

2018

Stationary Automata

Anaam Mutar Bidhan
ambidhan@mix.wvu.edu

Follow this and additional works at: <https://researchrepository.wvu.edu/etd>



Part of the [Mathematics Commons](#)

Recommended Citation

Bidhan, Anaam Mutar, "Stationary Automata" (2018). *Graduate Theses, Dissertations, and Problem Reports*. 3750.

<https://researchrepository.wvu.edu/etd/3750>

This Dissertation is protected by copyright and/or related rights. It has been brought to you by the The Research Repository @ WVU with permission from the rights-holder(s). You are free to use this Dissertation in any way that is permitted by the copyright and related rights legislation that applies to your use. For other uses you must obtain permission from the rights-holder(s) directly, unless additional rights are indicated by a Creative Commons license in the record and/ or on the work itself. This Dissertation has been accepted for inclusion in WVU Graduate Theses, Dissertations, and Problem Reports collection by an authorized administrator of The Research Repository @ WVU. For more information, please contact researchrepository@mail.wvu.edu.

Stationary Automata

Anaam Bidhan

Dissertation submitted to the
Eberly College of Arts and Sciences
at West Virginia University
in partial fulfillment of the requirements
for the degree of

Doctor of Philosophy

in

Mathematics

Jerzy Wojciechowski, Ph.D., Chair

John Goldwasser, Ph.D.

Rong Luo, Ph.D.

Elaine Eschen, Ph.D.

C.Q.Zhang, Ph.D.

Department of Mathematics

Morgantown, WV

2018

Keywords: Automata, Transfinite Words, Graph, Matching, Operations on Words

Copyright 2018 Anaam Bidhan

ABSTRACT

Stationary Automata

Anaam Bidhan

In this dissertation, we investigate new automata, we call it stationary automata or ST-automata. This concept is based on the definition of TF-automaton by Wojciechowski [7]. What is new in our approach is that we incorporate stationary subsets of limit ordinals of uncountable cofinality.

The first objective of the thesis is to motivate the new construction of automata. This concept of ST-automata allows us to make a connection with infinite graph theory. Aharoni, Nash-Williams, and Shelah [2] formulated a condition that is necessary and sufficient for a bipartite graph to have a matching. For a bipartite graph $G = (M, W, E)$, we define a language $\mathcal{L}(G)$ over the alphabet $\{M, W\}$. We construct an ST-automaton \mathcal{A} such that for each bipartite graph G , the automaton \mathcal{A} accepts an element of $\mathcal{L}(G)$ if and only if G has no matching. The theorem of Aharoni, Nash-Williams, and Shelah [2] is used to prove that \mathcal{A} has the above property.

The second objective is to compare the new ST-automata to TF-automata defined by Wojciechowski [7]. First, adding an extra condition, we define special ST-automata and prove that they are equivalent to TF-automata. Then we show that in general ST-automata are stronger. We give an example of a language accepted by an ST-automaton that is not accepted by any special ST-automaton.

In chapter four, we define operations on ST-automata over a fixed alphabet I as union, intersection, concatenation, raising to the powers, ω , $*$, and $\#$. We show that applying those operations to languages defined by ST-automata the obtained languages are also definable using ST-automata.

Acknowledgments

First and foremost, I am most indebted to my supervisor, Dr. Jerzy Wojciechowski, for his continued encouragement and support over these last few years. It is a pleasure to work under his supervision. Without him, this dissertation could not have come about.

I would also like to thank my other committee members: Dr. John Goldwasser, Dr. Elaine Eschen, Dr. Rong Luo, and Dr. C.Q.Zhang, for their help during my studies.

And finally, I would like to thank the Department of Mathematics and Eberly Collage of Arts and Science at West Virginia University for providing me with an excellent study environment and support during my study as a graduate student.

DEDICATION

To

My husband, Ali Kareem, my father, my mother in law, my mother, and

my lovely children Mustafa, and Amir

Contents

1	Introduction	1
1.1	Notation	1
1.2	Preliminaries	2
1.3	Main Definitions And Results	10
2	ST-automata	14
2.1	Basic Definitions	14
2.2	Motivation Result	16
3	Special ST-automata	36
3.1	Basic Definitions	36
3.2	Main Results	37
4	Operations on ST-automata	59
4.1	Union Operation	59
4.2	Intersection Operation	60
4.3	Concatenation Operation	68
4.4	* -Operation	82
4.5	ω -Operation	94
4.6	#-Operation	104
5	Future Work	117

Chapter 1

Introduction

1.1 Notation

Before defining the notion of the ST-automata, we introduce the terminology and present the basic facts concerning ordinal numbers and their arithmetic that is used in this dissertation.

The ordinal number β is the set of all ordinals less than β . Then the relation of order \leq is the inclusion relation \subseteq , and the operation of taking the upper bound on a set of ordinals is the operation of union.

The class of all ordinals will be denoted by ord , the class of all successor ordinals will be denoted by succ , and the class of all limit ordinals will be denoted by lim .

The smallest limit ordinal that is the set of natural numbers will be denoted by ω or \aleph_0 and the first uncountable ordinal will be denoted by ω_1 or \aleph_1 .

The arithmetical operations on ordinals are defined inductively as follows:

$$\begin{aligned}\alpha + 0 &= \alpha, \\ \alpha + 1 &= \alpha \cup \{\alpha\}, \\ \alpha + (\beta + 1) &= (\alpha + \beta) + 1, \\ \alpha + \zeta &= \sup \{\alpha + \beta : \beta < \zeta\},\end{aligned}$$

where $\alpha, \beta \in \text{ord}$, and $\zeta \in \text{lim}$.

Let $(\beta_\alpha)_{\alpha < \gamma}$ for $\gamma \in \text{ord}$ be a transfinite sequence of ordinals. The sum of that sequence $\sum_{\alpha < \gamma} \beta_\alpha$ is defined by induction on γ as follows:

if $\gamma = 0$, then

$$\sum_{\alpha < 0} \beta_\alpha = 0,$$

if $\gamma = \sigma + 1$, then

$$\sum_{\alpha < \gamma} \beta_\alpha = \sum_{\alpha < \sigma} \beta_\alpha + \beta_\sigma,$$

if $\gamma \in \text{lim}$, then

$$\sum_{\alpha < \gamma} \beta_\alpha = \sup \left\{ \sum_{\alpha < \theta} \beta_\alpha : \theta < \gamma \right\}, \text{ where } \theta \in \text{ord}.$$

We will denote by $\text{cf}(\beta)$ the cofinality of any limit ordinal β .

We will denote by $P(S)$ the family of all subsets of the set S .

$\sup(A) = \bigcup A$ is the supremum of the set A , where A is a set of ordinals.

$\text{dom}(f)$ is the domain of the function f .

$\text{rng}(f)$ is the range of the function f .

We denote by $f \upharpoonright \beta$ the restriction of f to β .

Finally, we will denote by \square the end of a proof.

1.2 Preliminaries

In this part, we show some lemmas, theorems, propositions, and definitions that will be used in the proofs of our main results.

The following definitions, theorems, and lemma are from [3].

Definition 1.2.1. A partial ordering on a class X is a binary relation that is anti-reflexive and transitive. A linear ordering on X is a partial ordering in which any two different elements are comparable. A well-ordering on a class X is a linear ordering on X such that for every $x \in X$ the class $\{y \in X : y < x\}$ is a set and each nonempty subset of X has the smallest element.

Definition 1.2.2. A set a is transitive if and only if every element of a is also a subset of a .

Definition 1.2.3. A set a is an ordinal number (or just an ordinal) if it is transitive and every element of a is also transitive.

Lemma 1.2.4. *Every nonempty class of ordinals has a smallest element.*

Theorem 1.2.5. *Every set can be well-ordered.*

Theorem 1.2.6. *Let $(X, <)$ be a well-ordering. Then there exists a unique ordinal δ , and a unique bijection $\phi : \delta \rightarrow X$, that preserves the order.*

Definition 1.2.7. A successor ordinal is any ordinal of the form $\beta = \alpha + 1$, for some ordinal α . If β is a nonzero ordinal that is not a successor ordinal, then we say that β is a limit ordinal.

Definition 1.2.8. Let X be a set. The cardinality of X , denoted by $|X|$ is the smallest ordinal α such that there exists a bijection $f : X \rightarrow \alpha$.

Definition 1.2.9. A cardinal number or just cardinal is an ordinal that is the cardinality of some set.

Definition 1.2.10. Let α be a limit ordinal and $A \subseteq \alpha$. We say that A is unbounded in α (cofinal in α) if and only if for every $\beta < \alpha$, there exists $\gamma \in A$, with $\beta < \gamma$.

Definition 1.2.11. Let α be any limit ordinal. The cofinality of α denoted by $\text{cf}(\alpha)$ is the smallest cardinal κ such that there is an unbounded subset A of α with $|A| = \kappa$.

Definition 1.2.12. Let κ be an infinite cardinal. We say that κ is regular if and only if the cofinality of κ is equal κ . Otherwise, κ is said to be singular.

Definition 1.2.13. Let θ be a limit ordinal with $\text{cf}(\theta) > \omega$ and $C \subseteq \theta$. We say that C is a closed in θ if and only if $\sup(D) \in C \cup \{\theta\}$, for every nonempty $D \subseteq C$. A club in θ is a subset of θ that is closed and unbounded in θ . If θ is a regular cardinal, then C is closed in θ if and only if for every $D \subseteq C$ with $|D| < \theta$, we have $\sup(D) \in C$.

Definition 1.2.14. Let $(A, <)$ be a well-ordering, α be an ordinal, and $f : \alpha \rightarrow A$, be any function. We say that f is a continuous function if and only if for every $B \subseteq \alpha$ that is bounded in α we have $f(\sup(B)) = \sup\{f(\beta) : \beta \in B\}$. If f is a strictly increasing, then f is continuous if and only if for every limit ordinal $\gamma < \alpha$, we have $f(\gamma) = \sup\{f(\beta) : \beta < \gamma\}$. We say that f is a normal function if and only if it is both continuous and strictly increasing (preserves the ordering).

Definition 1.2.15. Let θ be a limit ordinal with $\text{cf}(\theta) > \omega$. A subset S of θ is called a stationary set in θ if and only if it has a nonempty intersection with any club in θ .

The following definitions are from [6].

Definition 1.2.16. Let $G = (M, W, E)$ be a bipartite graph with $V = M \cup W$, and α be an ordinal. A string in G is a transfinite sequence $f : \alpha \rightarrow V$ that is injective.

Definition 1.2.17. Let $G = (M, W, E)$ be a bipartite graph with $V = M \cup W$, and $f : \alpha \rightarrow V$ be an injective function where α is an ordinal. We say that f is saturated at $\beta < \alpha$ if and only if

$$f(\beta) \in M \text{ implies that } E(f(\beta)) \subseteq \{f(\gamma) : \gamma < \beta\}.$$

We say that f is saturated in G iff it is saturated at every $\beta < \alpha$.

Definition 1.2.18. Let $G = (M, W, E)$ be a bipartite graph with $V = M \cup W$, and $E \subseteq M \times W$, and $f : \alpha \rightarrow V$ be a saturated injective sequence in G . The μ -margin of f (denoted $\mu(f)$) is an element of $\mathbb{Z}_\infty = \mathbb{Z} \cup \{\infty, -\infty\}$ defined by transfinite induction on α as follows:

If $\alpha = 0$, then $\mu(f) = 0$,

If $\alpha = \beta + 1$, then

$$\mu(f) = \begin{cases} \mu(f \upharpoonright_\beta) + 1 & \text{if } f(\beta) \in W; \\ \mu(f \upharpoonright_\beta) - 1 & \text{if } f(\beta) \in M. \end{cases}$$

If α is a limit ordinal, then $\mu(f) = \liminf_{\beta \rightarrow \alpha} \mu(f \upharpoonright_\beta)$.

Definition 1.2.19. Let G be a bipartite graph. G is a μ -admissible if and only if for every saturated string f in G we have $\mu(f) \geq 0$, where $\mu(f)$ is the μ -margin of f .

The following lemma is from [4].

Lemma 1.2.20. *If θ is a limit ordinal, $A \subseteq \theta$ is an unbounded set in θ , and $g : \eta \rightarrow A$ is an order preserving bijective function where $\eta \in \text{ord}$, then η is also a limit ordinal with $\text{cf}(\eta) = \text{cf}(\theta)$.*

The following definitions are from [3].

Definition 1.2.21. Let $G = (M, W, E)$ be a bipartite graph. A subgraph $G' = (M', W', E')$ of G is saturated if and only if it is induced ($E' = E \cap (M' \times W')$) and $E[M'] \subseteq W'$.

Definition 1.2.22. Let $G = (M, W, E)$ be a bipartite graph. If $M' \subseteq M$, and $W' \subseteq W$, then $G(M', W')$ is the subgraph of G induced by $M' \cup W'$.

Definition 1.2.23. Let $G = (M, W, E)$ be a bipartite graph, and $\mathcal{L} = \{L_\beta : \beta < \eta\}$ be a family of subgraphs $L_\beta = (M_\beta, W_\beta, E_\beta)$ of G for each $\beta < \eta$, where η is a limit ordinal. The union and join of \mathcal{L} are defined respectively by

$$\bigcup \{L_\beta : \beta < \eta\} = \left(\bigcup_{\beta < \eta} M_\beta, \bigcup_{\beta < \eta} W_\beta, \bigcup_{\beta < \eta} E_\beta \right),$$

and

$$\bigvee \{L_\beta : \beta < \eta\} = G \left(\bigcup_{\beta < \eta} M_\beta, \bigcup_{\beta < \eta} W_\beta \right).$$

Definition 1.2.24. Let G be a bipartite graph and $A = (M_1, W_1, E_1)$, $B = (M_2, W_2, E_2)$ be two subgraphs of G . The join, union, and difference of A and B are defined respectively by $A \vee B = G(M_1 \cup M_2, W_1 \cup W_2)$, $A \cup B = (M_1 \cup M_2, W_1 \cup W_2, E_1 \cup E_2)$, and $A \setminus B = A(M_1 \setminus M_2, W_1 \setminus W_2)$.

Definition 1.2.25. Let $G = (M, W, E)$ be a bipartite graph. A sequence $\mathcal{G} = (G_\alpha : \alpha \leq \xi)$ of saturated subgraphs of G is a ξ -tower in G if and only if G_α is a subgraph of G_β for each $\alpha < \beta \leq \xi$, and for every limit ordinal $\alpha \leq \xi$ the graph G_α is the union of all G_β with $\beta < \alpha$.

Definition 1.2.26. Let $G = (M, W, E)$ be a bipartite graph. A ξ -ladder in G is a sequence $\mathcal{L} = (L_\beta : \beta < \xi)$ of pairwise disjoint subgraphs of G such that $\mathcal{G} = (G_\alpha : \alpha \leq \xi)$ is a ξ -tower, where G_α is the join of all L_β for each $\beta < \alpha$.

$\mathcal{G} = (G_\alpha : \alpha \leq \xi)$ is called the tower associated with \mathcal{L} .

Every L_β is a rung of the ladder \mathcal{L} . If $\mathcal{G} = (G_\alpha : \alpha \leq \xi)$ is a ξ -tower, then $\mathcal{L} = (L_\beta : \beta < \xi)$ is the associated ξ -ladder where $L_\beta = G_{\beta+1} \setminus G_\beta$ for each $\beta < \xi$.

$\mathcal{G}(\mathcal{L})$ is called a tower (ladder) if it is a ξ -tower (ξ -ladder) for some ordinal ξ .

Definition 1.2.27. Let $G = (M, W, E)$ be a bipartite graph and $G' = (M', W', E')$ be a subgraph of G . We say G' is critical in G (or just critical) if and only if G' has a matching from W' to M' and each such matching using all vertices in M' .

Definition 1.2.28. Let $G = (M, W, E)$ be a bipartite graph and

$$K = \{1\} \cup \{\eta : \eta > \aleph_0, \text{ and } \eta \text{ is a regular cardinal}\}.$$

We will define a η -obstruction for each $\eta \in K$. The graph G is a 1-obstruction if $G \setminus \{a\}$ is critical for some $a \in M$. If $\eta \in K \setminus \{1\}$, then G is a η -obstruction if there is a η -ladder $\mathcal{L} = (L_\alpha : \alpha < \eta)$ in G such that G is the union of all rungs L_α for $\alpha < \eta$, and the following properties hold:

1. For each $\alpha < \eta$, the rung L_α of \mathcal{L} is either a μ -obstruction for some $\mu \in K \cap \eta$ or is of the form $(\emptyset, \{w\}, \emptyset)$ for some $w \in W$. We will say that L_α is trivial in the second case.
2. The set $S = \{\alpha < \eta : L_\alpha \text{ is a } \mu\text{-obstruction for some } \mu \in K \cap \eta\}$ is stationary in η .

G is said to be an obstruction if there exists a $\eta \in K$ such that G is a η -obstruction. We say that G' is an obstruction in the bipartite graph G if G' is a saturated subgraph of G and G' is an obstruction. We say that G has an obstruction if and only if there exists an obstruction G' in G .

The following definition is from [1].

Definition 1.2.29. Let $G = (M, W, E)$ be a bipartite graph. We say that G is c-admissible if and only if G has no 1-obstruction.

Nash-Williams defined the concept of q-admissibility for a bipartite graph G (see [1, 6]).

Theorem 1.2.30. [Aharoni [1]] *A bipartite graph is c-admissible if and only if it is q-admissible.*

Theorem 1.2.31. [Wojciechowski [6]] *A bipartite graph is q -admissible if and only if it is μ -admissible.*

The following definitions are from [7].

Definition 1.2.32. Let I be an alphabet. A word over I is a function $u : \alpha \rightarrow I$, where $\alpha \in \text{ord}$.

The following is a definition of the operation of concatenation for words.

Definition 1.2.33. Let $(u_\beta)_{\beta < \alpha}$ be a sequence of words over the alphabet I such that $u_\beta : \sigma_\beta \rightarrow I$, and σ_β is an ordinal for each $\beta < \alpha$. The concatenation of this sequence denoted by $\circ(u_\beta)_{\beta < \alpha}$ is a word $u : \bar{\sigma}_\alpha \rightarrow I$, where $\bar{\sigma}_\alpha = \sum_{\beta < \alpha} \sigma_\beta$, such that

$$u(\delta) = u_\gamma(\zeta) \text{ for each } \delta < \bar{\sigma}_\alpha,$$

where $\gamma = \max\{\beta < \alpha : \delta \geq \bar{\sigma}_\beta\}$ and ζ is such that $\delta = \bar{\sigma}_\gamma + \zeta$.

If $\alpha = n$ is a finite, then we will write $u_0 \circ u_1 \circ \dots \circ u_{n-1}$ instead of $\circ(u_\beta)_{\beta < \alpha}$.

The following is a definition of the operation of concatenation for classes of words.

Definition 1.2.34. Let A and B be two classes of words over the same alphabet. The class $A \circ B$ is defined as follows:

$$A \circ B = \{u_0 \circ u_1 : u_0 \in A, \text{ and } u_1 \in B\}.$$

The following are definitions of the operations of raising to the power $*$, and $\#$ for classes of words.

Definition 1.2.35. If A is a class of words and α is an ordinal, then

$$A^\alpha = \{u : u = \circ(u_\beta)_{\beta < \alpha}, u_\beta \in A \text{ for each } \beta < \alpha\},$$

$$A^* = \bigcup_{n < \omega} A^n,$$

$$A^\# = \bigcup_{\delta \in \text{ord}} A^\delta.$$

We prove the following lemmas and proposition and we use them in proving the main results of this thesis.

Lemma 1.2.36. *If D is any set of ordinals and $\text{sup}(D)$ is a successor ordinal, then $\text{sup}(D) \in D$.*

Proof. Assume that D is any set of ordinals with $\text{sup}(D)$ is a successor ordinal. Thus $\text{sup}(D) = \beta + 1$ for some ordinal β . Then β is not an upper bounded on D , so there is an element $\gamma \in D$ such that $\beta < \gamma$. Now since $\text{sup}(D) = \beta + 1$, we have $\gamma \leq \beta + 1$. Thus $\gamma = \beta + 1$ so $\text{sup}(D) \in D$. \square

Proposition 1.2.37. *Let $\theta = \sum_{\beta < \eta} \alpha_\beta$, such that $\eta \in \text{lim}$, and $\alpha_\beta \in \text{ord}$, $\alpha_\beta > 0$, for each $\beta < \eta$. Then $\text{cf}(\theta) = \text{cf}(\eta)$.*

Proof. Assume $\text{cf}(\eta) = \delta$ and we want to prove $\text{cf}(\theta) \leq \delta$. Since $\text{cf}(\eta) = \delta$, then there exists a function $f : \delta \rightarrow \eta$, such that $\text{rng}(f)$ is unbounded in η . Now we want to define a function $g : \delta \rightarrow \theta$, with range of g unbounded in θ . Define g as follows:

$$g(\gamma) = \sum_{\beta < f(\gamma)} \alpha_\beta, \text{ for each } \gamma < \delta.$$

Now we want to prove $\text{rng}(g)$ is unbounded in θ . Let $\alpha < \theta$ and since $\theta = \sum_{\beta < \eta} \alpha_\beta$, then $\alpha < \sum_{\beta < \delta} \alpha_\beta$, for some $\delta < \eta$, but $\text{rng}(f)$ unbounded in η , so we get $\sigma < \delta$, with $f(\sigma) > \delta$. Therefore

$$g(\sigma) = \sum_{\beta < f(\sigma)} \alpha_\beta > \sum_{\beta < \delta} \alpha_\beta > \alpha.$$

Therefore, $\text{rng}(g)$ is unbounded in θ . Thus implies $\text{cf}(\theta) \leq \delta$.

Now assume $\text{cf}(\theta) = \delta$ and we want to prove $\text{cf}(\eta) \leq \delta$. Since $\text{cf}(\theta) = \delta$, then there exists a function $f : \delta \rightarrow \theta$, with $\text{rng}(f)$ unbounded in θ . Now we want to define a function $g : \delta \rightarrow \eta$, with $\text{rng}(g)$ unbounded in η . Define g as follows:

$$g(\gamma) = \delta, \text{ if and if } f(\gamma) \geq \sum_{\beta < \delta} \alpha_\beta, \text{ for each } \gamma < \delta,$$

where $\delta < \eta$, smallest such ordinal.

Now we want to prove $\text{rng}(g)$ is unbounded in η . Let $\alpha < \eta$, then $\alpha' = \sum_{\beta < \alpha} \alpha_\beta < \theta$, but $\text{rng}(f)$ is unbounded in θ , so we get $\sigma < \delta$, with $f(\sigma) > \alpha'$. Assume

$$f(\sigma) = \sum_{\beta < \lambda} \alpha_\beta,$$

for some $\lambda < \eta$. Therefore, $g(\sigma) = \lambda$ and since

$$f(\sigma) = \sum_{\beta < \lambda} \alpha_\beta > \sum_{\beta < \alpha} \alpha_\beta = \alpha',$$

hence $g(\sigma) = \lambda > \alpha$. Therefore $\text{rng}(g)$ is unbounded in η . Thus implies $\text{cf}(\eta) \leq \delta$. Thus $\text{cf}(\eta) = \text{cf}(\theta)$. \square

Lemma 1.2.38. *If $\theta = \sum_{\beta < \eta} \theta_\beta$, where $\eta \in \text{lim}$, with $\text{cf}(\eta) > \omega$, and $\theta_\beta > 0$, for each $\beta < \eta$,*

then

$$\delta = \left\{ \sum_{\beta < \sigma} \theta_\beta : \sigma < \eta \right\}$$

is a club in θ .

Proof. We want to prove δ is unbounded and closed in θ . First we want to prove δ is unbounded in θ . Let $\alpha < \theta$. Then for some $\sigma < \eta$, $\alpha < \sum_{\beta < \sigma} \theta_\beta$, and since $\eta \in \text{lim}$, then there is $\gamma < \eta$ with $\gamma > \sigma$, so

$$\delta \ni \sum_{\beta < \gamma} \theta_\beta > \sum_{\beta < \sigma} \theta_\beta > \alpha.$$

Therefore δ is unbounded in θ .

Now we want to prove δ is closed in θ . Let $D \subseteq \delta$, with $|D| < \theta$. we want to prove $\sup(D) \in \delta$. If $\sup(D) \in D$, then done. Assume $\sup(D) \notin \delta$. Then $\sup(D) \in \text{lim}$, by lemma 1.2.36. Now, since $|D| < \theta$, then D is bounded by some ordinal in θ . Thus,

$$D = \left\{ \sum_{\beta < \sigma} \theta_\beta : \sigma \in \Gamma \right\}, \text{ for some } \Gamma \subseteq \eta.$$

So Γ is bounded by some ordinal in η , then $\sigma = \sup(\Gamma) < \eta$. Thus $\sup(D) = \sum_{\beta < \sigma} \theta_\beta \in \delta$. Hence, D is closed in θ . Therefore, D is a club in θ . \square

Lemma 1.2.39. Let α be a limit ordinal, S be a finite set, and $f : \alpha \rightarrow S$ be any function. If

$$\{\beta < \alpha : f(\beta) \in S\}$$

is unbounded in α , then there exists $s \in S$ such that

$$\{\beta < \alpha : f(\beta) = s\}$$

is unbounded in α .

Lemma 1.2.40. Let $\beta = \sigma + \theta$ be any ordinal such that $\theta \neq 0$. Then

1. $\beta \in \text{lim}$ if and only if $\theta \in \text{lim}$.
2. If β , and θ are limit ordinals, then $\text{cf}(\beta) = \text{cf}(\theta)$.

Proof. Assume $\beta = \sigma + \theta$ is an ordinal with $\theta \neq 0$.

Proof of (1). $\beta \in \text{lim}$ if and if $\theta \in \text{lim}$.

Assume first $\beta \in \text{lim}$, and we want to prove $\theta \in \text{lim}$. By way of a contradiction let $\theta \in \text{succ}$, then $\theta = \delta + 1$, for some $\delta \in \text{ord}$. Thus, $\beta = \sigma + \theta = \sigma + \delta + 1$, so $\beta \in \text{succ}$, which is a contradiction because $\beta \in \text{lim}$. Conversely, assume $\theta \in \text{lim}$, and we want to prove $\beta \in \text{lim}$. Again assume by way of a contradiction $\beta \in \text{succ}$, then $\beta = \delta + 1$, for some $\delta \in \text{ord}$. Thus, $\beta = \sigma + \theta = \delta + 1$, so $\theta = \gamma + 1$, for some $\gamma \in \text{ord}$, thus $\theta \in \text{succ}$, which is a contradiction because $\theta \in \text{lim}$.

Proof of (2). If β , and θ are limit ordinals, then $\text{cf}(\beta) = \text{cf}(\theta)$.

Assume β and θ are limit ordinals. First we want to prove

$$\text{cf}(\beta) \leq \text{cf}(\theta).$$

Assume $\text{cf}(\beta) = \delta$, for some $\delta \in \text{ord}$, and we want to prove $\delta \geq \text{cf}(\theta)$. Now since $\text{cf}(\beta) = \delta$, then there is a function $g : \delta \rightarrow \beta$, with $\text{ran}(g)$ unbounded in β . Now define $f : \delta \rightarrow \theta$, as follows:

$$f(\gamma) = \begin{cases} \sigma + 1 & \text{if } g(\gamma) \leq \sigma \\ g(\gamma) & \text{if } g(\gamma) > \sigma \end{cases}.$$

It remains to prove $\text{rng}(f)$ is unbounded in θ . Let $\alpha < \theta$ and we want to prove there is $\lambda \in \text{rng}(f)$, with $\lambda > \alpha$. Now since $\alpha < \theta$, then $\alpha < \beta = \sigma + \theta$, so there is an ordinal $\nu < \delta$, with $g(\nu) > \alpha$, because $\text{rng}(g)$ is unbounded in β . Then either $g(\nu) \leq \sigma$, or $g(\nu) > \sigma$.

Assume $g(\nu) \leq \sigma$, then $\lambda = f(\nu) = \sigma + 1 > \alpha$, and if $g(\nu) > \sigma$, then $\lambda = f(\nu) = g(\nu) > \alpha$. Hence, $\text{rng}(f)$ is unbounded in θ . Therefore, $\delta \geq \text{cf}(\theta)$.

Now we want to prove $\text{cf}(\beta) \geq \text{cf}(\theta)$.

Assume $\text{cf}(\theta) = \delta$, for some $\delta \in \text{ord}$, and we want to prove $\delta \leq \text{cf}(\beta)$. Now since $\text{cf}(\theta) = \delta$, then there is a function $g : \delta \rightarrow \theta$, with $\text{rng}(g)$ unbounded in θ . Now define $f : \delta \rightarrow \beta$, as follows:

$$f(\gamma) = g(\gamma), \text{ for each } \gamma < \delta.$$

It remains to prove $\text{rng}(f)$ is unbounded in β . Let $\alpha < \beta$ and we want to prove there is $\lambda \in \text{rng}(f)$, with $\lambda > \alpha$. Now since $\alpha < \beta = \sigma + \theta$, so either $\alpha > \sigma$, or $\alpha \leq \sigma$.

Assume first $\alpha > \sigma$, then $\alpha < \theta$, so there is an ordinal $\nu < \delta$, with $g(\nu) > \alpha$, because $\text{rng}(g)$ is unbounded in θ , thus $\lambda = f(\nu) = g(\nu) > \alpha$.

Now let $\alpha \leq \sigma$. $\sigma + 1 < \theta$ and since $\text{rng}(g)$ is unbounded in θ , then there is $\epsilon < \delta$ with $g(\epsilon) > \sigma + 1$, but $\lambda = f(\epsilon) = g(\epsilon) > \sigma + 1 > \sigma \geq \alpha$. Hence $\text{rng}(f)$ is unbounded in β . Therefore, $\delta \geq \text{cf}(\beta)$. \square

1.3 Main Definitions And Results

Let I be an alphabet. A word over I is a function $u : \alpha \rightarrow I$, where α is an ordinal. The class of all words u over I is denoted by $I^\#$.

There are different methods to characterize formal languages. In this thesis we indicate two of them, namely ST-automata and operations on ST-automata. We will have in view the generalization of those methods for describing subclasses of $I^\#$.

Several definitions of automata over words that are defined on ordinals have been proposed previously.

In chapter two in our thesis, we introduce a new concept of automata, namely, stationary automaton or ST-automata and this concept is analogous to the definition of TF-automaton by Wojciechowski [7].

In chapter two first we introduce the following definition:

Definition. (2.1.1): Let α be an ordinal, S be any set of states and $R : \alpha \rightarrow S$ be any function. Then for each $\beta \in \text{lim}$, $\beta \leq \alpha$, we define the following:

$$\text{sup}_\beta(R) = \{s \in S : \{\gamma < \beta : R(\gamma) = s\} \text{ is cofinal in } \beta\}.$$

Also, in chapter three we define the following:

(3.1.1): Let α be an ordinal, S be any set of states, $\mathcal{P}(S)$ be the set of all subsets of S and $H : \alpha \rightarrow S \cup \mathcal{P}(S)$ be any function. Then for each $\beta \in \text{lim}$, $\beta \leq \alpha$, we define

$$\text{sup}'_\beta(H) = \text{sup}_\beta(H) \cap S.$$

Wojciechowski [7], defined TF-quasiautomaton, that is, a TF-quasiautomaton is a system $Q = (S, I, T)$, where S is a finite set of states, I is a finite alphabet and

$$T \subset \{S \cup \mathcal{P}(S)\} \times I \times S$$

is the set of transitions. He also defined a TF-automaton, that is, a TF-automaton over I is a system $\mathcal{A} = (S, I, T, \psi, \mathcal{F})$, where $Q = (S, I, T)$ is a TF-quasiautomaton, $\psi \in S \cup \mathcal{P}(S)$ is the initial situation and $\mathcal{F} \subset S \cup \mathcal{P}(S)$ is the set of final situations. Also, he defined an accepting run of TF-automaton, that is, a run of \mathcal{A} on u where $u : \alpha \rightarrow I$, and $\alpha \in \text{ord}$, (called II-run in [7]) is a function $H : \alpha + 1 \rightarrow S \cup \mathcal{P}(S)$ such that:

1. $H(0) = \psi$.
2. $H(\beta) \in S$, for every successor ordinal $\beta < \alpha$.

3. $H(\beta) = \sup'_\beta(H)$ for every limit ordinal $\beta \leq \alpha$.
4. $(H(\beta), u(\beta), H(\beta + 1)) \in T$, for every $\beta < \alpha$.

A run H of \mathcal{A} on u is an accepting run if and only if $H(\alpha) \in \mathcal{F}$.

The language of \mathcal{A} denoted by $\mathcal{L}(\mathcal{A})$ is a class of all words u in $I^\#$ such that there exists an accepting run of \mathcal{A} on u .

In our work, we modify the TF-automaton definition and incorporate stationary subsets of limit ordinals of uncountable cofinality to get to the new concept to the automata.

In the following several chapters, we will present the following main definitions, and results.

1. In chapter two, first we introduce the following definitions, a ST-automaton over an alphabet concept (see definition 2.1.3), a word over an alphabet (see definition 2.1.4), an accepting run of ST-automaton, and the language of ST-automaton \mathcal{A} denoted by $\mathcal{L}(\mathcal{A})$ (see definition 2.1.5), and the language for a bipartite graph G denoted by $\mathcal{L}(G)$, (see definition 2.1.7).

Then we combining the results from Aharoni [1], Wojciechowski [6], and Aharoni, Nash-Williams, Shelah [2], we show that this new concept of the automaton allows us to make a connection with graph theory, and we prove the following main theorem by using Corollary 2.2.1, and Theorem 2.2.2.

Theorem (2.2.3): There exists an ST-automaton \mathcal{A} over an alphabet $I = \{M, W\}$ such that for every bipartite graph $G = (M, W, E)$ with $|M \cup W| \leq \aleph_1$ the following are equivalent:

- (a) G has a matching.
- (b) $\mathcal{L}(G) \cap \mathcal{L}(\mathcal{A}) = \emptyset$.

2. In chapter three, we introduce a new concept of automaton which we call a special ST-automaton (see definition 3.1.4). The aim of introducing this concept is to compare TF-automata in [7] and ST-automata. First we prove the following main result:

Theorem (3.2.1): Let I be a finite alphabet and \mathcal{C} be a subclass of $I^\#$. Then the following conditions are equivalent:

- (a) $\mathcal{C} = \mathcal{L}(\mathcal{A}')$ where \mathcal{A}' is a TF-automaton over I .
- (b) $\mathcal{C} = \mathcal{L}(\mathcal{A})$ where \mathcal{A} is a special ST-automaton over I .

Then we show that in general the concept of ST-automata is stronger than the concept of TF-automata by using the above theorem, giving a counter example 3.2.2, and proving the following theorem:

Theorem (3.2.3): There does not exist a special automaton $\mathcal{A} = (S, I, T, Z, F)$, over an alphabet $I = \{a\}$ such that $\mathcal{L}(\mathcal{A}) = \{u\}$, where $u : \omega_1 \rightarrow I$, and S is a countable set.

3. Wojciechowski [8], defined the operation of concatenation for classes of words (see definition 1.2.34), and the operations of raising to the power $*$ and $\#$ for classes of words (see definition 1.2.35). He also in [7] defined such operations on TF-automata as union, intersection, concatenation, and raising to the powers ω , $*$, $\#$. He used these definitions to prove the following theorem:

Theorem: If \mathcal{A} and \mathcal{A}' are TF-automata then:

- $\mathcal{L}(\mathcal{A} \cup \mathcal{A}') = \mathcal{L}(\mathcal{A}) \cup \mathcal{L}(\mathcal{A}')$.
- $\mathcal{L}(\mathcal{A} \cap \mathcal{A}') = \mathcal{L}(\mathcal{A}) \cap \mathcal{L}(\mathcal{A}')$.
- $\mathcal{L}(\mathcal{A} \circ \mathcal{A}') = \mathcal{L}(\mathcal{A}) \circ \mathcal{L}(\mathcal{A}')$.
- $\mathcal{L}(\mathcal{A}^*) = (\mathcal{L}(\mathcal{A}))^*$.
- $\mathcal{L}(\mathcal{A}^\omega) = (\mathcal{L}(\mathcal{A}))^\omega$.
- $\mathcal{L}(\mathcal{A}^\#) = (\mathcal{L}(\mathcal{A}))^\#$.

In chapter four, in an analogous way we introduce definitions of operations for ST-automata and we use these definitions to prove the corresponding theorem, that Wojciechowski proved in [7] on classes of words accepted by TF-automaton (above theorem). That means, we show that applying those operations to languages defined by ST-automata produce languages that are also definable using ST-automata as follows:

- First we define the union ST-automaton (see definition 4.1.1), and we prove the following main result:

Theorem (4.1.2): Let $\mathcal{A} = (S, I, T, Z, F)$ and $\mathcal{A}' = (S', I, T', Z', F')$ be two ST-automata over I , such that $S \cap S' = \emptyset$. Then $\mathcal{L}(\mathcal{A} \cup \mathcal{A}') = \mathcal{L}(\mathcal{A}) \cup \mathcal{L}(\mathcal{A}')$.

- Second we introduce definition of the intersection ST-automaton (see definition 4.2.1), and we prove the following main result:

Theorem (4.2.2): Let $\mathcal{A} = (S, I, T, Z, F)$ and $\mathcal{A}' = (S', I, T', Z', F')$ be two ST-automata over I such that S and S' are finite sets of states. Then $\mathcal{L}(\mathcal{A} \cap \mathcal{A}') = \mathcal{L}(\mathcal{A}) \cap \mathcal{L}(\mathcal{A}')$.

- Also, we define the concatenation ST-automaton (see definition 4.3.1), and we get to the following main theorem:

Theorem (4.3.2): Let $\mathcal{A} = (S, I, T, Z, F)$ and $\mathcal{A}' = (S', I, T', Z', F')$ be two ST-automata over I such that $S \cap S' = \emptyset$. Then $\mathcal{L}(\mathcal{A} \circ \mathcal{A}') = \mathcal{L}(\mathcal{A}) \circ \mathcal{L}(\mathcal{A}')$.

- After that, we introduce the definition of ST-automaton, that is, raising to the power $*$ of ST-automaton (see definition 4.4.1) and we prove the following theorem:

Theorem (4.4.2): If $\mathcal{A} = (S, I, T, Z, F)$ is a ST-automaton over I , with Z is a finite set, then $\mathcal{L}(\mathcal{A}^*) = (\mathcal{L}(\mathcal{A}))^*$.

- Subsequently, we define a ST-automaton, that is, raising to the power ω of ST-automaton (see definition 4.5.1) and we prove the following theorem:

Theorem (4.5.2): If $\mathcal{A} = (S, I, T, Z, F)$ is a ST-automaton over I , with Z is a finite set, then $\mathcal{L}(\mathcal{A}^\omega) = (\mathcal{L}(\mathcal{A}))^\omega$.

- Finally, we introduce the definition of a ST-automaton, that is, raising to the power $\#$ of ST-automaton (see definition 4.6.1) and we prove the following main result:

Theorem (4.6.2): If $\mathcal{A} = (S, I, T, Z, F)$ is a ST-automaton over I , with Z is a finite set, then $\mathcal{L}(\mathcal{A}^\#) = (\mathcal{L}(\mathcal{A}))^\#$.

Chapter 2

ST-automata

In this chapter, we introduce ST-automaton (stationary automaton) over an alphabet concept and define an accepting run of ST-automaton. Then we prove motivating result for ST-automata.

