(G) \cup F(G)$. We are going to discharge $c(x)$ according to the following rules:

R1. Every 2-vertex receives 1 from each of its neighbors;

R2. For each 3-vertex adjacent to three (≥ 5) vertices, it receives $\frac{1}{3}$ from each of its neighbors;

R3. For each 3-vertex adjacent to a (≤ 4)-vertex, it receives $\frac{1}{2}$ from each of the other two adjacent vertices.

Denote $c'(x)$ the new charge of x.

(1-1) For each face $x \in F(G)$, obviously, $c'(x) = c(x) \geq 4 - 4 = 0$.

For each 2-vertex x, $c'(x) = c(x) + 2 = 0$ since, by Corollary 1.2.2-(1), each 2-vertex is adjacent to two Δ-vertices.

For each 3-vertex x adjacent to three (≥ 5)-vertices, $c'(x) = c(x) + 3 \times \frac{1}{3} = 0$.

For each 3-vertex x adjacent to a (≤ 4)-vertex, by Corollary 1.2.2-(1), x is adjacent to two Δ-vertices. Therefore, $c'(x) = c(x) + 2 \times \frac{1}{2} = 0$.

Thus, for each 3-vertex x, $c'(x) \geq 0$.

For each 4-vertex x, $c'(x) = c(x) = 0$.

For each 5-vertex x, if x is adjacent to a 2-vertex, then, by Corollary 1.2.2-(3), the other neighbors of x are Δ-vertices. Therefore, $c'(x) = c(x) - 1 \geq 0$. If x is adjacent to a 3-vertex, then x is not adjacent to any 2-vertices and is adjacent to at most two 3-vertices by Lemma 1.2.1. Therefore, $c'(x) \geq c(x) - 2 \times \frac{1}{2} \geq 1 - 1 = 0$.

For each (≥ 6) vertex x, if it is adjacent to a 2-vertex, then the other neighbors of x are Δ-vertices, therefore, $c'(x) = c(x) - 1 \geq 6 - 4 - 1 = 1$. If it is adjacent to a 3-vertex, then x is not adjacent to any 2-vertices and is adjacent to at most two 3-vertices by Lemma 1.2.1. Therefore, $c'(x) \geq c(x) - 2 \times \frac{1}{2} \geq 6 - 4 - 1 = 1$.

By the above argument, we conclude that

(1-2) $c'(x) \geq 0$ *for each* $x \in V(G) \cup F(G)$. *In addition, for each face* x *of length at least 5 or for each vertex* x *of degree at least 6, we have that* $c'(x) > 0$.

If there exists $x \in V(G) \cup F(G)$ such that $c'(x) > 0$, then, by Equation (2.2), we have

$$0 \geq -4\chi(\Sigma) = \sum_{x \in V \cup F} c(x) = \sum_{x \in V \cup F} c'(x) > 0.$$

A contradiction.

Therefore,

(1-3) $c'(x) = 0$ *for each* $x \in V(G) \cup F(G)$.

(1-4) By (1-2) and (1-3), it is easy to see that $\Delta = 5$ *and the length of each face in* G *is 4.*

(1-5) We claim that *there are no 2-vertices in* G.

Assume that x is a 2-vertex. Then x is adjacent to two $\Delta-$vertices, say $y,\ z$. By Corollary 1.2.2-(3), y is adjacent to $(\Delta - 1)$ Δ-vertices. Let $w \in N(y) \setminus \{x,\ z\}$. Note that $d(x) + d(y) = \Delta + 2$ and w is not adjacent to x. By Lemma 1.2.3-(2), w is not adjacent to any $\leq \Delta - 2 = 3$ vertices. Since $d(w) = \Delta = 5$, by the discharging rules, we have that $c'(w) = c(w) = 5 - 4 = 1 > 0$ which contradicts (1-3).

(1-6) We claim that *any 3-vertex is adjacent to three 5-vertices.*

Assume that a 3-vertex x is adjacent to a (≤ 4)-vertex y. Since $d(x) + d(y) \geq \Delta + 2 = 5 + 2 = 7$, we have that $d(y) = 4$. By Lemma 1.2.3-(1), y is adjacent to at three $\Delta-$vertices. Let z be a Δ-vertex adjacent to y and not adjacent to x. Note that $d(x) + d(y) = \Delta + 2$ and $d(x),\ d(y) < \Delta$. By Lemma 1.2.3, every vertex in $N(z) \setminus \{x,\ y\} \subseteq N(x, y) \setminus \{x,\ y\}$ is a Δ-vertex. Note that $x \notin N(z)$. Hence, z is adjacent to one 4-vertex and four $\Delta-$vertices. Therefore, $c'(z) = c(z) = 5-4 = 1 > 0$ which contradicts (1-3).

(1-7) We claim that *there are no 3-vertices in* G.

Assume that x is a 3-vertex. Then, by (1-6), x is adjacent to three 5-vertices. Let $y \in N(x)$. Then, by Lemma 1.2.1, y is adjacent to at most two 3-vertices, one of which is x. By R1, y sends $\frac{1}{3}$ to x and at most $\frac{1}{2}$ to the other 3-vertex (if any). Therefore, $c'(y) \geq c(y) - \frac{1}{3} - \frac{1}{2} = 5 - 4 - \frac{1}{3} - \frac{1}{2} = \frac{1}{6} > 0$. A contradiction.

(1-8) By (1-5) and (1-7), it is obvious that *the minimum degree of* G *is at least 4.*

According to the discharging rules R1-R3, $c'(x) = c(x) \geq 0$. Since now $\Delta = 5$, there must exist a vertex x such that $c'(x) = c(x) = 5 - 4 = 1 > 0$ which contradicts (1-3).

This contradiction completes the proof of (1) of Theorem 2.1.4 .

2.3.2 Proof of (2) of Theorem 2.1.4

Let G be a counterexample to (2) of Theorem 2.1.4 with $|E(G)|$ as small as possible.

A path $v_0v_1\cdots v_r$ is called to be a *subdivided edge of length* r if $d(v_i)=2$ for each $i=1,\cdots,r-1$ and both $d(v_0)>2$ and $d(v_r)>2$. v_0 and v_r are called the *endvertices* of the subdivided edge. Two subdivided edges are said to be *adjacent* if they share at least one endvertex.

Then, it is obvious that

(2-1) $\delta(G)\geq 2$, $\Delta\geq 4$ and G is 2-connected.

(2-2) We claim that *the length of any subdivided edge is at most 2 and no two subdivided 2-edges are adjacent to each other*, because, by Corollary 1.2.2-(1), (3), every 2-vertex is adjacent to two Δ-vertices and every vertex is adjacent to at most one 2-vertex and $\Delta\geq 4$.

Denote $\overline{G}$ the *underlying graph* of G, the graph obtained from G by replacing every subdivided 2-edge with a single edge. For each edge $e=xy\in E(\overline{G})$, denote $\zeta(e)$ the corresponding subdivided edge of e in G . Note that $\zeta(e)$ is either e or a subdivided 2-edge with endvertices x and y.

Then, obviously,

(2-3) $\delta(\overline{G})\geq 3$ *and for each* $v\in V(\overline{G})\subseteq V(G)$, $d_{\overline{G}}(v)=d_G(v)$.

(2-4) We claim that *the girth of* $\overline{G}$ *is at least 4*.

Assume that $\overline{G}$ contains a cycle $\overline{C}$ of length at most 3. Denote C the corresponding cycle of $\overline{C}$ in G. Then, by (2-2), the length of C is at most 4 otherwise there are two adjacent subdivided 2-edges on the boundary of C. This contradicts the assumption that the girth of G is at least 5.

(2-5) For each edge $e=xy\in E(\overline{G})$ with $\min\{d_{\overline{G}}(x),d_{\overline{G}}(y)\}\leq\Delta-1$, we claim that $\zeta(e)=e$, because, by Corollary 1.2.2, any 2-vertex is adjacent to two $\Delta-$vertices.

(2-6) We claim that, *in* $\overline{G}$, *any 3-vertex is adjacent to at most one 3-vertex.*

Otherwise, assume that the 3-vertex x is adjacent to two 3-vertices $y,\ z$. By (2-5), $\zeta(xy)=xy$ and $\zeta(xz)=xz$. By (2-3), $x,\ y,\ z$ are all 3-vertices in G. Therefore, in G, the 3-vertex is adjacent to two 3-vertices. This implies that x is adjacent to at most one Δ-vertex because $\Delta(G)\geq 4$, which contradicts Corollary 1.2.2-(1).

(2-7) We claim that *for any 4-face* $f'=x_1x_2x_3x_4x_1$ *in* $\overline{G}$, $d_{\overline{G}}(x_i)\geq 4$ *for each* $i=1,\ 2,\ 3,\ 4$.

By way of contradiction, we assume that $d_{\overline{G}}(x_1)\leq 3$. Therefore, by (2-3), $d_{\overline{G}}(x_1)=3$. By (2-4), $\zeta(x_1x_2)=x_1x_2$ and $\zeta(x_1x_4)=x_1x_4$. Since the girth of G is at least 5, either $\zeta(x_2x_3)$ or $\zeta(x_3x_4)$ is a subdivided 2-edge in G. Without loss of generality, we assume that x_2x_3 is a subdivided 2-edge in G. Then, in G, x_2 is adjacent to a 2-vertex and a 3-vertex. Notice that $\Delta(G) \geq 4$. By Corollary 1.2.2, it is impossible.

(2-8) Since $\chi(\Sigma) \geq 0$, by Corollary 2.2.2-(1), $\overline{G}$ *must contain non-negative faces.*

(2-9) We claim that $\overline{G}$ *contains no positive faces.* Therefore, *all faces of* $\overline{G}$ *are zero faces.*

Assume that $\overline{G}$ contains a positive face f'. From Table 1, f' is of length either 4 or 5. If f' is of length 4, from Table 1, f' is adjacent to a 3-vertex which contradicts (2-7). If f' is of length 5, from Table 1, on the boundary of f', there exists a 3-vertex which is adjacent to two 3-vertices. This contradicts (2-6).

(2-10) We claim that *each face in* $\overline{G}$ *is of length 4 and is adjacent to four 4-vertices.* Therefore, $\overline{G}$ is 4-regular and the length of each face of $\overline{G}$ is 4.

Let $\overline{f} \in F(\overline{G})$. Then, by (2-9), $\overline{f}$ is a zero face. By (2-7) and from Table 2, $\overline{f}$ is of length 4 and each vertex adjacent to $\overline{f}$ is of degree 4.

Let $\overline{f}_1 = x_1x_2x_3x_4x_1$ be a 4-face of $\overline{G}$. Denote $f_1 \in F(G)$ the corresponding face of $\overline{f}_1$. Since the girth of G is 5, $\zeta(x_ix_{i+1})(Mod4)$ is a subdivided 2-edge for some $i = 1,\ 2,\ 3,\ 4$. Without loss of generality, assume that $\zeta(x_1x_2)$ is a subdivided 2-edge. Denote $\zeta(x_1x_2) = x_1yx_2$ where $d_G(y) = 2$. Let $\overline{f}_2 = x_2x_5x_6x_3x_2$ be the 4-face of $\overline{G}$ adjacent to $\overline{f}_1$ and sharing the edge x_2x_3 (see Figure 1). By (2-2), $\zeta(x_2x_5) = x_2x_5$ and $\zeta(x_2x_3) = x_2x_3$. Therefore either $\zeta(x_5x_6)$ or $\zeta(x_3x_6)$ is a subdivided 2-edge. Without loss of generality, assume that $\zeta(x_5x_6)$ is a subdivided 2-edge. Therefore x_5 is adjacent to a 2-vertex. Note that $d_G(x_2) + d_G(y) = \Delta + 2$. Since $x_5 \in N(x_2, y) \setminus \{x_2, y\}$, by Lemma 1.2.3, the neighbors of x_5 are of degree at least $\Delta - 1 \geq 4 - 1 = 3$. This contradiction completes the proof of (2) of Theorem 2.1.4.

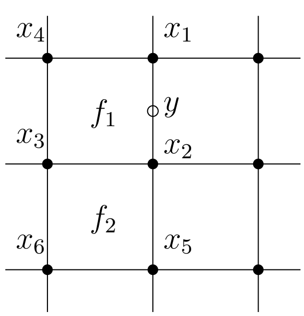

Figure 1

2.3.3 Proof of (3) of Theorem 2.1.4

Let G be a counterexample to (3) of Theorem 2.1.4 with $|E(G)|$ as small as possible.

Then, by (2), it is obvious that

(3-1) $\delta(G) \geq 2$, $\Delta = 3$ and G is 2-connected.

(3-2) We claim that *the length of any subdivided edge is at most 2 and no two subdivided 2-edges are adjacent to each other*, because, by Corollary 1.2.2-(1), (3), every 2-vertex is adjacent to two Δ-vertices and every vertex is adjacent to at most one 2-vertex and $\Delta = 3 > 2$.

Denote $\overline{G}$ the *underlying graph* of G, the graph obtained from G by replacing every subdivided 2-edge with a single edge. For each edge $e = xy \in E(\overline{G})$, denote $\zeta(e)$ the corresponding subdivided edge of e in G . Note that $\zeta(e)$ is either e or a subdivided 2-edge with endvertices x and y.

Then, obviously,

(3-3) $\delta(\overline{G}) \geq 3$ *and for each* $v \in V(\overline{G}) \subseteq V(G)$, $d_{\overline{G}}(v) = d_G(v) = 3$.

(3-4) We claim that *the girth of* $\overline{G}$ *is at least 6.*

Assume that $\overline{G}$ contains a cycle $\overline{C}$ of length at most 5. Denote C the corresponding cycle of $\overline{C}$ in G. Then, by (3-2), the length of C is at most 7 otherwise there are two adjacent subdivided 2-edges on the boundary of C. This contradicts the assumption that the girth of G is at least 8.

(3-5) We claim that $\chi(\Sigma) = 0$ and *the length of each face in* $\overline{G}$ *is 6.* Therefore, *the girth of* G *is at least 9.*

Otherwise, $\overline{G}$ has a positive face. From Table 1, we can see that the length of the positive face is at most 5. This contradicts (3-4). Therefore, $\overline{G}$ doesn't contain any positive faces. Thus, $\overline{G}$ only contains zero faces. By (3-4) and Table 2, each face of $\overline{G}$ is of length 6.

