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MOD $(2p+1)$-ORIENTATIONS AND $K_{1,2p+1}$-DECOMPOSITIONS*

HONG-JIAN LAI†

Abstract. In this paper, we establish an equivalence between the contractible graphs with respect to the mod $(2p+1)$-orientability and the graphs with $K_{1,2p+1}$-decompositions. This is applied to disprove a conjecture proposed by Barat and Thomassen that every 4-edge-connected simple planar graph G with $|E(G)| \equiv 0 \pmod 3$ has a claw decomposition.

Key words. nowhere zero flows, circular flows, mod $(2p+1)$-orientations, $K_{1,2p+1}$-decompositions

AMS subject classification. 05C99

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1. Introduction. Graphs in this paper are finite and loopless and may have multiple edges. See [2] for undefined notations and terminologies. In particular, $\kappa'(G)$ denotes the edge connectivity of a graph G, and if X is an edge subset or a vertex subset of a graph G, then $G[X]$ denotes the subgraph of G induced by X. A connected loopless graph with 3 edges and a vertex of degree 3 is called *a generalized claw*. When restricted to simple graphs, a generalized claw must be isomorphic to a $K_{1,3}$. A graph G with $|E(G)| \equiv 0 \pmod 3$ has a *claw decomposition* if $E(G)$ can be partitioned into disjoint unions $E(G) = X_1 \cup X_2 \cup \cdots \cup X_k$ such that, for each i with $1 \le i \le k$, $G[X_i]$ is a generalized claw. Barat and Thomassen [1] showed that the claw-decomposition problem is closely related to the nowhere zero 3-flow problem. In particular, the following conjecture is proposed.

CONJECTURE 1.1 (Barat and Thomassen [1]). *Every 4-edge-connected simple planar graph G with $|E(G)| \equiv 0 \pmod 3$ has a claw decomposition.*

The purpose of this note is to disprove this conjecture. In section 2, we shall introduce contractible graphs with respect to the mod $(2p+1)$-orientability and discuss their properties and their relationship to the graphs with $K_{1,2p+1}$-decompositions. In section 3, we disprove the conjecture above.

2. M^o_{2p+1} and $K_{1,2p+1}$-decompositions. Throughout this section, $p > 0$ denotes an integer. We shall extend the definition of claw decomposition to $K_{1,2p+1}$-decomposition as follows. A connected loopless graph with $2p+1$ edges and a vertex of degree $2p+1$ is called *a generalized* $K_{1,2p+1}$. A graph G with $|E(G)| \equiv 0$ $\pmod{2p+1}$ has *a $K_{1,2p+1}$-decomposition* if $E(G)$ can be partitioned into disjoint unions $E(G) = X_1 \cup X_2 \cup \cdots \cup X_k$ such that, for each i with $1 \le i \le k$, $G[X_i]$ is a generalized $K_{1,2p+1}$. In this case, we say that G has a $K_{1,2p+1}$-decomposition $\mathcal{X} = \{X_1, X_2, \ldots, X_k\}$.

Let $D = D(G)$ be an orientation of an undirected graph G. If an edge $e \in E(G)$ is directed from a vertex u to a vertex v, then let tail$(e) = u$ and head$(e) = v$. For a vertex $v \in V(G)$, let

$$E^+_D(v) = \{e \in E(D) : v = \text{ tail}(e)\} \text{ and } E^-_D(v) = \{e \in E(D) : v = \text{ head}(e)\}.$$

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We shall denote $d_D^+(v) = |E_D^+(v)|$ (the *out degree* of v) and $d_D^-(v) = |E_D^-(v)|$ (the *in degree* of v). The subscript D may be omitted when $D(G)$ is understood from the context. Let A be an (additive) Abelian group. If $f : E(G) \mapsto A$ is a function, then the *boundary* of f is a map $\partial f : V(G) \mapsto A$ such that

$$\partial f(v) = \sum_{e \in E_D^+(v)} f(e) - \sum_{e \in E_D^-(v)} f(e),\ \forall v \in V(G).$$

Let $k > 0$ be an integer, and assume that G has a fixed orientation D. A *mod k-orientation* of G is a function $f : E(G) \mapsto \{1, -1\}$ such that for all $v \in V(G)$, $\partial f(v) \equiv 0 \pmod{k}$. The collection of all graphs admitting a mod k-orientation is denoted by M_k. Note that, by definition, $K_1 \in M_k$. Jaeger has conjectured [7] that every $4k$-edge-connected graph is in M_{2k+1}. This conjecture is still open.