2.1 Basic Definitions

First we define stationary automata.

Definition 2.1.1. Let α be an ordinal, S be any set of states and $R : \alpha \rightarrow S$ be any function. For each $\beta \in \text{lim}$, $\beta \leq \alpha$ we define the following:

1. $\text{sup}_\beta(R) = \{s \in S : \{\gamma < \beta : R(\gamma) = s\} \text{ is cofinal in } \beta\}$.
2. $\text{stat}_\beta(R) = \{s \in S : \{\gamma < \beta : R(\gamma) = s\} \text{ is stationary in } \beta\}$, whenever $\text{cf}(\beta) > \omega$.

Definition 2.1.2. A ST-quasiautomaton over I is a system $Q = (S, I, T)$, where S is a set of states, I is an alphabet and

$$T \subseteq (S \times I \times S) \cup (\mathcal{P}(S) \times S) \cup (\mathcal{P}(S) \times \mathcal{P}(S) \times S)$$

is a set of transitions.

Definition 2.1.3. A stationary automaton over I (ST-automaton) is a system $\mathcal{A} = (S, I, T, Z, F)$, where $Q = (S, I, T)$ is a ST-quasiautomaton, denoted by $Q(\mathcal{A})$, $Z \subseteq S$ is a set of initial states and $F \subseteq S$ is a set of final states.

Definition 2.1.4. Let I be an alphabet. A word over I is a function $u : \alpha \rightarrow I$, where $\alpha \in \text{ord}$. The class of all words u over I is denoted by $I^\#$.

Now we define an accepting run of ST-automaton on a word and the language of ST-automaton.

Definition 2.1.5. Let $\mathcal{A} = (S, I, T, Z, F)$ be a ST-automaton over I and $u : \alpha \rightarrow I$ be a word over I , $\alpha \in \text{ord}$. A run of \mathcal{A} on u is a function $R : \alpha + 1 \rightarrow S$ such that:

1. For each $\beta < \alpha$, we have

$$(R(\beta), u(\beta), R(\beta + 1)) \in T.$$

2. For each $\beta \leq \alpha$, that is a limit ordinal with $\text{cf}(\beta) = \omega$, we have

$$(\sup_{\beta} R, R(\beta)) \in T.$$

3. For each $\beta \leq \alpha$, that is a limit ordinal with $\text{cf}(\beta) > \omega$, we have

$$(\sup_{\beta} R, \text{stat}_{\beta} R, R(\beta)) \in T.$$

A partial run of \mathcal{A} on u is a function $R : \alpha \rightarrow S$ such that (1) holds for all β such that $\beta + 1 < \alpha$ and (2),(3) hold for all $\beta < \alpha$.

A run $R : \alpha + 1 \rightarrow S$ of \mathcal{A} on u is an initial run if and only if $R(0) \in Z$, and a final run if and only if $R(\alpha) \in F$. A run of \mathcal{A} on u is an accepting run if and only if it is both initial and final.

Definition 2.1.6. The language of ST-automaton \mathcal{A} over I ($\mathcal{L}(\mathcal{A})$) is a class of all words u in $I^{\#}$ such that there exists an accepting run of \mathcal{A} on u .

The following is a definition of the language of a bipartite graph.

Definition 2.1.7. Let $G = (M, W, E)$ be a bipartite graph, where $E \subseteq M \times W$, $I = \{M, W\}$ be an alphabet and $\mathcal{L}(G)$ be a set of transfinite words over I define as follows:

let $f : \alpha \rightarrow V$ be an injective function from $\alpha \in \text{ord}$ into $V = M \cup W$. We say that f is a saturated at $\beta < \alpha$ if and only if

$$\text{if } f(\beta) \in M, \text{ implies that } E(f(\beta)) \subseteq \{f(\gamma) : \gamma < \beta\}.$$

We say that f is saturated in G if and only if it is saturated at every $\beta < \alpha$.

Now for each saturated sequence $f : \alpha \rightarrow V$, we assign the word $u = u(f) : \alpha \rightarrow I$ such that for each $\beta < \alpha$:

$$u(\beta) = \begin{cases} W & \text{if } f(\beta) \in W; \\ M & \text{if } f(\beta) \in M. \end{cases}$$

Define

$$\mathcal{L}(G) = \{u(f) : f \text{ is a saturated string in } G\}.$$

2.2 Motivation Result

We obtain the following corollary from definitions 1.2.19 and 1.2.29, and theorems 1.2.30 and 1.2.31.

Corollary 2.2.1. *Let G be a bipartite graph. Then G has a 1-obstruction if and only if there exists a saturated string f in G such that $\mu(f) < 0$.*

Theorem 2.2.2. *[Aharoni, Nash-Williams, Shelah [2]] A bipartite graph G has a matching if and only if there exists no obstruction in G .*

Now we prove the following motivating theorem of this thesis by using the above corollary and theorem.

Theorem 2.2.3. *There exists a ST-automaton \mathcal{A} over an alphabet $I = \{M, W\}$ such that for every bipartite graph $G = (M, W, E)$ with $|M \cup W| \leq \aleph_1$ the following are equivalent:*

1. G has a matching.
2. $\mathcal{L}(G) \cap \mathcal{L}(\mathcal{A}) = \emptyset$.

Proof. **Definition of \mathcal{A} .**

We define ST-automaton $\mathcal{A} = (S, I, T, Z, F)$ as follows :

The set of states is

$$S = \{\hat{0}, \hat{1}, \hat{2}, \dots\} \cup \{\bar{0}, \bar{1}, \bar{2}, \dots\} \cup \{\textcircled{S}, N, \textcircled{R}\},$$

the alphabet is

$$I = \{W, M\},$$

the set of initial states is

$$Z = \{\bar{0}, \textcircled{S}, N\},$$

the set of final states is

$$F = \{\textcircled{R}\},$$

and the set of transitions

$$T \subseteq (S \times I \times S) \cup (\mathcal{P}(S) \times S) \cup (\mathcal{P}(S) \times \mathcal{P}(S) \times S)$$

is defined as follows:

The fact that $(s, a, t) \in S \times I \times S$ belongs to T will be denoted by $s \xrightarrow{a} t$. We take:

$$N \xrightarrow{W} N, N \xrightarrow{W} \textcircled{S}, \textcircled{S} \xrightarrow{W} \hat{1}, \textcircled{S} \xrightarrow{M} N, \textcircled{S} \xrightarrow{M} \textcircled{S}, \bar{0} \xrightarrow{M} \textcircled{R}, \hat{0} \xrightarrow{M} N, \hat{0} \xrightarrow{M} \textcircled{S}.$$

For each $i \geq 0$, we take

$$\bar{i} \xrightarrow{W} \overline{i+1}, \text{ and } \hat{i} \xrightarrow{W} \widehat{i+1}.$$

For each $i > 0$, we define

$$\bar{i} \xrightarrow{M} \overline{i-1}, \text{ and } \hat{i} \xrightarrow{M} \widehat{i-1}.$$

If $C \subseteq \{0, 1, \dots\}$, then we will denote

$$\bar{C} = \{\bar{i} : i \in C\}, \text{ and } \hat{C} = \{\hat{i} : i \in C\}.$$

To denote that a pair $(A, s) \in \mathcal{P}(S) \times S$ belongs to T , we will write $A \longrightarrow s$.

If $C \neq \emptyset$ define:

$$\bar{C} \rightarrow \overline{\min(C)} \text{ and } \hat{C} \rightarrow \widehat{\min(C)}.$$

Moreover, if $A \subseteq S$ and $A \cap \{N, \textcircled{S}\} \neq \emptyset$, then

$$A \longrightarrow N \text{ and } A \longrightarrow \textcircled{S}.$$

To denote that a triple $(A, B, s) \in \mathcal{P}(S) \times \mathcal{P}(S) \times S$ belongs to T , we will write $(A, B) \longrightarrow s$.

Define

$$(\bar{C}, \bar{D}) \rightarrow \overline{\min(C)} \text{ and } (\hat{C}, \hat{D}) \rightarrow \widehat{\min(C)}.$$

Moreover, if $\textcircled{S} \in B$, then

$$(A, B) \rightarrow \textcircled{R}.$$

(2) implies (1). Assume that (1) is false, that is, G has no matching. We want to define a transfinite word $u \in I^\#$ such that $u \in \mathcal{L}(G) \cap \mathcal{L}(\mathcal{A})$; thus, we need a saturated string f in G such that $u(f) \in \mathcal{L}(\mathcal{A})$. By Theorem 2.2.2 there is a η -obstruction G' in G for some $\eta \in K$. Since G has \aleph_1 vertices, it follows that $\eta \in \{1, \aleph_1\}$. We will define a saturated string f in G for $\eta \in \{1, \aleph_1\}$.

The case when $\eta = 1$.

Assume that $\eta = 1$. Let f be a saturated string in G such that $\mu(f) < 0$. Such f exists by Corollary 2.2.1. Without loss of generality, we can assume that $\mu(f \upharpoonright \delta) \geq 0$ for each $\delta < \text{dom}(f)$. Let $\theta = \text{dom}(f)$ and $u = u(f) : \theta \rightarrow I$. Define an accepting run $R : \theta + 1 \rightarrow S$ of \mathcal{A}

on u as follows:

The choice of f implies that, θ is a successor ordinal. Let $\theta = \beta + 1$ and

$$R(\gamma) = \overline{\mu(f \upharpoonright \gamma)} \text{ for each } \gamma < \theta, \text{ and}$$

$$R(\theta) = \mathbb{R}.$$

It is clear that $R(0) = \bar{0} \in Z$. It remains to satisfy the following conditions:

1. For each $\alpha < \theta$ we have

$$(R(\alpha), u(\alpha), R(\alpha + 1)) \in T.$$

2. For each $\alpha \leq \theta$ that is a limit ordinal with $\text{cf}(\alpha) = \omega$, we have

$$(\sup_{\alpha}(R), R(\alpha)) \in T.$$

3. For each $\alpha \leq \theta$ that is a limit ordinal with $\text{cf}(\alpha) > \omega$, we have

$$(\sup_{\alpha}(R), \text{stat}_{\alpha}(R), R(\alpha)) \in T.$$

First we satisfy condition (1). For each $\alpha < \theta$ we have

$$(R(\alpha), u(\alpha), R(\alpha + 1)) = (\overline{\mu(f \upharpoonright \alpha)}, u(\alpha), \mu(\overline{f \upharpoonright (\alpha + 1)}))$$

and since $\mu(f \upharpoonright \delta) \geq 0$ for each $\delta < \text{dom}(f)$, so we get

$$(R(\alpha), u(\alpha), R(\alpha + 1)) = (\bar{i}, W, \overline{i + 1}) \in T$$

for each $i \geq 0$.

Now we want to prove condition (2). For each $\alpha \leq \theta$ that is a limit ordinal with $\text{cf}(\alpha) = \omega$, we want to prove

$$(\sup_{\alpha}(R), R(\alpha)) \in T.$$

Now since

$$\sup_{\alpha}(R) = \{s \in S : \{\gamma < \alpha : R(\gamma) = s\} \text{ is cofinal in } \alpha\},$$

then we get $\sup_{\alpha}(R) \subseteq \{\bar{0}, \bar{1}, \bar{2}, \dots\}$, and

$$R(\alpha) = \overline{\mu(f \upharpoonright \alpha)} = \overline{\liminf_{\beta \rightarrow \alpha} \mu(f \upharpoonright \beta)} = \overline{\min(\sup_{\alpha}(R))}.$$

Hence,

$$(\sup_\alpha(R), R(\alpha)) \in T.$$

It remains to prove condition three, that is, for each $\alpha \leq \theta$ that is a limit ordinal with $\text{cf}(\alpha) > \omega$, we have

$$(\sup_\alpha(R), \text{stat}_\alpha(R), R(\alpha)) \in T.$$

Since

$$\sup_\alpha(R) = \{s \in S : \{\gamma \in \alpha : R(\gamma) = s\} \text{ is cofinal in } \alpha\}$$

and

$$\text{stat}_\alpha(R) = \{s \in S : \{\gamma \in \alpha : R(\gamma) = s\} \text{ is stationary in } \alpha\},$$

whenever $\text{cf}(\alpha) > \omega$, so we get $\sup_\alpha(R) \subseteq \{\bar{0}, \bar{1}, \bar{2}, \dots\}$, $\text{stat}_\alpha(R) \subseteq \{\bar{0}, \bar{1}, \bar{2}, \dots\}$, and

$$R(\alpha) = \overline{\mu(f \upharpoonright \alpha)} = \overline{\liminf_{\beta \rightarrow \alpha} \mu(f \upharpoonright \beta)} = \overline{\min(\sup_\alpha(R))}.$$

Hence,

$$(\sup_\alpha(R), \text{stat}_\alpha(R), R(\alpha)) \in T.$$

The case when $\eta = \aleph_1$.

Let $\mathcal{L} = (L_\beta : \beta < \eta)$ be a η -ladder in G' . For each $\beta < \eta$, we have L_β is either a 1-obstruction or L_β is trivial. For each $\beta < \eta$, if L_β is a 1-obstruction, then let f_β be a saturated string in L_β such that $\mu(f_\beta) < 0$ (see 2.2.1). Let $\theta_\beta = \text{dom}(f_\beta)$, then θ_β is a successor ordinal, $\mu(f_\beta \upharpoonright \gamma) \geq 0$ for each $\gamma < \theta_\beta$, and $u_\beta = u(f_\beta) \in \mathcal{L}(\mathcal{A})$, as in case $\eta = 1$, and let f_β be the empty string when L_β is trivial.

For each $\beta < \eta$, let $L_\beta = (M_\beta, W_\beta, E_\beta)$ and $f'_\beta : \theta'_\beta \rightarrow M_\beta \cup W_\beta$ be a string in L_β for some $\theta'_\beta \geq \theta_\beta$ such that $f'_\beta \upharpoonright \theta_\beta = f_\beta$ and $f'_\beta(\gamma) \in W_\beta$ for $\gamma \geq \theta_\beta$ with $W_\beta \subseteq \text{rng}(f'_\beta)$. Clearly such f'_β is also saturated. Note that if L_β is trivial, then $\theta_\beta = 0$ and $\theta'_\beta = 1$.

To obtain the string f we combine all the strings f'_β together. Formally, the domain of f is the sum

$$\theta = \sum_{\beta < \eta} \theta'_\beta,$$

and if $\gamma < \theta$, and ρ is the smallest ordinal with $\sum_{\beta < \rho} \theta'_\beta \leq \gamma$, then

$$f(\gamma) = f'_\rho(\gamma'),$$

where $\gamma' < \theta'_\rho$ is such that

$$\gamma = \sum_{\beta < \rho} \theta'_\beta + \gamma'.$$

Let $u = u(f)$. We will show that $u \in \mathcal{L}(\mathcal{A})$. We have $\theta = \text{dom}(u)$. To obtain an accepting run $R : \theta + 1 \rightarrow S$ of \mathcal{A} on u , first we define a partial run $R_\beta : \theta'_\beta \rightarrow S$ of \mathcal{A} on $u_\beta = u(f'_\beta)$, for each $\beta < \eta$, and combine them together.

Now for each $\beta < \eta$, we define R_β as follows:

- If L_β is trivial, then $\theta_\beta = 0$.
Define $R_\beta : \{0\} \rightarrow S$ be such that $R_\beta(0) = N$.
- If L_β is a 1-obstruction, then $\theta_\beta \geq 1$.
Define $R_\beta : \theta'_\beta \rightarrow S$ as follows:

$$\begin{aligned} R_\beta(0) &= \textcircled{S}, \\ R_\beta(\gamma) &= \widehat{\mu(f_\beta \upharpoonright \gamma)}, \text{ for every } 0 < \gamma < \theta_\beta, \\ R_\beta(\gamma) &= N \text{ for every } \theta_\beta \leq \gamma < \theta'_\beta. \end{aligned}$$

Now we want to show R_β is a partial run of \mathcal{A} on u_β . The following three conditions have to be satisfied:

1. For each α such that $\alpha + 1 < \theta'_\beta$, we have

$$(R_\beta(\alpha), u_\beta(\alpha), R_\beta(\alpha + 1)) \in T.$$

2. For each $\alpha < \theta'_\beta$ that is a limit ordinal with $\text{cf}(\alpha) = \omega$, we have

$$(\sup_\alpha (R_\beta), R_\beta(\alpha)) \in T.$$

3. For each $\alpha < \theta'_\beta$ that is a limit ordinal with $\text{cf}(\alpha) > \omega$, we have

$$(\sup_\alpha (R_\beta), \text{stat}_\alpha (R_\beta), R_\beta(\alpha)) \in T.$$

First, we want to verify condition 1. If $\theta_\beta = \theta'_\beta = 1$, then there is nothing to verify. If $\theta_\beta = 1$, and $\theta'_\beta \geq 2$, then $u_\beta(0) = M$ and

$$(R_\beta(0), u_\beta(0), R_\beta(1)) = (\textcircled{S}, M, N) \in T.$$

If $2 \leq \alpha + 1 < \theta'_\beta$, then

$$(R_\beta(\alpha), u_\beta(\alpha), R_\beta(\alpha + 1)) = (N, W, N) \in T.$$

Assume $\theta_\beta \geq 2$, then $u_\beta(0) = W$ and

$$(R_\beta(0), u_\beta(0), R_\beta(1)) = ((\mathbb{S}), W, \hat{1}) \in T.$$

If $\alpha + 1 = \theta_\beta < \theta'_\beta$, then

$$(R_\beta(\alpha), u_\beta(\alpha), R(\alpha + 1)) = (\hat{0}, M, N) \in T$$

For each α such that $0 < \alpha + 1 < \theta_\beta$, we consider two cases:

- $u_\beta(\alpha) = W$.
- $u_\beta(\alpha) = M$.

If $u_\beta(\alpha) = W$, then $R_\beta(\alpha) = \hat{i}$, for some $i \geq 0$ and

$$(R_\beta(\alpha), u_\beta(\alpha), R_\beta(\alpha + 1)) = (\widehat{\mu(f_\beta \upharpoonright \alpha)}, u_\beta(\alpha), \widehat{\mu(f_\beta \upharpoonright (\alpha + 1))}) = (\hat{i}, W, \widehat{i + 1}) \in T.$$

If $u_\beta(\alpha) = M$, then $R_\beta(\alpha) = \hat{i}$, for some $i \geq 1$ and

$$(R_\beta(\alpha), u_\beta(\alpha), R_\beta(\alpha + 1)) = (\widehat{\mu(f_\beta \upharpoonright \alpha)}, u_\beta(\alpha), \widehat{\mu(f_\beta \upharpoonright (\alpha + 1))}) = (\hat{i}, M, \widehat{i - 1}) \in T.$$

when $\theta_\beta + 1 \leq \alpha + 1 < \theta'_\beta$ we have

$$(R_\beta(\alpha), u_\beta(\alpha), R_\beta(\alpha + 1)) = (N, W, N) \in T.$$

Now we want to verify condition 2. Let $\alpha < \theta'_\beta$ be a limit ordinal with $\text{cf}(\alpha) = \omega$. We want to prove

$$(\sup_\alpha(R_\beta), R_\beta(\alpha)) \in T.$$

Assume $\alpha < \theta_\beta$, then we have

$$\sup_\alpha(R_\beta) = \{s \in S : \{\gamma < \alpha : R_\beta(\gamma) = s\} \text{ is cofinal in } \alpha\} \subseteq \{\hat{0}, \hat{1}, \hat{2}, \dots\}$$

and

$$R_\beta(\alpha) = \widehat{\mu(f_\beta \upharpoonright \alpha)} = \liminf_{\gamma \rightarrow \alpha} \widehat{\mu(f_\beta \upharpoonright \gamma)} = \widehat{\min(\sup_\alpha(R_\beta))}.$$

Hence,

$$(\sup_{\alpha}(R_{\beta}), R_{\beta}(\alpha)) \in T.$$

Note that we can't have $\alpha = \theta_{\beta}$, since θ_{β} is a successor ordinal.

If $\theta_{\beta} < \alpha < \theta'_{\beta}$, then

$$(\sup_{\alpha}(R_{\beta}), R_{\beta}(\alpha)) = (\{N\}, N) \in T.$$

It remains to satisfy the condition 3. Let $\alpha < \theta'_{\beta}$ be a limit ordinal with $\text{cf}(\alpha) > \omega$, and we want to show

$$(\sup_{\alpha}(R_{\beta}), \text{stat}_{\alpha}(R_{\beta}), R_{\beta}(\alpha)) \in T.$$

Assume first that $\alpha < \theta_{\beta}$. For each $A, B \subseteq \{\hat{0}, \hat{1}, \hat{2}, \dots\}$ and $s \in S$ we have $(A, B, s) \in T$ if and only if $(A, s) \in T$. Therefore a similar argument as above shows that

$$(\sup_{\alpha}(R_{\beta}), \text{stat}_{\alpha}(R_{\beta}), R_{\beta}(\alpha)) \in T.$$

The remaining case is when $\theta_{\beta} < \alpha < \theta'_{\beta}$.

Now since $\theta'_{\beta} = \text{dom}(f'_{\beta})$, $f'_{\beta} : \theta'_{\beta} \rightarrow M_{\beta} \cup W_{\beta}$, and $|M \cup W| \leq \aleph_1$, then $\alpha < \theta'_{\beta} \leq \aleph_1$, but $\text{cf}(\alpha) > \omega$, thus there is no such α .

Therefore, for each $\beta < \eta$, R_{β} is a partial run of \mathcal{A} on u_{β} .

Definition of the run R.

Formally, R is defined as follows. The domain of R is $\theta + 1$, where

$$\theta = \sum_{\beta < \eta} \theta'_{\beta}.$$

If $\delta < \theta$, then let $\rho < \eta$ be the smallest ordinal with $\sum_{\beta < \rho} \theta'_{\beta} \leq \delta$ and let

$$R(\delta) = R_{\rho}(\delta'),$$

where $\delta' < \theta'_{\rho}$ is such that

$$(2.2.1) \quad \delta = \sum_{\beta < \rho} \theta'_{\beta} + \delta'.$$

Moreover, let $R(\theta) = \mathbb{R}$.

Now we want to prove R is an accepting run of \mathcal{A} on $u = u(f)$. That means we must prove that $R(0) \in Z$, $R(\theta) \in F$ and the following conditions hold:

1. For each $\delta < \theta$, we have

$$(R(\delta), u(\delta), R(\delta + 1)) \in T.$$

2. For each $\delta \leq \theta$ that is a limit ordinal with $\text{cf}(\delta) = \omega$, we have

$$(\sup_{\delta} R, R(\delta)) \in T.$$

3. For each $\delta \leq \theta$ that is a limit ordinal with $\text{cf}(\delta) > \omega$, we have

$$(\sup_{\delta} R, \text{stat}_{\delta} R, R(\delta)) \in T.$$

By the definition of R , we get $R(\theta) = \textcircled{\mathbb{R}} \in F$. Now we want to show that $R(0) \in Z$. If L_0 is trivial, then

$$R(0) = R_0(0) = N \in Z.$$

Assume that L_0 is a 1-obstruction, then

$$R(0) = R_0(0) = \textcircled{\mathbb{S}} \in Z.$$

It is remains to prove the above three conditions.

First, for each $\delta \leq \theta$, let δ' be as in (2.2.1).

Now we want to prove condition (1), that is, for each $\delta < \theta$, we have

$$(R(\delta), u(\delta), R(\delta + 1)) \in T.$$

Now, if $\delta' + 1 < \theta'_{\rho}$, then

$$(R(\delta), u(\delta), R(\delta + 1)) = (R_{\rho}(\delta'), u_{\rho}(\delta'), R_{\rho}(\delta' + 1)) \in T$$

because R_{ρ} is a partial run.

Assume $\delta' + 1 = \theta'_{\rho}$, then we consider the following cases:

- Assume L_ρ is trivial. then $\theta'_\rho = 1$, $\delta' = 0$ and

$$\begin{aligned} (R(\delta), u(\delta), R(\delta + 1)) &= (R_\rho(\delta'), u_\rho(\delta'), R_\rho(\delta' + 1)) \\ &= (R_\rho(0), u_\rho(0), R_{\rho+1}(0)) \\ &= \begin{cases} (N, W, \textcircled{S}) & L_{\rho+1} \text{ is nontrivial.} \\ (N, W, N) & L_{\rho+1} \text{ is trivial} \end{cases} \\ &\in T \end{aligned}$$

- Assume L_ρ is a 1-obstraction.

If $\theta'_\rho = \theta_\rho$, then

$$R_\rho(\delta') = \hat{0} \text{ and } u(\delta') = M,$$

and if $\theta'_\rho > \theta_\rho$, then

$$R_\rho(\delta') = N \text{ and } u(\delta') = W.$$

Moreover,

$$R_{\rho+1}(0) = \begin{cases} N & L_{\rho+1} \text{ is trivial} \\ \textcircled{S} & L_{\rho+1} \text{ is nontrivial} \end{cases}.$$

In each case we have

$$(R(\delta), u(\delta), R(\delta + 1)) = (R_\rho(\delta'), u_\rho(\delta'), R_{\rho+1}(0)) \in T.$$

Now we want to prove second condition: let $\delta \leq \theta$ be a limit ordinal with $\text{cf}(\delta) = \omega$, and we want to satisfy

$$(\sup_\delta(R), R(\delta)) \in T.$$

Since $\text{cf}(\theta) = \text{cf}(\eta) > \omega$, by lemma 1.2.37, so we must have $\delta < \theta$.

We consider the following cases:

1. If $\delta' > 0$, and δ' is a limit ordinal with $\text{cf}(\delta') = \omega$.
2. If $\delta' = 0$, $\rho = \xi + 1$, and θ'_ξ is a limit ordinal with $\text{cf}(\theta'_\xi) = \omega$.
3. If $\delta' = 0$, and ρ is a limit ordinal with $\text{cf}(\rho) = \omega$.

Case (1).

We have

$$(\sup_{\delta}(R), R(\delta)) = (\sup_{\delta}(R_{\rho}), R_{\rho}(\delta')) \in T,$$

where $\delta' < \theta'_{\rho}$ is as in (2.2.1), since R_{ρ} is a partial run.

Case (2).

Since θ'_{ξ} is a limit ordinal, then L_{ξ} is a 1-obstraction, thus θ_{ξ} is a successor ordinal so $\theta_{\xi} < \theta'_{\xi}$ and

$$\sup_{\delta}(R) = \sup_{\theta'_{\xi}}(R_{\xi}) = \{N\}.$$

Moreover,

$$R(\delta) = R_{\rho}(0) = \begin{cases} N & \text{if } L_{\rho} \text{ is trivial} \\ \textcircled{S} & \text{otherwise} \end{cases},$$

and since $(\{N\}, N) \in T$, and $(\{N\}, \textcircled{S}) \in T$, we have

$$(\sup_{\delta}(R), R(\delta)) \in T.$$

Case (3).

Assume $\delta' = 0$, and ρ is a limit ordinal with $\text{cf}(\rho) = \omega$. For each $\beta < \rho$, we have $R_{\beta}(0) \in \{N, \textcircled{S}\}$, thus

$$\sup_{\delta}(R) \cap \{N, \textcircled{S}\} \neq \emptyset.$$

Moreover,

$$R(\delta) = R_{\rho}(0) = \begin{cases} N & \text{if } L_{\rho} \text{ is trivial} \\ \textcircled{S} & \text{otherwise} \end{cases}.$$

Since $(A, N) \in T$ and $(A, \textcircled{S}) \in T$ whenever $A \cap \{N, \textcircled{S}\} \neq \emptyset$, we have

$$(\sup_{\delta}(R), R(\delta)) \in T.$$

Now we want to prove third condition: let $\delta \leq \theta$ be a limit ordinal with $\text{cf}(\delta) > \omega$, and we want to satisfy

$$(\sup_{\delta}(R), \text{stat}_{\delta}(R), R(\delta)) \in T.$$

Now since $\theta = \text{dom}(f)$, and $|W \cup M| \leq \aleph_1$, then $\theta \leq \aleph_1$. Thus $\delta \leq \aleph_1$, but $\text{cf}(\delta) > \omega$, then $\delta = \aleph_1$. Therefore, the only possible that we have $\delta = \theta$.

Now we will prove third condition when $\delta = \theta$. Assume δ is a limit ordinal with $\text{cf}(\delta) > \omega$, we want to satisfy

$$(\sup_{\delta}(R), \text{stat}_{\delta}(R), R(\delta)) \in T.$$

Now,

$$R(\delta) = R(\theta) = \mathbb{R},$$

and since $(A, B, \mathbb{R}) \in T$, whenever $\mathbb{S} \in B$, we have

$$(\sup_{\delta}(R), \text{stat}_{\delta}(R), R(\delta)) \in T,$$

so it is enough to show that $\mathbb{S} \in \text{stat}_{\delta}(R)$.

Thus by the definition of $\text{stat}_{\delta}(R) = \text{stat}_{\theta}(R)$, we must prove that the set

$$\{\lambda < \theta : R(\lambda) = \mathbb{S}\} \text{ is stationary in } \theta.$$

Therefore we must prove this set has nonempty intersection with every club in θ . Let $\Gamma \subseteq \theta$, be arbitrary club in θ . Now assume

$$\Delta = \left\{ \sum_{\beta < \sigma} \theta'_{\beta} : \sigma < \eta \right\}.$$

Thus Δ is a club in θ , by lemma 1.2.38.

Moreover, $\Gamma \cap \Delta$ is also a club in θ . Now, define the function $\varphi : \eta \rightarrow \theta$, by

$$\varphi(\sigma) = \sum_{\beta < \sigma} \theta'_{\beta} \text{ for each } \sigma < \eta.$$

Then φ is a normal function and $\varphi^{-1}(\Gamma \cap \Delta)$ is a club in η , (see [4]). Now, since G' is a η -obstruction in G , so we get

$$S' = \{\sigma < \eta : L_{\sigma} \text{ is 1-obstruction}\} \text{ is stationary in } \eta.$$

Thus,

$$S' \cap \varphi^{-1}(\Gamma \cap \Delta) \neq \emptyset.$$

Let $\sigma \in S' \cap \varphi^{-1}(\Gamma \cap \Delta)$. Thus, L_{σ} is a 1-obstruction, and $\varphi(\sigma) \in \Gamma \cap \Delta$.

By the definition of φ we get

$$\zeta = \varphi(\sigma) = \sum_{\beta < \sigma} \theta'_\beta,$$

and $R(\zeta) = R_\sigma(0) = \textcircled{\mathbb{S}}$, since L_σ is a 1-obstruction. Then,

$$\zeta \in \Gamma \cap \{\lambda < \theta : R(\lambda) = \textcircled{\mathbb{S}}\}.$$

Thus, for each club Γ in θ we get

$$\Gamma \cap \{\lambda < \theta : R(\lambda) = \textcircled{\mathbb{S}}\} \neq \emptyset,$$

which implying that,

$$\{\lambda < \theta : R(\lambda) = \textcircled{\mathbb{S}}\},$$

is a stationary set in θ . Therefore, $\textcircled{\mathbb{S}} \in \text{stat}_\delta(R)$. Hence,

$$(\text{sup}_\delta(R), \text{stat}_\delta(R), R(\delta)) \in T.$$

Then R is an accepting run of \mathcal{A} on u . Thus, $u \in \mathcal{L}(\mathcal{A})$. Therefore, $\mathcal{L}(\mathcal{A}) \cap \mathcal{L}(G) \neq \emptyset$.

(1) implies (2).

Assume that $\mathcal{L}(\mathcal{A}) \cap \mathcal{L}(G) \neq \emptyset$. We want to show G has no matching, so it is enough to prove that there exists an obstruction in G (see 2.2.2).

Let $u \in \mathcal{L}(\mathcal{A}) \cap \mathcal{L}(G)$ be any element such that $u : \theta \rightarrow I$, where $\theta \in \text{ord}$. Since

$$\mathcal{L}(G) = \{u(f) : f \text{ is a saturated string in } G\},$$

implies that, $u = u(f)$ for some saturated string $f : \theta \rightarrow M \cup W$ in G , and since $u \in \mathcal{L}(\mathcal{A})$ is a class of all words u in $I^\#$ such that there exists an accepting run of \mathcal{A} on u , implies that there exists an accepting run $R : \theta + 1 \rightarrow S$ of \mathcal{A} on u . Then $R(\theta) = \textcircled{\mathbb{R}}$. Now, we consider the following cases:

(A1) If θ is a successor ordinal, then we will construct a 1-obstruction in G

(A2) If θ is a limit ordinal with $\text{cf}(\theta) > \omega$, then we will construct a η -obstruction in G for some $\eta \in K$.

Case (A1).

Assume $\theta = \beta + 1$, for some $\beta \in \text{ord}$, then $R(\beta) = \bar{0}$, since the only transition in $S \times I \times S$ that we have with $R(\theta) = \bar{\mathbb{R}}$ is $\bar{0} \xrightarrow{M} \bar{\mathbb{R}}$, thus

$$R(\delta) \in \{\bar{i}, i \in \mathbb{N}\} \text{ for each } \delta \leq \beta,$$

and $u(\beta) = M$.

Now we will construct a 1-obstraction in G . Thus, it is enough to show that $\mu(f) < 0$, (see 2.2.1)

Claim that $\mu(f) < 0$. Now by induction on $\delta \leq \beta$, we want to prove

$$R(\delta) = \overline{\mu(f \upharpoonright \delta)}.$$

If $\delta = 0$, then $R(0) = \bar{0} = \overline{\mu(f \upharpoonright 0)}$ because

$$R(\delta) \in \{\bar{i}, i \in \mathbb{N}\} \text{ for each } \delta \leq \beta,$$

and the set of initial states is $Z = \{\bar{0}, \bar{\mathbb{S}}, N\}$.

Now assume the statement is true for δ , that is, $R(\delta) = \overline{\mu(f \upharpoonright \delta)}$.

We want to prove it is true for $\delta + 1 \leq \beta$. We know that

$$R(\delta), R(\delta + 1) \in \{\bar{i}, i \in \mathbb{N}\},$$

and there are just two transitions

$$(\bar{i}, W, \overline{i+1}), \text{ and } (\bar{i}, M, \overline{i-1}) \text{ in } S \times I \times S.$$

Now we consider the following two cases:

(B1) If $(R(\delta), u(\delta), R(\delta + 1)) = (\bar{i}, W, \overline{i+1})$, then

$$R(\delta) = \bar{i}, u(\delta) = W, \text{ and } R(\delta + 1) = \overline{i+1}.$$

Thus $\mu(f \upharpoonright \delta) = i$, by our assumption and $R(\delta) = \bar{i}$. Now since f is a saturated sequence in G and $u(\delta) = W$, so we get $f(\delta) \in W$, which implies to

$$\mu(f \upharpoonright (\delta + 1)) = \mu(f \upharpoonright \delta) + 1 = i + 1.$$

Therefore,

$$R(\delta + 1) = \overline{\mu(f \upharpoonright \delta + 1)}.$$

(B2) If $(R(\delta), u(\delta), R(\delta + 1)) = (\bar{i}, M, \bar{i} - 1)$, then

$$R(\delta) = \bar{i}, u(\delta) = M \text{ and } R(\delta + 1) = \bar{i} - 1.$$

Thus $\mu(f \upharpoonright \delta) = i$ by our assumption and $R(\delta) = \bar{i}$. Now since f is saturated in G and $u(\delta) = M$, so we get $f(\delta) \in M$, which implies to

$$\mu(f \upharpoonright (\delta + 1)) = \mu(f \upharpoonright \delta) - 1 = i - 1.$$

Therefore,

$$R(\delta + 1) = \overline{\mu(f \upharpoonright \delta + 1)}.$$

Thus, for each $\delta < \theta$, $\overline{\mu(f \upharpoonright \delta)} = R(\delta)$.

Now we want to prove our claim, that is, $\mu(f) < 0$. We know that

$$\overline{\mu(f \upharpoonright \beta)} = R(\beta) = \bar{0},$$

so we get $\mu(f \upharpoonright \beta) = 0$, and $f(\beta) \in M$ because $u(\beta) = M$, and f is a saturated string in G . Thus,

$$\mu(f) = \mu(f \upharpoonright \beta) - 1 = -1 < 0.$$

Therefore, by using corollary 2.2.1 there is a 1-obstruction in G .

Case (A2).

Assume θ is a limit ordinal, then $\text{cf}(\theta) > \omega$ because the only transitions leading to \mathbb{R} is either

$$(\bar{0}, M, \mathbb{R}) \in S \times I \times S, \text{ or}$$

$$(\sup_{\theta}(R), \text{stat}_{\theta}(R), R(\theta)) \in \mathcal{P}(S) \times \mathcal{P}(S) \times S,$$

so θ is either a successor ordinal or a limit ordinal with $\text{cf}(\theta) > \omega$. Then we will construct a \aleph_1 -obstruction G' in G .

Let

$$A = \{\delta < \theta : R(\delta) \in \{\mathbb{S}, N\}\}.$$

First we want to show A is a closed set in θ .

Assume $D \subseteq A$, with $|D| < \theta$. We want to show $\sup(D) \in A$. We only need to consider the case when $\sup(D) \notin D$. Then $\sup(D)$ is a limit ordinal (see 1.2.36). Assume $\delta = \sup(D)$. Since δ is a limit ordinal and R is an accepting run of \mathcal{A} on u , then

$$(\sup_\delta(R), R(\delta)) \in T,$$

so by the definition of T , we get

$$\sup_\delta(R) \cap \{N, \textcircled{S}\} \neq \emptyset, \text{ and } R(\delta) \in \{N, \textcircled{S}\}$$

so $\delta \in A$. Therefore, A is a closed set in θ .

Now for each $\alpha \in A$, α is either a successor ordinal or a limit ordinal with $\text{cf}(\alpha) = \omega$, by the definition of the transitions set T of \mathcal{A} .

A is a well ordered by $<$ on ordinals (see 1.2.5), so there is unique ordinal η and an order preserving bijection $g : \eta \rightarrow A$, (see 1.2.6).

Now we want to prove $\eta = \aleph_1$. Since R is an accepting run of \mathcal{A} on u and θ is a limit ordinal with $\text{cf}(\theta) > \omega$, then

$$(\sup_\theta(R), \text{stat}_\theta(R), R(\theta)) = (\sup_\theta(R), \text{stat}_\theta(R), \textcircled{R}) \in T,$$

thus

$$\textcircled{S} \in \text{stat}_\theta(R),$$

by the definition of T . Therefore,

$$\{\delta < \theta : R(\delta) = \textcircled{S}\},$$

is stationary in θ , so it is unbounded and since

$$\{\delta < \theta : R(\delta) = \textcircled{S}\} \subseteq A,$$

so we get A is unbounded, thus $|A| \geq \text{cf}(\theta)$ by the definition of $\text{cf}(\theta)$ (see 1.2.11), and since $\text{cf}(\theta) > \omega$, so we get $|A| \geq \aleph_1$. Now since g is a bijection function, so we get $|A| = |\eta|$, and since $|\eta| \leq \eta$ so we have $\aleph_1 \leq \eta$.