Let $\overline{f}_1 = x_1x_2x_3x_4x_5x_6x_1$ be a 6-face of $\overline{G}$. Denote $f_1 \in F(G)$ the corresponding face of $\overline{f}_1$. Since the girth of G is at least 9, there are ate least three subdivided 2-edges on the boundary of f_1. Since the length of f_1 is 6 and, by (3-2), no two subdivided 2-edges are adjacent, there are exactly three subdivided 2-edges on the boundary of f_1 and any two of them are not adjacent to each other. Without loss of generality, we assume that $\zeta(x_1x_6)$, $\zeta(x_5x_4)$, $\zeta(x_3x_2)$ are subdivided 2-edges.

Denote $\overline{f}_2 = x_5x_6y_1y_2y_3y_4x_5$ the face of $\overline{G}$ adjacent to $\overline{f}_1$ sharing the edge x_5x_6. We also denote $f_2 \in F(G)$ the corresponding face of $\overline{f}_2$. Since both $\zeta(x_1x_6)$ and

$\zeta(x_5x_4)$ are subdivided 2-edges. By (3-2), none of $\zeta(x_6y_1)$, $\zeta(x_5y_4)$, $\zeta(x_5x_6)$ is a subdivided edge. Since the length of f_2 is at least 9, f_2 must be adjacent to at least three subdivided 2-edges. Therefore, $\zeta(y_1y_2)$, $\zeta(y_2y_3)$, $\zeta(y_3y_4)$ are all subdivided 2-edges. Note that $\zeta(y_1y_2)$ and $\zeta(y_2y_3)$ are adjacent to each other. This contradicts (3-2). The contradiction completes the proof of (3).

Chapter 3

Edge coloring of graphs with small average degrees

3.1 Introduction

In 1968, Vizing [20] proposed another well-known conjecture concerning the size of critical graphs.

Conjecture 3.1.1 *If $G = (V,\ E)$ is a critical simple graph, then*

$$|E| \geq \frac{1}{2}(|V|(\Delta - 1) + 3).$$

Conjecture 3.1.1 can be expresses as follows.

If

$$\overline{d} < (\Delta - 1) + \frac{3}{|V|},$$

then G is not critical.

The best partial result to Conjecture 3.1.1 is the following theorem.

Theorem 3.1.2 *(Fiorini [5] 1975) If $G = (V,\ E)$ is a critical simple graph, then*

$$|E| \geq \frac{1}{4}(|V|(\Delta + 1)).$$

That is, if $\overline{d} < \frac{1}{2}(\Delta + 1)$, then G is not critical.

Up to now, Conjecture 3.1.1 is verified for $\Delta \leq 5$ (see Beineke and Fiorini [1], Jacobsen [7], Kayathri [9] and Yap [24]).

The following main theorems of this chapter are motivated by Conjecture 3.1.1 and the theorem of Fiorini (Theorem 3.1.2).

Theorem 3.1.3 *Let G be a graph with the maximum degree $\Delta \geq 8$ and the average degree $\overline{d} \leq 6$. Then G is not critical.*

Theorem 3.1.4 *Let G be a graph with the maximum degree $\Delta \geq 9$ and the average degree $\overline{d} \leq \frac{20}{3}$. Then G is not critical.*

An immediate corollary of Theorem 3.1.3 is the following theorem.

Theorem 3.1.5 *(Vizing [18] 1965, Mel'nikov [11] 1970 and Hind and Zhao [6] 1998) Let G be a graph which can be embedded on the surface S with the Euler characteristic $c_S \geq 0$. If the maximum degree $\Delta \geq 8$, then G is class one.*

Proof. It follows from the fact that the average degree of each component of G is at most 6 and Theorem 3.1.3. ■

Theorem 3.1.4 implies the following theorem due to Yan and Zhao [21].

Theorem 3.1.6 *(Yan and Zhao [21] 2000) Let $G = (V, E)$ be a graph embeddable on the surface S with the Euler characteristic $c_S = -1$. If the maximum degree $\Delta \geq 9$, then G is class one.*

Proof.

Let G be a counterexample to the theorem with $|E(G)|$ as small as possible. Then G is critical.

Let G be the embedding in the surface S. Denote F the set of faces of G. Since G is simple,

$$|F| \leq \frac{2|E|}{3}.$$

Thus, by Euler formula that

$$|V| + |F| \geq |E| - 1.$$

We have that

$$|V(G)| + \frac{2|E(G)|}{3} \geq |E(G)| - 1,$$

$$|V(G)| \geq \frac{|E(G)|}{3} - 1,$$

$$|V(G)| + 1 \geq \frac{|E(G)|}{3},$$
$$6(1 + \frac{1}{|V(G)|}) \geq \frac{2|E(G)|}{|V(G)|}.$$

Therefore,

$$\overline{d} = \frac{2|E|}{|V|} \leq 6(1 + \frac{1}{|V|}).$$

Since G is simple and $\Delta \geq 9$, we have that $|V| \geq 10$.

Therefore,

$$\overline{d} = \frac{2|E|}{|V|} \leq 6(1 + \frac{1}{|V|}) \leq 6.6 < \frac{20}{3}.$$

This contradicts Theorem 3.1.4.

■

In 1981, Yap [23] gave some lower bounds on the number of edges of critical graphs for small maximum degree.

Theorem 3.1.7 *(Yap [23] 1981) Let $G = (V, E)$ be a critical graph with the maximum degree Δ.*

(1) If $\Delta = 6$, then $|E| \geq \frac{(9n+1)}{4}$;

(2) If $\Delta = 7$, then $|E| \geq \frac{5n}{2}$.

Applying Theorems 3.1.3 and 3.1.4, we obtain some lower bounds on the number of edges of critical graphs as follows.

Corollary 3.1.8 *Let G be a critical graph.*

(1) If $\Delta \geq 8$, then $|E| \geq 3|V| + 1$;

(2) If $\Delta \geq 9$, then $|E| > \frac{10}{3}|V|$.

Proof. (1) By Theorem 3.1.3,

$$2|E| > 6|V|.$$

Therefore,

$$|E| \geq 3|V| + 1.$$

(2) By Theorem 3.1.4,

$$2|E| > \frac{20}{3}|V|.$$

Therefore,

$$|E| > \frac{10}{3}|V|.$$

■

Remark: Corollary 3.1.8 strengthens the bound $|E| \geq \frac{1}{4}(|V|(\Delta+1))$ obtained by Fiorini [5] for $8 \leq \Delta \leq 12$.

3.2 An adjacency Lemma

In this section, we prove an adjacency property which will be used in the proofs of our main theorems.

Lemma 3.2.1 *Let G be a critical graph with the maximum degree $\Delta \geq 5$ and x be a 3-vertex with $N(x) = \{u,\ v,\ w\} \subseteq V_\Delta$. Denote $m_y = \min\{d(z) : z \in N(y) \setminus \{x\}\}$ for each $y \in \{u,\ v,\ w\}$. Then,* $\max\{m_u,\ m_v,\ m_w\} \geq \Delta - 1$.

Proof. By contradiction, we assume that $m_y \leq \Delta - 2$ for each $y \in \{u,\ v,\ w\}$.

Let $G' = G - xw$. Then G' has a $\Delta-$edge coloring $\phi : E(G) \setminus \{xw\} \mapsto C = \{1,\ 2, \cdots,\ \Delta\}$.

The coloring ϕ of G' can be regarded as an edge coloring of G with the edge xw uncolored.

Assume that $\phi(xu) = 1$ and $\phi(xv) = \Delta$. For a vertex y in G, denote $\phi(y)$ the set of colors appearing at the edges incident with the vertex y and $\overline{\phi(y)} = C \setminus \phi(y)$.

I. We claim that

$$|\phi(w) \cap \phi(x)| = 1.$$

Since $d(w) = \Delta$, we have that $|\phi(w) \cap \{1,\ \Delta\}| \geq 1$.

Now assume that $|\phi(w) \cap \{1,\ \Delta\}| = 2$. Then $i \notin \phi(w)$ for some $i \in \{2, \cdots,\ \Delta - 1\}$. Therefore, the coloring ϕ can be extended to be a $\Delta-$edge coloring of G by coloring the edge xw with the color i, a contradiction. Thus $|\phi(w) \cap \{1,\ \Delta\}| = 1$. Therefore, we may assume that $\phi(w) = \{2,\ 3, \cdots, \Delta\}$.

Denote $N(u) = \{x,\ u_2,\ u_3, \cdots, u_\Delta\}$, $N(v) = \{x,\ v_1,\ v_2, \cdots, v_{\Delta-1}\}$ and $N(w) = \{x,\ w_2,\ w_3, \cdots, w_\Delta\}$. By I, without loss of generality, we may assume that $\phi(uu_{i+1}) = \phi(ww_{i+1}) = i+1$ and $\phi(vv_i) = i$ for each $i = 1,\ 2, \cdots,\ \Delta - 1$. An $(i-j)_\phi$ *edge chain* is a connected component of $\phi^{-1}(i) \cup \phi^{-1}(j)$. It is easy to see that an $(i-j)_\phi$ edge chain is either a path or an even cycle. For a vertex y of G, if $i \in \phi(y)$ and $j \in \overline{\phi(y)}$, we denote by $P_{i,j}(y)_\phi$ an $(i-j)_\phi$ edge chain starting from y.

Let $\phi_{1,i}$ be the edge-coloring obtained from ϕ by interchanging colors 1 and i along $P_{1,i}(x)_\phi$ and $\phi_{\Delta,i}$ be the edge-coloring obtained from ϕ by interchanging colors Δ and i along $P_{\Delta,i}(x)_\phi$ for each $i \in \{2, \; \cdots, \; \Delta - 1\}$. We also denote $\phi_{1,\ 1} = \phi_{\Delta,\ \Delta} = \phi$ for the sake of convenience.

II. We claim that for each color $i \in \phi(w) \setminus \{\Delta\}$, *any chain* $P_{1,i}(x)_\phi$ *ends at* w.

Suppose that $P_{1,i}(x)_\phi$ doesn't end at w. Since the color $1 \notin \phi(w)$, $P_{1,i}(x)_\phi$ doesn't contain the vertex w. Therefore, in the coloring $\phi_{1,i}$, the color $1 \notin \phi_{1,i}(x) \cup \phi_{1,i}(w)$. Hence, the coloring $\phi_{1,i}$ can be extended to be a $\Delta-$edge coloring of G by coloring the edge xw with the color 1, a contradiction.

III. From II, it is easy to see that for each vertex $z \notin V \setminus \{x, \; w\}$ and each $k \in \{2, \; \cdots, \; \Delta-1\}$, $\phi(z) = \phi_{1,k}(z)$, $\phi_{1,k}(x) = \{k, \; \Delta\}$ and $\phi_{1,k}(w) = \phi(w) \triangle \{1, \; k\}$.($A \triangle B$ is the symmetric difference of the two sets A and B.)

By applying the argument of II and by III,

IV. we claim that for each $k \in \{2, \cdots, \Delta - 1\}$ and each color $i \in \phi_k(w) \setminus \{\Delta\}$, *any chain* $P_{k,i}(x)_{\phi_k}$ *ends at* w.

V. We claim that $d(u_i) = \Delta$ *for each* $i = \{2, \cdots, \Delta - 1\}$ *and* $d(u_\Delta) = m_u \leq \Delta - 2$.

By II, in the coloring ϕ, u_i is incident with an edge colored by Δ and by IV, u_i is incident with an edge colored by k in each coloring $\phi_{1,\ k}$, for each $i \in \{2, \; \cdots, \; \Delta-1\}$. Therefore, by III, $\{1, \; \cdots, \; \Delta-1\} \subseteq \phi_{1,\ k}(u_i)$. Thus, $d(u_i) \geq \Delta-1$. By Lemma 1.2.1, u is adjacent to at most two minor vertices. Since $d(x) = 3$ and $m_u \leq \Delta-2$, $d(u_i) = \Delta$ for each $i = \{2, \cdots, \Delta - 1\}$ and $u_\Delta \leq \Delta - 2$.

VI. We claim that *for* $i \in \{2, \; \cdots, \Delta - 1\}$, $P_{\Delta,i}(x)_\phi$ *passes through* u.

Otherwise, we can interchange colors Δ and i along $P_{\Delta,i}(x)_\phi$ without changing the color of any edge at u. Let ϕ' be the new coloring. Then $\phi'(xv) = \phi'(uu_i)$. By the argument of I - V, $d(u_i) \leq \Delta - 2$, a contradiction.

VII. We claim that *for* $i \in \{2, \; \cdots, \Delta-1\}$, $P_{\Delta,i}(x)_\phi$ *passes through* w *and* $d(w_i) = \Delta$ *for each* $i \in \{2, \cdots, \Delta\}$ *and* $d(w_\Delta) = m_w \leq \Delta - 2$.

If we uncolor the edge xu and color the edge xw with 1, we get a new edge-coloring ϕ'. Notice that $\phi(e) = \phi'(e)$ for each edge $e \in E \setminus \{xu, \; xw\}$ and therefore, $P_{\Delta,i}(x)_\phi = P_{\Delta,i}(x)_{\phi'}$. By applying the argument of V and VI, we can prove this claim.

VIII. We claim that $d(v_i) = \Delta$ for each $i \in \{2, \; \cdots, \; \Delta-1\}$ and $d(v_1) = m_v \leq \Delta-2$.

By VI and VII, it is easy to see that $P_{\Delta,i}(x)_\phi$ passes through the vertex v_i and v_i is not an endvertex of the path. Therefore, in the coloring ϕ, $\Delta \in \phi(v_i)$ and for each $i, k \in \{2, \cdots, \Delta - 1\}$, $\phi_{\Delta,k}(v_i) = \phi(v_i)$. Repeating the argument of VI and VII with $\phi_{\Delta,k}$ in place of ϕ shows that $P_{k,i}(x)_{\phi_{\Delta,k}}$ passes through the vertex v_i for each $i \in \{2, \cdots, \Delta - 1\}$ and each $k \in \{2, \cdots, \Delta\}$ and therefore, $k \in \phi_{\Delta,k}(v_i) = \phi(v_k)$. Thus, $\{2, \cdots, \Delta\} \subseteq \phi(v_i)$ for each $i \in \{2, \cdots, \Delta\}$. Hence, $d(v_i) \geq \Delta - 1$. By Lemma 1.2.1, v is adjacent to at most two minor vertices. Since $d(x) = 3$ and $m_v \leq \Delta - 2$, $d(v_i) = \Delta$ and $d(v_1) = m_v \leq \Delta - 2$.

IX. We claim that *each $P_{1,i}(x)_\phi$ passes through the vertex v_1 and v_1 is not an endvertex for each $i \in \{2, \cdots, \Delta - 1\}$.*

Suppose that $P_{1,2}(x)_\phi$ doesn't pass through the vertex v_1. The in the coloring $\phi_{1,2}$, the edge vv_1 is still colored by the color 1 and the edge xu is colored by 2. Similar to VIII, we can find that $d(v_1) = \Delta$, a contradiction to VIII.

