ESSAY ON UNIFIED EXPOSITION OF FOUR MAJOR THEOREMS IN TOPOLOGY

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Abstract. This essay is an expository notes on four major theorems in Topology. The only prerequisites to comprehend this notes are basic knowledge in real analysis, elementary point set topology, and mathematical maturity.
Key words: Topology, Stone-Weierstrass Theorem

This essay set in mind at first as a personal note on expository of the Stone-Weierstrass Theorem, due to its importance in approximation theory. During the process of studying the materials, an article by J. Dydak & N. Feldman [1] came accross, where they shows the Stone-Weierstrass theorem become one of consequences in a unified fashion from topological perspectives. Hence, the aims become slightly bends, but still catered the original aim.

We organized the essay into two major sections. The first section is on preliminary material, this section consists of two parts. The first part will be stating some result without proof about real-valued functions algebras that plays as a basic foundation to build the exposition; the second part will be stating some results with which the main theorems will be stands on. At the end of the first section, the main theorem and its proof will be stated. The second section is only stating and proving the four major theorems in Topology, including the Stone-Weierstrass Theorem.

1. Preliminaries

1.1. **The function Algebra $\mathcal{C}(\mathcal{X}, \mathrm{R})$.** Let $\mathcal{X}$ be an arbitrary topological space. Let $\mathcal{C}(\mathcal{X}, \mathrm{R})$ be the set of all real-valued bounded continuous function on $\mathcal{X}$. This set equiped with point-wise addition operation and real scalar multiplication turned to be a real vector space.

An *algebra* $\mathcal{A}$ is a vector space equiped with vector-multiplication operation in such a way that:

1. $x \bullet (y \bullet z) = x(\bullet y) \bullet z$, for every $x, y, z \in \mathcal{A}$
2. $x \bullet (y + z) = x \bullet y + x \bullet z$ for every $x, y, z \in A$ and $(y + z) \bullet x = y \bullet x + z \bullet x$ for every $x, y, z \in A$.
3. $\alpha(x \bullet y) = (\alpha x) \bullet y = x \bullet (\alpha y)$, for every scalar α.

Moreover, the algebra $\mathcal{A}$ with identity if:

(4) There exist non-zero element in $\mathcal{A}$, denoted by e and called the identity element, such that :

$$e \bullet x = x \bullet e = x, \forall x \in \mathcal{A}$$

A *subalgebra* $\mathcal{A}_1$ of algebra $\mathcal{A}$ if $\mathcal{A}_1$ is subspace of $\mathcal{A}$ such that $\mathcal{A}_1$ is an algebra under operation inherited from $\mathcal{A}$.

In the case of $\mathcal{C}(\mathcal{X}, \mathrm{R})$, it is became an algebra if vector multiplication is defined point-wise. This algebra admits an identity element $e = \underline{1}$, which maps $\underline{1}(x) = 1$ for every $x \in \mathcal{X}$. Moreover this algebra is commutative.

Theorem 1.1. 1. *$\mathcal{C}(\mathcal{X}, \mathrm{R})$ is a Banach space with respect to point-wise addition and scalar multiplication and the norm defined by $\|f\| = \sup |f(x)|$.*
2. *If multiplication is defined point-wise, $\mathcal{C}(\mathcal{X}, \mathrm{R})$ is a commutative algebra with identity, in which:*

$$\|f\ g\| <= \|f\|\ \|g\| \qquad \textit{and} \quad \|\underline{1}\| = 1$$

3. *If $f <= g$ is definet to mean that $f(x) <= g(x)$ for all x, $\mathcal{C}(\mathcal{X}, \mathrm{R})$ is a lattice in which the greatest lower bound of a pair of functions f and g are given by $(f \vee g)(x) = \min\{f(x), g(x)\}$ and $(f \wedge g) = \max\{f(x), g(x)\}$.*

Let Y be a compact Haussdorf space. Let $f : X \to Y$ be a function such that $\overline{f(X)} = Y$. Then f induces a function $\hat{f}$ from the set $\mathcal{C}(Y)$ of all real-valued contnuous function to the set $\mathcal{C}(X, \mathrm{R})$ via the formula $\hat{f}(g) = g \circ f$. Let P_f denotes the image of $\hat{f}$.

