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Krzysztof Ciesielski
West Virginia University, krzysztof.ciesielski@mail.wvu.edu

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MARTIN'S AXIOM AND A REGULAR TOPOLOGICAL SPACE WITH UNCOUNTABLE NET WEIGHT WHOSE COUNTABLE PRODUCT IS HEREDITARILY SEPARABLE AND HEREDITARILY LINDELÖF

KRZYSZTOF CIESIELSKI[1]

In [1, p. 51] A. V. Arhangel'skiĭ, in connection with the problems of L-spaces and S-spaces, examined further the notions of hereditary separability and hereditary Lindelöfness. In particular he considered the following property P: "Every regular topological space has a countable net weight provided its countable product is hereditarily Lindelöf and hereditarily separable." He noticed that the continuum hypothesis implies negation of the property P and posed a question: "Do Martin's Axiom and the negation of the continuum hypothesis imply P?" The purpose of this paper is to give a negative answer to this question.

The set-theoretical and topological notation that we use is standard and can be found in [6] and [5] respectively.

Throughout the paper we will use the notation $H(X, Y)$ to denote the set of all finite functions from a set X to Y.

THEOREM. Con(ZFC) $\rightarrow$ Con(ZFC + MA + $\neg$CH + *there exists a 0-dimensional Hausdorff space X such that $nw(X) = \mathfrak{c}$ and $nw(Y) = \omega$ for any $Y \in [X]^{<\mathfrak{c}}$*).

PROOF. Let M be a model of ZFC satisfying CH and let F be an M-generic filter over the Cohen forcing $(H(\omega_2 \times \omega_2, 2), \supset)$. Then $f = \bigcup F$ is a function and $f: \omega_2 \times \omega_2 \rightarrow 2$.

In $M[f]$, define the functions $f_\zeta: \omega_2 \rightarrow 2$ by $f_\zeta(\eta) = f(\eta, \zeta)$ $(\zeta, \eta < \omega_2)$ and consider $X = \{f_\zeta: \zeta < \omega_2\}$ as a topological subspace of 2^{ω_2}.

For $\varepsilon \in H(\omega_2, 2)$, let $[\varepsilon]$ be the element of the standard basis of 2^{ω_2} (i.e. $[\varepsilon] = \{g \in 2^{\omega_2}: \varepsilon \subset g\}$). For $\alpha < \omega_2$, let

$$Q_\alpha = \{\langle A, \varepsilon\rangle: \varepsilon \in H(\omega_2, 2) \text{ and } A \subset ([\varepsilon] \cap \{f_\zeta: \zeta < \alpha\}) \text{ and } |A| < \omega\}$$

with ordering $\langle A, \varepsilon\rangle \leq \langle B, \delta\rangle$ if and only if $A \supset B$ and $\varepsilon \supset \delta$.

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It is easy to see that $Q_\alpha \in M[f \restriction \omega_2 \times \alpha]$ for $\alpha < \omega_2$. Moreover let Q be a direct product of the forcings Q_α, i.e.

$$Q = \prod\{Q_\alpha : \alpha < \omega_2\} = \{h : \mathrm{dom}(h) \in [\omega_2]^{<\omega} \ \&\ h(\alpha) \in Q_\alpha \text{ for every } \alpha \in \mathrm{dom}(h)\},$$

ordered by

$$h \leq h' \text{ if and only if } \mathrm{dom}(h) \supset \mathrm{dom}(h') \text{ and } h(\alpha) \leq h'(\alpha) \text{ for every } \alpha \in \mathrm{dom}(h').$$

Let G be an $M[f]$-generic filter over Q.

In [4, §5] (compare also [3, Theorems 2 and 4]) we proved that the forcing Q satisfies the ccc property in the model $M[f]$ and the model $M[f][G]$ satisfies

$$(*) \qquad nw(X) = \mathfrak{c} = \omega_2 \text{ and } nw(Y) = \omega \text{ for any } Y \in [X]^{<\mathfrak{c}}.$$

(The second part of $(*)$ follows immediately from the fact that if $Y \subset \{f_\zeta : \zeta < \alpha\}$ for some $\alpha < \omega_2$ and $N_k = \{f \in Y : f \in A \text{ for some element } \{\langle \alpha + k, \langle A, \varepsilon\rangle\rangle\} \in G\}$, then the family $\{N_k : k < \omega\}$ forms a countable network for Y.) We now give a generic extension of $M[f][G]$ which preserves the property $(*)$ and satisfies Martin's Axiom.