By II, the vertices x and w are the endvertices of the path $P_{1,i}(x)_\phi$. Therefore, v_1 can not be an endvertex of $P_{1,i}(x)_\phi$.

By IX, it is easy to see that $d(v_1) \geq \Delta - 1$. This contradicts VIII.

■

3.3 Proof of Theorem 3.1.3

Proof.

By contradiction, suppose that G is critical.

Since Theorem 3.1.4 implies Theorem 3.1.3 for $\Delta \geq 9$, we only need to prove Theorem 3.1.3 for $\Delta = 8$.

Denote $c(x) = d(x) - 6$ the *initial charge* of the vertex x.

Since the average degree $\overline{d} \leq 6$, we have that

$$\sum_{x \in V(G)} c(x) \leq 0 \tag{3.1}$$

with equality if and only if $\overline{d} = 6$.

We are going to use discharge method according to the following discharge rules:

(R1) Every 2-vertex receives 2 from each of its neighbors.

(R2) Every 3-vertex receives 1 from each of its neighbors if it is adjacent to three 8-vertices, or, receives $\frac{3}{2}$ from each adjacent 8−vertex if it is adjacent to a 7-vertex.

(R3) Every 4-vertex receives $\frac{2}{3}$ from each adjacent 8-vertex if it is adjacent to a 6-vertex, or, receives $\frac{1}{2}$ from each of its neighbors otherwise.

(R4) Every 5-vertex receives $\frac{1}{4}$ from each adjacent 8-vertex if it is adjacent to a 5-vertex, or, receives $\frac{1}{3}$ from each adjacent 8-vertex if it is adjacent to a 6-vertex, or, receives $\frac{1}{5}$ from each of its neighbors otherwise.

Let $c'(x)$ be the *new charge* of the vertex x.

I. We claim that $c'(x) = 0$ *for any vertex* x *with* $d(x) \leq 4$ *and* $d(x) = 6$.

If $d(x) = 2$, then by Lemma 1.2.1, x is adjacent to two 8-vertices. Therefore, by (R1), $c'(x) = c(x) + 4 = 0$.

If $d(x) = 3$, then x is either adjacent to three 8-vertices or is adjacent to two 8-vertices and one 7-vertices. Therefore, by (R2), $c'(x) = 0$.

If $d(x) = 4$, then x is either adjacent to a 6-vertex and three 8-vertices or is adjacent to four (≥ 7)-vertices. Therefore, by (R3), $c'(x) = 0$.

If $d(x) = 6$, it is easy to see that $c'(x) = c(x) = 0$.

II. For a 5-vertex x, let $y \in N(x)$ such that $d(y)$ is the smallest in $N(x)$. Then $c(x) = -1$ and by Lemma 1.2.1, $d(y) \geq 5$.

(II-1) If $d(y) = 5$, then, by Lemma 1.2.1, the other neighbors are all 8-vertices. Therefore, $c'(x) = c(x) + 4 \times \frac{1}{4} = 0$.

(II-2) If $d(y) = 6$, then by Lemma 1.2.1, x is adjacent to at most two 6-vertices and at least three 8-vertices. Therefore, by (R4), $c'(x) \geq c(x) + 3 \times \frac{1}{3} = 0$.

(II-3) If $d(y) \geq 7$, then by (R4), $c'(x) = c(x) + 5 \times \frac{1}{5} = 0$.

III. For a 7-vertex x, let $y \in N(x)$ such that $d(y)$ is the smallest in $N(x)$. Then $c(x) = 1$ and by Lemma 1.2.1, $d(y) \geq 3$.

(III-1) If $d(y) = 3$, then by Lemma 1.2.1, x is adjacent to six 8-vertices. Therefore, $c'(x) = c(x) = 1 > 0$.

(III-2) If $d(y) = 4$, then by Lemma 1.2.1, x is adjacent to at least five 8-vertices and at most two (≥ 4)-vertices. By (R3) and (R4), $c'(x) \geq c(x) - 2 \times \frac{1}{2} = 0$.

(III-3) If $d(y) \geq 5$, then by Lemma 1.2.1, x is adjacent to at least four 8-vertices and at most three (≥ 5)-vertices. Therefore, by (R4), $c'(x) \geq c(x) - 3 \times \frac{1}{5} = \frac{2}{5} > 0$.

IV. For an 8-vertex x, let $y \in N(x)$ such that $d(y)$ is the smallest in $N(x)$. Then $c(x) = 8 - 6 = 2$ and by Lemma 1.2.1, $d(y) \geq 2$.

(IV-1) If $d(y) = 2$, then by Lemma 1.2.1, the other neighbors of x other than y are 8-vertices. Therefore, by (R1), $c'(x) = c(x) - 2 = 0$.

(IV-2) If $d(y) = 3$ and y is adjacent to a 7-vertex , then by Lemma 1.2.1, x is adjacent to seven (≥ 7)-vertices and one 3-vertex. Therefore, by (R2), $c'(x) = c(x) - \frac{3}{2} \geq 2 - \frac{3}{2} = \frac{1}{2} > 0$.

(IV-3) If $d(y) = 3$ and no 3-vertex in $N(x)$ is adjacent to a 7-vertex, then by Lemma 1.2.1, $N(x)\backslash\{y\} \subseteq V_8 \cup V_7$. Therefore, by (R2) - (R4), $c'(x) \geq c(x) - 2\times 1 \geq 0$ with equality if and if only x is adjacent to two such 3-vertices.

(IV-4) If $d(y) = 4$ and y is adjacent to a 6-vertex, then by Lemma 1.2.1, x is adjacent to seven (≥ 6)$-$vertices and one 4-vertex. Therefore, by (R3), $c'(x) = c(x) - \frac{2}{3} \geq 2 - \frac{2}{3} = \frac{4}{3} > 0$.

(IV-5) If $d(y) = 4$ and no 4-vertex in $N(x)$ is adjacent to a 6-vertex, then by Lemma 1.2.1, x is adjacent to at least five 8-vertices and at most three 4-vertices. Therefore, by (R3), $c'(x) \geq c(x) - 3 \times \frac{1}{2} \geq 2 - \frac{3}{2} > 0$.

(IV-6) If $d(y) = 5$, then by Lemma 1.2.1, x is adjacent to at least four 8-vertices and therefore, is adjacent to at most four 5-vertices. Therefore, $c'(x) \geq c(x) - 4 \times \frac{1}{3} \geq 2 - \frac{4}{3} = \frac{2}{3} > 0$.

(IV-7) If $d(y) \geq 6$, then $c'(x) = c(x) \geq 2 > 0$ since any (≥ 6)-vertex receives zero from the adjacent 8-vertices.

By above argument, we conclude that

V. For any vertex x in G, the new charge $c'(x) \geq 0$.

By Equation (3.1), we have that

$$\sum_{x\in V(G)} c'(x) = \sum_{x\in V(G)} c(x) \leq 0.$$

VI. Since $c'(x) \geq 0$ (by I-IV), we have that $\sum_{x\in V(G)} c'(x) = 0$ *and therefore,* $c'(x) = 0$ *for each vertex* x *in* G.

VII. So, Cases (IV-2), (IV-4), (IV-5), (IV-6), (IV-7) can not happen since $c'(x) > 0$ for each of these cases and therefore, *every 8-vertex is adjacent to a 2-vertex or a 3-vertex.*

VIII. We claim that *there are no 3-vertices in* G.

Let x be a 3-vertex. If x is adjacent to a 7-vertex y, then by (III-1), $c'(y) > 0$. It contradicts to VI. Therefore, x is adjacent to three 8-vertices, say $x_1,\ x_2,\ x_3$. By VI and (IV-3), each of x_i is adjacent to exactly two 3-vertices. On the other hand, by Lemma 3.2.1, one of $x_1,\ x_2,\ x_3$ is adjacent to one 3-vertices and seven ≥ 7-vertices, a contradiction.

IX. By VII and VIII, *every 8-vertex is adjacent to a 2-vertex.*

X. The Final Step.

Let x be a 2-vertex and y_1 be a vertex adjacent to x. Let $y_2 \in N(y_1) \setminus \{x\}$ which is not adjacent to x. Then, by Lemma 1.2.1, $d(y_2) = 8$. By the choice of y_2 in $N(y_1)$, $y_2x \notin E(G)$ and therefore, by Lemma 1.2.3, $N(y_2) \setminus \{x\} \subseteq V_8 \cup V_7$ and $c(y_2) = c'(y_2) = 2 > 0$. It contradicts to (VI). This contradiction completes the proof of Theorem 3.1.3.

■

3.4 Proof of Theorem 3.1.4

Proof. Suppose that G is critical.

Let $c(x) = d(x) - \frac{20}{3}$ be the *initial charge* of the vertex x.

We are going to discharge according to the following rules.

(R1) Every 2-vertex receives $\frac{7}{3}$ from each of the adjacent $\Delta-$vertices.

(R2) Every 3-vertex receives $\frac{11}{6}$ from each of the adjacent $\Delta-$vertices if it is adjacent to a $(\Delta - 1)-$vertex.

(R3) If a 3-vertex x is adjacent to three Δ-vertices (described in Lemma 3.2.1), then x receives $\frac{7}{3}$ from each of those adjacent vertices whose neighbors are of degree at least $\Delta - 1$ except x and receives $\frac{2}{3}$ from each of the other adjacent vertices.

(R4) Every 4-vertex receives $\frac{8}{9}$ from each of the adjacent Δ-vertices if it is adjacent to a $(\Delta - 2)$-vertex, or, receives $\frac{2}{3}$ from each of its adjacent vertices otherwise.

(R5) Every 5-vertex receives $\frac{5}{12}$ from each of the adjacent Δ-vertices if it is adjacent to a $(\Delta - 3)$-vertex, or, receives $\frac{5}{9}$ from each of the adjacent Δ-vertices if it is adjacent to a $(\Delta - 2)$-vertex, or, receives $\frac{1}{3}$ from each of the adjacent vertices otherwise.

(R6) Every 6-vertex receives $\frac{2}{15}$ from each of the adjacent Δ-vertices if it is adjacent to a $(\Delta - 4)$-vertex or, receives $\frac{1}{6}$ from each of the adjacent Δ-vertices if it is adjacent to a $(\Delta - 3)$-vertex, or, receives $\frac{2}{9}$ from each of the adjacent Δ-vertices if it is adjacent to a $(\Delta - 2)$-vertex and not adjacent to any $(\Delta - 3)$-vertices, or, receives $\frac{1}{9}$ from each of the adjacent vertices otherwise.

Denote $c'(x)$ the *new charge* of the vertex x. We are going to show that $c'(x) \geq 0$ for every $x \in V(G)$.

I. $c'(x) = 0$ if $d(x) = 2$ and $c'(x) = c(x) > 0$ if $7 \leq d(x) \leq \Delta - 2$.

(I-1) If $d(x) = 2$, then by Lemma 1.2.1, x is adjacent to two $\Delta-$vertices. Therefore, $c'(x) = c(x) + 2 \times \frac{7}{3} = 0$.

(I-2) If $7 \leq d(x) \leq \Delta - 2$, then $c'(x) = c(x) > 0$ since $\Delta - 2 \geq 9 - 2 \geq 7$ and the vertex x is not affected by any rules above.

II. For a 3-vertex x, $c(x) = 3 - \frac{20}{3} = -\frac{11}{3}$. Let $y \in N(x)$ such that $d(y)$ is the smallest in $N(x)$. Then, by Lemma 1.2.1, $d(y) \geq \Delta - 1$.

(II-1) If $d(y) = \Delta - 1 \geq 8$, then by Lemma 1.2.1, x is adjacent to two Δ-vertices and, therefore, by (R2), $c'(x) = -\frac{11}{3} + 2 \times \frac{11}{6} = 0$.

(II-2) If $d(y) = \Delta$, then x is adjacent to three Δ-vertices, say $x_1,\ x_2,\ x_3$. Denote $\mu_i = \min\{d(z) : z \in N(x_i) \setminus \{x\}$ for each $i = 1, 2,\ 3$. By Lemma 3.2.1, we may assume that $\mu_1 \geq \Delta - 1$. Therefore, by (R3), $c'(x) \geq c(x) + \frac{7}{3} + 2 \times \frac{2}{3} = 0$ with the equality if and only if $\mu_i < \Delta - 1$ for $i = 2,\ 3$.

III. For a 4-vertex x, $c(x) = 4 - \frac{20}{3} = -\frac{8}{3}$. Let $y \in N(x)$ such that $d(y)$ is the smallest in $N(x)$. Then, by Lemma 1.2.1, $d(y) \geq \Delta - 2$.

(III-1) If $d(y) = \Delta - 2$, then by Lemma 1.2.1, the other neighbors of x are all of degree Δ. Therefore, by (R4), $c'(x) = -\frac{8}{3} + 3 \times \frac{8}{9} = 0$.

(III-2) If $d(y) \geq \Delta - 1$, then by (R4), $c'(x) = -\frac{8}{3} + 4 \times \frac{2}{3} = 0$.

IV. For a 5-vertex x, $c(x) = 5 - \frac{20}{3} = -\frac{5}{3}$. Let $y \in N(x)$ such that $d(y)$ is the smallest in $N(x)$. Then, by Lemma 1.2.1, $d(y) \geq \Delta - 3$.

(IV-1) If $d(y) = \Delta - 3$, then, by Lemma 1.2.1, the other neighbors of x are all of degree Δ. Therefore, by (R5), $c'(x) = -\frac{5}{3} + 4 \times \frac{5}{12} = 0$.

(IV-2) If $d(y) = \Delta - 2$, then, by Lemma 1.2.1, x is adjacent to at least three Δ-vertices. Therefore, by (R5), $c'(x) \geq -\frac{5}{3} + 3 \times \frac{5}{9} = 0$.

(IV-3) If $d(y) \geq \Delta - 1$, then, by (R5), $c'(x) = -\frac{5}{3} + 5 \times \frac{1}{3} = 0$.

V. For a 6-vertex x, $c(x) = 6 - \frac{20}{3} = -\frac{2}{3}$. Let $y \in N(x)$ such that $d(y)$ is the smallest in $N(x)$. Then, by Lemma 1.2.1, $d(y) \geq \Delta - 4$.

(V-1) If $d(y) = \Delta - 4$, then, by Lemma 1.2.1, the other neighbors of x are all of degree Δ. Therefore, by (R6), $c'(x) = -\frac{2}{3} + 5 \times \frac{2}{15} = 0$.