Theorem 1.2. *P_f is closed subalgebra of $\mathcal{C}(\mathcal{X}, \mathrm{R})$ and $\hat{f} : \mathcal{C}(\mathcal{Y}, \mathrm{R}) \to P_f$ is an isometry of algebras.*

Proof :
Clearly that, $\hat{f} : \mathcal{C}(\mathcal{Y}, R) \to P_f$ is surjective.
It is an isometry since $f(\mathcal{X})$ is dense in $\mathcal{Y}$, and

$$|g \circ f\,(x) - g' \circ f\,(x)| \leq a, \forall\, x \in \mathcal{X} \quad \text{implies: } |g\,(y) - g'(y)| \leq a,$$

$\forall\ y \in \dagger = \overline{f(\mathcal{X})}$. Since P_f is isometric to a complete space $\mathcal{C}(\mathcal{Y}, \mathrm{R})$, then P_f is complete, so it is closed in $\mathcal{C}(\mathcal{X}, \mathrm{R})$.

1.2. **Steps toward The Grand Unification Theorem.** To start with, defines the following.

Definition 1.3. Given subalgebra P of $\mathcal{C}(\mathcal{X}, \mathrm{R})$. Let $\mu(P)$ be the set of all subalgebras τ of P which are maximal with respect to the following:

$$\text{given} \quad \alpha \in \tau \quad \text{the set} \quad \alpha^{-1}(-\epsilon, \epsilon) \quad \text{is not empty for all} \quad \epsilon > 0. \tag{1.1}$$

Theorem 1.4. *If $\mathcal{X}$ is compact, then $\mu(P) = \{\tau_x; x \in \mathcal{X}\}$*

Proof :
For any finite k, let $f_1, f_2, \cdots, f_k$ be elements of $\tau \in \mu(P)$. The quadratic sum: $\sum_{i=1}^{k} f_i^2$ belongs to τ and attains its acsolute minimum at $x \in \mathcal{X}$. Since the values $m > 0$ would contradict condition (1.1), then $m = 0$. Hence $f_1, f_2, \cdots, f_k$ posses a mutual roots and all the functions member of τ have a mutual root, since $\mathcal{X}$ is compact.

Definition 1.5. Given $\alpha \in P$. Let $N(\alpha) = \{\tau \in \mu(P) : \alpha \notin \tau\}$

Theorem 1.6. $N(f) \cap N(g) = N(f \cdot g)$

Proof :
The theorem statement is equivalently stated as:
$\mu(P) - N(f \cdot g) = (\mu(P) - N(f)) \cup (\mu(P) - N(g))$, which is equivalently stated: if $f \cdot g \in \tau$ then $f \in \tau$ or $g \in \tau$.
Clearly, this statement easily recognized that τ is prime ideals of P.

If $f \in \tau$ and $g \in P$, choose $M > 1$ such that :

$$|g(x)| < M, \forall x \in \mathcal{X}$$

Given $\epsilon > 0, a \in \mathrm{R}$ and $h \in \tau$, since f and h are members of τ, there exists τ_0 such that $f^2(x_0) + h^2(x_0) < \min(\epsilon^2/4 \cdot M^2 a^2, \epsilon^2/4)$.
Then $|(a\, f \cdot g + h)(x_0)| < \epsilon$ which means that the subalgebra $\{af \cdot g + h; g \in P, h \in \tau, a \in \mathrm{R}\}$ satisfies (1.1). Hence: $\{af \cdot g + h\} \subset \tau$ and $f \cdot g \in \tau$.
Assume that $f \cdot g \in \tau$ but $f \notin \tau$ and $g \notin \tau$. Then, the subalgebra $\{af \cdot g + h; a \in \mathrm{R}, g \in P, h \in \tau\}$ does not satisfy (1.1) and $\inf_{x \in \mathcal{X}}\{|af(x) + h(x)|\} > 0$ for some $h \in \tau$ and $a \in \mathrm{R}$. Similarly, $\inf_{x \in \mathcal{X}}\{|af(x) + h'(x)]vert\}$ for some $h' \in \tau$ and $x \in \mathrm{R}$. Therefore, $\inf_{x \in \mathcal{X}}\{|(af(x)+h(x))(bg(x)+h'(x))|\} > 0$ contradicting $(af+h)(bg+h') = (ab)f \cdot g + af \cdot h' + bg \cdot h + h \cdot h' \in \tau$, since τ is an ideal.

The above theorem implies that the family $\{N(f)\}_{f \in P}$ can be used to generate a topology on $\mu(P)$.