It remains to prove $\eta \leq \aleph_1$. Assume by the way of a contradiction that $\eta > \aleph_1$, thus $g(\aleph_1) \in A$. Assume $g' = g \upharpoonright \aleph_1$, then $\text{rng}(g') \subseteq g(\aleph_1)$, since for each $\alpha \in \text{rng}(g')$, $\alpha = g(\beta)$ for

some $\beta < \aleph_1$, but g is an order preserving, then we get $\alpha = g(\beta) < g(\aleph_1)$. Now since

$$\aleph_1 = \sup\{\delta : \delta < \aleph_1\},$$

and g is a continuous function because g is strictly increasing and A is closed (see [4]), thus

$$g(\aleph_1) = \sup\{g(\delta) : \delta < \aleph_1\} = \sup(\text{rng}(g')),$$

therefore, no element of $g(\aleph_1)$ is an upper bounded on $\text{rng}(g')$, because $\sup(\text{rng}(g'))$ is the least upper bounded, thus $\text{rng}(g')$ is unbounded in $g(\aleph_1)$.

Now claim that,

$$\text{cf}(\aleph_1) = \text{cf}(g(\aleph_1)).$$

We want to prove the claim. Since g is an order preserving bijection so as $g' : \aleph_1 \rightarrow \text{rng}(g')$, and $\text{rng}(g')$ is unbounded in $g(\aleph_1)$, implies that

$$\text{cf}(\aleph_1) = \text{cf}(g(\aleph_1)),$$

by using lemma 1.2.20. Hence,

$$\text{cf}(g(\aleph_1)) = \aleph_1,$$

which is a contraction because $g(\aleph_1) \in A$ is either a successor ordinal or a limit ordinal with $\text{cf}(\alpha) = \omega$. Therefore, $\eta \leq \aleph_1$. Thus $\eta = \aleph_1$.

For each $\beta < \eta$, let L_β be a subgraph of G induced by

$$\{f(\delta) : g(\beta) \leq \delta < g(\beta + 1)\}.$$

Let

$$G' = \bigvee \{L_\beta : \beta < \eta\}.$$

We claim that, G' is a η -obstruction in G . We will show that:

(C1) G' is a saturated subgraph of G .

(C2) G' is a η -obstruction.

Proof of (C1).

Vertices set of $G' = (M', W', E')$ is a set $\{f(\delta) : \delta < \theta\}$ by the definition of G' . First, we want to show G' is a saturated subgraph of G , that is, G' is an induced ($E' = E \cap (M' \times W')$) and

$E[M'] \subseteq W'$. G' is an induced since it is a join and $E[M'] \subseteq W'$, since f is saturated in G .

Proof of (C2).

First, we want to construct a η -ladder \mathcal{L} in G' , claim that $\mathcal{L} = (L_\beta : \beta < \eta)$. Now we want to prove the claim, that is, we should prove the following:

$\mathcal{L} = (L_\beta : \beta < \eta)$ is a sequence of pairwise disjoint subgraphs of G such that $\mathcal{G} = (G_\alpha : \alpha \leq \eta)$ is a η -tower, where G_α is the join of all L_β for each $\beta < \alpha$.

Since f is a string, then L_β and L_α are disjoint when $\alpha \neq \beta$. It remains to show that, $\mathcal{G} = (G_\alpha : \alpha \leq \eta)$ is a η -tower, that is, the following conditions hold:

(D1) $\mathcal{G} = (G_\alpha : \alpha \leq \eta)$ is a sequence of saturated subgraphs of G .

(D2) G_α is a subgraph of G_β for each $\alpha < \beta \leq \eta$.

(D3) For every limit ordinal $\alpha \leq \eta$, the graph G_α is the union of all G_β with $\beta < \alpha$.

Proof of (D1).

$\mathcal{G} = (G_\alpha : \alpha \leq \eta)$ is a sequence of saturated subgraphs of G . Since for every $\alpha \leq \eta$, the vertices set of $G_\alpha = (M_\alpha, W_\alpha, E_\alpha)$ is

$$\{f(\delta) : \delta < g(\alpha)\}.$$

Then,

$$E_\alpha = E \cap (M_\alpha \times W_\alpha),$$

since

$$G_\alpha = \bigvee \{L_\beta : \beta < \alpha\},$$

and $E(M_\alpha) \subseteq W_\alpha$, since f is a saturated string in G .

Proof of (D2) and (D3) .

It is clear from the definitions.

In order to show that G' is a η -obstruction it remains to satisfy the following:

(E1) G' is the union of all rungs L_β of \mathcal{L} .

(E2) For each $\beta < \eta$, the rung L_β of \mathcal{L} is either a η_β -obstruction for some $\eta_\beta < K \cap \eta$ or L_β is trivial, that is, of the form $(\emptyset, \{w\}, \emptyset)$, for some $w \in W$.

(E3) The set $S' = \{\beta < \eta : L_\beta \text{ is a 1-obstruction}\}$ is stationary in η .

Proof of (E1).

It is clear from the definition of G' .

Proof of (E2).

First we want to prove for each $\beta < \eta$, the rung L_β of \mathcal{L} is a η_β -obstruction for some $\eta_\beta \in K \cap \eta$, when

$$R(g(\beta)) = \textcircled{S}.$$

Moreover, $\eta_\beta = 1$, because $\eta = \aleph_1$, $\eta < K \cap \eta$, and

$$K = \{1\} \cup \{\eta : \eta > \aleph_0 \text{ and } \eta \text{ is a regular cardinal}\}.$$

Now we want to prove that, for each $\beta < \eta$, the rung L_β of \mathcal{L} is a 1-obstruction, that is, we need to define a saturated string f' in L_β with $\mu(f') < 0$, by corollary 2.2.1.

First we want to define a saturated string f' in L_β . It follows from the definition of T that $g(\beta + 1)$ is a successor ordinal since if $g(\beta + 1)$ were a limit ordinal, then we must have

$$\sup_{g(\beta+1)}(R) \cap \{N, \textcircled{S}\} \neq \emptyset,$$

however,

$$\{R(\alpha) : g(\beta) < \alpha < g(\beta + 1)\} \subseteq \{\hat{0}, \hat{1}, \hat{2}, \dots\}.$$

Assume $g(\beta + 1) = g(\beta) + \sigma$, such that σ is the unique ordinal with $\sigma = \xi + 1$. Define the function $f' : \sigma \rightarrow V$ such that

$$f'(\nu) = f(g(\beta) + \nu) \text{ for every } \nu < \sigma.$$

It is clear that f' is a saturated string in L_β . We will show by transfinite induction on δ where $0 < \delta \leq \xi$, that

$$R(\delta) = \widehat{\mu(f' \upharpoonright \delta)}.$$

If $\delta = 1$, then $R(1) = \hat{1}$, because the only transitions that we have start with \textcircled{S} are

$$(\textcircled{S}, W, \hat{1}), (\textcircled{S}, M, N) \text{ and } (\textcircled{S}, M, \textcircled{S}),$$

and if $(R(0), u(0), R(1))$ is either (\textcircled{S}, M, N) , or $(\textcircled{S}, M, \textcircled{S})$, then $\xi = 0$, which is a contradic-

tion because $0 < \delta \leq \xi$. Therefore, the only possibility that we have

$$(R(0), u(0), R(1)) = ((\mathbb{S}), W, \hat{1}),$$

thus $R(1) = \hat{1}$, and since $u(0) = W$, then $f'(0) \in W$ implies that

$$\mu(f' \upharpoonright 1) = \mu(f' \upharpoonright 0) + 1 = 1,$$

therefore $R(1) = \widehat{\mu(f' \upharpoonright 1)}$. And the rest of the prove as in case (A1).

It is remains to show that, $\mu(f') < 0$.

Since $g(\beta + 1) \in A$, then $g(\beta + 1) \in \{N, \mathbb{S}\}$, and since

$$g(\beta + 1) = g(\beta) + \xi + 1,$$

thus $R(g(\beta) + \xi) = \hat{0}$, so $\mu(f' \upharpoonright g(\beta) + \xi) = 0$, but we have $f'(g(\beta) + \xi) = M$, then we get

$$\mu(f') = \mu(f' \upharpoonright g(\beta) + \xi) - 1,$$

which implies to $\mu(f') < 0$.

Second we want to prove L_β is trivial, that is, of the form $(\emptyset, \{w\}, \emptyset)$, for some $w \in W$ when

$$R(g(\beta)) = N$$

to prove that assume

$$R(g(\beta)) = N.$$

Now, since R is an accepting run of \mathcal{A} on u , so for each $\beta < \eta$,

$$(R(g(\beta)), u(g(\beta)), R(g(\beta) + 1)) = (N, a, t) \in T,$$

where $t \in \{\mathbb{S}, N\}$ and $a = W$ by the definition of transitions set T of \mathcal{A} , thus

$$g(\beta) + 1 \in A.$$

Now since L_β is the subgraph of G induced by

$$\{f(\delta) : g(\beta) \leq \delta < g(\beta) + 1\} = \{f(g(\beta))\},$$

but $f(g(\beta)) \in W$, since $u(g(\beta)) = W$. This implies to L_β is trivial.

Proof of (E3).

Since

$$(\sup_{\theta}(R), \text{stat}_{\theta}(R), \textcircled{R}) \in T,$$

then by the definition of T , we have

$$\textcircled{S} \in \text{stat}_{\theta}(R),$$

so

$$S'' = \{\beta < \theta : R(\beta) = \textcircled{S}\},$$

is a stationary set in θ . Note that $g : \eta \rightarrow \theta$ where $\text{rng}(g) = A$, is a continuous function because g is a strictly increasing function and A is a closed set in θ and if Δ is any club in η , thus $g(\Delta)$ is a club in θ (see [4]). Then,

$$g(\Delta) \cap S'' \neq \emptyset.$$

Thus, there is an element $\alpha \in \Delta$ such that $g(\alpha) \in S''$, so $R(g(\alpha)) = \textcircled{S}$, hence the rung L_{α} is a 1-obstruction as we show that before in proof of E2, which implies to $\alpha \in S'$, then $\alpha \in S' \cap \Delta$. Hence

$$S' \cap \Delta \neq \emptyset.$$

Therefore, S' is a stationary set in η . □

Chapter 3

Special ST-automata

In this part, we introduce a new concept of the automata, called a special ST-automata. We study the relation between the concept of TF-automata and the concept of ST-automata. First, we prove the equivalence relation between TF-automata and special ST-automata. Then, we show that the ST-automata are stronger by giving a counter example of a language accepted by ST-automata that is not accepted by any special ST-automata.

3.1 Basic Definitions

Definition 3.1.1. Let α be an ordinal, S be any set of states, $\mathcal{P}(S)$ be the set of all subsets of S , and $H : \alpha \rightarrow S \cup \mathcal{P}(S)$ be any function. Then for each $\beta \in \text{lim}$, $\beta \leq \alpha$ we define

$$\text{sup}'_{\beta}(H) = \text{sup}_{\beta}(H) \cap S.$$

The following definitions are from [7].

First we recall a TF-automaton concept.

Definition 3.1.2. A TF-quasiautomaton is a system $Q = (S, I, T)$, where S is a finite set of states, I is a finite alphabet and

$$T \subseteq (S \cup \mathcal{P}(S)) \times I \times S$$

is the set of transitions. A TF-automaton over I is a system $\mathcal{A} = (S, I, T, \psi, \mathcal{F})$, where $Q = (S, I, T)$ is a TF-quasiautomaton, denoted by $Q(\mathcal{A})$, $\psi \in S \cup \mathcal{P}(S)$ is the initial situation and $\mathcal{F} \subseteq S \cup \mathcal{P}(S)$ is the set of final situations.

The following is a definition of an accepting run of TF-automaton on a word.

Definition 3.1.3. Let $\mathcal{A} = (S, I, T, \psi, \mathcal{F})$ be a TF-automaton over I and $u : \alpha \rightarrow I$ be a word over I , $\alpha \in \text{ord}$. A run of \mathcal{A} on u (called II-run in [6]) is a function $H : \alpha + 1 \rightarrow S \cup \mathcal{P}(S)$ such that:

1. $H(0) = \psi$.
2. $H(\beta) \in S$, for every successor ordinal $\beta < \alpha$.
3. $H(\beta) = \sup'_\beta(H)$ for every limit ordinal $\beta \leq \alpha$.
4. $(H(\beta), u(\beta), H(\beta + 1)) \in T$, for every $\beta < \alpha$.

A run H of \mathcal{A} on u is an accepting run if and only if $H(\alpha) \in \mathcal{F}$.

Define $\mathcal{L}(\mathcal{A})$ to be the class of all words u in $I^\#$ such that there exists an accepting run of \mathcal{A} on u .

Now we define a special ST-automaton over an alphabet.

Definition 3.1.4. Let $\mathcal{A} = (S, I, T, Z, F)$ be a ST-automaton over I and $(A, B, s) \in \mathcal{P}(S) \times \mathcal{P}(S) \times S$ be a triple. \mathcal{A} is a special ST-automaton over I if and only if S is a finite set of states and the following condition holds:

$$(A, B, s) \in T \text{ if and only if } (A, s) \in T.$$

Now we show the equivalent relation between TF-automata and special ST-automata by the following main theorem.

3.2 Main Results

Theorem 3.2.1. Let I be a finite alphabet and \mathcal{C} be a subclass of $I^\#$. Then the following conditions are equivalent:

1. $\mathcal{C} = \mathcal{L}(\mathcal{A}')$ where \mathcal{A}' is a TF-automaton over I .
2. $\mathcal{C} = \mathcal{L}(\mathcal{A})$ where \mathcal{A} is a special ST-automaton over I .

Proof. (2) \implies (1).

Assume $\mathcal{C} = \mathcal{L}(\mathcal{A})$ such that $\mathcal{A} = (S, I, T, Z, F)$ is a special ST-automaton over I . We want to prove (1). That we should do the following:

(A1) Construct $\mathcal{A}' = (S', I, T', \psi, \mathcal{F})$, a TF-automaton over I .

(A2) $\mathcal{C} = \mathcal{L}(\mathcal{A}')$.

Proof of (A1).

Now we want to construct \mathcal{A}' a TF-automaton over I . Assume $\mathcal{A} = (S, I, T, \psi, \mathcal{F})$, such that $S' = S \times S \cup S$. Define the function $g : \mathcal{P}(S') \rightarrow \mathcal{P}(S)$ as the following:

$$g(A) = \{s \in S : s \in A, \text{ or there is } s' \in S \text{ such that } (s, s') \in A, \text{ or } (s', s) \in A\}.$$

Define $\mathcal{A}' = (S', I, T', \psi, \mathcal{F})$ as follows:

$$\psi = Z,$$

$$\mathcal{F} = F \cup A \cup \{B \subseteq S' : B \neq \emptyset \text{ and } (g(B), s) \in T \text{ for some } s \in F\} \cup \{(s, s') : s' \in F\}, \text{ where}$$

$$A = \begin{cases} \{\psi\} & \text{if } Z \cap F \neq \emptyset \\ \emptyset & \text{if } Z \cap F = \emptyset \end{cases}.$$

It is remains to define the transition relation T' such that $T' \subseteq (S' \cup \mathcal{P}(S')) \times I \times S'$. Assume

$$T' = T_1 \cup T_2 \cup T_3 \cup T_4,$$

where

$$T_1 = T \cap S \times I \times S,$$

$$T_2 = \{(\psi, a, s) : (s', a, s) \in T \text{ for some } s' \in \psi\},$$

$$T_3 = \{((s_1, s_2), a, s) : (s_2, a, s) \in T\},$$

$$T_4 = \{(B, a, (s_1, s_2)) : B \neq \emptyset, (g(B), s_1) \in T, \text{ and } (s_1, a, s_2) \in T\}.$$

It is clear that S' is a finite set. Now since $Z \subseteq S$, so we get $\psi \in S' \cup \mathcal{P}(S')$ and $\mathcal{F} \subseteq S' \cup \mathcal{P}(S')$ from the definition. Therefore $\mathcal{A}' = (S', I, T', \psi, \mathcal{F})$ is a TF-automaton over I .

Proof of (A2).

Now we want to show $\mathcal{C} = \mathcal{L}(\mathcal{A}')$, and since $\mathcal{C} = \mathcal{L}(\mathcal{A})$ by (1), that means we want to prove $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{A}')$. Thus we should prove the following:

(B1) $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{A}')$.

(B2) $\mathcal{L}(\mathcal{A}') \subseteq \mathcal{L}(\mathcal{A})$.

Proof of (B1): $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{A}')$.

Assume $u \in \mathcal{L}(\mathcal{A})$, such that $u : \alpha \rightarrow I$, where $\alpha \in \text{ord}$, and we want to prove $u \in \mathcal{L}(\mathcal{A}')$, so it is enough to prove there is an accepting run H of \mathcal{A}' on u .

If u is the empty word, then $\text{dom}(u) = \alpha = 0$, and $H : 1 \rightarrow S' \cup \mathcal{P}(S')$, such that

$$H(0) = H(\alpha) = \psi,$$

is an accepting run of \mathcal{A}' on u . Therefore, $u \in \mathcal{L}(\mathcal{A}')$.

Now assume u is a nonempty word. Since $u \in \mathcal{L}(\mathcal{A})$, then there is an accepting run $R : \alpha + 1 \rightarrow S$ of \mathcal{A} on u . Define $H : \alpha + 1 \rightarrow S' \cup \mathcal{P}(S')$ such that

$$H(\beta) = \begin{cases} \psi & \text{if } \beta = 0, \\ R(1) & \text{if } \beta = 1, \\ R(\beta) & \text{if } \beta = \delta + 1, \delta \in \text{succ}, \delta + 1 \leq \alpha, \\ (R(\delta), R(\delta + 1)) & \text{if } \beta = \delta + 1, \delta \in \text{lim}, \delta + 1 \leq \alpha, \\ \text{sup}'_{\beta}(H) & \text{if } \beta \in \text{lim}, \delta \leq \alpha. \end{cases}$$

Now we want to show that H is an accepting run of \mathcal{A}' on u , that is, we want to show $H(\alpha) \in \mathcal{F}$, and satisfy the following conditions:

1. $H(0) = \psi$.
2. $H(\beta) \in S'$, for every successor ordinal $\beta < \alpha$.
3. $H(\beta) = \text{sup}'_{\beta}(H)$, for every limit ordinal $\beta \leq \alpha$.
4. $(H(\beta), u(\beta), H(\beta + 1)) \in T'$, for every $\beta < \alpha$.

It is clear that 1, 2 and 3 are hold from the definition of H .

Now we want to prove condition (4).

First assume $\beta = 0$

$$(H(0), u(0), H(1)) = (\psi, u(0), R(1)) \in T',$$

because

$$T_2 = \{(\psi, a, s) : (s', a, s) \in T \text{ for some } s' \in \psi\} \subseteq T',$$

and

$$(R(0), u(0), R(1)) \in T, R(0) \in Z = \psi,$$

since R is an accepting run of \mathcal{A} on u .

Second let β be a successor ordinal such that $0 < \beta < \alpha$ and we want to prove

$$(H(\beta), u(\beta), H(\beta + 1)) \in T'.$$

Assume $\beta = \delta + 1$, for some $\delta \in \text{ord}$. Then we have the following cases:

- $\delta = 0$.
- δ is a successor ordinal.
- δ is a limit ordinal.

Assume $\delta = 0$, or δ is a successor ordinal, then we have

$$(H(\beta), u(\beta), H(\beta + 1)) = (R(\beta), u(\beta), R(\beta + 1)) \in T,$$

because R is an accepting run of \mathcal{A} on u , and since

$$T_1 = T \cap (S \times I \times S) \subseteq T',$$

implies to $(H(\beta), u(\beta), H(\beta + 1)) \in T'$.

Now let δ be a limit ordinal and we want to show $(H(\beta), u(\beta), H(\beta + 1)) \in T'$. We have

$$\begin{aligned} (H(\beta), u(\beta), H(\beta + 1)) &= (H(\delta + 1), u(\delta + 1), H(\delta + 2)) \\ &= ((R(\delta), R(\delta + 1)), u(\delta + 1), R(\delta + 2)), \end{aligned}$$

and since R is an accepting run of \mathcal{A} on u , so we get

$$(R(\delta + 1), u(\delta + 1), R(\delta + 2)) \in T,$$

but

$$T_3 = \{((s_1, s_2), a, s) : (s_2, a, s) \in T\} \subseteq T',$$

implies to $(H(\beta), u(\beta), H(\beta + 1)) \in T'$.

Finally, assume $\beta < \alpha$ be a limit ordinal and we want to prove $(H(\beta), u(\beta), H(\beta + 1)) \in T'$.

Claim that, $\sup_\beta(R) = g(\sup'_\beta(H))$, for every limit ordinal $\beta \leq \alpha$. Now let $\beta \leq \alpha$ be a limit ordinal and we want to prove our claim, assume $s \in \sup_\beta(R)$, where

$$\sup_\beta(R) = \{s' \in S : \{\delta < \beta : R(\delta) = s'\} \text{ is cofinal in } \beta\},$$

which implies to

$$C = \{\delta < \beta : R(\delta) = s\} \text{ is cofinal in } \beta.$$

Assume C' is the subset of C consisting of ordinals of the form $\zeta + 2$ for some ordinal ζ , C'' is the subset of C consisting of ordinals of the form $\zeta + 1$ for some limit ordinal ζ , and $C''' = C \cap \{\delta : \delta < \beta, \delta \in \text{lim}\}$.

For each $\delta \in C'$, we have $H(\delta) = R(\delta)$, so if C' is unbounded in β , then $s \in \text{sup}'_{\beta}(H)$. Thus $s \in \text{sup}'_{\beta}(H)$ and by the definition of g , we get $s \in g(\text{sup}'_{\beta}(H))$. Otherwise C'' is unbounded in β , or C''' is unbounded in β .

Assume C'' is unbounded in β . For each $\delta \in C''$ with $\delta = \zeta + 1$, and ζ is a limit ordinal we have

$$H(\delta) = (R(\zeta), R(\zeta + 1)) = (s', s)$$

for some $s' \in S$. Since S is a finite set, then there is $s' \in S$ such that

$$\{\delta \in C'' : H(\delta) = (s', s)\}$$

is unbounded in β by lemma 1.2.39. Thus,

$$(s', s) \in \text{sup}'_{\beta}(H)$$

and by the definition of g , we get $s \in g(\text{sup}'_{\beta}(H))$.

Now assume C''' is unbounded in β . For each $\delta \in C'''$, we have

$$H(\delta + 1) = (R(\delta), R(\delta + 1)) = (s, s')$$

for some $s' \in S$. Since S is a finite set, there is $s' \in S$ such that

$$\{\delta + 1 < \beta : H(\delta + 1) = (s, s')\}$$

is unbounded in β by lemma 1.2.39. Thus

$$(s, s') \in \text{sup}'_{\beta}(H)$$

and by the definition of g , we get $s \in g(\text{sup}'_{\beta}(H))$. From above we get $\text{sup}_{\beta}(R) \subseteq g(\text{sup}'_{\beta}(H))$.

It remains to prove $g(\text{sup}'_{\beta}(H)) \subseteq \text{sup}_{\beta}(R)$. Assume $s \in g(\text{sup}'_{\beta}(H))$, then by the definition of g we get either $s \in \text{sup}'_{\beta}(H)$, or $(s, s') \in \text{sup}'_{\beta}(H)$ for some $s' \in S$, or $(s', s) \in \text{sup}'_{\beta}(H)$ for some $s' \in S$.

1. Assume $s \in \text{sup}'_{\beta}(H)$, and let

$$C = \{\delta < \beta : H(\delta) = s\} \text{ is cofinal in } \beta.$$

Then $R(\delta) = H(\delta)$ for each $\delta \in C$, and $\delta = \gamma + 2$, for some $\gamma \in \text{ord}$ from the definition of H , which implies to

$$\{\delta < \beta : R(\delta) = s\}$$

is unbounded in β , therefore $s \in \text{sup}_\beta(R)$.

2. Assume $(s, s') \in \text{sup}'_\beta(H)$, for some $s' \in S$. Let

$$C = \{\delta < \beta : H(\delta) = (s, s')\} \text{ is cofinal in } \beta.$$

For each $\delta \in C$, there is a limit ordinal ζ such that $\delta = \zeta + 1$ by the definition of H . Then $R(\zeta) = s$ for each ζ such that $\zeta + 1 \in C$. Then

$$\{\zeta < \delta : R(\zeta) = s\}$$

is unbounded in β . Now since

$$\{\zeta < \delta : R(\zeta) = s\} \subseteq \{\gamma < \beta : R(\gamma) = s\},$$

which implies to

$$\{\gamma < \beta : R(\gamma) = s\},$$

is unbounded in β . Thus $s \in \text{sup}_\beta(R)$.

3. Assume $(s', s) \in \text{sup}'_\beta(H)$ for some $s' \in S$. Let

$$C = \{\delta < \beta : H(\delta) = (s', s)\} \text{ is cofinal in } \beta.$$

Then, $R(\delta) = s$ for each $\delta \in C$, $\delta \in \text{lim}$ and since

$$C \subseteq \{\gamma < \beta : R(\gamma) = s\}$$

so we get

$$\{\gamma < \beta : R(\gamma) = s\}$$

is unbounded in β . Thus $s \in \text{sup}_\beta(R)$.

From above we get $g(\text{sup}'_\beta(H)) \subseteq \text{sup}_\beta(R)$. Therefore $\text{sup}_\beta(R) = g(\text{sup}'_\beta(H))$.

Now, let $\beta < \alpha$ be a limit ordinal and we want to prove $(H(\beta), u(\beta), H(\beta + 1)) \in T'$. Then we have two cases: either $\text{cf}(\beta) = \omega$, or $\text{cf}(\beta) > \omega$.

First let $\beta < \alpha$ be a limit ordinal with $\text{cf}(\beta) = \omega$. Since

$$(H(\beta), u(\beta), H(\beta + 1)) = (\text{sup}'_{\beta}(H), u(\beta), (R(\beta), R(\beta + 1)))$$

and

$$(\text{sup}_{\beta}(R), R(\beta)) \in T, (R(\beta), u(\beta), R(\beta + 1)) \in T,$$

because R is an accepting run of \mathcal{A} on u . Now assume $\text{sup}'_{\beta}(H) \neq \emptyset$ and since

$$\text{sup}_{\beta}(R) = g(\text{sup}'_{\beta}(H)) \text{ and } T_4 = \{(B, a, (s_1, s_2)) : B \neq \emptyset, (g(B), s_1) \in T, (s_1, a, s_2) \in T\} \subseteq T',$$

thus we have

$$(\text{sup}'_{\beta}(H), u(\beta), (R(\beta), R(\beta + 1))) \in T',$$

which implies to $(H(\beta), u(\beta), H(\beta + 1)) \in T'$. It remains to prove $\text{sup}'_{\beta}(H) \neq \emptyset$. Assume by the way of a contradiction that

$$\text{sup}'_{\beta}(H) = \text{sup}_{\beta}(H) \cap S' = \emptyset,$$

and since S' is a finite set because S is finite, so for each $s \in S'$,

$$s \notin \text{sup}_{\beta}(H) = \{s' \in S' : \{\delta < \beta : H(\delta) = s'\} \text{ cofinal in } \beta\},$$

thus for each $s \in S'$,

$$\{\delta < \beta : H(\delta) = s\} \text{ bounded in } \beta.$$

Take $\gamma = \text{mix} \{\delta < \beta : H(\delta) = s, s \in S'\}$, thus $\gamma + 1 < \beta$ since $\beta \in \text{lim}$, and $H(\gamma + 1) \in S'$ by the definition of H . Now let $H(\gamma + 1) = t$ for some $t \in S'$, and

$$\{\delta < \beta : H(\delta) = t\} \text{ cofinal in } \beta,$$

then $t \in \text{sup}_{\beta}(H)$, therefore

$$t \in \text{sup}_{\beta}(H) \cap S' = \text{sup}'_{\beta}(H),$$

which is a contradiction. Therefore $\text{sup}'_{\beta}(H) \neq \emptyset$.

Second assume $\beta < \alpha$ is a limit ordinal with $\text{cf}(\beta) > \omega$.

Since R is an accepting run of \mathcal{A} on u , so we get

$$(\text{sup}_{\beta}(R), \text{stat}_{\beta}(R), R(\beta)) \in T,$$

which implies to

$$(\sup_{\beta}(R), R(\beta)) \in T,$$

because \mathcal{A} is a special ST-automaton over I , also we have $(R(\beta), u(\beta), R(\beta + 1)) \in T$ because R is an accepting run of \mathcal{A} on u , so we get $(H(\beta), u(\beta), H(\beta + 1)) \in T'$, as above.

Now we want to show $H(\alpha) \in \mathcal{F}$. Then we discuss the following cases:

1. Assume $\alpha = 0$.

If $\alpha = 0$, then $H(0) = \psi \in \mathcal{F}$.

2. If $\alpha = \beta + 1$, and β is a successor ordinal.

Then $H(\alpha) = R(\alpha)$, by the definition of H . Since R is an accepting run of \mathcal{A} on u , so we get $R(\alpha) \in F \subseteq \mathcal{F}$, thus implies to $H(\alpha) \in \mathcal{F}$.

3. If $\alpha = \beta + 1$, and β is a limit ordinal.

Then

$$H(\alpha) = H(\beta + 1) = (R(\beta), R(\beta + 1)),$$

by the definition of H and since R is an accepting run of \mathcal{A} on u , so we get $R(\alpha) = R(\beta + 1) \in F$, but $\{(s, s') : s' \in F\} \subseteq \mathcal{F}$, which implies to $H(\alpha) \in \mathcal{F}$.

4. If α is a limit ordinal with $\text{cf}(\alpha) = \omega$.

By the definition of H ,

$$H(\alpha) = \sup'_{\alpha}(H),$$

and since

$$\{B \subseteq S : B \neq \emptyset, (g(B), s) \in T \text{ for some } s \in F\} \subseteq \mathcal{F},$$

so it is enough to show that $\sup'_{\alpha}(H) \neq \emptyset$ and $(g(\sup'_{\alpha}(H)), s) \in T$, for some $s \in F$. By the same way above we can prove that $\sup'_{\alpha}(H) \neq \emptyset$. Now we know R is an accepting run of \mathcal{A} on u , then

$$(\sup_{\alpha}(R), R(\alpha)) \in T \text{ and } R(\alpha) \in F,$$

but

$$\sup_{\alpha}(R) = g(\sup'_{\alpha}(H)),$$

then $\sup'_{\alpha}(H) \in \mathcal{F}$, which implies to $H(\alpha) \in \mathcal{F}$.

5. If α is a limit ordinal with $\text{cf}(\alpha) > \omega$.

Since R is an accepting run of \mathcal{A} on u , we get

$$(\sup_{\alpha}(R), \text{stat}_{\alpha}(R), R(\alpha)) \in T,$$

and since \mathcal{A} is a special ST-automaton over I , so we get

$$(\sup_\alpha(R), R(\alpha)) \in T,$$

and as above we get to $H(\alpha) \in \mathcal{F}$.

Then H is an accepting run of \mathcal{A}' on u , so $u \in \mathcal{L}(\mathcal{A}')$. Therefore $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{A}')$.

Proof of (B2): $\mathcal{L}(\mathcal{A}') \subseteq \mathcal{L}(\mathcal{A})$.

Take $u \in \mathcal{L}(\mathcal{A}')$, such that $u : \alpha \rightarrow I$ where $\alpha \in \text{ord}$, we want to prove $u \in \mathcal{L}(\mathcal{A})$. Thus there is an accepting run $H : \alpha + 1 \rightarrow S' \cup \mathcal{P}(S')$ of \mathcal{A}' on u . That we should define an accepting run $R : \alpha + 1 \rightarrow S$ of \mathcal{A} on u .

Assume $u = \emptyset$, then $\text{dom}(u) = 0$. Therefore $\psi = H(0) = H(\alpha) \in \mathcal{F}$, because H is an accepting run of \mathcal{A}' on u , thus $Z \cap F \neq \emptyset$ by the definition of \mathcal{F} , so there exist an element $s \in Z \cap F$. Then define $R(0) = R(\alpha) = s$, which is an accepting run of \mathcal{A} on u .

Now assume $u \neq \emptyset$. Since $\emptyset \notin \mathcal{F}$ and $(\emptyset, a, s') \notin T'$ for some $s' \in S'$, $a \in I$, it follows that $\psi = H(0) \neq \emptyset$.

We define R as the following:

$$R(0) = \begin{cases} H(0) & \text{if } H(0) \in S \\ s_0 & \text{if } H(0) \in \mathcal{P}(S), \end{cases}$$

for some $s_0 \in H(0) = \psi$ is a fixed element and

$$R(\beta) = \begin{cases} H(1) & \text{if } \beta = 1, \\ H(\beta) & \text{if } \beta = \delta + 1, \delta \in \text{succ}, \text{ and } \beta \leq \alpha, \\ s & \text{if } H(\beta) = (s', s) \text{ for some } s' \in S, \beta = \delta + 1, \delta \in \text{lim}, \text{ and } \beta \leq \alpha, \\ s & \text{if } H(\beta) = (\beta + 1) = (s, s') \text{ for some } s' \in S, \beta \in \text{lim}, \text{ and } \beta < \alpha, \\ s & \text{if } (g(\sup'_\alpha(H)), s) \in T, \text{ for some } s \in F, \text{ and } \alpha \in \text{lim}. \end{cases}$$

Now we want to show that R is an accepting run of \mathcal{A} on u , that is, we want to show R is initial and final run, that is, $R(0) \in Z$ and $R(\alpha) \in F$ and satisfy the following conditions:

1. For each $\beta < \alpha$ we have

$$(R(\beta), u(\beta), R(\beta + 1)) \in T.$$

2. For each $\beta \leq \alpha$ that is a limit ordinal with $\text{cf}(\beta) = \omega$, we have

$$(\sup_{\beta}(R), R(\beta)) \in T.$$

3. For each $\beta \leq \alpha$ that is a limit ordinal with $\text{cf}(\beta) > \omega$, we have

$$(\sup_{\beta}(R), \text{stat}_{\beta}(R), R(\beta)) \in T.$$

It is clear from the definition of R that $R(0) \in Z$. Now we want to prove $R(\alpha) \in F$. Since $u \neq \emptyset$, then $\alpha \neq 0$. We will discuss the following cases:

1. Assume $\alpha = \delta + 1$, and δ is a successor ordinal.

Then $R(\alpha) = H(\alpha) \in \mathcal{F}$ because H is an accepting run of \mathcal{A}' on u , which is state and by the definition of \mathcal{F} we get $R(\alpha) \in F$.

2. Assume $\alpha = \delta + 1$, and δ is a limit ordinal.

Then,

$$R(\alpha) = s \text{ such that } H(\alpha) = H(\delta + 1) = (s', s) \text{ for some } s' \in S,$$

and since H is an accepting run of \mathcal{A}' on u , which implies to $H(\alpha) \in \mathcal{F}$, so we get $(s', s) \in \mathcal{F}$ and by the definition of \mathcal{F} , we get $s \in F$, therefore $R(\alpha) \in F$.

3. Assume α is a limit ordinal.

Then by the definition of R , we get

$$R(\alpha) = s \text{ where } s \in F \text{ and } (g(\sup'_{\alpha}(H)), s) \in T,$$

which implies to $R(\alpha) \in F$.

Now we want to prove first condition, that is, for each $\beta < \alpha$, we have

$$(R(\beta), u(\beta), R(\beta + 1)) \in T.$$

First assume $\beta = 0$, so we have two cases either $R(0) = H(0)$ when $H(0) \in S$, or $R(0) = s_0$, when $H(0) \in \mathcal{P}(S)$.

If $R(0) = H(0)$ when $H(0) \in S$, then

$$(R(0), u(0), R(1)) = (H(0), u(0), H(1)) \in T',$$

since H is an accepting run of \mathcal{A}' on u , and by the definition of T' , we get

$$(R(0), u(0), R(1)) \in T_1,$$

therefore $(R(0), u(0), R(1)) \in T$.

Now, if $R(0) = s_0$ when $H(0) \in \mathcal{P}(S)$, then

$$(R(0), u(0), R(1)) = (s_0, u(0), H(1)),$$

and since H is an accepting run of \mathcal{A}' on u , then $(\psi, u(0), H(1)) = (H(0), u(0), H(1)) \in T'$, and by the definition of T' , we get $(R(0), u(0), R(1)) \in T$.

Assume $\beta = \delta + 1$ such that δ is a successor ordinal $0 < \beta < \alpha$, we want to prove $(R(\beta), u(\beta), R(\beta + 1)) \in T$. We have

$$(R(\beta), u(\beta), R(\beta + 1)) = (H(\beta), u(\beta), H(\beta + 1)) \in T',$$

because H is an accepting run of \mathcal{A}' on u , thus we get by the definition of T' ,

$$(R(\beta), u(\beta), R(\beta + 1)) \in T_1,$$

therefore $(R(\beta), u(\beta), R(\beta + 1)) \in T$.

Now assume $\beta = \delta + 1$ such that δ is a limit ordinal $0 < \beta < \alpha$, and we want to show $(R(\beta), u(\beta), R(\beta + 1)) \in T$. From the definition of R we get

$$R(\beta) = s \text{ such that } H(\beta) = (s', s) \text{ for some } s' \in S$$

and since H is an accepting run of \mathcal{A}' on u , so we get

$$((s', s), u(\beta), R(\beta + 1)) = (H(\beta), u(\beta), H(\beta + 1)) \in T',$$

and since

$$T_3 = \{((s_1, s_2), a, s) : (s_2, a, s) \in T\} \subseteq T',$$

so we get

$$(s, u(\beta), R(\beta + 1)) \in T,$$

and since $R(\beta) = s$, because $H(\beta) = (s', s)$, then $(R(\beta), u(\beta), R(\beta + 1)) \in T$.

Finally assume $\beta < \alpha$ be a limit ordinal, and we want to show $(R(\beta), u(\beta), R(\beta + 1)) \in T$.

By the definition of R , we get

$$R(\beta) = s \text{ and } R(\beta + 1) = s' \text{ such that } H(\beta + 1) = (s, s') \text{ for some } s' \in S.$$

Now

$$(R(\beta), u(\beta), R(\beta + 1)) = (s, u(\beta), s'),$$

and since H is an accepting run of \mathcal{A}' on u , so we get

$$(H(\beta), u(\beta), H(\beta + 1)) \in T' \text{ and } H(\beta) = \sup'_\beta(H),$$

which implies to

$$(H(\beta), u(\beta), H(\beta + 1)) = (\sup'_\beta(H), u(\beta), (s, s')) \in T',$$

and by the definition of T_4 we get

$$(s, u(\beta), s') \in T,$$

and since $(R(\beta), u(\beta), R(\beta + 1)) = (s, u(\beta), s')$, $(R(\beta), u(\beta), R(\beta + 1)) \in T$.

Now we want to prove second condition, that is, for each $\beta \leq \alpha$ that is a limit ordinal with $\text{cf}(\beta) = \omega$,

$$(\sup_\beta(R), R(\beta)) \in T.$$

Since H is an accepting run of \mathcal{A}' on u , so we get

$$(H(\beta), u(\beta), H(\beta + 1)) \in T' \text{ and } H(\beta) = \sup'_\beta(H),$$

which implies to

$$(H(\beta), u(\beta), H(\beta + 1)) = (\sup'_\beta(H), u(\beta), H(\beta + 1)) \in T'.$$

Now by the definition of R , we get

$$R(\beta + 1) = s \text{ such that } H(\beta + 1) = (s', s) \text{ for some } s' \in S,$$

thus

$$(H(\beta), u(\beta), H(\beta + 1)) = (\sup'_\beta(H), u(\beta), (s', s)) \in T',$$

and by the definition of T' , we get

$$\left(\sup'_\beta(H), u(\beta), (s', s)\right) \in T_4,$$

which implies to

$$\left(g\left(\sup'_\beta(H)\right), s'\right) \in T,$$

and we know that $g\left(\sup'_\beta(H)\right) = \sup_\beta(R)$, therefore

$$\left(\sup_\beta(R), s'\right) \in T$$

and since

$$H(\beta + 1) = (s', s) = (R(\beta), R(\beta + 1))$$

then $(\sup_\beta(R), R(\beta)) \in T$.