(V-2) If $d(y) = \Delta - 3$, then, by Lemma 1.2.1, x is adjacent to at least four Δ-vertices. Therefore, by (R6), $c'(x) \geq -\frac{2}{3} + 4 \times \frac{1}{6} = 0$.

(V-3) If $d(y) = \Delta - 2$, then, by Lemma 1.2.1, x is adjacent to at least three Δ-vertices. Therefore, by (R6), $c'(x) \geq -\frac{2}{3} + 3 \times \frac{2}{9} = 0$.

(V-4) If $d(y) \geq \Delta - 1$, then, by (R6), $c'(x) = -\frac{2}{3} + 6 \times \frac{1}{9} = 0$.

VI. For a $(\Delta-1)$-vertex x, $c(x) \geq 8 - \frac{20}{3} = \frac{4}{3}$. Let $y \in N(x)$ such that $d(y)$ is the smallest in $N(x)$. Then $d(y) \geq 3$. Note that only the last subcases of each of R4, R5 and R6 affects the change of this $(\Delta-1)$-vertex

(VI-1) If $d(y) = 3$, then, by Lemma 1.2.1, the other neighbors of x are all of degree Δ. Therefore, by (R2), $c'(x) = c(x) \geq \frac{4}{3} > 0$.

(VI-2) If $d(y) = 4$, then, by Lemma 1.2.1, x is adjacent to at most two 4-vertices, therefore, by (R4) - (R6), $c'(x) \geq c(x) - 2 \times \frac{2}{3} = 0$ with equality if x is adjacent to two 4-vertices and $\Delta = 9$.

(VI-3) If $d(y) = 5$, then, by Lemma 1.2.1, x is adjacent to at most three 5-vertices, therefore, by (R5) and (R6), $c'(x) \geq c(x) - 3 \times \frac{1}{3} = \frac{1}{3} > 0$.

(VI-4) If $d(y) \geq 6$, then, by Lemma 1.2.1, x is adjacent to at most four 6-vertices, therefore, by (R6), $c'(x) \geq c(x) - 4 \times \frac{1}{9} = \frac{4}{9} > 0$.

VII. For a Δ-vertex x, $c(x) \geq 9 - \frac{20}{3} = \frac{7}{3}$. Let $y \in N(x)$ such that $d(y)$ is the smallest in $N(x)$. Then, by Lemma 1.2.1, $d(y) \geq 2$.

Let $\mu = \min\{d(z) : z \in N(y) \setminus \{x\}\}$. Then $\mu = d(z)$ for some $z \in N(y) \setminus \{x\}$ and $\mu + d(y) \geq \Delta + 2$.

(VII-1) If $d(y) \geq 7$, then $c'(x) = c(x) \geq \frac{7}{3} > 0$.

Now we assume that $d(y) \leq 6$.

(VII-2) If $\mu + d(y) = \Delta + 2$, then, by Lemma 1.2.3, $N(x) \setminus \{y,\ z\} \subseteq V_\Delta$. If $d(y) \leq 4$, then $d(z) \geq 7$ since $\Delta \geq 9$. Therefore, by (I-2) and rules (R1)-(R6), x sends at most $\frac{7}{3}$ to y. Thus. $c'(x) \geq c(x) - \frac{7}{3} \geq \frac{7}{3} - \frac{7}{3} = 0$ with equality if and only if $\Delta = 9$ and $d(y) = 2$. If $5 \leq d(y) \leq 6$, then $d(z) = \Delta + 2 - d(y) \geq 5$. Therefore, by (R5) and (R6), $c'(x) \geq \frac{7}{3} - 2 \times \frac{5}{12} > 0$.

Now we assume that $\mu + d(y) \geq \Delta + 3$.

(VII-3) Let $d(y) = 3$. then y is adjacent to three Δ-vertices. If $N(x) \setminus \{y\} \subseteq V_\Delta \cup V_{\Delta-1}$, then by (R3), $c'(x) \geq \frac{7}{3} - \frac{7}{3} = 0$ with equality if and only if $\Delta = 9$.

If there exists a vertex $z \in N(x) \setminus \{y\}$ such that $d(z) \leq \Delta - 2$, then $d(z) \geq 3$ and by Lemma 1.2.1, $N(x) \setminus \{y, z\} \subseteq V_\Delta$. Therefore, by (R3) - (R6), $c'(x) \geq \frac{7}{3} - 2 \times \frac{2}{3} = 1 > 0$.

(VII-4) If $d(y) = 4$, then $\mu \geq \Delta - 1$. By Lemma 1.2.1, x is adjacent to at most three 4-vertices. Therefore, by (R4)-(R6), $c'(x) \geq \frac{7}{3} - 3 \times \frac{2}{3} = \frac{1}{3} > 0$.

(VII-5) If $d(y) = 5$, then $\mu \geq \Delta - 2$. By Lemma 1.2.1, x is adjacent to at most four 5-vertices. Therefore, by (R5)-(R6), $c'(x) \geq \frac{7}{3} - 4 \times \frac{5}{9} = \frac{1}{9} > 0$.

(VII-6) If $d(y) = 6$, then $\mu \geq \Delta - 3$. By Lemma 1.2.1, x is adjacent to at most five 6-vertices. Therefore, by (R5)-(R6), $c'(x) \geq \frac{7}{3} - 5 \times \frac{2}{9} = \frac{11}{9} > 0$.

VIII. From (I)-(VII), we conclude that $c'(x) \geq 0$ *for each vertex* $x \in V$.

Therefore,

$$0 \leq \sum_{x \in V} c'(x) = \sum_{x \in V} c(x) \leq 0.$$

Hence,

$$c'(x) = 0, \tag{3.2}$$

for each vertex x in G.

IX. From (VII) and Equation (3.2), *every* Δ*-vertex is adjacent to a 2-vertex or a 3-vertex* since $c'(x) > 0$ in each of the cases (VII-1), (VII-4)-(VII-6).

X. We claim that *there are no 2-vertices in* G.

Let x be a 2-vertex and $y_1 \in N(x)$. Let $y_2 \in N(y_1) \setminus \{x\}$ that is not adjacent to x. Then, by Lemma 1.2.1, $d(y_2) = \Delta$. By the choice of y_2 in $N(y_1)$, $y_2 x \notin E(G)$ and therefore, by Lemma 1.2.3, $N(y_2) \setminus \{x\} \subseteq V_\Delta \cup V_{\Delta-1}$. Therefore, by (VII-1), $c'(y_2) = c(y_2) > 0$. This contradicts Equation (3.2).

XI. The final step.

By IX and X, every Δ-vertex is adjacent to a 3-vertex. Let y be a 3-vertex adjacent to x_1. Then, by (VII-2) and Equation (3.2), the vertex y is adjacent to three Δ-vertices, say $x_1,\ x_2,\ x_3$. Denote $\mu_i = \min\{d(z) : z \in N(x_i) \setminus \{x\}\}$ for each $i = 1, 2,\ 3$. By Lemma 3.2.1, we may assume that $\mu_1 \geq \Delta - 1$. Since $c'(y) = 0$, by (II-2), we have that both $\mu_2 \leq \Delta - 2$ and $\mu_3 \leq \Delta - 2$. Therefore, x_2 sends $\frac{2}{3}$ to the 3-vertex y. Denote $d(z) = \mu_2$ for some $z \in N(x_2)$. Then, $d(z) \leq \Delta - 2$. By Lemma 1.2.1, x_2 is adjacent to at least $\Delta - 2$ Δ-vertices and two ($\leq \Delta - 2$)-vertices, which are x and z. By Lemma 1.2.3, for any vertex $w \in N(z)$, $d(w) + d(z) \geq \Delta + 3$ since $y \in N(N(z, w)) \setminus \{z,\ w\}$ and $d(y) = 3 < \Delta - 1$. Therefore, by rules (R3)-(R6), y_2 sends at most $\frac{2}{3}$ to z. Since x and z are the only two ($\leq \Delta - 1$)-vertices, we have that $c'(y) \geq c(y) - 2 \times \frac{2}{3} = \frac{11}{3} - \frac{4}{3} > 0$. This contradicts to Equation (3.2). This contradiction completes the proof of Theorem 3.1.4.

■

Chapter 4

Edge-face chromatic number and edge chromatic number of simple plane graphs

4.1 Introduction

All graphs considered in this chapter are finite. For a plane graph G, denote $\chi_e(G)$, $\Delta(G)$, $\delta(G)$ *the edge chromatic number*, *the maximum degree* and *the minimum degree* of the graph G, respectively. Let $E(G)$, $V(G)$, $F(G)$ be *the edge set, the vertex set and the face set* of G, respectively. For $f \in F(G)$, let $B(f)$ be the *boundary* of the face f. An *element* of G is a member of $E(G) \cup F(G)$. Any two elements are *adjacent* if they are either adjacent to or incident with each other in the traditional sense. An *edge-face k-coloring* of the plane graph G is a function $\phi : E(G) \cup F(G) \mapsto \{1, \cdots, k\}$ such that for any two adjacent elements $a,\ b \in E(G) \cup F(G)$, $\phi(a) \neq \phi(b)$. Denote $\chi_{ef}(G)$ the *edge-face chromatic number* of G, i.e., the smallest integer k such that G has an edge-face k-coloring. This problem appears to have first been considered by Jucovič [8] and Fiamčík [4]. In 1975, Mel'nikov [12] made the following conjecture :

Conjecture 4.1.1 *(Mel'nikov [12], 1975) For any simple plane graph G, $\chi_{ef}(G) \leq \Delta(G) + 3$.*

In [22] and [14], Waller, Sanders and Zhao proved this conjecture independently. In [2], Borodin proved that for $\Delta(G) \geq 10$, $\chi_{ef}(G) \leq \Delta(G) + 1$. Also in [2], Borodin proposed the following problem: Characterize those simple plane graphs G having $\chi_e(G) = \chi_{ef}(G) = \Delta(G)$.

In this chapter, we investigate the relationship between $\chi_e(G)$ and $\chi_{ef}(G)$ for 2-connected simple plane graphs G.

Vizing [20] showed that an improvement to his edge coloring theorem is possible for planar graphs with large maximum degree.

Theorem 4.1.2 (Vizing [20]) *Let G be a simple planar graph. If $\Delta(G) \geq 8$, then $\chi_e(G) = \Delta(G)$.*

The main results of this chapter are the following theorem.

Theorem 4.1.3 *Let $k \geq 24$ be an integer and G be a 2-connected simple plane graph. If $\chi_e(G) \leq k$, then $\chi_{ef}(G) \leq k$.*

The following theorem is a corollary of Theorem 4.1.3 and Theorem 4.1.2.

Theorem 4.1.4 *For any 2-connected simple plane graph G with $\Delta(G) \geq 24$, $\chi_{ef}(G) = \Delta(G)$.*

4.2 Notation and terminology

A path $v_0v_1 \cdots v_r$ is called a *subdivided edge of length r* if $d(v_i) = 2$ for each $i = 1, \cdots, r-1$ and both $d(v_0) > 2$ and $d(v_r) > 2$.

Denote $C = \{1,\ 2, \cdots,\ k\}$ the color set. Let $\phi : E(H) \cup F(H) \mapsto C$ be an edge-face k-coloring of a plane graph H. For each vertex $v \in V(H)$, $\phi(v)$ is the set of all colors used by the edges incident with v.

Let $A \subseteq E(H) \cup F(H)$. A partial edge-face k-coloring of H on A is a function $\phi : A \mapsto C$ such that every pair of adjacent elements in A receive different colors. For a partial edge-face k-coloring ϕ of H on A, we denote $\phi(u)$ the set of colors used by the edges in $A \cap E(H)$ which are incident with the vertex $u \in V(H)$.

If there is no confusion, a face is usually denoted by the sequence of vertices that form the circuit (or cycle) around the face.

4.3 Euler contribution

Let G be a plane graph. The Euler contribution $\Phi(f)$ of a face f in G is defined as follows:

$$\Phi(f) = 1 - \frac{d(f)}{2} + \sum_{v \in B(f)} \frac{1}{d(v)}$$

where $B(f)$ is the boundary of the face f.

The following theorem by Lebesgue [10] will be applied here for finding some special configurations in a plane graph.

Theorem 4.3.1 *(Lebesgue [10]) Let G be a connected plane graph without loops and bridges. Then*

$$\sum_{f \in F(G)} \Phi(f) = 2.$$

Furthermore, there must be a face with a positive Euler contribution.

4.4 Lemmas

In this section, we are going to present two useful lemmas for the preparation of the proof of our main theorem.

Lemma 4.4.1 *Let A and B be two finite sets with the same cardinality $n \geq 2$. Then there exists a one-to-one mapping f from A to B such that $f(a) \neq a$ for any $a \in A$.*

Proof. Use induction on n. Denote $A = \{a_1, a_2, \cdots, a_n\}$ and $B = \{b_1, b_2, \cdots, b_n\}$ where $a_1 \neq b_1$. If $n = 2$, define the mapping f as follows: $f(a_i) = b_i, i = 1, 2$ if $a_2 \neq b_2$, or $f(a_1) = b_2, f(a_2) = b_1$ if $a_2 = b_2$. Now assume that $n \geq 3$. Then by the induction hypothesis, there is a one-to-one mapping from $A \setminus \{a_1\}$ to $B \setminus \{b_1\}$ such that $f(a) \neq a$ for any $a \in A \setminus \{a_1\}$. Extend f to the set A by defining $f(a_1) = b_1$. Then f satisfies the requirement. ■

Lemma 4.4.2 *Let G be a simple plane graph and k be a positive integer. Denote $F_s = \{f \in F(G) | d(f) \leq \frac{k-1}{2}\}$ and $E_s = \{uv \in E(G) | d(u) + d(v) \leq k - 1$ or $d(u) + d(v) = k$ and uv is adjacent to a face in $F_s\}$. Let $S \subseteq E_s \cup F_s$. If there is a partial edge-face k-coloring of G on $[E(G) \cup F(G)] \setminus S$, $\phi : [E(G) \cup F(G)] \setminus S \mapsto C$, then we can adjust (if necessary) and then extend the coloring ϕ to be an edge-face k-coloring of G.*

Proof. For $e = uv \in E_s \cap S$, if $d(u) + d(v) \leq k - 1$, then there are at least $k-(d(u)-1+d(v)-1)-2 = k-(d(u)+d(v)) \geq k-(k-1) = 1$ colors available for the uncolored edge e. If $d(u)+d(v) = k$, remove the color from the face f adjacent to uv whose length is at most $\frac{k-1}{2}$. Then there are at least $k-(d(u)-1+d(v)-1)-1 = k-(d(u)+d(v))+1 \geq k-k+1 = 1$ colors available for the uncolored edge e. We color this edge first and then color the face f because there are at least $k-2\times\frac{k-1}{2} = 1$ colors for this face.