Theorem 1.7.
1. *$\mu(P)$ with the topology $\{N(\alpha); \alpha \in P\}$ is compact.*
2. *$\iota_P : \mathcal{X} \to \mu(P)$ defined by $\iota_P(x) = \tau_x$ is continuous and ι_P is dense in $\mu(P)$.*

3. *If $\underline{1} \in P$ and P separates point, then $\imath_P$ is one-to-one. Moreover, if $(\alpha^{-1}(\mathrm{R} - \{0\}))_{\alpha\in P}$ is a base of $\mathcal{X}$, then $\imath_P : \mathcal{X} \longrightarrow \imath_P(x)$ is homeomorphism.*

Proof :

1. Assume $\mu(P) = \cup_{s\in S} N(\alpha_s)$ and $\mu(P) - \cup_{s\in A} N(\alpha_s) \neq 0$ for each finite subset A of S. Then $\{\alpha_s; s \in A\}$ is a subset for some $\tau_A \in \mu(P)$ for all finite subsets A of S.

   Let τ be the sub-algebra of P generated by $\{\alpha_s; s \in S\}$. Since each element of τ is contained in τ_A, τ satisfies (1.1). Thus $\tau \subset \tau' \in \mu(P)$ and $\tau' \in \mu(P) - \cup_{s\in S} N(\alpha_s)$, a contradiction.
2. Since $\imath_P^{-1}(N(\alpha)) = \alpha^{-1}(R - \{0\})$, so $\imath_P$ is continuous. Suppose $\alpha \in P$ and $N(\alpha) \cap \imath(\mathcal{X}) = \emptyset$. Then $\alpha \in \tau_x$ for all $x \in \mathcal{X}$, which means $\alpha \equiv 0$. In such a case $\alpha \in \tau$ for all $\tau \in \mu(P)$ and $N(\alpha) = \emptyset$. Thus $\imath_P(x)$ is dense in $\mu(P)$.
3. If $\underline{1} \in P$, then all the constant functions belong to $\mu(P)$, since P is a subspace of $\mathcal{C}(\mathcal{X}, \mathrm{R})$. Now, P separates point, then $\imath_P$ is injective. So $\imath_P^{-1}(N(\alpha)) = \alpha^{-1}(\mathrm{R} - \{0\})$ is equivalent to $N(\alpha) \cap \imath_P(\mathcal{X}) = \imath_P(\alpha^{-1}(R - \{0\}))$. Thus, if $\{\alpha^{-1}(R - \{0\})\}_{\alpha\in P}$ is a base of $\mathcal{X}$, then $\imath_P : \mathcal{X} \to \imath_P(\mathcal{X})$ is an open map.

Theorem 1.8. *Let $p : [0, 1] \to \mathrm{R}, p(x) \mapsto x + |x|$. Then there is a sequence of polynomials belong to $\mathcal{C}([0, 1], \mathrm{R})$ that converges uniformly to p.*

Proof :
Define the sequence $(p_n)_{n\in\mathrm{N}}$ of polynomials as follows: Let $p_1(x) = x^2$. Define recursively for each $n \in \mathrm{N}$:

$$p_{n+1}(x) = x - (x - p_n(x))(1 - p_n(x)/2), x \in [-1, 1]$$

The sequence $(p_n)_{n\in\mathrm{N}}$ is increasing for $x > 0$ and decreasing for $x < 0$. Given $0 < \varepsilon < 1$, then $|p(x) - p_n(x)| < \varepsilon$ for $|x| < \varepsilon$. If $x > \varepsilon$, then $(x - p_{n+1}) = (x - p_n(x))(1 - p_n(x)/2) \leq (x - p_n(x))(1 - \varepsilon^2/2)$, since $p_1(x) = x^2 \leq p_n(x)$. Also, for $x < -\varepsilon$ we have $p_{n+1}(x) = p_n(x)(1 + (x - p_n(x))/2) \leq p_n(x)(1 - \varepsilon/2)$ as $x - p_n(x) \leq x \leq -\varepsilon$. Thus, for n sufficiently large, $|p(x) - p_n(x)| < \varepsilon$, for all $x \in [-1, 1]$.

The previous theorem purpose is as an auxilliary steps to prove one part of the following 'small' theorem. The following theorem actually is a special case of Theorem 1.7.