Throughout this note, $\mathbf{Z}$ denotes the set of all integers. For integers $a_1, a_2, \dots a_k$ such that not all of them are zero, let $gcd(a_1, a_2, \dots, a_k)$ denote the greatest common divisor of $a_1, a_2, \dots a_k$. For an $m \in \mathbf{Z}$, $\mathbf{Z}_m$ denotes the set of integers modulo m, as well as the additive cyclic group on m elements. For a graph G, a function $b : V(G) \mapsto \mathbf{Z}_m$ is a *zero sum function* in $\mathbf{Z}_m$ if $\sum_{v \in V(G)} b(v) \equiv 0 \pmod{m}$. The set of all zero sum functions in $\mathbf{Z}_m$ of G is denoted by $Z(G, \mathbf{Z}_m)$. When $k = 2p+1 > 0$ is an odd number, we define M_{2p+1}^o to be the collection of graphs such that $G \in M_{2p+1}^o$ if and only if for all $b \in Z(G, \mathbf{Z}_{2p+1})$, $\exists f : E(G) \mapsto \{1, -1\}$ such that for all $v \in V(G)$, $\partial f(v) \equiv b(v)$ $\pmod{2p+1}$.

Note that if a function $f : E(G) \mapsto \{1, -1\}$ is given, then one can reverse the orientation of e for each $e \in E(G)$ with $f(e) = -1$ to obtain an orientation D' of G such that for all $v \in V(G)$, $d_{D'}^+(v) - d_{D'}^-(v) = \partial f(v)$. Thus we have the following proposition.

PROPOSITION 2.1. *$G \in M_{2p+1}^o$ if and only if for all $b \in Z(G, \mathbf{Z}_{2p+1})$, G has an orientation D with the property that for all $v \in V(G)$, $d_D^+(v) - d_D^-(v) \equiv b(v)$ (mod$2p+1$).*

For a subgraph H of G, define the set of *vertices of attachments* of H in G to be $A_G(H) = \{v \in V(H) : v \text{ is adjacent to a vertex in } G - V(H)\}$.

PROPOSITION 2.2. *For any integer $p \geq 1$, M_{2p+1}^o is a family of connected graphs such that each of the following holds.*

(C1) $K_1 \in M_{2p+1}^o$.

(C2) *If $e \in E(G)$ and if $G - e \in M_{2p+1}^o$, then $G \in M_{2p+1}^o$.*

(C3) *If H is a subgraph of G, and if $H, G/H \in M_{2p+1}^o$, then $G \in M_{2p+1}^o$.*

Proof. (C1) and (C2) are straightforward, and so we verify only (C3).

Suppose that G has a fixed orientation, H is a subgraph of G, and both $H \in M_{2p+1}^o$ and $G/H \in M_{2p+1}^o$. Thus the edges in both H and G/H are oriented by the orientation of G. By (C2), we may assume that H is an induced subgraph of G, and so $E(G)$ is the disjoint union of $E(H)$ and $E(G/H)$. Note that H is connected, and so H will be contracted to a vertex v_H (say) in G/H. Let $b : V(G) \mapsto \mathbf{Z}_{2p+1}$ such that $\sum_{v \in V(G)} b(v) \equiv 0 \pmod{2p+1}$, and let $a_0 = \sum_{v \in V(H)} b(v)$. Define $b_1 : V(G/H) \to A$ by setting $b_1(z) = b(z)$ if $z \neq v_H$, and $b_1(v_H) = a_0$. Then $\sum_{z \in V(G/H)} b_1(z) = \sum_{z \in V(G)} b(z) \equiv 0 \pmod{2p+1}$. Since $G/H \in M_{2p+1}^o$, there exists $f_1 : E(G/H) \mapsto \{1, -1\}$ such that $\partial f_1 = b_1$. For each $z \in V(H)$, define

$$b_2(z) = \begin{cases} b(z) + \sum_{e \in E_{G/H}^-(v_H) \cap E_G^-(z)} f_1(e) - \sum_{e \in E_{G/H}^+(v_H) \cap E_G^+(z)} f_1(e) & \text{if } z \in A_G(H), \\ b(z) & \text{otherwise.} \end{cases}$$