For $f \in F_s \cap S$, there are at least $k - 2 \times d(f) \geq k - 2 \times \frac{k-1}{2} = 1$ colors available for the uncolored face f. ■

4.5 Proof of Theorem 4.1.3

Let G be a counterexample to Theorem 4.1.3 with $|E(G)|$ as small as possible.

4.5.1 Operations

The Euler formula is one of the most useful methods in the study for plane graphs. However, if the minimum vertex degree or the minimum face degree of the graph is 2, the formula may not work effectively. Thus, we have to apply some operations to eliminate subdivided edges and digons of the graph so that Euler formula may be applied to the resulting graph that is of minimum vertex degree and minimum face degree at least 3.

Operation α: replacing each subdivided edge with a single edge.

Operation β: replacing each 2-face with a single edge recursively.

One may repeatedly apply these two operations to G. Since the graph is finite, with a finite many of operations, the resulting graph will be of minimum vertex degree and minimum face degree at least 3 (except for the case that G itself is a series parallel graph)

The operation sequence is recursively defined as follows.

$$\zeta_1 = \alpha,$$

$$\zeta_{2i} = \beta\zeta_{2i-1}, \quad \zeta_{2i+1} = \alpha\zeta_{2i}.$$

For any positive integer q, it is obvious that, for each edge $e \in E(\zeta_q(G))$ with endvertices x and y, $\zeta_q^{-1}(e)$ induces a series parallel subgraph in G with the terminal vertices x and y.

In the next few subsections, we will study the structure of $\zeta_q^{-1}(e)$ in G and we will also determine the smallest integer q so that $\zeta_q(G)$ is of minimum vertex degree and the minimum face degree is at least 3.

4.5.2 Reducible and irreducible configurations

Before the study of the structure of $\zeta_q^{-1}(e)$, we need a few basic structural results that will be used frequently in the rest of the chapter.

Proposition 1 (Configuration A) *(1) Every subdivided edge of G is of length at most* 2. *(The vertices u and w are called the terminal vertices of the configuration.)*
(2) Let uvw be a subdivided edge of G of length 2. If uw is not an edge of G, then

$$d(u) \geq k-2 \quad \text{and} \quad d(w) \geq k-2.$$

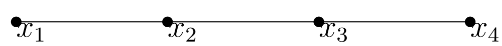

Figure 1

Proof. (1) Assume that $P = x_1x_2x_3 \cdots x_d$ is a subdivided edge of length $d-1 \geq 3$ in G . Consider the graph G_1 obtained from G by replacing the path $x_1x_2x_3$ with one edge x_1x_3. Then, G_1 remains 2-connected and simple and by Theorem 4.1.2, $\chi_e(G_1) \leq k$. Let ϕ be an edge-face k-coloring of G_1 with the color set C. Color the edge x_1x_2 with the color $\phi(x_1x_3)$. Then, ϕ can be viewed as a partial edge-face k-coloring of G on $[E(G) \cup F(G)] \setminus \{x_2x_3\}$. Since $d_G(x_2) = d_G(x_3) = 2$, there are at least $k - (d_G(x_2) + d_G(x_3) - 2 + 2) = k - 4 \geq 1$ colors of C available for the edge x_2x_3. Thus, G has an edge-face k-coloring, a contradiction.

(2) Since $uw \notin E(G)$, we may replace the path uvw with an edge uw, the resulting graph, denoted by G'_1, remains 2-connected and simple and by Theorem 4.1.2, $\chi_e(G'_1) \leq k$. Hence, G'_1 has an edge-face k-coloring ϕ. Then, ϕ can be viewed as a partial edge-face k-coloring of G on $[E(G) \cup F(G)] \setminus \{uv, vw\}$.

The coloring ϕ can be adjusted (if necessary) and then extended to be an edge-face k-coloring of G if $d_G(u) \leq k-3$ or $d_G(w) \leq k-3$ by Lemma 4.4.2. ■

Proposition 2 *G does not contain any of configurations illustrated in Figure 2 where $d(v_i) = 2$, for each $i = 0, 1, 2$ and uv_1wv_2u is a face, and, in (a) uv_1wu is a face, in (b) uv_1wv_0u is a face.*

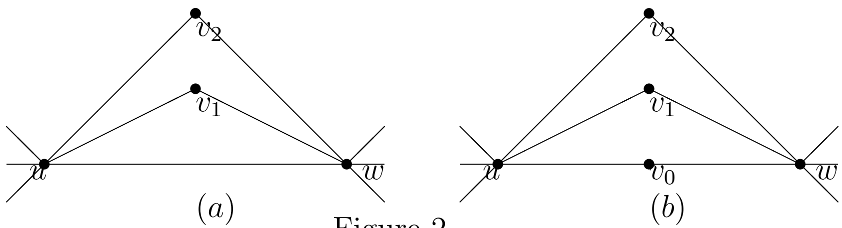

Figure 2

Proof. By way of contradiction, we assume that G contains the configuration (a) or (b). Let $G_2 = G \setminus \{v_1, v_2\}$. Then, G_2 remains 2-connected and simple and $\chi_e(G_2) \leq \chi_e(G) \leq k$. Thus, G_2 has an edge-face k-coloring: $\phi : E(G_2) \cup F(G_2) \mapsto C$. Denote $f_{uv_2w} \in F(G)$ the face adjacent to the face uv_1wv_2u and incident with the subdivided edge uv_2w and denote f_{uw} the face incident with the edge uw in (a) (or the subdivided edge uv_0w in (b)) other than the face bounded by the circuit uwv_1u (or uv_1wv_0u in (b)). We also use f_{uv_2w} and f_{uw} to denote the corresponding face in G_2. Then, ϕ can be viewed as a partial edge-face k-coloring of G on $[E(G) \cup F(G)] \setminus \{uv_1,\ uv_2,\ wv_1,\ wv_2,\ uv_1wv_2u,\ uv_1wu\}$ in (a) (or on $[E(G) \cup F(G)] \setminus \{uv_1,\ uv_2,\ wv_1,\ wv_2,\ uv_1wv_2u,\ uv_1wv_0u\}$ in (b))

Let $\{a, b\} \subseteq C \backslash \phi(u)$ and $\{c, d\} \subseteq C \backslash \phi(w)$ since $d_{G_2}(u) \leq k-2$ and $d_{G_2}(w) \leq k-2$. We consider two cases as follows:

Case 1: $\phi(f_{uv_2w}) \notin \{c, d\} \cap \{a, b\}$. Without loss of generality, we assume that $\phi(f_{uv_2w}) \notin \{c, d\}$ and $\phi(f_{uv_2w}) \neq a$.

By Lemma 4.4.1, there exists a one-to-one function $f : \{a, b\} \mapsto \{c, d\}$ such that $f(a) \neq a$ and $f(b) \neq b$. We can color uv_1, wv_1, uv_2, wv_2 with colors $b,\ f(b),\ a,\ f(a)$, respectively and then, by Lemma 4.4.2, we can color the faces uv_2wv_1u and uv_1wu properly. Thus, we obtain an edge-face k-coloring for the graph G. A contradiction.

Case 2: $\phi(f_{uv_2w}) \in \{c, d\} \cap \{a, b\}$. Without loss of generality, we assume that $a = \phi(f_{uv_2w}) = c$. We consider the following two subcases.

Subcase 2.1: G contains the configuration (a).

Denote $e = \phi(uw)$. Remove the color from the edge uw and color it with the color a. Then color the edges uv_1 and wv_2 with the color e and color the edges uv_2 and wv_1 with b, d respectively. By Lemma 4.4.2, we can further color the faces uv_2wv_1u and uv_1wu properly. Thus, we obtain an edge-face k-coloring for the graph G. A contradiction.

Subcase 2.2: G does not contain the configuration (a).

Then G contains the configuration (b).

Denote $g = \phi(uv_0)$ and $h = \phi(wv_0)$. Then $g \neq h$ and $a \notin \{g, h\}$. Remove the color g from the edge uv_0 and color it with the color a. We first color the edges wv_2 and wv_1 with colors d and a, respectively. Then color the edge uv_2 with a color in $\{b, g\} \setminus \{d\}$. And then color the edge uv_1 with the remaining color from $\{b, g\}$ since $a \notin \{b, g\}$. By Lemma 4.4.2, we can further color the faces uv_2wv_1u and uv_1wv_0u properly. Thus, we obtain an edge-face k-coloring for the graph G. A contradiction. ■

Proposition 3 (Configuration B) *If G contains a configuration illustrated in Figure 3 where $uvwu$ is a face and $d(u) \geq d(w)$ and $d(v) = 2$, then we have the following two conclusions:*

(1) For each partial edge-face k-coloring ϕ of G on $[E(G) \cup F(G)] \setminus \{uv,\ wv,\ uvwu\}$ which can be obtained from an edge-face k-coloring of $G \setminus \{v\}$, let $e = \phi(uw)$ and $a = \phi(f)$ where f is the face of G incident with the vertex v other than the triangle face $uvwu$. Then we have that

(a) $|C \setminus \phi(u)| = 1$ and consequently, $d(u) = k$. And

(b) by (a), let $b \in C \setminus \phi(u)$. Then ϕ must satisfy one of the following two cases:

Case 3.1. If $b = a$, then either $a \in \phi(w)$ or $\{a\} = C \setminus \phi(w)$,

or

Case 3.2. If $b \neq a$, then $C \setminus \phi(w) \subseteq \{a, b\}$.

(2)

$$\min\{d(u), d(w)\} \geq 4$$

(that is, $d(w) \geq 4$).

(The vertices u and w are called the terminal vertices of the configuration.)

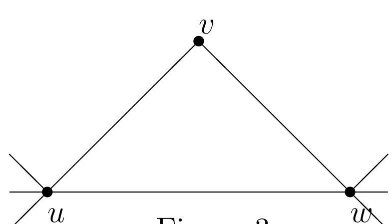

Figure 3

Proof. I. If one can adjust (if necessary) and then extend the coloring ϕ to the uncolored edges uv and vw, then, by Lemma 4.4.2, the partial edge-face k-coloring ϕ of G on $[E(G) \cup F(G)] \setminus \{uvwu\}$ can be further extended to the uncolored triangle face $uvwu$. Therefore, we only need to find a way to adjust (if necessary) and then extend the coloring ϕ to the uncolored edges uv and vw.

II. If there is a color $d \in C \setminus [\phi(u) \cup \{a\}]$ and a color $c \in C \setminus [\phi(w) \cup \{a\}]$ such that $d \neq c$, then the coloring ϕ can be easily extended to the uncolored edges uv and vw.

III. Assume that $|C \setminus \phi(u)| \geq 2$. Then $|C \setminus \phi(w)| \geq 2$ because $d(u) \geq d(w)$. By II, $|[[C \setminus \phi(u)] \cup [C \setminus \phi(w)]] \setminus \{a\}| \leq 1$, otherwise there is a pair of colors described in II. Therefore, $|C \setminus \phi(u)| = |C \setminus \phi(w)| = 2$ and $a \in C \setminus \phi(u) = C \setminus \phi(w)$. Let $\{a, c\} = C \setminus \phi(u) = C \setminus \phi(w)$. Denote $e = \phi(uw)$. Remove the color e from the edge uw and color it with the color a and then color the edges uv and wv with the colors e, c, respectively. This contradiction implies that $|C \setminus \phi(u)| = 1$ which is (1)-(a).

IV. Case 3.1. We assume that $b = a$. If $a \notin \phi(w)$ and $\{a\} \neq C \setminus \phi(w)$, then $a \in C \setminus \phi(w)$ and $|C \setminus \phi(w)| \geq 2$. Let $g \in C \setminus \phi(w)$ and $g \neq a$. Remove the color $e = \phi(uw)$ from the edge uw and color it with the color a and then color the edges uv and wv with the colors e and g, respectively. Therefore, either $a \in \phi(w)$ or $\{a\} = C \setminus \phi(w)$.

V. Case 3.2. We assume that $b \neq a$. Then, $C \setminus \phi(w) \subseteq \{a, b\}$ otherwise there is a color $h \in [C \setminus \phi(w)] \setminus \{a,\ b\}$ such that $\{h,\ b\}$ is a pair of colors described in II.

VI. Part (2) of the proposition can be proved easily by applying the conclusion of (1)-(b). If $d(w) \leq 3$, then $a \notin \phi(w)$ and $|C \setminus \phi(w)| = k - (d(w) - 1) \geq k - 2 > 2$. Obviously, it is neither Case 3.1 nor Case 3.2. ■

Proposition 4 (Configuration C) *If G contains the configuration illustrated in Figure 4, where $d(v_1) = d(v_2) = 2$ and uv_1wv_2u forms the boundary of a face, then*

$$d(u) = d(w) = k.$$

(The vertices u and w are called the terminal vertices of the configuration.)

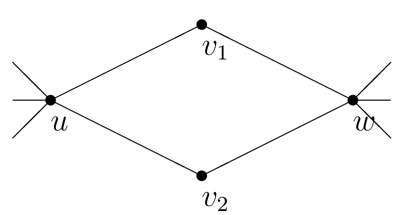

Figure 4

Proof. By way of contradiction, we assume that $d(u) \le k-1$. Let $G_4 = G \setminus \{v_1\}$. Then G_4 remains 2-connected and simple and $\chi_e(G_4) \le \chi_e(G) \le k$. Let $\phi : E(G_4) \cup F(G_4) \mapsto C$ be an edge-face k-coloring of G_4. Then ϕ can be viewed as a partial edge-face k-coloring of G on $[E(G) \cup F(G)] \setminus \{uv_1,\ wv_1,\ uv_1wv_2u\}$. Denote by f_{uv_iw} the face of G which is adjacent to the face uv_1wv_2u and incident with the subdivided edge uv_iw for each $i = 1, 2$. f_{uv_iw} is also considered as the corresponding face in G_4. Let $\{a, b\} \subseteq C \setminus \phi(u)$ and $c \in C \setminus \phi(w)$. Denote $d = \phi(uv_2)$, $e = \phi(wv_2)$, $f = \phi(f_{uv_1w})$ and $g = \phi(f_{uv_2w})$. We consider the following two cases:

Case 1: $c \neq f = \phi(f_{uv_1w})$.