Theorem 1.9.
1. *$i_p : \mathcal{X} \longrightarrow \mu(P)$ is a homeomorphism for any compact Haussdorf space $\mathcal{X}$ and $P = \mathcal{C}(\mathcal{X}, \mathrm{R})$.*
2. *$i_p : [a, b] \longrightarrow \mu(P)$ is a homeomorphism for any closed subalgebra P of $\mathcal{C}([a, b], \mathrm{R})$ containing all the polynomials.*

Proof :

1. By Theorem (1.4), i_P is onto and $i_P : \mathcal{X} \longrightarrow i_P = \mu(P)$ is homeomorphism by Theorem (1.7).
2. Since [a,b] and [-1,1] is homeomorphism by linear function, we may assume $a = -1$ and $b = 1$. Hence it is suffices to show, by Theorem (1.7), that for any (c, d) there is $\alpha \in P$ with $\alpha^{-1}(\mathrm{R} - \{0\}) = (c, d) \cap [-1, 1]$. Given c and d, the map $\alpha(x) = p(x - c) \cdot p(x - d)$, where p is function constructed in the previous theorem, satisfies $\alpha^{-1}(\mathrm{R} - \{0\}) = (c, d) \cap [-1, 1]$.

Now we turn to examining functorial property of our construction. Let $f : \mathcal{X} \longrightarrow \mathcal{Y}$ be a given map. Given P a subalgebra of $\mathcal{C}(\mathcal{X}, \mathrm{R})$ and Q a subalgebra of $\mathcal{C}(\mathcal{X}, \mathrm{R})$ such that for any $g \in Q$ the composition $g \circ f$ belongs to P, we would like to construct a map $\hat{f} : \mu(P) \longrightarrow \mu(Q)$ such that $f \circ i_p = i_Q \circ f$. The most natural choice for $\hat{f}(\tau)$ is $\tau' = \{g \in Q; g \circ f \in \tau\}$. To show that τ' is maximal is a bit difficult and only be able to show it for the case $\mathcal{Y}$ is compact.

Theorem 1.10. *Let $f : \mathcal{X} \longrightarrow \mathcal{Y}$ be a map, P is a subalgebra of $\mathcal{C}(\mathcal{X}, \mathrm{R})$ and Q a subalgebra of $\mathcal{C}(\mathcal{Y}, \mathrm{R})$ such that for any $g \in Q$ the composition $g \circ f$ belongs to P.*

1. *If for every $\tau \in \mu(P)$ the set $\tau' := \{g \in Q; g \circ f \in \tau\}$ belongs to $\mu(Q)$, the map $\hat{f} : \mu(P) \longrightarrow \mu(Q)$ defined by $\hat{f}(\tau) = \tau'$ is continuous and the diagram:*

$$\begin{array}{ccc} \mathcal{X} & \overrightarrow{f} & \mathcal{Y} \\ \downarrow i_P & & \downarrow i_Q \\ \mu(P) & \overrightarrow{\hat{f}} & \mu(Q) \end{array}$$

   is commutative.
2. *If $\mathcal{Y}$ is a compact Hausdorff space and $Q = \mathcal{C}(\mathcal{Y}, \mathrm{R})$ or $\mathcal{Y} = [a, b]$ and Q is the closure of polynomials on $[a, b]$, then for every $\tau \in \mu(P)$ the set $\tau' = \{g \in Q; g \circ f \in \tau\}$ equals $\tau_y =\in \mu(Q)$ for some $y \in \mathcal{Y}$.*

Proof :

1. In this case $\hat{f}^{-1}(N(\alpha)) = N(\alpha \circ f)$ for any $d \in Q$, so $\hat{f}$ is continuous.
   Also, if $x \in X$,
$$\begin{aligned} (\hat{f} \circ i_P)(x) &= \hat{f}(\tau_x) \\ &= \tau_{f(x)} \\ &= (i_Q \circ f)(x) \end{aligned}$$
2. Given τ we will show that there is a unique $y \in Y$ so that $\tau' \in \tau_y$. Suppose that for each $y \in Y$ there is a unique $\alpha_y \in Q$ such that $\alpha_y(y) \neq 0$ and $\alpha_y = \tau'$. Choose a neighbourhood $\mathcal{U}_y$ of y in $\alpha^{-1}(\mathrm{R} - \{0\})$. By compactness of $\mathcal{Y}$, there exist finitely many finctions $\alpha_1, \cdots, \alpha_m \in \tau'$ (by selecting finite subcovering of $(\mathcal{U}_y)_{y \in \mathcal{Y}}$),