Then $\sum_{z \in V(H)} b_2(z) \equiv 0 \pmod{2p+1}$. Since $H \in M_{2p+1}^o$, there exists $f_2 : E(G/H) \mapsto$

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$\{1,-1\}$ such that $\partial f_2 = b_2$. Now for each $e \in E(G)$, define $f(e) = f_1(e) + f_2(e)$. As $E(G)$ is a disjoint union of $E(H)$ and $E(G/H)$, it is routine to verify that $\partial f(z) \equiv b(z)$ (mod $2p+1$), and so $G \in M^o_{2p+1}$. □

Catlin [3] (see also [4], [5]) called families of connected graphs satisfying (C1), (C2), and (C3) complete families. Complete families seem to be useful in applying certain reduction methods ([3], [4], [5]).

For a subgraph H of a graph G, define

$$\partial(H) = \{uv \in E(G) : u \in V(H), v \in V(G) - V(H)\}.$$

Let D be an orientation of G. Let $d^+_D(H)$ denote the number of edges in $\partial(H)$ that are oriented in D from H to $G - V(H)$, and $d^-_D(H) = |\partial(H)| - d^+_D(H)$.

To demonstrate the relationship between M^o_{2p+1} and all of the graphs with $K_{1,2p+1}$-decompositions, we make the following definitions.

(i) $k_{c,2p+1}$ denotes the smallest integer $k > 0$ such that every k-edge-connected graph G is in M^o_{2p+1}.

(ii) $k^{c,2p+1}$ denotes the smallest integer $k > 0$ such that every k-edge-connected graph G with $|E(G)| \equiv 0$ (mod $2p+1$) has a $K_{1,2p+1}$-decomposition.

The main result of this section is the following relationship.

THEOREM 2.3. *For any positive integer $p > 0$, if one of $k_{c,2p+1}$ and $k^{c,2p+1}$ exists as a finite number, then $k_{c,2p+1} = k^{c,2p+1}$.*

To prove this theorem, we need to establish some lemmas. In each of the following lemmas, G is a graph and H is a subgraph of G. Suppose that G has a $K_{1,2p+1}$-decomposition $\mathcal{X} = \{X_1, X_2, \ldots, X_k\}$, where each $G[X_i]$ is a generalized $K_{1,2p+1}$ for all i. For each $G[X_i]$, we orient the edges from the vertex v_i of degree $2p+1$ in $G[X_i]$ to all other vertices of $G[X_i]$. This yields an orientation $D = D(\mathcal{X})$ induced by the decomposition $\mathcal{X}$. For each i, the vertex v_i is called the *center* of the oriented X_i.

LEMMA 2.4. *Suppose that G has a $K_{1,2p+1}$-decomposition $\mathcal{X} = \{X_1, X_2, \ldots, X_k\}$, and let $D = D(\mathcal{X})$. Then for any subgraph H of G,*

$$|E(H)| + d^+_D(H) \equiv 0 \pmod{2p+1}.$$

Proof. Let $[H, G - V(H)]$ denote the set of edges in $\partial(H)$ that are oriented in $D(\mathcal{X})$ from H to $G - V(H)$. Then $|[H, G - V(H)]| = d^+_D(H)$.