Color the edge wv_1 with the color c. If $\{a, b\} \neq \{c, f\}$, then color the edge uv_1 with a color from $\{a, b\} \setminus \{c, f\}$. Since the length of the face uv_1wv_2u is $4 \le \frac{k-1}{2}$, by Lemma 4.4.2, we can color it properly. Therefore, we get an edge-face k-coloring for G, a contradiction. Thus, $\{a, b\} = \{c, f\}$. Therefore, $d \neq c$. Since, in G_4, the faces f_{uv_1w}, f_{uv_2w} and the edges uv_2, wv_2 are pairwise adjacent to each other, we have that $|\{f, d, e, g\}| = 4$. Remove the color d from the edge uv_2 and color the edge uv_1 with the color d and color the edge uv_2 with the color f. Since the length of the face uv_1wv_2u is $4 \le \frac{k}{2} - 1$, by Lemma 4.4.2, we can further color it properly. Therefore, we get an edge-face k-coloring of G, a contradiction again.

Case 2: $c = f = \phi(f_{uv_1w})$.

Since $d = \phi(uv_2) \neq f = \phi(f_{uv_1w}) = c$ and $f \neq g = \phi(f_{uv_2w})$, we remove the color $e = \phi(wv_2)$ from the edge wv_2 and color it with the color f and color the edge wv_1 with the color e. If there is a color in $\{a,\ b\} \setminus \{e,\ f\}$, then we can color th edge uv_1 with this color. Therefore, $\{a,\ b\} = \{e,\ f\}$. Remove the color d from the edge uv_2 and color it with the color e and color the edge uv_1 with the color d. By Lemma 4.4.2, we may further extend the coloring ϕ to obtain an edge-face k-coloring of G, a contradiction again. ■

Proposition 5 (Configuration D) *If G contains the following configuration illustrated in Figure 5, where $d(v_i) = 2$, for each $i = 1, 2$ and the circuit uv_iwu forms a boundary of a face, for each $i = 1, 2$, then*

(1) for each edge-face k-coloring ϕ of $G \setminus \{v_1, v_2\}$ which can be viewed as a partial edge-face k-coloring of G on $[E(G) \cup F(G)] \setminus \{uv_1, uv_2,\ wv_1, wv_2,\ uv_1wu,\ uv_2wu\}$, let $a_i = \phi(f_i)$ where f_i is the face incident with the vertex v_i other than the triangle face uwv_iu for each $i = 1,\ 2$, we claim that

$$\{a_1, a_2\} = C \setminus \phi(u) = C \setminus \phi(w).$$

and consequently,
(2)

$$d(u) = d(w) = k.$$

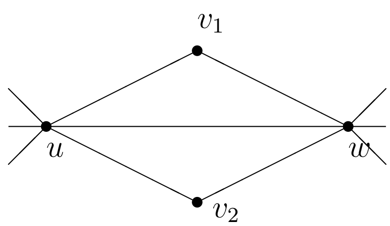

Figure 5

Proof.

Proof of (1). We first prove that $\{a_1, a_2\} = C \setminus \phi(u)$. Notice that $|C \setminus \phi(u)| = k - (d_G(u) - 2) \geq 2$. Therefore, it is sufficient to prove that $[C \setminus \phi(u)] \setminus \{a_1, a_2\} = \emptyset$. By way of contradiction, we assume that $[C \setminus \phi(u)] \setminus \{a_1, a_2\} \neq \emptyset$. Then, $|[C \setminus \phi(u)] \setminus \{a_1, a_2\}| \geq 1$.

I. If one can extend the partial coloring ϕ to the uncolored edges uv_1, v_1w, uv_2 and v_2w, then, by Lemma 4.4.2, the coloring ϕ can be further extended to the uncolored triangle face uv_iwu for each $i = 1,\ 2$. Therefore, we only need to find a way to adjust (if necessary) and then to extend the coloring ϕ to further color the uncolored edges uv_1, v_1w, uv_2 and v_2w.

II. We consider the following two cases:

Case 1: $|[C \setminus \phi(u)] \setminus \{a_1, a_2\}| \geq 2$.

Let $\{b_1,\ b_2\} \subseteq [C \setminus \phi(u)] \setminus \{a_1,\ a_2\}$ and $\{c_1,\ c_2\} \subseteq [C \setminus \phi(w)]$. By Lemma 4.4.1, there is a one-to-one function $f_1 : \{a_1,\ a_2\} \mapsto \{c_1,\ c_2\}$ such that $f_1(a_i) \neq a_i$ for each $i = 1,\ 2$. By Lemma 4.4.1 again, there is a one-to-one function $f_2 : \{f_1(a_1),\ f_1(a_2)\} \mapsto \{b_1,\ b_2\}$ such that $f_2(f_1(a_i)) \neq f_1(a_i)$ for each $i = 1, 2$. Therefore, we can color the edges uv_1, uv_2, wv_1 and wv_2 with the colors $f_2(f_1(a_1))$, $f_2(f_1(a_2))$, $f_1(a_1)$ and $f_1(a_2)$, respectively.

Case 2: $|[C \setminus \phi(u)] \setminus \{a_1,\ a_2\}| = 1$.

Notice that $|C \setminus \phi(u)| = k - (d_G(u) - 2) \geq 2$. Then either $a_1 \in C \setminus \phi(u)$ or $a_2 \in C \setminus \phi(u)$. Without loss of generality, we assume that $a_1 \in C \setminus \phi(u)$. Let $b \in [C \setminus \phi(u)] \setminus \{a_1,\ a_2\}$ and $\{c_1,\ c_2\} \subseteq C \setminus \phi(w)$. By Lemma 4.4.1, there is a one-to-one function $f : \{b,\ a_2\} \mapsto \{c_1,\ c_2\}$ such that $f(b) \neq b$ and $f(a_2) \neq a_2$. Therefore, if $a_1 \notin \{c_1,\ c_2\}$, we can color the edges uv_1, uv_2, wv_1 and wv_2 with the colors b, a_1, $f(b)$ and $f(a_2)$, respectively. Therefore, $a_1 \in \{c_1, c_2\}$. We assume that $a_1 = c_1$. Then,

$c_2 \neq a_1$. Denote $d = \phi(uw)$. Remove the color d from the edge uw and then color it with the color a_1. We further color the edges uv_1, uv_2, wv_1 and wv_2 with the colors $d,\ b,\ c_2,\ d$, respectively.

Therefore, $\{a_1, a_2\} = C \setminus \phi(u)$. Similarly, we can prove that $\{a_1, a_2\} = C \setminus \phi(w)$.

Proof of (2). By (1), we have that $|[C\setminus\phi(u)]| = |[C\setminus\phi(w)]| = 2 = k-(d_G(u)-2) = k-(d_G(w)-2)$. Thus, $d_G(u) = d_G(w) = k$. ■

Proposition 6 *G does not contain the configuration illustrated in Figure 6 where $d(v_i) = 2,\ i = 1,\ 2,\ 3,\ 4$ and $uvv_1u,\ uvv_2u,\ vwv_3v,\ vwv_4v$ are faces.*

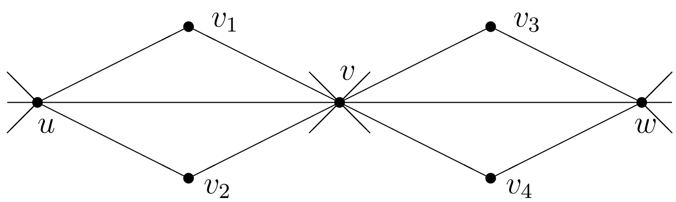

Figure 6

Proof. By way of contradiction, we assume that G contains this configuration. By Proposition 5, $d_G(u) = d_G(v) = d_G(w) = k$. Let $G_5 = G\setminus\{v_1, v_2, v_3, v_4\}$. Then G_5 remains 2-connected and simple and $\chi_e(G_5) \leq \chi_e(G) \leq k$. Let $\phi : E(G_5) \cup F(G_5) \mapsto C$ be an edge-face k-coloring of G_5. The coloring ϕ can be viewed as a partial edge-face k-coloring of G on $[E(G)\cup F(G)]\setminus\{uv_1,\ uv_2,\ vv_1,\ vv_2,\ vv_3,\ vv_4,\ wv_3,\ wv_4,\ uv_1vu,\ uv_2vu,\ vv_3wv,\ vv_4wv\}$. Then $d_{G_5}(v) = k-4$, $d_{G_5}(u) = d_{G_5}(w) = k-2$. Thus $|C \setminus \phi(v)| = 4$ and $|C \setminus \phi(u)| = |C \setminus \phi(w)| = 2$. Let $\{a, b, c, d\} = C \setminus \phi(v)$, $\{e, f\} = C \setminus \phi(u)$ and $\{g, h\} = C \setminus \phi(w)$ where $a \notin \{g,\ h\}$. Denote f_i the face incident with the vertex v_i other than the triangle incident with the vertex v_i and $a_i = \phi(f_i)$ for each $i = 1,\ 2,\ 3,\ 4$.

I. By Proposition 5 and Lemma 4.4.2, it is sufficient to find a way to adjust (if necessary) and then to extend the coloring ϕ to the edges $uv_1,\ uv_2,\ vv_1,\ vv_2$ such that the color set of the remaining two colors for the edges $vv_3,\ vv_4$ is not $\{g, h\}$.

II. We claim that $|\{b,\ c,\ d\} \setminus \{a_1,\ a_2\}| \leq 1$. Otherwise, assume that $\{b,\ c\} \cap \{a_1,\ a_2\} = \emptyset$. Therefore, we may first color the edges $uv_1,\ uv_2$ with the colors $e,\ f$ properly. By Lemma 4.4.1, we may further color the edges vv_1 and vv_2 with the colors $b,\ c$ properly. Hence, the remaining two colors for the edges $vv_3,\ vv_4$ are a and d where $\{a,\ d\} \neq \{g,\ h\}$ since $a \notin \{g,\ h\}$.

III. By II, we may conclude that $\{a_1,\ a_2\} \subset \{b,\ c,\ d\}$. Hence, we may assume that $c = a_1$ and $d = a_2$.

IV. If $\{e,\ f\} \cap \{a_1,\ a_2\} = \emptyset$, then we may first color the edges $vv_1,\ vv_2$ with the colors a_2 and a_1, respectively. By Lemma 4.4.1, we may further color the edges $uv_1,\ uv_2$ with the colors $e,\ f$ properly. Here, the remaining two colors for the edges $vv_3,\ vv_4$ are a and b where $\{a,\ b\} \neq \{g,\ h\}$ since $a \notin \{g,\ h\}$.

Therefore, $\{e,\ f\} \cap \{a_1,\ a_2\} \neq \emptyset$. Without loss of generality, we assume that $f = a_2$. Then, $e \neq a_2$. Denote $m = \phi(uv)$. Remove the color m from the edge uv and color it with the color $a_2\ (= d)$. Then we color the edges $uv_1,\ uv_2,\ vv_1,\ vv_2$ with the colors $m,\ e,\ b,\ m$, respectively. Hence, the remaining two colors for the edges $vv_3,\ vv_4$ are a and $a_1\ (= c)$ where $\{a,\ a_1\} \neq \{g,\ h\}$ since $a \notin \{g,\ h\}$. ■

Proposition 7 (Configuration E) *Assume that G contains the configuration illustrated in Figure 7 where $d(v_1) = d(v_2) = 2$, uv_1vu, vwv_2v are faces and the path uv_1vv_2w is on the boundary of a face, say, $f_{uv_1vv_2w}$. Then,*

$$d(v) \geq k - 2.$$

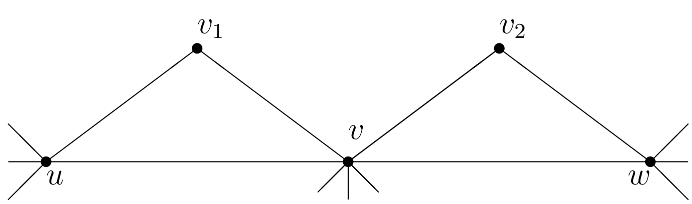

Figure 7

Proof. By way of contradiction, we assume that $d(v) \leq k - 3$. Let G_6 be the graph obtained from G by removing the two edges vv_1 and vv_2 and adding one edge v_1v_2. Then G_6 remains 2-connected and simple and $|E(G_6)| = |E(G)| - 1 < |E(G)|$. By Theorem 4.1.2, $\chi_e(G_6) \leq k$. Let ϕ be an edge-face k-coloring of the graph G_6. The coloring ϕ can be viewed as a partial edge-face k-coloring of G on $[E(G) \cup F(G)] \setminus \{vv_1,\ vv_2,\ uvv_1u,\ vv_2wv\}$. Since $d_G(v) + d_G(v_i) \leq k - 3 + 2 = k - 1$ for each $i = 1,\ 2$, by Lemma 4.4.2 ϕ can be extended to the graph G, a contradiction. ■

Proposition 8 *G does not contain the configuration illustrated in Figure 8 where $d(v) = 4$, $d(v_1) = d(v_2) = 2$ and uv_1vu, vwv_2v are faces.*

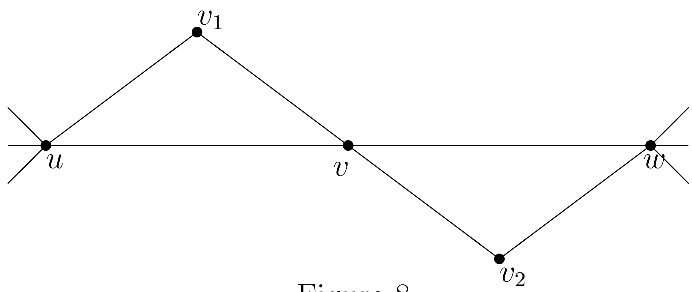

Figure 8

Proof. Let f be the face of G containing the path uv_1vw in its boundary. Then, the graph $G \setminus \{v_1\}$ has an edge-face k-coloring ϕ. The coloring ϕ can be viewed as a partial edge-face k-coloring of G on $[E(G) \cup F(G)] \setminus \{uv_1, \ vv_1, \ uv_1vu\}$ and let $a = \phi(f)$. By Proposition 3, $d_G(u) = k$. We may assume that $\phi(vv_2) \neq a$ otherwise we can replace it with another color from the color set $C \setminus \phi(v)$ since $|C \setminus \phi(v)| = k - (d_G(v) - 1) = k - 3 \geq 2$. Therefore, it is neither Case 3.1 nor Case 3.2 of Proposition 3 since $d(v) = 4 < k - 1$. ■

Proposition 9 *From Proposition 1, any subdivided edge of G is of length at most 2. Let $P = uvw$ or uw be a subdivided edge of G of length at most 2 and f be a face in G incident with P. Denote $G \setminus P = G \setminus E(P)$ if P is of length 1 or $G \setminus P = G \setminus \{v\}$ if P is of length 2. If $G \setminus P$ is 2-connected, then we have either*
(1)

$$d(u) + d(w) > k$$

or
(2)

$$d(f) \geq \frac{k}{2}.$$

Proof. Assume that both (1) and (2) are false. Let ϕ be an edge-face k-coloring of $G \setminus P$. The coloring ϕ can be viewed as a partial edge-face k-coloring of G on $[E(G) \cup F(G)] \setminus [E(P) \cup \{f\}]$. Then this coloring ϕ can be extended to $E(P) \cup \{f\}$ by Lemma 4.4.2 since the uncolored edges and face are in $E_s \cup F_s$ defined in Lemma 4.4.2. ■

4.5.3 The structure of $\zeta_q^{-1}(e)$

With the basic properties of previous subsection, we are ready to determine the structure of the subgraph $\zeta_q^{-1}(e)$ in G, for each positive integer q, and each $e \in E(\zeta_q(G))$.