with $\sum_{i=1}^{m} \alpha_i^2 > \varepsilon > 0$, which contradicts the facts that τ satisfies condition (1.1). Suppose $y \neq z$ and $\tau' \subset \tau_y, \tau' \subset \tau_z$. Select g, g' member of Q such that $g \cdot g' = 0$ and $g(y) \neq 0, g'(z) \neq 0$. Hence $g \notin i_y$ and $g' \notin \tau_z$. Since τ is prime ideal, then $g \circ f \in \tau$ or $g' \circ f \in \tau$; and $g \in \tau'$ or $g' \in \tau'$, a contradiction.
Now it left to shows that $\tau_y \subset \tau'$. Suppose $\mathcal{U}$ is open neighbourhood $\mathcal{V}$ of y and $g|_{\mathcal{U}} \equiv 0, g \in \mu(Q)$. Select a neighbourhood of $\mathcal{V}$ of y in $\mathcal{U}$ with $\bar{\mathcal{V}} \subset \mathcal{U}$. Let $h \in Q$ with $h(y) = 1$ and $h_| \mathcal{Y} - V \equiv 0$. Since $h \cdot g = 0$, and by τ is prime ideal, $h \circ f \in \tau$ or $g \circ f \in \tau$. In view of $h(y) = 1$ and $\tau' \subset \tau_y$, we have $g \circ f$ dna $g \in \tau'$. Lastly, since for any $g \in Q, g(y) = 0$ is a limit of $g_n \in Q, n \geq 1$, with each g_n vanishing on some neighbourhood of $\mathcal{Y}$. Thus, $g_n \in \tau'$. If $g \circ f \notin \tau$, there is $h \in \tau$ with $\inf_{x \in \mathcal{X}}\{|g \circ f(x) + h(x)|\} > \varepsilon$. Select g_n such that $|g - g_n| \leq \varepsilon 2$. Then $\inf_{x \in \mathcal{X}}\{|g_n \circ f(x) + h(x)|\} \geq \varepsilon 2$, a contradiction.

Now we will establish the fact that P_f contains P for $f = i_P : \mathcal{X} \longrightarrow \mu(P)$ provided P is closed and contains all the constant functions.

Corollary 1. Suppose P is a closed subalgebra of $\mathcal{C}(\mathcal{X}, \mathrm{R})$ containing identity element.

1. For any $\alpha \in P, \alpha : \mathcal{X} \longrightarrow [a, b]$ there is $\alpha' : \mu(P) \longrightarrow [a, b]$ with $\alpha' \circ i_P = \alpha$ and $N(\alpha) = (\alpha')^{-1}(R - \{0\})$. In particular $\mu(P)$ is Haussdorf.
2. If $\alpha : \mathcal{X} \longrightarrow \mathcal{Y}$ is a map from $\mathcal{X}$ to a compact Haussdorf space such that P_α is contained in P, then there is a unique map $\alpha' : \mu(P) \longrightarrow \mathcal{Y}$ with $\alpha = \alpha' \circ i_P$.

Proof :
Write $Q = \mathcal{C}(\mathcal{Y}, \mathcal{R})$ for the case (b) and put Q be the closure of all polynomials for the case (a). By theorem (1.9), i_Q is a homeomorphism.

1. Since P is an algebra containing constants function, $g \circ \alpha \in P$ for any polymials g. Hence $g \circ \alpha \in P$ for any $g \in Q$, since P is closed. By Theorem (1.10), there is a map $\hat{\alpha} : \mu(P) \longrightarrow \mu(Q)$ with $i_Q \circ \alpha = \hat{\alpha} \circ i_P$.
2. By theorem (1.10) construct a map $\hat{\alpha} : \mu(P) \longrightarrow \mu(Q)$ with $i_Q \circ \alpha = \hat{\alpha} \circ i_P$.
   Write $\alpha' = (i_Q)^{-1} \circ \alpha$. We will show that $\mu(P)$ is Haussdorff. Assume $\tau_1 \neq \tau_2 \in \mu(P)$ and choose $\alpha \in \tau_1 - \tau_2$. Then $\tau_1' = \{g \in Q; g \circ \alpha \in \tau_1\}$ does not contain id_X. Thus $\alpha'(\tau_1) \neq \alpha'(\tau_2)$, so $\mu(P)$ is Haussdorff.

Next we claim that P contains P_f.