Now we want to prove condition three, that is, for each $\beta \leq \alpha$ that is a limit ordinal with $\text{cf}(\beta) > \omega$, we have

$$(\sup_\beta(R), \text{stat}_\beta(R), R(\beta)) \in T.$$

Since \mathcal{A} is a special ST-automaton, then

$$(\sup_\beta(R), \text{stat}_\beta(R), R(\beta)) \in T \text{ if and only if } (\sup_\beta(R), R(\beta)) \in T,$$

and we show that $(\sup_\beta(R), R(\beta)) \in T$, for every limit ordinal $\beta \leq \alpha$. Then R is an accepting run of \mathcal{A} on u , thus $u \in \mathcal{L}(\mathcal{A})$. Therefore $\mathcal{L}(\mathcal{A}') \subseteq \mathcal{L}(\mathcal{A})$. Then from (B1) and (B2) we get $\mathcal{L}(\mathcal{A}') = \mathcal{L}(\mathcal{A})$, that is proof of (A2).

(1) \implies (2).

Assume $\mathcal{C} = \mathcal{L}(\mathcal{A}')$, such that $\mathcal{A}' = (S', I, T', \psi, \mathcal{F})$ is a TF-automaton over I . We want to prove (2), that we should do the following:

(A1) Construct $\mathcal{A} = (S, I, T, Z, F)$, a special ST-automaton over I .

(A2) $\mathcal{C} = L(\mathcal{A})$.

Proof of (A1).

We want to construct $\mathcal{A} = (S, I, T, Z, F)$, a special ST-automaton over I . Define

$$\begin{aligned} S &= S' \cup \mathcal{P}(S'), \\ Z &= \{\psi\}, \\ F &= \mathcal{F}, \\ T &= T' \cup T_1 \cup T_2, \end{aligned}$$

where

$$\begin{aligned} T_1 &= \{(A, B) : A \subseteq S \text{ and } B = A \cap S'\}, \quad \text{and} \\ T_2 &= \{(A, C, B) : A, C \subseteq S \text{ and } B = A \cap S'\}. \end{aligned}$$

First we want to show that, \mathcal{A} is a special ST-automaton. Since \mathcal{A}' is a TF-automaton over I , then S' is a finite set of states, so S is a finite set and from the definition of T_1 and T_2 , we get for each element $(A, B, s) \in \mathcal{P}(S) \times \mathcal{P}(S) \times S$,

$$(A, B, s) \in T \text{ if and only if } (A, s) \in T.$$

Therefore, \mathcal{A} is a special ST-automaton over I .

Proof of (A2).

Now we want to show $\mathcal{C} = \mathcal{L}(\mathcal{A})$, and since $\mathcal{C} = \mathcal{L}(\mathcal{A}')$ by (1), that means we want to prove $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{A}')$. Thus we should prove the following:

$$(B1) \quad \mathcal{L}(\mathcal{A}') \subseteq \mathcal{L}(\mathcal{A}).$$

$$(B2) \quad \mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{A}').$$

Proof of (B1): $\mathcal{L}(\mathcal{A}') \subseteq \mathcal{L}(\mathcal{A})$.

Assume $u \in \mathcal{L}(\mathcal{A}')$. We want to prove $u \in \mathcal{L}(\mathcal{A})$, so it is enough to show that there is an accepting run R of \mathcal{A} on u . Now since $u \in \mathcal{L}(\mathcal{A}')$, then there is an accepting run $H : \alpha + 1 \rightarrow S' \cup \mathcal{P}(S')$ of \mathcal{A}' on u . Define $R : \alpha + 1 \rightarrow S$ of \mathcal{A} on u , such that

$$R(\beta) = H(\beta) \text{ for each } \beta \leq \alpha.$$

Now we want to show that R is an accepting run of \mathcal{A} on u , that is, we want to show R is an initial run and a final run, that is, $R(0) \in Z$, $R(\alpha) \in F$, and satisfy the following conditions:

1. For each $\beta < \alpha$ we have

$$(R(\beta), u(\beta), R(\beta + 1)) \in T.$$

2. For each $\beta \leq \alpha$ that is a limit ordinal with $\text{cf}(\beta) = \omega$, we have

$$(\sup_{\beta}(R), R(\beta)) \in T.$$

3. For each $\beta \leq \alpha$ that is a limit ordinal with $\text{cf}(\beta) > \omega$, we have

$$(\sup_{\beta}(R), \text{stat}_{\beta}(R), R(\beta)) \in T.$$

First we want to show R is initial and final. By the definition of R , we get $R(0) = H(0) = \psi$, because H is an accepting run of \mathcal{A}' on u , and since $Z = \{\psi\}$, which implies to $R(0) \in Z$. Now we want to show R is final, again by the definition of R , we get $R(\alpha) = H(\alpha) \in \mathcal{F}$, because H is an accepting run of \mathcal{A}' on u and by the definition of \mathcal{A} , we get $F = \mathcal{F}$, which implies to $R(\alpha) \in F$.

Now we want to prove first condition, that is, for each $\beta < \alpha$, we have

$$(R(\beta), u(\beta), R(\beta + 1)) \in T.$$

Since H is an accepting run of \mathcal{A}' on u , then for each $\beta < \alpha$ we have

$$(H(\beta), u(\beta), H(\beta + 1)) \in T',$$

but $T' \subseteq T$, and $R(\beta) = H(\beta)$ for each $\beta \leq \alpha$, which implies to $(R(\beta), u(\beta), R(\beta + 1)) \in T$ for each $\beta < \alpha$.

We want to prove second condition, that is, for each $\beta \leq \alpha$ that is a limit ordinal with $\text{cf}(\beta) = \omega$ we have

$$(\sup_{\beta}(R), R(\beta)) \in T.$$

Let $\beta \leq \alpha$ be a limit ordinal with $\text{cf}(\beta) = \omega$ so by the definition of R , we get $R(\beta) = H(\beta) = \sup'_{\beta}(H)$, because H is an accepting run of \mathcal{A}' on u , but we know by the definition of $\sup'_{\beta}(H)$, that

$$\sup'_{\beta}(H) = \sup_{\beta}(R) \cap S',$$

and $T_1 = \{(A, B) : A \subseteq S \text{ and } B = A \cap S'\} \subseteq T$, thus $(\sup_{\beta}(R), R(\beta)) = (\sup_{\beta}(R), \sup'_{\beta}(H)) \in T$.

Finally the third condition holds because \mathcal{A} is a special ST-automaton and second condition holds.

Proof of (B2): $L(\mathcal{A}) \subseteq L(\mathcal{A}')$.

Assume $u \in L(\mathcal{A})$. We want to prove $u \in L(\mathcal{A}')$, so it is enough to show that there is an

accepting run H of \mathcal{A} on u . Now since $u \in L(\mathcal{A})$, then there is an accepting run $R : \alpha + 1 \rightarrow S$ of \mathcal{A} on u . Define $H : \alpha + 1 \rightarrow S' \cup \mathcal{P}(S')$ of \mathcal{A}' on u , such that

$$H(\beta) = R(\beta) \text{ for each } \beta \leq \alpha.$$

Now we want to show that H is an accepting run of \mathcal{A}' on u , that is, we want to show $H(\alpha) \in \mathcal{F}$ and satisfy the following conditions:

1. $H(0) = \psi$.
2. $H(\beta) \in S'$, for every successor ordinal $\beta < \alpha$.
3. $H(\beta) = \sup'_\beta(H)$ for every limit ordinal $\beta \leq \alpha$.
4. $(H(\beta), u(\beta), H(\beta + 1)) \in T'$, for every $\beta < \alpha$.

First we want to prove $H(\alpha) \in \mathcal{F}$. By the definition of H , we get $H(\alpha) = R(\alpha) \in F$, because R is an accepting run of \mathcal{A} on u , and since $F = \mathcal{F}$, which implies to $H(\alpha) \in \mathcal{F}$.

Now we want to prove first condition, that is, $H(0) = \psi$. By the definition of H , we get $H(0) = R(0) \in Z$, because R is an accepting run of \mathcal{A} on u , but $Z = \{\psi\}$, so we get $H(0) = \psi$.

The second condition holds since for every successor ordinal $\beta = \delta + 1$, $\delta \in \text{ord}$ and $\beta < \alpha$, we have

$$(R(\delta), u(\delta), R(\delta + 1)) \in T,$$

but $T = T' \cup T_1 \cup T_2$, which implies to

$$(R(\delta), u(\delta), R(\delta + 1)) \in T', \text{ and}$$

$$T' \subseteq (S' \cup \mathcal{P}(S')) \times I \times S',$$

thus we get $R(\beta) = R(\delta + 1) \in S'$.

We want to prove third condition, that is, $H(\beta) = \sup'_\beta(H)$ for every limit ordinal $\beta \leq \alpha$. Let β be any limit ordinal, $\beta \leq \alpha$. Then either $\text{cf}(\beta) = \omega$, or $\text{cf}(\beta) > \omega$.

First assume $\text{cf}(\beta) = \omega$, and since R is an accepting run of \mathcal{A} on u , then $(\sup_\beta(R), R(\beta)) \in T$, which implies to $(\sup_\beta(R), R(\beta)) \in T_1$, but $T_1 = \{(A, B) : A \subseteq S \text{ and } B = A \cap S'\}$, then

$$R(\beta) = \sup_\beta(R) \cap S', \text{ and } \sup_\beta(H) = \sup_\beta(R),$$

thus

$$R(\beta) = \sup_\beta(H) \cap S' = \sup'_\beta(H),$$

but we have by the definition of H , that $H(\alpha) = R(\alpha)$, therefore $H(\alpha) = \sup'_\beta(H)$.

Now let $\text{cf}(\beta) > \omega$, and since R is an accepting run of \mathcal{A} on u , then $(\sup_\beta(R), \text{stat}_\beta(R), R(\beta)) \in T$, which implies to $(\sup_\beta(R), \text{stat}_\beta(R), R(\beta)) \in T_2$, but $T_2 = \{(A, C, B) : A, C \subseteq S \text{ and } B = A \cap S'\}$, then

$$R(\beta) = \sup_\beta(R) \cap S', \text{ and } \sup_\beta(H) = \sup_\beta(R),$$

thus

$$R(\beta) = \sup_\beta(H) \cap S' = \sup'_\beta(H),$$

but we have by the definition of H , that $H(\alpha) = R(\alpha)$, therefore $H(\alpha) = \sup'_\beta(H)$.

Now we want to satisfy the last condition. Since R is an accepting run of \mathcal{A} on u , so we get for each $\beta < \alpha$

$$(R(\beta), u(\beta), R(\beta + 1)) \in T,$$

thus by the definition of T ,

$$(R(\beta), u(\beta), R(\beta + 1)) \in T'$$

and by the definition of H , we get $(H(\beta), u(\beta), H(\beta + 1)) \in T'$. Therefore H is an accepting run of \mathcal{A}' on u , then $u \in \mathcal{L}(\mathcal{A}')$. Thus $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{A}')$. From (B1) and (B2) we get $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{A}')$. □

The following example and theorem are showing that the concept of ST-automata is stronger than from the concept of TF-automata.

Example 3.2.2. Let $\mathcal{A} = (S, I, T, Z, F)$ be a ST-automaton over an alphabet $I = \{a\}$, such that

$$S = \{z, f\},$$

$$Z = \{z\},$$

$$F = \{f\}, \text{ and}$$

$$T = \{(z, a, z), (\{z\}, z), (\{z\}, \{z\}, f)\}.$$

Define $u : \omega_1 \rightarrow I$, by $u(\delta) = a$, for each $\delta < \omega_1$, and $R : \omega_1 + 1 \rightarrow S$, as the following:

$$R(\delta) = z, \text{ for each } \delta < \omega_1, \text{ and}$$

$$R(\omega_1) = f.$$

Then R is an accepting run of \mathcal{A} on u . Therefore, $\mathcal{L}(\mathcal{A}) = \{u\}$. But \mathcal{A} is not a ST-special automaton over I . Assume by the way of a contradiction that there exists a special ST-automaton,

that is, for any $A, B \subseteq S$ and $s \in S$, the following condition holds:

$$(A, s) \in T \text{ if and if } (A, B, s) \in T.$$

Therefore $(\{z\}, f) \in T$, because $(\{z\}, \{z\}, f) \in T$. Then there exists a limit ordinal $\alpha < \omega_1$, with $\text{cf}(\alpha) = \omega$, such that

$$(\sup_\alpha(R), R(\alpha)) = (\{z\}, f).$$

Now define $u' : \alpha \rightarrow I$, by $u(\delta) = a$, for each $\delta < \alpha$, and $R' : \alpha + 1 \rightarrow S$, as the following:

$$\begin{aligned} R'(\delta) &= z, \text{ for each } \delta < \alpha, \text{ and} \\ R'(\alpha) &= f. \end{aligned}$$

Hence R' is an accepting run of \mathcal{A} on u' . Therefore $u' \in \mathcal{L}(\mathcal{A})$ which is a contradiction because $\mathcal{L}(\mathcal{A}) = \{u\}$ and $u \neq u'$. Hence, \mathcal{A} is not a special ST-automaton over I .

Theorem 3.2.3. *There does not exist a special ST-automaton $\mathcal{A} = (S, I, T, Z, F)$ over an alphabet $I = \{a\}$ such that $\mathcal{L}(\mathcal{A}) = \{u\}$, where $u : \omega_1 \rightarrow I$, and S is a countable set.*

Proof. Assume by the way of a contradiction there is a special ST-automaton $\mathcal{A} = (S, I, T, Z, F)$ such that $\mathcal{L}(\mathcal{A}) = \{u\}$, $u : \omega_1 \rightarrow I$, $I = \{a\}$ and S is a countable set. Now since \mathcal{A} is a special ST-automaton, that is, for any $A, B \subseteq S$ and $s \in S$, the following condition holds:

$$(A, s) \in T \text{ if and if } (A, B, s) \in T,$$

and since $\mathcal{L}(\mathcal{A}) = \{u\}$, then there exists an accepting run $R : \omega_1 + 1 \rightarrow S$ of \mathcal{A} on u . Then $R(0) \in Z$, $R(\omega_1) \in F$, and the following three conditions hold:

1. For each $\beta < \omega_1$ we have

$$(R(\beta), u(\beta), R(\beta + 1)) \in T.$$

2. For each $\beta \leq \omega_1$ that is a limit ordinal with $\text{cf}(\beta) = \omega$, we have

$$(\sup_\beta(R), R(\beta)) \in T.$$

3. For each $\beta \leq \omega_1$ that is a limit ordinal with $\text{cf}(\beta) > \omega$, we have

$$(\sup_\beta(R), \text{stat}_\beta(R), R(\beta)) \in T.$$

Now since $\text{cf}(\omega_1) = \omega_1 > \omega$, then by third condition we get

$$(\sup_{\omega_1}(R), \text{stat}_{\omega_1}(R), R(\omega_1)) \in T,$$

and since \mathcal{A} is a special ST-automaton, then $(\sup_{\omega_1}(R), R(\omega_1)) \in T$.

Assume first S is a finite set.

Now since

$$\sup_{\omega_1}(R) = \{s \in S : \{\gamma < \omega_1 : R(\gamma) = s\} \text{ is cofinal in } \omega_1\},$$

then let

$$\begin{aligned} \sup_{\omega_1}(R) &= \{s_1, s_2, s_3, \dots, s_n\}, \quad \text{and} \\ S \setminus \sup_{\omega_1}(R) &= \{s_{n+1}, s_{n+2}, \dots, s_m\}. \end{aligned}$$

We know R is an accepting run of \mathcal{A} on u , and since $S \setminus \sup_{\omega_1}(R) = \{s_{n+1}, s_{n+2}, \dots, s_m\}$, so that for each $i = n+1, n+2, \dots, m$, there exists $\alpha_i < \omega_1$, such that $R(\alpha) \neq s_i$ for $\alpha > \alpha_i$.

Choose δ to be largest element in $\{\alpha_{n+1}, \alpha_{n+2}, \dots, \alpha_m\}$, and since

$$\sup_{\omega_1}(R) = \{s_1, s_2, s_3, \dots, s_n\},$$

so that there exists

$$\begin{aligned} &\beta_1 > \delta \text{ such that } R(\beta_1) = s_1, \\ &\beta_2 > \beta_1 \text{ such that } R(\beta_2) = s_2, \\ &\vdots \\ &\beta_n > \beta_{n-1} \text{ such that } R(\beta_n) = s_n, \\ &\beta_{n+1} > \beta_n \text{ such that } R(\beta_{n+1}) = s_1, \\ &\beta_{n+2} > \beta_{n+1} \text{ such that } R(\beta_{n+2}) = s_2, \\ &\vdots \\ &\beta_{n+n} > \beta_{n+(n-1)} \text{ such that } R(\beta_{n+n}) = s_n, \\ &\vdots \end{aligned}$$

and we will keep doing that infinitely many times we will get to the following sequence:

$$\beta_1 < \beta_2 < \dots < \beta_n < \beta_{n+1} < \dots < \beta_{n+n} < \beta_{n+n+1} < \dots$$

Define

$$\theta = \sup \{\beta_i : i = 1, 2, \dots\}.$$

It is clear that $\theta \in \text{lim}$, $\text{cf}(\theta) = \omega$ and $\sup_{\theta}(R) = \sup_{\omega_1}(R)$, and since $(\sup_{\omega_1}(R), R(\omega_1)) \in T$, so

we get

$$(\sup_{\theta}(R), R(\omega_1)) \in T.$$

Now define the word,

$$u' : \theta \rightarrow I \text{ such that } u'(\alpha) = a \text{ for each } \alpha < \theta, \text{ and}$$

$R' : \theta + 1 \rightarrow S$ such that

$$R'(\gamma) = R(\gamma) \text{ for each } \gamma < \theta, \text{ and}$$

$$R'(\theta) = R(\omega_1).$$

It remains to show R' is an accepting run of \mathcal{A} on u' , that is, $R'(0) \in Z$, $R'(\theta) \in F$, and the following three conditions hold:

1. For each $\beta < \theta$ we have

$$(R'(\beta), u'(\beta), R'(\beta + 1)) \in T.$$

2. For each $\beta \leq \theta$ that is a limit ordinal with $\text{cf}(\beta) = \omega$ we have

$$(\sup_{\beta}(R'), R'(\beta)) \in T.$$

3. For each $\beta \leq \theta$ that is a limit ordinal with $\text{cf}(\beta) > \omega$, we have

$$(\sup_{\beta}(R'), \text{stat}_{\beta}(R'), R'(\beta)) \in T.$$

It is clear that R' is an accepting run of \mathcal{A} on u' , by the definition of R and since R is an accepting run of \mathcal{A} on u . Therefore $u' \in \mathcal{L}(\mathcal{A})$ which is a contradiction because $\mathcal{L}(\mathcal{A}) = \{u\}$ and $u \neq u'$. Hence $\{u\}$ is not accepting by a special ST-automaton.

Second assume S is an infinite set and since

$$\sup_{\omega_1}(R) = \{s \in S : \{\gamma < \omega_1 : R(\gamma) = s\} \text{ is cofinal in } \omega_1\},$$

then we have the following cases:

1. The set $\sup_{\omega_1}(R)$, and $S \setminus \sup_{\omega_1}(R)$ are infinite.
2. The set $\sup_{\omega_1}(R)$ is finite and $S \setminus \sup_{\omega_1}(R)$ is infinite.
3. The set $\sup_{\omega_1}(R)$, is infinite and $S \setminus \sup_{\omega_1}(R)$ is finite.

First we will discuss case (1):

Assume,

$$\begin{aligned} \sup_{\omega_1}(R) &= \{s_1, s_2, s_3, \dots\}, \quad \text{and} \\ S \setminus \sup_{\omega_1}(R) &= \{t_1, t_2, t_3, \dots\}. \end{aligned}$$

We know R is an accepting run of \mathcal{A} on u , and since $S \setminus \sup_{\omega_1}(R) = \{t_1, t_2, t_3, \dots\}$, so that for each $i = 0, 1, 2, \dots$ there exists $\alpha_i < \omega_1$ such that $R(\alpha) \neq s_i$ for $\alpha > \alpha_i$.

Choose

$$\delta = \sup \{\alpha_i : i = 1, 2, 3, \dots\},$$

then $\delta \in \text{lim}$, and $\delta < \omega_1$, and since

$$\sup_{\omega_1}(R) = \{s_1, s_2, s_3, \dots\},$$

so that there exists

$$\begin{aligned} \beta_1 &> \delta \text{ such that } R(\beta_1) = s_1, \\ \beta_2 &> \beta_1 \text{ such that } R(\beta_2) = s_2, \\ \beta_3 &> \beta_2 \text{ such that } R(\beta_3) = s_3, \\ &\vdots \\ \beta_\omega &= \sup \{\beta_i : i < \omega\} \\ \beta_{\omega+1} &> \beta_\omega \text{ such that } R(\beta_{\omega+1}) = s_1, \\ \beta_{\omega+2} &> \beta_{\omega+1} \text{ such that } R(\beta_{\omega+2}) = s_2, \\ &\vdots \end{aligned}$$

and we will keep doing that we will get to the following sequence:

$$\beta_1 < \beta_2 < \beta_3 < \dots$$

Define

$$\theta = \sup \{\beta_\delta : \delta < \omega \cdot \omega\}.$$

It is clear that $\theta \in \text{lim}$, $\text{cf}(\theta) = \omega$ and $\sup_\theta(R) = \sup_{\omega_1}(R)$, and since $(\sup_{\omega_1}(R), R(\omega_1)) \in T$, so we get

$$(\sup_\theta(R), R(\omega_1)) \in T.$$

Now define the word,

$$u' : \theta \rightarrow I, \text{ such that } u'(\alpha) = a, \text{ for each } \alpha < \theta, \text{ and}$$

$R' : \theta + 1 \rightarrow S$, such that

$$R'(\gamma) = R(\gamma), \text{ for each } \gamma < \theta, \text{ and}$$

$$R'(\theta) = R(\omega_1).$$

It is clear that R' is an accepting run of \mathcal{A} on u' , by the definition of R and since R is an accepting run of \mathcal{A} on u . Therefore, $u' \in \mathcal{L}(\mathcal{A})$, which is a contradiction because $\mathcal{L}(\mathcal{A}) = \{u\}$ and $u \neq u'$. Hence, $\{u\}$ is not accepted by a special ST-automaton.

Similar to the cases above we can prove case (2) and (3). □

Chapter 4

Operations on ST-automata

In this chapter, we define the basic operations on ST-automata over an alphabet as union, intersection, concatenation, raising to the powers ω , $*$, and $\#$. Furthermore, we show that applying these operations to languages defined by ST-automata, the produced languages are also definable using ST-automata.

4.1 Union Operation

First we define the union ST-automaton as follows:

Definition 4.1.1. Let $\mathcal{A} = (S, I, T, Z, F)$ and $\mathcal{A}' = (S', I, T', Z', F')$ be two ST-automata over I such that $S \cap S' = \emptyset$. Define the union ST-automaton over I of \mathcal{A} and \mathcal{A}' denoted by $\mathcal{A} \cup \mathcal{A}'$ as follows:

let

$$\mathcal{A} \cup \mathcal{A}' = (S'', I, T'', Z'', F''),$$

such that

$$\begin{aligned} S'' &= S \cup S', \\ Z'' &= Z \cup Z', \\ F'' &= F \cup F', \text{ and} \\ T'' &= T \cup T'. \end{aligned}$$

Then we prove the following theorem that shows applying the union operation to languages defined by ST-automata, the produce language that is also definable using ST-automaton.

Theorem 4.1.2. Let $\mathcal{A} = (S, I, T, Z, F)$ and $\mathcal{A}' = (S', I, T', Z', F')$ be two ST-automata over I , such that $S \cap S' = \emptyset$. Then $\mathcal{L}(\mathcal{A} \cup \mathcal{A}') = \mathcal{L}(\mathcal{A}) \cup \mathcal{L}(\mathcal{A}')$.

Proof. Assume $\mathcal{A} = (S, I, T, Z, F)$ and $\mathcal{A}' = (S', I, T', Z', F')$ are two ST-automata over I such that

$$\mathcal{A} \cup \mathcal{A}' = \mathcal{A}'' = (S'', I, T'', Z'', F'').$$

We want to prove the following:

1. $\mathcal{L}(\mathcal{A} \cup \mathcal{A}') \subseteq \mathcal{L}(\mathcal{A}) \cup \mathcal{L}(\mathcal{A}')$.
2. $\mathcal{L}(\mathcal{A}) \cup \mathcal{L}(\mathcal{A}') \subseteq \mathcal{L}(\mathcal{A} \cup \mathcal{A}')$.

By the definition of \mathcal{A}'' we get the following:

$$\begin{aligned} S'' &= S \cup S', \\ Z'' &= Z \cup Z', \\ F'' &= F \cup F', \text{ and} \\ T'' &= T \cup T'. \end{aligned}$$

proof of (1) . $\mathcal{L}(\mathcal{A} \cup \mathcal{A}') \subseteq \mathcal{L}(\mathcal{A}) \cup \mathcal{L}(\mathcal{A}')$.

Assume $u \in \mathcal{L}(\mathcal{A} \cup \mathcal{A}')$, such that $u : \alpha \rightarrow I$ and $\alpha \in \text{ord}$. Then there is an accepting run $R : \alpha + 1 \rightarrow S''$ of \mathcal{A}'' on u . We want to prove $u \in \mathcal{L}(\mathcal{A}) \cup \mathcal{L}(\mathcal{A}')$. Since R is an accepting run of \mathcal{A}'' on u , then either $R(0) \in Z$, or $R(0) \in Z''$. If $R(0) \in Z$, then all values of R are in S because $S \cap S' = \emptyset$ and $T'' = T \cup T'$. Then R is an accepting run of \mathcal{A} on u . Thus $u \in \mathcal{L}(\mathcal{A})$, therefore $u \in \mathcal{L}(\mathcal{A}) \cup \mathcal{L}(\mathcal{A}')$. Similarly, if $R(0) \in Z'$, then $u \in \mathcal{L}(\mathcal{A}')$. Thus in either case we get $u \in \mathcal{L}(\mathcal{A}) \cup \mathcal{L}(\mathcal{A}')$.

proof of (2). $\mathcal{L}(\mathcal{A}) \cup \mathcal{L}(\mathcal{A}') \subseteq \mathcal{L}(\mathcal{A} \cup \mathcal{A}')$.

Assume $u \in \mathcal{L}(\mathcal{A}) \cup \mathcal{L}(\mathcal{A}')$, such that $u : \alpha \rightarrow I$ and $\alpha \in \text{ord}$. Then either $u \in \mathcal{L}(\mathcal{A})$, or $u \in \mathcal{L}(\mathcal{A}')$. Without loss of generality, we can assume $u \in \mathcal{L}(\mathcal{A})$. Then there is an accepting run $R : \alpha + 1 \rightarrow S$ of \mathcal{A} on u . It is clear that R is an accepting run of \mathcal{A}'' on u . Then $u \in \mathcal{L}(\mathcal{A} \cup \mathcal{A}')$. \square

4.2 Intersection Operation

Next we define the intersection ST-automaton as follows:

Definition 4.2.1. Let $\mathcal{A} = (S, I, T, Z, F)$ and $\mathcal{A}' = (S', I, T', Z', F')$ be two ST-automata over I such that S and S' are finite sets of states. If $A \subseteq S \times S'$, then we define $\pi_1(A)$ and $\pi_2(A)$ as

follows:

$$\begin{aligned}\pi_1(A) &= \{s \in S : (s, s') \in A \text{ for some } s' \in S'\} \text{ and} \\ \pi_2(A) &= \{s' \in S' : (s, s') \in A \text{ for some } s \in S\}.\end{aligned}$$

Define the intersection ST-automaton over I of \mathcal{A} and \mathcal{A}' denoted by $\mathcal{A} \cap \mathcal{A}'$ as follows:

let

$$\mathcal{A} \cap \mathcal{A}' = (S'', I, T'', Z'', F'').$$

such that

$$\begin{aligned}S'' &= S \times S', \\ Z'' &= Z \times Z', \\ F'' &= F \times F', \\ T'' &= T_1 \cup T_2 \cup T_3,\end{aligned}$$

where

$$\begin{aligned}T_1 &= \{(s, a, t) : s = (s_1, s_2), t = (t_1, t_2), (s_1, a, t_1) \in T, \text{ and } (s_2, a, t_2) \in T'\}, \\ T_2 &= \{(A, s) : A \subseteq S'', s = (s_1, s_2), (\pi_1(A), s_1) \in T, \text{ and } (\pi_2(A), s_2) \in T'\}, \text{ and} \\ T_3 &= \{(A, B, s) : A, B \subseteq S'', s = (s_1, s_2), (\pi_1(A), \pi_1(B), s_1) \in T, \text{ and } (\pi_2(A), \pi_2(B), s_2) \in T'\}.\end{aligned}$$

Then by proving the following theorem we show that, applying the intersection operation to languages defined by ST-automata, the produce language that is also definable using ST-automaton.

Theorem 4.2.2. *Let $\mathcal{A} = (S, I, T, Z, F)$ and $\mathcal{A}' = (S', I, T', Z', F')$ be two ST-automata over I such that S and S' are finite sets of states. Then $\mathcal{L}(\mathcal{A} \cap \mathcal{A}') = \mathcal{L}(\mathcal{A}) \cap \mathcal{L}(\mathcal{A}')$.*

Proof. Assume $\mathcal{A} = (S, I, T, Z, F)$ and $\mathcal{A}' = (S', I, T', Z', F')$ are two ST-automata over I , and S, S' are finite sets of states such that

$$\mathcal{A} \cap \mathcal{A}' = \mathcal{A}'' = (S'', I, T'', Z'', F'').$$

We want to prove the following:

$$(A1) \quad \mathcal{L}(\mathcal{A} \cap \mathcal{A}') \subseteq \mathcal{L}(\mathcal{A}) \cap \mathcal{L}(\mathcal{A}').$$

$$(A2) \quad \mathcal{L}(\mathcal{A}) \cap \mathcal{L}(\mathcal{A}') \subseteq \mathcal{L}(\mathcal{A} \cap \mathcal{A}').$$

By the definition of $\mathcal{A} \cap \mathcal{A}'$, we get to the following:

$$\begin{aligned} S'' &= S \times S', \\ Z'' &= Z \times Z', \\ F'' &= F \times F', \text{ and} \\ T'' &= T_1 \cup T_2 \cup T_3, \end{aligned}$$

where

$$\begin{aligned} T_1 &= \{(s, a, t) : s = (s_1, s_2), t = (t_1, t_2), (s_1, a, t_1) \in T, \text{ and } (s_2, a, t_2) \in T'\}, \\ T_2 &= \{(A, s) : A \subseteq S'', s = (s_1, s_2), (\pi_1(A), s_1) \in T, \text{ and } (\pi_2(A), s_2) \in T'\}, \text{ and} \\ T_3 &= \{(A, B, s) : A, B \subseteq S'', s = (s_1, s_2), (\pi_1(A), \pi_1(B), s_1) \in T, \text{ and } (\pi_2(A), \pi_2(B), s_2) \in T'\}. \end{aligned}$$

Proof of (A1). $\mathcal{L}(\mathcal{A} \cap \mathcal{A}') \subseteq \mathcal{L}(\mathcal{A}) \cap \mathcal{L}(\mathcal{A}')$.

Assume $u \in \mathcal{L}(\mathcal{A} \cap \mathcal{A}')$, such that $u : \alpha \rightarrow I$ and $\alpha \in \text{ord}$. Then there exists an accepting run $R : \alpha + 1 \rightarrow S''$ of \mathcal{A}'' on u . We want to prove $u \in \mathcal{L}(\mathcal{A}) \cap \mathcal{L}(\mathcal{A}')$. Then we should prove $u \in \mathcal{L}(\mathcal{A})$ and $u \in \mathcal{L}(\mathcal{A}')$.

First we want to prove $u \in \mathcal{L}(\mathcal{A})$. Thus we need to define an accepting run $H : \alpha + 1 \rightarrow S$ of \mathcal{A} on u . Now we define H as follows:

$$H(\beta) = s, \text{ when } R(\beta) = (s, t) \text{ for each } \beta \leq \alpha.$$

Now we want to prove H is an accepting run of \mathcal{A} on u . That we should prove the following: $H(0) \in Z$, $H(\alpha) \in F$, and satisfies the following conditions:

1. For each $\beta < \alpha$ we have

$$(H(\beta), u(\beta), H(\beta + 1)) \in T.$$

2. For each $\beta \leq \alpha$ that is a limit ordinal with $\text{cf}(\beta) = \omega$, we have

$$(\sup_{\beta} (H), H(\beta)) \in T.$$

3. For each $\beta \leq \alpha$ that is a limit ordinal with $\text{cf}(\beta) > \omega$, we have

$$(\sup_{\beta} (H), \text{stat}_{\beta}(H), H(\beta)) \in T.$$

Since R is an accepting run of \mathcal{A}'' on u , then $R(0) \in Z''$, and $R(\alpha) \in F''$, thus

$$R(0) = (s, s'), \text{ for some } (s, s') \in Z'', \text{ and}$$

$$R(\alpha) = (t, t'), \text{ for some } (t, t') \in F''.$$

By the definition of H we get $H(0) = s \in Z$, and $H(\alpha) = t \in F$.

It remains to prove the above three conditions. First we want to satisfy condition (1). Assume $\beta < \alpha$, and we want to show

$$(H(\beta), u(\beta), H(\beta + 1)) \in T.$$

Since R is an accepting run of \mathcal{A}'' on u , then $(R(\beta), u(\beta), R(\beta + 1)) \in T''$, so

$$(R(\beta), u(\beta), R(\beta + 1)) \in T_1,$$

let

$$(R(\beta), u(\beta), R(\beta + 1)) = (s, a, t) \text{ where}$$

$$s = (s_1, s_2), t = (t_1, t_2), (s_1, a, t_1) \in T \text{ and } (s_2, a, t_2) \in T'.$$

Then by the definition of H , we get

$$(H(\beta), u(\beta), H(\beta + 1)) = (s_1, a, t_1),$$

but $(s_1, a, t_1) \in T$, therefore $(H(\beta), u(\beta), H(\beta + 1)) \in T$.

Now we want to satisfy condition (2). Let $\beta \leq \alpha$ be a limit ordinal with $\text{cf}(\beta) = \omega$ and we want to prove

$$(\sup_{\beta}(H), H(\beta)) \in T.$$

Since R is an accepting run of \mathcal{A}'' on u , then

$$(\sup_{\beta}(R), R(\beta)) \in T'',$$

so $(\sup_{\beta}(R), R(\beta)) \in T_2$, but

$$T_2 = \{(A, s) : s = (s_1, s_2) \text{ and } (\pi_1(A), s_1) \in T, (\pi_2(A), s_2) \in T'\},$$

therefore $(\pi_1(\sup_\beta(R)), s_1) \in T$, where $R(\beta) = (s_1, s_2)$. Claim that

$$\sup_\beta(H) = \pi_1(\sup_\beta(R)),$$

therefore,

$$(\sup_\beta(H), H(\beta)) = (\pi_1(\sup_\beta(R)), s_1) \in T.$$

It is remains to prove our claim. We know that,

$$\pi_1(\sup_\beta(R)) = \{s \in S : (s, s') \in \sup_\beta(R) \text{ for some } s' \in S'\}, \text{ and}$$

$$\sup_\beta(H) = \{s \in S : \{\gamma < \beta : H(\gamma) = s\} \text{ is cofinal in } \beta\}.$$

Now, let $s \in \pi_1(\sup_\beta(R))$, then $(s, s') \in \sup_\beta(R)$ for some $s' \in S'$, implies to

$$D = \{\gamma < \beta : R(\gamma) = (s, s')\},$$

is cofinal in β , therefore $D \subseteq D'$, such that

$$D' = \{\gamma < \beta : H(\gamma) = s\},$$

is also cofinal in β , this implies to $s \in \sup_\beta(H)$. Therefore $\pi_1(\sup_\beta(R)) \subseteq \sup_\beta(H)$.

Now we want to prove $\sup_\beta(H) \subseteq \pi_1(\sup_\beta(R))$. Let $s \in \sup_\beta(H)$ be a fixed element. Then

$$D = \{\gamma < \beta : H(\gamma) = s\},$$

is cofinal in β . For each $s' \in S'$, let

$$D_{s'} = \{\gamma < \beta : R(\gamma) = (s, s')\},$$

then

$$D = \bigcup_{s' \in S'} D_{s'}.$$

Since S' is a finite set, then there exists $s' \in S'$ such that $D_{s'}$ is cofinal in β , to show that suppose by the way of a contradiction that, for each $s' \in S'$, $D_{s'}$ is not cofinal in β . Thus for each $s' \in S'$, let $\gamma_{s'} < \beta$ is an upper bounded on $D_{s'}$. Now choose

$$\gamma = \max\{\gamma_{s'} : s' \in S'\},$$

exists since S' is a finite set, therefore γ is an upper bounded on D , thus $\gamma < \beta$ and that is a

contradiction because D is cofinal in β .

That implies to $(s, s') \in \text{sup}_\beta(R)$, hence $s \in \pi_1(\text{sup}_\beta(R))$. Then $\text{sup}_\beta(H) \subseteq \pi_1(\text{sup}_\beta(R))$. Therefore $\pi_1(\text{sup}_\beta(R)) = \text{sup}_\beta(H)$.

Finally, we want to satisfy condition three. Let $\beta \leq \alpha$ be a limit ordinal with $\text{cf}(\beta) > \omega$, and we want to prove

$$(\text{sup}_\beta(H), \text{stat}_\beta(H), H(\beta)) \in T.$$

Since R is an accepting run of \mathcal{A}'' on u , then

$$(\text{sup}_\beta(R), \text{stat}_\beta(R), R(\beta)) \in T'',$$

therefore $(\text{sup}_\beta(R), \text{stat}_\beta(R), R(\beta)) \in T_3$, but

$$T_3 = \{(A, B, s) : s = (s_1, s_2) \text{ and } (\pi_1(A), \pi_1(B), s_1) \in T, (\pi_2(A), \pi_2(B), s_2) \in T'\},$$

which implies to

$$(\pi_1(\text{sup}_\beta(R)), \pi_1(\text{stat}_\beta(R)), s_1) \in T \text{ where } R(\beta) = (s_1, s_2).$$

Claim that

$$\text{sup}_\beta(H) = \pi_1(\text{sup}_\beta(R)), \text{ and } \text{stat}(H) = \pi_1(\text{stat}_\beta(R)),$$

therefore $(\text{sup}_\beta(H), \text{stat}_\beta(H), H(\beta)) \in T$.

