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Hegel's Theory of Quantity

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INTRODUCTION

What is quantity? Surprisingly, virtually no philosopher before Hegel ever hazarded a definition.[1] Rather, the concept is viewed as self-evident.

To my knowledge, Hegel is the first philosopher to provide a sustained definition. In common mathematical discourse, "[a] magnitude is usually defined as that which can be increased or diminished,"[2] Hegel remarks. This, Hegel finds, is a poor definition. What does "increase" mean? It means "make the magnitude more." What does "diminish" mean? It means "make the magnitude less." Covertly the word defined ("magnitude") appears in the definition. Nothing is learned from such a definition, except that magnitude is magnitude. To rectify the poverty of this definition, Hegel dedicated more pages of his monumental *Science of Logic* to this analytical task than to any other.

In the course of presenting philosophy's most rigorous definition of quantity, Hegel shows that numbers are not what they seem. Figures do lie after all. To illustrate, Hegel presents the well-known fact that a given natural number can, through well accepted mathematical laws, be expressed as an infinite series that merely approaches but never reaches

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1 The "analytical" philosophers give credit to Gottlob Frege (1848-1925) as the first to hazard number's definition. Roughly, Frege held number to be the universe of sets with the same number of members. For example, 3 is the set of all triads. For an excellent comparison of Frege and Hegel, see Alan L.T. Paterson, FREGE AND HEGEL ON CONCEPTS AND NUMBER (2000) (unpublished manuscript). This manuscript can be retrieved over the Google search engine on the internet. Professor Paterson proclaims the Fregean concept to be too "rigid" and suggests that Frege proceeded by abstraction while Hegel proceeded by deriving quantity from quality.

2 G.W.F. HEGEL, HEGEL'S SCIENCE OF LOGIC 186 (A.V. Miller trans., 1969) [hereinafter SCIENCE OF LOGIC].

the number. For example, $1/(1-a)$ can be expressed as $1 + a + a^2 + a^3 \ldots a^n$, where $a < 1$. If $a = 0.5$, the above expression approaches, but never reaches, 2.

The implication of this is profound. Apparently, numbers are never entirely present, and arithmetic is, to borrow the Derridean phrase, a "philosophy of presence."[3] In effect, every number is "filled in" by the mathematician, so that the ellipsis (" . . . ") is suppressed and the number seems complete. Numbers, Hegel will conclude, are constituted by an absence. Hegel will identify this absence as the very quality of quantity. For Hegel, quantity stands for absolute openness to determination by something external to itself. Quantity has no integrity—no content of its own. Quantity is that which absolutely refuses to define itself. Paradoxically, quantity defines itself precisely when it refuses to define itself. When this pure refusal is isolated, quantity has defined itself after all and has exhibited its true quality.

This article—second in a series of nine[4]—explicates Hegel's theory of quantity from his *Science of Logic*.[5] The article continues a series of pictographic conventions developed in the earlier essay. According to the conventions I have developed, logic is divided into three distinct moves, which repeat themselves over and over.

The first step belongs to what Hegel calls the Understanding. The Understanding begins by making a proposition about the universe. This proposition, however, is one-sided. The Understanding is basically stupid. It sees things immediately and one-sidedly. It refuses at first to see that everything is mediated (though, slowly, it is learning from its mistakes). Nevertheless, the Understanding is necessary to the logical system. There must be some proposition if the second and third steps are to do their work.

The second step is that of Dialectical Reason. Dialectical Reason knows that the Understanding's proposition is one-sided. It brings into the light what the Understanding has suppressed. Dialectical Reason is the voice of *experience*.[6] It retrieves what has been learned before and sets this suppressed material in opposition to the proposition of the Understanding.

Dialectical Reason brings forth what the Understanding has left

3 For a description of "philosophy of presence," see David Gray Carlson, *On the Margins of Microeconomics*, 14 CARDOZO L. REV. 1867, 1869 (1993).

4 For the first installment, see David Gray Carlson, *Hegel's Theory of Quality*, 22 CARDOZO L. REV. 425 (2001).

5 All numbers in parentheses refer to page numbers from SCIENCE OF LOGIC, *supra* note 2. I have also omitted ellipses at the end of any quoted phrase. An ellipsis signals that a sentence does not end with the quoted words. Hegel's sentences, however, never end, and so ellipses convey no useful information.

6 *See* KENNETH R. WESTPHAL, HEGEL'S EPISTEMOLOGICAL REALISM: A STUDY OF THE AIM AND METHOD OF HEGEL'S *Phenomenology of Spirit* 130 (1989).

out. But in positivizing this negated material, Dialectical Reason suffers from the same one-sidedness as the Understanding. Dialectical Reason is the pot calling the kettle black. It is as one-sided to emphasize what is *not* as it is to emphasize what *is*. Speculative Reason, in the third stage, sees this. It intervenes, like an exasperated parent mediating between two quarrelsome siblings. It points out that both sides are right *and* wrong. Speculative Reason maintains that the positive and negative must be thought together.

Once Speculative Reason reconciles the positions of the Understanding and Dialectical Reason, the Understanding again takes the stage to formulate a proposition about what it has learned. It reduces the lesson to a one-sided proposition. Dialectical Reason once again critiques the effort, and Speculative Reason mediates the dispute. These three steps replicate themselves again and again and again—until mediation exhausts itself and confesses (at the very end of the *Science of Logic*) that mediation is an immediacy after all.

By the time Hegel reaches the concept of Quantity,[7] much progress has been achieved. At the beginning of the Logic, the Understanding put forth Being as something positive or affirmative. But Dialectical Reason showed that Being is just as much Nothing. Speculative Reason has stated that both sides are right. Being is in flux. It turns into nothing; it "ceases to be." And Nothing is constantly "coming to be."

The Understanding gets smarter as the *Science of Logic* continues. Having learned that pure immediate Being is defeated by its origin in mediatedness, the Understanding begins to propose that Being is Nothing (just as Dialectical Reason predicted). A very key moment in this development is the idea of True Infinity—one of Hegel's most startling contributions to philosophy. The Understanding has proposed that Being is Finite. It must come to an end. Since Logic is timeless in quality, Finite Being is "always already" at an end. As Hegel memorably put it, "the being as such of finite things is to have the germ of decease as their being-within-self: the hour of their birth is the hour of their death." (129)

Speculative Reason teaches that, while this is true, Finitude is a one-sided point of view. Even if Finite things end, Finitude as such endures infinitely. There is an infinite memory of what was. The upshot of True Infinity is that Being (*i.e.*, immediate reality) repeals itself. But in abolishing itself, reality *preserves* itself (in thought). The slogan relevant to the True Infinite is that *it becomes something else*

7 Hegel performs certain "necessary" logical progressions, but also includes a great many asides and commentaries that are "for us" and not strictly part of the Logic. I capitalize any term to the extent that it qualifies as an official step in the Logic. Thus, Quality, Quantity and Measure, are the first three meta-steps in the *Science of Logic*.

while remaining what it is.[8] This formulation expresses the law of "sublation." Sublation means cancellation and preservation *simultaneously*.[9]

Upon reaching the realm of Quantity, a word of comfort is in order. Many readers will suffer from "math anxiety." Such readers will have nothing to fear from Hegel. With the exception of some notorious (and quite extraneous) remarks on calculus, very little in Hegel's Quantity chapters extends beyond rudimentary algebra. Hegel was no great champion of math[10]—though his education in it was formidable.[11] In fact, he had great contempt for its spiritual worth.[12] Nevertheless, Quantity has an important place in the *Science of Logic*. In this first chapter on Quantity (the fourth chapter in the Logic),[13] Hegel equates Pure Quantity with time, space and the ego—deeply metaphysical ideas. Everything that follows depends upon the logical attributes of Quantity.

For Hegel, Quality precedes Quantity. "[H]itherto," Hegel observes, "the determination of *quantity* has been made to precede *quality* . . . for no given reason." (79) Thus, Hegel reverses Kant's preferred order.[14] Errol Harris suggests why:

> Kant gives quantity precedence over quality but that is because he maintains that the categories are applicable only to sensuously intuited experience the *a priori* forms of which are space and time. Space and time, therefore, take precedence over that which fills

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8 Carlson, *supra* note 4, at 541. As Hegel will say later, "The qualitative finite *becomes* the infinite; the quantitative finite is in its own self its beyond and *points beyond itself*." (372) *See also* ANDREW HAAS, HEGEL AND THE PROBLEM OF MULTIPLICITY 79 (2000) ("within its determination, a being is always beyond its determination, always signifies that which transcends determination").

9 *See* Carlson, *supra* note 4, at 453.

10 *Science of Logic*, *supra* note 2, at 120; G.W.F. HEGEL, PHENOMENOLOGY OF SPIRIT ¶ 42 (A.V. Miller trans., 1977) [hereinafter PHENOMENOLOGY]; *see also* CHARLES TAYLOR, HEGEL 247 (1975) ("a low view of mathematics as a philosophical language").

11 The details of this education are set forth in Michael John Petry, *The Significance of Kepler's Laws*, *in* HEGEL AND NEWTONIANISM 439, 476-83 (Michael John Petry ed., 1993) [hereinafter HEGEL AND NEWTONIANISM].

12 *See* Carlson, *supra* note 4, at 471-73. Hegel calls mathematics a "subordinate field." (27) He refers to the "dead bones" of mathematical logic. (54) Its claim to "necessity" was inadequate, and its practitioners do nothing but ward off heterogeneity, an act itself tainted with heterogeneity. (40) In these remarks, and many others, Hegel will anticipate Gödel's critique of mathematics as inherently contingent and subjective. *See* Michael Kosok, *The Formalization of Hegel's Dialectical Logic: Its Formal Structure, Logical Interpretation and Intuitive Foundation*, *in* HEGEL: A COLLECTION OF CRITICAL ESSAYS 237, 263 (Alasdair MacIntyre ed., 1972) ("[D]ialectic logic can be taken as a way of generalizing Goedel's theorem, and instead of regarding it merely as a *limitation* to the expression of consistent systems in ordinary logical structures, it now becomes the *starting point* for a dialectic logic, which regards these limitations as the essence of its structure.").

13 Hegel starts renumbering his chapters after every section, so there is no "chapter 4" in the Quantity chapters.

14 IMMANUEL KANT, CRITIQUE OF PURE REASON 104-05 (J.M.D. Meiklejohn trans., 1990).

them, and space and time are quantitative schemata . . . .[15]

For Kant, space and time are subjective. They are added to the object by consciousness. For Hegel, however, space and time are Pure Quantities, which are derived from the very concept of Quality. This implies that "[t]he *externality of space and time* [exists] absolutely on its own account without the moment of subjectivity." (843) Space and time (Quantity) belong to the object (Quality) itself. Being derived from Quality, Quantity can impose itself on the qualitative realm of nature[16] and, to paraphrase G.R.G. Mure, Quantity is able to supervene on a world that is not "wholly unprepared."[17]

In the third chapter of the *Science of Logic*, Quality worked itself pure from a dependence on otherness. It became Being-for-self—being that was utterly indifferent to otherness and hence radically free. Yet "it cannot be conceived of as something which is entirely without relations . . . as was the more basic category of pure being."[18] In effect, Being-for-self is the pure idea of a relation without reference to any parts. In this mode, Being-for-self found out that its otherness was entirely outside itself. Ironically, it found itself completely dependent on otherness to define itself. Instead of being radically free, it was radically unfree. One can say of Being-for-self—now Quantity—that its "will is infinite . . . and [its] act a slave to limit."[19]

Quantity is Quality conceived as pure relation, divorced and separate from the "parts" which it relates. Quantity is devoid of all content. It is "indifferent to its affirmative determinateness." (372) Quantity represents the pure idea of simply *not being Quality*. The job of Quantity over the next three chapters is to recapture its own content. When it succeeds, it will pass over to Measure. Quality, Quantity, and Measure are the three parts of the Doctrine of Being. They may be drawn as follows:

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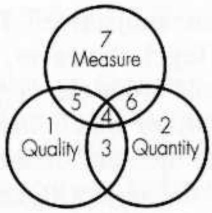
15 ERROL E. HARRIS, AN INTERPRETATION OF THE LOGIC OF HEGEL 124 (1983).

16 Nature—the material universe beyond thought—was shown to be that which is "non-spiritual" in Hegel's chapter on Determinate Being. Carlson, *supra* note 4, at 503-06. According to Hyppolite, nature is "the fall of the idea, a past of reason, rather than an absolute manifestation of reason." JEAN HYPPOLITE, GENESIS AND STRUCTURE OF HEGEL'S *PHENOMENOLOGY OF SPIRIT* 244 (Samuel Cherniak & John Heckman trans., 1974).

17 G.R.G. MURE, THE PHILOSOPHY OF HEGEL 117 (1965).

18 Merry Mule, *Hegel on the Interaction Between Science and Philosophy*, *in* HEGEL AND NEWTONIANISM, *supra* note 11, at 61,67.

19 WILLIAM SHAKESPEARE, TROILUS AND CRESSIDA Act 3 Scene 2.



The Doctrine of Being
Section Two: Magnitude (Quantity)

The preceding illustration invokes the conventions developed in the preceding Article on Quality. According to this convention, the left of the page stands for Being. Here is where the Understanding resides. Its proposition is qualitative. The right side of the page stands for negativity. Here is where Dialectical Reason resides. Its proposition is quantitative. Speculative Reason occupies the middle of the three circles. It is the true Measure of all things. Quantity is therefore a "rightward" leaning discourse, according to this convention. For this reason, Professor Clark Butler is correct to state that Quantity interrupts the development of Quality.[20] That is to say, Quantity is a dialectical critique of Quality.

What follows is a close analysis of how Hegel's logic of Quantity functions. I have replicated the subheadings in the *Science of Logic* precisely as they appear in A.V. Miller's translation of that work. Discussion under any given subhead roughly corresponds to Hegel's discussion. Every time Hegel locates a logical advance, I have memorialized it in a picture. Adding together the thirty official moves in the Quantity chapters, there are 51 logical progressions by the time the sixth chapter of the *Science of Logic* concludes. Twenty-one new steps are analyzed in the current installment. In all, 79 steps are needed to complete the *Science of Logic*.

Also included here—for the first time in English—is a detailed analysis of Hegel's critique of calculus. Readers of the *Science of Logic* usually conclude that these remarks are a complete digression and so they simply skip over them. In the main, they are perfectly right to do so. When the time comes, I will invite my readers to do the same. Nevertheless, for the sake of completeness and in the name of true science, and also out of sheer cussedness, I have analyzed what Hegel has to say about the calculus. For what it is worth, Hegel's critique of the calculus is acute indeed. Nevertheless, for the non-mathematical reader, please be assured that these commentaries can be safely skipped, in which case this article is significantly shorter than may seem to be the

20 CLARK BUTLER, HEGEL'S LOGIC: BETWEEN DIALECTIC AND HISTORY 91 (1996).

case.

I. FROM QUANTITY TO QUANTUM

Prior to the commencement of Hegel's first Quantity chapter, there is a short introductory essay entitled "Magnitude (Quantity)." Here, Hegel states broadly that Quality was "the first, immediate determinateness." (185) Quantity, in contrast, is

> the determinateness which has become indifferent to being, a limit which is just as much no limit, being-for-self which is absolutely identical with being-for-other—a repulsion of the many ones which is directly the non-repulsion, the continuity of them. (185)

A "determinateness" is a concept that combines the contradiction of Being and Nothing—thought together at the same time. A determinateness is therefore always mediated—never an immediate thought.[21] That Quantity is a determinateness can be seen directly as [4, 5, 6] in Figure[22] 10(c).

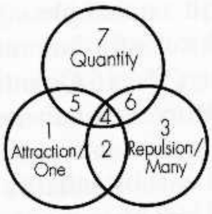


Figure 10 (c)
Quantity

That Quantity is indifferent to "being" was documented through chapter 3 of Quality. In effect, the Understanding constantly repulsed its own content—its own determinateness—until it had worked itself pure. When this process was complete, Quantity was pure indifference to its own Being.

Why, from Quantity's perspective, is "being" a "limit which is just as much no limit"? Limit was a step in Quality, portrayed in Figure

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21 *See* Carlson, *supra* note 4, at 480.

22 A "figure" represents an official move in the *Science of Logic*. In *Hegel's Theory of Quality*, I identified thirty logical moves across the first three chapters of the *Science of Logic*. Any figure labeled (a) represents the proposition of the Understanding. These figures will present a single "immediate" circle, which the Understanding has drawn from the preceding step. Any figure labelled (b) will be the step of Dialectical Reason. Dialectical Reason sees double, and so two overlapping circles are presented. Figure 10(c) is typical of Speculative Reason. Speculative Reason sees Being and Nothing together as a determinateness. Figure 10(c) was the last step in Quality and will be interpreted by the Understanding in Figure 11(a). All of the previous Figures are replicated in Appendix A to this Article.

4(c).[23] By the time Figure 10(a) was reached, Limit has been sublated. It has rendered itself into a mere ideality—a memory of a past reality. Quantity therefore is, on the one hand, distinguishable from "being" in general. But, on the other hand, "being" is no limit, and so Quantity suffuses or "continues" into the heart of external being with no opposition.

In the above-quoted passage, Quantity is said to be Being-for-self. This is obviously true on the law of sublation.[24] As a Being-for-self, it is identical with Being-for-other. This is just to say that Quantity has driven away all its otherness, and so now it has no content of its own. The other must supply *all* its content. Hence, Quantity is nothing *but* Being-for-other.

Finally, Quantity has repulsed the Many Ones (which were equated in chapter 3 of Quality with Attraction). But Quantity is the middle term between Attraction and Repulsion.[25] Hence, Quantity just as much attracts the Many ones. It is "continuous" with them. The idea of Quantity is therefore closely connected with and indeed *is* the idea of continuity.

Continuity. It is worth contemplating on our own the idea of continuity—recently the subject of a law review symposium under the name of "commensurability."[26] If Quantity is continuous, then we must be implying that Quantity is a substance, such that it exists for itself, but also flows—continues—into its other, so that the other can share in this substance. The substantiality of Quantity coheres with Hegel's notions. After all, Hegel has made Quantity the midsection of the Doctrine of Being. It must therefore be some species of "being." Quantity, however, is likewise very negative—though, on the law of sublation, full of inherent positivity. If all "things" contain negativity, then Quantity has continued from itself into these things. Quantity is, in short, the "universal" that all things have in common. For this reason, all "things" are "commensurable." This is the truth of Dialectical Reason, which reigns supreme in Hegel's theory of Quantity. But negativity is only one side of the story. Hence, all "things" are just as much incommensurable.

The law review symposium on commensurability entirely misses this point. What we find there are two groups of legal scholars each caught in the dogma of self-identity. Each side can only assert a one-sided view. Thus, one group thinks things are universally commensurable. These are the utilitarians, who wish to assert that preferences can be weighted and aggregated, by virtue of something

23 *See* Carlson, *supra* note 4, at 519.

24 That is, sublation always obliterates *and* preserves the prior logical steps. *Id.* at 453.

25 *See* Figure 10(c).

26 *See Law and Incommensurability*, 146 U. PA. L. REV. 1169 (1998).

universal that commensurates all things.[27] What commensurates for these scholars is money. Everything has its price. Without this commensurability of utilities in money, policy science would instantly implode. As this cannot be permitted to happen, the concept of commensurability is asserted as a one-sided truth.

The opponents of utilitarianism are rights-based dogmatists who insist that certain things are so sacred that they share nothing in common with other things. Thus, human dignity cannot be sold for cash.[28] These theorists are what Hegel calls "pantheists."[29]

Both sides suffer from self-identity. The utilitarians insist that preferences are quantities—commodities whose difference can be dissolved in the unified numéraire of money. The qualitative aspect of preferences is simply denied. The rights-based libertarians insist that certain values—chosen on an ad hoc basis according to the law of sentimentality which covertly governs their discourse—are complete qualities. The commensurable side of these sacred values is simply denied. Each side can only shout slogans at the other side. No solution is possible, and so the law review symposium on "commensurability" must be counted what Hegel would term a Spurious Infinity.[30]

Hegel provides the solution. Each side is partly right. Human values are commensurable, because they are in part negative and hence quantitative. The negative substance Hegel calls Pure Quantity continues in all discrete things—including the things the rights-based libertarians identify as sacred. Likewise, human values are incommensurable, because they are in part positive entities, just as the

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27 Perhaps the fringe extreme of this view is represented by Eric A. Posner, *The Strategic Basis of Principled Behavior: A Critique of the Incommensurability Thesis*, 146 U PA. L. REV. 1185 (1998), who goes so far to suggest that the very attribution of incommensurability is a strategic trick to obtain commensurable advantage over their fellows. Incommensuration is itself thus made a commodity commensurable with all other commodities. *See id.* at 1208.

28 Cass R. Sunstein, *Incommensurability and Valuation in Law*, 92 MICH. L. REV. 779 (1994).

29 "The maxim of Becoming, that Being is the passage into Nought, and Nought the passage into Being, is controverted by the maxim of Pantheism, the doctrine of the eternity of matter . . . ." G.W.F. HEGEL, HEGEL'S LOGIC § 80 Remark (William Wallace trans., 1975) [hereinafter LESSER LOGIC].

30 Hegel pointedly denounces such thinking when he remarks:

> [P]roceeding analytically, [common sense] now extracts especially identity and *then also again* obtains difference alongside it, is now a positing of likeness and *then also again* a positing of unlikeness—likeness when *abstraction* is made from difference, and unlikeness when abstraction is made from the positing of likeness. These assertions and opinions about what reason does must be completely set aside, since they are in a certain measure merely *historical*; the truth is rather that a consideration of everything that is, shows that *in its own self* everything is in its self-sameness different from itself and self-contradictory, and that in its difference, in its contradiction, it is self-identical, and is in its own self this movement of transition of one of these categories into the other, and for this reason, that each is in its own self the opposite of itself. (412)

rights-based theorists insist.

Where does that leave Hegel on the issue of human rights v. utilitarian contempt for rights? I think Hegel would recognize that each intuition has its moment. Human rights are prior. They set up the boundaries in between which utilitarian calculation is permissible and legitimate. Utilitarian calculus can never be permitted to set its own boundaries. It must not decide who lives and who dies, or who is a slave and who is free, but it might govern in relatively unimportant human institutions, such as market exchange. How shall the borderline between rights and utilitarianism be discovered? In *The Philosophy of Right*,[31] Hegel makes clear that Logic provides no clear answer. Rather, custom and tradition must set the precise border. It is useless for theorists to deduce the location of these borders from pure theory. In short, Hegel is ultimately a pragmatist in his politics, but, of course, when he philosophizes, he operates according to the dictates of necessity. Pragmatic politics is, in contrast, rife with contingency.[32]

Quantity's Indifference. To continue our analysis of Hegel's introductory essay on magnitude, Hegel reminds us that, at the end of chapter 3 of Quality, Being-for-self has been forced to admit that it is the ultimate Being-for-other: "that which is for itself is now posited as not excluding its other, but rather as affirmatively continuing itself into it." (185)

Quantity is therefore "otherness in so far as *determinate being* again appears in this continuity." (185) Of course, Quantity cannot be an otherness unless there is a determinate being into which Quantity can continue. When this other to Quantity appears, Quantity's determinateness will no longer be "in a simple self-relation." (185) That is, the relation will be overtly a relation with an external other. In this relation, Quantity "is posited as self-repelling, as in fact having the relation-to-self as a determinateness in another something (which is *for itself*)." (185) Or, in other words, Quantity will say, "I am not my radically external other." It will refuse to recognize itself in the "other something," but this refusal to recognize is the ultimate recognition.[33] Hence, Quantity is a slave to the other; only this other is truly "for itself."

Quantity and its other will pose as mutually indifferent to one

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31 G.W.F. HEGEL, ELEMENTS OF THE PHILOSOPHY OF RIGHT (Allen W. Wood trans., 1993) [hereinafter PHILOSOPHY OF RIGHT].

32 In warning that philosophers are not licensed to make policy suggestions, Hegel wrote: "Plato could well have refrained from recommending nurses never to stand still with children but to keep rocking them in their arms; and Fichte likewise need not have perfected his *passport regulations* . . . ." *Id.* at 21.

33 This was a major conclusion in Being-for-self, which is pure refusal to recognize the other. Refusal to consider the other was defined by Hegel as "the One." Carlson, *supra* note 4, at 557-59.

another (which is a lie, of course). From this perspective of utter independence, both entities are "indifferent, relationless limits reflected into themselves." (185) In this pose, each entity can say that "determinateness in general is outside itself." (185) These external determinatenesses are external "somethings," with which Quantity (in the false pose we are now considering) has nothing to do. Yet Quantity is also indifferent to its own Limit and so continues into these external somethings. This indifference "constitutes the *quantitative* determinateness of the something." (185)

Preview. Hegel next gives a preview to the first chapter on Quantity. As always, the true demonstration of these ideas must await their detailed unfolding. It is not expected that the reader will fully grasp the import of the preview that follows.

First we have Pure Quantity. This must be distinguished from its more complicated stage—Quantum. The challenge here is to remember that Quantum—*i.e.*, "Number"—is too advanced. We are aiming to isolate the deeper substance of numbers. Number appears only in the second chapter on Quantity.

Pure Quantity "develops a determinateness" and will become Quantum. (185) This determinateness of Quantum will be posited as *no* determinateness—as a determinateness which is both inside and outside of Quantity. Quantum is therefore "indifferent determinateness, that is, a self-transcending, self-negating determinateness." (185) These remarks should make some sense. As with everything that has appeared after the True Infinite, Quantum is an infinite being that erases itself and yet retains its being. How this actually unfolds must await the second chapter in Quantity. Hegel at this point predicts that, when Quantum self-erases, it lapses into a Spurious Infinity—now to be called the mathematical infinite, an idea with which common sense is quite familiar.

Earlier, we saw that the Spurious Infinite amounted to the pure act of self-erasure. This act of self-erasing is what the Finites did.[34] This unity between the Finite and the Spurious Infinite was precisely this self-abnegating activity, whose name is the True Infinite. Something similar will happen to the mathematical infinite. The self-erasure of the infinite integers will emerge as a True Infinite. When that happens, Quantity will have taken back its Quality.

Once Quality is back together with Quantity, we will have arrived at final chapter in Quantity, which Hegel names Quantitative Ratio. Here we will find that the qualitative other to Quantum is yet another Quantum—just as Being-for-other discovered that the repulsed other was yet another Being-for-other. Hence, quantum is in fact a ratio of

34 *See* Carlson, *supra* note 4, at 535-38.

two quanta.

In the ratio (by which Hegel means "relation in general"—not division or fractions), the quanta are still indifferent to each other. That is, the number 7 doesn't care if it is related to 8 or to 9 or any other quantum. It will accept any partner that the mathematician—an external force—cares to impose. We will discover, however, buried deep within the idea of "ratio" lies a true qualitative moment, in which the two quanta are *not* indifferent to each other after all. The relationship (which will turn out to be the square of a quantum)[35] will have an objective resistance from outside manipulation. This resistance constitutes the reappearance of Quality within Quantity. When that point is reached, we are ready to move onto Measure—the culmination of the Doctrine of Being.

Remark

The essay introducing Quantity terminates with a short Remark, in which Hegel reminds us that Limit, in Figure 4(c), *is* determinateness.[36] When a quality exceeds its Limit, it changes radically. Beyond the Limit was the Finite, whose fate (i.e., Being-in-itself) was to erase itself. Not so with mere quantitative limit:

> If, however, by limit we mean quantitative limit, then when, for example, a field alters its limit it still remains what it was before, a field. If on the other hand its qualitative limit is altered, then since this is the determinateness which makes it a field, it becomes a meadow, wood, and so on. (186)[37]

Hegel gives this example: "Red" is a quality of some thing—its color. Let's change its quantitative limit by making the thing brighter or paler red. It remains red all the same. But let's paint the thing blue. The thing has now undergone a *qualitative* change, not a mere quantitative change. In Marcuse's words, "A being which is immediately identical with its respective quality such as to remain the same throughout all its qualitative transformations, is no longer qualitatively but quantitatively

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35 That is, if we take 16 and contemplate $x \cdot x = 16$, then the ratio of $x \cdot x$ is largely (but not totally) immune to outside manipulation. The internal integrity (or Quality) of the ratio insists that $x = \{4, -4\}$. *See infra* text accompanying notes 294-300.

36 In Figure 4(c), Limit was the unity of Constitution—internal negativity implying change—and Determination of the in-itself. In effect, in Limit, inside and outside have switched places, and the self-destruction of "being" is launched. Carlson, *supra* note 4, at 519.

37 *Cf.* TAYLOR, *supra* note 10, at 247:

> [Q]ualitatively considered, the determinateness or limit of a thing is not a matter of indifference; if we alter the limit, we alter the nature of the thing; but considered purely quantitatively, the limits of a thing can be altered without changing its nature; it is 'indifferent' to them. It is thus a mark of the quantitative, says Hegel, that we are dealing with such indifferent limits, that the things can increase or decrease in extension without changing their nature.

defined."[38]

With regard to red that waxes bright or pale, Hegel states that the degree of redness is its magnitude. In magnitude, redness "has a permanent substratum of being *which is indifferent to its determinateness*." (186) In other words, red as such continues to be red even as the brightness or paleness (its determinateness) is manipulated by outside forces.

Magnitude. Hegel also, in this Remark, warns that "magnitude" means Quantum—not Quantity. Magnitude is too advanced for the concept of Pure Quantity, because it implies a determinateness that is beyond Pure Quantity. Thus, in common mathematical discourse, "[a] magnitude is usually defined as that which can be increased or diminished." (186) As we have seen,[39] Hegel finds this to be a lousy definition. Nothing is learned from such a definition, except that magnitude is magnitude. It is nevertheless clear, however, that, in this definition, "the *more* or *less* can be resolved into an affirmative addition" (or subtraction) which is externally added (or subtracted). (186) "It is this *external* form both of reality and of negation which in general characterizes the nature of *alteration* in quantum." (186) In other words, Quantum cannot alter itself. It requires an outside manipulator to make a thing *more* or *less* of what it is. Of "more or less," Hegel remarks:

> In that imperfect expression, therefore, one cannot fail to recognize the main point involved, namely the indifference of the alteration, so that the alteration's own *more* and *less*, its indifference to itself lies in its very Notion. (186)

In other words, the essence of Quantum is that it is indifferent to being changed by outside forces.

This last observation is significant. How many times have you heard someone, fearful of affirming something absolutely, refer to it as "more or less" true? What is aimed for here a switch from fragile qualitative Limit to robust quantitative Limit. If the speaker gains your acquiescence to this transition, then the speaker's proposition will be harder to refute. Of course, we should not fall for this trick. If the speaker is making a qualitative point, then the speaker is not entitled to the relative ease and comfort that mere quantitative Limit affords.

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38 HERBERT MARCUSE, HEGEL'S ONTOLOGY AND THE THEORY OF HISTORICITY 64 (Seyla Benhabib trans., 1987); *see also* LESSER LOGIC, *supra* note 29, § 80 Remark:

> Quality is . . . the character identical with being: so identical that a thing ceases to be what it is, if it loses its quality. Quantity. . . is the character external to being, and does not affect the being at all. Thus *e.g.* a house remains what it is, whether it be greater or smaller; and red remains red, whether it be brighter or darker.

Hegel further remarks that "in quantity we have an alterable, which in spite of alterations still remains the same." *Id.* § 106 Addition.

39 *See supra* notes 2-3 and accompanying text.

Meanwhile, the arguer remains in control of how much "more or less" can be tolerated before qualitative change must be conceded.[40]

Alteration of Quantum, then, is accomplished only externally—hence inessentially. This is the penalty Being-for-self pays for driving out all content. Only strangers can tell the Quantum what it is. The final chapter of Quantity, however, will reveal a moment of self-integrity within Quantum, from which will spring forth the slave-rebellion Hegel calls Measure.[41]

A. *Pure Quantity*

Hegel begins his first chapter on Quantity by reminding us of what unfolded in chapter 3 of Quality. There, Quantity was "the repelling one." (187) This can be seen in Figure 9(c).[42] Repulsion said of itself: "I am not *that*." In so announcing, it "treats the other as identical with itself, and in doing so has lost its determination." (187) The expelled content was then united in Attraction, as Figure 10(a) showed.[43] "The absolute brittleness of the repelling *one* has melted away" into Attraction. (187)[44] Attraction, however, "is at the same time determined by the immanent repulsion." (187) This was shown in Figure 10(b).[45] Of course, Quantity is the unity of Attraction and Repulsion, as shown in Figure 10(c).[46] Reminiscing about the relationship portrayed in

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[40] Some time ago I had cause to complain about a defense of legal determinism which asserted that the rule of law existed—more or less. David Gray Carlson, *Liberalism's Troubled Relation to the Rule of Law*, 62 U. TORONTO L.J. 257 (1993).

[41] A criticism of sorts is offered by Professor Terry Pinkard, who writes that Hegel

> followed the tradition of his time in his assumption that the elements and principles of mathematical thought were related to quantity and number . . . [M]ore recent developments show that a whole set of mathematical ideas must be defined without reference to quantity . . . To make matters worse for Hegel, the traditional quantitative conception of measurement to which he appeals is not necessarily tied up with a conception of quality per se.

TERRY PINKARD, HEGEL'S DIALECTIC 42 (1988). In fact, this development is exactly what Hegel wanted! In the end, his point is that mathematics had insufficiently appreciated the role of quality in the constitution of quantity. According to Tom Rockmore:

> But it would be a mistake to argue that if [Hegel's view of mathematics] can be refuted, which cannot be shown, the position as a whole could be rejected. For whatever fate of the critique of mathematics, it is no more than an illustration of the more general point that a form of thought which is divorced from the movement of reality, and hence feeds only on itself, is necessarily one-sided and abstract, or linear.

TOM ROCKMORE, HEGEL'S CIRCULAR EPISTEMOLOGY 9 (1986).

[42] Carlson, *supra* note 4, at 569.

[43] *Id.* at 571.

[44] It will be recalled that, in Figure 10(a), [7] posited the Void/Many Ones as not itself. But, covertly, [7]—Repulsion as an immediacy—was swept along and was not left behind, as it hoped to be. Hence, [7] entered into Attraction as an immediacy, but Dialectical Reason retrieved it in Figure 10(b).

[45] *Id.* at 575.

[46] *See supra* notes 21-22 and accompanying text.

Figure 10(c), the Understanding announces that Attraction is the moment of Continuity in Quantity.

This brings us to Figure 11(a):

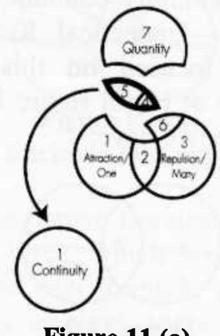

Figure 11 (a)
Continuity

The justification for this design is that Quantity has expelled all its content [4-7]. Therefore, the "content" of Figure 11(a) must be found amidst the expelled entities. Attraction [4, 5] is plucked from the exiles and made the Understanding's focus of attention. According to the Understanding, Quantity is the flow between self [5] and related other [4]. Absolute otherness [6] is suppressed. The name of this relation between [4] and [5] is Continuity [1].[47] Of [1], Hegel writes:

> Continuity is, therefore, simple, self-same self-relation, which is not interrupted by any limit of exclusion; it is not, however, an *immediate* unity, but a unity of ones which possess being-for-self. The *asunderness of the plurality* is still contained in this unity, but at the same time as not differentiating or *interrupting* it. (187)

The above passage shows a significant change of perspective. The first three chapters of the *Science of Logic* were the realm of "being"—the realm of *immediacy*. Hence, in those chapters, [1] was always immediate. Now, beyond the realm of immediacy, [1] is simple and not interrupted, but neither is it immediate. The Understanding continues to learn. It grasps [1] as a simple view of a complex "mediated" entity. Mediation as such now reigns in the extremes of Understanding and of Dialectical Reason. If immediacy exists at all within Continuity, it exists as a *moment*—a memory of its origin in reality. Indeed, Hegel will often use the word "immediate" in this and the following chapters. Understanding fully understands, however, that "immediacy" is always merely an ideal moment. The Understanding knows that it has left the crude realm of reality and exists now and forever in the realm of the

[47] Errol Harris usefully reminds us that continuity has the attributes of Attraction. That is, continuity is a plurality held together by an external will. HARRIS, *supra* note 15, at 126.

ideal.[48]

Dialectical Reason is now rather less insulting to and patronizing of the Understanding than before. Acknowledging that the Understanding sees that Continuity contains mediation within it—the distinction of Many Ones—Dialectical Reason proposes with due respect that attention be focused on this moment of difference. Discreteness is the recovery of [6] in figure 11(a). Discreteness is the *beyond* of continuity. Hence, we have:

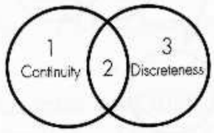

Figure 11 (b)
Discreteness

Of Figure 11(b), Hegel writes: "In continuity, therefore, magnitude immediately possesses the moment of *discreteness*—repulsion has now a moment in quantity." (187) The word "immediately" probably should not be invested with much significance here, for the reason just stated. We are beyond the realm of immediacy. Nevertheless, one could say that [3] is an "immediate" moment—but only in the ideal sense of remembering what [3] meant in the opening chapters of the *Science of Logic*. Hegel vindicates this judgment:

> Hence, discreteness on its side, is a coalescent discreteness, where the ones are not connected by the void, by the negative, but by their own continuity and do not interrupt this self-sameness in the many. (187)

In other words, [3] is not immediate, except in an ideal sense. The Many Ones are acknowledged in Discreteness; and they are acknowledged as connected by Continuity.[49]

Hegel next indicates that an enriched Quantity is the unity of Discreteness and Continuity:

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48 Reality ended and ideality began in the True Infinite. Carlson, *supra* note 4, at 538-48.