By the definition of $D(\mathcal{X})$, the edge subset $E(H) \cup [H, G - V(H)]$ is the disjoint union of the oriented X_i's whose centers are in $V(H)$. It follows that $|E(H)| + d^+_D(H) = |E(H) \cup [H, G - V(H)]| \equiv 0$ (mod $2p+1$). □

LEMMA 2.5. *Let $b \in \mathbf{Z}$ be a number, and let $d = |\partial(H)|$. Suppose that G has a $K_{1,2p+1}$-decomposition $\mathcal{X}$ and that H is a subgraph of G. If $2|E(H)| \equiv -d - b$ (mod $2p+1$), then, in the orientation $D = D(\mathcal{X})$,*

$$d^+_D(H) - d^-_D(H) \equiv b \pmod{2p+1}.$$

Proof. Let $d^+ = d^+_D(H)$ and $d^- = d^-_D(H)$. Then $d = d^+ + d^-$. By Lemma 2.4, $|E(H)| \equiv -d^+$ (mod $2p+1$), and so $b \equiv -d - 2|E(H)| \equiv -d + 2d^+ \equiv (-d + d^+) + d^+ \equiv d^+ - d^-$ (mod $2p+1$). □

The following below is well-known in number theory. For a reference, see Theorem 1.5 of [12].

LEMMA 2.6. *Let $a_1, a_2, \ldots, a_k$ be integers, not all zero. Then $gcd(a_1, a_2, \ldots, a_k) = 1$ if and only if there exist integers $x_1, x_2, \ldots, x_k$ such that $a_1x_1 + a_2x_2 + \cdots + a_kx_k = 1$.*

LEMMA 2.7. *Let $k, l, p \in \mathbf{Z}$ such that $k > 0$, $p > 0$, and $0 \le l \le 2p$. Each of the following holds.*

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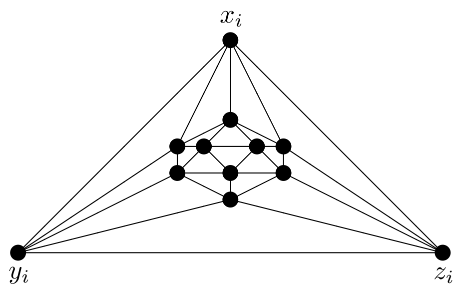

FIG. 1. $I_{12}(i)$ *(isomorphic to icosahedron) with specified* x_i, y_i, z_i.

(i) *There exists a planar graph H with $\kappa'(H) \geq k$ and* $2|E(H)| \equiv l \pmod{2p+1}$.

(ii) *There exists a simple graph H with $\kappa'(H) \geq k$ and* $2|E(H)| \equiv l \pmod{2p+1}$.

(iii) *If $2 \leq k \leq 5$, then there exists a simple planar graph H with $\kappa'(H) \geq k$ and* $2|E(H)| \equiv l \pmod{2p+1}$.

Proof. (i) For any integer $n > 0$, let nK_2 denote the connected loopless graph with two vertices and n multiple edges. Let $s > 0$ be an integer such that $s(2p+1) \geq k$. Define the desired H as follows:

$$H = \begin{cases} (2ps+s+t)K_2 & \text{if } l = 2t \text{ is even,} \\ ((2p+1)(s+1)-(p-t))K_2 & \text{if } l = 2t+1 \text{ is odd.} \end{cases}$$

(ii) Take an integer $m \geq 4p+2+k$, and let $H_v = K_m - W$ for some edge set $W \subset E(K_m)$ such that $|W| \leq 4p+1$ and $2(|E(K_m)| - |W|) \equiv l \pmod{2p+1}$.

(iii) Since $gcd(10, 18, 2p+1) = 1$, by Lemma 2.6, there are integers a_0, b_0, c_0 such that $10a_0 + 18b_0 + (2p+1)c_0 = 1$. Choose $x_0 = la_0 + l(|a_0|+1)(2p+1)$ and $y_0 = lb_0 + l(|b_0|+1)(2p+1)$. Then x_0, y_0 are positive integers such that