It is remains to prove our claim. We show early that $\text{sup}_\beta(H) = \pi_1(\text{sup}_\beta(R))$. Now we want to prove $\text{stat}_\beta(H) = \pi_1(\text{stat}_\beta(R))$. We know that,

$$\text{stat}_\beta(H) = \{s \in S : \{\gamma < \beta : H(\gamma) = s\} \text{ is stationary in } \beta\}, \text{ and}$$

$$\pi_1(\text{stat}_\beta(R)) = \{s \in S : (s, s') \in \text{stat}_\beta(R) \text{ for some } s' \in S'\}.$$

Let $s \in \pi_1(\text{stat}_\beta(R))$, then $(s, s') \in \text{stat}_\beta(R)$ for some $s' \in S'$, implies to

$$D = \{\gamma < \beta : R(\gamma) = (s, s')\},$$

is stationary in β , therefore $D \subseteq D'$, such that

$$D' = \{\gamma < \beta : H(\gamma) = s\},$$

is also stationary in β , this implies to $s \in \text{stat}_\beta(H)$. Therefore $\pi_1(\text{stat}_\beta(R)) \subseteq \text{stat}_\beta(H)$.

Now we want to prove $\text{stat}_\beta(H) \subseteq \pi_1(\text{stat}_\beta(R))$. Let $s \in \text{stat}_\beta(H)$ be a fixed element, then

$$D = \{\gamma < \beta : H(\gamma) = s\},$$

is a stationary set in β . For each $s' \in S'$, let

$$D_{s'} = \{\gamma < \beta : R(\gamma) = (s, s')\},$$

then

$$D = \bigcup_{s' \in S'} D_{s'}.$$

Since S' is a finite set, then there exists $s' \in S'$ such that $D_{s'}$ is stationary in β , to show that suppose by the way of a contradiction $D_{s'}$ is not stationary in β for every $s' \in S'$. Thus for every $s' \in S'$, there is a club $C_{s'}$ in β such that $C_{s'} \cap D_{s'} = \emptyset$. Let

$$C = \bigcap_{s' \in S'} C_{s'},$$

is also a club because S' is finite, which implies to

$$C \cap D = \emptyset,$$

which is a contradiction because C is a club in β and D is stationary in β . Then $(s, s') \in \text{stat}_\beta(R)$, which implies to $s \in \pi_1(\text{sup}_\beta(R))$. Then $\text{stat}_\beta(H) \subseteq \pi_1(\text{stat}_\beta(R))$.

Hence $\pi_1(\text{stat}_\beta(R)) = \text{stat}_\beta(H)$. Therefore, H is an accepting run of \mathcal{A} on u . Then $u \in \mathcal{L}(\mathcal{A})$. By the same way we can show that $u \in \mathcal{L}(\mathcal{A}')$. Therefore $\mathcal{L}(\mathcal{A} \cap \mathcal{A}') \subseteq \mathcal{L}(\mathcal{A}) \cap \mathcal{L}(\mathcal{A}')$.

Proof of (A2). $\mathcal{L}(\mathcal{A}) \cap \mathcal{L}(\mathcal{A}') \subseteq \mathcal{L}(\mathcal{A} \cap \mathcal{A}')$.

Assume $u \in \mathcal{L}(\mathcal{A}) \cap \mathcal{L}(\mathcal{A}')$, and $u : \alpha \rightarrow I$ where $\alpha \in \text{ord}$. Then $u \in \mathcal{L}(\mathcal{A})$ and $u \in \mathcal{L}(\mathcal{A}')$ therefore there exists an accepting run $R : \alpha + 1 \rightarrow S$ of \mathcal{A} on u and an accepting run $R' : \alpha + 1 \rightarrow S'$ of \mathcal{A}' on u . Now we want to define an accepting run H of \mathcal{A}'' on u . Define $H : \alpha + 1 \rightarrow S''$ as follows:

$$H(\beta) = (R(\beta), R'(\beta)), \text{ for each } \beta \leq \alpha.$$

Now we want to prove H is an accepting run of \mathcal{A}'' on u , that is, $H(0) \in Z''$, $H(\alpha) \in F''$ and satisfies the following conditions:

1. For each $\beta < \alpha$ we have

$$(H(\beta), u(\beta), H(\beta + 1)) \in T''.$$

2. For each $\beta \leq \alpha$ that is a limit ordinal with $\text{cf}(\beta) = \omega$, we have

$$(\sup_{\beta}(H), H(\beta)) \in T''.$$

3. For each $\beta \leq \alpha$ that is a limit ordinal with $\text{cf}(\beta) > \omega$, we have

$$(\sup_{\beta}(H), \text{stat}_{\beta}(H), H(\beta)) \in T''.$$

First we want to prove $H(0) \in Z''$ and $H(\alpha) \in F''$. We have R and R' are accepting runs of \mathcal{A} and \mathcal{A}' on u respectively and by the definition of H , we get $H(0) \in Z''$, and $H(\alpha) \in F''$.

Now we want to show condition (1). let $\beta < \alpha$, and we want to prove

$$(H(\beta), u(\beta), H(\beta + 1)) \in T''.$$

Again since R and R' are accepting runs of \mathcal{A} and \mathcal{A}' on u respectively, then $(R(\beta), u(\beta), R(\beta + 1)) \in T$ and $(R'(\beta), u(\beta), R'(\beta + 1)) \in T'$, and since

$$T_1 = \{(s, a, t) : s = (s_1, s_2), t = (t_1, t_2) (s_1, a, t_1) \in T \text{ and } (s_2, a, t_2) \in T'\},$$

thus

$$(H(\beta), u(\beta), H(\beta + 1)) = ((R(\beta), R'(\beta)), u(\beta), (R(\beta + 1), R'(\beta + 1))) \in T_1,$$

therefore, $(H(\beta), u(\beta), H(\beta + 1)) \in T''$.

Now we want to show condition two. Let $\beta \leq \alpha$ be a limit ordinal with $\text{cf}(\beta) = \omega$, and we want to prove

$$(\sup_{\beta}(H), H(\beta)) \in T''.$$

Since R and R' are accepting runs of \mathcal{A} and \mathcal{A}' on u respectively, then $(\sup_{\beta}(R), R(\beta)) \in T$ and $(\sup_{\beta}(R'), R'(\beta)) \in T'$. Since $H(\beta) = (R(\beta), R'(\beta))$ and

$$T_2 = \{(A, s) : s = (s_1, s_2) \text{ and } (\pi_1(A), s_1) \in T, (\pi_2(A), s_2) \in T'\},$$

so its enough to show that

$$(\pi_1(\sup_{\beta}(H)), R(\beta)) \in T \text{ and } (\pi_2(\sup_{\beta}(H)), R'(\beta)) \in T'.$$

Therefore, it suffices to show that

$$\pi_1(\sup_\beta(H)) = \sup_\beta(R) \text{ and } \pi(\sup_\beta(H)) = \sup_\beta(R'),$$

and by the same way early we can show that.

Now we want to prove condition (3). Let $\beta \leq \alpha$ be a limit ordinal with $\text{cf}(\beta) > \omega$, and we want to prove

$$(\sup_\beta(H), \text{stat}_\beta(H), H(\beta)) \in T''.$$

Since R and R' are accepting runs of \mathcal{A} and \mathcal{A}' on u respectively, then $(\sup_\beta(R), \text{stat}_\beta(R), R(\beta)) \in T$ and $(\sup_\beta(R'), \text{stat}_\beta(R'), R'(\beta)) \in T'$. Since $H(\beta) = (R(\beta), R'(\beta))$, and

$$T_3 = \{(A, B, s) : s = (s_1, s_2) \text{ and } (\pi_1(A), \pi_1(B), s_1) \in T, (\pi_2(A), \pi_2(B), s_2) \in T'\},$$

so its enough to show that

$$(\pi_1(\sup_\beta(H)), \pi_1(\text{stat}_\beta(H)), R(\beta)) \in T \text{ and } (\pi_2(\sup_\beta(H)), \pi_2(\text{stat}_\beta(H)), R'(\beta)) \in T'.$$

Therefore, it suffices to show that

$$\pi_1(\sup_\beta(H)) = \sup_\beta(R) \text{ and } \pi(\text{stat}_\beta(H)) = \text{stat}_\beta(R) \text{ and}$$

$$\pi_2(\sup_\beta(H)) = \sup_\beta(R') \text{ and } \pi(\text{stat}_\beta(H)) = \text{stat}_\beta(R'),$$

and by the same way early we can show that.

Then H is an accepting run of \mathcal{A}'' on u . Thus, $u \in \mathcal{L}(\mathcal{A} \cap \mathcal{A}')$. Therefore $\mathcal{L}(\mathcal{A}) \cap \mathcal{L}(\mathcal{A}') \subseteq \mathcal{L}(\mathcal{A} \cap \mathcal{A}')$. From proof of (A1), and (A2), we get $\mathcal{L}(\mathcal{A} \cap \mathcal{A}') = \mathcal{L}(\mathcal{A}) \cap \mathcal{L}(\mathcal{A}')$. \square

Then we define the concatenation ST-automaton as the following:

4.3 Concatenation Operation

Then we define the concatenation ST-automaton as the following:

Definition 4.3.1. Let $\mathcal{A} = (S, I, T, Z, F)$ and $\mathcal{A}' = (S', I, T', Z', F')$ be two ST-automata over I such that $S \cap S' = \emptyset$. Define the concatenation ST-automaton over I of \mathcal{A} and \mathcal{A}' denoted by $\mathcal{A} \circ \mathcal{A}'$ as follows:

let

$$\mathcal{A} \circ \mathcal{A}' = (S'', I, T'', Z'', F''),$$

such that

$$\begin{aligned} S'' &= S \cup S', \\ Z'' &= \begin{cases} Z & \text{if } \emptyset \notin \mathcal{L}(\mathcal{A}) \\ Z \cup Z' & \text{if } \emptyset \in \mathcal{L}(\mathcal{A}) \end{cases}, \\ F'' &= \begin{cases} F' & \text{if } \emptyset \notin \mathcal{L}(\mathcal{A}') \\ F \cup F' & \text{if } \emptyset \in \mathcal{L}(\mathcal{A}') \end{cases}, \\ T'' &= T \cup T' \cup T_1 \cup T_2 \cup T_3, \end{aligned}$$

where

$$\begin{aligned} T_1 &= \{(s, a, s') : s \in S, a \in I, s' \in Z' \text{ such that } (s, a, t) \in T \text{ for some } t \in F\}, \\ T_2 &= \{(A, s') : A \subseteq S, s' \in Z' \text{ such that } (A, t) \in T \text{ for some } t \in F\}, \text{ and} \\ T_3 &= \{(A, B, s') : A, B \subseteq S, s' \in Z' \text{ such that } (A, B, t) \in T \text{ for some } t \in F\}. \end{aligned}$$

By proving the following theorem we show that, applying the concatenation operation to languages defined by ST-automata, the produce language that is also definable using ST-automaton.

Theorem 4.3.2. *Let $\mathcal{A} = (S, I, T, Z, F)$ and $\mathcal{A}' = (S', I, T', Z', F')$ be two ST-automata over I such that $S \cap S' = \emptyset$. Then $\mathcal{L}(\mathcal{A} \circ \mathcal{A}') = \mathcal{L}(\mathcal{A}) \circ \mathcal{L}(\mathcal{A}')$.*

Proof. Assume $\mathcal{A} = (S, I, T, Z, F)$ and $\mathcal{A}' = (S', I, T', Z', F')$ are two ST-automata over I such that $S \cap S' = \emptyset$. Let $\mathcal{A} \circ \mathcal{A}' = \mathcal{A}'' = (S'', I, T'', Z'', F'')$. By the definition of $\mathcal{A} \circ \mathcal{A}'$, we get

$$\begin{aligned} S'' &= S \cup S', \\ Z'' &= \begin{cases} Z & \text{if } \emptyset \notin \mathcal{L}(\mathcal{A}) \\ Z \cup Z' & \text{if } \emptyset \in \mathcal{L}(\mathcal{A}) \end{cases}, \\ F'' &= \begin{cases} F' & \text{if } \emptyset \notin \mathcal{L}(\mathcal{A}') \\ F \cup F' & \text{if } \emptyset \in \mathcal{L}(\mathcal{A}') \end{cases}, \\ T'' &= T \cup T' \cup T_1 \cup T_2 \cup T_3, \end{aligned}$$

where

$$\begin{aligned} T_1 &= \{(s, a, s') : s \in S, a \in I, s' \in Z' \text{ such that } (s, a, t) \in T \text{ for some } t \in F\}, \\ T_2 &= \{(A, s') : A \subseteq S, s' \in Z' \text{ such that } (A, t) \in T \text{ for some } t \in F\}, \text{ and} \\ T_3 &= \{(A, B, s') : A, B \subseteq S, s' \in Z' \text{ such that } (A, B, t) \in T \text{ for some } t \in F\}. \end{aligned}$$

We want to prove the following:

$$(A1) \mathcal{L}(\mathcal{A} \circ \mathcal{A}') \subseteq \mathcal{L}(\mathcal{A}) \circ \mathcal{L}(\mathcal{A}').$$

$$(A2) \mathcal{L}(\mathcal{A}) \circ \mathcal{L}(\mathcal{A}') \subseteq \mathcal{L}(\mathcal{A} \circ \mathcal{A}').$$

Proof of (A1). $\mathcal{L}(\mathcal{A} \circ \mathcal{A}') \subseteq \mathcal{L}(\mathcal{A}) \circ \mathcal{L}(\mathcal{A}')$.

Assume $u \in \mathcal{L}(\mathcal{A} \circ \mathcal{A}')$ such that $u : \alpha \rightarrow I$, and $\alpha \in \text{ord}$. Then there is an accepting run $R : \alpha + 1 \rightarrow S''$ of \mathcal{A}'' on u . We want to prove $u \in \mathcal{L}(\mathcal{A}) \circ \mathcal{L}(\mathcal{A}')$. Since $\mathcal{L}(\mathcal{A})$ and $\mathcal{L}(\mathcal{A}')$ are two classes of words over the same alphabet I , then

$$\mathcal{L}(\mathcal{A}) \circ \mathcal{L}(\mathcal{A}') = \{u_0 \circ u_1 : u_0 \in \mathcal{L}(\mathcal{A}) \text{ and } u_1 \in \mathcal{L}(\mathcal{A}')\},$$

so we need to show that

$$u = u_0 \circ u_1,$$

for some $u_0 \in \mathcal{L}(\mathcal{A})$, and $u_1 \in \mathcal{L}(\mathcal{A}')$, where $u_0 : \alpha_0 \rightarrow I$, and $u_1 : \alpha_1 \rightarrow I$, with $\alpha = \alpha_0 + \alpha_1$, and $\alpha_0, \alpha_1 \in \text{ord}$. Now since R is an accepting run of \mathcal{A}'' on u , and $S'' = S \cup S'$, so we have the following two cases:

(B1) Either $R(\alpha) \in S$,

(B2) Or $R(\alpha) \in S'$.

Case (B1). If $R(\alpha) \in S$.

Since R is an accepting run of \mathcal{A}'' on u , then $R(\alpha) \in F''$, but $R(\alpha) \in S$, and $S \cap S' = \emptyset$ thus we must have $R(\alpha) \in F$, therefore $\emptyset \in \mathcal{L}(\mathcal{A}')$. Hence choose

$$\alpha_0 = \alpha, \alpha_1 = 0 \text{ and } u_0 = u, u_1 = \emptyset.$$

It is clear that $H : \alpha_0 + 1 \rightarrow S$, with $H = R$ is an accepting run of \mathcal{A} on u_0 . Therefore $u_0 \in \mathcal{L}(\mathcal{A})$.

Now since $\emptyset \in \mathcal{L}(\mathcal{A}')$, then H' is an accepting run of \mathcal{A}' on u_1 . Then $u_1 \in \mathcal{L}(\mathcal{A}')$.

Therefore $u_0 \in \mathcal{L}(\mathcal{A})$ and $u_1 \in \mathcal{L}(\mathcal{A}')$. Hence $u \in \mathcal{L}(\mathcal{A}) \circ \mathcal{L}(\mathcal{A}')$. Therefore, $\mathcal{L}(\mathcal{A} \circ \mathcal{A}') \subseteq \mathcal{L}(\mathcal{A}) \circ \mathcal{L}(\mathcal{A}')$.

Case (B2). If $R(\alpha) \in S'$.

Assume $B = \{\gamma \leq \alpha : R(\gamma) \in S'\}$. Then $B \neq \emptyset$ because $R(\alpha) \in S'$, therefore choose

$$\alpha_0 = \min(B)$$

$$\alpha_1 \text{ such that } \alpha = \alpha_0 + \alpha_1 \text{ and}$$

$$u_0 = u \upharpoonright_{\alpha_0}, \text{ and } u_1(\delta) = u(\alpha_0 + \delta) \text{ for each } 0 \leq \delta \leq \alpha_1.$$

Now we want to define $H : \alpha_0 + 1 \rightarrow S$ and $H' : \alpha_1 + 1 \rightarrow S'$ to be accepting runs of \mathcal{A} on u_0 and \mathcal{A}' on u_1 respectively. Then we have two cases:

(C1) Either $\alpha_0 = 0$,

(C2) Or $\alpha_0 > 0$.

Case (C1). Assume $\alpha_0 = 0$.

Since $\alpha_0 = \min(B)$ and $\alpha_0 = 0$, then the only possibility that we have $R(0) \in S'$ by the definition of B . Since R is an accepting run of \mathcal{A}'' on u , then $R(0) \in Z''$, but $R(0) \in S'$, so we must have $R(0) \in Z'$, and by the definition of \mathcal{A}'' we get that $\emptyset \in \mathcal{L}(\mathcal{A})$. Thus there is $t \in Z \cap F$ such that $H(0) = H(\alpha_0) = t$, therefore H is an accepting run of \mathcal{A} on u_0 . Then $u_0 \in \mathcal{L}(\mathcal{A})$.

Define $H' = R$, it is clear that H' is an accepting run of \mathcal{A}' on u_1 . Thus $u_1 \in \mathcal{L}(\mathcal{A}')$.

Therefore $u_0 \in \mathcal{L}(\mathcal{A})$ and $u_1 \in \mathcal{L}(\mathcal{A}')$. Hence $u \in \mathcal{L}(\mathcal{A}) \circ \mathcal{L}(\mathcal{A}')$. Therefore, $\mathcal{L}(\mathcal{A} \circ \mathcal{A}') \subseteq \mathcal{L}(\mathcal{A}) \circ \mathcal{L}(\mathcal{A}')$.

Case (C2). Assume $\alpha_0 > 0$.

We need to define H and H' . Now since $\alpha_0 = \min(B)$, then $R(\alpha_0) \in S'$, by the definition of B , therefore

$$R(\delta) \in S \text{ for each } \delta < \alpha_0.$$

Then we will discuss the following cases:

(D1) If α_0 is a successor ordinal.

(D2) If α_0 is a limit ordinal.

Case (D1). If α_0 is a successor ordinal.

Assume $\alpha_0 = \sigma + 1$, for some $\sigma \in \text{ord}$. Now Since R is an accepting run of \mathcal{A}'' on u , then

$$(R(\sigma), u(\sigma), R(\sigma + 1)) = (R(\sigma), u(\sigma), R(\alpha_0)) \in T''$$

but $R(\sigma) \in S$, $u(\sigma) \in I$, and $R(\alpha_0) \in S'$, thus we must have $(R(\sigma), u(\sigma), R(\sigma + 1)) \in T_1$, by the definition of T'' , but

$$T_1 = \{(s, a, s') : s \in S, a \in I, s' \in Z' \text{ such that there is } t \in F \text{ with } (s, a, t) \in T\},$$

therefore,

$$R(\sigma + 1) = R(\alpha_0) \in Z' \text{ and } (R(\sigma), u(\sigma), t) \in T,$$

for some $t \in F$.

Now we define H and H' as the following:

First define $H : \alpha_0 + 1 \rightarrow S$ by

$$H(\beta) = \begin{cases} R(\beta) & \text{for each } \beta < \alpha_0. \\ t & \beta = \alpha_0. \end{cases}$$

Define $H' : \alpha_1 + 1 \rightarrow S'$ by

$$H'(\delta) = R(\alpha_0 + \delta) \text{ for } 0 \leq \delta \leq \alpha_1.$$

Now we want to prove H is an accepting run of \mathcal{A} on u_0 , that is, $H(0) \in Z$, $H(\alpha_0) \in F$ and satisfies the following conditions:

1. For each $\beta < \alpha_0$ we have

$$(H(\beta), u_0(\beta), H(\beta + 1)) \in T.$$

2. For each $\beta \leq \alpha_0$ that is a limit ordinal with $\text{cf}(\beta) = \omega$, we have

$$(\sup_{\beta} H, H(\beta)) \in T.$$

3. For each $\beta \leq \alpha_0$ that is a limit ordinal with $\text{cf}(\beta) > \omega$, we have

$$(\sup_{\beta} H, \text{stat}_{\beta} H, H(\beta)) \in T.$$

First we want to show that $H(0) \in Z$, and $H(\alpha_0) \in F$. Since R is an accepting run of \mathcal{A}'' on u , then $R(0) \in Z''$ but $\alpha_0 > 0$ and as we show early that

$$R(\delta) \in S \text{ for each } \delta < \alpha_0,$$

therefore $R(0) \in S$ but $R(0) \in Z''$, this implies to $R(0) \in Z$, hence $H(0) \in Z$, since $H(0) = R(0)$ by the definition of H and by the definition of H it is clear that $H(\alpha_0) \in F$.

It remains to prove the three conditions.

Now we want to prove condition (1). Let $\beta < \alpha_0$, and we want to prove

$$(H(\beta), u_0(\beta), H(\beta + 1)) \in T.$$

Since R is an accepting run of \mathcal{A}'' on u , then $(R(\beta), u(\beta), R(\beta + 1)) \in T''$, for each $\beta < \alpha$, but

$$(H(\beta), u_0(\beta), H(\beta + 1)) = (R(\beta), u \upharpoonright_{\alpha_0}(\beta), R(\beta + 1)) = (R(\beta), u(\beta), R(\beta + 1))$$

for each $\beta < \alpha_0$, by the definition of H and u_0 and $R(\delta) \in S$ for each $\delta < \alpha_0$, therefore

$$(H(\beta), u_0(\beta), H(\beta + 1)) \in T.$$

Hence, for each $\beta < \alpha_0$, we get $(H(\beta), u_0(\beta), H(\beta + 1)) \in T$.

Next we want to prove condition (2). Let $\beta \leq \alpha_0$ be a limit ordinal with $\text{cf}(\beta) = \omega$ and we want to prove

$$(\sup_{\beta}(H), H(\beta)) \in T.$$

Again since R is an accepting run of \mathcal{A}'' on u , then $(\sup_{\beta}(R), R(\beta)) \in T''$ for each limit ordinal β , where $\beta \leq \alpha$, with $\text{cf}(\beta) = \omega$. We know that,

$$\sup_{\beta}(H) = \{s \in S : \{\gamma < \beta : H(\gamma) = s\} \text{ is cofinal in } \beta\} \quad \text{and}$$

$$\sup_{\beta}(R) = \{s \in S'' : \{\gamma < \beta : R(\gamma) = s\} \text{ is cofinal in } \beta\}.$$

It remains to show that for any limit ordinal $\beta \leq \alpha_0$ we have that $\sup_{\beta}(H) = \sup_{\beta}(R)$ and this is clear for each $\beta < \alpha_0$, since $H(\beta) = R(\beta)$ for $\beta < \alpha_0$, by the definition of H and in this case $\sup_{\beta}(R) \subseteq \mathcal{P}(S)$, and $R(\beta) \in S$, but $(\sup_{\beta}(R), R(\beta)) \in T''$ for each limit ordinal, $\beta \leq \alpha$, therefore for any limit ordinal $\beta < \alpha_0$ we have that

$$(\sup_{\beta}(H), H(\beta)) = (\sup_{\beta}(R), R(\beta)) \in T,$$

and for $\beta = \alpha_0$,

$$(\sup_{\beta}(H), H(\beta)) = (\sup_{\beta}(H), t) = (\sup_{\beta}(R), t) \in \mathcal{P}(S) \times S,$$

which implies to for any limit ordinal $\beta \leq \alpha_0$, $(\sup_{\beta}(H), H(\beta)) \in T$.

Now we want to prove condition (3). Let $\beta \leq \alpha_0$ that is a limit ordinal with $\text{cf}(\beta) > \omega$, we want to prove

$$(\sup_{\beta}(H), \text{stat}_{\beta}(H), H(\beta)) \in T.$$

Again since R is an accepting run of \mathcal{A}'' on u , then $(\sup_{\beta}(R), \text{stat}_{\beta}(R), R(\beta)) \in T''$. We know that,

$$\sup_{\beta}(H) = \{s \in S : \{\gamma < \beta : H(\gamma) = s\} \text{ is cofinal in } \beta\}, \quad \text{and}$$

$$\text{stat}_\beta(H) = \{s \in S : \{\gamma < \beta : H(\gamma) = s\} \text{ is stationary in } \beta\},$$

so it is enough to show that for any limit ordinal $\beta \leq \alpha_0$, we have that $\sup_\beta(H) = \sup_\beta(R)$ and $\text{stat}_\beta(H) = \text{stat}_\beta(R)$.

By the same way above we can see that $\sup_\beta(H) = \sup_\beta(R)$, and $\text{stat}_\beta(H) = \text{stat}_\beta(R)$. Therefore, for each limit ordinal $\beta \leq \alpha_0$, with $\text{cf}(\beta) > \omega$, we get $(\sup_\beta(H), \text{stat}_\beta(H), H(\beta)) \in T$.

Therefore H satisfies the three conditions. Hence H is an accepting run of \mathcal{A} on u_0 . Therefore $u_0 \in \mathcal{L}(\mathcal{A})$.

As we define before $H' : \alpha_1 + 1 \rightarrow S'$ with

$$H'(\delta) = R(\alpha_0 + \delta) \text{ for } 0 \leq \delta \leq \alpha_1.$$

Now we want to prove H' is an accepting run of \mathcal{A}' on u_1 , that is, $H'(0) \in Z'$, $H'(\alpha_1) \in F'$ and satisfies the following conditions:

1. For each $\beta < \alpha_1$ we have

$$(H'(\beta), u_1(\beta), H'(\beta + 1)) \in T'.$$

2. For each $\beta \leq \alpha_1$ that is a limit ordinal with $\text{cf}(\beta) = \omega$, we have

$$(\sup_\beta(H'), H'(\beta)) \in T'.$$

3. For each $\beta \leq \alpha_1$ that is a limit ordinal with $\text{cf}(\beta) > \omega$, we have

$$(\sup_\beta(H'), \text{stat}_\beta(H'), H'(\beta)) \in T'.$$

First we want to prove $H'(0) \in Z'$, and $H'(\alpha_1) \in F'$. By the definition of H' we get $H'(0) = R(\alpha_0) \in Z'$ as we shown early and $H'(\alpha_1) = R(\alpha_0 + \alpha_1) = R(\alpha) \in F'$ because R is an accepting run of \mathcal{A}'' on u but we have $R(\alpha) \in S'$, so we must have $R(\alpha) \in F'$. Hence $H'(\alpha_1) \in F'$.

It is remains to prove the three conditions.

Now we want to prove condition (1). Let $\beta < \alpha_1$ and we want to prove $(H'(\beta), u_1(\beta), H'(\beta + 1)) \in T'$. Since R is an accepting run of \mathcal{A}'' on u , then $(R(\beta), u(\beta), R(\beta + 1)) \in T''$, for each $\beta < \alpha$, but

$$(H'(\beta), u_1(\beta), H'(\beta + 1)) = (R(\alpha_0 + \beta), u(\alpha_0 + \beta), R(\alpha_0 + \beta + 1))$$

for each $\beta \leq \alpha_1$, by the definition of H' and u_1 and we have $(H'(\beta), u_1(\beta), H'(\beta + 1)) \in S' \times I \times S'$,

so we must have

$$(H'(\beta), u_1(\beta), H'(\beta + 1)) \in T'.$$

Therefore, for each $\beta < \alpha_1$, we get $(H'(\beta), u_1(\beta), H'(\beta + 1)) \in T'$.

Next we want to prove condition (2). Let $\beta \leq \alpha_1$ be a limit ordinal with $\text{cf}(\beta) = \omega$ and we want to prove

$$(\sup_{\beta}(H'), H'(\beta)) \in T'.$$

Again since R is an accepting run of \mathcal{A}'' on u , then $(\sup_{\alpha_0+\beta}(R), R(\alpha_0 + \beta)) \in T''$ for each $\beta \leq \alpha_1$.

$$\sup_{\beta}(H') = \{s \in S' : \{\gamma < \beta : H'(\gamma) = s\} \text{ is cofinal in } \beta\} \text{ and}$$

$$\sup_{\beta}(R) = \{s \in S'' : \{\gamma < \beta : R(\gamma) = s\} \text{ is cofinal in } \beta\}.$$

By lemma 1.2.40, we get $\alpha_0 + \beta \leq \alpha$ is a limit ordinal with $\text{cf}(\alpha_0 + \beta) = \omega$, and since $\sup_{\beta}(H') = \sup_{\alpha_0+\beta}(R)$ and in this case $\sup_{\alpha_0+\beta}(R) \subseteq \mathcal{P}(S')$, and $R(\alpha_0 + \beta) \in S'$, but $(\sup_{\alpha_0+\beta}(R), R(\alpha_0 + \beta)) \in T''$ for each $\beta < \alpha$, therefore for any limit ordinal $\beta \leq \alpha_1$ we have that

$$(\sup_{\beta}(H'), H'(\beta)) = (\sup_{\alpha_0+\beta}(R), R(\alpha_0 + \beta)) \in T',$$

which implies that, for any limit ordinal $\beta \leq \alpha_1$, with $\text{cf}(\beta) = \omega$, $(\sup_{\beta}(H'), H'(\beta)) \in T'$.

Now we want to prove condition (3). Let $\beta \leq \alpha_1$ be a limit ordinal with $\text{cf}(\beta) > \omega$, we want to prove

$$(\sup_{\beta}(H'), \text{stat}_{\beta}(H'), H'(\beta)) \in T'.$$

By lemma 1.2.40, we get $\alpha_0 + \beta < \alpha$ is a limit ordinal with $\text{cf}(\alpha_0 + \beta) > \omega$, and since R is an accepting run of \mathcal{A}'' on u , then $(\sup_{\alpha_0+\beta}(R), \text{stat}_{\alpha_0+\beta}(R), R(\alpha_0 + \beta)) \in T''$. We have

$$\sup_{\beta}(H') = \{s \in S : \{\gamma < \beta : H'(\gamma) = s\} \text{ is cofinal in } \beta\}, \text{ and}$$

$$\text{stat}_{\beta}(H') = \{s \in S : \{\gamma < \beta : H'(\gamma) = s\} \text{ is stationary in } \beta\},$$

so for any limit ordinal $\beta \leq \alpha_1$, we have that $\sup_{\beta}(H') = \sup_{\alpha_0+\beta}(R)$ and $\text{stat}_{\beta}(H') = \text{stat}_{\alpha_0+\beta}(R)$. Therefore, for each limit ordinal $\beta \leq \alpha_1$, with $\text{cf}(\beta) > \omega$, we get $(\sup_{\beta}(H'), \text{stat}_{\beta}(H'), H'(\beta)) \in T'$, by the same way in condition (2). Thus, H' satisfies the three conditions.

Hence, H' is an accepting run of \mathcal{A}' on u_1 . Therefore $u_1 \in \mathcal{L}(\mathcal{A}')$. Hence $u \in \mathcal{L}(\mathcal{A}) \circ \mathcal{L}(\mathcal{A}')$. Therefore, $\mathcal{L}(\mathcal{A} \circ \mathcal{A}') \subseteq \mathcal{L}(\mathcal{A}) \circ \mathcal{L}(\mathcal{A}')$.

Case (D2). If α_0 is a limit ordinal.

Assume α_0 is a limit ordinal and we want to define H and H' . Then either $\text{cf}(\alpha_0) = \omega$, or $\text{cf}(\alpha_0) > \omega$.

Assume first $\text{cf}(\alpha_0) = \omega$, and since $(\sup_{\alpha_0}(R), R(\alpha_0)) \in T''$ because R is an accepting run of \mathcal{A}'' on u , and we have $\sup_{\alpha_0}(R) \subseteq \mathcal{P}(S)$ because $R(\delta) \in S$ for each $\delta < \alpha_0$, and $R(\alpha_0) \in Z'$ since $\alpha_0 = \min(B)$, and R is an accepting run of \mathcal{A}'' on u , therefore the only possibility that we have $(\sup_{\alpha_0}(R), R(\alpha_0)) \in T_2$. Then there an element $t \in F$ such that $(\sup_{\alpha_0}(R), t) \in T$.

Now we define $H : \alpha_0 + 1 \rightarrow S$ as the following:

$$H(\beta) = \begin{cases} R(\beta) & \text{for each } \beta < \alpha_0, \\ t & \beta = \alpha_0. \end{cases}$$

Define $H' : \alpha_1 + 1 \rightarrow S'$ as follows:

$$H'(\delta) = R(\alpha_0 + \delta) \text{ for each } 0 \leq \delta \leq \alpha_1.$$

And by the same way as in case α_0 is a successor ordinal we can show that H and H' are accepting runs of \mathcal{A} on u_0 and \mathcal{A}' on u_1 respectively. Therefore, $u_0 \in \mathcal{L}(\mathcal{A})$ and $u_1 \in \mathcal{L}(\mathcal{A})'$. Then $u \in \mathcal{L}(\mathcal{A}) \circ \mathcal{L}(\mathcal{A}')$. Therefore $\mathcal{L}(\mathcal{A} \circ \mathcal{A}') \subseteq \mathcal{L}(\mathcal{A}) \circ \mathcal{L}(\mathcal{A}')$.

Second assume $\text{cf}(\alpha_0) > \omega$. Since $(\sup_{\alpha_0}(R), \text{stat}(R), R(\alpha_0)) \in T''$ because R is an accepting run of \mathcal{A}'' on u , and we have $\sup_{\alpha_0}(R) \subseteq \mathcal{P}(S)$ and $\text{stat}_{\alpha_0}(R) \subseteq \mathcal{P}(S)$ because $R(\delta) \in S$ for each $\delta < \alpha_0$, and $R(\alpha_0) \in Z'$ since $\alpha_0 = \min(B)$, and R is an accepting run of \mathcal{A}'' on u , therefore the only possibility that we have $(\sup_{\alpha_0}(R), \text{stat}_{\alpha_0}(R), R(\alpha_0)) \in T_2$. Then there exists an element $t \in F$ such that $(\sup_{\alpha_0}(R), \text{stat}_{\alpha_0}(R), t) \in T$.

Now we define $H : \alpha_0 + 1 \rightarrow S$ as the following:

$$H(\beta) = \begin{cases} R(\beta) & \text{for each } \beta < \alpha_0, \\ t & \beta = \alpha_0. \end{cases}$$

Define $H' : \alpha_1 + 1 \rightarrow S'$ as follows:

$$H'(\delta) = R(\alpha_0 + \delta) \text{ for } 0 \leq \delta \leq \alpha_1.$$

And by the same way as in case α_0 is a successor ordinal we can show that H and H' are accepting runs of \mathcal{A} on u_0 and \mathcal{A}' on u_1 respectively. Therefore $u_0 \in \mathcal{L}(\mathcal{A})$ and $u_1 \in \mathcal{L}(\mathcal{A})'$. Then $u \in \mathcal{L}(\mathcal{A}) \circ \mathcal{L}(\mathcal{A}')$. Therefore $\mathcal{L}(\mathcal{A} \circ \mathcal{A}') \subseteq \mathcal{L}(\mathcal{A}) \circ \mathcal{L}(\mathcal{A}')$.

Proof of (A2). $\mathcal{L}(\mathcal{A}) \circ \mathcal{L}(\mathcal{A}') \subseteq \mathcal{L}(\mathcal{A} \circ \mathcal{A}')$.

Assume $u \in \mathcal{L}(\mathcal{A}) \circ \mathcal{L}(\mathcal{A}')$, such that $u : \alpha \rightarrow I$, and $\alpha \in \text{ord}$. Then

$$u = u_0 \circ u_1, \text{ such that}$$

$$u(\delta) = u_0(\delta) \text{ for } \delta < \alpha_0 \text{ and } u(\alpha_0 + \delta) = u_1(\delta) \text{ for } 0 \leq \delta \leq \alpha_1$$

for some $u_0 \in \mathcal{L}(\mathcal{A})$, $u_0 : \alpha_0 \rightarrow I$ and $u_1 \in \mathcal{L}(\mathcal{A}')$, $u_1 : \alpha_1 \rightarrow I$, $\alpha_0, \alpha_1 \in \text{ord}$ with $\alpha = \alpha_0 + \alpha_1$.

We want to prove $u \in \mathcal{L}(\mathcal{A} \circ \mathcal{A}')$, so we need to define an accepting run R of \mathcal{A}'' on u . Now since $u_0 \in \mathcal{L}(\mathcal{A})$ and $u_1 \in \mathcal{L}(\mathcal{A}')$, then there are accepting runs $H : \alpha_0 + 1 \rightarrow S$ of \mathcal{A} on u_0 and $H' : \alpha_1 + 1 \rightarrow S'$ of \mathcal{A}' on u_1 .

Now we want to define an accepting run $R : \alpha + 1 \rightarrow S''$, of \mathcal{A}'' on u . Define R as follows:

$$\begin{aligned} R(\delta) &= H(\delta), \text{ if and if } \delta < \alpha_0, \\ R(\alpha_0 + \delta) &= H'(\delta), \text{ if and if } 0 \leq \delta \leq \alpha_1. \end{aligned}$$

We want to prove R is an accepting run of \mathcal{A}'' on u , that we should prove the following $R(0) \in Z''$, $R(\alpha) \in F''$ and satisfies the following conditions:

1. For each $\beta < \alpha$ we have

$$(R(\beta), u(\beta), R(\beta + 1)) \in T''.$$

2. For each $\beta \leq \alpha$ that is a limit ordinal with $\text{cf}(\beta) = \omega$, we have

$$(\sup_{\beta} R, R(\beta)) \in T''.$$

3. For each $\beta \leq \alpha$ that is a limit ordinal with $\text{cf}(\beta) > \omega$, we have

$$(\sup_{\beta} R, \text{stat}_{\beta} R, R(\beta)) \in T''.$$

We will consider the following two cases:

(B1) If $\alpha_0 = 0$.

(B2) If $\alpha_0 > 0$.

Case (B1). If $\alpha_0 = 0$.

Then $\alpha_1 = \alpha$. Then $u_0 = \emptyset$ and $u_1 = u$ and since H is an accepting run of \mathcal{A} on u_0 , then $u_0 = \emptyset \in \mathcal{L}(\mathcal{A})$, which implies to $R(0) \in Z'$, by the definition of \mathcal{A}'' . Therefore $R = H'$ and

there is nothing to prove.

Case (B2). If $\alpha_0 > 0$.

Second assume $\alpha_0 > 0$. Then we will discuss the following two cases:

(C1) If α_0 is a successor ordinal.

(C2) If α_0 is a limit ordinal.

Case (C1). If α_0 is a successor ordinal.

We want to prove R is an accepting run of \mathcal{A}'' on u . Then assume $\alpha_0 = \sigma + 1$, for some $\sigma \in \text{ord}$. Now we want to prove $R(0) \in Z''$, and $R(\alpha) \in F''$.

By the definition of R we get $R(0) = H(0) \in Z$, and $R(\alpha) = H'(\alpha_1) \in F'$ because H is an accepting run of \mathcal{A} on u_0 , and H' is an accepting run of \mathcal{A}' on u_1 , then $R(0) \in Z''$ and $R(\alpha) \in F''$, by the definition of \mathcal{A}'' .