49 Hegel gives this useful example of Continuity and Discreteness in the *Lesser Logic*:

> It may be said, the space occupied by this room is a continuous magnitude, and the hundred men assembled in it form a discrete magnitude. And yet the space is continuous and discrete at the same time; hence we speak of points of space, or we divide space, a certain length, into so many feet . . . which can be done on the hypothesis that space is also potentially discrete. Similarly . . . the discrete magnitude, made up of a hundred men, is also continuous; and the circumstance on which this continuity depends is the common element, the species man, which pervades all the individuals and unites them with each other.

LESSER LOGIC, *supra* note 29, § 100 Remark.

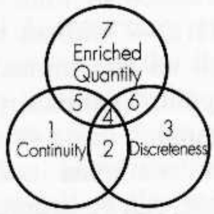

Figure 11 (c)
Enriched Quantity

Hegel does not use the phrase Enriched Continuity. I have added this to distinguish Figure 11(c), which brings Continuity to the fore, from Figure 10(c).[50] In the next Remark, we learn that this Enriched Quantity is the same thing as time, space, the ego, and many other quantitative ideas.

Of Figure 11(c), Hegel writes: "Quantity is the unity of these moments of continuity and discreteness, but at first it is so in the *form* of one of them, *continuity*, as a result of the dialectic of being-for-self, which has collapsed into the form of self-identical immediacy." (187) This is a direct reference to Quantity as portrayed in Figure 11(a). Here we have reference to [1] in Figure 11(a) as an *immediacy*, when we said that our days of immediacy were over—except as an ideal "moment." Of Quantity in this guise, Hegel states: "Quantity is, as such, this simple result in so far as being-for-self has not yet developed its moments and posited them within itself." (187) In other words, Hegel agrees that Quantity, taken as a mere immediacy, is retrogressive—a throwback to the last part of chapter 2 of Quality. This immediacy, however, is precisely what Being-for-self expelled by the end of chapter 3. Thus, Quantity, as portrayed in Figure 11(a), *contains* the moments of Being-for-self

> posited as it is in truth. The determination of being-for-self was to be a self-sublating relation-to-self; a perpetual coming-out-of-itself. But what is repelled is itself; repulsion is, therefore, the creative flowing away of itself. (187-88)

This "creative" flowing of content out of Being-for-self is precisely what Continuity is. Thus, Being-for-self flows into all the other Ones: "On account of the self-sameness of what is repelled, this distinguishing or differentiation is an uninterrupted continuity." (188)

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50 Could I have said that Figure 11(c) is *the same* as Figure 10(c), but with the names of the extremes changed? I did something similar once before. In chapter 1, Figure 1(c) was Pure Being, Pure Nothing, and Becoming. Then the names changed, without an advance. Figure 1(c) became coming-to-be, ceasing-to-be, and Determinate Being. Carlson, *supra* note 4, at 438-48 & n.58. Nevertheless, an advance to Figure 11(c) is justified. In Figure 10(a), [1] was Attraction—precisely a stubborn unity that nevertheless covertly implies Repulsion. In Figure 11(c), Continuity shows no such stubbornness.

Hegel finishes his discussion of Pure Quantity by restating that Continuity—[1] in Figure 11(a)—"without being interrupted, is at the same time a plurality, which no less immediately remains in its self-identicalness." (188) Once again, "immediacy" must be taken only as an ideal moment. The Understanding has a simple, yet mediated, view of Quantity as a substance that continues itself in all things. (For this reason, *everything*, except God, can be enumerated.)

Remark 1: The Conception of Pure Quantity

In Remark 1, Hegel emphasizes that Pure Quantity does not yet have any Limit. Even when it becomes Quantum, it will not be bounded by Limit "but, on the contrary, consists precisely in not being bounded by limit." (188) In Figure 11(b), Discreteness has appeared, but this is no Limit:

> The presence in it of discreteness as a moment can be expressed by saying that quantity is simply the omnipresent *real possibility* within itself of the one, but conversely that the one is no less absolutely continuous. (188)

That is, Quantity holds the *promise* of self-limitation—of being "the one." This is what Continuity's relation to Discreteness portends. For now, Quantity is absolutely continuous.

In bad philosophy—what Hegel calls "thinking that is not based on the Notion"—Continuity quickly devolves into "mere *composition*, that is, an *external* relation of the ones to one another, in which the one is maintained in its absolute brittleness and exclusiveness." (188) For Hegel, "composition" is a derogatory term, suggesting that the unity is not immanent to the entities but is imposed upon them from the outside.[51] Compositional philosophies fail to see that the One "essentially and spontaneously (*an und für sich selbst*)" passes over into ideality. (188) This spontaneous action was documented at the end of chapter 2 of Quality (when True Infinity appeared) and throughout chapter 3. This action proves that Continuity belongs to the One—here, Enriched Quantity in Figure 11(c).

Atomism—much denounced in chapter 3[52]—holds that Continuity is external to the One, an idea that "ordinary thinking finds it difficult to forsake." (188) (Here, as we shall soon discover, Hegel is thinking about the concept of time and space.) Mathematics, however, rises above this naive view. It "rejects a metaphysics which would make

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[51] According to Hegel, composition is "the worst form in which anything can be considered . . . That the form of the untruest existence should be assigned, above all, to the ego, to the Notion, that is something we should not have expected and that can only be described as inept and barbarous." (615)

[52] Carlson, *supra* note 4, at 564-66.

time *consist* of points of time [or space] . . . It allows no validity to such discontinuous ones." (188) A plane may consist of the sum of infinitely many lines, but the Discreteness of the lines is only a moment. The sublation of this moment is implied by the infinite plurality of the lines.

Time, space, "matter as such,"[53] the ego—these are to be taken as examples of Pure Quantity. These things are "expansions, pluralities which are a coming-out-of-self, a flowing which, however, does not pass over into its opposite, into quality or the one." (189) Thus, space is "absolute *self-externality* which equally is absolutely uninterrupted, a perpetual becoming-other which is self-identical." (189)[54] Time likewise "is an absolute coming-out-of-itself." (189) It generates the "now"—the present—but then immediately annihilates it. Time is the "continuous annihilation of this passing away" and the "spontaneous generating of non-being." (189) In its pure destructivity, self-devouring time is "a simple self-sameness and self-identity." (189)[55]

The ego is also Pure Quantity. It is "an absolute becoming-other, an infinite removal or all-round repulsion to the negative freedom of being-for-self." (190) In short, the ego constantly states, "I am not *that*." No proposition ever captures all of the ego, which is nothing at all but Continuity over time—"utter simple continuity." (190) That the ego is Continuity (which *is* time itself) Hegel expresses this way: the ego is

> the continuity of universality or being-with-self uninterrupted by the infinitely manifold limits, by the content of sensations, intuitions, and so forth. (190)

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[53] In the *Philosophy of Nature*, Hegel identifies matter as the unity of Attraction and Repulsion, which is, of course, exactly what Quantity is. Host-Heino von Borzeszkowski, *Hegel's Interpretation of Classical Mechanics*, *in* HEGEL AND NEWTONIANISM, *supra* note 11, at 73, 79 (citing HEGEL'S PHILOSOPHY OF NATURE § 262 (A.V. Miller trans., 1970)).

Hegel also distinguishes between Pure Quantity and matter. Quantity is a determination of pure thought. Matter is the same thing, but in outer existence. Hegel quotes Leibniz for this: *Non omnino improbabile est, materiam et quantitatem esse realiter idem.* (189) ("Not every thing is improbable, matter and quantity being the same reality.").

[54] Space (Pure Quantity) will be the starting point for Hegel's *Philosophy of Nature*, just as consciousness is the starting point for the *Phenomenology* and the autonomous individual is the starting point for the *Philosophy of Right*. *See* Lawrence S. Stepelevich, *Hegel's Conception of Space*, 1 NATURE AND SYSTEM 111 (1979).

In chapter 2, we saw Hegel derive nature as other to Spirit taken as other. We now may add that nature so expelled by Spirit is Pure Quantity. MURE, *supra* note 17, at 116 ("Quantity is conspicuous in Nature, since self-externality as opposed to the self-possession of spirit is the distinctive character of Nature.").

[55] *See* Richard Dien Winfield, *Space, Time and Matter: Conceiving Nature Without Foundations*, 29, 61-62, *in* HEGEL AND THE PHILOSOPHY OF NATURE (Stephen Houlgate ed., 1998) (calling time "this self-devourer"). Andrew Haas describes Hegel's view of time nicely:

> [T]he "now" (and "this"), exemplifying the immediacy of sense-certainty (that does not yet think "time and space"), is not the now—for the now is no longer at precisely the moment when it is now; it is far more a not-now, a having-been: "'*now*'"; it has already ceased to be in being shown; the *now* that *is*, is another now than the one shown, and we see that the now is just this; already when it is, to be no more.

HAAS, *supra* note 8, at 252.

The equation of the ego with being-with-self (which I interpret to be the same as "being-within-self") is very significant. In chapter 2 of Quality, we saw that being-within-self equates with [4]—the sole entity that always appears in all three circles.[56] It connoted immanence and hence freedom from outside compulsion. The birth of being-within-self in chapter 2 was therefore also the birth of human self-consciousness—though that concept as such was far too advanced for us in chapter 2 or even now.[57]

The ego continues through its content—"sensations, intuitions, and so forth." None of these things, however, is adequate to the ego. The ego is always *beyond* these things and so never fully present to itself. But neither is the ego Pure Nothing. In fact, the ego is always suspended between its content and Pure Nothing. For this very reason, it is constantly restless.

Those familiar with Jacques Lacan's theory of the subject can glimpse it prefigured in Hegel's theory of Pure Quantity. Lacan thought the subject was "split" between the realm of the Symbolic—the external realm of "being"—and the Real, the obliterative concept of Pure Nothing. The Lacanian subject constantly tries to fill in the gaps so that it can fully "be." This is precisely what "desire" is—the drive to be complete and whole. Yet desire must fail. For the subject to be whole would be for it to surrender its very essence—Continuity that stays forever free from the external realm of "being."[58]

Remark 2: The Kantian Antinomy of the Indivisibility and the Infinite Divisibility of Time, Space and Matter

In this long Remark, Hegel makes his famous criticism that there are not merely four antinomies, as Kant alleged,[59] but infinitely numerous antinomies; *every* concept is a union of opposites—as Becoming implies.[60]

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56 Carlson, *supra* note 4, at 495.

57 Self-conciousness is the theme of the Subjective Logic, which commences in Hegel's 19th chapter.

58 These thoughts summarize JEANNE L. SCHROEDER, THE VESTAL AND THE FASCES: HEGEL, LACAN, PROPERTY, AND THE FEMININE (1998). In this book, Professor Schroeder draws rigorous parallels between Lacanian and Hegelian thought.

59 Kant's antinomies are:

Thesis	Antithesis
1. The world is limited in time and space.	The world is not limited in time and space.
2. There are simples.	Everything is divisible.
3. Freedom exists.	Everything is caused.
4. There is a god.	There is no god.

CRITIQUE OF PURE REASON, *supra* note 14, at 241-63.

60 Figure 1(c) shows Becoming as the contradictory unity of Being and Nothing. Carlson, *supra* note 4, at 438.

Kant's second antinomy is (1) there is a simple that cannot be further subdivided, and (2) there are no simples, because everything can be further subdivided.[61] In Remark 2, Hegel states that it is Figure 11(c) that gives rise to this antinomy, which "consists solely in the fact that discreteness must be asserted just as much as continuity. The one-sided assertion of discreteness gives infinite or absolute dividedness, hence an indivisible, for principle; the one-sided assertion of continuity, on the other hand, gives infinite divisibility." (190) In other words, Discreteness implies an indivisible. Continuity implies divisibility. Figure 11(c) shows *both* to be necessary moments. It diagrams the antinomy itself. Kant thought that both sides of the antinomy are false, because each can be disproved by *apagogic reasoning*—that is, reason by process of elimination. The Kantian solution to the antinomies, Hegel says, was to make the contradiction subjective, where it remained unresolved.[62] The genuine solution, however, is to recognize that each side of an antinomy is one-sided and hence not valid on its own. "[O]n the contrary, they are true only as sublated." (192)

Before demolishing the antinomies, Hegel praises them as "the downfall of previous metaphysics." (190) They helped to produce the conviction that finite things are null in content. Nevertheless, they are far from perfect. Hegel in effect accuses Kant of choosing these antinomies (from the infinite collection that could have been chosen) to match his four categories of the understanding, earlier developed in the *Critique of Pure Reason*.[63] This was done, Hegel remarks, to provide a mere "show of completeness." (191)[64]

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61 CRITIQUE OF PURE REASON, *supra* note 14, at 248.

62 Harris states that the understanding holds the two sides of the antinomy "incommunicado," and that the result is "logomachy"—a war on words. HARRIS, *supra* note 15, at 128.

63 Here is how the categories of understanding match up with the antinomies:

Categories Of the Understanding	Antinomies
Quantity	Beginning/No Beginning in Time
Quality	Infinite Divisibility/Simple
Relation	Freedom/Causation
Modality	Absolutely necessary God/No God

The categories are said to belong *a priori* to the understanding. CRITIQUE OF PURE REASON, *supra* note 14, at 62. According to Kant, we cannot think any object except by means of the categories. We cannot cognize any thought except by means of intuitions corresponding to these conceptions. *Id.* at 94. They are the mere forms of thought for the construction of cognitions from intuitions. *Id.* at 153.

64 Ironically, I will later conclude that Hegel will render his theory of Judgment quadratic (not triadic) only so that it conforms with Kant's Table of Logical Functions in Judgement. This occurs in Hegel's 20th chapter, in the Subjective Logic. For Kant's table, *see* CRITIQUE OF PURE REASON, *supra* note 14, at 56.

Hegel provides us with this memorable denunciation of Kant:

> The Kantian antinomies on closer inspection contain nothing more than the quite simple categorical assertion of *each* of the two opposed moments of a determination, each being taken on its own in isolation from the other. But at the same time this simple categorical, or strictly speaking assertoric[65] statement is wrapped up in a false, twisted scaffolding of reasoning which is intended to produce a semblance of proof and to conceal and disguise the merely assertoric character of the statement . . . . (192)

To make good on this criticism, Hegel paraphrases one side of Kant's second antinomy as follows: "Every composite substance in the world consists of simple parts, and nowhere does there exist anything but the simple or what is compounded from it." (192) In the "thesis," Kant opposes the atom to the composite, "a very inferior determination compared to the continuous." (192)[66] Yet both the atom and the composite have a substrate (or common denominator)—substance.

The truth of the thesis is to be established by apagogic reason—or reasoning by process of elimination. Thus, if Kant can prove that infinite divisibility is impossible, he has proved apagogically that a "simple" exists. Hegel, however, claims that the apagogic demonstration is superfluous. He accuses Kant of bringing forth the very presuppositions that Kant introduced into the model, so that nothing is achieved. Here is Hegel's appraisal of Kant's *real* argument for proving that the indivisibly simple exists: (1) Assume there is such a thing as substance. (2) Now assume that the composites do not have simple parts. (3) Now think away all composition. Nothing remains. (4) This contradicts the assumption that there is substance. (5) Ergo, there must be atoms. This, Hegel complains, does not move the argument. Kant could have begun this way: Composition is merely a contingent relation of the substance. By "contingent" is meant that the relation is externally imposed on substance and therefore not immanent to it and of no concern to it. If composition is external, then substance is simple. In short, substance is a "thing-in-itself," which, in chapter 2, Hegel suggested, was a simplex.[67]

But this mode of arguing is likewise unsatisfactory, Hegel says. In it, the contingency of composition is assumed—not proved. Hence, the presence of a simplex is tautological. In other words, the structure of Kant's argument is: (1) Assume there is a simplex. (2) That would

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65 That is, dogmatic, merely asserted.

66 Hegel also complains that to oppose the composite to the simple is tautological, since the simple itself might be a "*relatively* simple" and hence another composite. (193) Hence, the thesis does not exclude the antithesis.

67 In chapter 2, with regard to Figure 3(b), Hegel suggested that Being-in-itself/Being-for-other amounted to the thing-in-itself, which had to be taken as a simplex. Carlson, *supra* note 4, at 507-10.

imply that composition is external and contingent. (3) Wish away composition, since it is only *wished for* in the first place (*i.e.*, is subjective). (4) A simplex remains.

Composition. Hegel likewise attacks the demonstration that everything is infinitely divisible, which he calls "a whole *nest* (to use an expression elsewhere employed by Kant) of faulty procedure." (195)[68] To disprove the existence of simplicity, Kant's apagogy proceeds as follows: (1) Composites exist in space. (2) Space is infinitely divisible. (3) Since a simplex can occupy only one space at a time, it too must be equally divisible, to conform to the many spaces it occupies. (4) Ergo, simplicity does not exist.

Hegel complains that this argument assumes that whatever is substantial is spatial. It also assumes that space is infinitely divisible, which is by no means proven. Indeed, space *is* Quantity, in Hegel's view.[69] As such, it is Continuous *and* Discrete.[70] Furthermore, the second move ("composites exist in space") suggests that simplicity is *not* spatial. Simplicity, by definition, does not have complexity within it. Composition is outside it. If composition is outside the simple, so is space. Thus, simplicity is not spatial. Only composition is. Yet, if simplicity is not spatial, Kant's demonstration falls apart.

Hegel accuses Kant of *quaternio terminorum*: "there is also involved here a clash between the continuity of space and composition; the two are confused with each other. [Space is] substituted for [composition] (which results in a *quaternio terminorum* in the conclusion)." (196) A *quaternio terminorum* is a syllogism with four terms instead of three. A syllogism should have this form:

A=B

B=C

A=C

A *quaternio terminorum* has the following form:

A=B

C=D

A=D

For example, (A) all politicians are (B) managers; (C) all administrators are (D) sybarites; therefore all (A) politicians are (D) sybarites.[71]

How is Kant guilty of *quaternio terminorum*? The critique seems to be that Kant changes the meaning of "space." Earlier in the *Critique of Pure Reason*, Kant said that space is sole and single. It does not have

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68 Hegel refers here to Kant's critique of the cosmological proof of God, which Kant calls a "perfect nest of dialectical assumptions." CRITIQUE OF PURE REASON, *supra* note 14, at 340.

69 *See supra* text accompanying notes 54-55.

70 Hegel says that the second antimony, which applies to substance, could have been applied to time or space. (192) Hence, it is possible space is not infinitely divisible.

71 *See* IRVING M. COPI & CARL COHEN, INTRODUCTION TO LOGIC 206 (11th ed. 2002).

discrete parts.[72] There, Kant properly equates space with Continuity, as Hegel would do. But in the demonstration with regard to the second antimony, this point has been forgotten. Now space has infinite parts and is itself a composition of them. Hence, Kant's proposition is (A) all composites are in (B) space (conceived as continuous); (C) space (conceived as made up of parts) is (D) infinitely divisible. Therefore, (A) composites are (D) infinitely divisible.

Furthermore, in his discussion of the second antimony, Kant reminds his readers that we know only phenomena. Space is a condition of possibility for phenomena. Hence, Hegel reasons, if "substance" means sensuous material, we are discussing only phenomenal substance, not substance-in-itself. Thus, the disproof of simplicity amounts to this: sensual experience shows us only what is composite. What is simple is not empirically discoverable.

When Kant's argument is liberated from "all pointless redundancy and tortousness," (197) the proof of the antithesis ("everything is divisible") assumes space is Continuity, because substance is placed in space. In the proof of the thesis, however, space is not continuous. Rather, "substance are *absolute ones*." (197) The thesis asserts Discreteness. The antithesis asserts Continuity. When substance, space, time, etc. are taken as discrete, their principle is the indivisible One. When they are taken as continuous, division is possible.

Continuity contains the atom within it, however. If division is always a possibility, there must be something to divide—the atom. That is, a discrete thing must exist before divisibility, with its golden axe, cleaves it in twain. Likewise, Discreteness contains Continuity. In it, the ones are purely simple and hence identical to each other. The sameness of the ones is precisely Continuity. As Figure 11(b) shows, "each of the two opposed sides contains its other within itself and neither can be thought without the other." (197) Neither side, taken alone, has the truth. The truth lies only in their unity—which is shown in Figure 11(c).

In the end, Kant leaves the solution of the antimony to one side. According to Hegel, each side of the antinomy should have nullified itself (as each is by now a True Infinite). In this activity, each side is "in its own self only the transition into its other, the unity of both being *quantity* in which they have their truth." (199)[73]

The Eleatics were "[i]nfinitely more ingenious and profound" than poor, benighted Kant. (197) Hegel forgoes analyzing them, except to criticize the empirical procedure of the notorious Diogenes. Thus,

> when a dialectician pointed out the contradiction in motion, made no effort to reason it out but, by silently walking up and down, is

72 CRITIQUE OF PURE REASON, *supra* note 14, at 23.

73 This would be the Enriched Quantity of Figure 11(c).

supposed to have referred to the evidence of sight for an answer. Such assertion and refutation is certainly easier to make than to engage in thinking and to hold fast and resolve by thought alone the complexities originating in thought . . . . (198)[74]

Hegel claims that Aristotle was genuinely speculative about space, time and motion. He opposed divisibility to continuity. Of course, Hegel has said divisibility *is* continuity. But Aristotle understood that divisibility implies atoms—there must be something for divisibility to divide. He saw that discreteness and continuity each imply the other. Each constitutes the condition of possibility for the other. Pierre Bayle[75] did not see this. He assumed Aristotle was claiming that everything *actually* contains infinite parts—one side of the Kantian antinomy. Aristotle saw that *both* sides were possibilities.

B. *Continuous and Discrete Magnitude*

We have seen that Continuity "requires the other moment, discreteness, to complete it.' (199) Yet Continuity is not merely the *same as* but is also *distinct from* Discreteness. Hence, we must extract difference from the middle term and consider it in isolated form:

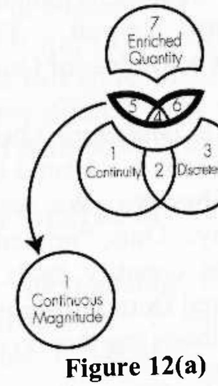

Figure 12(a)
Continuous Magnitude

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74 In the *Phenomenology*, Hegel states that Diogenes liked to defeat Plato by kicking a rock and thereby proving it "existed." But all this showed was the *utility* of the rock—its status as an object for actual consciousness; or the "being-for-other" of the rock. PHENOMENOLOGY, *supra* note 10, ¶¶ 389, 579. Such a reality is one-sided, in that it emphasizes the negativity (being-for-other) of the thing and excludes the side of being-for-self. Such an insistence on the *factum brutum*—the "being-for-us" of the rock—paradoxically renders the rock entirely subjective and denies the rock the very integrity that the attribute of "reality" should have provided for it. PHILOSOPHY OF RIGHT, *supra* note 31, §275. This is, incidentally, the posture of law-and-economics toward law. ALAN BRUDNER, THE UNITY OF THE COMMON LAW 22-23 (1996).

75 Bayle was the author of a philosophical dictionary, "which was then the sourcebook for philosophy used by all rational and free-thinking men." STUART HAMPSHIRE, SPINOZA 27 (1951).

Of Figure 12(a), Hegel writes:

> But quantity is a concrete unity only in so far as it is the unity of *distinct* moments. These are . . . not to be resolved again into attraction and repulsion, but are to be taken as . . . remaining in its unity with the other, that is, remaining in the *whole*. (199)

Here Hegel emphasizes that Figure 12(a) is more advanced than Figure 10(b), which featured Attraction and Repulsion. Attraction and Repulsion exhibited Being-for-self. Each expelled its other so that each could be by itself. Now Continuous Magnitude humbly realizes it is part of a community, even though it asserts its individuality within that community. Posited as Continuous Magnitude, Continuity is "no longer only a moment but the whole of quantity." (199) The addition of the word "magnitude," then, signifies "determinateness in quantity." (201) Because this is so, Figure 12(b) will show an advance over Figure 11(b), where the positedness of the extremes was not yet manifest. This justifies the isolation of Figures 12(a) and (b) as separate official steps in the Logic.[76]

Continuous and Discrete Magnitude are "*species* of quantity." (200) By this Hegel means that each extreme is Quantity as such, and is a determinateness in light of its own "moments." (201) The use of the phrase "moment" signifies that determinateness is but a memory, conjured forth by Dialectical Reason. This determinateness now appears within the context of a whole—of Quantum portrayed in Figure 12(c).

Continuous Magnitude is *immediate* Quantity—taken as a whole. But, of course, immediacy is only a sublated immediacy. Immediacy as such was the province of Quality. We are beyond that now. We partake of an *ideal* immediacy. Thus, "immediacy is a determinateness the sublatedness of which is quantity itself." (200) In other words, quantity as a whole has sublated Determinateness and rendered it ideal.

When we place the emphasis on this recollected Determinateness, we obtain Discrete Magnitude.

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76 There is counter-evidence, however. With regard to Figure 11(c), Hegel states that space and time are represented by the enriched Quantity shown there. *See supra* notes 52-56, 58 and accompanying text. In the Remark now under discussion, he says that space and time are Continuous Magnitudes. That space and time are represented by both Figure 11(c) and by Figure 12(a) suggests that no advance has been made. Nevertheless, Continuous Magnitude brought to the fore something not present within mere Continuity—an acknowledged membership in a larger community. Furthermore, the middle term will be Quantum (*a* Quantity). Hence, I have declared Continuous and Discrete Magnitude to be official steps.

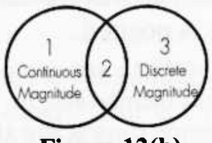

Figure 12(b)
Discrete Magnitude

Like Continuous Magnitude, Discrete Magnitude is to be taken as a unified whole, with a double moment of Continuity and Discreteness within it:

> Quantity is in itself asunderness, and continuous magnitude is this asunderness continuing itself without negation as an internally self-same connectedness. But discrete magnitude is this asunderness as discontinuous, as interrupted . . . . (200)

The relations between the extremes are now much more genteel than they were in the first three chapters. Each extreme admits to its subordinate role within a community, whereas, earlier, the extremes selfishly insisted on being "for themselves."

Hegel emphasizes that, if Continuous Magnitude is "the manifold one in general," Discrete Magnitude is "posited as the *many of a unity*." (200) That is, just as [3] in Figure 9(c)[77] was both the Void and the Many Ones, and in Figure 10(c) [3] was Repulsion (of each One from the other), so Discreteness in Figure 11(b) and Discrete Magnitude in Figure 12(b) represent *many* discrete ones which nevertheless continue into each other by virtue of their complete sameness.

Remark : The Usual Separation of These Magnitudes

There is a "usual" interpretation of Continuous and Discrete Magnitude that Hegel disfavors. It suppresses the fact that each extreme contains its fellow inside it.[78] The only proper distinction

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77 *See* Carlson, *supra* note 4, at 569.

78 Terry Pinkard, who calls for a complete rewriting of Hegel's analysis of Quantity, is guilty of this fault. Professor Pinkard denies that Hegel's Continuity is connected to the modern mathematical notion. PINKARD, *supra* note 41, at 44. This is, I think, precisely wrong. The continuity of a curve (which makes the curve differentiable) is exactly what is at stake here. Pinkard attempts to recast Hegel in the language of Bertrand Russell's obsession with set theory. Thus, "[c]ontinuity would only be another way of talking of the one, and discreteness would only be another way of talking of the many." *Id.* This misses the main point. Continuity is the activity of a thing going outside of itself and into the other while remaining itself. This is the hallmark of True Infinity, which is missing from Pinkard's account.

In an earlier essay, Pinkard called for a reformulation of Hegel's entire theory of quantity, but, in this essay, Pinkard betrays a desire to keep the analysis in the realm of self-identity—precisely the realm that Hegel's Logic wishes to implode. Thus, he writes: "The least one could do is reformulate Hegel's doctrine into saying that the two concepts defining numbers are those between Continuous and Discrete Magnitude is that, in Continuous Magnitude, determinateness is merely implicit, while in Discrete Magnitude, determinateness is posited.

> Space, time, matter, and so forth are continuous magnitudes in that they are repulsions from themselves, a streaming forth out of themselves which at the same time is not their transition or relating of themselves to a qualitative other. (200)

Each one of these possesses the *possibility* that, at any time, the One may be posited in them. Thus, time's One would be *the present*. As a Continuous Magnitude, time implies that it can be frozen. (Indeed, borrowing from earlier points Hegel makes, since time annuls all moments, it must indeed have a moment before it to annul.) Discrete Magnitude, on the other hand, *expressly* posits presence as a necessary component of time.

Hegel finishes this section by saying a few words about genus and species. Ordinary thinkers organize species into genera "according to some *external* basis of classification." (201) For example, mammals are a genera because we choose to emphasize milk production by females as the organizing principle. But Continuous and Discrete Magnitude produce their own genus in Quantum, described in Figure 12(c). This is equally true for each stage of Speculative Reason.[79]

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of *unity* and *multiplicity*; numbers would then be multiplicities of units which we count." Terry Pinkard, *Hegel's Philosophy of Mathematics*, 41 PHIL. & PHENOMENOGICAL RES. 453, 460 (1980-81). Thus, numbers are self-identical units which are fused together only subjectively through counting. To these units Pinkard denies any inherent continuity or True Infinity:

> He should begin with the notion of units . . . as members of classes and then proceed to show how construction rules which involve these units can be given for numbers . . . One could then use the categorial notion of a unit (a member of a class, represented by a variable), proceed to counting units (i.e., adopt construction rules), thus introducing the concepts of numbers, then define magnitude in the way mentioned, and *then* one could define quantity (i.e., that which is capable of relations of quantitative equality) . . . at the end of the series not at the beginning.

Id. at 460-61. This suggestion repeals the whole notion of the True Infinite and is definitely un-Hegelian in approach. Hegel is keen to show that Quantity is the *activity* of the True Infinite, and so he *begins* (not ends) with the concept of Pure Quantity.

[79] Joining Walter Kauffman, who denies the triune structure of the Logic, Andrew Haas complains that "'Continuous and Discrete Magnitude' has only two moments and not three." HAAS, *supra* note 8, at 79. This is technically accurate. The opening chapter on Quantity has principal three sections but only two revolutions. The truth of Continuous and Discrete Magnitude will be revealed in the last section of the chapter. The test, however, should not be whether each revolution is honored by a subhead, but whether the revolutions, wherever they occur, are driven by a rigorous alternation of the Understanding, Dialectical Reason, and Speculative Reason. We have already seen that Hegel does not always dedicate a subsection to a single revolution. Several subsections have already had multiple revolutions. Some—such as this one—have less than one full revolution.

C. *Limitation of Quantity*

As we saw earlier, Discrete Magnitude is One. It is also a plurality of Ones which repel each other. But each of these Ones is quite the same as any other. Hence, the Ones "continue" from one into the other.

When we focus on the oneness of Discrete Magnitude, we behold an "*excluding* one, a limit in the unity." (201) But Limit has long been sublated. Accordingly, Hegel adds, Discrete Magnitude is

> immediately not limited; but as distinguished from continuous magnitude [1] it is a determinate being [2, 3], a something, with the one [3] as its determinateness and also as its first negation and limit. (201)

Thus, not only is Discrete Magnitude plainly a determinateness, considered as [2, 3], but even in its isolated form [3] it is still a determinateness, because Discrete Magnitude fully remembers its ideal moment of being the Many Ones. Furthermore, even as [3] is posited as the Many Ones, still it is One and, as such, is Limit and first negation to its own being-in-itself [2].

If we take [3], in Figure 12(b), as "enclosing, encompassing limit," (201) [3] is self-related and is the negation in Discrete Magnitude [2, 3]. [3] is "the negative point itself." (201) But Discrete Magnitude [2] is likewise Continuity [1,2], "by virtue of which it passes beyond the limit, beyond this one [3], to which it is indifferent." This speculative moment leads us to Figure 12(c):

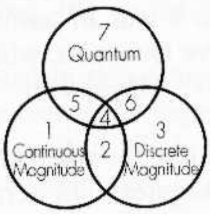

Figure 12 (c)
Quantum

Of Figure 12(c), Hegel writes: "Real discrete quantity is thus *a* quantity, of quantum—quantity as a determinate being and a something." (201) Thus, Quantum is to Pure Quantity what Determinate Being was to Pure Being,[80] and the second chapter of Quantity is to Quantity as a whole what the second chapter of Quality was to Quality as a whole—a display of Dialectical Reason. Quantum is, in effect, determinate Quantity.

Has Speculative Reason worked on Figure 12(b) in the same way it

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[80] As Hegel specifically emphasizes. LESSER LOGIC, *supra* note 29, § 101 Remark.

did in the Quality chapters? Recall that, at first, the extremes modulated back and forth.[81] Speculative Reason then named the movement and produced the middle term.[82] Later, the extremes turned on themselves and self-erased (the Finites). Speculative Reason named this self-erasure as the True Infinite.[83] Now it appears that Speculative Reason has operated on [2, 3] without considering the role of [1].

Hegel ends the chapter by correcting this misapprehension. Reverting back to [3] for a moment, Hegel holds that this "one which is a limit includes within itself the many ones of discrete quantity." But these Many Ones are sublated. [3] serves as a limit to Continuity, which Continuity leaps over with ease. Since Continuity [1] leaps over [2] and enters into [3] with ease, [3] likewise leaps back into [1], which is just as much Discrete Magnitude as it was Continuous Magnitude. The extremes equally leap out of themselves, and so Speculative Reason, like a sportscaster, still names the activity it witnesses in the extremes.

II. From Number to Quantitative Infinity

We now commence what is, by far, the longest, most maddening chapter in the *Science of Logic*—Quantum.

At the end of the first chapter of Quantity, Hegel had derived Quantum. Quantum becomes Number—"quantity with a determinateness or limit in general." (202) Quantum/Number will melt, thaw, and resolve itself into a pair of terms unfamiliar to the modern eye—Extensive and Intensive Quantum, which Hegel also indifferently calls Extensive and Intensive Magnitude. Intensive Quantum is also called Degree.

Extensive Quantum stands for the entire set of numbers which a single number—say, 10—negates. In terms of natural numbers, it stands for 1-9 and 11 and higher. Meanwhile, Intensive Quantum (Degree) stands for 10, in the above example. Intensive Quantum will resist outside manipulation in a way that Pure Quantity—which had its being outside it—could not. Quantum's intensity will quickly yield to Quantitative Infinity and the infinitely small or large number, which can never be named. When we reach this unnameable thing, Quantum has truly recaptured its Being.

None of this is very helpful at this stage. Suffice it to say that, whereas as the middle chapter of Quality saw Being chasing away its own content, the middle chapter of Quantity will do the opposite—it

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81 Carlson, *supra* note 4, at 445.
82 *Id.* at 446-47.
83 *Id.* at 538-43.

will recapture some measure of its content.

A. *Number*

Hegel starts with the premise that Continuous Magnitude and Discrete Magnitude are the same, at this point. Each is Quantum, and Quantum has Limit. But Limit exists only in its ideal form:

> The very nature of quantity as sublated being-for-self is *ipso facto* to be indifferent to its limit. But equally, too, quantity is not unaffected by the limit or by being quantum; for it contains within itself . . . the one, which . . . is its limit . . . . (202)

In short, quanta have discreteness. Three is distinct from four. But three what? The number three has no content *except that* it is not two or four. In three's insistence upon its independence from two or four we witness that three is "not unaffected by the limit" which exists in Quantum as an ideal moment.

Quantum, then, contains within itself the moment of the One.[84] "This one is thus the principle of quantum." (202) But this One is more advanced than the One of chapter 3.[85] First, it is continuous with all the other quanta. That is, it is a *unity* of Continuity and Discreteness. Second, it is discrete and hence different from all the other quanta. And third, Quantum is a negation of the negation. As such, it has exceeded the ideal Limit which Discrete Magnitude represented. It is an ideal being that excludes its otherness from itself. "Thus the one [of Quantum] is (α) *self-relating*, (ß) *enclosing* and (μ) *other-excluding* limit." (202)

When completely posited in these three determinations, Quantum is Number. Thus, with reference to Figure 12(c), Number includes "limit as a *plurality*" (203)—or [4, 5, 6]. In its analysis of Quantum, the Understanding first isolates this plurality as Amount, and so we get:

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84 One commentator goes so far to suggest that the first three chapters of the *Science of Logic* are entirely dedicated to establishing this one proposition. Petry, *supra* note 11, at 485.

85 *See* Carlson, *supra* note 4, at 558-59.

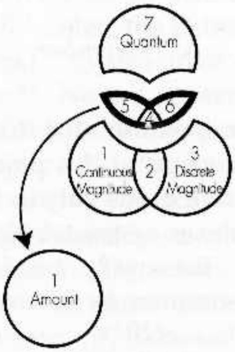

Figure 13 (a)
Amount

In Figure 13(a), the Understanding sees Quantum as containing the Many Ones. But Quantum "does not contain them in an indeterminate manner, for the determinateness of limit falls in[side] them." (203) In Amount, Quantum determines itself as unique from other pluralities. In short, "three" proudly boasts that it is uniquely "three" and *not* some other number like two or four.

Amount is a plurality—of what? Units! Hence, "three" is really always three units, or 3 = 3·1. Hence, we immediately derive:

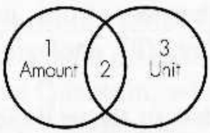

Figure 13 (b)
Unit

"*Amount* and *unit* constitute the *moments* of number." (203) This brings us quickly to Figure 13(c):

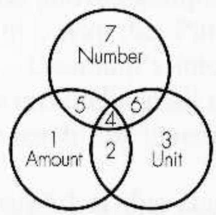

Figure 13 (c)
Number

Hegel says of Figure 13(c), "Quantum is limited generally; its limit is an abstract, simple determinateness of it. But in quantum as number, this limit is posited as *manifold within itself*." (203) The "manifold" is Number's Amount.