4.5.3.1 $q = 1$

It is obvious that $\zeta_1^{-1}(e)$ must be a subdivided edge of length at most 2 (by Proposition 1).

4.5.3.2 $q = 2$

For each $e \in E(\zeta_2(G))$ with endvertices u and w, if the multiplicity of e in $\zeta_1(G)$ is at least 2 then $\zeta_2^{-1}(e)$ must be the union of a few subgraphs $J_1, \cdots, J_t$ where $t \geq 2$ is the multiplicity of e in $\zeta_1(G)$, each of which is a subdivided edge of length at most 2 with the endvertices u, w (a Configuration A, by 4.5.3). Hence, by Proposition 2, $\zeta_2^{-1}(e)$ must be one of the Configurations B, C and D described in Propositions 3, 4 and 5.

4.5.3.3 $q = 3$

We claim that $\zeta_3(G) = \zeta_2(G)$. That is, operations stop at $q = 2$.

Proof of the Claim. It is sufficient to prove that there are no subdivided edges of length at least 2 in $\zeta_2(G)$. By way of contradiction, let $P = u_0u_1 \cdots u_r$ be a subdivided edge of length $r \geq 2$ in $\zeta_2(G)$. Denote $e_1 = u_0u_1$ and $e_2 = u_1u_2$. Since $\delta(\zeta_1(G)) \geq 3$, we have that $\zeta_2^{-1}(e_1)$ must be the union of a few subgraphs $J_1, \cdots, J_t$, each of which is a subdivided edge of length at most 2 with the endvertices u_0, u_1 and $\zeta_2^{-1}(e_2)$ must be the union of a few subgraphs $I_1, \cdots, I_s$, each of which is a subdivided edge of length at most 2 with the endvertices u_1 and u_2 where $max\{s, t\} \geq 2$. We consider the following two cases:

Case 1: Either $t = 1$ or $s = 1$. Without loss of generality, we assume that $s = 1$.

Since $s = 1$, we have that $t \geq 2$. By 4.5.3, $\zeta_2^{-1}(e_1)$ is one of the configurations B, C and D described in Propositions 3, 4, and 5 with the terminal vertices u_0 and u_1. On the other hand, by Proposition 2, $t \leq 3$. Therefore, $d_G(u_1) \leq 3 + 1 = 4$. If $\zeta_2^{-1}(e_1)$ is Configuration C or D, we must have that $d_G(u_1) = k$. Hence, $\zeta_2^{-1}(e_1)$ must be Configuration B. In this case, $t = 2$ and therefore, $d_G(u_1) = 2 + 1 = 3 < 4$. This contradict (2) of Proposition 3.

Case 2: Both $t \geq 2$ and $s \geq 2$.

By Proposition 2, we have that $t \leq 3$ and $s \leq 3$. Therefore, $d_G(u_1) \leq 3 + 3 = 6 < k$. Hence, neither $\zeta_2^{-1}(e_1)$ nor $\zeta_2^{-1}(e_2)$ is Configuration C or D. Thus, both of them must be Configuration B. This implies that $d_G(u_1) = 2 + 2 = 4 \leq k - 3$. By

Proposition 8, $\zeta_2^{-1}(u_0u_1u_2)$ must be Configuration E. By Proposition 7, $d_G(u_1) \geq k-2$, a contradiction.

Now, we have proved that $q = 2$, that is, $\zeta_2(G) = \zeta_3(G) = \cdots$. Denote $\zeta_2 = \zeta$ and $\zeta(G) = H$.

Let f be a face in G. We say that the face f' in H is the *corresponding face of f* if f' can be obtained from f by replacing subdivided edge of length 2 with single edges in G. In this sense, we also call f to be the *corresponding face of f'*.

4.5.4 Some further structures of H

4.5.4.1 Classification of edges of H

By the discussion of the previous subsection, we can see that for each edge $e \in H$, $\zeta_2^{-1}(e)$ is one of the Configurations B, C and D, otherwise $\zeta_2^{-1}(e)$ is either a single edge or a single subdivided edge of length 2. Therefore, the edge of H can be partitioned into three classes:

$$E_3 = \{e \in E(H) : \zeta_2^{-1}(e) \text{ is one of the Configurations B, C and D}\},$$

$$E_2 = \{e \in E(H) : \zeta_2^{-1}(e) \text{ is a subdivided edge of length 2}\},$$

$$E_1 = \{e \in E(H) : \zeta_2^{-1}(e) \text{ is a subdivided edge of length 1}\}.$$

Obviously,

$$E_1 \subseteq E(G), \quad E_2 \subseteq E(\zeta_1(G)), \quad E_3 \subseteq E(\zeta_2(G)) = E(H).$$

Furthermore, each edge $e \in E_3$ is called a B-edge, a C-edge, or, a D-edge if $\zeta^{-1}(e)$ is a B-configuration, a C-configuration, or, a D-configuration, respectively; and, each edge $e \in E_i$ $(i = 1, 2)$ is called an E_i-edge ($\zeta^{-1}(e)$ is subdivided edge of length i).

4.5.4.2 Some further structures of H

(I) The relation between $d_G(v)$ and $d_H(v)$ $(v \in V(H))$ is to be discussed here. We claim that, for each $v \in V(H) \subseteq V(G)$,

$$d_G(v) \leq 2d_H(v) + 1; \tag{4.1}$$

$$\text{if} \quad d_G(v) < k, \quad \text{then} \quad d_G(v) \leq 2d_H(v); \tag{4.2}$$

consequently,

$$\text{if} \quad d_G(v) = k, \quad \text{then} \quad d_H(v) \geq \frac{k-1}{2}; \tag{4.3}$$

or, equivalently,

$$\text{if} \quad d_H(v) < \frac{k-1}{2}, \quad \text{then} \quad d_G(v) < k. \tag{4.4}$$

The degrees of a vertex v would be different in the graphs G and H if v is incident with some B-, C- or D-edges in H. However, by Proposition 6, no vertex is incident with more than one D-edge. This proves Inequality (4.1). Furthermore, if $d_G(v) < k$, then by Proposition 5 the vertex v is not incident with any D-edge in H. This proves Inequality (4.2).

Inequalities (4.3) and (4.4) are immediate consequences of Inequality (4.1).

(II) It is obvious that H *is loopless, 2-connected and* $\delta(H) \geq 3$*, and every face is of degree at least* 3. Note that the graph H may have some parallel edges, but they do not form degree 2 faces.

(III) We claim that

$$\text{if} \quad e = uv \in E_3, \quad \text{then} \quad \max\{d_G(u),\ d_G(v)\} = k, \tag{4.5}$$

and,

$$\text{if} \quad \max\{d_H(u),\ d_H(v)\} < \frac{k-1}{2}, \quad \text{then} \quad e \in E_1 \cup E_2. \tag{4.6}$$

Inequality (4.5) is a corollary of Propositions 3, 4 and 5 since $\zeta^{-1}(e)$ is a Configuration B, C or D if $e \in E_3$. Inequality (4.6) is an immediate consequence of Inequalities (4.4) and (4.5).

(IV) By Theorem 3.1, a face of H with positive Euler contribution must be in the following list:

$d_H(f)$	degree sequence around the face
5	3, 3, 3, 3, ≤ 5
4	3, 3, 3, $\leq \Delta$
4	3, 3, 4, ≤ 11
4	3, 3, 5, ≤ 7
4	3, 4, 4, ≤ 5
3	5, 6, ≤ 7
3	5, 5, ≤ 9
3	4, 7, ≤ 9
3	4, 6, ≤ 11
3	4, 5, ≤ 19
3	4, 4, $\leq \Delta$
3	3, 11, ≤ 13
3	3, 10, ≤ 14
3	3, 9, ≤ 17
3	3, 8, ≤ 23
3	3, 7, ≤ 41
3	3, ≤ 6, $\leq \Delta$

A face of H with positive Euler contribution is called a *positive face.*

With further investigation, we will prove that the length of a positive face in H is exactly 3 and the maximum degree of the vertices on its boundary is very large (See (V)).

(V) Let $f' = x_1 \cdots x_d x_1$ be a positive face in H . We claim that

(1)

$$d = 3;$$

(2)

$$\max\{d_H(x_1), \cdots, d_H(x_3)\} \geq 12;$$

(3) Let $d_H(x_3) \geq d_H(x_2) \geq d_H(x_1)$. Then, $d_H(x_2) \leq 11 < \frac{k-1}{2}$, $d_H(x_1) \leq 4 < \frac{k-1}{2}$ and $d_H(x_1) + d_H(x_2) \leq 14$.

Proof of (V) (1) Let f be the corresponding face of f' in G. For each $1 \leq i \leq d$, since H is 2-connected and $\delta(H) \geq 3$, it is obvious that either $H \setminus \{x_{i-1}x_i\}$ or $H \setminus \{x_i x_{i+1}\}$ remains 2-connected. Therefore, if $d \geq 4$, then, by (IV), there exits an edge uv adjacent to f' with $d_H(u) \leq 4$ and $d_H(v) \leq 5$ such that $H \setminus \{uv\}$ is 2-connected. Therefore, $G \setminus E(\zeta_2^{-1}(uv))$ is also 2-connected and simple. By Inequality (6), $uv \in E_1 \cup E_2$. Obviously, $d_G(f) \leq 2 \times d_H(f') \leq 2 \times 5 = 10 \leq \frac{k-1}{2}$. By Inequality (4), we have that $d_G(u) < k$ and $d_G(v) < k$ and therefore, by Inequality (2), $d_G(u) + d_G(v) \leq 2 \times (d_H(u) + d_H(v)) \leq 2 \times (4+5) = 18 < 24 \leq k$. It contradicts to Proposition 9. Therefore, $d = 3$.

(2) We may assume that $d_H(x_1) \leq d_H(x_2) \leq d_H(x_3)$. By way of contradiction, we assume that $d_H(x_3) \leq 11$. Then, $d_H(x_i) < \frac{k-1}{2}$ and, by Inequality (4), we have that $d_G(x_i) < k$ for each $i = 1,\ 2,\ 3$. Therefore, in H, no C-edges or D-edges are incident with the vertex x_i and, by Inequality (6), $x_i x_{i+1} \in E_1 \cup E_2$ for each $i = 1,\ 2,\ 3$.

Denote by m_i the number of B-edges incident with the vertex x_i in H. Then $d_G(x_i) = d_H(x_i) + m_i$. It is obvious that either $G \setminus E(\zeta_2^{-1}(x_1x_2))$ is 2-connected and simple or $G \setminus E(\zeta_2^{-1}(x_1x_3))$ is 2-connected and simple. Without loss of generality, we assume that $G \setminus E(\zeta_2^{-1}(x_1x_3))$ is 2-connected and simple. Let ϕ be an edge-face k-coloring of $G \setminus E(\zeta_2^{-1}(x_1x_3))$. Remove the colors from those edges which have endvertices x_i and a 2-vertex for each $i = 1,\ 3$. Notice that there are at least m_i 2-vertices adjacent to x_i for each $i = 1,\ 2,\ 3$. If $\zeta_2^{-1}(x_1x_3) = x_1x_3$, there are at least $k - (d_G(x_1) + d_G(x_3) - 2) + m_1 + m_3 - 1 = k - (d_H(x_1) + m_1 + d_H(x_3) + m_3 - 2) + m_1 + m_2 - 1 = k - (d_H(x_1) + d_H(x_3)) + 1 \geq 24 - 14 + 1 = 11$ colors available for the edge x_1x_3. If $\zeta_2^{-1}(x_1x_3) = x_1x_0x_3$ where $d_G(x_0) = 2$. Then ϕ can be viewed as a partial edge-face k-coloring of G on $[E(G) \cup F(G)] \setminus \{x_1x_0,\ x_0x_3,\ f\}$. Notice that the uncolored face f is the corresponding face of f' whose length is at most $3 \times 2 = 6$ and the uncolored edges are in E_s (defined in Lemma 4.4.2). Therefore, by Lemma 4.4.2, ϕ can be extended to the graph G. A contradiction.

(3) is obvious by the table of (IV).

Let $f' = uvwu$ be a positive face in H with $d_H(u) \leq d_H(v) \leq d_H(w)$. The edge uv is called *special* and the face in H incident with the special edge uv other than the face f' is also called *special* with respect to the edge uv.

The strategy and the outline of the remaining part of the proof. We will re-assign the Euler contribution of the graph H (or, commonly called *charge/discharge*) in subsection 4.5.4 as follows: The Euler contribution of every positive face will be discharged to a special face by crossing a special edge. Consequently, we will show that, after re-assignment, H will have no face with positive charge. It is obvious that the new charges of the non-special faces are non-positive. We will prove that the new charge of each special face remains non-positive. Notice that each special face receives some charge from adjacent positive faces sharing special edges.

In order to keep the new charge of a special face non-positive, it is sufficient to prove that the initial charge of a special face is negative and that the magnitude of its initial charge is very large. By Theorem 4.3.1, the initial charge (Euler contribution) of a face is determined by its length and the degrees of the vertices on its boundary. Therefore, it is sufficient to prove that the length of each special face is large enough (see (VII)) and that there is enough number of vertices with large degree (see (VIII)).