$$10x_0 + 18y_0 \equiv l \mod (2p+1)$$

holds. Let $t = (2p+1)(x_0+y_0+1)$, and let $I_{12}(i)$, $1 \leq i \leq t-1$, be a graph isomorphic to icosahedron defined below (see Figure 1). Define H to be the graph obtained from $I_{12}(1), I_{12}(2), \ldots, I_{12}(t)$ by identifying z_i and y_{i+1}, $1 \leq i \leq t-1$, and by adding $x_0 + 2y_0$ new vertices $u_1, u_2, \ldots, u_{x_0}, v_1, v_2, \ldots, v_{y_0}, w_1, w_2, \ldots, w_{y_0}$ with $N(u_k) = \{x_{5k-4}, x_{5k-3}, x_{5k-2}, x_{5k-1}, x_{5k}\}$, $N(v_{k'}) = \{x_{5x_0+8k'-7}, x_{5x_0+8k'-6}, x_{5x_0+8k'-5}, x_{5x_0+8k'-4}, w_{k'}\}$, and $N(w_{k'}) = \{x_{5x_0+8k'-3}, x_{5x_0+8k'-2}, x_{5x_0+8k'-1}, x_{5x_0+8k'}, v_{k'}\}$, where $1 \leq k \leq x_0$ and $1 \leq k' \leq y_0$. So H is a simple planar graph with $\kappa(H) \geq k$ and $2|E(H)| = 60t + 10x_0 + 18y_0 = 60(2p+1)(x_0+y_0+1) + 10x_0 + 18y_0 \equiv l \mod (2p+1)$. ∎

LEMMA 2.8. (i) *Let $k > 0$ be an integer. If every k-edge-connected (simple) graph G with $|E(G)| \equiv 0 \pmod{2p+1}$ has a $K_{1,2p+1}$-decomposition, then every k-edge-connected (simple) graph $L \in M^o_{2p+1}$.*

(ii) *Let $k > 0$ be an integer. If every k-edge-connected planar graph G with $|E(G)| \equiv 0 \pmod{2p+1}$ has a $K_{1,2p+1}$-decomposition, then every k-edge-connected planar graph $L \in M^o_{2p+1}$.*

(iii) *Let $2 \leq k \leq 5$. If every k-edge-connected simple planar graph G with $|E(G)| \equiv 0 \pmod{2p+1}$ has a $K_{1,2p+1}$-decomposition, then every k-edge-connected simple planar graph $L \in M^o_{2p+1}$.*

Proof. We shall prove (i) and assume first that every k-edge-connected (simple) graph G with $|E(G)| \equiv 0 \pmod{2p+1}$ has a $K_{1,2p+1}$-decomposition. By contradiction, we assume that there exists a k-edge-connected (simple) graph L such that $L \notin M^o_{2p+1}$.

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Therefore, $\exists b \in Z(L, \mathbf{Z}_{2p+1})$ such that L does not have an orientation D satisfying $d_D^+(v) - d_D^-(v) \equiv b(v) \pmod{2p+1}$ for all $v \in V(L)$.

Let $l_v \in \mathbf{Z}$, with $0 \le l_v \le 2p$ such that $l_v \equiv -b(v) - d_L(v) \pmod{2p+1}$ for all $v \in V(L)$. By Lemma 2.7 (ii), there exists a simple graph H_v with $2|E(H_v)| \equiv l_v \equiv -b(v) - d_L(v) \pmod{2p+1}$ such that H_v is also k-edge-connected. For each $v \in V(L)$, replace v by H_v in such a way that the resulting graph G is also a k-edge-connected (simple) graph.

Since $b \in Z(L, \mathbf{Z}_{2p+1})$, $2|E(G)| = \sum_{v\in V(L)} 2|E(H_v)| + 2|E(L)| = -\sum_{v\in V(L)} b(v) - \sum_{v\in V(L)} d_L(v) + 2|E(L)| \equiv -\sum_{v\in V(L)} b(v) \equiv 0 \pmod{2p+1}$. By the fact that 2 and $2p+1$ are relatively prime, $|E(G)| \equiv 0 \pmod{2p+1}$. By the assumption of this lemma, G has a $K_{1,2p+1}$-decomposition $\mathcal{X}$. By the construction of G, $|\partial(H_v)| = d_L(v)$. Since $2|E(H_v)| \equiv l_v \equiv -b(v) - d_L(v) \pmod{2p+1}$, it follows by Lemma 2.5 that, in the orientation $D = D(\mathcal{X})$ for all $v \in V(L)$, $d_D^+(H_v) - d_D^-(H_v) \equiv b(v) \pmod{2p+1}$, contrary to the assumption that L is a counterexample.