Number is a "complete positedness" (203)—that is, a complex—when the plural limit [4, 5, 6] is considered together with the immediate unity [7]. So considered, Number is a Discrete Magnitude, or a Unit. That is, in [7] it is unmediated. But [7] just as much continues into [4, 5, 6]. Because it is continuous, it is a "complete *determinateness*, for in it the limit [4,5,6] is present as a specific *plurality* which has for its principle the one [7], the absolutely determinate." (203)[86]

What is the difference between Number as a complete positedness and Number as a complete determinateness? Positedness represents what a True Infinite presupposes by self-erasing—that there is an other that controls its content. Thus, the number three is simply *not* four or five; it *posits* its being in all the other numbers.

Determinateness represents a cruder stage—"being" which admits that it is in unity with non-being but which refuses to self-erase:

> In the sphere of determinate being, the relation of the limit to [Determinate Being, or, here, Amount] was primarily such that the determinate being persisted as the affirmative on this side of its limit, while the limit, the negation, was found outside of the border of the determinate being . . . . (203)

Putting these points together, Number is a True Infinite. It becomes something other (positing); its being is determined externally, by all the other numbers it is not. Yet it also stays what it is (Determinate Being); for this very reason, 100 does not change into 99 or 101.[87]

Hegel has said that Number is a "complete determinateness" because of continuity. How so? Because, just as Attraction fused the Many into One,[88] so Continuity fuses the plurality into One. Hence, Number [4-7] is made into One by Continuity. Yet this One refers to both being [4, 5] and to nothing [4, 6]. Equally, this One's being might be viewed as continuous plurality [4, 5, 6] or the negative unity [7] that holds it together. Either way, because it is complex, Number is a determinateness. Quantum is beginning to recapture some of the content that Being-for-self shed from itself in Repulsion.

With regard to Amount in Figure 13(c), Hegel asks how the Many Ones (of which Amount consists) are present in Number. In effect, Amount assumes an external "counter," who breaks off Amount for his

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86 Mure puts it this way: "Any whole number is the 'discerning' of a sum within a continuous multiplicity of self-equal units, within an endless flow in which the unit endlessly repeats itself." MURE, *supra* note 17, at 119.

87 Andrew Haas remarks, "The concept of quantum . . . is not merely quantitative. Indeed, if number can show itself as qualitative, then it is because every quantitative difference (numerical e-quality and ine-quality;arithmetic and geometry) is always also a qualitative difference . . . ." HAAS, *supra* note 8, at 117.

88 Carlson, *supra* note 4, at 571-72.

own purposes and isolates it from the many other Amounts that could have been isolated.[89] For example, the counter, for reasons of his own, counts to 100 and breaks off the counting there. This amount is thus isolated from 99 or 101, by some external "counting" force.

Of counting to 100, Hegel writes:

> In the sphere a number, say a hundred, is conceived in such a manner that the hundredth one alone limits the many to make them a hundred . . . but none of the hundred ones has precedence over any other for they are only equal—each is equally the hundredth; thus they [i.e., the units] all belong to the limit which makes the number a hundred and the number cannot dispense with any of them for its determinateness. (203-04)

In other words, Unit is Limit to Amount. 100 is simultaneously *one* Unit, but it also implies 100 equal units contained therein, each one of which lays equal claim to being the 100th.[90]

Number has a limiting Unit—the 100th Unit. By this, 100 differs from 99 or 101. The distinction, however, is not qualitative. Qualitative distinctions are self-generated. Quantitative distinction is externally imposed. The units do not count *themselves* to 100. They require "comparing *external* reflection"—a mathematician—to do the counting. (204) 100 is thus externally derived. Once this is accomplished, 100 "remains returned into itself and indifferent to others." (204)

Hegel finishes his analysis of Number by emphasizing that it is an "absolutely determinate" Unit, "which at the same time has the form of simple immediacy and for which, therefore, the relation to other is completely external." (204) If some things are numerically determined, the things themselves are qualitatively unaffected by Number. Thus, if I say I have three things and you say you have four things, our "things" have not yet been distinguished in and of themselves.[91] They are still homogeneous "things" in spite of the numerical difference. Conversely, Number is blithely indifferent to the things to which we apply them. Within Number is a complete openness to externally imposed content. In short, we can use Number to count *any* qualitative thing.

Number, then, displays its *own* immediacy not imposed by the outside thing. Besides being this immediacy, Number is also a determinateness—a mediation. Its moments are Amount and Unit.

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[89] As Hegel puts it, "the breaking off [of the counting] of the many ones and the exclusion of other ones appears as a determination falling outside the enclosed ones." (203).

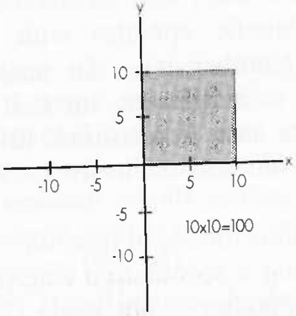
[90] A related point was made by Hegel earlier with regard to Attraction. In chapter 3, Hegel stated that the Many Ones were fused into One by Attraction. We were not, however, to assume that, amidst the Many Ones, a single Caesar had risen to become the imperial One. Rather, each of the Many Ones had an equal claim to the crown of One. So it is with the Units in Number. *See* Figure 10(a); Carlson, *supra* note 4, at 578.

[91] Of course, put just this way, the things are distinguished because some of the things are mine and some are yours. But this "quality" of the things is purely external to the things themselves. They are utterly indifferent to whom their owners are.

This contradiction—Number as an immediacy and as a determinateness—is said to be "the quality of quantum," (204) which will lead to further development.

Remark 1: The Species of Calculation in Arithmetic; Kant's Synthetic Propositions *a priori* of Intuition

Geometry. In this long Remark, Hegel distinguishes and also relates geometry and arithmetic. Hegel identifies the science of spatial magnitude as geometry, which has Continuous Magnitude as its subject matter. In contrast, arithmetic trafficks in Discrete Magnitude. Perhaps this can be seen in the Cartesian plane.



Cartesian Plane

On the Cartesian plane, 100 is a rectangle and so is continuous through its allotted space. But the arithmetical 100 is, like valor, the better part of discreteness. It is simply neither 99, 101, nor any other Number.

Hegel emphasizes that geometry does not measure spatial figures. It only compares them. When it trafficks in equality of sides or equidistance of points from a center, it owes no debt to number. Thus, a circle is the set of equidistant points from a given center. But if geometry wishes to treat of triangles or non-square rectangles, number is requisite. Whereas, before geometry was driven solely by the external force of the geometer, now the geometry of triangles and rectangles relies on Number, which contains a moment of Discreteness. Where Number appears, mere comparison by the geometer no longer has exclusive jurisdiction.

Spatial geometry nevertheless implies and "continues into" arithmetic. Hegel returns to the "point" that from the geometric point springs the line of its own accord.[92] This is because the Zeus-like point was the Limit of the Athenian line. Since Limit is a correlative term

92 Carlson, *supra* note 4, at 522-23.

requiring two subparts in need of correlation, the minute we designated the point as Limit, we had to think of point's correlate—the line. Hegel admits that this demonstration indicates that spatial magnitude—*i.e.*, geometry free and clear of Number—generates numerical magnitude. The One of spatial magnitude immediately sublates itself and continues on to become the line of many Ones. Furthermore, to the extent a line is limited, the Limit of the line—the spatial point—must be viewed as a Number that limits the line of many ones. In the point, the line's self-determinedness is located. Hence, from the line's perspective, its self-determinedness is self-external. That is, the line *repels* from itself its Limit. The point seems to be that geometry is never entirely isolated from arithmetic, just as Continuity is never entirely isolated from Discreteness.

Arithmetic. Arithmetic operates with Number (but does not speculate as to what Number is). To arithmetic, Number is "the determinateness which is indifferent, inert; it must be actuated from without and so brought into a relation." (205) This is arithmetic's function. Numbers do not add themselves. Arithmetic is the tool of some outside will.

Arithmetic has various modes of relation—addition, multiplication, etc. Arithmetic not being a speculative enterprise, the transition from one of these modes to another is not made prominent. These modes can, however, all be derived from the very concept of Number. "Number has for its principle the one and is, therefore, simply an aggregate externally put together, a purely analytic figure devoid of any inner connectedness." (205) Thus, an external "counter" breaks off the counting at, say, 100, thereby isolating this Number from the infinite others the counter may have preferred. All calculation is essentially counting.[93]

Suppose we have two numbers chosen by the counter. Whatever relation these two numbers have must also be supplied by the counter. The counter must decide whether to subtract or divide these numbers. Number has a qualitative difference within it—Unit and Amount. But the identity or difference between two given Numbers is entirely external.

Numbers can be produced in two ways. We can count up the units and produce a number. Or we can subdivide from an aggregate already given. That is, given 100, we can negate 70 of the Units and isolate 30. In both cases, counting is implicated. One is positive counting. The other is negative counting.

Addition and Multiplication. In counting Units, the Amount of the Unit is set arbitrarily. We can count five single Units. Then we can

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93 In the *Lesser Logic*, Hegel refers to the mathematical operations as "telling a tale" about numbers. LESSER LOGIC, *supra* note 29, § 103.

decide to count some more—seven more units are added. Hence, we get 7+5=12. In "addition," the relation of 7 and 5 is a complete contingency. These two Numbers are quite indifferent to each other. They were simply put together by the mathematicians for their own private purposes—an arranged, not a romantic, marriage.[94]

We can also count six Units of two (multiplication). Hence, multiplication is the same as counting. What counts as a Unit (one, two, ten, etc.) is externally decided by the mathematician. All this counting, however, is tedious and so, to save time, we learn by rote what the sums and products of two numbers are.

Kantian Arithmetic. The sum 7+5=12 is chosen by Hegel because Kant used this very sum to demonstrate that arithmetic is a synthetic proposition.[95] Hegel denounces this conclusion of synthesis to be meaningless:

> The sum of 5 and 7 means the mechanical [*begrifflose*] conjunction of the two numbers, and the counting from seven onwards thus mechanically continued until the five units are exhausted can be called a putting together, a synthesis, just like counting from one onwards; but it is a synthesis wholly analytical in nature, for the connection is quite artificial, there is nothing in it or put into it which is not quite externally given. (207-08)

It is not clear to me why Hegel was so heated in denouncing Kant's invocation of synthesis with regard to arithmetic. Was Kant not simply saying that 5 and 7 do not add themselves? And is not Hegel in complete agreement that addition is a matter for the external counter? In short, "synthesis" to Kant is what "externality of content" is for Hegel.[96]

Hegel also objects to Kant's conclusion that arithmetic is a *a priori*. By *a priori*, Kant meant not derived from experience.[97] If we synthesize our *experiences*, then our knowledge is merely empirical and contingent, or *a posteriori*.[98] Hegel attacks the very distinction of *a priori* and *a posteriori*. He asserts that every sense or impulse "has in it the *a priori* moment, just as much as space and time, in the shape of spatial and temporal existence, is determined *a posteriori*." (208) This plaint is related to Hegel's criticism of the unknowable thing-in-itself.

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94 In analyzing "analysis," Hegel will summarize this point by announcing that arithmetic is basically "one"—magnitude as such. If this "one" is rendered plural, or unified into a sum, this is done externally. "How *numbers* are further combined and separated depends solely on the positing activity of the cognizing subject." (790)

95 CRITIQUE OF PURE REASON, *supra* note 14, at 10.

96 Very much later, Hegel returns to his view that aritmetic is analytic, not synthetic. He will say that 5+7 already contains the command to count 7 more beyond 5. The result contains nothing more than what was in 5+7—the command to keep counting. Therefore, arithmetic is analytic only. *Science of Logic* at 791-92.

97 CRITIQUE OF PURE REASON, *supra* note 14, at 60.

98 *Id.* at 44.

In effect, Hegel believes that our knowledge of objects is always a unity of our perception (*a posteriori*) and the authentic integrity of the object (*a priori*).[99]

Hegel praises, after a fashion, Kant's notion of the synthetic *a priori* judgment as belonging "to what is great and imperishable in his philosophy." (209) But what he likes about it is the speculative content Kant never brought to light. In the synthetic *a priori* judgment, "something differentiated . . . equally is inseparable." (209) Identity is "in its own self an inseparable difference." (209) In other words, if arithmetic is *a priori* synthetic, then 7+5 can be kept apart and also *not* kept apart simultaneously. Difference and identity each have their moments in 7+5=12. But this identity of identity and difference[100] is no mere property of the *a priori* synthetic judgment. It is just as much present in intuition—*a posteriori* judgment. Hence, the compliment to Kant is, at best, ironically tendered.

In any case, Hegel attacks Kant's assertion that geometry is grounded in synthesis. Kant conceded that some of its axioms are analytic, but he also held as synthetic the proposition that the shortest line between two points is a straight line.[101] In contrast, Hegel has held that, at least if "point" is thought together with Limit, the line generates itself. This self-generated line is inherently simple. "[I]ts extension does not involve any alteration in its determination, or reference to another point or line outside itself." (208) Simplicity is the very Quality of the line, which springs forth from its Limit in the point. Euclid therefore was correct in listing amongst his postulates the purely analytical proposition that the shortest line between two points is a straight line.[102] Because this definition includes nothing heterogeneous to geometry, Euclid's proposition is analytic, not synthetic.[103]

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99 These demonstrations are made in the early chapters of the *Phenomenology*, *supra* note 10.

100 The identity of identity and difference—a key Hegelian slogan—has already been discussed in Remark 2 following "The Unity of Being and Nothing" in chapter 1. It will be expressly considered as an important part of the Doctrine of Reflection. *See Science of Logic* at 408-33.

101 CRITIQUE OF PURE REASON, *supra* note 14, at 10.

102 Euclid gave these four postulates upon which all geometry is based:

> (1) a straight line segment can be drawn joining any two points.
> (2) Any straight line segment can be extended indefinitely in a straight line.
> (3) Given any straight line segment, a circle can be drawn having the segment as radius and one end point as center.
> (4) All right angles are congruent.

DOUGLAS R. HOFSTADTER, GÖDEL, ESCHER, BACH: AN ETERNAL GOLDEN BRAID 90 (1979). A fifth was added, but it turned out to be subjective, not objective.

> (5) If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough.

Id. The suspension of the fifth postulate leads to non-Euclidean geometry.

103 Antonio Moretto, *Hegel on Greek Mathematics and the Modern Calculus*, *in* HEGEL AND NEWTONIANISM, *supra* note 11, at 149, 154.

Subtraction and Division. Subtraction and division are negative counting. In subtraction (*i.e.*, 12-5=7), the Numbers are indifferent or "generally unequal" to each other. That is, given a line segment of 12 units, we could have subdivided the line as 7 and 5, or 9 and 3, or 11 and 1, *etc*. The two Numbers into which a line of 12 units is subdivided bear no relation to each other.

If we make the two Numbers (qualitatively) equal, then we have entered the province of division. Suppose we count up a Unit—say, 6. The Number 12 now has a Unit of 6 and an Amount of 2.[104]

Division is different from multiplication, however. In multiplication, where 6·2=12, it was a matter of indifference whether 6 counted as Amount or Unit.[105] Division would seem to operate on another principle. After all, if we solve the above for 2, then 2=12/6. 12/6 is not the same as 6/12. But, remembering that "negative counting" takes 12 as given, it is likewise immaterial whether the divisor (6) or quotient (2) is Unit or Amount. If we say 6 is Unit, we ask how often 6 is contained in 12. If we say that the quotient (2) is Unit, then "the problem is to divide a number [12] into a given amount of equal parts [here, 6] and to find the magnitude of such part." (210)

Exponents. In multiplication and division, the two Numbers are related to each other as Unit and Amount. Yet Unit and Amount are likewise "still immediate with respect to each other and therefore simply *unequal*." (210) If we insist that Unit and Amount be equal, we will complete the determinations immanent within Number. This last mode of counting is the raising of a Number to a power.

Take 6^2=36. Here, "the several numbers to be added are the same." (210) Should not Hegel have said the *two* numbers [6 and 6] to be *multiplied* are the same? No. Hegel has already said that multiplication is counting, just like addition. Hence, we shall count six units. Each unit has six in it. In short, we count from 1 to 6. Next we count from 7 to 12, and so forth. Eventually we reach 36. The point is that in squaring 6, Amount equals Unit.

If we advance from 6^2=36 to 6^3=216, "inequality enters again." The new factor (6), is equal to the former Unit (6) and Amount (6). But this "new factor" must now be taken as Unit. The prior square (6^2) is now Amount. Hence, Unit and Amount are now not equal.[106] But, at least if we stick with squares:

> [w]e have here in principle those determinations of amount and unit which, as the essential difference of the Notion, have to be equalized

104 Qualitative equality means that Unit and Amount have a kind of discreteness to them. Of course, we external reflectors must decide which of the two numbers is Unit and which is Amount. These numbers do not yet determine themselves.

105 This is the "commutative" property of multiplication, according to which *ab* = *ba*.

106 Of course, we could likewise say that 36 is Unit and 6 is Amount.

> before number as a going-out-of-itself has completely returned into self . . . [T]he arithmetical square alone contains an immanent absolute determinedness . . . . (211)

Here we have a preview of what, in the last chapter of Quantity, will be called the "Ratio of Powers."[107] The premise is that if we insist that Unit equals Amount, the number shows resistance to outside manipulation. The Ratio of Powers will represent the last stage of Quantity. It is here that Quantum recaptures its integrity and wins its independence from the counters who have so tyrannized it prior to that point.

The self-integrity that squares enjoy explains various mathematical phenomena, according to Hegel. Thus, "higher equations"—equations involving powers higher than two[108]—must be reduced to quadratic equations, which only involve squares.[109] This also explains why "equations with odd exponents can only be formally determined." (211) By this Hegel seems to mean as follows: if I consider a higher equation involving an odd exponent, I can calculate the "root"[110] only by the use of imaginary numbers, such as $-\sqrt{1}$.[111] This route to the root is taken to be a bit of a mathematical imperialism, from which mere squares are immune.[112]

A last example of the dominance of the square is that, in geometry, only "right" triangles have immanent integrity. In a right triangle, where *c* is the hypotenuse, $a^2+b^2 = c^2$—Pythagoras's theorem. Only in this figure is there "absolute determinedness." (211) For this reason, all geometric figures must be reduced to right triangles for their complete determination.

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107 *See infra* notes 246-63 and accompanying text.

108 An example:

$$ax^3 + bx^2 + cx + d = 0$$

where $a \neq 0$.

109 A quadratic equation has this form:

$$ax^2 + bx + c = 0$$

where $a \neq 0$.

110 In quadratic equations, there are always two different solutions, or roots, though occasionally the roots are equal to each other (when $b^2 = 4ac$).

111 The entire sentence I am interpreting asserts:

> [E]quations with odd exponents can only be formally determined and, just when roots *are* rational they cannot be found otherwise than by an imaginary expression, that is, by the opposite of that which the roots are and express. (211)

On solutions to the cubic equation, *see* CARL B. BOYER, A HISTORY OF MATHEMATICS 284-86 (rev. ed. 1991) [hereinafter BOYER, MATHEMATICS].

112 Later, Hegel will say that the solution to the higher equations is synthetic, not analytic, because the relevance of imaginary numbers must be proven and is not simply analytic "counting," as arithmetic is. *Science of Logic* at 792-93.

Hegel has a mysterious paragraph on "graded instruction." (212) By this he presumably means ordinary high school math courses.[113] Hegel states instructors teach about powers before they teach "proportions." I take "proportion" to mean ordinary division of numbers—fractions. Proportions are connected with the *difference* between Unit and Amount. That is, 6/2 is not the same as 2/6—it rather matters which is the dividend and which the divisor. The study of "proportions" thus goes beyond immediate quantum, where Unit and Amount are mere moments. Any such study is external to Quantum. In Quantitative Ratio—to be considered in the final chapter of Quantity—Number is also no longer immediate quantum. Rather, ratio possesses a determinateness of its own.[114]

Hegel has spent considerable time deriving addition, etc., from the very concept of Number. But, he warns:

> It cannot be said that the progressive determination of the species of calculation here given is a philosophy of them or that it exhibits, possibly, their inner significance. (212)

Rather, Hegel suggests that we must distinguish what is self-external to Number. When we identify what is external to Number, then we know that what the Notion accomplishes happens in an external manner. Thus, any idea of equality or inequality of Numbers is external to the concept of Number as such.

Hegel concludes the Remark with this observation:

> It is an essential requirement when philosophizing about real objects to distinguish those spheres to which a specific form of the Notion belongs . . . [O]therwise the peculiar nature of a subject matter which is external and contingent will be distorted by Ideas, and similarly these Ideas will be distorted and made into something merely formal. (212)

Presumably this warning means that speculative philosophy has its sphere, and higher mathematics has its sphere. Each should be wary of permitting the other field from unduly interfering the project at hand.

Remark 2: The Employment of Numerical Distinction for Expressing Philosophical Notions

Hegel has already shown that Number is the "absolute determinateness of quantity, and its element is the difference which has

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113 Significantly, Hegel wrote these remarks while serving as a high school principal in Nuremburg, waiting impatiently for a university to offer him a professorship. TERRY PINKARD, HEGEL: A BIOGRAPHY 332-51 (2000).

114 Of course, earlier Hegel has suggested that "proportions" are simply negative counting, involving Unit and Amount, just like addition or multiplication. *See supra* text accompanying notes 103-04. Here Hegel reverses field and states that division is more "advanced" than positive counting. Perhaps this is true only from a pedagogical point of view.

become indifferent." (212) The indifference of Number implies that Number finds its content imposed upon it from the outside. Thus, arithmetic is an analytical science. It does not contain the Notion. Arithmetic combinations are not intrinsic to the concept of Number "but are effected on it in a wholly external manner." (212) It is therefore "no problem for speculative thought, but is the antithesis of the Notion." (212) When thought engages in arithmetic, it is involved in activity which is the

> extreme externalization of itself, an activity in which it is forced to move in a realm of thoughtlessness and to combine elements which are incapable of any necessary relationships. (213)

Mathematics is "the abstract thought of *externality* itself." (213) For this very reason, Number is the abstract version of sense (also external to thought). In Number, "sense is brought closest to thought: number is the *pure thought* of thought's own externalization." (213)

Hegel relates the Many Ones to sensual material. The Many "is in its own self external and so proper to sense." (213)[115] When thought—"what is most alive and most active"—is translated to Number, then what is concrete turns into what is abstract—"dead, inert determinations." (214) The ancients knew that Number stands midway between sense and thought. They knew that philosophy was not fit for mere numbers—something Hegel's contemporaries had forgotten.

Numbers are supposed to be educational for students, but Hegel thinks this is over-rated.

> Number is a non-sensuous object, and occupation with it and its combinations is a non-sensuous business; in it mind is held to communing with itself . . . a matter of great though one-sided importance. (216)

But occupation with numbers "is an unthinking, mechanical one. The effort consists mainly in holding fast what is devoid of the Notion and in combining it purely mechanically." (216) Calculation dulls the mind and empties it of substance. It is so thoroughly debased, Hegel notes, "that it has been possible to construct machines which perform arithmetical operations with complete accuracy." (216; *see also* 791)

B. *Extensive and Intensive Quantum*

(a) Their Difference

In Figure 13(c), Number can be interpreted as a having its

[115] Presumably this means that the Many Ones were expelled from the One in Figure 9(b) and therefore became other to thought. *See* Carlson, *supra* note 4, at 560-61.

determinateness isolated in Amount [4, 5, 6].[116] [7] is Number's Unit, which can be taken, in its Discreteness, as a plurality, since Amount continues right through it. Number is nothing *but* Limit—five is nothing but *not* six or four.

Quantum "with its limit, which [Limit] is in its own self a plurality, is *extensive magnitude*." (217)

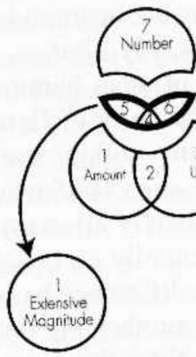

Figure 14(a)
Extensive Magnitude
(Extensive Quantum)

Extensive Magnitude represents the Understanding's realization that a number is nothing but *not* all the other numbers. In short, the Understanding proposes that Number is a metonym,[117] with all its Being outside of itself. Thus, if our number is three, Extensive Magnitude is $-\infty \rightarrow \{2.999\ldots\}$ and $\{3.000\ldots001\} \rightarrow \infty$.

Extensive Magnitude is to be distinguished from the earlier stage of Continuous Magnitude in Figure 12(a). Continuous Magnitude was a component part of Quantum. It was derived by extracting [4, 5, 6] from Quantity, as Enriched by the development of Continuity and Discreteness. In Continuous Magnitude, Quantum was conceived as identical with its Limit [4, 5, 6]. "Continuous magnitude is not yet truly determined as being for itself because it lacks the one [7] (in which being-for-selfness is implied) and number." (217) Magnitude does not receive this "one" until Dialectical Reason arrives to bring out the merely implicit idea of Discrete Magnitude. Continuous Magnitude

116 [6] is beyond Amount proper. But, since Unit is just as much Amount as Amount, [6] can be included as part of the determinateness which Hegel names as Amount.

117 Metonymy is the inability to name the thing directly, but only the context surrounding the thing. Metonymy so defined recognizes the negative constitution of things. *See* Michel Rosenfeld, *The Identity of the Constitutional Subject*, *in* LAW AND THE POSTMODERN MIND: ESSAYS ON PSYCHOANALYSIS AND JURISPRUDENCE 157-65 (Peter Goodrich & David Gray Carlson eds., 1998); Jeanne L. Schroeder, *The Midas Touch: The Lethal Effect of Wealth Maximization*, 1999 WIS. L. REV. 687, 763 ("In metonymy, the signified always remains hidden, and negative.").

therefore represents "only one of the two sides which together make quantum fully determined and a *number*." (217)

Discrete Magnitude was more advanced. It brought out what was merely implicit in Continuous Magnitude, but it also suffered from the same fault. In Discrete Magnitude, there was a discreteness [3] which did not *expressly* admit its unity with Continuous Magnitude.

What was merely in-itself in these earlier stages is now made express. "Extensive and intensive magnitudes are determinatenesses of the quantitative *limit* itself." (217) That is, Extensive Magnitude is a more adequate interpretation of what Number is—mere Limit fending off all the other numbers. The unity of Extensive Magnitude "has the moment of continuity present within itself," (217) but Extensive Magnitude presents itself as *one*. Referring to Figure 14(a), Hegel states that Number "is immediately an *extensive quantum*—the *simple* determinateness which is essentially an *amount*, but an amount of one and the same *unit*." (217) The difference between Extensive Magnitude and Number is only this: "in number the determinateness is expressly posited as a plurality." (218) Now the *unity* of Number comes to the fore. But the unity presented is the unity of what Number is *not*—all the other numbers that metonymically described what Number *is*.

Dialectical Reason comes forward to complain that Extensive Magnitude suppresses [7]—the truly *internal* "quality" of Number. Thus, the number three is -∞ →{2.999 . . .} and {3.000 . . . 001}→ ∞ (its Extensive Magnitude), but it is also three. In other words, Number is *not just* a metonym but also an affirmative being.

This "intensive" quality of Number is Intensive Magnitude or Degree:

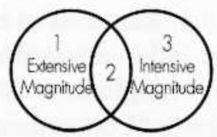

Figure 14 (b)
Intensive Magnitude (Degree)

Of Figure 14(b), Hegel writes, "the limit of quantum, which as extensive had its real determinateness in the self-external amount, passes over into *simple determinateness*." (218) Here, I think, we have self-erasure. Extensive Magnitude says, in effect, "I am not a unity." Unity therefore flees the precinct of Extensive Magnitude and takes sanctuary in Intensive Magnitude.[118]

[118] Much later, in mediating between Kant and Moses Mendelssohn on the immortality of the soul, Hegel states that, if the soul is viewed as purely simple, it has no "extensive magnitude." (779) Extensive Magnitude will represent "parts" in the dialectic of whole and parts.

Earlier, Number was Amount—a plurality made into one. Within Amount, each of the Many Ones was the same as any other. None was *primus inter pares*. If Amount is 100, each One could claim to be the determining 100th. Hence, Amount did not exhibit determinateness as such (Limit). Amount thus collapsed into Unit.

Degree, in contrast, is a specific magnitude. For example, it is the 100th One. As such, it "is not an aggregate or plural *within itself*." (218) Rather, it is a "plurality only in principle." (218) In Degree, "determinate being has returned into being-for-self." (218)

The determinateness of Degree must be expressed by a number. It must be, for example, the 100th One. In this expression, 100 is not Amount. It is only unitary (or a degree). Now, a single One emerges as *primus inter pares* over all the other Ones.[119]

Degree enjoys Being-for-self. It resists continuity in a way that, in the earlier stages of Quantity, the Many Ones could not. But, at the same time, Degree's content is external to itself. If it is the 100th, the "100" is outside of it. It is Extensive Magnitude that owns plurality—the externality of Degree.[120] Yet this plurality is likewise One. In effect, Extensive Magnitude has already turned itself into One, when it said, "I am plurality itself." In this posture, Extensive Magnitude, by announcing itself not a unity, unified the plurality in [1] of Figure 14(a). Hence, Degree and Extensive Magnitude are doing the same thing—expelling their own content, which we can now interpret as [2].[121] "A plurality [1, 2] external to the degree [3] constitutes the determinateness

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119 Perhaps some of this is missed by Justus Hartnack in his description of Extensive Magnitude:

> To say about a quantum that it is an extensive magnitude is to say that it is measurable. If I say about a quantum that its length is ten yards, this means that yard follows upon yard until one reaches the end, the tenth yard (ten being the limiting number). . . . By performing the act of counting, I treat the quantum as an extensive quantum.

JUSTUS HARTNACK, AN INTRODUCTION TO HEGEL'S LOGIC 32 (Lars Aagaard-Mogensen trans., 1998). My problem with this account is that it does not quite capture Extensive Magnitude's role in making Degree coherent. Thus, if we think of the tenth yard, Extensive Magnitude is the plurality of numbers which are *not* tenth. Hence, Extensive Magnitude is the first nine yards and the eleventh yard and beyond. Extensive Magnitude is therefore not a Quantum on which we focus but rather the background which makes Degree coherent.

Hartnack goes on to say:

> If we talk about . . . a room temperature of 20° C, then the degrees below the 20° never formed an extensive magnitude that was absorbed in that degree of temperature . . . . The degree cannot be verified by adding the degrees below 20°—as we can add the yards . . .

Id. I think this is absolutely wrong. The Extensive Magnitude of 20° is *precisely* all the degrees that 20° excludes.

120 *See* HAAS, *supra* note 8, at 118 ("Multiplicity . . . is interior to extensive quantum . . . . Here number reveals the quality of quantum: determinate *indifference* . . . to is own multiplicity (within or without).").

121 *See id.* at 118-19 ("Although extensive and intensive quanta are differentiated according to the ways in which they express the multiplicity that forms their other, they are identical insofar as they both are characterized by qualitative indifference.").

of a simple limit which the degree is for itself." (219)

This expulsion of [2] produces a middle term, of which Hegel writes:

> Number as a one, being posited as self-relation reflected into itself, excludes from itself the indifference and externality of the amount [i.e., the plurality] and is self-relation as *relation through itself to an externality*. (219)

Notice the return of self-erasure. In Figures 13(a) and 13(b), the left extreme announced an immediacy (Amount). The right extreme brought out the ideal moment (Unit) that the immediacy did not fully emphasize. Now, in Figure 14(a), Extensive Magnitude [1] sheds its unity and insists on being plural as such (and in so doing unifies the plural). Meanwhile the unity it shed [2] was secretly plural [1, 2]. On the other side, Degree does the same. It sheds its plurality [2] and insists on being One [3]. The middle term names this self-relation.

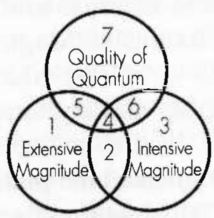

Figure 14 (c)
The Quality of Quantum

In this middle term, "quantum as a reality [is] conformable to its Notion." (219) The word "reality," perhaps, connotes "being" in conjunction with its non-being,[122] although, since reality has long since given way to ideality, "reality" must be taken in the sublated sense. But non-being here must be understood as containing all the content that Quantity has shed into the external realm.

Speculative reason, in picking up [2] from Figure 14(b), emphasizes a determinateness that is indifferent to its extremes. That is to say, [2] consists of material shed by the extremes in their indifference. This indifferent determinateness is precisely the *quality* of Quantity—that Quantity is indifferent to its content.[123]

Hegel concludes this section by dropping back and describing Degree as different from Extensive Magnitude. Degree is a unitary *determinateness* [3]. But it is unitary by grace of a self-external

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122 "Reality" is reflected in Figure 2(b)—quality as opposed to negation of quality. It represents a one-sided view of Being as it exists prior to thought. Carlson, *supra* note 4, at 479-82.

123 HAAS, *supra* note 8, at 123 ("In other words, the qualitative aspect of quantity means qualitative opposition, that is, having determinateness in another, by means of its non-being, having its being by virtue of its nothing . . . .").

plurality [1, 2]. A Degree differs from every other Degree, but the Degrees are likewise "essentially interrelated so that each has its determinateness in this continuity with the others." (219) There is a continuity running through the Degrees, which makes possible an

> ascent and descent in the scale of degrees of a continuous progress, a flux, which is an uninterrupted, indivisible alteration; none of the various distinct degrees is separate from the others but each is determined only through them. (219)

Degree is what it is because of what is external to it. It is therefore not indifferent to its content—even though it actually shed this same content in an act of indifference. As always, by showing indifference to its content, Degree demonstrates how absolutely dependent it is on its Extensive Magnitude. This very contradiction is the Quality of Quantum, as shown in Figure 14(c).

(b) Identity of Extensive and Intensive Magnitude

The last section discussed the *difference* between Extensive and Intensive Magnitude. Ironically, difference was gathered together in a middle term: the very Quality of Quantum is indifference to content. Now we shall explore this Quality/indifference and discover the identity lurking within Figure 14(c)—as if that were not already plainly visible there.

Our next move is as follows:

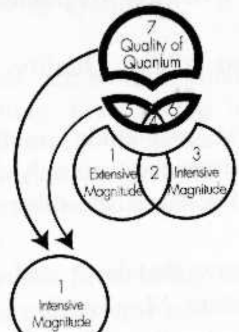

Figure 15(a)
Intensive Magnitude (Degree)

The Understanding now takes up one side of the middle term. But it fully sees that it takes up the one side. Hence, it moves the whole of the middle term, even as it isolates Degree.

With regard to Intensive Magnitude taken positively, we learn that "Degree is not external to itself within itself." (220) That is, [1] is taken as a simple immediacy. It is, however, more advanced than the "*indeterminate* one, the principle of number as such," (220) *i.e.*, Discreteness. Degree is also to be distinguished from its ancestor, Amount, "save in the negative sense of not being any particular amount." (220) Rather, Degree is

> primarily a unitary one of a plurality; there are many degrees, but they are determined neither as a simple one nor as a plurality, but only in the relation of this self-externality . . . If, therefore, the many as such are indeed outside the simple unitary degree, nevertheless the determinateness of the degree consists in its relation to them; it thus contains amount. (220)

In other words, Degree [1, 2] sheds its content [2]—plurality—but, by shedding [2], it contains [2].

What it sheds, of course, is other degrees, with which it is continuous. Thus, the twentieth degree sheds all the other degrees, even while it retains for itself the "twenty"—which uniquely distinguishes the twentieth degree from all others. These excluded degrees can be called, collectively, Extensive Magnitude (or Extensive Quantum)—this time taken negatively.

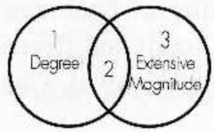

Figure 15(b)
Extensive Magnitude

Thanks to this exploration of the Quality of Quantum, we can see clearly that

> [e]xtensive and intensive magnitude are thus one and the same determinateness of quantum; they are only distinguished by the one having amount within itself and the other having amount outside itself. (220)

Similarly, we previously saw that Unit and Amount were the same—also Continuous and Discrete Magnitude. Throughout Quantity, the extremes end up being each other—here expressly, as Intensive and Extensive Magnitude have literally swapped places.

The middle term between the obversely charged extremes is:

Figure 15 (c)
Qualitative Something

This unity is an "identity . . . which is self-related through the *negation of its differences*." (221) In short, it is the standard move of Speculative Reason, as developed in and after the True Infinite. It names the very act of the extremes in erasing themselves and stating what they are not.

"Something" was the name of Figure 2(c)—a unity between Determinate Being (or Quality) and Negation.[124] Furthermore, "Quality" has long since been sublated. Why does Hegel use the phrase Qualitative Something here?[125]

The answer is that Degree (in both its forms of plural and unique) still has its content outside itself. The Qualitative Something is precisely that content—but taken negatively as simply the opposite of Quantum. Degree depends on that Qualitative Something to define what it is. Meanwhile, as Degree changes, the Qualitative Something remains what it is. Such a Something is indifferent to its quantitative limit.

To be sure, the Qualitative Something is a Quantum. But it is Quantum that is indifferent to Quantum. It is substrate to the more primitive quanta.

> Quantum, number as such, and so forth could be spoken of without any mention of its having a something as substrate. But the something now confronts . . . its determinations, through the negation of which it is *mediated* with itself, as *existing for itself* and, since it has a quantum, as something which has an extensive and an intensive quantum. (221)

As Quantum, the Qualitative Something is subsistent, whereas the original Something of Figure 2(c) was not.[126] Quantum is, after all, a True Infinite, which stays what it is even as it becomes something different. In the Quantitative Something, we have a something that positively resists the transgression of its Limit—which the original Something could not achieve.[127]

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124 Carlson, *supra* note 4, at 495.