It remains to prove the three conditions.

Now we want to prove condition (1). Assume $\beta < \alpha$. Then we have two cases either $\beta \leq \alpha_0$ or $\alpha_0 < \beta < \alpha$ and we want to prove

$$(R(\beta), u(\beta), R(\beta + 1)) \in T''.$$

First assume $\beta \leq \alpha_0$. By the definition of R , we get for each $\beta < \alpha_0$,

$$(R(\beta), u(\beta), R(\beta + 1)) = (H(\beta), u_0(\beta), H(\beta + 1)).$$

Since H is an accepting run of \mathcal{A} on u_0 , then $(R(\beta), u(\beta), R(\beta + 1)) \in T$, hence for each $\beta < \alpha_0$, we get $(R(\beta), u(\beta), R(\beta + 1)) \in T''$, by the definition of T'' .

Second assume $\beta = \alpha_0 = \sigma + 1$, for some $\sigma \in \text{ord}$, then

$$(R(\sigma), u(\sigma), R(\sigma + 1)) = (H(\sigma), u_0(\sigma), H'(0)).$$

So it is enough to show

$$(H(\sigma), u_0(\sigma), H'(0)) \in T'',$$

in particular we will show that

$$(H(\sigma), u_0(\sigma), H'(0)) \in T_1.$$

Now since H' is an accepting run of \mathcal{A}' on u_1 , then $H'(0) \in Z'$, and since H is an accepting

run of \mathcal{A} on u_0 , then $H(\alpha_0) \in F$, and $(H(\sigma), u_0(\sigma), H(\sigma + 1)) = (H(\sigma), u_0(\sigma), H(\alpha_0)) \in T$, so we are done.

Now assume $\alpha_0 < \beta < \alpha$. Hence $\beta = \alpha_0 + \delta$, for $0 \leq \delta < \alpha_1$ and by the definition of R we get

$$(R(\beta), u(\beta), R(\beta + 1)) = (H'(\delta), u_1(\delta), H'(\delta + 1))$$

Since H' is an accepting run of \mathcal{A}' on u_1 , then $(R(\beta), u(\beta), R(\beta + 1)) \in T'$, hence for each $\alpha_0 < \beta < \alpha$, we get $(R(\beta), u(\beta), R(\beta + 1)) \in T''$, by the definition of T'' .

Therefore, for each $\beta < \alpha$, we have $(R(\beta), u(\beta), R(\beta + 1)) \in T''$.

Next we want to prove condition (2). Assume $\beta \leq \alpha$, that is a limit ordinal with $\text{cf}(\beta) = \omega$, then we have two cases either $\beta < \alpha_0$ or $\alpha_0 < \beta \leq \alpha$ and we want to prove

$$(\sup_{\beta}(R), R(\beta)) \in T''.$$

Not that $\beta \neq \alpha_0$, because α_0 is a successor ordinal.

First assume $\beta < \alpha_0$. By the definition of R we get for each $\beta < \alpha_0$, $\sup_{\beta}(R) = \sup_{\beta}(H)$ and $R(\beta) = H(\beta)$, therefore

$$(\sup_{\beta}(R), R(\beta)) = (\sup_{\beta}(H), H(\beta)) \in T,$$

since H is an accepting run of \mathcal{A} on u_0 , hence for each $\beta < \alpha_0$, we get $(\sup_{\beta}(R), R(\beta)) \in T''$ by the definition of T'' .

Now assume $\alpha_0 < \beta \leq \alpha$. Hence $\beta = \alpha_0 + \delta$, for $0 < \delta \leq \alpha_1$ and by the definition of R we get

$$\sup_{\beta}(R) = \sup_{\delta}(H'), \text{ and } R(\beta) = H'(\delta),$$

therefore

$$(\sup_{\beta}(R), R(\beta)) = (\sup_{\delta}(H'), H'(\delta)) \in T',$$

since H' is an accepting run of \mathcal{A}' on u_1 , and $\delta \leq \alpha_1$ is a limit ordinal with $\text{cf}(\delta) = \omega$, by lemma 1.2.40, then by the definition of T'' , we get $(\sup_{\beta}(R), R(\beta)) \in T''$ for each $\alpha_0 < \beta \leq \alpha$. Therefore $(\sup_{\beta}(R), R(\beta)) \in T''$, for every limit ordinal $\beta \leq \alpha$ with $\text{cf}(\beta) = \omega$.

Now we want to prove condition (3). Let $\beta \leq \alpha$ be a limit ordinal with $\text{cf}(\beta) > \omega$, we want to prove

$$(\sup_{\beta}(R), \text{stat}_{\beta}(R), R(\beta)) \in T''.$$

We have two cases either $\beta < \alpha_0$ or $\alpha_0 < \beta \leq \alpha_1$, note that also we don't need $\beta = \alpha_0$, since α_0 is a successor ordinal.

First assume $\beta < \alpha_0$. By the definition of R we get $\sup_\beta(R) = \sup_\beta(H)$, $\text{stat}_\beta(R) = \text{stat}_\beta(H)$ and $R(\beta) = H(\beta)$, therefore

$$(\sup_\beta(R), \text{stat}_\beta(R), R(\beta)) = (\sup_\beta(H), \text{stat}_\beta(H), H(\beta)) \in T,$$

since H is an accepting run of \mathcal{A} on u_0 , hence for each $\beta < \alpha_0$, we get $(\sup_\beta(R), \text{stat}_\beta(R), R(\beta)) \in T''$ by the definition of T'' .

Now assume $\alpha_0 < \beta \leq \alpha$. Hence $\beta = \alpha_0 + \delta$, for $0 < \delta \leq \alpha_1$, and by the definition of R we get

$$\sup_\beta(R) = \sup_\delta(H'), \text{stat}(R) = \text{stat}_\delta(H') \text{ and } R(\beta) = H'(\delta),$$

therefore

$$(\sup_\beta(R), \text{stat}_\beta(R), R(\beta)) = (\sup_\delta(H'), \text{stat}_\delta(H'), H'(\delta)) \in T',$$

since H' is an accepting run of \mathcal{A}' on u_1 , and δ is a limit ordinal with $\text{cf}(\delta) > \omega$, by lemma 1.2.40, then by the definition of T'' , we get $(\sup_\beta(R), \text{stat}_\beta(R), R(\beta)) \in T''$ for each $\alpha_0 < \beta \leq \alpha$. Thus for every limit ordinal $\beta \leq \alpha$ with $\text{cf}(\beta) > \omega$, $(\sup_\beta(R), \text{stat}_\beta(R), R(\beta)) \in T''$.

Therefore, R satisfies the three conditions. Then R is an accepting run of \mathcal{A}'' on u . Therefore $u \in \mathcal{L}(\mathcal{A} \circ \mathcal{A}')$.

Case (C2). If α_0 is a limit ordinal.

Then we want to prove R is an accepting run of \mathcal{A}'' on u . By the same way when $\alpha_0 \in \text{succ}$ we can prove $R(0) \in Z''$, and $R(\alpha) \in F''$.

It remains to prove the three conditions.

First we want to prove condition (1), that is, For each $\beta < \alpha$, we have

$$(R(\beta), u(\beta), R(\beta + 1)) \in T''.$$

Then we have two cases either $\beta \leq \alpha_0$, or $\alpha_0 < \beta \leq \alpha$. We can show that by the same way when $\alpha_0 \in \text{succ}$ except for $\beta = \alpha_0$.

Now assume $\beta = \alpha_0$, then

$$(R(\beta), u(\beta), R(\beta + 1)) = (R(\alpha_0), u(\alpha_0), R(\alpha_0 + 1)) = (H'(0), u_1(0), H'(1)) \in T',$$

since H' is an accepting run of \mathcal{A}' on u_1 . Therefore, for each $\beta < \alpha$, we get $(R(\beta), u(\beta), R(\beta + 1)) \in T''$

Next we want to prove condition (2). Assume $\beta \leq \alpha$ that is a limit ordinal with $\text{cf}(\beta) = \omega$, then we have two cases either $\beta \leq \alpha_0$, or $\alpha_0 < \beta \leq \alpha$ and we want to prove

$$(\sup_{\beta}(R), R(\beta)) \in T''.$$

Also we can show that by the same way when $\alpha_0 \in \text{succ}$ except for $\beta = \alpha_0$.

Now assume $\beta = \alpha_0$, then we want to prove

$$(\sup_{\alpha_0}(R), R(\alpha_0)) \in T''.$$

In particular we want to prove $(\sup_{\alpha_0}(H), H'(0)) \in T_2$, because

$$(\sup_{\alpha_0}(R), R(\alpha_0)) = (\sup_{\alpha_0}(H), H'(0)).$$

It is clear by the definition of R , that $\sup_{\alpha_0}(R) = \sup_{\alpha_0}(H) \subset S$, and since H' is an accepting run of \mathcal{A}' on u_1 , then $R(\alpha_0) = H'(0) \in Z'$, but H is an accepting run of \mathcal{A} on u_0 , and α_0 is a limit ordinal then $H(\alpha_0) \in F$, and $(\sup_{\alpha_0}(H), H(\alpha_0)) \in T$. Thus by the definition of T_2 we get $(\sup_{\alpha_0}(R), R(\alpha_0)) \in T_2$. Therefore, for each $\beta \leq \alpha$, that is a limit ordinal with $\text{cf}(\beta) = \omega$, we get $(\sup_{\beta}(R), R(\beta)) \in T''$.

Finally we want to prove condition (3). Assume $\beta \leq \alpha$, that is a limit ordinal with $\text{cf}(\beta) > \omega$, then we have two cases either $\beta \leq \alpha_0$ or $\alpha_0 < \beta \leq \alpha$ and we want to prove

$$(\sup_{\beta}(R), \text{stat}_{\beta}(R), R(\beta)) \in T''.$$

Also we can show that by the same way when $\alpha \in \text{succ}$ except for $\beta = \alpha_0$.

Now assume $\beta = \alpha_0$, then we want to prove

$$(\sup_{\alpha_0}(R), \text{stat}_{\alpha_0}(R), R(\alpha_0)) \in T''.$$

In particular we want to prove $(\sup_{\alpha_0}(H), \text{stat}_{\alpha_0}(H), H'(0)) \in T_3$, because

$$(\sup_{\alpha_0}(R), \text{stat}_{\alpha_0}(R), R(\alpha_0)) = (\sup_{\alpha_0}(H), \text{stat}_{\alpha_0}(H), H'(0)).$$

It is clear by the definition of R , that $\sup_{\alpha_0}(R) = \sup_{\alpha_0}(H) \subset S$, and $\text{stat}_{\alpha_0}(R) = \text{stat}_{\alpha_0}(H) \subset S$ and since H' is an accepting run of \mathcal{A}' on u_1 , then $R(\alpha_0) = H'(0) \in Z'$, but H is an accepting run of \mathcal{A} on u_0 , and α_0 is a limit ordinal then $H(\alpha_0) \in F$, and $(\sup_{\alpha_0}(H), \text{stat}_{\alpha_0}(H), H(\alpha_0)) \in T$.

Thus by the definition of T_3 we get $(\sup_{\alpha_0}(R), \text{stat}_{\alpha_0}(H), R(\alpha_0)) \in T_3$, where

$$T_3 = \{(A, B, s') : A, B \subseteq S, s' \in Z' \text{ such that } (A, B, t) \in T \text{ for some } t \in F\}.$$

Therefore, for each $\beta \leq \alpha$ that is a limit ordinal with $\text{cf}(\beta) > \omega$, we show $(\sup_{\beta}(R), \text{stat}_{\alpha_0}(H), R(\beta)) \in T''$. Thus, R satisfies the three conditions. Hence $u \in \mathcal{L}(\mathcal{A} \circ \mathcal{A}')$, which implies to $\mathcal{L}(\mathcal{A}) \circ \mathcal{L}(\mathcal{A}') \subset \mathcal{L}(\mathcal{A} \circ \mathcal{A}')$.

From (A1), and (A2) we get $\mathcal{L}(\mathcal{A}) \circ \mathcal{L}(\mathcal{A}') = \mathcal{L}(\mathcal{A} \circ \mathcal{A}')$. \square

4.4 * -Operation

Next, we define the *-ST-automaton as follows:

Definition 4.4.1. Let $\mathcal{A} = (S, I, T, Z, F)$ be a ST-automaton over I with $Z \neq \emptyset$. Define the *-ST-automaton over I of \mathcal{A} , or $\mathcal{A}^* = (S', I, T', Z', F')$ as follows:

$$\begin{aligned} S' &= S \times \{0, 1\}, \\ Z' &= Z \times \{1\}, \\ F' &= Z', \\ T' &= T_1 \cup T_2 \cup T_3, \end{aligned}$$

where

$$\begin{aligned} T_1 &= \{((s, 0), a, (t, 0)) : (s, a, t) \in T\} \cup \\ &\quad \{(A \times \{0\}, (t, 0)) : A \subseteq S, \text{ and } (A, t) \in T\} \cup \\ &\quad \{(A \times \{0\}, B \times \{0\}, (t, 0)) : A, B \subseteq S, \text{ and } (A, B, t) \in T\}, \\ T_2 &= \{((s, 0), a, (t, 1)) : (s, a, t') \in T \text{ for some } t' \in F, \text{ and } (t, 1) \in F'\} \cup \\ &\quad \{(A \times \{0\}, (t, 1)) : A \subseteq S, (A, t') \in T \text{ for some } t' \in F, \text{ and } (t, 1) \in F'\} \cup \\ &\quad \{(A \times \{0\}, B \times \{0\}, (t, 1)) : A, B \subseteq S, (A, B, t') \in T \text{ for some } t' \in F, \text{ and } (t, 1) \in F'\}, \\ T_3 &= \{((s, 1), a, (t, 0)) : (s, a, t) \in T\}. \end{aligned}$$

If $Z = \emptyset$, then define $\mathcal{A}^* = (\{s\}, I, \emptyset, \{s\}, \{s\})$, for any $s \in S$.

Then we prove the following theorem that show, applying the * operation to languages defined by ST-automata, the produce language that is also definable using ST-automaton.

Theorem 4.4.2. *If $\mathcal{A} = (S, I, T, Z, F)$ is a ST-automaton over I , with Z is a finite set, then $\mathcal{L}(\mathcal{A}^*) = (\mathcal{L}(\mathcal{A}))^*$.*

Proof. Assume $\mathcal{A} = (S, I, T, Z, F)$ is a ST-automaton over I , with Z is a finite set and we want to prove $\mathcal{L}(\mathcal{A}^*) = (\mathcal{L}(\mathcal{A}))^*$, such that $\mathcal{A}^* = (S', I, T', Z', F')$. We want to prove the following:

$$(A1) \quad \mathcal{L}(\mathcal{A}^*) \subseteq (\mathcal{L}(\mathcal{A}))^*.$$

$$(A2) \quad (\mathcal{L}(\mathcal{A}))^* \subseteq \mathcal{L}(\mathcal{A}^*).$$

If $Z \neq \emptyset$, then by the definition of \mathcal{A}^* we get to the following:

$$\begin{aligned} S' &= S \times \{0, 1\}, \\ Z' &= Z \times \{1\}, \\ F' &= Z', \\ T' &= T_1 \cup T_2 \cup T_3, \end{aligned}$$

where

$$\begin{aligned} T_1 &= \{(s, 0), a, (t, 0) : (s, a, t) \in T\} \cup \\ &\quad \{(A \times \{0\}, (t, 0)) : A \subseteq S, \text{ and } (A, t) \in T\} \cup \\ &\quad \{(A \times \{0\}, B \times \{0\}, (t, 0)) : A, B \subseteq S, \text{ and } (A, B, t) \in T\}, \end{aligned}$$

$$\begin{aligned} T_2 &= \{(s, 0), a, (t, 1) : (s, a, t') \in T \text{ for some } t' \in F, \text{ and } (t, 1) \in F'\} \cup \\ &\quad \{(A \times \{0\}, (t, 1)) : A \subseteq S, (A, t') \in T \text{ for some } t' \in F, \text{ and } (t, 1) \in F'\} \cup \\ &\quad \{(A \times \{0\}, B \times \{0\}, (t, 1)) : A, B \subseteq S, (A, B, t') \in T \text{ for some } t' \in F, \text{ and } (t, 1) \in F'\}, \end{aligned}$$

$$T_3 = \{(s, 1), a, (t, 0) : (s, a, t) \in T\}.$$

If $Z = \emptyset$, then define $\mathcal{A}^* = (\{s\}, I, \emptyset, \{s\}, \{s\})$ for any $s \in S$.

Proof of (A1). $\mathcal{L}(\mathcal{A}^*) \subseteq (\mathcal{L}(\mathcal{A}))^*$.

First we want to prove (A1), that is, $\mathcal{L}(\mathcal{A}^*) \subseteq (\mathcal{L}(\mathcal{A}))^*$. We can assume $Z \neq \emptyset$, since if $Z = \emptyset$, then $\mathcal{A}^* = (\{s\}, I, \emptyset, \{s\}, \{s\})$, for some $s \in S$, hence $\mathcal{L}(\mathcal{A}^*) = \{\emptyset\}$, and since $\mathcal{L}(\mathcal{A}) = \emptyset$, then $(\mathcal{L}(\mathcal{A}))^* = \{\emptyset\}$, therefore $\mathcal{L}(\mathcal{A}^*) = (\mathcal{L}(\mathcal{A}))^*$.

So we can assume that $Z \neq \emptyset$. Now let $u \in \mathcal{L}(\mathcal{A}^*)$, such that $u : \alpha \rightarrow I$, and $\alpha \in \text{ord}$. Since $\emptyset \in (\mathcal{L}(\mathcal{A}))^*$, so we can assume $u \neq \emptyset$. Then there exists an accepting run $R : \alpha + 1 \rightarrow S'$ of

\mathcal{A}^* on u . We want to prove $u \in (\mathcal{L}(\mathcal{A}))^*$, and since

$$(\mathcal{L}(\mathcal{A}))^* = \bigcup_{n < \omega} (\mathcal{L}(\mathcal{A}))^n,$$

such that for each $n < \omega$,

$$(\mathcal{L}(\mathcal{A}))^n = \{u : u = o(u_\beta)_{\beta < n}, u \in \mathcal{L}(\mathcal{A}) \text{ for each } \beta < n\}.$$

We will find $n < \omega$, such that $u \in (\mathcal{L}(\mathcal{A}))^n$. First we show that

$$C = \{\beta \leq \alpha : R(\beta) \in F'\},$$

is finite.

Suppose by the way of a contradiction that C is infinite. Let $C = \{\alpha_i : i < \eta\}$, where η is an infinite ordinal and $\alpha_i < \alpha_j$, for each $i < j < \eta$. Let

$$\sigma = \sup\{\alpha_i : i < \omega\},$$

then $\sigma \in \text{lim}$, so

$$\text{sup}_\sigma(R) = \{s \in S' : \{\gamma < \sigma : R(\gamma) = s\} \text{ is cofinal in } \sigma\}.$$

Since Z is finite so F' is finite because $F' = Z \times \{1\}$, then there is $(t, 1) \in F'$ such that $(t, 1) \in \text{sup}_\sigma(R)$ which is a contradiction since R is an accepting run of \mathcal{A}^* on u , then either $(\text{sup}_\sigma(R), R(\sigma)) \in T'$, when $\text{cf}(\sigma) = \omega$, or $(\text{sup}_\sigma(R), \text{stat}_\sigma(R), R(\sigma)) \in T'$, when $\text{cf}(\sigma) > \omega$, and there are no $(A, s) \in T'$ and $(A, B, s) \in T'$, with $(s, 1) \in A$. Therefore C is a finite set. Let

$$n = |C| - 1.$$

Now define that, for each i and $0 \leq i < n$, $\theta_i \in \text{ord}$ such that

$$\alpha_i + \theta_i = \alpha_{i+1}, \text{ and}$$

$$u_i : \theta_i \rightarrow I, \text{ is such that}$$

$$u_i(\delta) = u(\alpha_i + \delta), \text{ for all } \delta < \theta_i,$$

then we get,

$$u = o(u_i)_{i < n}, \text{ and } \alpha = \sum_{i < n} \theta_i.$$

Now, since for each i with $0 \leq i < n$, $\alpha_i \in C$, then $R(\alpha_i) \in F'$, and $F' = Z \times 1$, so define $R_i : \theta_i + 1 \rightarrow S$, as the following:

$$\begin{aligned} R_i(0) &= s, \text{ such that } R(\alpha_i) = (s, 1), \text{ for some } (s, 1) \in F', \\ R_i(\delta) &= s, \text{ such that } R(\alpha_i + \delta) = (s, 0), \text{ for all } 0 < \delta < \theta_i, \text{ and} \\ R_i(\theta_i) &= t', \end{aligned}$$

such that we get to t' as the following:

we consider two cases either α_{i+1} is a successor ordinal, or a limit ordinal.

First assume α_{i+1} is a successor ordinal.

Then let $\alpha_{i+1} = \sigma + 1$, for some $\sigma \in \text{ord}$, and since R is an accepting run of \mathcal{A}^* on u , then

$$(R(\sigma), u(\sigma), R(\alpha_{i+1})) \in T',$$

but for each $0 \leq i < n$, $R(\alpha_{i+1}) \in F' = Z \times \{1\}$, because $\alpha_{i+1} \in C$, then let $R(\alpha_{i+1}) = (t, 1)$, thus

$$(R(\sigma), u(\sigma), R(\alpha_{i+1})) \in T_2,$$

by the definition of T_2 , which implies to

$$(R(\sigma), u(\sigma), R(\alpha_{i+1})) = ((s, 0), u(\sigma), (t, 1)),$$

such that $(s, u(\sigma), t') \in T$, for some $t' \in F$.

Second assume α_{i+1} is a limit ordinal. Then either $\text{cf}(\alpha_{i+1}) = \omega$, or $\text{cf}(\alpha_{i+1}) > \omega$.

First assume $\text{cf}(\alpha_{i+1}) = \omega$, then

$$\left(\sup_{\alpha_{i+1}} (R), R(\alpha_{i+1}) \right) \in T',$$

since R is an accepting run of \mathcal{A}^* on u , but $R(\alpha_{i+1}) \in F' = Z \times \{1\}$, because $\alpha_{i+1} \in C$, then let $R(\alpha_{i+1}) = (t, 1)$, thus

$$\left(\sup_{\alpha_{i+1}} (R), R(\alpha_{i+1}) \right) \in T_2,$$

by the definition of T_2 , which implies to

$$\left(\sup_{\alpha_{i+1}} (R), R(\alpha_{i+1}) \right) = (A \times \{0\}, (t, 1)),$$

for some $A \subseteq S$ and $(A, t') \in T$, for some $t' \in F$.

Now assume $\text{cf}(\alpha_{i+1}) > \omega$, then

$$\left(\sup_{\alpha_{i+1}}(R), \text{stat}_{\alpha_{i+1}}(R), R(\alpha_{i+1})\right) \in T',$$

since R is an accepting run of \mathcal{A}^* on u , but $R(\alpha_{i+1}) \in F' = Z \times \{1\}$, because $\alpha_{i+1} \in C$, then let $R(\alpha_{i+1}) = (t, 1)$, thus

$$\left(\sup_{\alpha_{i+1}}(R), \text{stat}_{\alpha_{i+1}}(R), R(\alpha_{i+1})\right) \in T_2,$$

by the definition of T_2 , which implies to

$$\left(\sup_{\alpha_{i+1}}(R), \text{stat}_{\alpha_{i+1}}(R), R(\alpha_{i+1})\right) = (A \times \{0\}, B \times \{0\}, (t, 1)),$$

for some $A, B \subseteq S$ and $(A, B, t') \in T$, for some $t' \in F$.

So it remains to prove for each $0 \leq i < n$, R_i is an accepting run of \mathcal{A} on u_i . That is, we must prove that, for each $0 \leq i < n$, $R_i(0) \in Z$, $R_i(\theta_i) \in F$ and satisfies the following conditions:

1. For each $\beta < \theta_i$, we have

$$(R_i(\beta), u_i(\beta), R_i(\beta + 1)) \in T.$$

2. For each $\beta \leq \theta_i$ that is a limit ordinal with $\text{cf}(\beta) = \omega$, we have

$$(\sup_{\beta}(R_i), R_i(\beta)) \in T.$$

3. For each $\beta \leq \theta_i$ that is a limit ordinal with $\text{cf}(\beta) > \omega$, we have

$$(\sup_{\beta}(R_i), \text{stat}_{\beta}(R_i), R_i(\beta)) \in T.$$

Choose i , such that $0 \leq i < n$ and we want to prove R_i is an accepting run of \mathcal{A} on u_i . First, we want to prove $R_i(0) \in Z$, and $R_i(\theta_i) \in F$, and that is clear by the definition of R_i . It remains to prove above three conditions.

Now we want to prove condition (1). Let $\beta < \theta_i$, and we want to prove

$$(R_i(\beta), u_i(\beta), R_i(\beta + 1)) \in T.$$

Since R is an accepting run of \mathcal{A}^* on u , then $(R(\alpha_i + \beta), u(\alpha_i + \beta), R(\alpha_i + \beta + 1)) \in T'$. Now we will discuss the following cases:

(B1) If $\beta = 0$.

(B2) If $0 < \beta < \theta_i$.

Case (B1). If $\beta = 0$.

Then

$$(R(\alpha_i + \beta), u(\alpha_i + \beta), R(\alpha_i + \beta + 1)) = (R(\alpha_i), u(\alpha_i), R(\alpha_i + 1)) \in T',$$

and since $\alpha_i \in C$, then $R(\alpha_i) \in F' = Z \times \{1\}$, so

$$(R(\alpha_i + \beta), u(\alpha_i + \beta), R(\alpha_i + \beta + 1)) = (R(\alpha_i), u(\alpha_i), R(\alpha_i + 1)) \in T_3,$$

so by the definition of T_3 , we get

$$(R(\alpha_i), u(\alpha_i), R(\alpha_i + 1)) = ((s, 1), a, (t, 0)),$$

for some $(s, a, t) \in T$ and by the definition of R_i and u_i we get

$$(R(0), u(0), R(1)) = (s, a, t) \in T.$$

Case (B2). If $0 < \beta < \theta_i$.

Then $\alpha_i < \alpha_i + \beta < \alpha_{i+1}$, and

$$(R(\alpha_i + \beta), u(\alpha_i + \beta), R(\alpha_i + \beta + 1)) \in T',$$

since R is an accepting run of \mathcal{A}^* on u , and since $R(\alpha_i + \beta) = (s, 0)$, $R(\alpha_i + \beta + 1) = (t, 0)$ for some $t, s \in S$, thus

$$(R(\alpha_i + \beta), u(\alpha_i + \beta), R(\alpha_i + \beta + 1)) = ((s, 0), a, (t, 0)) \in T_1,$$

by the definition of T_1 , which implies to

$$(R_i(\beta), u_i(\beta), R_i(\beta + 1)) = ((s, a, t)) \in T,$$

by the definition of R_i , u and T_1 .

Therefore, from (B1), and (B2) we get for each $\beta < \theta_i$, we have

$$(R_i(\beta), u_i(\beta), R_i(\beta + 1)) \in T.$$

Next we want to prove condition (2). Assume $\beta \leq \theta_i$, that is a limit ordinal with $\text{cf}(\beta) = \omega$, and we want to prove

$$(\sup_{\beta}(R_i), R_i(\beta)) \in T.$$

We have two cases:

(C1) Either $\beta < \theta_i$,

(C2) Or $\beta = \theta_i$.

Case (C1). Assume $\beta < \theta_i$.

Now, since β is a limit ordinal in θ_i , with $\text{cf}(\beta) = \omega$, then $\alpha_i + \beta$ is a limit ordinal in α , with $\text{cf}(\alpha_i + \beta) = \omega$, by lemma 1.2.40, and we know R is an accepting run of \mathcal{A}^* on u , then $(\sup_{\alpha_i + \beta}(R), R(\alpha_i + \beta)) \in T'$ and since $R(\alpha_i + \beta) = (s, 0)$, then $(\sup_{\alpha_i + \beta}(R), R(\alpha_i + \beta)) \in T_1$ and since

$$(\sup_{\alpha_i + \beta}(R), R(\alpha_i + \beta)) = (\sup_{\beta}(R_i) \times \{0\}, (R_i(\beta), 0)),$$

then $(\sup_{\beta}(R_i), R_i(\beta)) \in T$ by the definition of T_1 .

Case (C2). Assume $\beta = \theta_i$.

Since β is a limit ordinal with $\text{cf}(\beta) = \omega$, then $\alpha_i + \theta_i = \alpha_{i+1}$ is a limit ordinal in α , with $\text{cf}(\alpha_{i+1}) = \omega$, by lemma 1.2.40, thus $(\sup_{\alpha_{i+1}}(R), R(\alpha_{i+1})) \in T'$ and since $R(\alpha_{i+1}) = (t, 1)$, then $(\sup_{\alpha_{i+1}}(R), R(\alpha_{i+1})) \in T_2$ and

$$(\sup_{\alpha_{i+1}}(R), R(\alpha_{i+1})) = (\sup_{\theta_i}(R_i) \times \{0\}, (t, 1)),$$

and since α_{i+1} is a limit ordinal with $\text{cf}(\alpha_{i+1}) = \omega$, so $R_i(\theta_i) = t'$ and $(\sup_{\theta_i}(R_i), R_i(\theta_i)) \in T$ by the definition of T_2 .

Therefore, from (C1), and (C2) we get, for each $\beta \leq \theta_i$ that is a limit ordinal with $\text{cf}(\beta) = \omega$, we have

$$(\sup_{\beta}(R_i), R_i(\beta)) \in T.$$

Finally we want to prove condition (3). Assume $\beta \leq \theta_i$, that is a limit ordinal with $\text{cf}(\beta) > \omega$, and we want to prove

$$(\sup_{\beta}(R_i), \text{stat}_{\beta}(R_i), R_i(\beta)) \in T.$$

Then we have the following cases:

(D1) Either $\beta < \theta_i$,

(D2) Or $\beta = \theta_i$.

Case (D1). Let $\beta < \theta_i$.

Since R is an accepting run of \mathcal{A}^* on u , and β is a limit ordinal in θ_i , with $\text{cf}(\beta) > \omega$, then $\alpha_i + \beta$ is a limit ordinal in α , with $\text{cf}(\alpha_i + \beta) > \omega$, by lemma 1.2.40, thus

$$(\sup_{\alpha_i + \beta}(R), \text{stat}_{\alpha_i + \beta}(R), R(\alpha_i + \beta)) \in T',$$

and since $R(\alpha_i + \beta) = (s, 0)$, then $(\sup_{\alpha_i + \beta}(R), \text{stat}_{\alpha_i + \beta}(R), R(\alpha_i + \beta)) \in T_1$, and since

$$(\sup_{\alpha_i + \beta}(R), \text{stat}_{\alpha_i + \beta}(R), R(\alpha_i + \beta)) = (\sup_{\beta}(R_i) \times \{0\}, \text{stat}_{\beta}(R_i) \times \{0\}, (R_i(\beta), 0)),$$

then by the definition of T_1 we get $(\sup_{\beta}(R_i), \text{stat}_{\beta}(R_i), R_i(\beta)) \in T$.

Case (D2). Let $\beta = \theta_i$.

Since β is a limit ordinal with $\text{cf}(\beta) > \omega$, then $\alpha_i + \theta_i = \alpha_{i+1}$ is a limit ordinal in α , with $\text{cf}(\alpha_{i+1}) > \omega$, by lemma 1.2.40, thus $(\sup_{\alpha_{i+1}}(R), \text{stat}_{\alpha_{i+1}}(R), R(\alpha_{i+1})) \in T'$ and since $R(\alpha_{i+1}) = (t, 1)$, then $(\sup_{\alpha_{i+1}}(R), \text{stat}_{\alpha_{i+1}}(R), R(\alpha_{i+1})) \in T_2$, and since

$$(\sup_{\alpha_{i+1}}(R), \text{stat}_{\alpha_{i+1}}(R), R(\alpha_{i+1})) = (\sup_{\theta_i}(R_i) \times \{0\}, \text{stat}_{\theta_i}(R_i) \times \{0\}, (t, 1)).$$

but α_{i+1} is a limit ordinal with $\text{cf}(\alpha_{i+1}) > \omega$, so $R_i(\theta_i) = t'$ and $(\sup_{\theta_i}(R_i), \text{stat}_{\theta_i}(R_i), R_i(\theta_i)) \in T$ by the definition of T_2 .

Therefore, from (D1), and (D2) we get, for each $\beta \leq \theta_i$ that is a limit ordinal with $\text{cf}(\beta) > \omega$, $(\sup_{\beta}(R_i), \text{stat}_{\beta}(R_i), R_i(\beta)) \in T$.

Hence, the three conditions are achieved. Then, R_i is an accepting run of \mathcal{A} on u_i . Therefore, R_i is an accepting run of \mathcal{A} on u_i , for each $0 \leq i < n$. Hence $u \in (\mathcal{L}(\mathcal{A}))^*$. Therefore, $\mathcal{L}(\mathcal{A}^*) \subseteq (\mathcal{L}(\mathcal{A}))^*$.

Proof of (A2). $(\mathcal{L}(\mathcal{A}))^* \subseteq \mathcal{L}(\mathcal{A}^*)$.

Assume $u \in (\mathcal{L}(\mathcal{A}))^*$, such that $u : \alpha \rightarrow I$, for some $\alpha \in \text{ord}$.

If $u = \emptyset$, then $u \in \mathcal{L}(\mathcal{A}^*)$, since $Z \neq \emptyset$, then there exist $s \in Z$ with $(s, 1) \in Z' = F'$, so define $R : 1 \rightarrow S'$ such that $R(0) = R(\alpha) = (s, 1)$, and that is an accepting run of \mathcal{A}^* on u .

Now assume $u \neq \emptyset$, and we need to define an accepting run $R : \alpha + 1 \rightarrow S'$ of \mathcal{A}^* on u . Now since $u \in (\mathcal{L}(\mathcal{A}))^*$, and

$$(\mathcal{L}(\mathcal{A}))^* = \bigcup_{n < \omega} (\mathcal{L}(\mathcal{A}))^n,$$

such that for each $n < \omega$,

$$(\mathcal{L}(\mathcal{A}))^n = \{u : u = \circ(u_{\beta})_{\beta < n}, u_{\beta} \in \mathcal{L}(\mathcal{A}) \text{ for each } \beta < n\}.$$

Then there exist $n < \omega$ such that $u \in (\mathcal{L}(\mathcal{A}))^n$ and

$$u = \circ(u_i)_{i < n}, \quad u_i \in \mathcal{L}(\mathcal{A}) \text{ for each } i < n.$$

Since $u \neq \emptyset$, then $n > 0$, and we can assume that for all $i < n$, $u_i \neq \emptyset$. Let $u_i : \theta_i \rightarrow I$, $\theta_i \in \text{ord}$, $\theta_i \neq 0$, and $\alpha_i = \sum_{j=0}^{i-1} \theta_j$, for each $i < n$. And

$$u(\alpha_i + \delta) = u_i(\delta), \text{ for all } \delta < \theta_i.$$

Then $\alpha = \sum_{i < n} \theta_i$.

Now, since for each $i < n$, $u_i \in \mathcal{L}(\mathcal{A})$, then for each $i < n$, there exist an accepting run $R_i : \theta_i + 1 \rightarrow S$ of \mathcal{A} on u_i .

We want to prove $u \in \mathcal{L}(\mathcal{A}^*)$. Define $R : \alpha + 1 \rightarrow S'$ as follows:

$$\begin{aligned} R(\alpha_i) &= (R_i(0), 1), \text{ for each } i < n, \\ R(\alpha_i + \delta) &= (R_i(\delta), 0), \text{ for each } 0 < \delta < \theta_i, \quad i < n \\ R(\alpha) &= (t, 1), \end{aligned}$$

for an element $(t, 1) \in F'$ such element exist because $Z \neq \emptyset$, then $F' = Z \times \{1\} \neq \emptyset$.

So it is remains to prove R is an accepting run of \mathcal{A}^* on u . That is, we must prove that $R(0) \in Z'$, $R(\alpha) \in F'$ and satisfies the following conditions:

1. For each $\beta < \alpha$, we have

$$(R(\beta), u(\beta), R(\beta + 1)) \in T'.$$

2. For each $\beta \leq \alpha$ that is a limit ordinal with $\text{cf}(\beta) = \omega$, we have

$$(\sup_{\beta} (R), R(\beta)) \in T'.$$

3. For each $\beta \leq \alpha$ that is a limit ordinal with $\text{cf}(\beta) > \omega$, we have

$$(\sup_{\beta} (R), \text{stat}_{\beta}(R), R(\beta)) \in T'.$$

First we want to prove $R(0) \in Z'$, and $R(\alpha) \in F'$.

By the definition of R we get $R(0) = R(\alpha_0) = (R_0(0), 1)$ but $R_0(0) \in Z$ because R_0 is an accepting run of \mathcal{A} on u_0 , so we get $R(0) \in Z'$ and $R(\alpha) \in F'$, it is clear by the definition of R .

It is remains to prove the three conditions.

Now we want to prove condition (1). Let $\beta < \alpha$ and we want to prove $(R(\beta), u(\beta), R(\beta + 1)) \in T'$. We have the following cases:

(B1) Either $\beta = 0$,

(B2) Or $0 < \beta < \alpha$.

Case (B1). Assume $\beta = 0$.

Since R_0 is an accepting run of \mathcal{A} on u_0 , then

$$(R_0(0), u_0(0), R_0(1)) \in T,$$

but by the definition of R , we get

$$(R(0), u(0), R(1)) = ((R_0(0), 1), u_0(0), (R_0(1), 0)),$$

hence $(R(0), u(0), R(1)) \in T_3$, by the definition of T_3 , then $(R(0), u(0), R(1)) \in T'$.

Case (B2). Assume $0 < \beta < \alpha$.

Now, assume i – smallest ordinal such that $\beta < \alpha_{i+1}$, then there is unique $\delta < \theta_i$ such that $\beta = \alpha_i + \delta$.

We want to prove

$$(R(\beta), u(\beta), R(\beta + 1)) \in T'.$$

Then we have two cases either $\delta = 0$, or $\delta > 0$.

First assume $\delta = 0$, then $\beta = \alpha_i$, so by the definition of R , we get

$$(R(\beta), u(\beta), R(\beta + 1)) = (R(\alpha_i), u(\alpha_i), R(\alpha_i + 1)) = ((R_i(0), 1), u_i(0), (R_i(1), 0))$$

and since for each $n > i$, R_i is an accepting run of \mathcal{A} on u_i , then $(R_i(0), u_i(0), R_i(1)) \in T$, hence

$$(R(\beta), u(\beta), R(\beta + 1)) \in T_3,$$

by the definition of T_3 . Therefore $(R(\beta), u(\beta), R(\beta + 1)) \in T'$.

Second assume $\delta > 0$, and we want to prove $(R(\beta), u(\beta), R(\beta + 1)) \in T'$. Since for each $n > i$, R_i is an accepting run of \mathcal{A} on u_i , and $\delta < \theta_i$, then $(R_i(\delta), u_i(\delta), R_i(\delta + 1)) \in T$, and by the definition of R we get

$$(R(\beta), u(\beta), R(\beta + 1)) = ((R_i(\delta), 0), u_i(\delta), (R_i(\delta + 1), 0)) \in T_1.$$

Thus $(R(\beta), u(\beta), R(\beta + 1)) \in T'$.