Some notations:

Denote $SPE(H)$, the set of all special edges of H and $SPE_1(H)$, the set of all such special edges both of whose endvertices are of degree 3 in H. Denote $SPE_2(H) = SPE(H) \setminus SPE_1(H)$

(VI) For each special edge $uv = e \in SPE(H)$ with uvw as the adjacent positive face, we have that

(1) $e \in E_1 \cup E_2$;

(2) For any $A \subseteq E(G) \cup F(G)$, any partial edge-face k-coloring ϕ of G on A can be adjusted and then extended to $A \cup \zeta^{-1}(e)$;

(3) $G \setminus E(\zeta^{-1}(e))$ is not 2-connected, and the vertex w is the cut-vertex of the graph $G \setminus E(\zeta_2^{-1}(e))$;

(4) $e \in E_1$.

Proof (VI) (1) It is obvious by Inequality (6) and (V)-(3) that $e \in E_1 \cup E_2$.

(2) It is sufficient to show that for each $e = uv \in Q'$, the coloring ϕ can be adjusted and then, extended to the edges in $E(\zeta^{-1}(e))$. By (V)-(3), $d_H(u) \leq d_H(v) \leq 11 < \frac{k-1}{2}$. Therefore, by Inequality (4), $\max\{d_G(u),\ d_G(v)\} < k$. Thus, the vertices u, v are not incident with any $C-$ or $D-$edges by Prosit ions 4 and 5.

Denote by m_1 the number of B-edges incident with u and m_2 the number of B-edges incident with v. Then $d_G(u) = d_H(u) + m_1$ and $d_G(v) = d_H(v) + m_2$. Let $E' =$ the set of edges in G with endvertices u and a 2-vertex and $E'' =$ the set of edges in G with endvertices v and a 2-vertex. Notice that, by (V)-(3), $d_H(u) + d_H(v) \leq 14$.

Remove the colors from the edges in $E' \cup E''$ and then color the edges in $\zeta^{-1}(e)$ since there are at most $d_H(u) - 1 + d_H(v) - 1 + 2 \leq 14$ forbidden colors for each of those edges. Since $e = uv \in E_1 \cup E_2$, there are at most $d_H(u) - 1$ B-edges incident with the vertex u. Therefore, $d_G(u) = d_H(u) + m_1 \leq d_H(u) + d_H(u) - 1 \leq 11 + 11 - 1 = 21$ since $d_H(u) \leq 11$. Similarly, $d_G(v) = d_H(v) + m_2 \leq 21$. Therefore, for each edge xy

in $E' \cup E''$, $d_G(x) + d_G(y) \leq 2 + 21 = 23 \leq k - 1$. By Lemma 4.4.2, we can recolor the edges in $E' \cup E''$.

(3) If $G \setminus \zeta^{-1}(e)$ is 2-connected, then it has an edge-face k-coloring ϕ. By (2), the coloring ϕ can be adjusted and then extended to the edges of $\zeta^{-1}(e)$. Since $d_G(uvw) \leq 6$, by Lemma 4.4.2, the coloring ϕ can be further extended to the positive face $uvwu$ and therefore the entire G.

(4) By (1), assume that $e \in E_2$. By (3), since $G \setminus \zeta^{-1}(e)$ has a cut-vertex w, it is impossible that $G \setminus \zeta^{-1}(e)$ has an edge joining u and v. By Inequality (1) and (V)-(3), one of $d_G(u)$ and $d_G(v)$ is at most $2 \times 4 + 1 = 9 < k - 2$. This contradicts Proposition 1-(2) that the degree of each of $\{u, v\}$ must be at least $k - 2$ in G.

(VII) For any special face f'', let f be its corresponding face in G. Denote s the number of special edges in the boundary of f''. Then, we claim that

$$2d_H(f'') \geq d_G(f) + s; \tag{4.7}$$

$$d_G(f) \geq \frac{k}{2} + s; \tag{4.8}$$

$$d_H(f'') \geq \frac{k}{4} + s. \tag{4.9}$$

Proof of (VII) (4.7) Let $e \in E(f'')$ in H. If $\zeta^{-1}(e)$ is not an edge in G, then the subgraph of G induced by $\zeta^{-1}(e)$ must be an E_i-, B-, C-, or D-edge. Thus, the edge e in H corresponds to a subdivided edge of length 1 or 2 around the boundary of f in G. By (VI)-(4), every special edge is an original edge in G. Therefore,

$$d_G(f) \leq 2d_H(f'') - s.$$

(4.8) Let uv be a special edge incident with f''. By (VI)-(3), $G \setminus \{uv\}$ is not 2-connected. Let $f' = vuw$ be the positive face adjacent to the special edge $e = uv$ in H. By (VI)-(3), w is a cut vertex in $G \setminus \{e\}$. Moreover, w separates $G \setminus \{e\}$ into two blocks, say, G' and G'', and each block is 2-connected and they share the face f and the vertex w. Thus, G' and G'' both have edge-face k-colorings ϕ' and ϕ'' such that $\phi'(w) \cap \phi''(w) = \emptyset$. Denote F' the set of all faces of G adjacent to the face f'' in H whose corresponding faces in H are positive, and E' the set of all special edges incident with the face f. We remove the colors from the faces and edges of $E' \cup F' \cup \{f\}$. Then we can combine the colorings ϕ' and ϕ'' into a partial edge-face-coloring ϕ of G: $\phi : [E(G) \cup F(G)] \setminus [E' \cup F' \cup \{f\}] \mapsto C$.

If the partial coloring ϕ can be extended to the special face f, we can further color the edges in E' by (VI)-(2) and the faces in F' by Lemma 4.4.2 since by (V), the length of each positive face is at most $2 \times 3 = 6 < \frac{k-1}{2}$. So, the partial coloring ϕ can not be extended to the face f.

Obviously,

$$|E' \cup F'| \geq 2s.$$

Thus, there are at most

$$2d_G(f) - |E' \cup F'| \leq 2d_G(f) - 2s$$

forbidden colors for the face f.

Assume that

$$2d_G(f) - 2s \leq k - 1.$$

Then there are at least $k-(2d_G(f)-2s) \geq 1$ colors available for the face f. Therefore, ϕ can be extended to the face f. A contradiction. Hence, we must have that

$$2d_G(f) - 2s \geq k.$$

(4.9) By Inequalities (4.7) and (4.8), we have that

$$2d_H(f'') \geq d_G(f) + s \geq [\frac{k}{2} + s] + s.$$

Hence,

$$d_H(f'') \geq \frac{k}{4} + s.$$

(VIII) For each special edge $uv \in SPE_1(H)$, let $f' = uvw$ be the positive face adjacent to the edge uv. Let u_1 be the vertex in H adjacent to u other than v and w, and, v_1 be the vertex in H adjacent to v other than u and w. We claim that

$$\max\{d_H(u_1), d_H(v_1)\} \geq \frac{k-4}{2}.$$

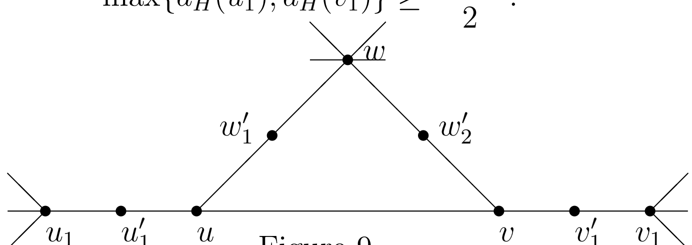

Figure 9

Proof of (VIII) Notice that $d_H(u) = d_H(v) = 3$ since $uv \in SPE_1(H)$.

(a) By way of contradiction, we assume that both $d_H(u_1) < \frac{k-4}{2}$ and $d_H(v_1) < \frac{k-4}{2}$. Then, by Inequality (6), the edges uu_1 and vv_1 are all in $E_1 \cup E_2$ and by (VI)-(4), $uv \in E_1$. Let $uu_1' \in E(\zeta^{-1}(uu_1))$ and $vv_1' \in E(\zeta^{-1}(vv_1))$. Note that either $u_1' = u_1$ or $d_G(u_1') = 2$ and either $v_1' = v_1$ or $d_G(v_1') = 2$. Denote f'' the face in H adjacent to the face f' and incident with the edge uw. Denote f_1, f_2 the corresponding faces of f' and f'' in G, respectively.

(b) We claim that both $uw \in E_1 \cup E_2$ and $vw \in E_1 \cup E_2$.

By way of contradiction, we assume that $uw \in E_3$. Since $d_H(u) \leq 11 < \frac{k-1}{2}$, by Inequality (4), $d_G(u) < k$. Therefore $\zeta^{-1}(uw)$ must be a B-edge. Let w_3 be the only 2-vertex in $\zeta^{-1}(uw)$. If w_3 is on the boundary of f_1. Then, ww_3 is adjacent to two faces with length at most 6 and $d_G(w_3) + d_G(u) \leq 2 + 2 \times 11 = 24 \leq k$ and uw_3 is also adjacent to two faces with length at most 6. Therefore, by Lemma 4.4.2, any edge-face k-coloring of the graph $G \setminus \{w_3\}$ can be extended to the graph G. Therefore, w_3 must be on the boundary of f_2.

Clearly, $G \setminus \{uw\}$ remains 2-connected and simple. Let ϕ be an edge-face k-coloring of $G \setminus \{uw\}$. Remove the colors from the edges uu_1', uv and uw_3 and from the face f_1. Denote $S = \{uw,\ uu_1,\ uv,\ uw_3,\ f_1, uw_3wu\}$. Then ϕ can be viewed as a partial edge-face k-coloring of G on $[E(G) \cup F(G)] \setminus S$. Obviously, there is at least one color available for the edge uw and color it. Since $S \setminus \{uw\}$ is a subset of $E_s \cup F_s$ defined in Lemma 4.4.2, by Lemma 4.4.2 ϕ can be adjusted and then extended to G. This contradiction shows that $uw \in E_1 \cup E_2$. Similarly, we can also prove that $vw \in E_1 \cup E_2$.

(c) Note that $u_1u,\ uw,\ uv \in E_1 \cup E_2$. We have $d_G(u) = d_H(u) = 3$. Similarly, $d_G(v) = d_H(v) = 3$.

(d) Let $ww_1' \in E(\zeta^{-1}(uw))$ and $ww_2' \in E(\zeta^{-1}(vw))$. Note that either $w_1' = u$ or $d_G(w_1') = 2$, and that either $w_2' = v$ or $d_G(w_2') = 2$. By (VI)-(3), $G \setminus E(\zeta_2^{-1}(uv))$ is not 2-connected. Therefore, $G \setminus E(\zeta_2^{-1}(uw))$ remains 2-connected and simple. Let ϕ be an edge-face k-coloring of the graph $G \setminus E(\zeta_2^{-1}(uw))$. Remove the colors from the edges uu_1', uv, vv_1' and vw_2' (if any). Then, ϕ can be viewed as a partial edge-face k-coloring of G with the elements ww_1', uw_1', uu_1', uv, vv_1', vw_2' (if any) and f_1 uncolored.

Denote $a = \phi(ww_2')$, $b = \phi(f_2)$ and $c \in C \setminus \phi(w)$. If $c \neq b$, we can color the edge ww_1' with the color c. If $c = b$, remove the color a from the edge ww_2' and then color it with the color b and then color the edge ww_1' with the color a. The remaining uncolored elements are the edges uu_1', vv_1', uw_1' (if any) and vw_2' (if any) and the face f_1. Notice that these elements are all in $E_s \cup F_s$ defined in Lemma 4.4.2. Therefore,

ϕ can be extended to the graph G. A contradiction.

4.5.4.3 Charge and Discharge

Consider Φ, the Euler contribution of H, as the initial charge of the face set of H. We will reassign a new charge Φ' to each face of H as follows. Each positive face f' sends its total amount of its Euler contribution $\Phi(f')$ to the adjacent special face sharing the special edge with it by crossing the special edge.

We now check the new charge $\Phi'(f')$.

(a) For each non-special face f^* with $\Phi(f^*) \leq 0$, the charge remains the same. That is,

$$\Phi'(f^*) = \Phi(f^*) \leq 0.$$

(b) For each positive face f' in H,

$$\Phi'(f') = \Phi(f') - \Phi(f') = 0,$$

and if the positive face f' is adjacent to a special edge in SPE_1, then

$$\Phi(f') \leq 1 - \frac{3}{2} + 2 \times \frac{1}{3} + \frac{1}{12} = \frac{1}{4}.$$

If the positive face f' is adjacent to a special edge in SPE_2, then

$$\Phi(f') \leq 1 - \frac{3}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{12} = \frac{1}{6}.$$

In summary, each positive face in H discharges either $\leq \frac{1}{6}$ or $\leq \frac{1}{4}$ to an adjacent special face sharing the special edge with it by crossing a special edge $e \in SPE_2$ or $e \in SPE_1$, respectively.

(c) For each special face f', denote $r = d_H(f')$ and s_i the number of special edges in $SPE_i(H)$ adjacent to f' for each $i = 1, 2$.

(d) By (VIII), there are at least $\frac{s_1}{2}$ vertices in $B(f')$ with degrees at least $\frac{k-4}{2}$.

(e) By Inequality (9), we have that

$$r \geq \frac{k}{4} + (s_1 + s_2). \tag{4.10}$$

Therefore

$$\begin{aligned}
\Phi'(f') &\leq \Phi(f') + \tfrac{s_1}{4} + \tfrac{s_2}{6} \\
&= 1 - \tfrac{r}{2} + \textstyle\sum_{v \in B_H(f')} \tfrac{1}{d_H(v)} + \tfrac{s_1}{4} + \tfrac{s_2}{6} \\
&\leq 1 - \tfrac{r}{2} + \left[\tfrac{s_1}{2} \times \tfrac{2}{k-4} + (r - \tfrac{s_1}{2}) \times \tfrac{1}{3}\right] + \tfrac{s_1}{4} + \tfrac{s_2}{6} \quad \text{(by (d))} \\
&= 1 - \tfrac{r}{6} + \tfrac{s_1}{k-4} + \tfrac{s_1}{12} + \tfrac{s_2}{6} \\
&\leq 1 - \left[\tfrac{k}{24} + \tfrac{s_1+s_2}{6}\right] + \tfrac{s_1}{12} + \tfrac{s_2}{6} + \tfrac{s_1}{k-4} \quad \text{(by (e))} \\
&= 1 - \tfrac{k}{24} - \tfrac{s_1}{12} + \tfrac{s_1}{k-4} \\
&\leq 0 \quad \text{(since } k \geq 24\text{)}.
\end{aligned}$$

Thus,

$$2 = \sum_{f' \in F(H)} \Phi(f') = \sum_{f' \in F(H)} \Phi'(f') \leq 0$$

A contradiction.

This completes the proof of Theorem 4.1.3.

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