125 "With this identity, the *qualitative something* makes its appearance." (221)

126 *See* Carlson, *supra* note 4, at 494-500.

127 *See* Haas, *supra* note 8, at 119-20 ("a quantum is infinite self-negation, production, creation, self-expulsion, a coming-out of itself as increase/decrease, that is, the moving, growing, living limit . . . that determines quantity insofar as it infinitely supersedes, determines and takes

Remark 1: Example of This Identity

All quanta have both Extensive and Intensive Magnitude. Yet, Hegel complains, these are sometimes treated as separate, so that one thing is Extensive and another thing is Intensive. Density, for example, should not be treated merely as Extensive, as in so many atoms per cubic centimeter. It is also Intensive. As such it is dynamic—"*not* a certain *aggregate* and *amount* of material parts in a quantum of space, but as a certain degree of the space-filling *force* of matter." (221)

It is not clear what Hegel means by this.[128] Perhaps the point is that density is not a just plurality of units in space but is itself a force—mass. "The mass of a body is its inertia or resistance to change of motion. More precisely, it is a property of the body that determines the body's acceleration under the influence of a given force."[129] The mechanical point of view, Hegel says, leaves density as the concept of separately existing, independent parts, "which are only externally combined into a whole." (222) But the dynamic point of view holds density to be a space-filling force. These consideration of (mechanical) whole and parts or of (dynamic) force and expression are too advanced for Quantum and will be considered later on, Hegel assures us.[130] Nevertheless, it can be said now that the relation of force and its expression corresponds to Intensive and Extensive Magnitude respectively. In other words, like Degree, which cannot be considered separately from Extensive Magnitude, Force is one-sided and cannot be considered separately from its expression.

With regard to Intensive Magnitude, Hegel gives the example of the circle with its 360 degrees. The determinateness of any one degree "derives essentially from the many parts outside it." (222) One degree of the circle depends on its relation with the other 359.

More concrete objects exhibit the dual aspect of being both extensive and intensive. Extensive Magnitude represents the outer being of such an object. Intensive Magnitude represents the inwardness of it. Hegel gives the example of mass as weight. It is an Extensive Magnitude in so far as it constitutes an amount of pounds. It is an Intensive Magnitude in so far as it exerts a certain pressure. Pressure is

care of itself.").

128 He could mean that bodies generate gravitational force, which can be observed when a second body with gravitational force comes within its field. He could mean that, when one body exerts force on another, the second body likewise exerts force on the first. In other words, for every action there is a reaction—Newton's third law of motion. Finally, if a second body exerts a force on a first body, the first body accelerates and acquires a velocity which continues until some third object slows it down.

129 *Mass*, *in* 5 ENCYCLOPEDIA OF PHILOSOPHY 177 (1967).

130 This will occur in chapter 15.

expressed as a degree on a scale.

As exerting pressure, mass is manifested as a being-within-self, as a subject to which belongs a difference of intensive magnitude. Conversely, that which exerts this *degree* of pressure is capable of displacing a certain *amount* of pounds, etc., and its magnitude is measured by this. (223)

Heat famously has a Degree. But it also has Extensive Magnitude—the expansion of mercury in a thermometer or the expansion of air. Musical notes have a Degree—pitch—and Extensive Magnitude—the number of vibrations.

Meanwhile, in the sphere of spirit, "high intensity of character, of talent or genius, is bound up with a correspondingly far-reaching reality in the outer world." (223) In short, there is no such thing as talent or genius unless it is actually manifested—*i.e.*, becomes "extensive."

Remark 2: The determination of degree as applied by Kant to the soul

Kant applied Intensive Magnitude to the metaphysical determination of the soul, Hegel says. Kant considers the inference of the soul's immortality from the soul's simplicity, an inference Kant opposes.[131] Kant proceeds as follows: Admit the soul is simple. It thus has no Extensive Magnitude—no plurality to it. Nevertheless, the soul has Intensive Magnitude—a Degree of reality. This degree can diminish gradually and eventually vanish. Hence, the soul is not immortal, just because it is simple.

Kant's mistake is to consider the soul a "thing." If it were so, then to it could be attributed Quantum. But, Hegel, protests, the soul is Spirit, and Spirit forever exceeds the bounds of mere thinghood.[132]

(c) Alteration of Quantum

Middle terms have proved to be names for *activities*. Accordingly the Qualitative Something names the self-erasure of Extensive and Intensive Magnitude. It is the "difference" between them. This "difference"—the Something—is indifferent to Quantum. Quantum has negated itself and is ineffectual against the Qualitative Something.

The Qualitative Something of Figure 15(c) is self-contradictory. It

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131 *See* CRITIQUE OF PURE REASON, *supra* note 14, at 221.

132 According to Marcuse, "The 'existing thing,' for example, can never be Idea in the Hegelian sense, for its particularities still appear as relatively independent and separable properties. It does not exist in continuous 'sameness with itself.'" MARCUSE, *supra* note 38, at 151.

is "posited as being the simple, *self-related* determinateness which is the negation of itself, having its determinateness not within itself but in another quantum." (225) In other words, Extensive Magnitude in Figure 15(b) erased itself, and a new Quantum was produced. Taken as Quantum, the Qualitative Something [7] has its entire determinateness [4-6] outside of itself—in Degree and Extensive Magnitude. Yet it is simultaneously quite immune from these determinatenesses.

If we focus on the fact that the Qualitative Something has its entire determinateness outside of itself, we can say in fairness that it is "in absolute continuity with its externality, with its otherness." (225) From this perspective (even while admitting that the Qualitative Something is immune from other quanta), the Qualitative Something can *both* "transcend every quantitative determinateness" *and* be altered. (225) In fact, Hegel says it *must* alter.

In the Qualitative Something, Quantum reveals the "express character" (225) of impelling itself beyond itself into its external character, thereby becoming an other. That is, the Qualitative Something is quantitative determinateness. As such, it consists in undergoing increase or decrease:

> The quantitative determinateness continues itself into its otherness in such a manner that the determination has its being only in this continuity with an other; it is not a *simply affirmative* limit, but a limit which *becomes*. (225)

When Quantum impels itself beyond itself, it becomes another Quantum. But this new Quantum is "a limit which does not stay." (225) The new Quantum becomes yet another Quantum, "*and so on to infinity*." (225)[133] With this we are ready to move onto Hegel's monumental treatment of Quantitative Infinity, an untravelled country from whose bourne few travelers have ever returned.

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133 Professor Alan Paterson complains that the "successor function" and "induction principle" are not accounted for in Hegel's logic. Alan L.T. Paterson, *The Successor Function and Induction Principle in a Hegelian Philosophy of Mathematics*, 30 IDEALISTIC STUD. 25 (2002). But are not these very ideas the stuff of the Qualitative Something? (The successor function is "the function that sends a natural number to its successor. . . . The induction principle (roughly) says that we can assert a fact about *all* natural numbers if we know that it is true for the first one (0) and that its truth is preserved when we go from a number to its successor." *Id.* at 26).

Does Hegel have an account of ordinality in his theory of number? Paterson suggests the answer is no, and he may be right. Perhaps the concept of "more or less" is simply supplied by external reflection of the counter. *See supra* text accompanying notes 37-42.

C. *Quantitative Infinity*

(a) Its Notion

The nature of Quantum is to alter itself into another Quantum *ad infinitum*. As it alters, it conveys to its *other* the very status of "Quantum-ness." "[T]he other is thus also a quantum." (225) Yet, it is simultaneously the very *negation* of Quantum-ness, "the negative of quantum as limited." (225)[134]

The Understanding now isolates the very act of Quantum going outside itself:

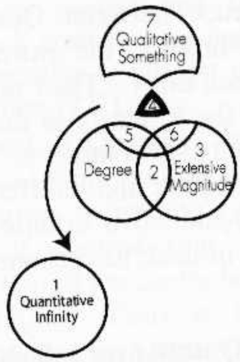

Figure 16 (a)
Quantitative Infinity

Hegel writes that Quantitative Infinity

> is an ought-to-be; it is by implication determined as being for itself, and this being-determined-for-itself is rather the being-determined-in-an-other, and, conversely, it is the sublation of being-determined-in-an-other, is an *indifferent* subsisting for itself. (226)

Are all of these things true? Yes. Recall that the Ought was the self-erasure of the Finite.[135] Now Quantum erases itself and becomes Other. This was the quintessential move of Being-for-self. Being-for-self turned itself into Quantity, which was total being-determined-in-an-other. But now Quantum, as Quantitative Infinity, sublates (*i.e.*, includes) all the other quanta. It is *all* the quanta. As such, it is indifferent to externality, because it has swallowed every Number. It is therefore indifferently subsisting for itself and no other.

134 Errol Harris puts it this way: "The contradiction of Quantum is that its internal determination rests in a limit which in its very nature posits an external other, on which the precise magnitude of the quantum is as much dependent as it is on what precedes the limit." HARRIS, *supra* note 15, at 138.

135 Carlson, *supra* note 4, at 527-28.

Hegel now compares Quantitative Infinity to the Quantum of earlier stages. The Quantum was finite but impelled beyond itself. In contrast, Quantitative Infinity is "unlimitedness" and also "returnedness into itself, its indifferent being-for-self . . . [I]n the infinite, quantum possesses its final determinateness." (226)

Yet this Infinity likewise is the "impulse to go beyond itself to an other in which its determination lies." (226)[136] Therefore, Quantitative Infinity is a Spurious Infinite—a Finite that propels itself to yet another Finite, which in turn propels itself to yet another Finite.[137]

The upshot of these contradictions is that Quantum has both Finitude and Quantitative Infinity in it at the same time. "[I]n other words, the concept of quantitative infinity thinks the process through which the finite finds its other (the infinite) in itself . . . ."[138]

What is the difference between Qualitative and Quantitative Infinity? In Qualitative Infinity, the extremes—[1] and [3]—stood "abstractly opposed" to each other. Their unity was only "in-itself"—implicit. This relation of the Finites was their transition (self-erasure) outside themselves. Quantity, in contrast is "sublated determinateness; it is *posited* as being unlike itself and indifferent to itself, consequently as alterable." (226) The Quantitative Infinite *expressly* continues itself into its other. In short, the in-itself has become for-itself.[139]

(b) The Quantitative Infinite Progress

Our next stage is drawn as follows:

136 *Id.* at 136. ("And as the continuity of quantum expresses itself equally in endless extensity and in endless diminution, the progression is interminable either way, though neither the infinitesimal nor the infinite is ever attainable.").

137 The return trip is not to the original Number but to some larger or smaller Number *ad infinitum*. This conclusion is compelled by the lesson learned in the One and the Many. *See* Carlson, *supra* note 4, at 560-69. There, the entity out of which the new entity springs does not go out of existence. Rather, the new entity springs out of itself and into yet another entity, creating infinitely Many Ones. The same result happens in Quantitative Infinity, though Hegel nowhere says so explicitly.

138 HAAS, *supra* note 8, at 120-21.

139 It is possible to quibble with Errol Harris's remark that, to resolve Quantum's contradiction, "the externality of the other must somehow be internalized to produce a true infinity." HARRIS, *supra* note 15, at 138. At this stage, the extremes each have long since been True Infinites. Precisely what Quantum must express is that it is as much its other as it is its own self. Hence, Harris is right that the external must be internalized, but the external must also *stay* external as it becomes internal. Furthermore, it is already a True Infinite and therefore need not, at this late stage, become one.

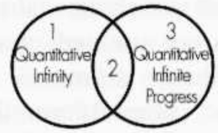

Figure 16 (b)
Quantitative Infinite Progress

In this stage of Dialectical Reason, the extremes fall into a Spurious Infinity—a senseless modulation back and forth, but this time with a quantitative flavor. Hegel describes this flavor as follows:

> [I]n the sphere of quantity the limit in its own self dispatches and continues itself into its beyond and hence, conversely, the quantitative infinite too is posited as having quantum within it; for quantum in its self-externality is also its own self, its externality belongs to its determination. (227)

In other words, at the level of Quantity, the Infinite self-consciously goes beyond itself *and* stays within itself as it travels into this beyond.

Hegel says the "progress to infinity" is implicit in "quantum as such," its "expression of contradiction." (227)[140] This progress, however, is not the *resolution* of the contradiction. (This must await the middle term in Figure 16(c).) There is, however, a mere *show* of resolution, which Hegel blames on Continuity of one extreme into the other.

As Hegel sees it, the Quantitative Infinite Progress promises the infinite but never delivers. "[I]t does not get beyond quantum, nor does the infinite become positively present." (227) The problem is that Quantum, by its nature, always has a beyond and is never fully present.

This beyond, considered on its own, is the non-being of Quantum. By its own act, Quantum vanishes into this beyond. Nevertheless, Quantum contains a qualitative moment in which it does *not* vanish into its beyond. But, simultaneously, Quantum continues *into* the beyond—its quantitative moment. Thus, "quantum consists precisely in being the other of itself, in being external to itself; this [beyond] is, therefore, no more an other than quantum itself." (227) In short, this beyond is *itself* another Quantum. "In this way, the beyond is recalled from its flight and the infinite is attained." (227) But such an infinite is spurious. (228) It is just another Quantum. "[W]hat has been posited is only a fresh limit." (227) This generates the familiar modulation back and forth between extremes.

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140 The presence of spurious infinity in number will become vividly apparent in the Maclaurin series, where a fixed number is expanded into an uncompletable infinite series. *See infra* text accompanying note 213.

The two extremes are unified in the expression "infinitely great or infinitely small." (227) The infinitely Great/Small are precisely what the extremes are *not*. Any *fixed* notion (or "absolute determinateness") of the infinitely Great/Small is not attained. Each extreme is thus posited as self-external. There is always a "more" or "less." This beyond to any given expression of the infinitely small or great is a moment of qualitative opposition in every Quantum. This means that a decrease of the infinitely small or an increase of the infinitely large brings us *no closer* to infinity. Infinity is thus a liar. The infinitely great "is supposed to be *great*, that is, a quantum, and *infinite*, that is, not a quantum." (228) Infinity, however, is Quantum only.

Accordingly, Quantitative Infinity is spurious:

> Like the qualitative spurious infinite, it is the perpetual movement to and fro from one term of the lasting contradiction to the other, from the limit to its non-being, and from this back again to the limit. (228)

There is nevertheless a progressive aspect of Quantitative Infinity, compared to the qualitative infinite. In the Spurious Infinite, the movement was towards "an abstract *other* in general." (228) Now it is towards "an explicitly different quantum." (228) But a qualitative moment prevents Quentitative Infinity from reaching completion.[141] Hence, the Quantitative Infinite Progress is

> not a real advance but a repetition of one and the same thing, a positing, a sublating, and then again a positing and again a sublating, an impotence of the negative, for what it sublates is continuous with it, and in the very act of being sublated returns to it. (228)

What is the bond between the two extremes of Figure 16(b)? Simply that each flees from the other, "and in fleeing from each other they cannot become separated but are joined together even in their flight." (228)

Remark 1: The High Repute of the Progress to Infinity

Hegel is no admirer of Quantitative Infinity. No doubt it is held to be sublime, and "in philosophy it has been regarded as ultimate." (228) With Kant obviously in mind, Hegel remarks:

> [T]his *modern* sublimity does not magnify the *object*—rather does this take flight—but only the *subject* which assimilates such vast quantities. (229)

In *The Critique of Judgment*, Kant defined sublimity as a subjective feeling that one could actually know the thing-in-itself (which is

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141 Lacanians will recognize this qualitative moment as structurally similar to trauma—a stumbling block, or piece of the Real, which prevents the patient from completing his fantasy. *See* BRUCE FINK, THE LACANIAN SUBJECT: BETWEEN LANGUAGE AND *JOUISSANCE* 26 (1995).

impossible).[142] Hence, the sublime definitely does exalt the subject (and not the object) in Kant's work.

What makes thought succumb to the awe of the Quantitative Infinite Progress, Hegel remarks,

> is nothing else but the wearisome repetition which makes a limit vanish, reappear, and then vanish again . . . giving only the feeling of the *impotence* of this infinite or this ought-to-be, which *would* be master of the finite and *cannot.* (229)[143]

Kant compares the sublime to the withdrawal of the individual into his ego, where the individual opposes his absolute freedom to all the terrors of tyranny and fate. At this moment, Kant says, the individual knows himself to be equal to himself.[144]

Of this withdrawn ego, Hegel agrees that it is "the reached beyond; it has *come to itself*, is *with* itself, here and now." (230) This highly negative thing—the ego—has "determinate reality . . . confronting it as a beyond." (231) In this withdrawal of the ego,

> [w]e are faced with that same contradiction which lies at the base of the infinite progress, namely a returnedness-into-self which is at the same time immediately an out-of-selfness, a relation to its other as to its non-being . . . (231)

How is this so? It will be recalled that Quantitative Infinity stayed within itself, but this "in-itself" had no content. All the content was in the beyond. Simultaneous with its being-for-self, Quantitative Infinity was pure flight into the beyond and hence a constant modulation between these moments of flight and return. Now Hegel says that the ego is the same thing. Here we have the Lacanian view of the subject as suspended between the realm of the Symbolic (*i.e.*, "being") and the Real (*i.e.*, nothing).[145]

That the subject finds part of its selfhood in its beyond is the structure of desire itself. The subject seeks wholeness but cannot

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142 IMMANUEL KANT, CRITIQUE OF JUDGMENT 100-01 (J.H. Bernhard trans., 1951) [hereinafter CRITIQUE OF JUDGMENT]; *see also* IMMANUEL KANT, CRITIQUE OF PRACTICAL REASON 142 (T.K. Abbott trans., 1996) [hereinafter CRITIQUE OF PRACTICAL REASON].

143 Carl Sagan, the telegenic Cornell astronomer, produced a popular TV program on astronomy in which he frequently advertised his astonishment at the "billions and billions of stars" in the universe. Hegel reserves special scorn on such astronomers.

> The shallow astonishment to which they surrender themselves, the absurd hopes of wandering in another life from one star to another . . . this they declare to be a cardinal factor in the excellence of their science . . . . (230)

See also LESSER LOGIC, *supra* note 29, § 94 Remark ("the infinity of [space] has formed the theme of barren declamation to astronomers with a talent for edification").

144 This is probably a reference to CRITIQUE OF PRACTICAL REASON, *supra* note 142, at 191.

145 FINK, *supra* note 109, at 59 ("[t]he subject is nothing but this very split"). Kant, in turn, describes the "I"—the pure universal aspect of personality, which Lacanians insist is *not* the subject. Mladen Dolar, *The Cogito as the Subject of the Unconscious*, in SIC 2: COGITO AND THE UNCONSCIOUS 11, 12 (Slavoj Žižek ed., 1998).

achieve it. This is what Lacanians called symbolic castration.[146] Hegel, however, sees this precisely some 150 years before Lacan. Thus, Hegel writes that the relation of the subject to its non-being (*i.e.*, the Symbolic realm, where the subject is accorded the privilege of "being"),

> remains a *longing*, because on the one side is the unsubstantial, untenable void of the ego fixed as such by the ego itself, and on the other, the fulness which though negated remains present, but is fixed by the ego as its beyond. (231)

Hegel specially complains that morality has been equated with Quantitative Infinity, and once again the target is Kant. The antithesis just described—ego *v.* reality—was a qualitative opposition. In this opposition, the ego determines nature by distinguishing itself. That is, ego announces, "I am not *that*." *That* ends up being nature in general—that which opposes the ego. In this opposition, the ego is singular. External reality, however, is "manifold and quantitative." (231) But the *relation* between qualitative ego and quantitative nature is *itself* quantitative. This relation is morality itself—in Kantian terms the power of the universal "I" over nature (over what Kant would tend to call "inclination" or "pathology"). Thus:

> the power of the ego over the non-ego, over sense and outer nature, is consequently so conceived that morality can and ought continually to increase, and the power of sense continually to diminish. But the perfect adequacy of the will to the moral law is placed in the unending progress to infinity, that is, is represented as an *absolutely unattainable* beyond, and this unattainableness is supposed to be the true sheet-anchor and fitting consolation; for morality is supposed to be a struggle, but such it can be only if the will is inadequate to the moral law which thus becomes a sheer beyond for it. (231)

Here is a concise critique of Kant's doctrine of "radical evil."[147] According to Kant, the ego is forever tainted with pathology. It can never finally purge itself of pathology but can only struggle for moral

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[146] Slavoj Žižek describes symbolic castration as follows:

> by means of the Word, the subject finally *finds* himself, comes to himself: he is no longer a mere obscure longing for himself since, in the Word, he directly attains himself, posits himself as such. The price, however, is the irretrievable *loss* of the subject's self-identity: the verbal sign that stands for the subject—in which the subject posits himself as self-identical—bears the mark of an irreducible dissonance; it never fits the subject. This paradoxical necessity on account of which the act of returning-to-oneself, of finding oneself, immediately, in its very actualization, assumes the form of its opposite, of the radical loss of one's self-identity, displays the structure of what Lacan calls "symbolic castration." This castration involved in the passage to the Word can also be formulated as the redoubling, the splitting, of an element into itself and its place in the structure.

SLAVOJ ŽIŽEK, THE INDIVISIBLE REMAINDER: AN ESSAY ON SCHELLING AND RELATED MATTERS 46-47 (1996).

[147] For a description of Kant's theory of radical evil, see Jeanne L. Schroeder & David Gray Carlson, *Kenneth Starr: Diabolically Evil?*, 88 CAL. L. REV. 653 (2000).

purity. Kant even goes so far as to deduce the immortality of the soul from the very fact that all eternity is required for the soul to reach the state of perfection.[148] Hence, Kant is quite guilty as charged. He has reduced morality to Quantitative Infinity.

With regard to Kant's opposition of ego-pure-will-moral-law and nature-sensuousness-inclination, Hegel complains that they are put forth as "completely self-subsistent and mutually indifferent." (231) "At the same time, however, both are moments of *one and the same being*, the ego." (232) Hence, the very constitution of the Kantian subject is the Lacanian split.[149] This contradiction is never resolved in the infinite progress. "[O]n the contrary, it is represented and affirmed as unresolved and unresolvable." (232)

This Kantian standpoint is "powerless to overcome the qualitative opposition between the finite and infinite and to grasp the idea of the true will which is substantial freedom." (232) Instead, this standpoint uses *quantity* to mediate. Quantity (sublated quality) is "the difference which has become indifferent." (232) Hence, the qualitative moments of pure ego and nature are quite indifferent to the alteration of their quanta. In other words, the subject counts it as nothing that it has progressed toward the perfection of pure morality.[150]

"That all opposition is only quantitative was for some time a cardinal thesis of recent philosophy," Hegel complains. (233) Oppositions were in effect reduced to polarities. In these polarities:

> the opposed determinations have the same nature, the same content; they are real sides of the opposition in so far as each of them has within it both determinations, both factors of the opposition, only that on one side one of the factors *preponderates*, on the other side the other . . . is present in as *greater quantity* or in an *intenser degree* . . . But in so far as substances or activities are presupposed, the quantitative difference rather confirms and completes their externality and indifference to each other and to their unity. (233)

In other words, at the base of any claimed polarity is a self-identical qualitative moment that Hegel finds to be an unjustified presupposition. Polarity is only the "first negation" (Dialectical Reason), not the "negation of the negation"—(Speculative Reason). (233) In fixed polar oppositions, being and thought "become completely external to each other and unrelated." (233) In short, fixed polarity is a species of

148 CRITIQUE OF PRACTICAL REASON, *supra* note 142, at 148, 155.

149 Incidentally, Lacanians give Kant the greatest credit for this. Schroeder & Carlson, *supra* note 147, at 671-80.

150 Fichte, Hegel's predecessor as professor of philosophy at the University of Berlin, is also singled out for relying on Quantitative Infinity in his theory of personality. Fichte saw the subject as a unity between self-identity and self-difference. MURE, *supra* note 17, at 30-32. The difference between self-identity and self-difference is likewise said to rest on Quantitative Infinity, in which the beyond remains forever beyond.

atomism, much criticized in chapter 3 of Quality.[151] In polarity, "[i]t is a third, an external reflection, which abstracts from their difference and recognizes their unity, but a unity which is *inner*, *implicit* only, not *for itself*." (233) What is needed is an immanent sublation of the extremes by Speculative Reason.[152]

Remark 2: The Kantian Antinomy of the Limitation and Non-limitation of the World in Time and Space

We have seen that Hegel has small regard for the four antinomies of reason that Kant presents in the *Critique of Pure Reason*.[153] Now he repeats his conclusion "that the Kantian antinomies are expositions of the opposition of finite and infinite in a *more concrete* shape, applied to more specific substrata of conception." (234) That is, the antinomies are spurious qualitative infinities. Each side of a given antinomy is merely a one-sided view of the truth. By "specific substrata of conception" Hegel means that Kant has taken his four categories of understanding and developed four antinomies with regard to them in order to produce the illusion that the antinomies are complete.

The antinomy Hegel now discusses is Kant's first one—whether the world is limited in time and space.[154] This antinomy is the one Kant associated with the category of quantity[155] (which is why Hegel discusses it here). According to Kant's thesis: (1) The world has a beginning in time and is limited in space. According to the antithesis: (2) The world has no beginning in time and no limit in space.[156]

In terms of *time*, Kant proves the thesis (the world has a beginning) by showing that the antithesis is impossible. If time has no beginning, then at any given point of time, an "eternity"—an infinite series of temporal measures—has lapsed. But an infinite series already lapsed is impossible. Therefore, time must have a beginning. In terms of *space*, Kant proves the thesis by showing that, if the universe was unlimited in space, then it would consist of infinite co-existing things. We cannot think of an infinite quantity of things.[157] Rather, there must be a finite

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151 *See* Carlson, *supra* note 4, at 564-66.

152 This will occur in chapter 11, when Hegel explains why oppositions must cancel each other out and fall to the ground.

153 *See* Carlson, *supra* note 4, at 465-66, 478, 554-56; *see also* text accompanying notes 59-74.

154 Of this antimony, Henry Allsion remarks, "These are the most widely criticized of Kant's arguments . . . ." HENRY E. ALLISON, KANT'S TRANSCENDENTAL IDEALISM: AN INTERPRETATION AND DEFENSE 36 (1983). Allison offers a tepid defense and suggests that Kant's point is, in effect, a warning against the inductive fallacy. *Id.* at 42-43.

155 *See supra* note 62. Kant called time and space "the two primitive quanta of all intuition." CRITIQUE OF PURE REASON, *supra* note 14, at 233.

156 *Id.* at 241-45.

157 If we think of the "whole thing," we are in effect assuming space is limited and does not surpass this "whole."

number of things.

The antithesis is also proved by ruling out the opposite. In terms of *time*, suppose the world has a beginning. Before the beginning, the world does not exist. An existing thing, however, cannot originate from nothing. Nothing comes from nothing, as King Lear and Spinoza discovered. In terms of *space*, suppose the world is finite. Space, however, has no limit. Hence, there must be a "void space." We thus have a relation of the finite world to void space. But this is a relation of things to *no object*. A thing cannot have a relation to nothing. Consequently the world is not limited in space—nothing is not a thing.

Hegel's first proposition about this antinomy is that the "world" could have been left out of the discussion. Kant could have addressed time as such and space as such.[158]

Hegel's second proposition is that Kant could have restated his antinomy as follows: (1) there is a limit, and (2) limit must be transcended—two things Hegel says are true of Quantity generally.

The Thesis. Hegel next proposes that Kant's entire proof of the thesis was unnecessary. The proof is itself only the direct assertion of what was to be proved. With regard to the thesis about time (it has a beginning), the very assertion that time has points introduces the idea that time is already limited. "In the proof therefore, a limit of time is *presupposed* as actual; but *that* is just what *was to be proved*." (235)

One point in time is, of course, "now." It designates an end of the past and is also the beginning of the future. With regard to limiting the past, "now" represents a qualitative limit. But why, Hegel implicitly asks, should "now" be a *qualitative* limit? Suppose, however, we say that "now" is a *quantitative* limit. Time would then continue on from the past, over the "now," and into the future, because Quantitative Infinity always leaps o'er the vaunts and firstlings of any limit. Quantitative Infinity "not only must be transcended but *is* only as the transcending of itself." (235) If time is a Quantitative Infinity, "then the infinite time series would not have *passed away* in it, but would continue to flow on." (235) A switch from qualitative to quantitative limit would therefore destroy Kant's argument.

But, Hegel continues, let us concede the qualitative nature of "now" as a limit to the past. In such a case it is also the beginning of the future. But this is precisely the thesis to be proved—that time has a beginning. What if this beginning was preceded by a deceased past? This does not affect the argument. The past is conceived as radically separate from the future. Hence, the very introduction of "now"—a point in time—presupposes that time has a beginning.

Suppose we say the past is related to the future through the "now."

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158 Professor Allison disagrees and thinks that synthesis of a world out of infinite moments is central to Kant's argument. ALLISON, *supra* note 154, at 39.

In this case, "now" is a mere quantitative limit. "[T]he infinite time series would continue itself in what was called future and would not be, as was assumed, *completed*." (236)

Hegel now repeats his own theory of time. It is Pure Quantity. A point in time which supposedly interrupts time "is really only the *self-sublating* being-for-self of the now." (236) This harkens back to Hegel's description of time as "an absolute coming-out-of-itself." (189) Time constantly generates the "now" but then simultaneously annihilates it.[159]

The antithesis. Hegel sees Kant's argument for the thesis as merely asserting that the "now" is a qualitative limit to time—the very thesis to be proved. The antithesis fares no better, in Hegel's opinion. It likewise merely asserts what must be proved.

In order to prove the antithesis (time has no beginning), Kant considers and dismisses the *thesis*. Assume a null, empty time (prior to the beginning). Kant insists upon the continuance of the world into this empty time, "with the result that the existence of the world is continued into infinity." (236) As this continuance (into void time) is impossible—*i.e.*, nothing can come from nothing—Kant rejects the thesis and proves the antithesis. Because time can continue back forever, there must be no beginning.

According to Hegel, this argument presupposes that, just because the world exists, it must have "an *antecedent condition* which is in time." (236) But this is the very antithesis to be proved. Furthermore, when Kant insists that nothing can come from nothing—when "the *condition* is sought in empty time"—this means that the world is taken as temporal and hence limited. (236) Something always precedes the "now" of the world. There is always a yesterday. All of this, Hegel charges, is presupposed. It is the antithesis itself.

Kant's demonstration of the antithesis in terms of space is likewise rejected. For purposes of the antithesis, Kant assumed that space was *no* object and was unlimited. If the world were finite (and space infinite), space would exceed it. The world (an object) would have a relation with the void space beyond the world. But how could an object have a relation with *no* object?

Hegel finds again that Kant has merely restated the proposition—not proved it. Kant assumes that space is not an object, and that, in order to prevent the impossible relation of object to non-object, the object must continue itself as far as space does. This means that Kant thinks space must never be empty—the world must continue into it. Yet this is precisely the antithesis restated.

Hegel concludes this remark by criticizing Kant for "subjectivizing" contradiction.[160] That is, the four antinomies do not

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159 *See supra* text accompanying notes 54-55.

160 *See* Daniel O. Dahlstrom, *Hegel's Appropriation of Kant's Account of Teleology in Nature*,

occur in nature. Rather, they occur in consciousness. (Time and space, Kant says, are the very conditions of possibility for subjective intuitions).[161] Of this subjectivization of the first antinomy, Hegel writes:

> It shows an excessive tenderness for the world to remove contradiction from it and then to transfer the contradiction to spirit, to reason, where it is allowed to remain unresolved. In point of fact it is spirit which is so strong that it can endure contradiction, but it is spirit, too, that knows how to resolve it. (237-38)

The "so-called world" *is* contradictory, Hegel insists. (238) The world "is unable to endure it and is, therefore, subject to coming-to-be and ceasing-to-be." (238)

(c) The Infinity of Quantum

The middle term between Quantitative Infinity and the Quantitative Infinite Progress is the Infinitely Great and/or Infinitely Small. The Infinitely Small, at least, is what mathematicians would call the differential—δx in the derivative δy/δx.

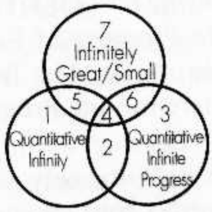

Figure 16 (c)
Infinitely Great and Infinitely Small

The Infinitely Great/Small is the destination that the Quantitative Infinite Progress implies. It is a Quantum, but

> at the same time it is the non-being of quantum. The infinitely great and infinitely small are therefore pictorial conceptions which, when looked at more closely, turn out to be nebulous shadowy nullities. (238)

This should be clear to even the non-speculative readers who have survived this far into this text. In the Quantitative Infinite Progress, the counting mathematician is aiming to reach infinity. That infinity has "being" is thus presupposed by the counter who is aiming to reach this end. Yet this end will never be reached. It is a non-being.

This contradiction—the non-being of infinity—is now explicitly present, and so is the very nature of Quantum. When Quantum reached

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167, 176, *in* HEGEL AND THE PHILOSOPHY OF NATURE (Stephen Houlgate ed., 1998).

161 CRITIQUE OF PURE REASON, *supra* note 14, at 23-24, 32-33, 85, 279.

Intensive Magnitude (Degree), Quantum "attained its reality." (238) But now the very notion of Quantum manifests itself.

As Degree, Quantum was "unitary, self-related and determinate within itself." (238) As unitary, Degree sublated (*i.e.*, negated) its otherness and its determinateness. These were now external to Degree. This self-externality was the "*abstract non-being* of quantum generally, the spurious infinity." (238) In other words, Degree in Figure 15(a) yielded the Qualitative Something which in turn yielded Quantitative Infinity in Figure 16(a). If we now examine Figure 16(b), we witness each of the extremes—Quantitative Infinity and the Quantitative Infinite Progress—erasing itself and establishing its non-being in the other, while expressly continuing itself in the other, so that each was a Quantum as well as not a Quantum. Hence, "this non-being of quantum, infinity, is thus limited, that is, this beyond is sublated, is itself determined as quantum which, therefore, in its negation is with itself." (238)

The in-itself of Quantum is therefore to be external to itself.[162] Its externality determines what Quantum is. The Infinitely Great/Small thus illustrates the very notion of Quantum. It is "not there" and yet treated as if it *is* there. Hegel writes, "In the infinite progress, therefore, the *Notion* of quantum is *posited*." (238) This must be taken to mean that the Quantitative Infinite Progress of Figure 16(b) shows what its content is—to be external to itself. The Infinitely Great/Small is the very beyond of the Quantitative Infinite Progress.

Figure 16(c) claims:

> In the infinite progress as such, the only reflection usually made is that every quantum, however, great or small, must be capable of vanishing, of being surpassed; but not that this self-sublating of quantum, the beyond, the spurious infinite itself also vanishes. (239)

How is this claim justified? Why has the spurious infinite vanished? Consider what the Infinitely Great/Small is: the end that the Quantitative Infinite Progress could never reach. If we have that end before us, then we do *not* have the Quantitative Infinite Progress before us. In short, we can take Figure 16(c) in terms of [7]—which is isolated from the vanished Quantitative Infinite Progress. This isolation is a sign that Quantity is beginning to recapture its Quality.

Quantum summarized. Hegel next reflects upon Quantum generally. Quantum (via Quantity) is the negation/sublation of Quality. Considered immediately by the Understanding, as in, say, Figure 11(a) or Figure 13(a), it is already the first negation—in *positivized form*. But Quantum is only the first negation *in principle*. It is posited as a "being," and "its negation is fixed as the infinite, as the beyond of

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162 *Cf.* the Ought in Figure 5(c), where the in-itself of the Finite was that it must cease to be and become other. Carlson, *supra* note 4, at 526-29.

quantum, which remains on this side as an *immediate*." (239) In this guise, it is the "beyond" that is *overtly* the first negation, as shown in Figure 16(b). Now, in the Infinitely Great/Small, we have "quantum determined in conformity with its Notion, which is different from quantum determined in its immediacy." (239) The Infinitely Great/Small is externality itself, brought inward as a moment of Quantity. For this reason, Hegel can say that

> externality is now the opposite of itself, posited as a moment of quantity itself—quantum is posited as having its determinateness in another quantum by means of its non-being, of infinity. (239)

Because Quantum has brought its externality inward, "it is *qualitatively* [what] it is." (239) But, Hegel warns, to the extent we compare Quantum's recaptured quality to its Notion, this characterization is "for us." It "belongs more to our reflection, to a relationship which is not yet present here." (239) (Notion as such is strictly the province of the Subjective Logic, which Hegel alternatively names the Doctrine of the Notion).

For *itself*, however, "quantum has reverted to *quality* [and] is from now on qualitatively determined." (239) Its quality (or, to use Hegel's term, its "peculiarity") is that its determinateness (or content) is external. Quantum in Figure 16(c) is "indifferent" to its determinateness. But the outside is now in. Thus, "Quantum has infinity, self-determinedness, no longer outside it but within itself." (239)

In Figure 16(c), Quantum is "posited as repelled from itself, with the result that there are two quanta which, however, are sublated, are only as moments of *one unity*." (240) In the final chapter of Quantity, Quantum will now appear as a double—as Quantitative Ratio. In Quantitative Ratio, the content of Quantum will be external to itself (yet within itself). This externality will itself be a relation of quanta, "each of which is as such a unity." (240) This unity is not a mere "comparison" by an external reflection. Rather, this unity is Quantity's own qualitative determination. In Quantitative Ratio, Quality comes back into partnership with Quantity. The middle term between this partnership is Measure.[163]

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[163] Charles Taylor expresses his dissatisfaction with Hegel's entire discussion of Quantum, and we are now in a position to answer his queries. Taylor writes:

> But one might think that Hegel is a little cavalier in his transitions here. Granted that Quantity is the realm in which things are indifferent to their limit, how does that show that quanta must go beyond themselves, and change? (whatever that means). And even if they do so endlessly, even granted Hegel's dislike for the "bad" infinity of the endless progress, does this show a contradiction requiring resolution by a higher category?