Therefore from (B1), and (B2) we get for each $\beta < \alpha$, $(R(\beta), u(\beta), R(\beta + 1)) \in T'$.

Next we want to prove condition (2). Assume $\beta \leq \alpha$, that is a limit ordinal with $\text{cf}(\beta) = \omega$, and we want to prove

$$(\sup_{\beta}(R), R(\beta)) \in T'.$$

We have two cases:

(C1) Either $\beta < \alpha$,

(C2) Or $\beta = \alpha = \sum_{i < n} \theta_i$.

Case (C1). If $\beta < \alpha$.

Let i —smallest ordinal such that $\beta < \alpha_{i+1}$, then there is unique $\delta < \theta_i$ such that $\beta = \alpha_i + \delta$. Then we have two cases either $\delta > 0$, or $\delta = 0$.

Now let $\delta = 0$, then $\beta = \alpha_i = \theta_0 + \theta_1 + \cdots + \theta_{i-1}$. We want to prove $(\sup_{\beta}(R), R(\beta)) \in T_2$. By the definition of R , we get

$$R(\beta) = R(\alpha_i) = (R_i(0), 1) \in F',$$

since for each $n > i$, R_i is an accepting run of \mathcal{A} on u_i , and $\theta_{i-1} \in \text{lim}$, with $\text{cf}(\theta_{i-1}) = \omega$ because $\beta \in \text{lim}$, with $\text{cf}(\beta) = \omega$, by lemma 1.2.40, which give us $R_{i-1}(\theta_{i-1}) \in F$ and

$$(\sup_{\theta_{i-1}}(R_{i-1}), R_{i-1}(\theta_{i-1})) \in T, \text{ and}$$

$$(\sup_{\beta}(R), R(\beta)) = (\sup_{\theta_{i-1}}(R_{i-1}) \times \{0\}, (R_i(0), 1)) \in T_2,$$

by the definition of T_2 , therefore $(\sup_{\beta}(R), R(\beta)) \in T'$.

Assume $\delta > 0$, and we want to prove $(\sup_{\beta}(R), R(\beta)) \in T$. Then $\delta < \theta_i$, and $\delta \in \text{lim}$ with $\text{cf}(\delta) = \omega$ because $\beta \in \text{lim}$, with $\text{cf}(\beta) = \omega$ by lemma 1.2.40. Now, since for each $n > i$, R_i is an accepting run of \mathcal{A} on u_i , then

$$(\sup_{\delta}(R_i), R_i(\delta)) \in T,$$

and since

$$(\sup_{\beta}(R), R(\beta)) = (\sup_{\delta}(R_i) \times \{0\}, (R_i(\delta), 0)) \in T_1,$$

then $(\sup_{\beta}(R), R(\beta)) \in T'$.

Case (C2). Assume $\beta = \alpha = \sum_{i < n} \theta_i$.

We want to prove $(\sup_{\beta}(R), R(\beta)) \in T$. By the definition of R , we get

$$R(\beta) = R(\alpha) = (t, 1) \in F',$$

and R_{n-1} is an accepting run of \mathcal{A} on u_{n-1} , and $\theta_{n-1} \in \text{lim}$, with $\text{cf}(\theta_{n-1}) = \omega$ because $\beta \in \text{lim}$, with $\text{cf}(\beta) = \omega$, by lemma 1.2.40, which give us $R_{n-1}(\theta_{n-1}) \in F$ and

$$(\sup_{\theta_{n-1}}(R_{n-1}), R_{n-1}(\theta_{n-1})) \in T, \text{ and}$$

$$(\sup_{\beta}(R), R(\beta)) = (\sup_{\theta_{n-1}}(R_{n-1}) \times \{0\}, (t, 1)) \in T_2,$$

by the definition of T_2 , therefore $(\sup_{\beta}(R), R(\beta)) \in T'$.

Therefore, from (C1), and (C2) we get, for each $\beta \leq \alpha$ that is a limit ordinal with $\text{cf}(\beta) = \omega$, $(\sup_{\beta}(R), R(\beta)) \in T'$.

Finally we want to prove condition (3). Assume $\beta \leq \alpha$, $\beta = \alpha_i + \delta$, that is a limit ordinal with $\text{cf}(\beta) > \omega$, and we want to prove

$$(\sup_{\beta}(R), \text{stat}_{\beta}(R), R(\beta)) \in T'.$$

We have two cases:

(D1) Either $\beta < \alpha$,

(D2) Or $\beta = \alpha = \sum_{i < n} \theta_i$.

Case (D1). $\beta < \alpha$.

Let i –smallest ordinal such that $\beta < \alpha_{i+1}$, then there is unique $\delta < \theta_i$ such that $\beta = \alpha_i + \delta$. Then we have two cases either $\delta > 0$, or $\delta = 0$.

Now let $\delta = 0$, then $\beta = \alpha_i = \theta_0 + \theta_1 + \dots + \theta_{i-1}$. We want to prove $(\sup_{\beta}(R), \text{stat}_{\beta}(R), R(\beta)) \in T$. By the definition of R , we get

$$R(\beta) = R(\alpha_i) = (R_i(0), 1) \in F',$$

since for each $n > i$, R_i is an accepting run of \mathcal{A} on u_i , and $\theta_{i-1} \in \text{lim}$, with $\text{cf}(\theta_{i-1}) > \omega$ because $\beta \in \text{lim}$, with $\text{cf}(\beta) > \omega$, by lemma 1.2.40, which gives us $R_{i-1}(\theta_{i-1}) \in F$ and

$$(\sup_{\theta_{i-1}}(R_{i-1}), \text{stat}_{\theta_{i-1}}(R_{i-1}), R_{i-1}(\theta_{i-1})) \in T, \text{ and}$$

$$(\sup_{\beta}(R), \text{stat}_{\beta}(R), R(\beta)) = (\sup_{\theta_{i-1}}(R_{i-1}) \times \{0\}, \text{stat}_{\theta_{i-1}}(R_{i-1}) \times \{0\}, (R_i(0), 1)) \in T_2,$$

by the definition of T_2 , therefore $(\sup_{\beta}(R), \text{stat}_{\beta}(R), R(\beta)) \in T'$.

Assume $\delta > 0$. $\beta = \alpha_i + \delta$, and we want to prove $(\sup_\beta(R), \text{stat}_\beta(R), R(\beta)) \in T'$. Then $\delta < \theta_i$, and $\delta \in \text{lim}$ with $\text{cf}(\delta) > \omega$ because $\beta \in \text{lim}$, with $\text{cf}(\beta) > \omega$ by lemma 1.2.40. Since for each $n > i$, R_i is an accepting run of \mathcal{A} on u_i , then

$$(\sup_\delta(R_i), \text{stat}_\delta(R_i), R_i(\delta)) \in T,$$

and since

$$(\sup_\beta(R), \text{stat}_\beta(R), R(\beta)) = (\sup_\delta(R_i) \times \{0\}, \text{stat}_\delta(R_i) \times \{0\}, (R_i(\delta), 0)),$$

then $(\sup_\beta(R), \text{stat}_\beta(R), R(\beta)) \in T_1$. Therefore, $(\sup_\beta(R), \text{stat}_\beta(R), R(\beta)) \in T'$.

Case (D2). Assume $\beta = \alpha = \sum_{i < n} \theta_i$.

We want to prove $(\sup_\beta(R), \text{stat}_\beta(R), R(\beta)) \in T_2$. By the definition of R , we get

$$R(\beta) = R(\alpha) = (t, 1) \in F'$$

and R_{n-1} is an accepting run of \mathcal{A} on u_{n-1} , and $\theta_{n-1} \in \text{lim}$, with $\text{cf}(\theta_{n-1}) > \omega$ because $\beta \in \text{lim}$, with $\text{cf}(\beta) > \omega$, by lemma 1.2.40, which give us $R_{n-1}(\theta_{n-1}) \in F$ and

$$(\sup_{\theta_{n-1}}(R_{n-1}), \text{stat}_{\theta_{n-1}}(R_{n-1}), R_{n-1}(\theta_{n-1})) \in T, \text{ and}$$

$$(\sup_\beta(R), \text{stat}_\beta(R), R(\beta)) = (\sup_{\theta_{n-1}}(R_{n-1}) \times \{0\}, \text{stat}_{\theta_{n-1}}(R_{n-1}) \times \{0\}, (t, 1)),$$

then by the definition of T_2 , we get $(\sup_\beta(R), \text{stat}_\beta(R), R(\beta)) \in T_2$. Therefore $(\sup_\beta(R), \text{stat}_\beta(R), R(\beta)) \in T'$.

Therefore, from (D1), and (D2) we get, for each $\beta \leq \alpha$, that is a limit ordinal with $\text{cf}(\beta) > \omega$, $(\sup_\beta(R), \text{stat}_\beta(R), R(\beta)) \in T'$. Hence, R satisfies the three conditions.

Thus, R is an accepting run of \mathcal{A}^* on u . Hence, $u \in \mathcal{L}(\mathcal{A}^*)$. This implies to $(\mathcal{L}(\mathcal{A}))^* \subseteq \mathcal{L}(\mathcal{A}^*)$. From proof of (A1), and (A2) we get $\mathcal{L}(\mathcal{A}^*) = (\mathcal{L}(\mathcal{A}))^*$. \square

4.5 ω -Operation

The following is a definition of the ω -ST-automata.

Definition 4.5.1. Let $\mathcal{A} = (S, I, T, Z, F)$ be ST-automaton over I . Then define the ω -ST-automaton

over I of \mathcal{A} , or $\mathcal{A}^\omega = (S', I, T', Z', F')$ as follows:

$$\begin{aligned} S' &= S \times \{0, 1\} \cup \{s_f\}, \text{ such that } s_f \notin S \times \{0, 1\}, \\ Z' &= Z \times \{1\}, \\ F' &= \left\{ \begin{array}{ll} \{s_f\} & \text{if } \emptyset \notin \mathcal{L}(\mathcal{A}) \\ \{s_f\} \cup Z' & \text{if } \emptyset \in \mathcal{L}(\mathcal{A}) \end{array} \right\}, \text{ and} \\ T' &= T_1 \cup T_2 \cup T_3 \cup T_4, \end{aligned}$$

where

$$\begin{aligned} T_1 &= \{((s, 0), u, (t, 0)) : (s, a, t) \in T\} \cup \\ &\quad \{(A \times \{0\}, (t, 0)) : A \subseteq S, \text{ and } (A, t) \in T\} \cup \\ &\quad \{(A \times \{0\}, B \times \{0\}, (t, 0)) : A, B \subseteq S, \text{ and } (A, B, t) \in T\}, \end{aligned}$$

$$\begin{aligned} T_2 &= \{((s, 0), a, (t, 1)) : (s, a, t') \in T \text{ for some } t' \in F, \text{ and } (t, 1) \in Z'\} \cup \\ &\quad \{(A \times \{0\}, (t, 1)) : A \subseteq S, (A, t') \in T \text{ for some } t' \in F, \text{ and } (t, 1) \in Z'\} \cup \\ &\quad \{(A \times \{0\}, B \times \{0\}, (t, 1)) : A, B \subseteq S, (A, B, t') \in T \text{ for some } t' \in F, \text{ and } (t, 1) \in Z'\}, \end{aligned}$$

$$T_3 = \{((s, 1), a, (t, 0)) : (s, a, t) \in T\}, \text{ and}$$

$$T_4 = \{(A, s_f) : A \subseteq S \times \{0, 1\} \text{ such that there is } s \in S \text{ with } (s, 1) \in A\}.$$

Then we prove the following theorem that show, applying the ω operation to languages defined by ST-automaton, the produce language that is also definable using ST-automaton.

Theorem 4.5.2. *If $\mathcal{A} = (S, I, T, Z, F)$ is a ST-automaton over I , with Z is a finite set, then $\mathcal{L}(\mathcal{A}^\omega) = (\mathcal{L}(\mathcal{A}))^\omega$.*

Proof. Assume $\mathcal{A} = (S, I, T, Z, F)$ is a ST-automaton over I , with Z is a finite set and we want to prove $\mathcal{L}(\mathcal{A}^\omega) = (\mathcal{L}(\mathcal{A}))^\omega$, such that $\mathcal{A}^\omega = (S', I, T', Z', F')$. Thus we should prove the following:

$$(A1) \quad \mathcal{L}(\mathcal{A}^\omega) \subseteq (\mathcal{L}(\mathcal{A}))^\omega.$$

$$(A2) \quad (\mathcal{L}(\mathcal{A}))^\omega \subseteq \mathcal{L}(\mathcal{A}^\omega).$$

By the definition of \mathcal{A}^ω , we get

$$\begin{aligned} S' &= S \times \{0, 1\} \cup \{s_f\}, \quad \text{such that } s_f \notin S \times \{0, 1\}, \\ Z' &= Z \times \{1\}, \\ F' &= \left\{ \begin{array}{ll} \{s_f\} & \text{if } \emptyset \notin \mathcal{L}(\mathcal{A}) \\ \{s_f\} \cup Z' & \text{if } \emptyset \in \mathcal{L}(\mathcal{A}) \end{array} \right\}, \quad \text{and} \\ T' &= T_1 \cup T_2 \cup T_3 \cup T_4, \end{aligned}$$

where

$$\begin{aligned} T_1 &= \{((s, 0), u, (t, 0)) : (s, a, t) \in T\} \cup \\ &\quad \{(A \times \{0\}, (t, 0)) : A \subseteq S, \text{ and } (A, t) \in T\} \cup \\ &\quad \{(A \times \{0\}, B \times \{0\}, (t, 0)) : A, B \subseteq S, \text{ and } (A, B, t) \in T\}, \end{aligned}$$

$$\begin{aligned} T_2 &= \{((s, 0), a, (t, 1)) : (s, a, t') \in T \text{ for some } t' \in F, \text{ and } (t, 1) \in Z'\} \cup \\ &\quad \{(A \times \{0\}, (t, 1)) : A \subseteq S, (A, t') \in T \text{ for some } t' \in F, \text{ and } (t, 1) \in Z'\} \cup \\ &\quad \{(A \times \{0\}, B \times \{0\}, (t, 1)) : A, B \subseteq S, (A, B, t') \in T \text{ for some } t' \in F, \text{ and } (t, 1) \in Z'\}, \end{aligned}$$

$$T_3 = \{((s, 1), a, (t, 0)) : (s, a, t) \in T\}, \quad \text{and}$$

$$T_4 = \{(A, s_f) : A \subseteq S \times \{0, 1\} \text{ such that there is } s \in S \text{ with } (s, 1) \in A\}.$$

Proof of (A1). $\mathcal{L}(\mathcal{A}^\omega) \subseteq (\mathcal{L}(\mathcal{A}))^\omega$.

We can assume $Z \neq \emptyset$, since if $Z = \emptyset$, then $Z' = \emptyset$ and $\mathcal{L}(\mathcal{A}) = \emptyset$, and so $(\mathcal{L}(\mathcal{A}))^\omega = \emptyset$, and $\mathcal{L}(\mathcal{A}^\omega) = \emptyset$, therefore $\mathcal{L}(\mathcal{A}^\omega) = (\mathcal{L}(\mathcal{A}))^\omega$.

So we can assume $Z \neq \emptyset$. Let $u \in \mathcal{L}(\mathcal{A}^\omega)$, such that $u : \alpha \rightarrow I$, and $\alpha \in \text{ord}$. Then there is an accepting run $R : \alpha + 1 \rightarrow S'$ of \mathcal{A}^ω on u . We want to prove $u \in (\mathcal{L}(\mathcal{A}))^\omega$, when

$$(\mathcal{L}(\mathcal{A}))^\omega = \{u : u = \circ(u_i)_{i < \omega}, u_i \in \mathcal{L}(\mathcal{A}) \text{ for each } i < \omega\}.$$

Now if $u = \emptyset$, then $\emptyset \in \mathcal{L}(\mathcal{A}^\omega)$, therefore $Z' \cap F' \neq \emptyset$, so $R(0) = R(\alpha) = (s, 1)$, for some $(s, 1) \in Z' \cap F'$, hence $F' = \{s_f\} \cup Z'$, thus $\emptyset \in \mathcal{L}(\mathcal{A})$, by the definition of F' . Therefore $\emptyset \in (\mathcal{L}(\mathcal{A}))^\omega$. Thus $u \in (\mathcal{L}(\mathcal{A}))^\omega$.

Now, assume $u \neq \emptyset$. So the only we need to discuss the following two cases:

(B1) $R(\alpha) \in Z'$.

(B2) $R(\alpha) = s_f$.

Case (B1). If $R(\alpha) \in Z'$.

In this case $\emptyset \in \mathcal{L}(\mathcal{A})$. Then by an argument similar to the proof of the previous theorem we can write $u = u_0 \circ u_1 \circ \dots \circ u_n$, such that $u_i \neq \emptyset$, for each $i \leq n$ and $n \geq 0$. Then $u \in (\mathcal{L}(\mathcal{A}))^\omega$.

Case (B2). If $R(\alpha) = s_f$.

We will show that

$$u = \circ(u_i)_{i < \omega}, \quad u_i \neq \emptyset, \quad u_i \in \mathcal{L}(\mathcal{A}) \text{ for each } i < \omega.$$

Define

$$C = \{\sigma < \alpha : R(\sigma) \in Z'\} = \{\alpha_\beta : \beta < \eta\},$$

where η is an ordinal with $\alpha_\beta < \alpha_\gamma$, for each $\beta < \gamma < \eta$. Now we want to prove $\eta = \omega$. Since $R(\alpha) = s_f$, the only way to get s_f in our transition in T_4 , so we get $\alpha \in \text{limit}$ with $\text{cf}(\alpha) = \omega$. Therefore $\sup_\alpha(R) = A$, for some $A \subseteq S \times \{0, 1\}$, such that there is $s \in S$ with $(s, 1) \in A$. Thus $\eta \geq \omega$. Now suppose $\eta > \omega$. Let $C' = \{\alpha_i : i < \omega\}$, and $\delta = \sup(C')$, then $\delta < \alpha$, since if $\delta = \alpha$, then α_ω is an upper bound on C' , so $\delta \leq \alpha_\omega < \alpha$, hence $R(\delta) = s_f$ and by the definition of T_4 , we get $\sup_\delta(R) = B$, for some $B \subseteq S \times \{0, 1\}$, such that there is $s \in S$ with $(s, 1) \in B$ which is a contraction so $C' = C$ and $\eta = \omega$.

Let $C = \{\alpha_i : i < \omega\}$. Now define that, for each i and $0 \leq i < \omega$, $\theta_i \in \text{ord}$ such that

$$\alpha_i + \theta_i = \alpha_{i+1}, \text{ and}$$

$$u_i : \theta_i \rightarrow I, \text{ is such that}$$

$$u_i(\delta) = u(\alpha_i + \delta), \text{ for all } \delta < \theta_i,$$

then we get,

$$u = \circ(u_i)_{i < \omega}, \text{ and } \alpha = \sum_{i < \omega} \theta_i.$$

Now since for each i with $i < \omega$, $\alpha_i \in C$, then $R(\alpha_i) \in Z'$, and $Z' = Z \times 1$, so define $R_i : \theta_i + 1 \rightarrow S$, as the following:

$$R_i(0) = s, \text{ such that } R(\alpha_i) = (s, 1),$$

$$R_i(\delta) = s, \text{ such that } R(\alpha_i + \delta) = (s, 0), \text{ for all } 0 < \delta < \theta_i, \text{ and}$$

$$R_i(\theta_i) = t',$$

such that we get to t' as the following: we consider two cases either α_{i+1} is a successor ordinal or limit ordinal. First assume α_{i+1} is a successor ordinal, then let $\alpha_{i+1} = \sigma + 1$, for some $\sigma \in \text{ord}$, and since R is an accepting run of \mathcal{A}^ω on u , then

$$(R(\sigma), u(\sigma), R(\alpha_{i+1})) \in T',$$

but $R(\alpha_{i+1}) \in Z' = Z \times \{1\}$, because $\alpha_{i+1} \in C$, then let $R(\alpha_{i+1}) = (t, 1)$, for some $t \in Z$, thus

$$(R(\sigma), u(\sigma), R(\alpha_{i+1})) \in T_2,$$

by the definition of T_2 , which implies to

$$(R(\sigma), u(\sigma), R(\alpha_{i+1})) = ((s, 0), u(\sigma), (t, 1)),$$

such that $(s, u(\sigma), t') \in T$, for some $t' \in F$.

Second assume α_{i+1} is a limit ordinal, then either $\text{cf}(\alpha_{i+1}) = \omega$ or $\text{cf}(\alpha_{i+1}) > \omega$.

First assume $\text{cf}(\alpha_{i+1}) = \omega$, then

$$\left(\sup_{\alpha_{i+1}} (R), R(\alpha_{i+1}) \right) \in T',$$

since R is an accepting run of \mathcal{A}^ω on u , but $R(\alpha_{i+1}) \in Z' = Z \times \{1\}$, because $\alpha_{i+1} \in C$, then let $R(\alpha_{i+1}) = (t, 1)$, for some $t \in Z$, thus

$$\left(\sup_{\alpha_{i+1}} (R), R(\alpha_{i+1}) \right) \in T_2,$$

by the definition of T_2 , which implies to

$$\left(\sup_{\alpha_{i+1}} (R), R(\alpha_{i+1}) \right) = (A \times \{0\}, (t, 1)),$$

for some $A \subseteq S$, and $(A, t') \in T$ for some $t' \in F$.

Now assume $\text{cf}(\alpha_{i+1}) > \omega$, then

$$\left(\sup_{\alpha_{i+1}} (R), \text{stat}_{\alpha_{i+1}} (R), R(\alpha_{i+1}) \right) \in T',$$

since R is an accepting run of \mathcal{A}^ω on u , but $R(\alpha_{i+1}) \in Z' = Z \times \{1\}$, because $\alpha_{i+1} \in C$, then let $R(\alpha_{i+1}) = (t, 1)$, for some $t \in Z$, thus

$$\left(\sup_{\alpha_{i+1}} (R), \text{stat}_{\alpha_{i+1}} (R), R(\alpha_{i+1}) \right) \in T_2,$$

by the definition of T_2 , which implies to

$$\left(\sup_{\alpha_{i+1}}(R), \text{stat}_{\alpha_{i+1}}(R), R(\alpha_{i+1})\right) = (A \times \{0\}, B \times \{0\}, (t, 1)),$$

for some $A, B \subseteq S$, and $(A, B, t') \in T$ for some $t' \in F$.

Now by the same way in previous theorem we can prove that R_i is an accepting run of \mathcal{A} on u_i , for each $i < \omega$. Therefore $u_i \in \mathcal{L}(\mathcal{A})$, for each $i < \omega$, hence $u \in (\mathcal{L}(\mathcal{A}))^\omega$. Therefore $\mathcal{L}(\mathcal{A}^\omega) \subseteq (\mathcal{L}(\mathcal{A}))^\omega$.

Proof of (A2). $(\mathcal{L}(\mathcal{A}))^\omega \subseteq \mathcal{L}(\mathcal{A}^\omega)$.

Assume $u \in (\mathcal{L}(\mathcal{A}))^\omega$, such that $u : \alpha \rightarrow I$, for some $\alpha \in \text{ord}$.

If $u = \emptyset$, then $\emptyset \in \mathcal{L}(\mathcal{A})$, since

$$(\mathcal{L}(\mathcal{A}))^\omega = \{u : u = \circ(u_i)_{i < \omega}, u_i \in \mathcal{L}(\mathcal{A}) \text{ for each } i < \omega\}.$$

Then there exists $s \in F \cap Z$, so define $R : 1 \rightarrow S'$ such that $R(0) = R(\alpha) = (s, 1)$, and that is an accepting run of \mathcal{A}^ω on u .

Now assume $u \neq \emptyset$, and we need to define an accepting run $R : \alpha + 1 \rightarrow S'$ of \mathcal{A}^ω on u . Now since $u \in (\mathcal{L}(\mathcal{A}))^\omega$, then

$$u = \circ(u_i)_{i < \omega}, u_i \in \mathcal{L}(\mathcal{A}) \text{ for each } i < \omega.$$

Since $u \neq \emptyset$, then we can assume that for each $i < \omega$, $u_i \neq \emptyset$. Let $u_i : \theta_i \rightarrow I$, $\theta_i \in \text{ord}$, $\theta_i \neq \emptyset$, for each $i < \omega$ and $\alpha_i = \sum_{j=0}^{i-1} \theta_j$, for each $i < \omega$.

$$u(\alpha_i + \delta) = u_i(\delta), \text{ for all } \delta < \theta_i \text{ and } i < \omega.$$

Then $\alpha = \sum_{i < \omega} \theta_i$. Since for each $i < \omega$, $u_i \in \mathcal{L}(\mathcal{A})$, then there are accepting runs $R_i : \theta_i + 1 \rightarrow S$ of \mathcal{A} on u_i , for each $i < \omega$. Then we have two cases:

(B1) $\emptyset \notin \mathcal{L}(\mathcal{A})$.

(B2) $\emptyset \in \mathcal{L}(\mathcal{A})$.

Case (B1). If $\emptyset \notin \mathcal{L}(\mathcal{A})$.

Now, we want to prove $u \in \mathcal{L}(\mathcal{A}^\omega)$. Define $R : \alpha + 1 \rightarrow S'$ as follows:

$$\begin{aligned} R(\alpha_i) &= (R_i(0), 1), \text{ for each } i < \omega, \\ R(\alpha_i + \delta) &= (R_i(\delta), 0), \text{ for each } 0 < \delta < \theta_i, i < \omega \\ R(\alpha) &= s_f. \end{aligned}$$

Now we want to prove R is an accepting run of \mathcal{A}^ω on u . That is we must prove that, $R(0) \in Z'$, $R(\alpha) \in F'$ and satisfies the following conditions:

1. For each $\beta < \alpha$, we have

$$(R(\beta), u(\beta), R(\beta + 1)) \in T'.$$

2. For each $\beta \leq \alpha$ that is a limit ordinal with $\text{cf}(\beta) = \omega$, we have

$$(\sup_\beta(R), R(\beta)) \in T'.$$

3. For each $\beta \leq \alpha$ that is a limit ordinal with $\text{cf}(\beta) > \omega$, we have

$$(\sup_\beta(R), \text{stat}_\beta(R), R(\beta)) \in T'.$$

First we want to prove $R(0) \in Z'$, and $R(\alpha) \in F'$. By the definition of R we get $R(0) = R(\alpha_0) = (R_0(0), 1)$ but $R_0(0) \in Z$ because R_0 is an accepting run of \mathcal{A} on u_0 so we get $R(0) \in Z'$ and $R(\alpha) \in F'$, it is clear by the definition of R .

It remains to prove the three conditions. Now we want to prove condition (1). Let $\beta < \alpha$ and we want to prove

$$(R(\beta), u(\beta), R(\beta + 1)) \in T'.$$

We discuss the following cases:

(C1) If $\beta = 0$.

(C2) If $0 < \beta < \alpha$.

Case (C1). If $\beta = 0$.

Since R_0 is an accepting run of \mathcal{A} on u_0 , then

$$(R_0(0), u_0(0), R_0(1)) \in T,$$

but by the definition of R , we get

$$(R(0), u(0), R(1)) = ((R_0(0), 1), u_0(0), (R_0(1), 0)),$$

hence $(R(0), u(0), R(1)) \in T_3$, by the definition of T_3 , then $(R(0), u(0), R(1)) \in T'$.

Case (C2). If $0 < \beta < \alpha$.

Let i —smallest ordinal such that $\beta < \alpha_{i+1}$, then there is unique $\delta < \theta_i$, such that $\beta = \alpha_i + \delta$. We want to prove

$$(R(\beta), u(\beta), R(\beta + 1)) \in T'.$$

Then we have two cases either $\delta = 0$, or $\delta > 0$.

First assume $\delta = 0$, then $\beta = \alpha_i$, so by the definition of R , we get

$$(R(\beta), u(\beta), R(\beta + 1)) = (R(\alpha_i), u(\alpha_i), R(\alpha_i + 1)) = ((R_i(0), 1), u_i(0), (R_i(1), 0))$$

and since for each $i < \omega$, R_i is an accepting run of \mathcal{A} on u_i , then $(R_i(0), u_i(0), R_i(1)) \in T$, hence

$$(R(\beta), u(\beta), R(\beta + 1)) \in T_3,$$

by the definition of T_3 . Therefore $(R(\beta), u(\beta), R(\beta + 1)) \in T'$.

Second assume $\delta > 0$, and we want to prove $(R(\beta), u(\beta), R(\beta + 1)) \in T'$. Since for each $i < \omega$, R_i is an accepting run of \mathcal{A} on u_i , and $\delta < \theta_i$, then $(R_i(\delta), u_i(\delta), R_i(\delta + 1)) \in T$, and by the definition of R we get

$$(R(\beta), u(\beta), R(\beta + 1)) = ((R_i(\delta), 0), u_i(\delta), (R_i(\delta + 1), 0)) \in T_1.$$

Thus $(R(\beta), u(\beta), R(\beta + 1)) \in T'$. Thus, from (C1), and (C2) we get for each $\beta < \alpha$, we have $(R(\beta), u(\beta), R(\beta + 1)) \in T'$.

Next we want to prove condition (2). Assume $\beta \leq \alpha$, $\beta = \alpha_i + \delta$, that is a limit ordinal with $\text{cf}(\beta) = \omega$, and we want to prove

$$(\sup_{\beta} (R), R(\beta)) \in T'.$$

We have two cases:

(D1) $\beta < \alpha$.

(D2) $\beta = \alpha$.

Case (D1). If $\beta < \alpha$.

Let i —smallest ordinal such that $\beta < \alpha_{i+1}$, then there is unique $\delta < \theta_i$ such that $\beta = \alpha_i + \delta$. Then we have two cases either $\delta > 0$, or $\delta = 0$.

Now let $\delta = 0$, then $\beta = \alpha_i = \theta_0 + \theta_1 + \cdots + \theta_{i-1}$. We want to prove $(\sup_\beta(R), R(\beta)) \in T_2$. By the definition of R , we get

$$R(\beta) = R(\alpha_i) = (R_i(0), 1) \in Z',$$

since for each $i < \omega$, R_i is an accepting run of \mathcal{A} on u_i , and $\theta_{i-1} \in \text{lim}$, with $\text{cf}(\theta_{i-1}) = \omega$ because $\beta \in \text{lim}$, with $\text{cf}(\beta) = \omega$, by lemma 1.2.40, which gives us $R_{i-1}(\theta_{i-1}) \in F$ and

$$(\sup_{\theta_{i-1}}(R_{i-1}), R_{i-1}(\theta_{i-1})) \in T, \text{ and}$$

$$(\sup_\beta(R), R(\beta)) = (\sup_{\theta_{i-1}}(R_{i-1}) \times \{0\}, (R_i(0), 1)) \in T_2,$$

by the definition of T_2 , therefore $(\sup_\beta(R), R(\beta)) \in T'$.

Assume $\delta > 0$, where $\beta = \alpha_i + \delta$, and we want to prove $(\sup_\beta(R), R(\beta)) \in T_1$. Then $\delta < \theta_i$, and $\delta \in \text{lim}$ with $\text{cf}(\delta) = \omega$ because $\beta \in \text{lim}$, with $\text{cf}(\beta) = \omega$, by lemma 1.2.40. Since for each $\omega > i$, R_i is an accepting run of \mathcal{A} on u_i , then

$$(\sup_\delta(R_i), R_i(\delta)) \in T,$$

and since

$$(\sup_\beta(R), R(\beta)) = (\sup_\delta(R_i) \times \{0\}, (R_i(\delta), 0)) \in T_1,$$

then $(\sup_\beta(R), R(\beta)) \in T'$.

Case (D2). If $\beta = \alpha$.

Now assume $\beta = \alpha = \sum_{i < \omega} \theta_i$. We want to prove $(\sup_\beta(R), R(\beta)) \in T'$. By the definition of R , we get

$$R(\beta) = R(\alpha) = s_f \in F'.$$

It is clear that, $\sup_\beta(R) = A \subseteq S \times \{0, 1\}$, and there is $s \in S$ with $(s, 1) \in A$, since Z' is finite and $R(\alpha_i) = (R_i(0), 1)$, for each $i < \omega$. Therefore, $(\sup_\beta(R), R(\beta)) \in T_4$, so $(\sup_\beta(R), R(\beta)) \in T'$. Therefore, from (D1), and (D2) we get for every limit ordinal $\beta \leq \alpha$, with $\text{cf}(\beta) = \omega$, $(\sup_\beta(R), R(\beta)) \in T'$.

Now we want to prove condition (3). Notes that, $\beta \neq \alpha$, because $\text{cf}(\alpha) = \omega$ by lemma 1.2.37, so assume $\beta < \alpha$, such that i – smallest ordinal with $\beta < \alpha_{i+1}$, then there is unique $\delta < \theta_i$ such that $\beta = \alpha_i + \delta$ that is a limit ordinal with $\text{cf}(\beta) > \omega$, and we want to prove

$$(\sup_\beta(R), \text{state}_\beta(R), R(\beta)) \in T'.$$

Then we have two cases either $\delta > 0$, or $\delta = 0$.

First, let $\delta = 0$, then $\beta = \alpha_i = \theta_0 + \theta_1 + \dots + \theta_{i-1}$. We want to prove $(\sup_\beta(R), \text{state}_\beta(R), R(\beta)) \in T_2$. By the definition of R , we get

$$R(\beta) = R(\alpha_i) = (R_i(0), 1) \in Z',$$

since for each $i < \omega$, R_i is an accepting run of \mathcal{A} on u_i , and $\theta_{i-1} \in \text{lim}$, with $\text{cf}(\theta_{i-1}) > \omega$ because $\beta \in \text{lim}$, with $\text{cf}(\beta) > \omega$, by lemma 1.2.40, which gives us $R_{i-1}(\theta_{i-1}) \in F$ and

$$(\sup_{\theta_{i-1}}(R_{i-1}), \text{stat}_{\theta_{i-1}}(R_{i-1}), R_{i-1}(\theta_{i-1})) \in T, \text{ and}$$

$$(\sup_\beta(R), \text{state}_\beta(R), R(\beta)) = (\sup_{\theta_{i-1}}(R_{i-1}) \times \{0\}, \text{state}_{\theta_{i-1}}(R_{i-1}) \times \{0\}, (R_i(0), 1)) \in T_2,$$

by the definition of T_2 , therefore $(\sup_\beta(R), \text{state}_\beta(R), R(\beta)) \in T'$.

Assume $\delta > 0$, where $\beta = \alpha_i + \delta$, and we want to prove $(\sup_\beta(R), \text{state}_\beta(R), R(\beta)) \in T_1$. Then $\delta < \theta_i$, and $\delta \in \text{lim}$ with $\text{cf}(\delta) > \omega$ because $\beta \in \text{lim}$, with $\text{cf}(\beta) > \omega$, by lemma 1.2.40. Since for each $i < \omega$, R_i is an accepting run of \mathcal{A} on u_i , then

$$(\sup_\delta(R_i), \text{state}_\delta(R_i), R_i(\delta)) \in T,$$

and since

$$(\sup_\beta(R), \text{state}_\beta(R), R(\beta)) = (\sup_\delta(R_i) \times \{0\}, \text{state}_\delta(R_i) \times \{0\}, (R_i(\delta), 0)) \in T_1,$$

then $(\sup_\beta(R), \text{state}_\beta(R), R(\beta)) \in T'$.

Therefore, R satisfies the three condition, hence R is an accepting run of \mathcal{A}^ω on u . Then $u \in \mathcal{L}(\mathcal{A}^\omega)$. Thus, $(\mathcal{L}(\mathcal{A}))^\omega \subseteq \mathcal{L}(\mathcal{A}^\omega)$.

Case (B2). If $\emptyset \in \mathcal{L}(\mathcal{A})$.

We want to prove $u \in \mathcal{L}(\mathcal{A}^\omega)$. Then we have two cases either

$$u = \circ(u_i)_{i < \omega}, u \in \mathcal{L}(\mathcal{A}) \text{ for each } i < \omega, \text{ or}$$

$$u = u_0 \circ u_1 \circ \dots \circ u_n, \text{ such that } u_i \neq \emptyset \text{ for each } i \leq n \text{ and } n \geq 0.$$

If $u = \circ(u_i)_{i < \omega}$, $u_i \in \mathcal{L}(\mathcal{A})$ for each $i < \omega$, then by the same proof of case (B1), we can show that $u \in \mathcal{L}(\mathcal{A}^\omega)$.

Now assume that $u = u_0 \circ u_1 \circ \cdots \circ u_n$. Define $R : \alpha + 1 \rightarrow S'$ as follows:

$$R(\alpha_i) = (R_i(0), 1), \text{ for each } i < n,$$

$$R(\alpha_i + \delta) = (R_i(\delta), 0) \text{ for each } 0 < \delta < \theta_i, i < n$$

$$R(\alpha) = (t, 1),$$

for an element $(t, 1) \in Z'$ such element exist because $Z \neq \emptyset$, then $Z' = Z \times \{1\} \neq \emptyset$. Then $u \in \mathcal{L}(\mathcal{A}^\omega)$, also by the same proof for the previous theorem. Therefore, from (A1) and (A2) we get $\mathcal{L}(\mathcal{A}^\omega) = (\mathcal{L}(\mathcal{A}))$. \square

4.6 #–Operation

Finally, we define the #-ST-automata as follows:

Definition 4.6.1. Let $\mathcal{A} = (S, I, T, Z, F)$ be a ST-automaton over I with $Z \neq \emptyset$. Define the #-ST-automaton over I of \mathcal{A} , or $\mathcal{A}^\# = (S', I, T', Z', F')$ as follows:

$$\begin{aligned} S' &= S \times \{0, 1\}, \\ Z' &= Z \times \{1\}, \\ F' &= Z', \\ T' &= T_1 \cup T_2 \cup T_3 \cup T_4, \end{aligned}$$

where

$$\begin{aligned} T_1 &= \{((s, 0), a, (t, 0)) : (s, a, t) \in T\} \cup \\ &\quad \{(A \times \{0\}, (t, 0)) : A \subseteq S, \text{ and } (A, t) \in T\} \cup \\ &\quad \{(A \times \{0\}, B \times \{0\}, (t, 0)) : A, B \subseteq S, \text{ and } (A, B, t) \in T\}, \\ T_2 &= \{((s, 0), a, (t, 1)) : (s, a, t') \in T \text{ for some } t' \in F, \text{ and } (t, 1) \in Z'\} \cup \\ &\quad \{(A \times \{0\}, (t, 1)) : A \subseteq S, (A, t') \in T \text{ for some } t' \in F, \text{ and } (t, 1) \in Z'\} \cup \\ &\quad \{(A \times \{0\}, B \times \{0\}, (t, 1)) : A, B \subseteq S, (A, B, t') \in T \text{ for some } t' \in F, \text{ and } (t, 1) \in Z'\}, \\ T_3 &= \{((s, 1), a, (t, 0)) : (s, a, t) \in T\}, \end{aligned}$$

$$T_4 = \{(A, (t, 1)) : A \subseteq S' \text{ such that there is } s \in S \text{ with } (s, 1) \in A, \text{ and } (t, 1) \in Z'\} \cup \\ \{(A, B, (t, 1)) : A, B \subseteq S' \text{ such that there is } s \in S \text{ with } (s, 1) \in A, \text{ and } (t, 1) \in Z'\}.$$

If $Z = \emptyset$, then define $\mathcal{A}^\# = (\{s\}, I, \emptyset, \{s\}, \{s\})$, for any $s \in S$.