Theorem 1.11. *Suppose P is a closed subalgebra of $\mathcal{C}(\mathcal{X}, \mathrm{R})$ containing $\underline{1}$. Then for any $g : \mu(P) \longrightarrow \mathrm{R}$ the map $g \circ i_P$ belongs to P.*

Proof :
Let $h \in P$ be a map. By corollary (1), there exist a unique map

$\hat{h} : \mu(P) \longrightarrow$ R satisfying $\hat{h} \circ i_P = h$. We will show that $\{\hat{h}; h \in P\} = \mathcal{C}(\mu(P), \mathrm{R})$.
Suppose $\varepsilon > 0$ and $g : \mu(P) \longrightarrow$ R is continuous. Select for each $y \in \mu(P)$ a neghbourhood $\mathcal{U}_y = N(\alpha_y)$ of y such that $|g(z) - g(z')| < \varepsilon$ for $y \in \mu(P)$. Select finitely many points $y_1, \cdots, y_k$ with $\bigcup_{i=1}^{k} = \mathcal{Y}$, where $\mathcal{U}_i = \mathcal{U}_{y_1}$ and $\alpha_i = \alpha_{y_i}$, for $i \leq k$. Observe that $|g(y_i) - g \circ i_P(x)| < \varepsilon$ for $i_P(x)\mathcal{U}_i$ and $\alpha_i(x) = 0$ otherwise. Hence:

$$|\sum(g(y_i) - g \circ i_P(x) \cdot \alpha_i(x))| < \varepsilon \cdot \sum |\alpha_i(x)|, \qquad \forall x \in \mathcal{X}.$$

If we can select the functions $\{\alpha_i\}_i$ in such a way that $\sum |\alpha_i| = 1$ and $\alpha_i > 0$, then the task will be done. Clearly, $g' = \sum g(y_i) \cdot \alpha_i$ belongs to P and :

$$|g'(x) - g \circ i_P(x)| = |\sum(g(y_i) - g \circ i_P(x)) \cdot \alpha(x)|\varepsilon \sum |\alpha_i(x)| = \varepsilon.$$

By Theorem (1.6), we may replace each α_i by α_i^2, in view $N(\alpha_i^2) = N(\alpha_i) \cap N(\alpha_i) = N(\alpha_i)$. Since $\mu(P)$ is compact, $\alpha = (\widehat{\sum \alpha_i})$ is bounded. For any $y \in \mu(P)$, note that $\alpha(y) \neq 0$. Clearly, $\{h; \hat{h}(y) = 0\} \subset N(\alpha_i)$ for some i means $\hat{\alpha}_i(y) \neq= 0$. Hence $\alpha : \mu(P) \longrightarrow [a, b]$, where $a > 0$. Note that the map $r(x) = 1x, x \in [a, b]$, is the limit of polynomial:

$$1x = 1b \cdot \frac{1}{1 - \dfrac{b - x}{b}} = 1b \cdot \sum_{k=1}^{\infty} (b - xb)^n .$$

Consequently $\beta = 1/(\sum \alpha_i) \in P$ and $N(\beta) = \mu(P)$. (i.e. $\mu(P) = N(\underline{1}) = N(\beta) \cap N(\sum \alpha_i)$, so $N(\beta) = \mu(P)$.)
Now, each α_i can be replaced by $\beta \cdot \alpha_i$ in view of:

$$N(\beta \cdot \alpha_i) = N(\alpha_i) \cap N(\beta) = N(\alpha_i).$$

At last, we arrive to stating and proving *the grand unification theorem* of the following.

Theorem 1.12 (The Grand Unification Theorem)**.** *Suppose P is a closed subalgebra of $\mathcal{C}(\mathcal{X}, \mathrm{R})$ containing identity $\underline{1}$.*
Then there exist a compact Haussdorf space $\mu(P)$ and a map $i_P : \mathcal{X} \longrightarrow \mu(P)$ such that the function $\hat{i}_P : \mathcal{C}(\mu(P), \mathrm{R}) \longrightarrow \mathcal{C}(\mathcal{X}, \mathrm{R})$ given by $\hat{i}_P(g) = g \circ i_P$ is injective and its image is P.
The space $\mu(P)$ is unique in the following sense : for each map $f : \mathcal{X} \longrightarrow \mathcal{Y}$ with $f(\bar{\mathcal{X}}) = \mathcal{Y}$ being compact Haussdorf space and $P_f = P$, there is a homeomorphism $h : \mathcal{Y} \longrightarrow \mu(P)$ with $h \circ f = i_P$.
Moreover, if $f : \mathcal{X} \longrightarrow \mathcal{Y}$ is a map and Q is a closed subalgebra of $\mathcal{C}(\mathcal{Y}, \mathrm{R})$ such that for any $g \in Q$ the composition $g \circ f$ belongs to P and $\underline{1} \in Q$, then there is a unique map $\hat{f} : \mu(P) \longrightarrow \mu(Q)$ making the diagram :