TAYLOR, *supra* note 10, at 248. The answer to the first question is, since quanta are True Infinites, their very function is to go beyond their limit (while staying what they are). This very act is the Quality of the Quantum. But this does not necessarily mean that numbers change and

Before we can move on to Quantity's final chapter, however, we must suffer through three long Remarks, the first two of which are by far the longest Remarks Hegel will include in the *Science of Logic*. Both cover the subject of calculus, which endlessly fascinated Hegel, because the "differential"—the Infinitely Small—embodied his slogan that nothing is something.

Undoubtedly, Hegel will criticize nineteenth century calculus for its lingering dependence on geometrical ideas, and for the quantification of δx, which Hegel views as an undefined quality.[164] Future generations of mathematicians would tend to agree with this estimate.[165]

The calculus remarks are usually dismissed as "digression," at best.[166] Undoubtedly, this is a fair observation. I have found few

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that arithmetic is promiscuous and subjective. Quanta have limits within themselves. Three does not melt into two. If the limit external to a quantum is exceeded, it is exceeded spiritually, not empirically. The answer to the second question is that the bad infinity's modulation between quanta is itself the higher category. As always, Speculative Reason names the autistic modulation of Dialectical Reason and underwrites progress to a higher level.

Taylor's own response to his inquiries is to interpret the entire chapter on Quantum as an attack on atomism. Taking atomism to its extreme, Taylor sees it as the assertion that all things are mere aggregates of indistinguishable units. But if so, then how do atomists determine that one aggregate has 50 units but another has 100 units? Some non-quantitative criteria must operate, Taylor opines, and therefore the atomists are defeated. Taylor writes, "what drives the quantum on to its endless alterations is the search for an adequate specification in purely quantitative terms, a search whose object always eludes it, and which for this reason is endless." *Id.* at 250. While these are good arguments against atomism, it is hard to draw this moral from Hegel's discussion of Quantum. What seems to drive the progress on is the act of the True Infinite to erase itself while remaining within itself. It is the very erasure of quantitativity that produces the Infinitely Great/Small—the qualitative beyond of Quantum.

164 The differential δx stands for change in the variable x. As such, it is indefinable, because it is supposed to be infinitely close to (but distinguishable from) zero. Yet δx given δy (or δy/δx) is fully determinate. This point is important in understanding why the Infinitely Great/Small is a ratio. It has its being in δy, and *vice versa*. The two become visible only when brought in conjunction.

165 In this regard, one recalls Hegel's early remark that mathematical necessity is inadequate. Mathematicians do nothing, he says, but ward off heterogeneous elements—an effort that is itself "tainted" with heterogeneity. (40) Perhaps the heterogeneous element warded off by mathematicans is the qualitative nature of δx.

An astute commentator views the point of the calculus discussion as follows: Calculus cannot "yield the 'mathematics of nature which Hegel was looking for. [S]uch a mathematics can only take over what is qualitative from experience, it cannot develop it out of itself." Borzeszkowski, *supra* note 53, at 76.

166 MURE, *supra* note 17, at 118. Those inclined to accept scorn as substitute for thought will interest themselves in the sentiments of Bertrand Russell:

> Hegel (especially in his *Greater Logic*) made a quite different use of mathematics . . . Hegel fastened upon the obscurities in the foundations of mathematics, turned them into dialectical contradictions, and resolved them by nonsensical syntheses. It is interesting that some of his worst absurdities in this field were repeated by Engels in the *Antidüring*, and that, in consequence, if you live in the Soviet Union and take account of what has been done on the principles of mathematics during the last one hundred years, you run a grave risk of being liquidated.

Bertrand Russell, *Logical Positivism*, *in* LOGIC AND KNOWLEDGE: ESSAYS 1902-1950, at 368-69 (Robert Charles Marsh ed., 1950).

references to Hegel's views on the calculus, which nevertheless seem prescient for his day. Readers are invited at this point to skip to Part III of this Article, as virtually all Hegelians have done for generations.[167] Nothing great will be lost if this is done. However, for the intrepid reader who wishes a "scorched earth" understanding of Hegel's *Science of Logic*, I summarize and simplify as best I can the thrust of Hegel's lengthy critique of the calculus.

Remark 1: The Specific Nature of the Notion of the Mathematical Infinite

Mathematics exploits the mathematical infinite for pragmatic reasons. Calculus works. But Hegel denounces the utilitarian attitude of mathematicians as unscientific. At least as of his time, "mathematics has not yet succeeded in justifying its use of this infinite by the Notion." (240)[168] Unless the matter is corrected, mathematics will be "unable to determine the scope of its application and to secure itself against the misuse of it." (241)[169]

Often mathematicians defend themselves by denying the competence of metaphysics to comment on mathematical notions. They assert that, so long as mathematical concepts operate consistently in

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[167] *See, e.g.*, BUTLER, *supra* note 20, at 110-11 ("suspecting Hegel of wishing in part to demonstrate his mastery of mathematics and science to contemporaries and colleagues ...").

[168] Or, as Hegel will put it later, "mathematics to this day has never succeeded in justifying by its own means, that is, mathematically, the operations that rest on that transition, because the transition is not of a mathematical nature." (793)

[169] Hegel memorably condemns the illusion that mathematics owns the fee simple of academic rigor, in a passage that the law-and-economic movement in American law schools should take to heart.

> If quantity is not reached through the action of thought, but taken uncritically from our generalized image of it, we are liable to exaggerate the range of its validity, or even to raise it to the height of an absolute category. And that such a danger is real, we see when the title of exact science is restricted to those sciences the objects of which can be submitted to mathematical calculation. Here we have another trace of the bad metaphysics . . . which replace the concrete idea by partial and inadequate categories of understanding. Our knowledge would be in a very awkward predicament if such objects as freedom, law, morality, or even God himself, because they cannot be measured or calculated, or expressed in a mathematical formula, were to be reckoned beyond the reach of exact knowledge . . . And this mere mathematical view, which identifies with the Idea one of its special stages . . . is no other than the principle of Materialism . . . Matter . . . is just what . . . has that form only as an indifferent and external attribute.

LESSER LOGIC, *supra* note 29, § 99 Remark. Also, from the *Phenomenology*, the following complaint about dogmatism:

> The evident character of this defective condition of which mathematics is proud, and on which it plumes itself before philosophy, rests solely on the poverty of its purpose and the defectiveness of its stuff, and is therefore of a kind that philosophy must spurn. Its *purpose* or Notion is *magnitude*. It is just this relationship that is unessential, lacking the Notion.

PHENOMENOLOGY, *supra* note 10, ¶ 45.

their own sphere, they need not concern themselves with metaphysics. Hegel paraphrases the attitude of the mathematicians: "Metaphysics, though disagreeing with the use of the mathematical infinite, cannot deny or invalidate the brilliant results obtained from it." (241)[170]

If the difficulty were solely with the Notion as such, mathematics could dispense with it, because math's project is not to generate the very being of the objects which interest it. The problem is that the calculus is contradictory. The Notion of a concept is much more than a precise determination of it. But the calculus poses a special challenge to precise definition.

> [T]he infinitesimal calculus permits and requires modes of procedure which mathematics must wholly reject when operating with finite quantities, and at the same time it treats these infinite quantities as if they were finite and insists on applying to [the Infinitely Great/Small] the same modes of operation which are valid for [finite quanta]. (241-42)

In other words, mathematics does not condone dividing by zero, but it condones dividing by non-numbers that are infinitely close to zero ($\lim_{x\to 0} \delta y/\delta x$). And, once it condones this, such differentials can be multiplied or subtracted as if they really were finite numbers.

Hegel states that the track record of the Infinitely Small is mixed.[171] Hegel finds the calculus is

> burdened with a seeming inexactitude, namely, having increased finite magnitudes by an infinitely small quantity, this quantity is in the subsequent operation in part retained and in part ignored. The peculiarity of this procedure is that in spite of the admitted inexactitude, a result is obtained which is not merely fairly close and such that the difference can be ignored, but is perfectly exact. (242)

If I may intercede with an example that illustrates this last passage, suppose Δx represents a given change in x—not necessarily an infinitesimally small change. Suppose further that y is a function of x (or $y = f(x)$). If x_0 represents x at a particular value, and if $x_1 = f(x_0 + \Delta x)$, then $\Delta y = f(x_1) - f(x_0) = f(x_0 + \Delta x) - f(x_0)$. We can define the "difference quotient" by dividing each side of the equation by Δx to obtain

$$\frac{\Delta y}{\Delta x} = \frac{(f(x_0) + \Delta x) - f(x_0)}{\Delta x}$$

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[170] Michael John Petry reads this passage as meaning that "in this context it is metaphysics which has a lesson or two to learn from mathematics, not vice versa." Petry, *supra* note 11, at 486. But the thrust of Hegel's remarks is that it is the mathematicians who have not done their homework.

[171] Terry Pinkard rightfully criticizes Bertrand Russell who saw Hegel as wedded to the Infinitely Small as the basis of calculus. PINKARD, *supra* note 41, at 41. The entire point here is to *attack* any reliance by the calculus on such a notion.

This difference quotient illustrates Hegel's assertion that the calculus exists by inexactitude. Let us apply this "difference quotient" to a concrete example. Suppose $y = 3x^2 - 4$. We can write:

$$\frac{\Delta y}{\Delta x} = \frac{3(x_0 + \Delta x)^2}{\Delta x} = \frac{6\, x_0 \Delta x + 3(\Delta x)^2}{\Delta x} = 6\, x_0 + 3\Delta x$$

Thus, if $x = 2$, and if $\Delta x = 3$, then, as x changes from 2 to 5, y changes from 8 to 71 (or 21 units of y per unit of x).

Where Δx is infinitesimally small, the calculus feels licensed simply to ignore Δx in the above calculation. Thus, the derivative of $3x^2 - 4$ is supposed to be $6x$, not $6x + 3\delta x$.[172] The remainder $3\delta x$ is simply dropped. This erasure shows that the calculus, as Hegel charges, is burdened with inexactitude.

Hegel remarks:

> In the operation itself, however, which precedes the result, one cannot dispense with the conception that a quantity is not equal to nothing, yet is so inconsiderable that it can be left out of the account. (242)

In other words, in the expression $\Delta y/\Delta x$, one can see that Δx is not zero, because one cannot divide by zero. Yet simultaneously Δx is "left out" *as if* it were zero.

In modern times, mathematicians would deny that $3\delta x$ is simply erased. Rather, they would say that $6x$ is the *limit* past which they may not go, when

$$\lim_{x \to 0} \frac{\delta x}{\delta y}$$

On this view, $3\delta x$ does not vanish. It is simply unnecessary to refer to it when identifying $6x$ as a limit. In fact, the limit is never reached, because δx never does reach zero.[173] This is a procedure of which Hegel would have approved. The "limit" of $6x$ is the qualitative "beyond" of Quantum.[174]

Infinitesimals. Hegel next considers the nature of the Infinitely Great/Small according to the mathematical point of view:

> The usual definition of the mathematical infinite is that it is a magnitude than which there is no greater (when it is defined as the

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172 According to the familiar "power rule," $\delta x^n/\delta x = nx^{n-1}$.

173 *See* Borzeszkowski, *supra* note 53, at 76; *see* A.W. Moore, *The Method of Exhaustion as a Model for the Calculus*, *in* HEGEL AND NEWTONIANISM, *supra* note 11, at 147-48.

174 *See* Kosok, *supra* note 12, at 254 ("the very notion of a mathematical limit entails the *negative* presence of *that* which is limited . . . .").

> infinitely large), or no smaller (when it is defined as the infinitely small). (243)

Meanwhile, mathematics defines "magnitude" as that which can be increased or diminished. Since the Infinitely Great/Small cannot be increased/diminished, then the Infinitely Great/Small is no longer a Quantum as such. This is so on mathematical terms.

What mathematics cannot comprehend is that the mathematical infinite is simultaneously Quantum *and* not Quantum. It is "something which is not a quantum but yet retains its quantitative character." (243)[175]

What mathematics cannot comprehend is that the mathematical infinite is simultaneously Quantum *and* not Quantum. It is "something which is not a quantum but yet retains its quantitative character." (243)

An Attack on Kant. Hegel returns to his criticism of Kant. Kant says of the mathematical infinite that it is a magnitude beyond which none is greater. We can never name this amount. Some other magnitude could always be named that is greater, defeating our pretensions. But to say "infinite," we do not invoke the concept of a maximum. Rather we express only a beyond of any given, named magnitude. The infinite is therefore always a *relation* to a fixed number—a beyond of it.[176]

Thus, Kant declines to regard the infinite whole as a maximum. The maximum would be a mere quantum, which can always be exceeded. Rather, Kant sees that the mathematical infinite is *beyond* Quantum. Hegel complains that Kant thinks the mathematical infinite can never be completed. This, he finds, is "nothing but an expression of the progress to infinity." (243) It is represented as transcendental, by which Kant means (says Hegel) psychologically subjective. That is, subjective opinion burdens any given magnitude with an unreachable beyond—a species of the thing-in-itself.

> Here, therefore, there is no advance beyond the contradiction contained in quantity; but the contradiction is distributed between the object and the subject, limitedness being ascribed to the [object], and to the [subject] the progress to infinity, in its spurious sense, beyond every assigned determinateness. (243)

That is, in Kant's critique, the proposed maximum is objective. The burden of the beyond is subjective. By subjective, Hegel seems to invoke the presupposition that we can never know the infinite; it is always beyond our experience. Hegel, however, believes the nature of

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[175] Carl Boyer has suggested that the infinitesimally small was "atomic" in nature—a self-identity that could not be further subdivided. CARL B. BOYER, THE HISTORY OF THE CALCULUS AND ITS CONCEPTUAL DEVELOPMENT 12 (1949) [hereinafter BOYER, CALCULUS]. If so, Hegel—a virulent opponent of atomism—would also oppose any use of the infinitesimal in the calculus.

[176] CRITIQUE OF PURE REASON, *supra* note 14, at 243-44.

the Infinitely Great/Small can be known precisely.

Mathematical v. Speculative Notions. Returning to the mathematical notion of infinity, Hegel says that, for mathematicians, the mathematical infinite is not a Quantum but a beyond of Quantum—a conclusion Hegel endorses (but only as a one-sided view, since the Infinitely Great/Small is just as much Quantum). Hegel compares this attitude to the "speculative" point of view.

According to Hegel, the Infinitely Great/Small is "*in its own self* infinite." (244) That is, the Infinitely Great/Small of Figure 16(c) has sublated/negated both the quantum *and* its beyond. In the Infinitely Great/Small, the entire spurious infinity has vanished. The Infinitely Great/Small, viewed as [7] in Figure 16(c), is a simple unity.

Extensive Quantum in Figure 14(a) was also a simple unity, but the Infinitely Great/Small is an advance over this more primitive unity. Extensive Quantum erased itself and became Intensive Quantum. In this act, it determined itself only implicitly. Extensive Quantum saw itself as entirely separate and isolated from Intensive Quantum—the usual delusion of the Understanding. The Infinitely Great/Small, however, *expressly* sees itself as the beyond of the Quantitative Infinite Progress. It expressly says, "I am not that, *and that is what I am*." It sees that it is simultaneously a unity of opposites. It is both Quantum and *not* Quantum—something the Understanding cannot grasp.

The Infinitely Great/Small is no longer finite Quantum. Finite Quantum is determined by all the other quanta. In comparison, the Infinitely Great/Small is simple. As simplicity is the very hallmark of "being," the Infinitely Great/Small is "quantitative determinateness in *qualitative* form." (245) What it expresses is "its essential unity with its other." (245) By "essential" Hegel means that "it has meaning solely with reference to that which stands in *relation* to it. *Apart from this relation* it is a *nullity*." (245)[177] Quantum as such is indifferent to the relation expressed in Figure 16(c). That is, if I propose one trillion as a candidate for the Infinitely Great, that number is indifferent to the fact that it has a beyond—an even larger number. Yet the Infinitely Great is nothing but the *beyond* of any given number. Hence, it is a nullity without the idea of a fixed, inert number.

But the Infinitely Great/Small is only a moment. If number is indifferent to it, the Infinitely Great/Small is not likewise indifferent to number. Hence, "the quantum in its infinity is a *being-for-self*." (245) In other words, it is qualitative, but has its content outside itself. But it is also a *Being-for-one*.

Why is the Infinitely Great/Small a Being-for-one? It will be recalled that Being-for-one "expresses the manner in which the finite is

[177] Essence, as we shall see, is always correlative. *See* LESSER LOGIC, *supra* note 29, §112.

present in its unity with the infinite." (159)[178] That is, Being-for-one is a memory embedded within a unity that once there was disunity. Hence, the Infinitely Great/Small is a unity that likewise appreciates its history—it was generated when the Quantitative Infinite Progress vanished, leaving only the bare idea of a beyond.

Fractions and Infinite Series. Hegel provides an example of Quantitative Infinity. Quantum, he reminds us, is covertly a ratio. Thus, Number was the unity of Amount and Unit, as shown in Figure 13(c). Also, in the next chapter, ratio will become further developed as the qualitative moment of Quantity.[179]

Hegel proposes to analyze the fraction 2/7. This fraction does not appear even superficially to be a unity, like a whole number does. Rather, it "is directly determined by two other numbers which are related to each other as amount and unit, the unit itself being a specific amount." (245)[180] But consider the extremes of the ratio—2 and 7. These are indifferent to being in the ratio (which Hegel here calls the "exponent").[181] This exponent is a third to the extremes of 2 and 7. Once 2 and 7 are in the relation, however, they no longer count as 2 and 7 but they count according to the fraction in which they participate. Relation is now paramount. To prove this, Hegel points out that 4 and 14 or 6 and 21 could serve just as well to express the exponent. The ratio of 2 and 7 therefore has a qualitative character, separate and apart from the sides of the ratio.

This qualitative character, Hegel says, is "a moment of infinity." (246) This qualitative moment survives quantitative change, as when 2/7 becomes 4/14. If the ratio has this qualitative "infinite" moment, it is only imperfectly expressed. The 2 and the 7 can be removed from the ratio, in which case they revert back to ordinary quanta. "[T]heir connection as moments of the ratio is an external circumstance which does not directly concern them." (246) Furthermore, the ratio of 2/7 is likewise an ordinary Quantum.

That the fraction is an ordinary Quantum can be seen if we express 2/7 as 0.285714 . . . So expressed, 2/7 generates an infinite series.[182]

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178 *See* CARLSON, *supra* note 4, at 551-53.

179 *See infra* notes 246-63 and accompanying text.

180 Earlier, Hegel stated that, in Number, what is deemed unit and what is deemed amount is arbitrary. *See supra* text accompanying notes 93-94. Thus, with regard to 2/7, we have either two units of 1/7 or 1/7 units of two.

181 Hegel's usage is unusual. The exponent is usually defined as measure of power of a base. But in 6^4, the exponent is 4, and it raises 6 to the fourth power. Thus, it would be an error to assume Hegel that the exponent of 2 and 7 is $\log_2{}^7$. Rather the exponent of 2 and 7 seems to mean the quotient of unit and amount. Andrew Haas, one of the few commentators to focus on Hegel's analysis of calculus, manages to miss this point. HAAS, *supra* note 8, at 129-30.

182 Hegel also states that $1/(1\text{-}a)$ can be expressed as $1 + a + a^2 + a^3$ etc. This, however, is true only if $a < 1$. More generally, if $|r| < 1$, the geometric series

The quotient of 0.285714 . . . (which Hegel calls the "sum") is the finite expression of the ratio. This Hegel characterizes as "an *aggregate* of units added together, as an amount." (247) True, the magnitudes of which this expression consists (2, 8, 5 etc.) are each a decimal fraction and hence each is a ratio, but this is irrelevant, "for this circumstance concerns the particular kind of *unit* of these magnitudes, not the magnitudes as constituting an *amount*." (247) In other words, 8 (in 0.285714 . . .) is really 8/100. The "particular unit" of the amount 8 is 1/100. Any such consideration isolates the 8 from the entire expression of 0.285714 . . . This last expression can be viewed as an aggregate of indifferent parts in the sense that it is 2/10 + 8/100 + 5/1000, etc.

The infinite series contains Spurious Infinity, "because what the series is meant to express remains an *ought-to-be* and what it does express is burdened with a beyond which does not vanish and *differs* from what was meant to be expressed." (248) The series is actually only something finite—"something *which is not what it ought to be*." (248) In the infinite series, the negative is outside its terms. That is, if 2/7 is expressed as 0.2857, the defect of the expression is that (0.000014 . . .) is left out. In comparison, in the expression 2/7, the negative is "immanent as the *reciprocal* determining of the sides of the ratio and this is an accomplished return-into-self." (248) That is to say, both sides of 2/7 are merely moments of the quantum expressed. As a mere moment, each side (2 and 7, taken immediately) is the negative of the ratio. The "self-related unity" that 2/7 represents is "a negation of the negation" and "consequently has *within it* the determination of infinity." (248) The fraction 2/7 is therefore notionally superior to the infinite series, because the qualitative moment of 2/7 survives the quantitative increase to 4/14 or 6/21. This internalization of infinity is a sign that Quantity is beginning to recapture its own substance, which has been imposed on it by an other. Thus, "the so-called *finite expression* [2/7] is the truly *infinite expression*." In 0.285714 . . . , however, the infinite is expressly missing.[183] (It is what the ellipsis tries to capture.) As a taunt at the infinite series, Hegel remarks:

> The word *infinite* even as used in infinite series, is commonly fancied to be something lofty and exalted; this is a kind of superstition, the superstition of the understanding; we have seen how, on the contrary, it indicates only a deficiency. (249)

Hegel next calls attention to the fact that "the existence of the finite

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$$a + ar + ar^2 \ldots ar^{n-1} \ldots$$

converges to a sum a/(1-r).

183 The usually astute Professor Mure thus gets it wrong when he comments that "the logical principle of the convergent infinite series" is the True Infinite. MURE, *supra* note 17, at 119. Rather, the True Infinite lies in the rational expression of the number, not in the infinite series in which any number can be expressed, as Mure recognizes elsewhere. *Id.* at 120.

series which cannot be summed is an external and contingent circumstance with respect to the form of the series as such." (249) That is, if the ordinary division of 2 into 7 generates an infinite series, the division of 1 into 4 (1/4) does not. Nevertheless, the infinite series expresses "a higher kind of infinity than do those which can be summed"—*i.e.*, 1/4 can be summed and hence is speculatively inferior. (249) 0.285714 . . . at least expresses "an incommensurability, or the impossibility of representing the quantitative ratio contained in them as a quantum." (249) This incommensurability is even more pronounced in irrational numbers—numbers that cannot even be expressed as fractions (*i.e.*, √5). In any case, a series capable of summation (such as 1/4) likewise contains the same spurious infinity that an inexpressible series contains.[184] Hegel claims that a similar terminological inversion occurs in the work of an unnamed philosopher, who designates the mathematical infinite—in the sense of the True Infinite—as the relative infinite. This philosopher gives the name "absolute" to the Spurious Infinite. "But in point of fact it is this metaphysical [*i.e.*, absolute] infinite which is merely relative, because the negation which it expresses is opposed to a limit only in such a manner that this limit *persists* outside it." (249) In other words, Spurious Infinity is always a Finite facing another Finite which is its beyond. The True Infinite encompasses both the finites, as Figure 7(c) showed in chapter 2 of Quality.[185] The mathematical infinite, properly viewed, is a True Infinite.[186] It "has within itself truly sublated the finite limit because the *beyond* of the latter is united with it." (249)

Spinoza. Spinoza recognized the True Infinite and profitably compared it to the Spurious Infinite. According to Spinoza, the infinite is "the absolute affirmation of any kind of natural existence." (249) Absolute affirmation "is to be taken as its relation to itself, its not being dependent on an *other*." (250) The mere finite, for Spinoza, is "a determinateness, as a negation . . . a ceasing-to-be in the form of a *relation* to an *other* which begins *outside* it." (250)

These are sentiments with which Hegel is in accord, but Hegel also thinks that "the absolute affirmation of an existence does not . . . exhaust the notion of infinity." (250) An infinity is not merely an immediacy. Rather, it is "restored by the reflection of the *other* into itself, or as negation of the negative." (250) In short, the True Infinite is

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184 Thus, 1/4 can be expressed as $(1 + a + a^2 + a^3 \ldots)/8$, where a = .5. This is on the formula x=1/(1-a), where a < 1. *See supra* note 182.

185 *See* CARLSON, *supra* note 4, at 538-43.

186 Justus Hartnack, who admits that he struggles with Measure, sees that more advanced state as showing "why the alleged bad [mathematical] infinity is a true infinity." HARTNACK, *supra* note 119, at 35. But it must be recognized that the concepts here have been True Infinites ever since the end of Chapter 2. What Measure will do is to show the sublation of both Quality and Quantity, and the establishment of a true self-subsistence of things.

a middle term. With Spinoza, however, substance is an inert unity—"a fixity or rigidity in which the Notion of the negative unity of the self, i.e subjectivity, is still lacking." (250) Nevertheless, Spinoza at least recognized that the True Infinite (Spinoza's "infinite of thought") was "complete and present within itself." (250) The Spurious Infinite (Spinoza's "infinite of the imagination") "definitely lacks something." (251) Thus, according to Hegel's reading of Spinoza, 2/7 is what the infinite series (0.285714 . . .) *ought to be.* Meanwhile, imagination, in contemplating the Spurious Infinite, "stops short at quantum as such and does not reflect on the qualitative relation which constitutes the ground of the existing incommensurability." (251) In other words, Speculative Reason sees that the Infinitely Great/Small cannot be named as such and so it is both qualitative—independent of outside manipulation—and a relation between the alternating finites of the Spurious Infinite. As the name of the alternating activity, the Infinitely Great/Small is thus incommensurate with the finites it unites.

Arithmetic v. Calculus. Hegel next wishes to consider the incommensurability between arithmetic and calculus, which manifests itself in functions of curved lines (*e.g.*, $y^2 = ax$). Such a function is said to involve *variables.* These variables are different in character than the variability of 2 in 2/7, which equally can be 4 or 8, if the denominator becomes 14 or 28. In contemplating y = ax, x and y can be any magnitude. Hegel complains, "The expression 'variable magnitudes' is therefore very vague and ill-chosen for those determinations of magnitude whose interest and manner of treatment lie in something quite distinct from their mere variability." (251-52) Hegel's basic complaint is that, because the same terminology ("variable") is used in both arithmetic and calculus, a qualitative metaphysical difference between the two practices remains hidden.

What is our interest in x and y, as these appear in the function y = ax? Recall that, in 2/7 or y/x, the numerator is an independent quantum with regard to the denominator. The relation of numerator to denominator is not essential to the quanta that are made to participate in the ratio. But 2/7 and y/x are also "a fixed quantum, a quotient." (252) This observation does not hold if we consider the function $y^2/x = a$. This function has a determinate quotient, to be sure, but, within $y^2/x = a$, y/x has no fixed relation with y^2/x. In other words, y/x is irrelevant and indifferent (or, as Hegel puts it, "variable") to the ratio y^2/x. Thus, in y^2/x, x has a relation to y^2, but not to y. This leads Hegel to observe that "[t]he relation of a magnitude to a *power* is not a *quantum*, but essentially a *qualitative* relation." (252).

What does Hegel mean by this? If we map y/x=7/2 on a Cartesian plane, a straight line is generated. On this straight line, the quotient never changes. This is no longer true with regard to y^2/x. $7^2/2$ is not the same as $14^2/4$; y^2/x enjoys a qualitative moment—a complete independent "variability" from y/x. In comparison, y/x is "only *formally* a function of variable magnitudes." (252) In the ratio y/x, y and x are "not in that determination in which the differential and the integral calculus considers them." (252)[187] Presumably, what Hegel means by this is that, in y/x, the ratio is dependent on otherness—y and x are in an indifferent relation. But calculus trafficks in δy/δx. Where δy or δx are the infinitely small changes in y or x, these entities are not even quanta, as Hegel is about to emphasize.[188]

Given the qualitative moment in y^2/x, which is not present in y/x, "it would have been fitting to have introduced both a special name for them." (252) There is "an essential difference between those magnitudes and such quanta which are merely unknown, but are in themselves completely determined or are a definite range of determinate quanta." (252-53) Mathematics should have seen what a radical break calculus is, compared to the "equation of the straight line." (253)[189] "A great deal of formalism would, indeed, have been avoided if it had been perceived that the calculus is concerned not with variable magnitudes as such but with *relations of powers*." (253)[190]

The Differential Calculus. Suppose x and y are in a power relation, such as y^2/x. In this relation, x and y still signify quanta. But "this significance is altogether and completely lost in the so-called *infinitesimal differences*." (253) Take the expression δx/δy, where δx stands for some change in x and hence some fixed change in y. In this

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187 If differential calculus studies δy given δx, integral calculus goes backwards. It contemplates x as a differential, and it derives the primitive formula of which x is the differential. For example, if differential calculus states that the derivative of y = 5x is 5, integral calculus contemplates 5 and deduces that it is the derivative of y = 5x + c.

188 Hegel elsewhere emphasizes that the Infinitely Great/Small is simultaneously Quantum and the beyond of Quantum. *See supra* text accompanying notes 161-66. Here Hegel obviously means a more primitive Quantum. That is, the calculus of Hegel's time viewed δx and δy as Numbers, not qualitative entities.

189 According to one commentator:

> Mathematics is essentially the science of operating with finite quantities. Calculation in respect of the infinite requires procedures that are clearly at odds with this. At one and the same time, procedures relevant to computation in respect of finite quantities are being used in connection with infinite quantities. This notionless procedure apparent in the differential calculus, shows that this kind of mathematics is incapable of dealing with qualitative differences, and such a calculus is therefore quite unsuitable for physics.

Borzeszkowski, *supra* note 53, at 76.

190 Modern textbooks reflect this notion that application of the calculus to linear functions masks the true qualitative significance of the practice and even take the point farther. One exemplar refers to the "degenerate case of a function of one variable," and states, "The notion of the differential of a function does not appear in its true light in the theory of functions of one variable." R. CREIGHTON BUCK, ADVANCED CALCULUS 243 (2d ed. 1965). In considering 2x as the derivative of $y = x^2$, this textbook advises: "one must draw the subtle distinction between a number c [*i.e.*, 2x] and the 1-by-1 matrix [c]." *Id.* These remarks are entirely Hegelian in their thrust.

expression δx and δy "are no longer quanta, nor are they supposed to signify quanta; it is solely in their relation to each other that they have any meaning, *a meaning merely as moments*." (253) Hegel states that δx and δy "are no longer *something* (something taken as a quantum), not finite differences; but neither are they *nothing*; not empty nullities." (253) Apart from their relation to each other, they *are* nullities, but as moments of δx/δy, each is highly significant.

In δx/δy, Hegel says, Quantum is "genuinely completed into a qualitative reality; it is posited as actually infinite; it is sublated not merely as this or that quantum but as quantum generally." (253) In other words, neither δx nor δy is a quantum on its own. Rather, each is a "vanishing magnitude" and hence no particular quantum. (254) What we have is pure ratio, no longer a Quantity determined by outside forces. The ratio posits (announces) itself as infinite. It is a negation of the negation. It has sublated its finite parts and has genuine being-for-self. Because δx/δy exceeds finite quanta, it stands for Quantum generally, just as the True Infinite stood for all the Finites. Nevertheless, δx/δy is still a determinateness. Mathematics takes δx/δy as "not nothing" but as "an intermediate state . . . between being and nothing." (254) This state does not exist, however, just as Becoming does not exist. It is erroneous to think of Becoming or δx/δy as a *state*.

The nature of Quantum is that "it is supposed to have a completely indifferent existence apart from its ratio." (254) That is, 2 and 7 have meaning on their own apart from 2/7. But δx/δy "has being *solely* in the ratio" (254) and hence is not even a Quantum.

The True Infinite nature of δx/δy has been a target, even for mathematicians, Hegel says. But these attacks come from an inability to digest the Notion. Nevertheless, anyone wishing to practice the calculus—which converts curved into straight lines and the like—must come to grips with the fact that the practice exceeds "the nature of merely finite determinations." (254)

Newton. Hegel undertakes to show how the originators of the calculus did not adequately grasped the nature of the True Infinite. As a result, they "found it necessary in the application to resort again to expedients which conflict with their better cause." (255)

Newton correctly saw a differential as, not an indivisible, but as a vanishing divisible—not as a sum and a ratio but as the *limit* of a sum and ratio. For Newton there are no indivisibles. Indivisibles would imply "a leap again from the abstract ratio to its sides as supposedly having an independent value of their own as indivisibles outside their relation." (256)

Hegel quotes Newton's reference to δx/δy as a "final ratio." (255, 256) Is this attribution of finality fair, when the ratio itself is in the business of vanishing—*i.e.*, approaching zero? Hegel thinks so, because the "ratio of vanishing magnitudes is to be understood not [as] the ratio *before which* and *after which* they vanish, but *with which* they vanish." (255) In other words, the ratio is "final" only in this odd, contradictory state of ceasing-to-be. The phrase δx/δy is therefore a species of Becoming.

Newton saw that this final ratio (δx/δy) is not to be taken as a ratio of final magnitudes, but as a limit to which the ratio of the "magnitudes decreasing without limit are nearer than any *given* . . . difference." (256) If Newton had been attentive to the Notion, however, "there would have been no need for the *decreasing without limit* into which Newton converts the quantum and which only expresses the progress to infinity." (256)

The ratio δx/δy is therefore in a state of continuity between being and vanishing. Hegel approves of the phrase *continuity*, "if the continuity of the quantum is not understood to be the continuity which it has in the finite progress where the quantum is continued in its vanishing." (257) This is only Spurious Infinity. But where transition is made, not to another finite quantum, but to the True Infinite, the usage is appropriate:

> so *continuous* is it, so completely is it preserved, that the transition may be said to consist solely in throwing into relief the pure ratio and causing the non-relational determination—i.e., that a quantum which is a side of the ratio is still a quantum outside this relation—to vanish. (257)

This purification of the quantitative ratio—the loss of indifferent quanta as the determining sides—"is thus analogous to grasping an empirical reality in terms of its Notion." (257) Still, the very expression δx is

> the fundamental vice in these methods—the permanent obstacle to disengaging the determination of the qualitative moment of quantity in its purity from the conception of the ordinary quantum. (258)

That is, the very reference to x—a fixed quantum—in δx leads away from proper appreciation of the Notion inherent in the obviously fascinating concept of the derivative.

Hegel dislikes the word "infinitesimal." "The nature of these magnitudes is supposed to be such that they may be *neglected*." (258)[191] This neglect, "along with a gain in facility," gives the calculus "the appearance of inexactitude and express incorrectness in its method of procedure." (258) Hegel criticizes Christian Wolff[192] for comparing the

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191 For example, where $f(x) = 3x^2 - 4$, it is said that $\delta y/\delta x = 6x$. In truth, it equals $6x + 3\delta x$. But, since δx approaches zero, the calculus feels entitled to treat 3δx as if it has vanished. *See, e.g.*, ALPHA CHIANG, FUNDAMENTAL METHODS OF MATHEMATICAL ECONOMICS 129 (3d ed., 1984).

192 Christian Wolff (1679-1754) was professor of Mathematics and Philosophy at Marburg and a "popularizer and systematizer of Liebniz." *See* André Mense, *Hegel's Library: The Works on Mathematics, Mechanics, Optics and Chemistry*, *in* HEGEL AND NEWTONIANISM, *supra* note 11,

calculus to a surveyor "who, in measuring the height of a mountain is no less accurate if meanwhile the wind has blown away a grain of sand from the top."[193] (258) Hegel rejects a common sense tolerance of such inexactitude. "[I]n the science of mathematics there cannot be any question of such empirical accuracy." (258)

Euler. Leonhard Euler,[194] Hegel says, "insists that the differential calculus considers the *ratios of the increments* of a magnitude, but that the *infinite difference* as such is to be considered as wholly *nil*." (259) In truth, Hegel responds, the infinite difference is a nil "only of quantum, not a qualitative nil." (259) The infinitesimal is perhaps not a quantitative difference. For this reason, it is wrong, in Hegel's opinion, to speak of these moments as "increments or decrements and as *differences*." (259) Such terms imply that "something is *added to* or *subtracted from* the initially given finite magnitude." (259) Such arithmetical operations are quite external to the essence of the calculus. "[T]he transition of the variable magnitude into its differential is of quite a different nature." (259) Rather than reducing a quantum through subtraction, $\delta y/\delta x$ "is to be considered as a reduction of the finite function to the qualitative relation of its quantitative determinations." (259)

In any case, we must not reduce δx or δy to zero, "for a zero no longer has any determinateness at all." (259) True, zero negates the quantum, which is useful, since δx and δy are not quanta. But zero fails to capture the positive significance of the negation of quantum.

The calculus as practiced, then, neglects the True Infinite and transforms δy and δx into the "finite determinateness of quantity and the operation cannot dispense with the conception of a quantum which is merely *relatively small*." (260) These quanta are then subject to ordinary arithmetical operations, as if they were finite magnitudes.

Vanishing magnitudes. How does mathematics justify the transformation of the True Infinite into mere magnitudes? The

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at 670, 690. At a time when he was considered Germany's leading philosopher, he was discharged for heresy from the University of Halle. PINKARD, *supra* note 113, at 90. According to Michael Inwood, Wolff was "Hegel's stock example of an arid metaphysician of the pre-Kantian sort." M.J. INWOOD, HEGEL 528 n.26 (1983).