The following theorem shows that, applying the $\#$ operation to languages defined by ST-automaton, the produce language that is also definable using ST-automaton.

Theorem 4.6.2. *If $\mathcal{A} = (S, I, T, Z, F)$ is a ST-automaton over I , with Z is a finite set, then $\mathcal{L}(\mathcal{A}^\#) = (\mathcal{L}(\mathcal{A}))^\#$.*

Proof. Assume $\mathcal{A} = (S, I, T, Z, F)$ is a ST-automaton over I , with Z is a finite and we want to prove $\mathcal{L}(\mathcal{A}^\#) = (\mathcal{L}(\mathcal{A}))^\#$, such that $\mathcal{A}^\# = (S', I, T', Z', F')$. We want to prove the following:

$$(A1) \quad \mathcal{L}(\mathcal{A}^\#) \subseteq (\mathcal{L}(\mathcal{A}))^\#.$$

$$(A2) \quad (\mathcal{L}(\mathcal{A}))^\# \subseteq \mathcal{L}(\mathcal{A}^\#).$$

If $Z \neq \emptyset$, then by the definition of $\mathcal{A}^\#$, we get to the following:

$$S' = S \times \{0, 1\}, \\ Z' = Z \times \{1\}, \\ F' = Z', \\ T' = T_1 \cup T_2 \cup T_3 \cup T_4,$$

where

$$T_1 = \{((s, 0), a, (t, 0)) : (s, a, t) \in T\} \cup \\ \{(A \times \{0\}, (t, 0)) : A \subseteq S, \text{ and } (A, t) \in T\} \cup \\ \{(A \times \{0\}, B \times \{0\}, (t, 0)) : A, B \subseteq S, \text{ and } (A, B, t) \in T\},$$

$$T_2 = \{((s, 0), a, (t, 1)) : (s, a, t') \in T \text{ for some } t' \in F, \text{ and } (t, 1) \in Z'\} \cup \\ \{(A \times \{0\}, (t, 1)) : A \subseteq S, (A, t') \in T \text{ for some } t' \in F, \text{ and } (t, 1) \in Z'\} \cup \\ \{(A \times \{0\}, B \times \{0\}, (t, 1)) : A, B \subseteq S, (A, B, t') \in T \text{ for some } t' \in F, \text{ and } (t, 1) \in Z'\},$$

$$T_3 = \{((s, 1), a, (t, 0)) : (s, a, t) \in T\},$$

$$T_4 = \{(A, (t, 1)) : A \subseteq S' \text{ such that there is } s \in S \text{ with } (s, 1) \in A, \text{ and } (t, 1) \in Z'\} \cup \\ \{(A, B, (t, 1)) : A, B \subseteq S' \text{ such that there is } s \in S \text{ with } (s, 1) \in A, \text{ and } (t, 1) \in Z'\}.$$

If $Z = \emptyset$, then define $\mathcal{A}^\# = (\{s\}, I, \emptyset, \{s\}, \{s\})$, for any $s \in S$.

We can assume $Z \neq \emptyset$, since if $Z = \emptyset$, then $\mathcal{A}^\# = (\{s\}, I, \emptyset, \{s\}, \{s\})$, for some $s \in S$, hence $\mathcal{L}(\mathcal{A}^\#) = \{\emptyset\}$, and since $\mathcal{L}(\mathcal{A}) = \emptyset$, then $(\mathcal{L}(\mathcal{A}))^\# = \{\emptyset\}$, therefore $\mathcal{L}(\mathcal{A}^\#) = (\mathcal{L}(\mathcal{A}))^\#$.

So we can assume $Z \neq \emptyset$.

Proof of (A1). $\mathcal{L}(\mathcal{A}^\#) \subseteq (\mathcal{L}(\mathcal{A}))^\#$.

Let $u \in \mathcal{L}(\mathcal{A}^\#)$, such that $u : \alpha \rightarrow I$, and $\alpha \in \text{ord}$. Since $\emptyset \in (\mathcal{L}(\mathcal{A}))^\#$, so we can assume $u \neq \emptyset$. Then there is an accepting run $R : \alpha + 1 \rightarrow S'$ of $\mathcal{A}^\#$ on u . We want to prove $u \in (\mathcal{L}(\mathcal{A}))^\#$ and since

$$(\mathcal{L}(\mathcal{A}))^\# = \bigcup_{\gamma \in \text{ord}} (\mathcal{L}(\mathcal{A}))^\gamma,$$

such that for each $\gamma \in \text{ord}$,

$$(\mathcal{L}(\mathcal{A}))^\gamma = \{u : u = \circ(u_\beta)_{\beta < \gamma}, u_\beta \in \mathcal{L}(\mathcal{A}) \text{ for each } \beta < \gamma\}.$$

We will find $\gamma \in \text{ord}$, such that $u \in (\mathcal{L}(\mathcal{A}))^\gamma$. Let

$$C = \{\beta \leq \alpha : R(\beta) \in Z'\} = \{\alpha_\beta : \beta < \gamma\}, \text{ where } \gamma \in \text{ord}.$$

Now define that, for each i and $0 \leq i < \gamma$, $\theta_i \in \text{ord}$ such that

$$\alpha_i + \theta_i = \alpha_{i+1}, \text{ and}$$

$$u_i : \theta_i \rightarrow I, \text{ is such that}$$

$$u_i(\delta) = u(\alpha_i + \delta), \text{ for all } \delta < \theta_i,$$

then we get,

$$u = \circ(u_i)_{i < \gamma}, \text{ and } \alpha = \sum_{i < \gamma} \theta_i.$$

Now since for each i with $0 \leq i < \gamma$, $\alpha_i \in C$, then $R(\alpha_i) \in Z'$, and $Z' = Z \times 1$, so define $R_i : \theta_i + 1 \rightarrow S$, as the following:

$$R_i(0) = s \text{ such that } R(\alpha_i) = (s, 1), \text{ for some } (s, 1) \in Z',$$

$$R_i(\delta) = s \text{ such that } R(\alpha_i + \delta) = (s, 0), \text{ for all } 0 < \delta < \theta_i, \text{ and}$$

$$R_i(\theta_i) = t',$$

such that we get to t' as the following: we consider two cases either α_{i+1} is a successor ordinal

or limit ordinal. First assume α_{i+1} is a successor ordinal, then let $\alpha_{i+1} = \sigma + 1$, for some $\sigma \in \text{ord}$, and since R is an accepting run of $\mathcal{A}^\#$ on u , then

$$(R(\sigma), u(\sigma), R(\alpha_{i+1})) \in T',$$

but $R(\alpha_{i+1}) \in Z' = Z \times \{1\}$, because $\alpha_{i+1} \in C$, then let $R(\alpha_{i+1}) = (t, 1)$, thus

$$(R(\sigma), u(\sigma), R(\alpha_{i+1})) \in T_2,$$

by the definition of T_2 , which implies to

$$(R(\sigma), u(\sigma), R(\alpha_{i+1})) = ((s, 0), u(\sigma), (t, 1)),$$

such that $(s, u(\sigma), t') \in T$, for some $t' \in F$.

Second assume α_{i+1} is a limit ordinal, then either $\text{cf}(\alpha_{i+1}) = \omega$ or $\text{cf}(\alpha_{i+1}) > \omega$.

First assume $\text{cf}(\alpha_{i+1}) = \omega$, then

$$(\sup_{\alpha_{i+1}}(R), R(\alpha_{i+1})) \in T',$$

since R is an accepting run of $\mathcal{A}^\#$ on u , but $R(\alpha_{i+1}) \in Z' = Z \times \{1\}$, because $\alpha_{i+1} \in C$, then let $R(\alpha_{i+1}) = (t, 1)$, thus

$$(\sup_{\alpha_{i+1}}(R), R(\alpha_{i+1})) \in T_2,$$

by the definition of T_2 , which implies to

$$(\sup_{\alpha_{i+1}}(R), R(\alpha_{i+1})) = (A \times \{0\}, (t, 1)),$$

for some $A \subseteq S$ and $(A, t') \in T$, for some $t' \in F$.

Now assume $\text{cf}(\alpha_{i+1}) > \omega$, then

$$(\sup_{\alpha_{i+1}}(R), \text{stat}_{\alpha_{i+1}}(R), R(\alpha_{i+1})) \in T',$$

since R is an accepting run of $\mathcal{A}^\#$ on u , but $R(\alpha_{i+1}) \in Z' = Z \times \{1\}$, because $\alpha_{i+1} \in C$, then let $R(\alpha_{i+1}) = (t, 1)$, thus

$$(\sup_{\alpha_{i+1}}(R), \text{stat}_{\alpha_{i+1}}(R), R(\alpha_{i+1})) \in T_2,$$

by the definition of T_2 , which implies to

$$(\sup_{\alpha_{i+1}}(R), \text{stat}_{\alpha_{i+1}}(R), R(\alpha_{i+1})) = (A \times \{0\}, B \times \{0\}, (t, 1)),$$

for some $A, B \subseteq S$ and $(A, B, t') \in T$, for some $t' \in F$.

So it remains to prove for each $i < \gamma$, R_i is an accepting run of \mathcal{A} on u_i . That is, we must prove that, for each $i < \gamma$, $R_i(0) \in Z$, $R_i(\theta_i) \in F$ and satisfies the following conditions:

1. For each $\beta < \theta_i$ we have

$$(R_i(\beta), u_i(\beta), R_i(\beta + 1)) \in T.$$

2. For each $\beta \leq \theta_i$ that is a limit ordinal with $\text{cf}(\beta) = \omega$, we have

$$(\sup_{\beta} R_i, R_i(\beta)) \in T.$$

3. For each $\beta \leq \theta_i$ that is a limit ordinal with $\text{cf}(\beta) > \omega$, we have

$$(\sup_{\beta} R_i, \text{stat}_{\beta} R_i, R_i(\beta)) \in T.$$

Let $i < \gamma$, and we want to prove R_i is an accepting run of \mathcal{A} on u_i .

First we want to prove $R_i(0) \in Z$, and $R_i(\theta_i) \in F$, and that is clear by the definition of R_i .

It remains to prove the three conditions. Now we want to prove condition (1). Let $\beta < \theta_i$, and we want to prove

$$(R_i(\beta), u_i(\beta), R_i(\beta + 1)) \in T.$$

Since R is an accepting run of $\mathcal{A}^{\#}$ on u , then $(R(\alpha_i + \beta), u(\alpha_i + \beta), R(\alpha_i + \beta + 1)) \in T'$.

We have two cases:

(B1) $\beta = 0$.

(B2) $0 < \beta < \theta_i$.

Case (B1). $\beta = 0$.

Then

$$(R(\alpha_i + \beta), u(\alpha_i + \beta), R(\alpha_i + \beta + 1)) = (R(\alpha_i), u(\alpha_i), R(\alpha_i + 1)) \in T',$$

and since $\alpha_i \in C$, then $R(\alpha_i) \in F' = Z \times \{1\}$, so

$$(R(\alpha_i + \beta), u(\alpha_i + \beta), R(\alpha_i + \beta + 1)) = (R(\alpha_i), u(\alpha_i), R(\alpha_i + 1)) \in T_3,$$

so by the definition of T_3 , we get

$$(R(\alpha_i), u(\alpha_i), R(\alpha_i + 1)) = ((s, 1), a, (t, 0)),$$

for some $(s, a, t) \in T$ and by the definition of R_i and u_i we get

$$(R_i(0), u_i(0), R_i(1)) = (s, a, t) \in T.$$

Case (B2). $0 < \beta < \theta_i$.

Now assume $0 < \beta < \theta_i$. Then $\alpha_i < \alpha_i + \beta < \alpha_{i+1}$ and

$$(R(\alpha_i + \beta), u(\alpha_i + \beta), R(\alpha_i + \beta + 1)) \in T',$$

since R is an accepting run of $\mathcal{A}^\#$ on u , and since $R(\alpha_i + \beta) = (s, 0)$, $R(\alpha_i + \beta + 1) = (t, 0)$ for some $t, s \in S$, thus

$$(R(\alpha_i + \beta), u(\alpha_i + \beta), R(\alpha_i + \beta + 1)) = ((s, 0), a, (t, 0)) \in T_1,$$

by the definition of T_1 , which implies to

$$(R_i(\beta), u_i(\beta), R_i(\beta + 1)) = ((s, a, t)) \in T,$$

by the definition of R_i , u_i and T_1 .

Therefore from (B1) and (B2), we get for each $\beta < \theta_i$, $(R_i(\beta), u_i(\beta), R_i(\beta + 1)) \in T$.

Next we want to prove condition (2). Assume $\beta \leq \theta_i$, that is a limit ordinal with $\text{cf}(\beta) = \omega$, and we want to prove

$$(\sup_\beta(R_i), R_i(\beta)) \in T.$$

We have two cases:

(C1) $\beta < \theta_i$.

(C2) $\beta = \theta_i$.

Case (C1). $\beta < \theta_i$.

Let $\beta < \theta_i$. Now since β is a limit ordinal in θ_i , with $\text{cf}(\beta) = \omega$, then $\alpha_i + \beta$ is a limit ordinal in α , with $\text{cf}(\alpha_i + \beta) = \omega$, by lemma 1.2.40, and we know R is an accepting run of $\mathcal{A}^\#$ on u , then $(\sup_{\alpha_i + \beta}(R), R(\alpha_i + \beta)) \in T'$ and since $R(\alpha_i + \beta) = (s, 0)$, then $(\sup_{\alpha_i + \beta}(R), R(\alpha_i + \beta)) \in T_1$ and since

$$(\sup_{\alpha_i + \beta}(R), R(\alpha_i + \beta)) = (\sup_\beta(R_i) \times \{0\}, (R_i(\beta), 0)),$$

then $(\sup_\beta(R_i), R_i(\beta)) \in T$, by the definition of T_1 .

Case (C2). $\beta = \theta_i$.

Let $\beta = \theta_i$, and since β is a limit ordinal with $\text{cf}(\beta) = \omega$, then $\alpha_i + \theta_i = \alpha_{i+1}$ is a limit ordinal in α , with $\text{cf}(\alpha_{i+1}) = \omega$, by lemma 1.2.40, thus $(\sup_{\alpha_{i+1}}(R), R(\alpha_{i+1})) \in T'$ and since $R(\alpha_{i+1}) = (t, 1)$, then $(\sup_{\alpha_{i+1}}(R), R(\alpha_{i+1})) \in T_2$ and

$$(\sup_{\alpha_{i+1}}(R), R(\alpha_{i+1})) = (\sup_{\theta_i}(R_i) \times \{0\}, (t, 1)),$$

and since α_{i+1} is a limit ordinal with $\text{cf}(\alpha_{i+1}) = \omega$, so $R_i(\theta_i) = t'$ and $(\sup_{\theta_i}(R_i), R_i(\theta_i)) \in T$ by the definition of T_2 .

Then from (C1), and (C2) we get, for every $\beta \leq \theta_i$, that is a limit ordinal with $\text{cf}(\beta) = \omega$, $(\sup_{\beta}(R_i), R_i(\beta)) \in T$.

Finally we want to prove condition (3). Assume $\beta \leq \theta_i$, that is a limit ordinal with $\text{cf}(\beta) > \omega$, and we want to prove

$$(\sup_{\beta}(R_i), \text{stat}_{\beta}(R_i), R_i(\beta)) \in T.$$

Then we have two cases:

(D1) $\beta < \theta_i$.

(D2) $\beta = \theta_i$.

Case (D1). $\beta < \theta_i$.

First let $\beta < \theta_i$. Since R is an accepting run of $\mathcal{A}^\#$ on u , and β is a limit ordinal in θ_i , with $\text{cf}(\beta) > \omega$, then $\alpha_i + \beta$ is a limit ordinal in α , with $\text{cf}(\alpha_i + \beta) > \omega$, by lemma 1.2.40, thus

$$(\sup_{\alpha_i + \beta}(R), \text{stat}_{\alpha_i + \beta}(R), R(\alpha_i + \beta)) \in T'$$

and since $R(\alpha_i + \beta) = (s, 0)$, then

$$(\sup_{\alpha_i + \beta}(R), \text{stat}_{\alpha_i + \beta}(R), R(\alpha_i + \beta)) \in T_1$$

and since

$$(\sup_{\alpha_i + \beta}(R), \text{stat}_{\alpha_i + \beta}(R), R(\alpha_i + \beta)) = (\sup_{\beta}(R_i) \times \{0\}, \sup_{\beta}(R_i) \times \{0\}, (R_i(\beta), 0)),$$

then by the definition of T_1 , we get $(\sup_{\beta}(R_i), \text{stat}_{\beta}(R_i), R_i(\beta)) \in T$.

Case (D2). $\beta = \theta_i$.

Now let $\beta = \theta_i$, and since β is a limit ordinal with $\text{cf}(\beta) > \omega$, then $\alpha_i + \theta_i = \alpha_{i+1}$ is a limit ordinal in α , with $\text{cf}(\alpha_{i+1}) > \omega$, by lemma 1.2.40, thus $(\sup_{\alpha_{i+1}}(R), \text{stat}_{\alpha_{i+1}}(R), R(\alpha_{i+1})) \in T'$

and since $R(\alpha_{i+1}) = (t, 1)$, then $(\sup_{\alpha_{i+1}}(R), \text{stat}_{\alpha_{i+1}}(R), R(\alpha_{i+1})) \in T_2$ and since

$$(\sup_{\alpha_{i+1}}(R), \text{stat}_{\alpha_{i+1}}(R), R(\alpha_{i+1})) = (\sup_{\theta_i}(R_i) \times \{0\}, \text{stat}_{\theta_i}(R_i) \times \{0\}, (t, 1)).$$

and since α_{i+1} is a limit ordinal with $\text{cf}(\alpha_{i+1}) > \omega$, so $R_i(\theta_i) = t'$ and $(\sup_{\theta_i}(R_i), \text{stat}_{\theta_i}(R_i), R_i(\theta_i)) \in T$ by the definition of T_2 . Thus, from (D1), and (D2) we get, for each $\beta \leq \theta_i$ that is a limit ordinal with $\text{cf}(\beta) > \omega$, $(\sup_{\beta}(R_i), \text{stat}_{\beta}(R_i), R_i(\beta)) \in T$. Hence, R_i is satisfying the three conditions. Therefore, R_i is an accepting run of \mathcal{A} on u_i , for each $i < \gamma$. Then $u \in (\mathcal{L}(\mathcal{A}))^\#$. Therefore $\mathcal{L}(\mathcal{A}^\#) \subseteq (\mathcal{L}(\mathcal{A}))^\#$

Proof of (A2). $(\mathcal{L}(\mathcal{A}))^\# \subseteq \mathcal{L}(\mathcal{A}^\#)$.

Assume $u \in (\mathcal{L}(\mathcal{A}))^\#$, such that $u : \alpha \rightarrow I$, for some $\alpha \in \text{ord}$. If $u = \emptyset$, then $\emptyset \in \mathcal{L}(\mathcal{A}^\#)$, since $Z \neq \emptyset$, then there exist $s \in Z$ with $(s, 1) \in Z' = F'$, so define $R : 1 \rightarrow S'$ such that $R(0) = R(\alpha) = (s, 1)$, and that is an accepting run of $\mathcal{A}^\#$ on u . Now assume $u \neq \emptyset$ and we need to define an accepting run $R : \alpha + 1 \rightarrow S'$ of $\mathcal{A}^\#$ on u . Now since $u \in (\mathcal{L}(\mathcal{A}))^\#$, and

$$(\mathcal{L}(\mathcal{A}))^\# = \bigcup_{\gamma \in \text{ord}} (\mathcal{L}(\mathcal{A}))^\gamma,$$

such that for each $\gamma \in \text{ord}$,

$$(\mathcal{L}(\mathcal{A}))^\gamma = \{u : u = \circ(u_\beta)_{\beta < \gamma}, u_\beta \in \mathcal{L}(\mathcal{A}) \text{ for each } \beta < \gamma\}.$$

Then there exists $\gamma \in \text{ord}$, such that $u \in (\mathcal{L}(\mathcal{A}))^\gamma$ and

$$u = \circ(u_i)_{i < \gamma}, u_i \in \mathcal{L}(\mathcal{A}) \text{ for each } i < \gamma.$$

Since $u \neq \emptyset$, then $\gamma > 0$, and we can assume that for all $i < \gamma$, $u_i \neq \emptyset$. Let $u_i : \theta_i \rightarrow I$, $\theta_i \in \text{ord}$, $\theta_i \neq 0$ for each $i < \gamma$ and $\alpha_i = \sum_{j < i} \theta_j$, for each $i < \gamma$.

$$u(\alpha_i + \delta) = u_i(\delta), \text{ for all } \delta < \theta_i, i < \gamma.$$

Then $\alpha = \sum_{i < \gamma} \theta_i$. Since for each $i < \gamma$, $u_i \in \mathcal{L}(\mathcal{A})$, then there is an accepting run $R_i : \theta_i + 1 \rightarrow S$ of \mathcal{A} on u_i . Now we want to prove $u \in \mathcal{L}(\mathcal{A}^\#)$. Define $R : \alpha + 1 \rightarrow S'$ as follows:

$$\begin{aligned} R(\alpha_i) &= (R_i(0), 1), \text{ for each } i < \gamma, \\ R(\alpha_i + \delta) &= (R_i(\delta), 0), \text{ for each } 0 < \delta < \theta_i, i < \gamma, \\ R(\alpha) &= (t, 1), \end{aligned}$$

such element $(t, 1) \in F'$, exists since $Z \neq \emptyset$. So it remains to prove R is an accepting run of $\mathcal{A}^\#$ on u . That is we must prove that, $R(0) \in Z'$, $R(\alpha) \in F'$ and satisfies the following conditions:

1. For each $\beta < \alpha$ we have

$$(R(\beta), u(\beta), R(\beta + 1)) \in T'.$$

2. For each $\beta \leq \alpha$ that is a limit ordinal with $\text{cf}(\beta) = \omega$, we have

$$(\sup_{\beta} R, R(\beta)) \in T'.$$

3. For each $\beta \leq \alpha$ that is a limit ordinal with $\text{cf}(\beta) > \omega$, we have

$$(\sup_{\beta} R, \text{stat}_{\beta} R, R(\beta)) \in T'.$$

First we want to prove $R(0) \in Z'$, and $R(\alpha) \in F'$. By the definition of R we get $R(0) = R(\alpha_0) = (R_0(0), 1)$ but $R_0(0) \in Z$ because R_0 is an accepting run of \mathcal{A} on u_0 so we get $R(0) \in Z'$ and $R(\alpha) \in F'$, it is clear by the definition of R . It remains to prove the three conditions.

Now we want to prove condition (1). Let $\beta < \alpha$ we want to prove

$$(R(\beta), u(\beta), R(\beta + 1)) \in T'.$$

Then we have the following cases:

(B1) $\beta = 0$.

(B2) $0 < \beta < \alpha$.

Case (B1). $\beta = 0$.

Assume $\beta = 0$. Since R_0 is an accepting run of \mathcal{A} on u_0 , then

$$(R_0(0), u_0(0), R_0(1)) \in T,$$

but by the definition of R , we get

$$(R(0), u(0), R(1)) = ((R_0(0), 1), u_0(0), (R_0(1), 0)),$$

hence $(R(0), u(0), R(1)) \in T_3$, by the definition of T_3 , then $(R(0), u(0), R(1)) \in T'$.

Case (B2). $0 < \beta < \alpha$.

Let i —smallest ordinal such that $\beta < \alpha_{i+1}$, then there is unique $\delta < \theta_i$ such that $\beta = \alpha_i + \delta$. Now, we want to prove

$$(R(\beta), u(\beta), R(\beta + 1)) \in T'.$$

Then we have two cases either $\delta = 0$, or $\delta > 0$.

First assume $\delta = 0$, then $\beta = \alpha_i$, so by the definition of R , we get

$$(R(\beta), u(\beta), R(\beta + 1)) = (R(\alpha_i), u(\alpha_i), R(\alpha_i + 1)) = ((R_i(0), 1), u_i(0), (R_i(1), 0))$$

and since for each $i < \gamma$, R_i is an accepting run of \mathcal{A} on u_i , then $(R_i(0), u_i(0), R_i(1)) \in T$, hence

$$(R(\beta), u(\beta), R(\beta + 1)) \in T_3,$$

by the definition of T_3 . Therefore $(R(\beta), u(\beta), R(\beta + 1)) \in T'$.

Second assume $\delta > 0$, and we want to prove $(R(\beta), u(\beta), R(\beta + 1)) \in T'$. Since for each $i < \gamma$, R_i is an accepting run of \mathcal{A} on u_i , and $\delta < \theta_i$, then $(R_i(\delta), u_i(\delta), R_i(\delta + 1)) \in T$, and by the definition of R we get

$$(R(\beta), u(\beta), R(\beta + 1)) = ((R_i(\delta), 0), u_i(\delta), (R_i(\delta + 1), 0)) \in T_1.$$

Thus $(R(\beta), u(\beta), R(\beta + 1)) \in T'$.

Therefore from (B1) and (B2), we get for each $\beta < \alpha$, $(R(\beta), u(\beta), R(\beta + 1)) \in T'$.

Next we want to prove condition (2). Assume $\beta \leq \alpha$, that is a limit ordinal with $\text{cf}(\beta) = \omega$ and we want to prove

$$(\sup_{\beta} R), R(\beta)) \in T'.$$

We have two cases:

(C1) $\beta < \alpha$.

(C2) $\beta = \alpha$.

Case (C1). $\beta < \alpha$.

First let $\beta < \alpha$ and i —smallest ordinal such that $\beta < \alpha_{i+1}$, then there is unique $\delta < \theta_i$ such that $\beta = \alpha_i + \delta$. Then we have two cases either $\delta > 0$, or $\delta = 0$.

Now let $\delta = 0$, then $\beta = \alpha_i = \sum_{j < i} \theta_j$, and also we have two cases either $i \in \text{succ}$, or $i \in \text{lim}$.

First assume $i \in \text{succ}$, so $i = \sigma + 1$. We want to prove $(\sup_{\beta} R), R(\beta)) \in T_2$. By the definition of R , we get

$$R(\beta) = R(\alpha_i) = (R_i(0), 1) \in F',$$

since for each $i < \gamma$, R_i is an accepting run of \mathcal{A} on u_i , and $\theta_\sigma \in \text{lim}$, with $\text{cf}(\theta_\sigma) = \omega$ because $\beta \in \text{lim}$, with $\text{cf}(\beta) = \omega$, by lemma 1.2.40, which give us $R_\sigma(\theta_\sigma) \in F$ and

$$(\sup_{\theta_\sigma}(R_\sigma), R_\sigma(\theta_\sigma)) \in T, \quad \text{and}$$

$$(\sup_\beta(R), R(\beta)) = (\sup_{\theta_\sigma}(R_\sigma) \times \{0\}, (R_i(0), 1)) \in T_2,$$

by the definition of T_2 , therefore $(\sup_\beta(R), R(\beta)) \in T'$.

Now assume $i \in \text{lim}$, and we want to prove $(\sup_\beta(R), R(\beta)) \in T_4$. Then

$$(\sup_\beta(R), R(\beta)) = (\sup_{\alpha_i}(R), R(\alpha_i)) = (\sup_{\alpha_i}(R), (R_i(0), 1)),$$

and $\sup_\beta(R) = A \subseteq S'$ and there is $s \in S$ with $(s, 1) \in A$, since Z' is finite and $R(\alpha_j) = (R_j(0), 1)$, for each $j < i$, $i \in \text{lim}$. Therefore $(\sup_\beta(R), R(\beta)) \in T_4$, so $(\sup_\beta(R), R(\beta)) \in T'$.

Assume $\delta > 0$, where $\beta = \alpha_i + \delta$, and we want to prove $(\sup_\beta(R), R(\beta)) \in T_1$. Then $\delta < \theta_i$, and $\delta \in \text{lim}$ with $\text{cf}(\delta) = \omega$ because $\beta \in \text{lim}$, with $\text{cf}(\beta) = \omega$ by lemma 1.2.40. Since for each $i < \gamma$, R_i is an accepting run of \mathcal{A} on u_i , then

$$(\sup_\delta(R_i), R_i(\delta)) \in T,$$

and since

$$(\sup_\beta(R), R(\beta)) = (\sup_\delta(R_i) \times \{0\}, (R_i(\delta), 0)) \in T_1,$$

then $(\sup_\beta(R), R(\beta)) \in T'$.

Case (C2). $\beta = \alpha$.

Now assume $\beta = \alpha = \sum_{i < \gamma} \theta_i$. We want to prove $(\sup_\beta(R), R(\beta)) \in T'$. Then we have two cases, either $\gamma = \sigma + 1$, or $\gamma \in \text{lim}$. First assume $\gamma = \sigma + 1$, and we want to prove $(\sup_\beta(R), R(\beta)) \in T_2$. By the definition of R , we get

$$R(\beta) = R(\alpha) = (t, 1) \in F' = Z',$$

and since for each $i < \gamma$, R_i is an accepting run of \mathcal{A} on u_i , and $\theta_\sigma \in \text{lim}$, with $\text{cf}(\theta_\sigma) = \omega$ because $\beta \in \text{lim}$, with $\text{cf}(\beta) = \omega$, by lemma 1.2.40, which give us $R_\sigma(\theta_\sigma) \in F$ and

$$(\sup_{\theta_\sigma}(R_\sigma), R_\sigma(\theta_\sigma)) \in T, \quad \text{and}$$

$$(\sup_\beta(R), R(\beta)) = (\sup_{\theta_\sigma}(R_\sigma) \times \{0\}, (t, 1)) \in T_2,$$

by the definition of T_2 , therefore $(\sup_\beta(R), R(\beta)) \in T'$.

Now assume $\gamma \in \text{lim}$, and we want to prove $(\sup_{\beta}(R), R(\beta)) \in T_4$. Then by the definition of R , we get

$$R(\beta) = R(\alpha) = (t, 1) \in F' = Z'.$$

It is clear that, $\sup_{\beta}(R) = A \subseteq S'$ and there is $s \in S$ with $(s, 1) \in A$, since Z' is finite and $R(\alpha_i) = (R_i(0), 1)$, for each $i < \gamma$, $\gamma \in \text{lim}$. Therefore $(\sup_{\beta}(R), R(\beta)) \in T_4$, so $(\sup_{\beta}(R), R(\beta)) \in T'$.

Then from (C1), and (C2) we get for each $\beta \leq \alpha$ that is a limit ordinal with $\text{cf}(\beta) = \omega$, $(\sup_{\beta}(R), R(\beta)) \in T'$.

Finally we want to prove condition (3). Assume $\beta \leq \alpha$, that is a limit ordinal with $\text{cf}(\beta) > \omega$, and we want to prove

$$(\sup_{\beta}(R), \text{stat}_{\beta}(R), R(\beta)) \in T'.$$

We have two cases:

(D1) $\beta < \alpha$.

(D2) $\beta = \alpha$.

Case (D1). $\beta < \alpha$.

Let i –smallest ordinal such that $\beta < \alpha_{i+1}$, then there is unique $\delta < \theta_i$ such that $\beta = \alpha_i + \delta$. Then we have two cases either $\delta > 0$, or $\delta = 0$.

Now let $\delta = 0$, then $\beta = \alpha_i = \sum_{j < i} \theta_j$, and also we have two cases either $i \in \text{succ}$, or $i \in \text{lim}$.

First assume $i \in \text{succ}$, so $i = \sigma + 1$. We want to prove $(\sup_{\beta}(R), \text{stat}_{\beta}(R), R(\beta)) \in T_2$. By the definition of R , we get

$$R(\beta) = R(\alpha_i) = (R_i(0), 1) \in F' = Z',$$

and since for each $i < \gamma$, R_i is an accepting run of \mathcal{A} on u_i , and $\theta_{\sigma} \in \text{lim}$, with $\text{cf}(\theta_{\sigma}) > \omega$ because $\beta \in \text{lim}$, with $\text{cf}(\beta) > \omega$, by lemma 1.2.40, which give us $R_{\sigma}(\theta_{\sigma}) \in F$ and

$$(\sup_{\theta_{\sigma}}(R_{\sigma}), \text{stat}_{\theta_{\sigma}}(R_{\sigma}), R_{\sigma}(\theta_{\sigma})) \in T, \text{ and}$$

$$(\sup_{\beta}(R), \text{stat}_{\beta}(R), R(\beta)) = (\sup_{\sigma}(R_{\sigma}) \times \{0\}, \text{stat}_{\theta_{\sigma}}(R_{\sigma}) \times \{0\}, (R_i(0), 1)) \in T_2,$$

by the definition of T_2 , therefore $(\sup_{\beta}(R), \text{stat}_{\beta}(R), R(\beta)) \in T'$.

Now assume $i \in \text{lim}$, and we want to prove $(\sup_{\beta}(R), \text{stat}_{\beta}(R), R(\beta)) \in T_4$. Then

$$(\sup_{\beta}(R), \text{stat}_{\beta}(R), R(\beta)) = (\sup_{\alpha_i}(R), \text{stat}_{\alpha_i}(R), R(\alpha_i)) = (\sup_{\alpha_i}(R), \text{stat}_{\alpha_i}(R), (R_i(0), 1)),$$

and $\sup_{\beta}(R) = A \subseteq S'$ and there is $s \in S$ with $(s, 1) \in A$, since Z' is a finite set and $R(\alpha_j) = (R_j(0), 1)$, for each $j < i$, $i \in \text{lim}$. Therefore $(\sup_{\beta}(R), \text{stat}_{\beta}(R), R(\beta)) \in T_4$, so $(\sup_{\beta}(R), \text{stat}_{\beta}(R), R(\beta)) \in T'$.

Assume $\delta > 0$, and we want to prove $(\sup_\beta(R), \text{stat}_\beta(R), R(\beta)) \in T_1$. Then $\delta < \theta_i$, and $\delta \in \text{lim}$ with $\text{cf}(\delta) > \omega$ because $\beta \in \text{lim}$, with $\text{cf}(\beta) > \omega$ by lemma 1.2.40. Since for each $i < \gamma$, R_i is an accepting run of \mathcal{A} on u_i , then

$$(\sup_\delta(R_i), \text{stat}_\delta(R_i), R_i(\delta)) \in T,$$

and since

$$(\sup_\beta(R), \text{stat}_\beta(R), R(\beta)) = (\sup_\delta(R_i) \times \{0\}, \text{stat}_\delta(R_i) \times \{0\}, (R_i(\delta), 0)),$$

then $(\sup_\beta(R), \text{stat}_\beta(R), R(\beta)) \in T_1$. Therefore $(\sup_\beta(R), \text{stat}_\beta(R), R(\beta)) \in T'$.

Case (D2). $\beta = \alpha$.

Now assume $\beta = \alpha = \sum_{i < \gamma} \theta_i$. We want to prove $(\sup_\beta(R), \text{stat}_\beta(R), R(\beta)) \in T'$. Then we have two cases, either $\gamma = \sigma + 1$, or $\gamma \in \text{lim}$.

First assume $\gamma = \sigma + 1$, and we want to prove $(\sup_\beta(R), \text{stat}_\beta(R), R(\beta)) \in T_2$. Then by the definition of R , we get

$$R(\beta) = R(\alpha) = (t, 1) \in F' = Z',$$

and since for each $i < \gamma$, R_i is an accepting run of \mathcal{A} on u_i , and $\theta_\sigma \in \text{lim}$, with $\text{cf}(\theta_\sigma) > \omega$ because $\beta \in \text{lim}$, with $\text{cf}(\beta) > \omega$, by lemma 1.2.40, which give us $R_\sigma(\theta_\sigma) \in F$ and

$$(\sup_{\theta_\sigma}(R_\sigma), \text{stat}_{\theta_\sigma}(R_\sigma), R_\sigma(\theta_\sigma)) \in T, \text{ and}$$

$$(\sup_\beta(R), \text{stat}_\beta(R), R(\beta)) = (\sup_{\theta_\sigma}(R_\sigma) \times \{0\}, \text{stat}_{\theta_\sigma}(R_\sigma) \times \{0\}, (t, 1)) \in T_2,$$

by the definition of T_2 , therefore $(\sup_\beta(R), R(\beta)) \in T'$.

Now assume $\gamma \in \text{lim}$, and we want to prove $(\sup_\beta(R), \text{stat}_\beta(R), R(\beta)) \in T_4$. Then by the definition of R , we get

$$R(\beta) = R(\alpha) = (t, 1) \in F' = Z'.$$

It is clear that, $\sup_\beta(R) = A \subseteq S'$, $\text{stat}_\beta(R) = B \subseteq S'$, and there is $s \in S$ with $(s, 1) \in A$, since Z' is a finite and $R(\alpha_i) = (R_i(0), 1)$, for each $i < \gamma$, $\gamma \in \text{lim}$.

Therefore $(\sup_\beta(R), \text{stat}_\beta(R), R(\beta)) \in T_4$, so $(\sup_\beta(R), \text{stat}_\beta(R), R(\beta)) \in T'$. Hence from (D1), and (D2) we get, for each $\beta \leq \alpha$ that is a limit ordinal with $\text{cf}(\beta) > \omega$, $(\sup_\beta(R), \text{stat}_\beta(R), R(\beta)) \in T'$.

Therefore R is an accepting run of $\mathcal{A}^\#$ on u , hence $u \in \mathcal{L}(\mathcal{A}^\#)$. Therefore $(\mathcal{L}(\mathcal{A}))^\# \subseteq \mathcal{L}(\mathcal{A}^\#)$. Hence $(\mathcal{L}(\mathcal{A}))^\# = \mathcal{L}(\mathcal{A}^\#)$. \square

Chapter 5

Future Work

The first objective is to try to prove main theorem 2.2.3 with a bound on $|W \cup M|$ that is larger than \aleph_1 or without any bound, that is, prove that there exists an ST-automaton \mathcal{A} over $\{M, W\}$ such that every bipartite graph $G = (M, W, E)$ has a matching if and only if $L(G) \cap L(\mathcal{A}) = \emptyset$.

Another direction for future work could be defining analogs of Büchi automata, Muller automata, deterministic automata and finding relationships with the original definitions (see [5]).

Bibliography

- [1] R. Aharoni, *On the equivalence of two conditions for the existence of transversals*, J. Combin. Theory Ser. A 34 (1983) 202-214.
- [2] R. Aharoni, C. St. J. A. Nash-Williams, S. Shelah, *A general criterion for the existence of transversals*, Proc. London Math. Soc. 47 (1983), 43–68.
- [3] M. Holz, K.-P. Podewski, and K. Steffens, *Injective choice functions*, Lecture Notes in Mathematics, 1238, Springer-Verlag, Berlin-New York, 1987.
- [4] T. Jech, *Set Theory, 3rd millennium (revised) ed.*, 2003, Springer Monographs in Mathematics, Springer.
- [5] D. Perrin, J.-E'. Pin, *Infinite Words: Automata, Semigroups, Logic and Games*, Pure and Applied Mathematics vol. 141 Elsevier, 2004.
- [6] J. Wojciechowski, *A criterion for the existence of transversals of set systems*, J. London Math. Soc. (2) 56 (1997) 491–503.
- [7] J. Wojciechowski, *Classes of transfinite sequences accepted by finite automata*, Fundamenta Informaticae VII 2 (1984), 191-223.
- [8] J. Wojciechowski, *Finite automata on transfinite sequences and regular expressions*, Fundamenta Informaticae VIII 3--4 (1985), 379-396.