$$\begin{array}{ccc} \mathcal{X} & \overrightarrow{f} & \mathcal{Y} \\ \downarrow\ i_P & & \downarrow\ i_Q \\ \mu(P) & \overrightarrow{\hat{f}} & \mu(Q) \end{array}$$

commutative.

Proof :
By theorem (1) and theorem (1.11), $P_{i_P} = P$ is established. If $f : \mathcal{X} \longrightarrow \mathcal{Y}$ is a map such that $f(\mathcal{X}) = \mathcal{Y}$ being compact Haussdorff space and $P_f = P$, then the map $f' : \mu(P) \longrightarrow \mathcal{Y}$ with $f = f' \circ i_P$ must be homeomorphism by corollary 1 part b. Clearly $(\hat{f'}) : \mathcal{C}(\mathcal{Y}, \mathrm{R}) \longrightarrow \mathcal{C}(\mu(P), \mathrm{R})$ is an isomorphism of algebras, so f' must be surjective (otherwise $g \circ f' = 0$ for a nontrivial $g : \mathcal{Y} \longrightarrow [0, 1]$), and it must be injective (otherwise a pair of points in $\mu(P)$ could not be separated by a real-valued function).
Let $f : \mathcal{X} \longrightarrow \mathcal{Y}$ be a map and Q is closed subalgebra of $\mathcal{C}(\mathcal{Y}\mathrm{R})$ such that for any $g \in Q$ the composition $g \circ f$ belongs to P and $\underline{1} \in Q$. Consider $\alpha = i_Q \circ f : \mathcal{X} \longrightarrow \mu(Q)$. If $g : \mu(Q) \longrightarrow \mathrm{R}$, then $g \circ i_Q \in Q$ by theorem (1.11), so $g \circ \alpha \in P$. By corollary 1 [art b, there exist a unique map $\hat{f} : \mu(P) \longrightarrow \mu(Q)$ such that $\hat{f} \circ i_P = \alpha$. Then the diagram:

$$\begin{array}{ccc} \mathcal{X} & \overrightarrow{f} & \mathcal{Y} \\ \downarrow\ i_P & & \downarrow\ i_Q \\ \mu(P) & \overrightarrow{\hat{f}} & \mu(Q) \end{array}$$

is commutative.

2. The Four Major Theorems

This section, as stated in introductory notes, only stating and proving The Stone-Weierstrass Theorem, The Stone-Ĉech compactification theorem, The Tietze-Urysohn extension Theorem, and the Tychonoff Theorem as consequences of the grand unification theorem.

Theorem 2.1 (Stone Weierstrass Theorem). *Suppose $\mathcal{X}$ is a compact Haussdorf space.*
If P is a closed subalgebra of $\mathcal{C}(\mathcal{X}, \mathrm{R})$ which contains $\underline{1}$ and separates the points of $\mathcal{X}$, then $P = \mathcal{C}(\mathcal{X}, \mathrm{R})$.

Proof :
The map $i_P : \mathcal{X} \longrightarrow \mu(P)$ is surjective (since its image is dense in $\mu(P)$) and is injective (otherwise: $f(x) = f(y)$ for all $f \in P$ ans som $x \neq y$). Thus i_P is homeomorphism and $P = \mathcal{C}(\mathcal{X}, \mathrm{R})$.

Theorem 2.2 (Stone-Ĉech Compactification Theorem). *Suppose $\mathcal{X}$ is a completely regular space.*
Then there is a compact Haussdorf space $\beta\ \mathcal{X}$ containing $\mathcal{X}$ as a dense set such that any map $f : \mathcal{X} \longrightarrow \mathcal{Y}$ from $\mathcal{X}$ to a compact Haussdorf space $\mathcal{Y}$ extends over $\beta\ \mathcal{X}$.

Proof :
Write $\beta\ \mathcal{X} = \mu(\mathcal{C}(\mathcal{X}, \mathrm{R}))$. By theorem 1.7 , $\mathcal{X}$ may considered as subset of $\beta\ \mathcal{X}$. Then $f(\mathcal{X}) = \mu(P)$ (up to homeomorphism) for some $P \subset \mathcal{C}(\mathcal{X}, \mathrm{R})$ so there is a map from $\beta\ \mathcal{X}$ to $\mu(P)$ extending f.