[193] According to one commentator the point is that Wolff

> combines empirical and analytical argumentation and therefore is not conclusive. Wolff's analogy is inadequate and inconsistent because it identifies two logically different arguments: mathematical inference and measurement. The limited accuracy of measurement is not a mathematical proof.

Wolfgang Neuser, *The Difference Between* Begrifflicher Spekulation *and Mathematics in Hegel's Philosophy of Nature*, 226, 236, *in* HEGEL AND MODERN PHILOSOPHY (David Lamb ed., 1987).

[194] Leonhard Euler (1707-83) was a Swiss mathematician who lived mostly in St. Petersburg. He worked heavily on infinite series, but his chief legacy is the designation of the mysterious e=2.71828. . . , a concept important to economists. Hegel calls him "[t]he great *Euler*, who displayed an infinitely fertile and acute mind in seizing and combining the deeper relations of algebraic magnitudes." (616)

precursors of calculus (Fermat and Barrow)[195] "frankly believed that they were entitled to omit the products of infinitesimal differences and their higher powers, solely on the ground that they *vanish relatively* to the lower order." (262) For example, x^2, by definition, is always greater than δx^2. The differential therefore vanishes "relatively" to the variable x.

The attitude toward curves demonstrates a like methodology of omitting that which is taken as insignificant.[196] In effect, where x_1 and x_2 are points on the curve that are infinitely close together, calculus assumes that the two points are connected by a straight line. Area exists between the straight line and the curve, but this space is ignored.

Hegel presents a demonstration by Newton as an example of the inexactitude from which calculus suffers.[197] This involves Newton's derivation of the "product rule" of the differential calculus. According to this rule, where $y = f(x)$, $\delta xy/\delta x = x\delta y + y\delta x$.[198]

Newton derived the product rule as follows. Take the product xy. First, reduce each element by half its infinitesimal difference ($x - \delta x/2$ and $y - \delta y/2$). Second, multiply these reduced quanta together. Thus, $(x - \delta x/2)(y - \delta y/2) = xy - x\delta y/2 - y\delta x/2 + \delta x\delta y/4$. Now do the opposite: increase each element by half its infinitesimal difference, to obtain $xy + x\delta y/2 + y\delta x/2 + \delta x\delta y/4$. Now subtract one result from the other:

$$(xy + x\delta y/2 + y\delta x/2 + \delta x\delta y/4) - (xy - x\delta y/2 - y\delta x/2 + \delta x\delta y/4) = y\delta x + x\delta y.$$

This last formulation ($y\delta x + x\delta y$) is, of course, the product rule. This product rule is the surplus when the first product was subtracted from the second. The surplus, Hegel says, is "the difference between the two products" and "therefore the differential of xy." (263)

Hegel retorts, "in spite of the name of Newton it must be said that such an operation although very elementary, is incorrect." (263) Hegel

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[195] Pierre de Fermat (1601-65) was a lawyer and amateur number theorist who worked in Provence. He is chiefly famous for Fermat's last theorem—his claim to a proof that, where x to the n + y to the n = z to the n, n=2 (the Pythagorean theorem) but no number higher than 2. Fermat never set forth the proof, but it was apparently proven recently by a Princeton mathematician, Andrew Wiles. On Fermat, see W.W. ROUSE BALL, A SHORT ACCOUNT OF THE HISTORY OF MATHEMATICS 311-12 (4th ed. 1960) (1908). Isaac Barrow was Newton's teacher at Oxford and vacated his professorship so that Newton could have it in 1669. *Id.* at 310.

[196] Calculus holds that $\delta y/\delta x = y/a$, where x is distance on the abscissa, y is the distance on the ordinate between the tangent and the abscissa, and a is the subtangent (*i.e.*, the line segment on the abscissa between the tangent point and the place where the tangent (as hypotenuse) meets the ordinate.) In particular, this describes the technique of Isaac Barrow, Newton's teacher. *See id.*; *see also infra* text accompanying notes 229-32.

[197] Apparently, Hegel "had no very high opinion of Newton's ability to deal with thoughts." Renate Wahsner, *The Philosophical Background to Hegel's Criticism of Newton*, *in* HEGEL AND NEWTONIANISM, *supra* note 11, at 81.

[198] For example, if $x = 2a + 3$, and $y = 3a^2$, then $\delta x/\delta a = 2$ and $\delta y/\delta a = 6a$. According to the product rule, $\delta xy/\delta x = (2)(3a^2) + (2a + 3)6a = 18(a^2 + a)$.

thinks Newton's proof of $y\delta x + x\delta y$ fairly implies the following:

$$(x + \delta x)(y + \delta y) - xy$$

That is, Newton's procedure was to increase x by δx and y by δy and multiply them. Then Newton subtracted xy and was supposedly left with the product rule. Yet, if the above expression is expanded (or multiplied out), we obtain

$$(x + \delta x)(y + \delta y) - xy = y\delta x + x\delta y + \delta x\delta y$$

Hence, the product rule (by which we calculate the derivative of $\delta xy/\delta x$) leaves out the product of the differentials ($\delta x\delta y$)—the usual imprecision of which 18th century calculus was guilty. "It can only have been the need to establish the all-important fluxional calculus which could bring Newton to deceive himself with such a proof," Hegel remarks. (263)[199]

Many of Newton's proofs, Hegel complains, involve the infinite series. His equations can only be solved by approximations. These omissions, Hegel says, "gave his opponents the occasion of a triumph of their method over his." (263) Thus, in mechanics,[200] the function of motion is developed from a series. The series is given a specific meaning. The first term refers to the moment of velocity. The second refers to accelerating force. The third refers to the resistance of forces. The terms of these series are not to be regarded as parts of a whole, but are rather "*qualitative moments of a whole determined by the concept*." (264) Thus, Hegel says,

> the omission of the rest of the terms belonging to the spuriously infinite series acquires an altogether different meaning from omission on the ground of their relative smallness. The error in the Newtonian solution arose, *not* because terms of the series were neglected only as *parts of a sum*, but because the *term containing the qualitative determination*, which is the essential point, was ignored. (264)

In other words, Newtonian physics omits the True Infinite, which was expelled from the analysis when the Spurious Infinite was transformed into a fixed sum.

In a lengthy footnote, Hegel, quotes Lagrange[201] as demonstrating how parts of infinite series are left out in Newton's demonstrations. In

199 Newton called δx or δy "fluxions." Fluxions were "evanescent quantities" and had a significance separate and apart from the ratio $\delta y/\delta x$. BOYER, CALCULUS, *supra* note 175, at 255.

200 Hegel elsewhere accuses Newton of having "flooded mechanics with monstrous metaphysics." HEGEL'S PHILOSOPHY OF NATURE § 270 Remark (A.V. Miller trans., 1970).

201 Joseph Louis Lagrange was the foremost mathematician in the eighteenth century. He worked in Turin until Frederick the Great offered him a position in Berlin in 1766. BALL, *supra* note 195, at 404.

this demonstration, space traversed is considered as a function of time elapsed (x = ft). When developed as f(t + δt), an infinite series is developed: ft + δf't + δ2f"t/2 . . . Motion is therefore said to be "composed"—a Hegelian swear word[202]—of various partial motions. (264 n.1) Each part is likewise expressed by the same infinite series.

In the infinite series, the first derivative is associated with velocity. The second derivative is associated with acceleration.[203] The third derivative and the rest are simply ignored.[204]

Newtonian procedure, Hegel remarks, "is made to depend on the qualitative *meaning*." (265) By this, Hegel probably means that, since the procedure is exact, it is not entirely quantitative and hence is therefore qualitative. Thus, Newton and his followers suppose they omit the tail end of the series because it is an insignificant sum, "while the reason for omitting them is made to consist in the relativity of their *quantum*." (266) The relativity in question justifies the belief that the omissions are based on a quantitative insignificance, not a qualitative one.

What physics should do, then, is to state the qualitative meaning and make the procedure depend on it. This would displace the formalism of Newtonian method. Speculative Reason has no difficulty in ending the series with the first derivative (f't), because Speculative Reason names this the True Infinite.

> Thus the omission of the rest of the terms is not on account of their relative smallness; and so there is no assumption of an inexactitude, an error or mistake which could be compensated or rectified by another error . . . . (265)

If Speculative Reason were in control of physics, it would recognize the first derivative as a relation, not a sum. Physics would be saved from the Spurious Infinite.

Limit. The mathematical notion of limit, Hegel says, is qualitative in nature. It implies "the true category of the *qualitatively* determined relation of variable magnitudes." (266) Thus δx and δy, which represent the infinitesimally small changes of x and y, "are supposed to be taken simply and solely as moments of δy/δx." (266-67) Indeed, the ratio δy/δx is to be taken as indivisible. Indivisibility (*i.e.*, simplicity) is the hallmark of Quality.[205]

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202 Composed things—*i.e.*, 2/7—are brought together by an outside finite will and are therefore doomed to de-compose.

203 *See, e.g.*, GEORGE B. THOMAS, JR., CALCULUS AND ANALYTIC GEOMETRY 60 (3d ed. 1962).

204 In other words, "negligibility of terms of order three and higher in the Taylor-development of the path as a function of time, is an empirical, not a mathematical fact." Louik Fleischhacker, *Hegel on Mathematics and Experimental Science*, *in* HEGEL AND NEWTONIANISM, *supra* note 11, at 207, 211.

205 *See* HAAS, *supra* note 8, at 123 ("inseparability . . . marks the quality of the quantitative

Limit here is the limit of a given function. For example, given y = $3x^2$

$$\lim_{x \to 1} y = 3$$

In mathematics, the limit of $y = 3x^2$ bears no relation to $y = 3x^2$ as such. But the very use of the phrase "limit" suggests "limit of *something*." (267) "It is supposed to be the limit of the *ratio* between the two *increments* by which the two variable magnitudes connected in an equation are supposed to have been *increased*." (267) In short, it is supposed to be the limit of $\delta y/\delta x$. Limit need not entail the use of the infinitely small. Even so, the "way in which the limit is found involves the same inconsistencies as are contained in the other methods." (267) Hegel gives this example: suppose $y = f(x)$. Consider $y+k= f(x+h)$. Any constant (k, to be taken here as Δx)) can be expressed as an infinite series. Hence, $k = ph + qh^2 \ldots$; and $y + k = f(x) + ph + qh^2 \ldots$ If we divide both sides of this equation by h, we get $k/h = p + qh + rh^2 \ldots$ If k and h vanish, because of their insignificance, the right side of this equation also vanishes, with the exception of p. This p is the limit of the ratio of the two increments (k/h). In short, for "vanishing" purposes, $h = 0$. Here h is a quantum. Yet k/h cannot equal 0/0. It must remain a ratio, so, for this purpose, $h > 0$.

The idea of limit (p) was to avoid the inconsistency in which h is implicated. This limit is not 0/0, but only an infinite approximation. This limit (the infinitely small) is no longer a quantitative difference. But we have not gotten away from $\delta y/\delta x = 0$. If $\delta y/\delta x = p$—a quantitative ratio—then how could $h = 0$—an indispensable assumption if $p = \delta y/\delta x$?

> To this there is at once an obvious answer, the simple, meagre answer that it is a coefficient derived in such and such a way—the first function, derived in a certain specific manner, of an original function. (268)

If this suffices as an answer, the theory of limits would be rid of the troublesome increments. But what meaning, then, does p have—"apart from the meagre definition, quite adequate for this theory, that it is simply a function derived from the expansion of a binomial"? (269) Limit should have a more than merely formal significance, which only philosophy can supply.

Hegel next addresses the "confusion which the concept of *approximation* . . . has occasioned in the understanding of the true, qualitative determinateness" of $\delta y/\delta x$. (269) The "so-called

ratio . . . .").

infinitesimals express the vanishing of the sides of the ratio as quanta. [W]hat remains is their quantitative relation solely as qualitatively determined." (269) There is no loss of the qualitative relation here. On the contrary, "it is just this relation which results from the conversion of finite into infinite magnitudes." (269)

Hegel complains that the ordinate and abscissa each vanish into a yet smaller ordinate or abscissa. But the abscissa never seems to convert itself into the ordinate or vice versa. This is evidence of qualitative determinations of δy or δx.

The calculus, however, insists that δy, for example, is a quantum—an "element of the ordinate." (270) In fact, "the limit here does not have the meaning of ratio; it counts only as the final value to which another magnitude of a similar kind continually approximates in such a manner that it can differ from it by as little as we please." (270) In truth, δx or δy are not even quanta, and, because of this, it makes no sense to speak of δx or δy expressing a distance between two quanta. For this reason, the phrase "*approximation* of a magnitude to its limit" is rankly abused. (270) δx is in fact incommensurable with x_0 or x_1.

Calculus and the physical world. Hegel accuses physics of extrapolating forces of nature from calculus instead of vice versa:

> It is announced as a triumph of science that by means of the calculus alone, laws are found *transcending experience*, that is, proposition about existence which have no existence. (272)

Of this practice, Hegel remarks, "I do not hesitate to regard this affectation as nothing more than mere jugglery and window-dressing." (273) Newton is expressly named as guilty of jugglery.[206]

Mathematics is proclaimed "altogether incapable of proving determinations of the physical world in so far as they are laws based on the *qualitative* nature of the moments [of the subject matter]." (270) Hence, science is less than philosophy, because it "does not *start from the Notion*." (273) In science, the "qualitative element, in so far as it is

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206 One commentator opines:

> Treating one of the greatest minds ever to have devoted itself to the natural sciences in this manner, naturally led many of Hegel's contemporaries, just as it has led so many of his later interpreters, into thinking that [Hegel's] manner of philosophizing was fundamentally at odds with Newton's mathematico-mechanical approach . . . Since Newton's dynamics have proved themselves in the course of time to be immensely superior to the competing approaches of Descartes and Leibniz, the conclusion has been drawn that there is really no point in paying any attention to Hegel's arguments.

Karl-Norbert Ihmig, *Hegel's Rejection of the Concept of Force*, *in* HEGEL AND NEWTONIANISM, *supra* note 11, at 399, 399-400. But Borzeszkowski proclaims Hegel "quite right" on this score. "One has to agree with him completely when he objects to basing the calculus on, 'an increment from the force of gravity,' or the argument of the 'unimportance of the difference.'" Borzeszkowski, *supra* note 53, at 76 (footnotes omitted) (citing *Science of Logic* at 272, 259, 262). Cauchy, Heine and Weierstrass, among others, would, more or less contemporaneously with Hegel's time, put calculus on the firmer footing of "limit." *Id*; Moore, *supra* note 173; Moretto, *supra* note103, at 160.

not taken lemmatically[207] from experience, lies outside its sphere." (270) Science has a desire to "uphold the honour of mathematics" and so it forgets its limits. "[T]hus it seemed against its honour to acknowledge simply experience as the course and sole proof of empirical propositions." (270) In Hegel's view, experience is a poor source for truth.[208]

Hegel predicts the downfall of Newton: "Without doubt, however, the same justice will be done to that framework of Newtonian proof as was done to another baseless and artificial Newtonian structure of optical experiments." (270) Here, Hegel refers to the alleged rejection of Newton's optical theories.[209]

Remark 2: The Purpose of the Differential Calculus Deduced from its Application

Hegel moves from the nature of the infinitesimal in the calculus to its applications, a subject he finds more difficult.

Hegel states that "the whole method of the differential calculus is complete in the proposition $\delta x^n = nx^{n-1}\delta x$, or $[f(x + i) - fx]/i = P$." (274) The former expression denotes the power rule. The latter is the difference quotient presented (by me) at the beginning of the last remark to illustrate Hegel's comment that calculus is burdened with inexactitude.[210] In both these formulae, where a binomial formula has the form of $(x + d)$, δx is the coefficient of the first term (e.g., where $y = 5x + c$, $\delta y/\delta x = 5$). Of calculus, Hegel sniffs, "[t]here is no need to learn anything further." (274) The product rule[211] or the power rule of calculus follows mechanically from this. It takes a half hour to learn calculus, Hegel claims:

> What takes longer is simply the effort to understand . . . how it is that, after so easily . . . finding the differential, analytically, i.e. purely arithmetically, by the expansion of the function of the variable after this has received the form of a binomial by the addition of an increment;[212] how it is that the *second stage* can be correct, namely the omission of all the terms except the first, of the [infinite] series arising from the expansion. (274)[213]

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207 *I.e.*, deductively.

208 To paraphrase Kant, experience has insufficient vouchers for the truth. CRITIQUE OF JUDGMENT, *supra* note 142, at 74.

209 These remarks perhaps reflect Hegel's partisanship for Goethe's polar theory of color as a mixture of light and darkness—views discredited today. *See* PINKARD, *supra* note 86, at 560.

210 *See supra* text accompanying notes 169-73. In Hegel's formulation, $P = \Delta y/\Delta x$. The letter i stands for "increment."

211 $\delta xy/\delta x = x\delta y + y\delta x$.

212 In the difference quotient, this expansion consists in $f(x + i)$.

213 Such an omission can be witnessed in the Maclaurin series and the Taylor series, developed in the first half of the eighteenth century. The Maclaurin series can be described as follows:

Calculus, Hegel claims, was not invented for its own sake. Only after it was invented did mathematicians reflect on the nature of the practice. In the previous Remark, Hegel showed how the differential (δx) was qualitative in nature. Hegel's interest in this demonstration was to show the Notion present in the practice. Now it is time to consider the transition from this origin to its application.

Relation of Powers. Calculus has its spiritual significance when it deals with the relations of powers. As emphasized in the prior Remark, $\delta y/\delta x = 5$, where $y = 5x + c$, but this is not particularly interesting to speculative philosophy. On the other hand, $\delta y/\delta x = 3x^2$, which implies $y = x^3 + c$—this is spiritually significant, in Hegel's view.[214] In the next chapter, Hegel will show how, in the relation of powers (*e.g.*, $x^2 = y$), Quantity recaptures its Quality.

To be sure, mere algebra deals with the higher powers, as when the roots of quadratic formulae are "extracted"[215] or when logarithms[216] are

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Suppose $y = f(x) = a_0 + a_1x + a_2x^2 \ldots + a_nx^n$. Where f(x) passes through the origin, x = 0 when y = 0. Hence, $f(x) = a_0$, or $f(0) = a_0$. Furthermore

$$f'(x) = a_1 + 2a_2x + 3a_3x^2 \ldots + na_nx^{n-1}$$
$$f''(x) = 2a_2 + 6a_3x + 12a_4x^2 \ldots + na_nx^{n-2}$$
$$f'''(x) = 6a_3 + 24a_4x + 60a_3x^2 \ldots + na_nx^n$$

Where x = 0, "all the terms except the first" vanish, as Hegel says. On this assumption

$$f_n^{(n)}(x) = n!a_n$$

where (!) stands for $1 \cdot 2 \cdot 3 \ldots \cdot n$. In other words, the nth derivative of f(x), where x = 0, is a_n multiplied by the factorial of n.

Setting x at zero (f(0)) and solving all the above formulae for a, we obtain

$$a_1 = f'(x)$$
$$a_2 = f''(x)/2!$$
$$a_3 = f'''(x)/3!$$
$$a_n = f_n^{(n)}(x)/n!$$

Substituting these values back into $y = a_0 + a_1x + a_2x^2 \ldots + a_nx^n$, we obtain

$$y = f(0) + f'(0)x + f''(0)x^2/2! + f'''(0)x^3/3! \ldots + f_n^{(n)}(0)x^n/n!$$

This is Maclaurin's series. It calculates a power series for values of x near zero. Taylor's series works for values of x that are not near zero.

214 Hegel remarks: "the express qualitative nature of quantity is essentially connected with the forms of powers, and . . . the specific interest of the differential calculus is to operate with qualitative forms of magnitude." (276) Only when calculus deals with the higher powers does calculus operate overtly with "qualitative forms of magnitude."

215 A quadratic equation (*i.e.*, one with a "square" in it), has the form of $ax^2 + bx + c = 0$. The "root" is x, and it is the privilege of a quadratic equation to have two different roots (where $b^2 \neq 4c$). Thus, in $x^2 = 25$ (or, to use the quadratic form, in $x^2 + 0x + 25 = 0$), x is either 5 or -5. Hegel, in the text, is saying that solving quadratic equations is spiritually unrewarding, compared to the operation of calculus on the relation of powers.

216 A logarithm is the exponent that ties two known quanta together. Thus, in $4^t = 16$, t is the logarithm and, of course, t = 2. Logarithms are subject to their own strictly mathematical laws.

used. But "which of the various relations in which the determinations of powers can be put is the peculiar interest and subject matter of the differential calculus." (276)

The previous Remark showed "the futility of the search for principles which would . . . solve the contradiction revealed by the method instead of excusing it or covering up merely by the insignificance of what is here to be omitted." (276) But perhaps from applications adequate principles could be derived.

In his search for the speculative truth, Hegel examines two kinds of subject matter—(a) second degree equations and (b) infinite series (which Hegel calls functions of potentiation).[217]

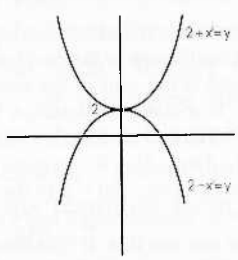
(a) *Second degree equations*. Hegel mentions equations in the form of $y^2 = x$.[218] Such an equation is indeterminate. If, however, one of the variables is assigned a fixed value, then the other has one also. Thus, one is a function of the other. When such formulae are rendered determinate in this way, such formulas are "simple, unimportant, easy determinations." (278) They are made difficult, however, "by importing into them what they do not contain in order that this may then be derived from them—namely, the specific determination of the differential calculus." (278) By this Hegel presumably means that $y^2 = x$ is *qualitatively different* from its derivative function and, in that sense, the primitive does not "contain" the derivative.

Hegel considers the relation of constants to variables. Of these constants, Hegel writes, "it is . . . an indifferent empirical magnitude determining the variables only with respect to their empirical quantum as limit of their minimum and maximum." (278) Thus, to change Hegel's principal example a bit, take $x^2 + 2 = y$. The constant determines the minimum of the parabola. Or, if $-x^2 - 2 = y$, 2 becomes the maximum of the parabola.[219] No matter what values y or x take, 2 is

217 Hegel's text subdivides the Remark into paragraph (a) and paragraph (b), which I have followed in my own text.

218 This is in fact a quadratic equation, and can be expressed as $y^2 + 0y - x = 0$.

219 These parabolas can be drawn as follows:



Constant as Maximum/ Minimum of Parabola

I have changed Hegel's standard example of $x = y^2$ because such a parabola does not properly yield a function at all and simultaneously has a minimum and a maximum. This parabola is not a unaffected. Yet 2 itself is related to y^2 by calculus. For instance, a straight line (e.g., x = 2y) is made into a parabola ($x = y^2 + c$) by integration.[220] The "expansion of the binomial generally" shows the constant to be related to the roots (278).[221] Hegel also writes, "Where, in the integral calculus, the constant is determined from the given formula, it is to that extent treated as a function of this." (278) "This" refers to the "root." Presumably this is illustrated by the fact that the "primitive" of the constant 2 is 2x + c, thereby showing that the constant 2 is related to the root x (not to mention the additional constant c). This indifferent empirical magnitude is actually a relation to what is otherwise taken as diverse and unrelated.

Calculus, Hegel thinks, is most significant for speculative purposes when applied to equations of higher powers. The significance of this has to do with the major point of Hegel's third chapter of Quantity ("The Quantitative Relation or Quantitative Ratio"). According to that chapter, Quantum recaptures its integrity in this kind of relationship: x·x = 16. In such a "ratio" of x to x, no outside mathematician can manipulate the value of x, so long as 16 holds fast. The variable x thus

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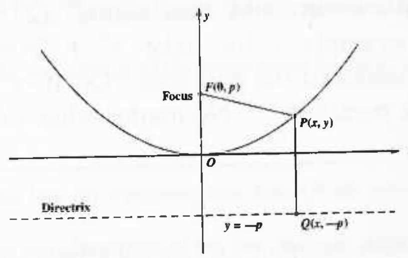
function because each value of x yields more than one value of y.

220 Or, in other words, $\delta y/\delta x = 2x$, where $y = x^2$. 2x is a straight line, and $y = x^2$ is a parabolic curve. The exact phrase Hegel writes here is "a straight line, for example, has the meaning of being the parameter of a parabola." (278) It is also possible that Hegel has in mind the "directrix" of a parabola. A parabola is defined is the points equidistant from a straight line (directrix) and a "focus"—a point:



Parabola, Focus and Directrix

In any case, both these ideas relate the line to the parabolic curve.

221 Specifically Hegel states, "the coefficient of the first term of the development is the sum of the roots, the coefficient of the second is the sum of the products, in pairs." (278) Presumably, this remark is explained as follows. In a quadratic equation in the form $ax^2 + bx + c = 0$, x has two solutions which are the "roots." Thus, if we have $x^2 - 8x + 15 = 0$, x = {3, 5}. The above formula also can be expressed as (x - 5)(x - 3) = 0. In more general terms, if $3 = r_1$ and $5 = r_2$, then $(x - r_1)(x - r_2) = 0$. If we convert $(x - r_1)(x - r_2) = 0$ to the quadratic form, we obtain $x^2 - r_1x - r_2x + r_1r_2 = 0$, or $x^2 - (r_1 + r_2)x + r_1r_2 = 0$. Substituting 3 and 5 back in, $(x - 5)(x - 3) = x^2 - (3 + 5)x + (3\cdot5)$. In this last formulation, one can see that, if "the development" of the expansion of the binomial excludes x^2 and includes only $(r_1 + r_2)x + r_1r_2$, then the coefficient of the first term is the sum of the roots. The second term (which, however, is not a function of x) is a number (or "sum") that is the product of the roots. Hence, the constants (a, b, c) are related to the roots (the two magnitudes of x). This, at least, is what I think Hegel is driving at. In any case, Hegel is right that the constants are related to the roots.

has "being-for-self," and Quantum has recaptured Quality.[222]

These moves from Quantity's final chapter explain why Hegel emphasizes the significance of variables when they are in a relation of powers (as x is in $x \cdot x = 16$). In such a relation "[t]he alteration of variables is . . . *qualitatively* determined, and hence continuous." (278) By "continuous" Hegel means that x remains what it is (hence qualitatively determined) even as it influences the other x.

In this relationship, it is very important, Hegel says, that what he calls the "exponent" x (what we would call the variable) be in a relationship with itself. The variable "raised" to the first power ($x^1 = x$) may have significance in relation with other, higher powers, but on its own, "x is merely any indeterminate quantum." (279) Calculus is (spiritually) pointless when applied to equations of the first order (as in $y = ax^1 + c$). To be sure, we can write $\delta y/\delta x = a$, but nothing is gained from this, in terms of developing the Notion of Quantum.

(b) *Infinite series*. In equations of the second order or higher (e.g., $y^2 = x$), "the power is taken as being *within itself* a relation or a *system of relations*." (279) Hegel defines power as "number which has reached the stage where it *determines its own alteration*." (279) In such a relation, the "moments of unit and amount are identical." (280)

Power is always a number. Thus, when 4 is raised to the second power, the result is 16, or when cubed, 4 becomes 64. These "powers" (16 or 64) could itself be "analysed into an arbitrary amount of numbers which have no further determination relatively to one another or their sum, other than that together they are equal to the sum." (280) If we take 16, these can be split into indifferent parts: $15 + 1 = 16$, or $7 + 9 = 16$. Such a procedure has no philosophic significance. But the power could likewise be "split into a *sum* of differences which are determined by the *form of the power*." (280) That is, if $x \cdot x = 16$, x has a certain qualitative integrity of its own, comparatively immune from outside manipulation.

As Hegel is interested in quadratic formulas at this point, Hegel suggests that 16 should be viewed as a sum, or $(y + z)^2 = y^2 + 2yz + z^2 = 16$. Thus, each "radical root" is a binomial $(y + z)$. "[G]enuine universality" is on full display with the binomial. (280) The roots could be taken as polynomial, but such "further increase in the number of terms is a mere *repetition* of the same determination and therefore meaningless." (280) Once we have a binomial "the *law* is found." (280 n.1) The law in question, presumably, is the "*qualitative determinateness* of the terms resulting from the *raising to a power* of the root taken as a sum." (280)

Hegel says, "This determinateness lies solely in the alteration

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222 Even here, $x=\{4,-4\}$, so that x is not completely immune from subjectivity.

which the potentiation is." (280) Potentiation may be defined as "the state of being rendered more potent, or more active." (280) Hegel uses the phrase "function of potentiation" to describe the following series: $x^n = (y + z)^n = y^n + ny^{n-1}z + n(n-1)y^{n-2}z^2/2 + n(n-1)(n-2)y^{n-3}z^3/3! \ldots$[223] Thus, "potentiation" reveals any Quantum to be qualitative at heart—a power to some binomial.

If x (which stands for some magnitude) is rendered into a series, x can be shown to contain within it a power relation.

> But in this connexion it is essential to distinguish another object of interest, namely the *relation of the fundamental magnitude* itself (whose determinateness, since it is a complex, *i.e.* here an equation, includes *within itself* a power) *to the functions of potentiation*. This relation, taken in complete abstraction from the previously mentioned interest of the *sum*, will show itself to be the sole standpoint yielded by the practical aspect of the science. (280-81)

To translate, the "fundamental magnitude" would be the power (16 in the expression $x^2 = 16$). The function of potentiation reveals the qualitative core (x) at the heart of any Quantum. Divorced from mere arithmetic, the qualitative nature of the relation of powers thus emerges from a "spiritual" study of calculus.

Before this qualitative relation is considered, Hegel wishes to dispel a possible implication of what has been said. The variable that is self-determined in the power relation (x in $x^2 = 16$) is in fact a system of terms. Thus, $x^n = (y + z)^n = y^n + ny^{n-1}z + n(n-1)y^{n-2}z^2/2 + n(n-1)(n-2)y^{n-3}z^3/3! \ldots$. What matters here, Hegel asserts, is not the sum as such, but the power relation revealed in the above series. The power relation as such can be isolated or abstracted from the "plus" signs of the above series.

But every power likewise has an express "plus sign" in it as the preceding power series reveals. This "plus sign" stands for indeterminacy, or quantitative difference. Or, in other words, the power relation may be an advance over simpler Quantum, but not *that* much of an advance such that it is entirely immune from outside manipulation. Even the power relation has (some of its) quality outside itself, as the chapters on Measure will emphasize. For this reason, Hegel remarks:

> To treat an equation of the powers of its variables as a relation of the functions developed by potentiation can, in the first place, be said to

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223 This is the binomial theorem discovered by Isaac Newton. *See* BOYER, MATHEMATICS, *supra* note 111, at 393-96; Niccolò Guicciardini, *Newton and British Newtonians on the Foundation of the Calculus*, *in* HEGEL AND NEWTONIANISM, *supra* note 11, at 167, 170. The text reflects a slight change and extrapolation from the Miller translation on which this commentary is largely based. According to that source, $x^n = (y + z)^n = y + ny^{n-1}z \ldots$ (281) In other words, the exponent n is left off the y variable, which is incorrect. The Johnston-Struthers translation does not make this error. HEGEL'S SCIENCE OF LOGIC 298 (W.H. Johnston & L.G. Struthers trans., 1929).

> be just a *matter of choice* or a *possibility*; the utility of such a transformation has to be indicated by some further *purpose* or use . . . (281)

In other words, quanta do not transform *themselves* to power series. Some outside force must make it happen. The Hegelian motive to do so, of course, would be to push in the Logic beyond Quantity into Measure. Objective progress in the Logic still depends so far on subjective intervention—the so-called "silent fourth" discussed in connection with Determinate Being.[224]

In the function of potentiation given by Hegel, $x^n = (y + z)^n = y + ny^{n-1}z + n(n-1)y^{n-2}z^2/2 + n(n-1)(n-2)y^{n-3}z^3/3! \ldots$, every term beyond y is the derivative of $(y + z)^n$ multiplied by z and divided by n! Because the derivative is involved in this expansion, the increment ($z = \delta x$) is added to the original variable y. According to the mathematicians, δx "is supposed to be not a quantum but only a *form*, the whole value of which is that it *assists* the development." (282)[225] Euler and Lagrange admit that the expansion is intended to produce the coefficients of the variables.[226] But setting z at 1 instead of an increment would likewise preserve coefficients, if that is all that is required.[227] Meanwhile, the use of $\delta x = z$ is to be criticized because δx "is burdened with the false idea of a quantitative difference" which must later be "removed and left out." (282) In any case, "the essential point of interest" is the revelation of the power relation inherent in any Quantum. (282) This "*power determination is immediate*." (282) That is, it resists officious intermeddling by the mathematician and shows a moment of integrity within any Quantum.

Nor should this "power determination" be defined as the coefficient of the first term.[228] Apparently, this "quantifies" δx illegitimately. Instead the series should be described as a "derived

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224 Carlson, *supra* note 4, at 485-88.

225 In general, Hegel accuses mathematics of quantifying δx instead of leaving it an unnameable quality. Here he perhaps suggests that the practice of mathematics is inconsistent: it justifies the addition of the increment because the increment is pure form (qualitative, not quantitative).

226 Hegel's exact sentence: "it is admitted—most categorically by Euler and Lagrange and in the previously mentioned concept of limit—that what is wanted is only the resulting power determinations of the variables, the so-called *coefficients*, namely, of the increment and its powers, according to which the series is ordered and to which the different coefficients belong." (282)

227 Hegel seems to suggest that Euler and Lagrange hold that $y = x - \delta x$, and that $z = \delta x$, but in fact the formula works for any value of z. Hegel also states in this regard: "In order to retain the form of a series expanded on the basis of powers, the designations of the exponents as indices could equally well be attached to the one." (282) I take this remark to mean the following: In the power series $x^n = (y + z)^2 = y + ny^{n-1}z^1 + n(n-1)y^{n-2}z^2/2 + n(n-1)(n-2)y^{n-3}z^3/3! \ldots$ the exponent to z indexes the terms of the expansion—the amount added to y. If z = 1, Hegel is saying, the exponent to z would still effectively index the terms of the expansion.

228 This can be discerned *supra* in note 213.

function of a power." (283) Presumably, this would signal the qualitative nature of δx.

Hegel asks, what is to be made of the power relation revealed in the function of potentiation? He observes that the series involves a decrease in the magnitude of the exponent: thus, n-1 yields to n-2, which in turn yields to n-3, as we travel from the first, second and third derivative. This series reflects the nature of space. Thus, x^3 describes a cube, with height, space and width. The first derivative ($3x^2$) reduces the cube to a plane. The second derivative ($6x$) reduces the plane to a line. The calculus amounts to a relation between these various dimensions. "The straight line [$y = 6x$] has an empirical quantum," Hegel writes (283). But the plane [$3x^2$] is qualitative; it contains a power relation.

Similarly, with regard to motion, the function of space traversed to time elapsed is a quantitative relation—that is, a straight line with no power relation. But accelerating or decelerating speed involves a power relation and hence is qualitative.

The differential calculus as applied to these relations appears arbitrary, but this would not be the case if one is aware of "the nature of the spheres in which its application is permissible." (284) Hegel implies that some consideration of a higher order equation to its derivative will reveal something on this score.

Hegel invokes "the simplest example from curves determined by an equation of the second degree." (284) For instance, $f(x) = x^2 + c$. The first derivative of such a formula produces the slope of the line tangent to this curve ($2x$). Other relevant lines to this curve are the "normal," which is perpendicular to the tangent[229] and the subtangent.[230] "The problem," Hegel writes, "consists in finding the connection between the *relation* of these lines and the *equation* of the curve."(285)

Tangents. Hegel then launches into a history of the relationship between the parabolic curve and the straight line. At first, this relation was discovered empirically. Newton's teacher Isaac Barrow set forth a method for finding the slope of lines tangent to curves that was distinct from Newton's calculus. Barrow would consider a point on a curve—say a parabolic curve described as $y^2 = x$, to use Hegel's favorite example. He then would take a second point on the curve very close to this point. This second point, if below $y^2 = x$, could be described as $(y - \delta y)^2 = (x - \delta x)$.[231] If $y^2 = x$ is subtracted from $(y - \delta y)^2 = (x - \delta x)$, the

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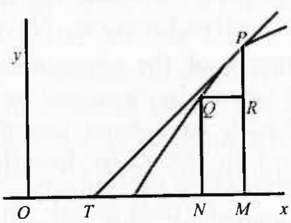
[229] The normal has a slope that is the negative reciprocal of the slope of the tangent line. Thus, if the slope of the tangent is $f'(x)$, the slope of the normal is $-1/f'(x)$.

[230] The subtangent is the distance on the abscissa, as measured from the line that proceeds directly downward from the intersection of the normal and the tangent, to the point where the tangent line meets the abscissa. In short, it is the horizontal base of a right triangle. It will become important in the pre-calculus method of Isaac Barrow, which is about to be discussed.

[231] These are "the tiny little lines afterwards known as increments in the characteristic triangle

result is $2(\delta y)(y) - \delta y^2 = \delta x$. Since δy is an infinitesimal, δy^2 is infinitely smaller. Therefore, Barrow "gives the instruction, in the form of a mere *rule, to reject as superfluous* the terms which, as a result of the expansion of the equations, appears as powers of the said increments or as products." (285) δy^2 is the increment to the second power, and is also the product of increments. If we choose to ignore δy^2, then $2(\delta y)(y) = \delta x$. Dividing both sides of this last equation by δy, we obtain $\delta x/\delta y = 2y$.

Consider the following diagram:



Barrow's Method

Hegel writes that, in Barrow's method, "for the increments of the ordinate and abscissa, the ordinate itself and the subtangent respectively are to be substituted." (285) In other words, PR is the increment of the ordinate and QR is the increment of the abscissa. The ratio PR/QR is equal to the ratio of the "ordinate itself" (285) (PM, or y^2 in the above formulation) and the subtangent (TM). Or, $PR/QR = y^2/TM$.