In $M[f][G]$ let R be an iteration with finite supports of ccc forcings of the form $T_\alpha = \langle \omega_1, \leq_\alpha\rangle$ such that $R \Vdash \mathrm{MA} + (\mathfrak{c} = \omega_2)$ (compare for example [2, §6, pp. 444–451]). Hence $R = H(\omega_2, \omega_1)$ and, for $g, g' \in R$, $g \leq g'$ if and only if $\mathrm{dom}(g) \supset \mathrm{dom}(g')$ and $g \restriction \alpha \Vdash g(\alpha) \leq_\alpha g'(\alpha)$ for every $\alpha \in \mathrm{dom}(g')$.

Let H be an $M[f][G]$-generic filter over R. Clearly

$$M[f][G][H] \models \mathrm{MA} + (\mathfrak{c} = \omega_2).$$

Moreover for every $Y \subset 2^{\omega_1}$ the sentence "N is a network of Y" is absolute; by the ccc property of R, for every $Z \in [X]^{\omega_1}$ from $M[f][G][H]$ there exists $Y \in [X]^{\omega_1}$ from $M[f][G]$ such that $Z \subset Y$. Hence

$$M[f][G][H] \models nw(Y) = \omega \text{ for every } Y \in [X]^{<\mathfrak{c}}.$$

Note that since $hL(Y) \leq nw(Y)$ we have (in $M[f][G][H]$) $hL(Y) = \omega$ for every $Y \in [X]^{<\mathfrak{c}}$. This implies $M[f][G][H] \models hL(X) = \omega$ (cf. [5] or [4]).

To complete the proof it suffices to show that $nw(X) = \mathfrak{c}$ holds in $M[f][G][H]$. The original idea of the proof is that, for $M[f][G]$, $\models nw(X) = \omega$ (see [4] or [3]).

By way of contradiction assume that there exists a family $\mathscr{F} = \{F_\zeta : \zeta < \omega_1\}$ in $M[f][G][H]$ such that

$$M[f][G][H] \models \text{"}\mathscr{F} \text{ is a network for } X\text{"}.$$

By the regularity of X the family $\{\mathrm{cl}_X F_\zeta : \zeta < \omega_1\}$ is also a network for X, so we can assume that the sets F_ζ are closed in X. Hence, by hereditary Lindelöfness of X, there exists in $M[f][G][H]$ a sequence $\mathscr{E} = \langle \varepsilon_n^\zeta \in H(\omega_2, 2) : \zeta < \omega_1 \ \&\ n < \omega\rangle$ such that

$$X \backslash F_\zeta = X \cap \bigcup\{[\varepsilon_n^\zeta] : n < \omega\} \quad \text{for all } \zeta < \omega_1.$$

Let R_α be an iteration of length α of the forcings T_β for $\beta < \alpha$ (i.e. $R_\alpha = \{g \restriction \alpha : g \in R\}$). Then $H_\alpha = R_\alpha \cap H$ is an $M[f][G]$-generic filter over R_α.

Since the forcing R is ccc and $|\mathscr{E}| \leq \omega_1$, there exists $\alpha < \omega_2$ such that $\mathscr{E} \in M[f][G][H_\alpha]$.

For $\beta < \omega_1$ put $\bar{Q}^\beta = \prod\{Q_\gamma: \beta \leq \gamma < \omega_2\}$, $G^\beta = G \cap \bar{Q}^\beta$ and $G_\beta = G \cap \prod\{Q_\gamma: \gamma < \beta\}$. We know that Q is ccc in $M[f]$ and $|R_\alpha| \leq \omega_1$; thus there exists $\beta < \omega_2$ such that $R_\alpha \in M[f][G_\beta]$.

Hence, by the product lemma,

$$M[f][G][H_\alpha] = M[f][G_\beta][G^\beta][H_\alpha] = M[f][G_\beta][H_\alpha][G^\beta],$$

i.e.

$$\mathscr{E} \in M[f][G_\beta][H_\alpha][G^\beta].$$

But in $M[f][G_\beta]$ we have $\bar{Q}^\beta \Vdash R_\alpha$ is ccc; consequently $\bar{Q}^\beta \times R_\alpha$ is ccc and $R_\alpha \Vdash \bar{Q}^\beta$ is ccc. Hence in $M[f][G_\beta][H_\alpha]$, the forcing $\bar{Q}^\beta$ is ccc and $|\mathscr{E}| \leq \omega_1$.