Theorem 2.3 (Tietze-Urysohn)**.** *If A is a closed subset of a normal space $\mathcal{X}$, then any continuous function $f : A \longrightarrow \mathrm{R}$ extends over $\mathcal{X}$.*

Proof :
The proof is split into two cases, first in the case of $\mathcal{X}$ being compact, and secondly $\mathcal{X}$ is not compact.

In the case of $\mathcal{X}$ being compact, this exactly means $P = \{f \circ \imath f : \mathcal{X} \rightarrow \mathrm{R}\}$ equals $\mathcal{C}(A, \mathrm{R})$, where $\imath : A \rightarrow \mathrm{R}$ is inclusion map. By Stone-Wierstrass theorem, one gets $\bar{P} = \mathcal{C}(A, \mathrm{R})$, so it suffices to show that P is closed.

Suppose $f_n \circ \imath$ converges uniformly to f. We may assume $|f_{n+1}(a) - f_n(a)| < 2^{-n}$ for all $a \in A$. Let $r_n : \mathrm{R} \rightarrow [-2^{-n}, 2^{-n}]$ defined by :

$$r_n(x) = \begin{cases} x, & -2^{-n} \leq x \leq 2^{-n} \\ -2^{-n}, & -2^{-n} > x \\ 2^{-n}, & 2^{-n} < x \end{cases}$$

Then $g_n = f_1 + \sum k = 1^n r_k(f_{k+1} - f_k)$ converges uniformly to $g : \mathcal{X} \rightarrow \mathrm{R}$ with $g(a) = f(a)$ for $a \in A$.

Note that $\hat{\imath} : \beta_A \longrightarrow \beta\mathcal{X}$ is injective. Let $x \neq y$ in βA, select two disjoint closed set C, D in βA with $x \in C^{\circ}$ and $y \in \beta$. Let $g : \mathcal{X} \rightarrow [0, 1]$ be a map with $g(C \cap A) = \{0\}$ and $g(D \cap A) = \{1\}$. There is an extension $g' : \mathcal{X} \rightarrow [0, 1]$ of g and an extension $g'' : \mathcal{X} \rightarrow [0, 1]$ of $g|_A$. Since A is dense in βA, we have $g'' = g' \circ \hat{\imath}$ and $\hat{\imath}(x) \neq \hat{\imath}(y)$.

If $\mathcal{X}$ is not compact and $f : A \rightarrow \mathrm{R}$ is bounded we extend f over $\beta A = \mu(\mathcal{C}(A, \mathrm{R}))$. By the previous part, we can extend over βX and the restriction of this extension to $\mathcal{X}$ is the desired extension of f.

If $f : A \rightarrow \mathrm{R}$ is not bounded, we identify R with $(-1, 1)$ and chose an extension $g : \mathcal{X} \rightarrow [-1, 1]$ of f. Then consider $\alpha : \mathcal{X} \rightarrow [0, 1]$ with $\alpha(g^{-1}(\{-1, 1\})) \subset \{0\}$ and $\alpha(A) = \{1\}$. Define $f'(x) = \alpha(x) \cdot g(x)$.

Theorem 2.4 (Tychonoff Theorem)**.** *If $\{\mathcal{X}_i\}_{\mathcal{I}}$ is a family of compact Haussdorf spaces, then their cartesian product $\prod_{\mathcal{I}} \mathcal{X}_i$ is compact Haussdorf.*

Proof :
Put $\mathcal{X} = \prod_{\mathcal{I}} \mathcal{X}_i$ and $P = \mathcal{C}(\mathcal{X}, \mathrm{R})$. For each $i \in I$ there is a map $g_i : \mu(P) \longrightarrow \mathcal{X}_i$ such that $g_i \circ i_P = \prod_i$ is the projection $\prod_i \mathcal{X}_i \longrightarrow \mathcal{X}_i$. Then $g = \prod_{\mathcal{I}} g_i : \mu(P) \longrightarrow \mathcal{X}$ is continuous and $g \circ i_P = id_{\mathcal{X}}$. Since $i_P \circ g$ is identity on a dense subset $i_P(\mathcal{X})$ of $\mu(P), i_P \circ g = id$ and g is homeomorphism.

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