Of this method, Hegel writes:

> The procedure, if one may say so, can hardly be set forth in a more schoolmasterlike manner; the latter substitution [$PR/QR = y^2/TM$] is the assumption of the proportionality of the increments of the ordinate and the abscissa with the ordinate and the subtangent, an assumption on which is based the determination of [the slope of] the tangent in the ordinary differential method; in Barrow's rule this assumption appears in all its naïve nakedness. (285-86)

In the pre-calculus days of Barrow and Fermat, "[i]t was a mathematical craze of those times to find so-called *methods*, i.e. rules of that kind and to make a secret of them." (286) Indeed, Barrow's technique was not even a *method*. Nothing was derived from established principles. "[T]he inventors had found only a empirical external rules, not a method." (286) Leibniz and Newton generalized the form of such empirical rules and thereby "opened up new paths for the sciences." (286)[232]

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of a curve." (285)

232 Leibniz, however, is singled out as having made the transition from magnitude to calculus "in the most inadequate manner possible, a manner that is as completely unphilosophical . . . as it was unmathematical." (793)

The more genuine way of proceeding (compared to Barrow's method) is as follows. First, "the power forms (of the variables of course) contained in the equation are reduced to their first functions." (286) The value of the terms of the equation, however, is altered. The two functions do not equal each other. Rather, they are simply in a relation.[233] The "primitive" function is a curve; the derivative is a line. "*But with all this nothing is as yet known,*" Hegel insists. (286) Even the ancients understood Barrow's seventeenth-century method of finding the slope of the tangent line by taking the ratio of the ordinate (y) to the subtangent. What the moderns added is the direct mode of producing the derivative from the primitive function. Nevertheless:

> the imaginary increments of the co-ordinates and an imaginary characteristic triangle formed by them and by an equally imaginary increment of the tangent, have been invented in order that the proportionality of the ratio found by lowering the degree of the equation to the ratio formed by the ordinate and subtangent, may be represented, not as something only empirically accepted as an already familiar fact, but as something demonstrated. (287)

In other words, $\delta y/\delta x$ is designed to look familiar and comfortable to those familiar with Barrow's method.

Lagrange rejected "this pretence and took the genuinely scientific course." (287) He dispensed with "infinitely small arcs, ordinates and abscissae" and hence with $\delta y/\delta x$.[234] With regard to $\delta y/\delta x$, however, the line derived "is determined only in so far as it forms the side of a triangle." (287) The unique point that conjoins the line and the curve also forms a part of the triangle. The tangent line thus has the form $p = aq$. This "determination does not require the additional term, + b which is added only on account of the fondness for generality." (288)[235] Hegel also draws attention to the fact that $a = p/q$, and the coefficient of a (here, 1) is the derivative of dp/dq. Thus, a is "the essential determination of the straight line which is applied as tangent to the curve."(288)

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233 Hegel writes: "Instead of $px = y^2$ we have p:2y." (286) This is so on the rules of implicit differentiation (a version of the "chain rule"). According to this method, we take the derivative of both sides of the equation in terms of x. Hence, $dpx/\delta x = (\delta y^2/\delta y)(\delta y/\delta x)$, or $p = 2y\delta y/\delta x$. Hence, $\delta y/\delta x = p/2y$. Hegel also poses $2ax - x^2 = y^2$ and suggests that the derivative is a - x:y. This must be read as (a - x)/y. That is:

$$d2ax/\delta x - \delta x^2/\delta x = (\delta y^2/\delta y)(\delta y/\delta x)$$
$$2a - 2x = 2y(\delta y/\delta x)$$
$$\delta y/\delta x = (a - x)/y$$

234 *Cf.* BOYER, CALCULUS, *supra* note 175, at 13 ("The calculus has therefore been gradually emancipated from geometry and has been made dependent through the definitions of the derivative and the integral, on the notion of the natural numbers . . . .").

235 Thus, if $x = y^2$, or if $x = y^2 + b$, the derivative is 2y regardless. The "+ b," Hegel charges, is added for sentimental reasons.

Descartes. In order to show that the straight line produced by derivation is the same straight line as the tangent, Descartes (who lived more than a century before Lagrange) had recourse to increments of the ordinate and the abscissa. "Thus here, too, the objectionable increment also makes its appearance." (288) But Descartes must be acquitted of the sins of calculus. Descartes was justified because he was acting as a geometer, when he asserted that a point on a curve has a unique tangent line. "For, as thus determined, the quality of tangent or not-tangent is reduced to a *quantitative difference*." (289) The tangent line is simply the smallest line (perhaps in terms of its difference between itself and the parabolic curve, which the derived line is supposed to represent). Such a relative smallness "contains no empirical element whatever" and nothing dependent on a quantum as such." (289) Yet, although reduced to quantitative difference, the line is qualitative, if the line is derived from a "difference in powers." (289) Apparently referring to the type of expansion associated with Isaac Barrow,[236] Hegel observes that the tangent line (when expanded to discover the slope of the line) reveals a difference of i and i^2 (in Cartesian terms)[237] or δy and δy^2 (in Leibnizian terms). That i^2 is comparatively smaller than i is logically true—a qualitative relation. Hence, any attribution of a quantum to i is "superfluous and in fact out of place." (289) Hegel thus acquits Descartes of relying on infinitesimals in his analysis of the "greater smallness" of the tangent line (compared to the parabolic curve).

Hegel regrets that the Cartesian tangential method is "nowadays mostly forgotten." (289) Hegel quotes Descartes as stating that this method is "the most useful and most general problem that I know but even that I ever desired to know in geometry."[238] Hegel rather cryptically describes Descartes' method of finding the slope of the tangent. Here is an example of how it worked, with regard to the tangent of a given parabolic formula, say $y^2 = px$, to use a formula Hegel favors. Descartes first imagined an unknown point on that curve—some value of x and y. For ease of illustration, suppose {x, y} = p. Descartes then imagined a circle whose center was on the abscissa with a distance of h from the origin. If the circle has "equal roots,"[239]

236 It will be recalled that Barrow started with $y^2 = x$ and expanded both x and y by an increment: $(y - \delta y)^2 = x - \delta x$. The original formula, $y^2 = x$, was subtracted from $(y - \delta y)^2 = x - \delta x$, and the result was $2(\delta y)(y) - \delta y^2 = \delta x$. Since δy is an infinitesimal, δy^2 is infinitely smaller. This authorized Barrow to simply ignore δy^2. *See supra* text accompanying notes 229-32.

237 Here i stands for an increment.

238 Miller leaves this quoted in untranslated French. The translation provided follows BOYER, MATHEMATICS, *supra* note 111, at 166.

239 The roots of a quadratic equation are equal when the "discriminant" is equal to zero. The discriminant is $b^2 - 4ac$ in the standard solution to quadratic equations:

the radius of this circle is the "normal," which is perpendicular to the tangent. If we know the slope of the normal, we know the slope of the tangent, which is merely its inverse reciprocal.

The slope of the normal is defined as the ratio of the ordinate ($y = p$) and the "subnormal" ($h - p$).[240] Hence, h is the unknown that must be calculated to solve the problem.

The general formula for a circle whose center is not at the origin is $x^2 + y^2 - 2hx - 2ky + k^2 + h^2 - r^2 = 0$.[241] Because the center of this circle on the abscissa, $k = 0$, thereby simplifying the formula. Meanwhile, $r^2 = (x - h)^2 + (y - k)^2$, by the Pythagorean theorem. Since $\{x, y\} = p$, then $r^2 = (h - p)^2 + p^2$. Substituting this expression of r^2, we obtain $x^2 + y^2 - 2hx + h^2 - [(h - p)^2 + p^2] = 0$, or, more simply, $x^2 + y^2 - 2hx + 2ph - 2p^2 = 0$. From the given parabola, we know that $y^2 = px$. Substituting, we have $x^2 + px - 2hx + 2ph - 2p^2 = 0$. Rearranged in the form of a quadratic, we have $x^2 + (p - 2h)x + (2ph - 2p^2) = 0$. Where $h = 3p/2$,[242] the quadratic equation just given has equal roots (p, p). Therefore the slope of the normal is $-p/(h-p) = -p/(1.5p)-p = -2$.[243] Since the slope of the tangent is

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$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Quadratic Equation

For instance, given $(x - 3)^2 = 0$, the two roots are obviously 3. The discriminant is zero, because $b = 6$, $a = 1$, and $c = 9$.

240 Just as the subtangent is space on the abscissa underneath the tangent line, so the subnormal is like space under the normal.

241 The formula for a circle whose center is the origin is the Pythagorean theorem: $r^2 = x^2 + y^2$, where x and y are points on the circle. If the center is not the origin, the formula becomes $(x - h)^2 + (y - k)^2 = r^2$, where h is the distance from the center to the ordinate and k is the distance from the center to the abscissa. Multiplied out, the formula becomes $x^2 + y^2 - 2hx - 2hx - 2ky + k^2 + h^2 = r^2$. Subtracting both sides of the formula by r^2, we obtain the "standard form" of the circle's formula: $x^2 + y^2 - 2hx - 2hx - 2ky + k^2 + h^2 - r^2 = 0$.

242 This conclusion is reached by use of the discriminant, $b^2 = 4ac$, which holds when the quadratic equation has equal roots. *See supra* note 239. In the expression $x^2 + (p - 2h)x + (2ph - 2p^2) = 0$, $a = 1$, $b = (p - 2h)$, $c = (2ph - 2p^2)$. Substituting this into $b^2 = 4ac$:

$$(p - 2h)^2 = 4(2ph - 2p^2)$$
$$p^2 - 4ph + 4h^2 = 8ph - 8p^2$$
$$9p^2 - 12hp + 4h^2 = 0$$

Once again we exploit the fact that, where roots are equal, $b^2 - 4ac = 0$. Because this is so, the standard solution to the quadratic equation reduces to $x = -b/2a$. In the last quadratic expression, $x = p$, $a = 9$, $b = -12h$. Hence, $p = 12h/18$ and $h = 3p/2$.

243 The slope is negative because x and y are in reciprocal relations with regard to the normal. Below is a diagram of the Cartesian progress in question.

the negative reciprocal, the tangent's slope is 1/2.

Is this the result that calculus obtains? Using the technique of implicit differentiation on $y^2 = px$, we obtain $\delta y/\delta x = p/2y$. But since it is given that $y = p$, we obtain $\delta y/\delta x = 1/2$.[244]

This ability to obtain the tangent algebraically and without any use of the increment is, according to Hegel:

> the brilliant device of a genuinely analytical mind, in comparison with which the dogmatically assumed proportionality of the subtangent and the ordinate with postulated infinitely small, so-called increments, of the abscissa and ordinate drops into the backgrounds. (290)[245]

Hegel complains that "it is by no means self-evident that such a derivative equation is also correct." (290) The derivative "yields only a *proportion*" between δy and δx. (290) Yet y and x are quanta. These can be made into infinite series—"functions of potentiation." When this is done, the values of x and y are altered. Now it is no longer certain that the proportion that previously governed in δy/δx still holds. "All that the equation $\delta y/\delta x = P$ expresses is that P is a *ratio* and no other real meaning can be ascribed to δy/δx. But even so, we still do not know of this ratio = P, to what other ratio it is equal." (291)

Furthermore, Hegel charges, calculus claims, for instance, that, where $(x-3)^2=0$, $\delta(x - 3)^2/\delta x = 2x - 6$, but it fails to validate this conclusion. It is validated "from another source"—from the Cartesian

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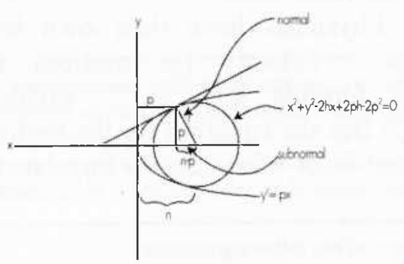

Cartesian Tangential Method

244 Hegel points out that, in a quadratic equation with equal roots, the coefficient of the term containing the unknown in the first power is twice the single root (in terms of its absolute value). This can be seen easily in $(x - 3)^2 = x^2 - 6x + 9 = 0$. Obviously, the unique answer is $x = 3$, and this is half the coefficient of the second term (|6|). Consider also the derivative of the above formula—$2x - 6 = 0$. Calculus obviously agrees that the derivative is related to a quadratic equation in which a unique root is half the coefficient of the second term. (To be precise, Hegel uses the examples of $d(x^2 - ax - b)/\delta x = 2x - a$, and $d(x^3 - px\ p\ q)/\delta x = 3x^2 - p$. I have changed the example to something more easily digested.)

245 Brilliant Descartes' method may have been, but the above example involves $x = y = p$. For any other value of x and y, Descartes' method becomes, at best, monstrously complex. It is no wonder that Descartes' method is "nowadays mostly forgotten," (289) as calculus finds the slope of the tangent with delightful ease, for any point on the curve.

algebraic method of equal roots. (291) Instead what calculus does with $(x - 3)^2 = 0$, is to equate zero with y and proceed accordingly.

Calculus has this further fault. Given that quanta are equally functions of potentiation, it ought to explain that any magnitude is the function of other magnitudes. It does not do so. It simply leaves the magnitudes as given.[246]

The Omitted Constant. With regard to $x^2 - 6x + 9 = (x - 3)^2 = 0$, the derivative function is 2x - 6. The constant (9) is omitted without discussion. Hegel thinks that this omission means that, according to calculus, the constant plays no part in the determination of the roots if these roots are equal. The determination of the roots was exhausted by the coefficient of the second term of the quadratic equation. But this is not so. In Descartes's example (and in the example just given), the constant was the square of the roots, which therefore can be determined from the constant as well as from the coefficients. The constant is thus a function of the roots.

Terminology. Hegel offers an observation about the names "differentiation" and "integration." The character of these operations belie their names. To differentiate is to posit differences. But the result of differentiating is to reduce the dimensions of an equation, and to omit the constant is to remove an element of difference. The roots of the variables are made equal. Their difference is canceled.

Meanwhile, in integration, the constant must be added again. The previously canceled difference is restored. The names assigned to these operations help to obscure the essential nature of the matter—the qualitative nature of the increment.

Mechanics.[247] Physicists have their own interpretation of the differential calculus. Velocity (or motion) has already been mentioned.[248] The formula s = ct,[249] "offers no meaning for differentiation." (292) But the equation for the motion of a falling body $(s = at^2)$[250] does so, and $\delta s/\delta t = 2at$. 2at "is translated into language, and

[246] This is my interpretation of the following passage:

> The functional calculus, it is true, is supposed to deal with functions of potentiation and the differential calculus with differentials; but it by no means follows from this alone that the magnitudes from which the differentials or functions of potentiation are taken, are themselves supposed to be *only* functions of *other* magnitudes. Besides, in the theoretical part, in the instruction to derive the differentials, i.e. the functions of potentiation, there is no indication that the magnitudes which are to be subjected to such treatment are themselves supposed to be functions of other magnitudes. (291)

[247] "'Mechanics' refers to the study of the rest and motion of bodies . . . under the action of normal mechanical forces . . . Its principal concerns are space and time, motion and flow, force and energy, mass and inertia, equilibrium and disequilibrium, impact and elasticity." Ivor Grattan-Guinness, *Hegel's Heritage in Applied Mathematics*, *in* HEGEL AND NEWTONIANISM, *supra* note 11, at 201, 202.

[248] *See supra* text accompanying notes 197-99.

[249] Where s is distance, t is time, and c is velocity.

[250] In this formula, a stands for the acceleration effect of gravity.

also into existence" (293) as a factor in a sum. The sum is the attractive force of gravity and 2at is supposed to be "the force of inertia, i.e. of a simply uniform motion." (293) 2at implies "that in *infinitely small* parts of time the motion is uniform, but in *finite* parts of time, i.e. in actually existent parts of time, it is non-uniform." (293) 2at implies "that *if* gravity ceased to act, the body, with the velocity reached at the *end* of its fall, would cover twice the distance it had traversed, in the same period of time as its fall." (293) This, Hegel proclaims, is unsatisfactory metaphysics. "[T]he end of the period of time in which the body has fallen, is itself still a period of time; if it were *not*, there would be assumed a state of *rest* and hence no velocity." (293)

When physics uses the differential in arenas in which there is no motion, "even more illegitimate formalism of inventing an existence" occurs. (293) Hegel thinks this occurs in the analysis of the behavior of light ("apart from what is called its propagation in space") and in the application of quantitative determinations to colors. (293)

The motion described by $s = at^2$ is found empirically in falling bodies. The next simplest motion is $s = ct^3$, but no such motion is found in nature. Yet $s^3 = at^2$ is Kepler's third law of the motion of planets in the solar system, Hegel says.[251] Now $\delta s/\delta t = 2at/3s^2$.[252] Hegel suggests that a theory that explains the motions of the planets from the starting point of $2at/3s^2$ "must indeed present an interesting problem in which analysis would display a brilliance most worthy of itself." (294) Perhaps Hegel is being sarcastic here.[253]

The application of calculus to physics is not interesting, Hegel complains.[254] But the analysis of trajectory (in ballistics) is significant,

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251 Here, s stands for the semimajor axis of an ellipse—*i.e.*, the farthest distance possible between the planet and the center of the sun. The variable t stands for the period of the orbit (for earth, one year).

252 This is so on the implied differential method, wherein we differentiate both sides of Kepler's formula by t. Hence $\delta s^3/\delta t = (\delta s^3/\delta s)(\delta s/\delta t) = 3s^2(\delta s/\delta t)$, and $\delta at^2/\delta t = 2at$. Combining these results:

$$3s^2(\delta s/\delta t) = 2at$$
$$\delta s/\delta t = 2at/3s^2$$

253 Hegel believed Newton's fame was unjustified—that he merely reformulated Kepler's third law, and that Kepler deserved the crown. *Science of Logic* at 343-44; *see also id.* at 365 (complaining that Newton's great reputation protects his theories from criticism); PHILOSOPHY OF NATURE, *supra* note 200, § 270 ("It has subsequently become customary to speak as if *Newton* were the first to have discovered the proof of these laws. The credit for a discovery has seldom been denied a man with more unjustness."). This, commentators complain, is unfair to Newton, who unified Kepler's and Galileo's laws in a single theory of gravitation. Borzeszkowski, *supra* note 53, at 78; Wahsner, *supra* note 197, at 87-88. One commentator suggests that Hegel was guilty of "grotesquely exaggerated patriotism." Ihmig, *supra* note 206, at 400. *But see* Robert Weinstock, *A Worm in Newton's Apple*, *in* HEGEL AND NEWTONIANISM, *supra* note 11, at 430 (strongly agreeing that Newton is overrated).

254 Borzeszkowski disagrees and states that Hegel "ignores the fact that [δx/δy] creates a new

if trajectory is a curve defined by the higher powers. To construct such a curve, transitions are required from "rectilinear functions"—*i.e.*, straight lines. (294) In other words, a cannon ball exits the muzzle of the cannon in a straight line but, thanks to gravity, converts its trajectory into a parabolic curve. These rectilinear functions, "as functions of potentiation," are derivatives that must be "obtained from the original equation of motion containing the factor of time." (294) The factor of time, however, is eliminated when the rectilinear functions are derived, and the powers in the original equation are "reduced to lower functions of development." (294) Such considerations lead to "the interesting feature of the other part of the differential calculus" (294)—presumably the part that deals with infinite series.

The Integral Calculus. Hegel has now concluded his comments on the differential calculus.[255] His next subject is the integral calculus. Hegel thinks it to be an advance that the integral calculus no longer views itself as a method of summation.[256]

The integral calculus, "as everyone knows," is the converse of the differential calculus. (295) The starting point is the derived function, and one travels back to the "primitive"[257] function from which the first function is derived.

What is the meaning of the primitive formula that the integral calculus discovers? The ratio of the abscissa and the ordinate. The differential calculus, on the other hand, deals only with the ratio of $\delta y/\delta x$.

The usual method of the integral calculus entails the use of the

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quantity, and that on account of the duality in space-time and velocity, it also created the possibility of representing physically the fact that "to move means . . . to be in this place and not to be in, at one and the same time." Borzeszkowski, *supra* note 52, at 78 (footnote omitted) (citing 1 G.W.F. HEGEL, LECTURES IN THE HISTORY OF PHILOSOPHY 273 (E.S. Haldane trans., 1963)). Borzeszkowski also chastises Hegel, who spends much time on Lagrange, for having neglected Lagrange's *Analytical Mechanics*, which, Borzeszkowski feels, would have been instructive. Finally, Hegel wrongly accused physics of asserting the self-identicality of rest and motion and an inability to describe how one changes into another. Borzeszkowski asserts that physics views "rectilinear uniform motion as being equivalent to rest." Borzeszkowski, *supra*, at 80.

255 He terminates with this grand summary:

> Its nature has been found to consist in this, that from an equation of power functions the coefficient of the term of the expansion, the so-called first function, is obtained, and the *relation* which this first function represents is demonstrated in moments of the concrete subject matter, these moments being themselves determined by the equation so obtained between the two relations. (294)

To translate, calculus reveals a relation—$\delta y/\delta x$. The differentials (δx and δy) are the qualitative moments in Quantum.

256 Integral calculus stands basically for the area under a curve. One view of it is that integral calculus adds together the infinitely narrow rectangles that run from the curve to the abscissa. This vision of summation, however, is what Hegel is rejecting. According to Boyer, "[t]he definite integral is defined in mathematics as the limit of an infinite sequence and not as the sum of an infinite number of points, lines, or surfaces." BOYER, CALCULUS, *supra* note 175, at 50.

257 Hegel calls this the "original formula." (297)

infinitesimal difference. Thus, the area under a curve is infinitely divided up into "trapezia" (296)—that is, trapezoids whose parallel sides are the ordinate (y_0) and another ordinate (y_1) separated only by δx. The unparallel sides of the trapezoid are the abscissa and the curve itself. The area under the curve is thus the rectangle entirely under the curve, plus the right triangle sitting atop the rectangle, formed by the abscissa, the ordinate and the arc. "[T]he square of the arc element is supposed to be equal to the sum of the squares of the two other infinitely small elements." (296)[258]

The primitive formula derived from a function taken as a derivative is the area under the curve that the formula expresses.[259] The derivative constitutes the "quadrated curve" at the top of infinitely narrow trapezoid under the curve. But, Hegel complains, the integral calculus mechanically notes the relation between the derivative and the primitive—that these constitute a proportion. It "spares itself the trouble of demonstrating the truth of what it simply presupposes as a fact." (297) The integral calculus has "found out from results already known elsewhere, that certain specific aspects of a mathematical object stands in the relation to each other of the original to the derived function." (297)

In the integral calculus, the primitive function is derived. The derivation is given. But it is

> not directly given, nor is it at once evident which part or element of the mathematical object is to be correlated with the derived function in order that by reducing this to the original function there may be found that other part or element, whose magnitude is required to be determined. (297)

In other words, a ratio is formed between the derivation and the primitive, but the sides of the ratio are not really described by the integral calculus. "The usual method" is to assign to the derivative the status of the infinitely small. (297) This derivative (taken as the top of the trapezoid whose sides are the ordinate and whose bottom is δx) produces a right triangle of three infinitely small sides—the derivative as hypotenuse, the ordinate and abscissa as the other two sides. This triangle, together with the rectangle below it, make up the area under

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258 This is an extremely loose interpretation of Hegel's rather mystifying interpretation. In order to compensate for the curve's failure to be parallel to the abscissa, the integral calculus takes the average of the two ordinates and forms a rectangle, which is roughly bisected by the curve. This corresponds to taking the area of the rectangle entirely below the curve, plus the "right triangle" that sits atop this rectangle.

259 This can be seen as follows. Take a primitive function, such as $y = x^2$. $\delta y/\delta x = 2x$. Integral calculus now describes the primitive as $y = \int 2x\delta x$. This last expression ($\int 2x\delta x$) represents the ordinate ($y=x^2 + c$) times the abscissa (δx). In short, the area under the curve is divided into infinite quasi-rectangles, as defined by the ordinate times the abscissa. Such is the meaning of the expression $\int 2x\delta x$. The integral sign ($\int$) indicates summation of all these quasi-rectangles, thereby encompassing all the area under the curve.

the curve, once the totality of such trapezoids are summed.

> The transition from such so-called elements of the area, the arc, etc., to the magnitude of the total area or the whole arc itself, passes merely for the ascent from the infinite expression to the finite expression, or to the *sum* of the infinitely many elements of which the required magnitude is supposed to consist. (298)

That is, δx or δy represent infinite, qualitative moments. Yet the integral calculus makes finite what is truly infinite—a Notional fault. For this reason, "[it] is therefore merely superficial to say that the integral calculus is simply the converse of the differential calculus." (298)

Lagrange. Lagrange did not smooth out these problems. "The declared object of his method" was to "provide an independent *proof* of the fact that between particular elements of a mathematical whole, for example, of a curve, there exists a relation of the original to the derived function." (298) In other words, Lagrange undertook to prove the truth of integral calculus, but could not proceed directly, because the derivative contains "terms which are *qualitatively* distinct"—that is, δx or δy, which are not quantities. All that can be shown is "the mean between a *greater* and a *less*." (298) In other words, the integral calculus always takes a rectangle defined by δx at the base and two infinitely close ordinates as the vertical sides. These sides are averaged, so that, in the resulting rectangle, the average ordinate is always too great or too small for the area under the curve. From this it is "deduced" that "the function of the ordinate is the derived, first function of the function of the area." (298)[260]

Archimedes. Hegel sees Lagrange translating Archimedean principles into modern terms. Archimedes taught that

> the arc of a curve is greater than its chord and smaller than the sum of the two tangents drawn through the end points of the arc and contained between these points and the point of intersection of the tangents. (299)

Archimedes' method was, through repetition, to render the difference between arc and the chords or tangents smaller and smaller through subdivision.[261]

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260 Hegel puts it this way:

> From the development of the condition that the required magnitude is greater than the one easily determinable limit and smaller than the other, it is then deduced that, e.g. the function of the ordinate is the derived, first function of the function of the area. (299)

It is generally true that the differential calculus views the curve as the first derivative of the area under the curve. That is, the plane is rendered into a line by differential calculus.

261 Archimedes was important in the history of the calculus. *See* BOYER, CALCULUS, *supra* note 175, at 50-53. Archimedes calculated the area under an arc by finding the area of the triangle, which we will call A. He then took the lines XZ and YZ as the bases of two new triangles with their vertex on the curve. The areas of these triangles were found to be ¼ of A. Again, each side of these new triangles became the bases of newer triangles, whose area was

"[T]he formalism of the infinitesimal directly presents us with the equation $\delta z^2 = \delta x^2 + \delta y^2$," Hegel writes (299). This must be taken as an example of the Pythagorean theorem, and it reflects the fact that the integral calculus measures infinitesimal changes at the top of each infinitely narrow trapezoid beneath a curve. Lagrange, starting from this premise, showed that "the length of the arc is the original function to a derived function whose characteristic term is itself a function coming from the relation of a derived function to the original function of the ordinate." (299) In other words, there is a circular relation to differential and integral calculus. Differential calculus produces the derivative from the primitive, and integral calculus produces the primitive from the derivative.[262]

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calculated to be ¼ of the prior triangles. This was continued so long as patience held out. The sum of all the triangles constituted the area under the arc. Archimedes calculated this to be

$$A(1/4 + 1/4^2 + 1/4^3 \ldots 1/4^{n-1}) = (4/3)A$$

The higher the value of n, the more the answer approached (4/3)A.

262 Antonio Moretto reads Hegel as pointing to Lagrange's apagogic reasoning (reasoning by process of elimination). Lagrange assumed a curve, $y = f(x) > 0$. The area under this curve, bounded by the ordinate, abscissa and some value of y, is F(x). Assume y is increasing between x and (x + i). Now let us isolate the area under (x + i) - x as F(x + i) - F(x). It is true that

$$(1)\ if(x) < F(x + i) - F(x) < if(x + i)$$

This expresses the fact that the height of the ordinate—f(x)—times the *i* (the increment in x) is less than the height of the ordinate further on times the increment—or if(x + i). In other words, f(x) is rising over this interval. Meanwhile, the area of the interval—F(x + i) - F(x)—is an average of the rectangles described solely by if(x) or if(x + i).

Lagrange next uses the Newtonian binomial expansion, *supra* note 181, but truncates the series. He stipulates:

$$(2)\ f(x + i) = f(x) + if'(x + j)$$

Strictly speaking, the binomial expansion method implies that the very last term of (2) is if'(x), but j is defined as $0 < j < i$. In other words, j is an increment smaller than the smallest increment. Basically, (2) as rearranged states:

$$(3)\ [f(x + i) - f(x)]/i = f'(x + j)$$

The expression in (3) simply says that the slope (f'(x + j)) is the ratio of the difference between the ordinates f(x + i) - f(x) divided by the abscissa (i).

Next, Lagrange expands F(x + i) but truncates the series at the end of the third term of the expansion.

$$(4)\ F(x + i) = F(x) + iF'(x) + i^2F''(x + j)/2$$

(2) and (4) are now substituted into (1) to obtain

$$(5)\ i[F'(x) - f(x)] + i^2F''(x + j)/2 < i^2f'(x + j)$$

Lagrange now draws attention to [F'(x) - f(x)]. He reasons (apagogically) that either [F'(x) - f(x)] = 0, or it does not. If not, (4) does not hold universally. Hence, by apagogic reasoning, F'(x) -

Both Archimedes's method and Kepler's "treatment of stereometric objects"[263] entail use of the infinitesimal, and this prestigious heritage, Hegel complains, "has often been cited as an authority for the employment of this idea in the differential calculus." (299)[264] "The infinitesimal signifies, strictly, the negation of quantum as quantum" (299)—the proposition that, Hegel repeatedly charges, mathematicians do not confront. The methods of Valerio[265] and Cavalieri[266] get better marks. Their work centered on the relations between geometrical objects. "[T]he fundamental principle is that the *quantum* as such of the objects concerned, which are primarily considered only in their constituent relations, is for this purpose to be left out of account, the objects thus being taken as *non-quantitative*." (299-300) They fall short, however, of bringing to the fore the "affirmative aspect" or "*qualitative* determinateness" of δx (300).[267] This aspect will be made explicit, Hegel promises, in his discussion of the Ratio of Powers, with which Quantity finally concludes.[268] Lagrange, however, is credited with bringing this affirmative aspect to notice, "with the result that the procedure which is burdened with an unlimited progression is given its proper limit." (300)

Hegel concludes this second longest of all remarks by reaffirming

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f(x) = 0, or F'(x) = f(x), and thus integral calculus is vindicated. *See* Moretto, *supra* note 103, at 158-59.

The expression in (5) can be usefully interpreted if we divide both sides of the inequality by i and then solve for i. We get:

$$(6)\ i > [F'(x) - f(x)]/[f'(x + j) - (F''(x + j)/2)]$$

If I can think of an i larger than the right side of the inequality, then (1)—true by definition—is true only if F'(x) - f(x) = 0.

That Lagrange was guilty of alternation between a segment too large and a segment too small can be seen in (1), where if(x) < if(x + i). Nevertheless, Hegel praises Lagrange's method

> because it proves an insight into the *translation* of the Archimedean method into the principle of modern analysis, thus enabling us to see into the inner, true meaning of the procedure which in the other method is carried out mechanically. (299)

In other words, Archimedes' method was literally exhaustive and mechanical, but Lagrange was able, albeit through apagogic reasoning, to calculate the area under a curve without exhaustion.

263 Stereometry is the art of measuring solids.

264 Hegel may be thinking of Simon L'Huilier, who, in a "prize-winning essay" published in 1787, proposed that "the method of the ancients, known under the name of Method of Exhaustion, conveniently extended, suffices to establish with certainty the principles of the new calculus." BOYER, CALCULUS, *supra* note 175, at 255 (citing L'Huilier's *Exposition élementaire des principes des calculs supérieurs*).

265 Luca Valerio is the author of *De Centro Gravitatis Solidorum*. In this work, he anticipated the notion of limit later adopted in calculus. *See* BOYER, CALCULUS, *supra* note 175, at 104-07.

266 Buonaventura Cavaliere was Galileo's student who favored use of the indivisible. *Id.* at 117. Boyer describes Cavalieri as not sharing "the Aristotelian view of infinity as indicating a potentiality only . . . ." *Id.*

267 The affirmative nature of Quantum's quality becomes a major theme in Hegel's chapter on Quantitative Relation. *See infra* text accompanying notes 287-305.

268 *See infra* text accompanying notes 296-300.

that his goal is to describe the Notion, not to reform calculus as such. In any case, a review of calculus for its appreciation of Quantum's Notion would have been inductive only (and hence of poor truth content).

The subject matter of the calculus, Hegel says, is "the relation between a power function and the function of its expansion or potentiation, because this is what is most readily suggested by an insight into the nature of the subject matter." (298) The calculus readily exploits addition, logarithms, and "circular functions," but these are merely convenient to the enterprise—not essential. The calculus has "a more particular interest in common with the form of series namely, to determine those functions of expansion which in the series are called coefficients of the terms." (301)[269] The calculus, however, concerns itself with the relation of the original function to the coefficient of the first term. The series aims at exhibiting a number in the form of a sum. The infinite on display in the series has nothing in common with the affirmative qualitative determination on display in the calculus.

In the calculus, Hegel complains, an expansion occurs by means of the "the infinitesimal in the shape of the *increment*." (301)[270] But this is achieved "externally," by the will of the mathematician. Mathematicians do not develop the notional implication of δx. The series, "which in fact is not what is wanted" by consumers of the calculus, has the fault of producing "an excess"—a remainder "the elimination of which causes the unnecessary trouble." (301) Lagrange favored the series and so had this difficulty. But Lagrange's method at least brought to notice "what is truly characteristic of the calculus." (301) The forms of δx and δy are not forced into objects by Lagrange. Lagrange "directly demonstrated to which part of the object the determinateness of the derived function (function of expansion) belongs." (301)

Remark 3: Further Forms Connected With the Qualitative Determinateness of Magnitude

In the previous Remark, Hegel emphasized the qualitative nature of the infinitesimal. This qualitative dimension was present in the power function—a function involving x^2 or a higher power. There is a weaker form as well, which is the subject of this remark. This form appears in the context of geometry.

From the analytical side, power relations are formal and

269 It will be recalled that the derivative is, in effect, the coefficient of the first term, as in $\delta y/\delta x = 5$, where $f(x) = 5x + c$. The Maclaurin series was likewise an infinite series of the coefficient of the first term. *See supra* note 170.

270 This can be observed in the difference quotient, described *supra* in the text accompanying notes 169-73.

homogeneous. "[T]hey signify *numerical* magnitudes which as such do not possess that qualitative difference from each other." (302) But when these concepts are used by geometers, qualitative determinateness is "manifested as the transition from linear to planar determinations, from determinations of straight lines to those of curves, and so on." (302)

Spatial objects, as Hegel had earlier emphasized,[271] are by nature "given in the form of *continuous* magnitudes." (302) But they likewise are to be taken as discrete. Thus, a plane is an aggregate of lines, the line an aggregate of points.

Integral calculus derives the point from the line, or the line from the plane. (For example, given a point designated as 5, the line 5x + c can be derived through the integral calculus.) The starting point is simple, compared to the concrete, continuous magnitude that is derived. It is important, however, that the starting point be self-determined. That is, the point is without dimensions. It is not "determined," but rather determines itself. The point, then, like δx, is qualitative.[272]

Hegel calls the summation of points into a line or of lines into a plane the "direct method." (303) This may be compared to the indirect method that begins with limits; between these limits lie the self-determined element—the goal toward which the method advances. For example, if we may speak of the area of the circle, π is an infinite series. Hence the area of a circle (πr^2) only approaches the limit because π is never complete. Between the limit and πr^2 there is always a remainder. The result in both methods comes to the same thing—the law for progressively determining the required magnitude without the possibility of reaching the perfect, finite determination demanded.

Kepler was the first to develop the direct method, starting with the discrete as the starting point. He expresses this in his analysis of Archimedes' first proposition of cyclometry.[273] According to this first proposition, "a circle is equal to a right-angled triangle having one of the sides enclosing the right angle equal to the [radius] and the other to the circumference of the circle." (303)[274] Kepler interpreted this to mean that the circumference has infinite points in it, each of which could be

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271 *See supra* text accompanying note 92.

272 It will be recalled that the point as automatically generative of the line first appeared in Hegel's discussion of Limit. *See* Carlson, *supra* note 4, at 521-22. The point that immediately goes outside of itself may fairly be called quantitative, but to the extent it logically produces the line proves that the point is immune from external reflection and therefore is also qualitative. On the point as leading from the derivation of time and space in Hegel's philosophy of nature, see Lawrence S. Stepelevich, *Hegel's Geometric Theory*, *in* HEGEL AND THE PHILOSOPHY OF NATURE (Stephen Houlgate ed., 1998).