Therefore there exists $\gamma < \omega_2$ such that $\beta < \gamma$ and

$$\mathscr{E} \in M[f][G_\beta][H_\alpha][G_\gamma^\beta] = M[f][G_\gamma][H_\alpha]$$

where $G_\gamma^\beta = G^\beta \cap \prod\{Q_\delta: \beta \leq \delta < \gamma\}$. Moreover $\prod\{Q_\delta: \delta < \gamma\} \in M[f \upharpoonright \omega_2 \times \gamma]$, i.e.

$$M[f][G_\gamma] = M[f \upharpoonright \omega_2 \times \gamma][G_\gamma][f \upharpoonright \omega_2 \times (\omega_2 \backslash \gamma)].$$

Hence there exists $\delta < \omega_2$, $\delta > \gamma$, such that

$$R_\alpha \in M[f \upharpoonright \omega_2 \times \gamma][G_\gamma][f \upharpoonright \omega_2 \times (\delta \backslash \gamma)] = M[f \upharpoonright \omega_2 \times \delta][G_\gamma].$$

In particular we have

$$\mathscr{E} \in M[f \upharpoonright \omega_2 \times \delta][G_\gamma][H_\alpha][f \upharpoonright \omega_2 \times (\omega_2 \backslash \delta)].$$

Let $N = M[f \upharpoonright \omega_2 \times \delta][G_\gamma][H_\alpha]$, $a \in \omega_2 \backslash \bigcup\{\mathrm{dom}(\varepsilon_n^\zeta): \zeta < \omega_1 \ \&\ n < \omega\}$ and $f_\delta(a) = i$, and put

$$\phi \equiv \text{“}(\forall \zeta < \omega_1)(f_\delta \notin F_\zeta \text{ or } F_\zeta \not\subset [\{\langle a, i\rangle\}])\text{”}.$$

We next prove

$$(**) \qquad N[f \upharpoonright \omega_2 \times (\omega_2 \backslash \delta)] \models \phi.$$

Fix $\zeta < \omega_1$ and assume that $f_\delta \in F_\zeta$. Then there exists $s \in H(\omega_2 \times (\omega_2 \backslash \delta), 2)$ such that

$$N \models s \Vdash (f_\delta \in F_\zeta = X \backslash \bigcup\{[\varepsilon_n^\zeta]: n < \omega\}).$$

But $\langle \varepsilon_n^\zeta: n < \omega\rangle \in N$, i.e. the last statement is equivalent to

$$X \cap [\varepsilon] \subset X \backslash \bigcup\{[\varepsilon_n^\zeta]\} = F_\zeta,$$

where $\varepsilon = \{\langle \xi, i\rangle: \langle \xi, \delta, i\rangle \in s\}$. Hence, for $\varepsilon' = \varepsilon \upharpoonright \bigcup\{\mathrm{dom}(\varepsilon_n^\zeta): n < \omega\}$, $X \cap [\varepsilon'] \subset F_\zeta$. Put $\varepsilon'' = \varepsilon' \cup \{\langle a, 1-i\rangle\}$. Then

$$\varnothing \neq [\varepsilon''] \cap X \subset [\varepsilon'] \cap X \subset F_\zeta \quad \text{and} \quad [\varepsilon''] \cap X \cap [\{\langle a, i\rangle\}] = \varnothing;$$

i.e. $F_\zeta \not\subset [\{\langle a, i\rangle\}]$. This completes the proof of $(**)$.

To finish the proof of the theorem first observe that the sentence ϕ is absolute, hence $M[f][G][H] = \text{“}\phi\text{”}$ (i.e. $M[f][G][H] \models \mathscr{F}$ is not a network for X). This contradicts our assumption.

COROLLARY. Con(ZFC) $\rightarrow$ Con(ZFC + MA + $\neg$CH + *there exists a regular space* X *such that* $nw(X) = \mathfrak{c}$ *but* $hL(X^{\omega}) = hd(X^{\omega}) = \omega$).

PROOF. It is enough to show that the condition "$nw(Y) = \omega$ for every $Y \in [X]^{\omega_1}$" implies $hL(X^{\omega}) = hd(X^{\omega}) = \omega$. So let us assume that $nw(Y) = \omega$ for every $Y \in [X]^{\omega_1}$. Then for every $n < \omega$ and $Y \in [X]^{\omega_1}$ we have $hL(Y^n)\, hd(Y^n) \leq nw(Y^n) = \omega$. Therefore $hL(X^n) = hd(X^n) = \omega$ and hence (see for example [1, p. 51]) $hL(X^{\omega}) = hd(X^{\omega}) = \omega$.

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DEPARTMENT OF MATHEMATICS
WARSAW UNIVERSITY
WARSAW, POLAND

DEPARTMENT OF MATHEMATICS
BOWLING GREEN STATE UNIVERSITY
BOWLING GREEN, OHIO 43403

The Warsaw address is the permanent one.