273 Cyclometry is the measurement of circles.

274 Miller erroneously writes "diameter" instead of radius. Since the circumference is $2\pi r$ and the radius is r, and since the area of the right triangle is $\frac{1}{2}(2\pi r)(r)$, the area of the right triangle is πr^2, the area of a circle.

regarded as the base of an isosceles triangle.[275] The apex of each triangle is the center of the circle. In this vision, the circle becomes an infinite set of extremely thin "pie slices," and the area could thus be calculated.[276] "[H]e thus gives expression to the resolution of the continuous into the form of the discrete." (303) This description of the infinite, however, "is still far removed from the definition it is supposed to have in the differential calculus." (303) Discrete elements

> can only be *externally* summed up . . . [T]he analytic transition from these ones is made only to their *sum* and is not simultaneously the geometrical transition from the point to the line or from the line to the plane. (304)

Hegel implies that only speculative philosophy can draw from discrete points or line the continuous quality that they have with lines or planes.[277]

A moment of qualitative transition occurs, which entails recourse to the infinitely small. This recourse, Hegel says, is the difficulty. To dispense with this expedient, "it would have to be possible to show that the analytic procedure itself which appears as a mere *summation*, in fact already contains a *multiplication*." (304)

But such an admission involves a fresh assumption about the application of arithmetical relations to geometrical figures. According to this assumption, arithmetical multiplication constitutes a transition to a higher dimension. Thus, the multiplication of lines produces a plane. (For example, the area of a square with side x is x^2.) Here multiplication is not merely an alteration of magnitude, but the production of a qualitative spatial character. Repeating themes from earlier chapters,[278] Hegel insists that "the transition of the line into a plane must be understood as the *self-externalization* of the line." (304) Likewise the point externalizes itself into the line and the plane into a volume.[279]

With regard to the transition from plane to volume, Hegel remarks that the self-externality of a plane (two dimensions) should involve the multiplication of a plane by a plane, thereby creating a four-dimensional object. "[G]eometrical determination," however, reduces the dimensions to three." (305) This is because space, "represented as an expansion outward from the point," is "a *concrete* determinateness

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275 An isosceles triangle is one that has two equal sides.

276 *See* Moore, *supra* note 173, at 139-41.

277 It can be noted here that Archimedes' simple method does not rely on the infinitesimal, but Kepler's method does so rely—a Hegelian fault. *See* Moretto, *supra* note 103, at 160.

278 *See* Carlson, *supra* note 4, at 520-23.

279 "This," Hegel states, "is the same as the representation of the line as the *motion* of the point, and so forth." (304-05) Motion, however, includes a determination of time and thus appears in this representation rather as merely a contingent, external alteration of state." (305) This is not so "from the standpoint of the Notion which was expressed as a self- externalization." (305)

beyond the line in the third dimension." (305) Hegel suggests that Kepler's law ($s^3 = at^2$)[280] has a spatial side, which is geometrical, and a temporal side which is merely arithmetical.

"It will now be evident," Hegel observes, "without further comment, how the qualitative element here considered differs from the subject of the previous Remark." (305) In the power relation at the heart of notional calculus, "the qualitative element lay in the determinateness of power." (305) This point relates to what Hegel will say of the Ratio of Powers in the next chapter: when $x \cdot x = 16$, x determines itself, so long as 16 stays fixed. This self-determination is qualitative in nature and stands for Quantum's recapture of its being, which was entirely external at the beginning of Quantity.

The Quality present in geometry is different. "[H]ere, like the infinitely small, it is only the factor as arithmetically related to the product, or as the point to the line or the line to the plane, and so on." (305) The qualitative transition from the discrete to the continuous "is effected as a process of summation." (305)

This summation, however, does imply multiplication. This comes into view when the area of a trapezoid (or "trapezium," as Hegel's translator calls it) is said to be the sum of two opposite horizontal, parallel lines, divided by two, times the height.[281] The height is represented to be the set of infinite lines which must be summed up. These lines "must at the same time be posited with negation." (306) That is, they are so infinitely narrow that "they have their determination solely in the linear quality of the parallel limits of the trapezium." (306) These trapezoids "can be represented as the terms of an arithmetical progression, having a simply uniform difference which does not, however, require to be determined, and whose first and last terms are these two parallel lines."[282] (306) The sum of such a series, Hegel, explains is "the *product* of the parallels and half the *amount* or *number* of terms." (306) This product is "the specific magnitude of something which is *continuous*"—the height of trapezoids without width. (306) This sum can be viewed as the "*multiplication* of lines by lines." (306) That is, the sum is a geometric area—"something having the quality of a plane." (306) Implicated is "the qualitative element of the transition from the dimension of line to that of plane." (306)

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280 It will be recalled that s stands for the semimajor axis of an ellipse, and t stands for the time of the orbit. *See supra* text accompanying note 205.

281 Hegel's formula is "the product of the sum of two opposite parallel lines and half the height." (306) Where the parallel lines are {a, b} and distance between a and b is h, the formula is

$$\frac{(a + b)h}{2}$$

282 Hegel does not have in mind the trapezoid that forms an element of a definite integral here. He has in mind a trapezoid with real area and, furthermore, tipped on its side so that the parallel lines are horizontal.

The method of representing planes as sums of lines is also used when multiplication is not entailed. Hegel considers this formula

$$\frac{\text{circle}}{\text{ellipse}} = \frac{\text{major axis}}{\text{minor axis}}$$

where the diameter of the circle is the same length as the major axis.[283] Each ordinate of the circle thus corresponds to an ordinate of the ellipse. The relation of the corresponding ordinates is the same as the proportion of the major axis to the minor axis. Therefore, the sum of all the ordinates must be in like proportion. Hence, proportionality likewise makes the leap from discreteness to continuity. "[T]o be swayed by the representation of a plane," Hegel remarks, "or to help it out by adding the idea of *sum* to this *one* moment, is really to fail to recognize the essential mathematical element here involved." (306)

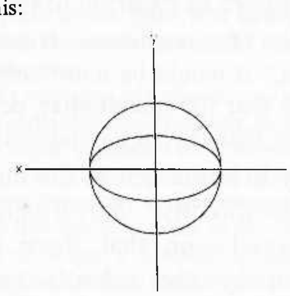
Cavalieri. Hegel returns to Cavalieri, who received relatively good marks for resisting the quantification of δx.[284] Cavalieri used indivisibles (*i.e.*, qualities), rather than infinitesimals. The indivisibles were lines when he considered a plane, and squares or circles when he considered a three dimensional object. These indivisibles he called the regula.

Hegel quotes Cavalieri as follows:

> all figures, both plane and solid, are *proportionate* to all their indivisibles, these being compared with each other collectively and, if there is a common proportion in the figures, distributively. (307)

By this means, Cavalieri proved the proposition that parallelograms of equal height are proportional to their bases.[285] Two lines from these figures that are equidistant from and parallel to the base have the same

283 Such a figure looks like this:



Circle and Ellipse with Common Diameter and Major Axis

284 *See supra* text accompanying notes 265-68.

285 Actually, any line within the parallelogram, will of course be of the same length as the base. Therefore, it follows that lines equidistant from the base of their parallelogram would, of course, bear the same proportion.

proportion as the two bases have. The line, however, is not presented by Cavalieri as "the whole content of the figure." (308) Rather, the line is the content only "in so far as it is to be arithmetically *determined*." (308) Properly, "it is the line which is the element of the content and through it alone must be grasped the specific nature of the figure." (308)

Hegel now reflects on "the difference which exists with respect to that feature into which the determinateness of a figure falls." (308) This is the figure's external limit. Where the determinateness of a figure is an external limit, the continuity of the figure "*follows upon* the quality or the proportion of the limit." (308) When the boundary of two figures coincide, the figures are equal. In parallelograms of equal height and base (and hence of equal area), however, only the base is an external limit. The height, upon which proportion depends, "introduces a second principle of determination additional to the external limits." (308) To prove that parallelograms are equal when they have the same base and height, Euclid reduced them to triangles—*continuous* figures *limited externally*." (308) Cavalieri, in his proof of the proportionality of parallelograms, was careful to state that we never know the *amount* of lines in a parallelogram—an amount Hegel names "an empty idea assumed in support of the theory." (308) Cavalieri spoke only of the magnitude of proportional lines. Because the space of the parallelogram is enclosed within limits, the magnitude of the lines is likewise enclosed within the same limits. Hegel paraphrases Cavalieri as saying, "*the continuous figure is nothing other than the indivisibles themselves* . . . if it were something apart from them it would not be comparable." (309)

According to Hegel, Cavalieri meant to distinguish "what belongs to the *outer existence* of the continuous figure from what constitutes its *determinateness*." (309) In constructing theorems about the figure, we must attend to the determinateness alone. In stating that "the continuous is *composed* or *consists* of indivisibles," Cavalieri implicitly located continuity of the figure as external to the figure.

> [I]nstead of saying that "the continuous is nothing other than the indivisibles themselves," it would be more correct and also directly self-explanatory to say that the quantitative determinateness of the continuous is none other than that of the indivisibles themselves.

In other words, continuity is immanent to the indivisible itself, a fact to which Figure 11(a) speaks directly. Nevertheless "Cavalieri does not support the erroneous conclusion that there are greater and lesser infinites which is drawn *by the schools* from the idea that the indivisibles constitute the continuous." (309) Cavalieri says that he took the aggregate of indivisibles not as an infinite number of lines, "but in so far as they possess a specific kind of limitedness." (309) Any proofs by Cavalieri are free from "any admixture of infinity." (309) His method reduces to "the conception of determinateness as an external spatial limit." (309)

With regard to the coincidence of geometric figures, it is, Hegel says, a "childish aid for sense perception." (309) In fact, if triangles are congruent, we have only one triangle before us. This singularity of the triangle is its true qualitative determinateness, "in its distinction from what is given in intuition." (310)[286]

With parallelograms, Hegel observes that the height and equality of the angles are distinct from the sides ("the external limits") of the figure. (310) This gives rise to an uncertainty. Besides the base, are we to take the vertical side of the parallelogram as an external limit, or the height? If we compare a square with an extremely acute parallelogram with the same base and height, the parallelogram[287] may look bigger than the square. The side of such a parallelogram is indeed longer than the side of the square. Such a longer line may seem to launch "more" infinite lines than the shorter side of the square. "Such a conception, however, is no argument against Cavalieri's method." (310) The aggregate of parallel lines imagined in the two parallelograms presupposes the equidistance of the compared lines from their base. From this it follows that the height, not the side of the parallelogram, is, with the base, the determining moment.[288]

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286 The point here is that Euclidean geometry "has not entirely freed itself from being embroiled in what is sensory: the congruence of plane figures is reduced to the possibility of superposition, and simply by means of the figures' being moved with a rigid motion." Moretto, *supra* note 103, at 153. According to Moretto:

> Nowadays we can appreciate the datedness as well as the validity of his observation. By means of the modern procedures of abstract algebra, we can now say that when two sides of a triangle and the angle between them are given, the class of all the infinitely many triangles congruent to that given, is unambiguously defined.

Id.

287 The Miller translation errs by calling the acute parallelogram a "triangle." (310)

288 Hegel bids us to compare two parallelograms having the same height and base but not lying in the same plane. Rather, the plane of one figure is at an angle from the plane of the second figure. If a third plane cuts through the parallelograms, the lines from the one parallelogram are not equidistant from the lines of the other.

This image leads to Hegel's mediation of a dispute over indivisibles between Cavalieri and Andreas Tacquet (who lived in early seventeenth century Holland). According to Carl Boyer, Tacquet denied that objects of a higher dimension could be viewed as made up of elements of a lower dimension. *See* BOYER, CALCULUS, *supra* note 175, at 139-40 (calling this criticism justified). With regard to a cone formed by a right triangle encompassing the axis, the two apparently disagreed as to which line should be taken as the discrete element that determines the surface of the cone. According to Tacquet's objection, Cavalieri's atomistic method represents the triangle of the cone to be composed of lines parallel to the base and perpendicular to the axis. These lines are radii of the circles of which the surface of the cone is made. If this surface is taken to be the sum of all the circles, "such a result clashes with the truth formerly taught and demonstrated by Archimedes" (311)—that a cone is formed by revolving the hypotenuse in a circle around the axis. Isaac Barrow (whose work was "tainted with the assumption . . . that a curvilinear triangle . . . may be equated with a rectilinear triangle if both are infinitely, that is, *very* small" (311)) answers this objection by Tacquet. To determine the surface of the right angled cone, it is not the axis but the hypotenuse of the triangle of the cone which must be taken as the line that, when it spins around in a circle, generates the surface. Presumably, the point here

Hegel finally concludes. The intention of foregoing remarks on the calculus, he says, "has been to bring to notice the *affirmative* meanings which, in the various applications of the infinitely small in mathematics, remain so to speak in the background." (312) In the infinite series, as well as in Archimedean cyclometry, the infinite means this: "the reduction of the arc to the straight line cannot be effected." (312) Presumably this means that calculus cannot be achieved by purely arithmetic means.

A distinction, Hegel says, is introduced between the continuous and the discrete which makes the continuous appear as if it does not possess any quantum.[289] (For example, the number of points in a line segment cannot possibly be assigned, because there are infinite points there.) By breaking down continuous objects into discrete infinitesimals, the difference appears to be quantitative. In truth, the difference is qualitative. If the magnitude of one line is multiplied by the magnitude of another, we have "the qualitative alteration of the transition from line into plane; and to that extent a negative determination comes into play." (313) That is, qualitative alteration obliterates and negates, whereas quantitative alteration does not. For this reason, the introduction of the infinite, thereby quantifying what should be qualitative, "only serves to aggravate [the difficulty] and prevent its solution." (313)

If I have any readers left at this point, they may finally move on to consider a mercifully short chapter on quantitative ratio, which will stand for a qualitative relation.

III. QUANTITATIVE RELATION

Quantum is an infinite being. It changes *quantitatively* but it

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is that, since the axis is shorter than the hypotenuse, it generates fewer lines than the hypotenuse, and this is what contradicts Archimedes' privileging of the hypotenuse.

One may fairly ask why Hegel, in a work that unfolds the very spirit of the universe, felt compelled to mediate disputes such as this.

289 At this point, Hegel sets forth this apparently mystifying sentence:

> This difference appears arithmetically as a purely quantitative one, that of the root and power, or whatever degree of powers it may be; however, if the expression is to be taken only c.uantitatively, for :xample, $a:a^2$ or $d.a^2 = 2a:a^2 = 2:a$, or for the law of descent of a falling body, $t:a^2$, then it yields the meaningless ratios of 1:a, 2:a, 1:at; in supersession of their merely quantitative aspect, the sides would have to be held apart by their different qualitative significance, as $s = at^2$, the magnitude in this way being expressed as a quality, s a function of the magnitude of another quality.

In the above sentence "$d.a^2$" is peculiar. I have interpreted "d." to mean $\delta a^2/\delta a = 2a$, and "$d.a^2$" to mean $2a:a^2$. Furthermore, I take "$t:a^2$" to be a misprint, stemming from the German addition. Properly, this should be "$t:at^2$," or time as compared with acceleration. The entire sentence states that a and a^2, for instance, or t and at^2 are qualitatively different, but if these are made into quantitative ratios, they are the meaningless expressions of 1/a, or 1/at.

remains what it is *qualitatively*.[290] Accordingly, "[t]he infinity of quantum has been determined to the stage where it is the negative beyond of quantum, which beyond, however, is contained within the quantum itself. This beyond is the qualitative moment as such."(314) At this stage, Quantum is a unity of the qualitative and the quantitative. Thus, the third chapter of Quantity is a chapter of Speculative Reason, just as first chapter represented the Understanding and second chapter represented Dialectical Reason.

Quantum at this advanced stage is *ratio*. Ratio is "the contradiction of externality and self-relation, of the affirmative being of quanta and their negation." (315) Its distinct feature is that it is "qualitatively determined as simply related to its beyond." (314) Quantum is continuous with this beyond, and the beyond is another Quantum. The relation between Quanta, however, is no longer externally imposed. These Quanta have recaptured an integrity that more primitive Quantum did not have. Their integrity is that they have no integrity.

In becoming other, these quanta show their true selves: because they are as much Other as they are themselves.

> [T]he other constitutes the determinateness of each. The flight of quantum away from and beyond itself has now therefore this meaning, that it changed not merely into an other, or into its abstract other, into its negative beyond, but that in this other it reached its determinateness, finding *itself* in its beyond, which is another quantum. (314)

Here Hegel implies that Quantum cannot distinguish itself without the aid of the Other. Therefore, the Other is as much the stuff of self as it is Other. Hence, in distinguishing the other, Quantum finds itself.[291] The quality of Quantum, then, is "its externality as such." (314).

At stake here is not just one Quantum and its beyond (another Quantum), but the *relation* between these two quanta. Thus Quantum "is not only *in* a ratio, but it is itself *posited as a ratio*." (314) Each extreme then has to be taken as a singularity and *also* as a mediation.

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290 MARCUSE, *supra* note 38, at 64 ("A being which is immediately identical with its respective quality such as to remain the same throughout all its qualitative transformations, is no longer qualitatively but quantitatively determined.").

291 In this passage, Hegel echoes perhaps the most famous passages he ever wrote—the Lord-Bondsman dialectic in the *Phenomenology*. PHENOMENOLOGY, *supra* note 10, ¶178-215. In this dialectic, two warriors try to subjugate each other. One succeeds and becomes the master, the other the slave. But the master discovers that the other is truly himself. The master is thus reduced to dependency. Likewise, in ratio, Quantum attempts to distinguish itself by expelling the Other, only to find that the Other is as much itself as itself is.

Errol Harris calls "The Quantitative Relation or Qualitative Ratio" a chapter that is "more technical than philosophical." HARRIS, *supra* note 15, at 140. But perhaps he underestimates its importance. In any case, the Ratio of Powers, with which the chapter ends, is a very lucid and powerful demonstration of the qualitative moment in the heart of Quantity.

The extremes have grown concrete.

Hegel concludes his short introduction to ratio by describing the three sections into which the present chapter is divided. First, we have Direct Ratio (A/B = C). Here the qualitative moment is not yet explicit. Rather, it still shows the retrogressive mode of having its externality outside itself. Direct Ratio shows all the defects of the Understanding. Second is Indirect Ratio, or Inverse Ratio (AB = C). Here Dialectical Reason holds forth. A modulation occurs here between the quanta as they negate each other. Third, we have the Ratio of Powers ($A^2 = C$). Here Quantum (A) reproduces itself. When this middle term is posited as a simple determination, we have reached Measure—the unity of Quantity and Quality. At this moment, the rightward leaning chapters of Quantity will give way to the centrist chapters of Measure.

The culmination of this chapter, then, is the Ratio of Powers—$A^2 =$ C. The middle term, however, is a definition of the absolute.[292] Shall we say, then, that the universe (C) is A^2? Yes, in a sense, if A stands for some "thing" (or Unit). This chapter—Quantitative Relation—in effect argues that all "things" define all other things, even while remaining a thing-in-itself. Hegel is therefore describing a universe of deeply contextual unitary "things."[293]

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292 Every proposition of the Understanding and Speculative Reason is a vision of the Absolute. Dialectic Reason, in contrast, is purely a critique of the Understanding's proposition. Carlson, *supra* note 4, at 533-34.

293 Professor Terry Pinkard is scathing about the ratio of powers. He proclaims it

> so idiosyncratic to Hegel's system that it offers little insight to anyone who has not accepted the entire Hegelian outlook—lock, stock and barrel. It is not one of the things that even Hegelians have seen fit to develop, and there is good reason for this lack of interest.

PINKARD, *supra* note 33, at 52 (citation omitted). This criticism strikes me as out of order. If Hegel is developing a theory of metonymic meaning, this chapter—substrate to the concept of Measure—should not necessarily be expected to yield "useful" dividends to common sense. The chapter does indeed further develop what Quality and even freedom are.

A. *The Direct Ratio*

Direct Ratio can be drawn as follows:

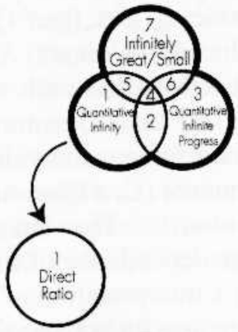

Figure 17 (a)
Direct Ratio

In Figure 17(a), Direct Ratio is immediate. Yet the ratio is nevertheless expressly a relation of *quanta*. These quanta are *other* to the Direct Ratio, and so the determinateness of the ratio lies in an other. Yet Direct Ratio also has its otherness inside itself as well, as it is an infinite being.

Direct Ratio (A/B = C) is itself a Quantum. Hegel insists on calling this the exponent—a confusing choice of words. For mathematicians, C is a quotient. Where $A^2 = C$, 2 is the exponent. What Hegel means by exponent, however, is simply the relation between the two quanta making up the ratio. Thus "the exponent, simply as product, is implicitly the unity of unit and amount." (320-21).

As a Quantum, C is the unity of Unit and Amount, per the laws of sublation. Unit stands for Being-for-self, and Amount stands for "the indifferent fluctuation of the determinateness, the external indifference of quantum." (315) Earlier, Amount and Unit were moments of Quantum. Now, each of them are quanta on their own. Hence, an infinite regress is before us. Every quantum is in turn an Amount and Unit, which are in turn quanta. In short, we have passed into the realm of the quantitative infinite.

As Figure 17(a) indicates, C is a "simple determinateness" (316)—a paradox because determinatenesses are complex. Nevertheless, this coheres with the complex-but-simple nature of the extremes at this stage of the Logic. Thus, the exponent is a Quantum. As such it is complex—an Amount. The exponent is also simple and hence qualitative—a Unit. Hegel explains the qualitative nature of the exponent as follows. Take A/B = 2. If an outside force determines that B = 5, then the exponent determines that A = 10. This power of the exponent over its parts is the exponent's qualitative nature at work. Thus, Hegel can write that the determinateness of the sides of the ratio (A or B) lie beyond themselves. There is but one determinateness common to both sides of the ratio, and it is located in the exponent.

The two sides of the ratio (A/B) thus work to constitute C. It follows, then, that each side is less than Quantum, compared to the exponent. A and B are reduced to Unit and Amount.

But didn't Hegel just tell us that each side was a Quantum? Why now say that each side is *less* than Quantum? In order to emphasize that, at this late stage, outside force cannot simply have its way with the sides of the ratio. The exponent (C, a Quantum) disciplines the sides of the ratio (less than Quantum). The integrity of the Direct Ratio therefore implies the servile dependence of the sides. Recall that Direct Ratio is the Understanding's interpretation of the Infinite Great/Small in Figure 16(a). The infinitely Small (δx) was unnameable. But, when δx is part of a Direct Ratio ($C=\delta y/\delta x$), it is perfectly determinate. C therefore reigns over A/B.

Hegel refers to this incompleteness of the sides as a negation. What this means is that the sides of the ratio are no longer independent. Only the exponent of the ratio lays claim to quality. The sides of the ratio are the negative to that quality. They embody quantitative difference.

But, if the sides are incomplete, this does not mean that the exponent[294] is itself complete. C is still Unit or Amount and so it is incomplete. If $A/B = C$, then $A = B/C$. The quotient C can take A's place with ease. When it does, it is incomplete. C, therefore, is "not posited as what it ought to be . . . the ratio's qualitative unity." (317)

B. *Inverse Ratio*

If Figure 17(a) emphasizes the immediacy of ratio, Figure 17(b) emphasizes its incompleteness, which Hegel has named a sign of negativity.[295]

294 Here Hegel calls it "exponent as quotient." (316)

295 In this sense, I disagree with Mure's analysis of Direct and Indirect Ratio, the sum total of which is as follows: "In Direct Ratio . . . the two quanta unified in the constant exponent increase or diminish together. In Indirect Ratio they vary inversely and so in closer relation . . . ." MURE, *supra* note 17, at 120 (1965). It is hard to sustain the view that the two sides of the Indirect Ratio are in "closer relation," when the function of the Indirect Ratio is to emphasize the qualitative difference between either side and the exponent.

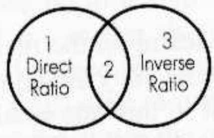

Figure 17 (b)
Inverse Ratio

The fault of Figure 17(a) was its failure to be immediate and immune from outside manipulation. An external reflection had to determine whether C was the exponent or whether it was one of the subordinate sides. Thus, in C = A/B, C is exponent. But it is likewise Unit/Amount in A/C = B.

In the Inverse Ratio, the exponent is some *fixed* Quantum. That is, the exponent stays put and does *not* migrate over to the other side of the equal sign. Apparently, we are not to multiply A = BC by 1/C, which would reveal the exponent to be no different from the Unit/Amount. Rather, we are to consider A as fixed.

Certainly it is odd that the fault of Direct Ratio—its openness to outside manipulation—becomes the virtue of Inverse Ratio, which depends on outside fixity to differentiate it from Direct Ratio. But recall that Quantum is by now the Quality of the Quantum, which precisely was Quantum's openness to outside manipulation. In other words, the integrity of Quantum is its lack of integrity. That is what the fixity of the exponent represents.

When C = A/B, A and B had a *direct* relationship. If A is increased and C stayed constant, B also increased. Now, when A = BC and the exponent stays fixed, B and C are in an inverse relationship. If B increases, C must fall in value. When the exponent is fixed, B and C can fluctuate wildly. Fluctuation "is their distinctive character—in contrast to the qualitative moment as a *fixed* limit; they have the character of *variable* magnitudes, for which the said fixed limit is an infinite beyond." (320) But there is a limit to this inverse relationship. Whereas the mathematician can cause B to fall and C to rise, the mathematician cannot force B to zero. Otherwise, the exponent, which is supposed to be fixed, is destroyed.[296] This resistance of B (or C) to the mathematician's will is a sign that Inverse Ratio has recaptured its Quality.

Speculative Reason seizes upon the resistance of B and C. In this refusal of B and C to go to zero, they are equal *qualities*—immune from outside manipulation at least to this one small extent. Since B = C as a qualitative matter, the in-itself of A = BC is A = BB, or A = B^2—the Ratio of Powers. Here the first B determines the value of the second B. The second reciprocally determines the value of the first. If A = 16, the mathematician has no option but to admit that B = {4, -4}.

Each side of the Inverse Ratio, then, "limits the other and is simultaneously limited by it." (319) Yet once the side of the ratio achieves its in-itself—its potential—it establishes its independence from the other side. "[T]he *other* magnitude become[s] zero." (319) It vanishes. Obviously this last point cannot be taken mathematically. If one side of the ratio (B) is zero, the other side (also B) must likewise be zero, and B^2 is no longer equal to A > 0. Rather, the point is that the first B enjoys Being-for-self. If so, then it is indifferent to the second B, from whom it is a nothing—a void. But since B = B qualitatively, *both* B's are zero. They erase themselves and remove their being to a middle term.

Hegel summarizes the Inverse Ratio as follows:

> The general result can be indicated by saying that the whole, as exponent, is the limit of the reciprocal limiting of both terms and is therefore posited as *negation of the negation*, hence as infinity, as an *affirmative* relation to itself. (320)

In other words, the exponent is limit to the sides and the sides are likewise limit to the exponent. The negation of the negation is, precisely, the refusal of either side to disappear and becomes zero. A limit is now located in the sides of the ratio. These sides negate the superiority of the exponent. Now the sides speak for themselves as to what they are (within the confines of the externally fixed exponent).

Of the sides, Hegel makes two final points: (α) Ratio has an

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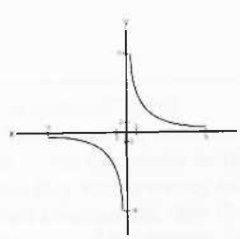
296 Geometrically, *B* and *C* are in hyperbolic relation and could be portrayed as follows:



Hyperbola

"affirmative aspect," (321) which is presumably immunity or fixity in general. Yet, because each side of the ratio cannot be raised to equality with the exponent, each side is, in a sense, "fixed." This fixity—the refusal of B or C to equal A—means that each side is "*implicitly* the whole of the exponent,"(321) since the Inverse Ratio is about fixity.[297] Yet (β) the same refusal is a negative moment. B and C are constituted by a Spurious infinite, which frustrates the mathematician who strives to make them equal to the exponent. This resistance to manipulation is the "negation of the self-externality of the exponent." (321) This very resistance is the resultant middle term, which is therefore "posited as preserving itself and uniting with itself in the negation of the indifferent existence of the quanta, thus being the determinant of its self-external otherness." (321)[298]

C. *The Ratio of Powers*

The Ratio of Powers is shown in Figure 17(c):

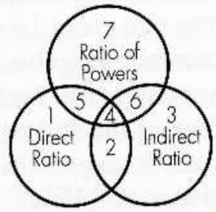


Figure 17 (c)
Ratio of Powers

Hegel says of the Ratio of Powers ($B^2 = 16$, for example) that it is a "quantum which, in its otherness, is identical with itself and which determines the beyond of itself." (321) That is, given the requirement of B^2 and the exponent of 16, the one B determines itself and its other. At this point, Quantum "has reached the stage of being-for-self." (321) Here "quantum is posited as returned to itself." (322)

In earlier stages, we could never tell whether B or C was Unit or

297 This "equality" of a given side of the ratio with the exponent justifies Professor Mure's remark:

> In Ratio of Powers, where one [*i.e.*, the exponent] is a higher power of the other [*i.e.*, a side of the ratio], they relate, if I follow Hegel, so closely that they are fully equivalent to the exponent, and the total expression is true infinity.

MURE, *supra* note 17, at 120. A True Infinite becomes other and remains the same. Hence, the sides become the exponent, in the sense that each is *fixed*. Fixity stands for quality here.

298 Andrew Haas unsuccessfully points to Inverse Ratio as evidence that Hegel's Logic does not have a triune structure. He writes, "in the 'Inverse Ratio'. . . the second moment has only two sub-moments . . . ." HAAS, *supra* note 8, at 79. This is not a fair point. The *entire chapter* is triune. The three moments are Direct Ratio (Understanding), Inverse Ratio (Dialectical Reason), and the Ratio of Powers (Speculative Reason).

Amount. Now B = B, so that Unit *is* Amount. For this reason the Ratio of Powers is "posited as determined only by the unit." (322) The quantum (B) may undergo alteration, separate and apart from the Ratio of Powers, "but in so far as this alteration is a raising to a power, this its otherness is limited purely by itself." (322)

Hegel refers to the Ratio of Powers as qualitative yet external—an apparent contradiction. Where the exponent is fixed, the variable B is determined only by the other B. Hence, its determinateness is external. But B is equally *internal*: "this externality is now posited in conformity with the Notion of quantum, as the latter's own self-determining, as its relation to its own self, as its *quality*." (323) "[I]n so far as the externality or indifference of its determining counts," (323) the Ratio of Powers is still Quantum. At this moment it "is posited *simply* or *immediately*." (323) But also at this moment "it has become the other of itself, namely, quality." (323) In going outside itself, Quantum stays within itself, "so that *in this very externality quantum is self-related*" and hence "is *being* as quality." (323)

We have presented $B^2 = 16$ as an example of the Ratio of Powers. In it, B is unalterable, and thus B has recaptured its Quality. But is it not the case that outside forces can erase 16 and choose, say, 25 instead, thereby changing B? Of course, Hegel admits, but nevertheless the Ratio of Powers "has a closer connection with the *Notion* of quantum." (322) In it, Quantum has reached the full extent of its Notion "and has completely realized it." (322) It expresses the distinctive feature of Quantum, which Hegel describes as follows:

> Quantum is the *indifferent* determinateness, i.e., *posited* as *sublated*, determinateness as a limit which is equally no limit, which continues itself into its otherness and so remains identical with itself therein. (322)

Why is Quantum a determinateness? It will be recalled that Determinateness was another name for Limit.[299] It stands for a unity of being and nothing. So Quantum, as Number, is the unity of Amount (being) and Unit (nothing). Number—an early version of Quantum—was indifferent to its Quality. It depended on external reflection to determine which of its parts was Amount and which was Unit. But now that indifference is sublated. Number is now the Ratio of Powers, which resists outside manipulation. Nevertheless, Amount and Unit are indistinguishable precisely because they are equal (B = B). Each side of the ratio stays what it is and yet it determines itself in its other. It both "remains identical with itself" and goes outside itself.[300]

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299 Carlson, *supra* note 4, at 519.

300 Trying hard to tart up an otherwise dry chapter, Andrew Haas writes that the Power Relation is

> a relation of *potencies* . . . The mathematician cannot help but speak (always) the

Quantum is also said to be "the difference of itself from itself." (322) How is this so? If we contemplate, BB = 16, clearly the first B is distinguishable from the second B, if only by the very difference of location each has on this printed page. Nevertheless, B = B, and so, if the first B is different from the second B, it is different *from itself*. It is no self-identical entity, of which Hegel is so thoroughly critical.

To have a selfhood that is different from itself is what it means for Quantum to be a ratio. At first, in Direct Ratio, ratio showed itself in an immediate form. There, "its self-relation which it has as exponent, in contrast to its differences, counts only as the fixity of an amount of the unit." (322) Presumably, this means that, in Direct Ratio, where the Unit is fixed, Amount is fixed. Yet the exponent itself was not qualitatively different from Unit or Amount. Direct Ratio was not what it ought to have been. In the Inverse Ratio, the exponent is only *in principle* the determinant of the sides of the ratio. In fact, B and C can fluctuate greatly, but they never quite become zero. For this reason the exponent is affirmative in that it has an independence from its sides. That is, the Quantum which is exponent relates itself to itself. In the Ratio of Powers, however, self-relation extends to the sides of the Ratio as well as the exponent.

A summary. Hegel now summarizes the entire journey that Quantity has made—a journey that is now at an end. Quantity was at first opposed to Quality. But Quantity was itself a Quality—"a purely self-related determinateness distinct from the determinateness of its other, from quality as such." (323) Ironically, Quantity learned to resist Quality, and in its resistance, it showed itself to be a Quality. By hating its other, it *became* its other. "Quantity . . . is in its truth the externality which is no longer indifferent but has returned into itself." (323)[301]

But Quantity is not just a Quality. "[I]t is the truth of quality itself." (323) Without Quantity, there could be no Quality.

On the brink of Measure, Hegel notes that a double transition was necessary. Not only does one determinateness continue into the other but the other determinateness continues into the original one.[302] Thus, Quality is contained in Quantity, "but this is still a one-sided determinateness." (323) The converse is true as well—Quantity is

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> words of the philosopher and the sexologist: the decent, respectable power ratio is (also) the indecent language of fornication, the multiple obscenities of the polygamist.

HAAS, *supra* note 8, at 131. At least we can go along so far as to confirm that the Ratio of Powers is the inseparable unity of two quality-quantities.

301 Professor Mure remarks, "In Ratio the endless impotent self-externalizing of Quantity became a self-relation as near to true infinity as Quantity can rise to. It has developed an internal systematicness, which is only thinkable as qualitative." MURE, *supra* note 17, at 121. Mure errs, however, in implying that Quantity is not itself a True Infinite. *See supra* notes 6-8, 28 and accompanying text.

302 That is, in $x^2 = y$, x stays an x and determines the other side of the ratio as x.

contained in Quality. "This observation on the necessity of the *double* transition," Hegel remarks, "is of great importance throughout the whole compass of scientific method." (323)[303]

The name of the partnership between Quality and Quantity is Measure.

Remark

Hegel's study of Quantity ends with a short remark. Here Hegel criticizes an unnamed philosopher's description of the Notion. In this philosophy, immediate Notion was named the first power. Rendered determinate, Notion was called the second power. In its return to itself, wherein it is a totality, it was the third power. Power used here, Hegel states, belongs to Quantum. They do not correspond to Aristotle's dynamic notions.

The power relation "expresses determinateness in the form or difference which has reached its truth." (324) But this Notional truth is appropriate only for the primitive stage of Quantum. It is not appropriate for the Notion as such. "Differences which are proper to quantum are superficial determinations for the Notion itself and are still far from being determined as they are in the notion." (324) In the infancy of philosophy, Pythagoras used numbers to designate universal distinctions, but the first through third powers referred to above are little better than numbers.

> [T]o retrogress from [thought] determinations to those of number is the action of a thinking which feels its own incapacity, a thinking which . . . makes itself ridiculous by pretending that this impotence is something new, superior, and an advance. (324-25)

It is unhelpful, Hegel suggests, to borrow mathematical terms to describe the Notion. If these are merely symbols for the true Notion, then the Notion would first have to be derived logically and *then* symbolized. But, upon deriving the Notion, the symbols become superfluous, as we would have before us the direct Notion. Use of mere forms simply evades "the task of grasping the determinations of the Notion." (325)

CONCLUSION

"A main result of the science of logic is to repudiate quantitative

303 This double exchange is what Slavoj Žižek named the "chiasmic exchange of properties." Carlson, *supra* note 4, at 468 (citing SLAVOJ ŽIŽEK, FOR THEY KNOW NOT WHAT THEY DO: ENJOYMENT AS A POLITICAL FACTOR 39-41 (1991)).

definition of the absolute, and to retrieve qualitative definition."[304] Across the "quantity" chapters, we have seen how an exclusive quantitative perspective falls apart.

At first, Being expelled otherness so that it could be all by itself—independent from the negative. But it discovered that, in this mode, Being expelled all its content and became Quantity. Quantity stands for the very act of expelling all content.

Quantity discovers, however, that it has an integrity that it cannot expel—a limit that preserves its content within itself. "This inability to reach its bourne Hegel describes as *eine Ohbmacht des Negativen*—a weakness of the negative—in that what it abolishes by its own cancelling immediately reasserts itself."[305]

This reassertion of what is canceled is nevertheless "other." Hence, in Quantitative Infinity (Figure 16(a)), Quantum goes outside itself to a beyond. The infinitely big or small number can never be named. Yet, in going beyond its limit, Quantum discovers that its *own content* is beyond the limit. In this sense, Quantum returns to itself when it exports its content to the other. It shows what it is when it shows it is nothing. This return will later be called reflection-into-self—the hallmark of Essence. For now, it can be noted that the nature of being has changed. Whereas in the first three chapters Being constituted expelling the negative, now Being constitutes expelling its own self and therefore, in this act of expulsion, accomplishes a return to itself. This return to self is still implicit and will remain so in the last installment of the Doctrine of Being—Measure. It becomes express in Hegel's theory of reflection.

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[304] BUTLER, *supra* note 20, at 143.

[305] HARRIS, *supra* note 15, at 136. Theodor Adorno refuses to accept any weakness of the negative and so disagrees with Hegel's entire system. But this amounts to a dogmatic insistence on the self-identity and irreducibility of the negative, an irony of which he seems to be unaware. THEODOR W. ADORNO, NEGATIVE DIALECTICS 119 (E.B. Ashton trans., 2000).