The proofs for (ii) and (iii) are similar except that we shall use Lemma 2.7 (i) and (iii) instead of Lemma 2.7 (ii). Thus we omit the detailed proofs. □

LEMMA 2.9. *If G has an orientation D such that for all $v \in V(G)$, $d_D^+(v) \equiv 0 \pmod{2p+1}$, then G is $K_{1,2p+1}$-decomposable.*

Proof. Note that if D is an orientation of G, then $E(G) = \cup_{v\in V(G)} E_D^+(v)$ is a disjoint union. As for all $v \in V(G)$, $d_D^+(v) \equiv 0 \pmod{2p+1}$, each $E_D^+(v)$ is a disjoint union of generalized $K_{1,2p+1}$'s, and so G is $K_{1,2p+1}$-decomposable. □

LEMMA 2.10. *Suppose that $G \in M_{2p+1}^o$. If $|E(G)| \equiv 0 \pmod{2p+1}$, then G has a $K_{1,2p+1}$-decomposition.*

Proof. For all $v \in V(G)$, pick an $x(v) \in \{0, 1, \ldots, 2p\}$ such that $d(v) \equiv x(v) \pmod{2p+1}$. Define $b(v) = d(v) - 2x(v)$. First, we shall show that $b \in Z(G, \mathbf{Z}_{2p+1})$. Since $x(v) \equiv d(v) \pmod{2p+1}$, we have $d(v) - 2x(v) \equiv -x(v) \equiv -d(v) \pmod{2p+1}$. Note also that $\sum_{v\in V(G)} d(v) = 2|E(G)| \equiv 0 \pmod{2p+1}$. Thus

$$\sum_{v\in V(G)} b(v) = \sum_{v\in V(G)} (d(v) - 2x(v)) = -\sum_{v\in V(G)} d(v) \equiv 0 \pmod{2p+1}.$$

Hence $b \in Z(G, \mathbf{Z}_{2p+1})$.

Since $G \in M_{2p+1}^o$, there exists an orientation D of G such that, under this orientation, at each vertex $v \in V(G)$, $d^+(v) - d^-(v) = b(v) = d(v) - 2x(v)$. Since $d^+(v) + d^-(v) = d(v)$, we have $2d^+(v) = 2d(v) - 2x(v) = 2(d(v) - x(v))$. Since 2 and $2p+1$ are relatively prime, $d^+(v) \equiv d(v) - x(v) \equiv 0 \pmod{2p+1}$. Therefore, by Lemma 2.9, G has a $K_{1,2p+1}$-decomposition. □

Now we can easily prove Theorem 2.3. By Lemma 2.8, $k_{c,2p+1} \le k^{c,2p+1}$ and by Lemma 2.10, $k_{c,2p+1} \ge k^{c,2p+1}$. Thus Theorem 2.3 follows.

By (ii) and (iii) of Lemma 2.8 and by Lemma 2.10, and noting that the edge connectivity of a simple planar graph cannot exceed 5 (Corollary 9.5.3 of [2]), we also have the following corollary.

COROLLARY 2.11. (i) *Let k' denote the smallest positive integer such that every k'-edge-connected planar graph G with $|E(G)| \equiv 0 \pmod{2p+1}$ has a $K_{1,2p+1}$-decomposition, and let k'' denote the smallest positive integer such that every k''-edge-connected planar graph is in M_{2p+1}^o. Then $k' = k''$.*

(ii) *Let l' denote the smallest positive integer such that every l'-edge-connected simple planar graph G with $|E(G)| \equiv 0 \pmod{2p+1}$ has a $K_{1,2p+1}$-decomposition, and let l'' denote the smallest positive integer such that every l''-edge-connected simple planar graph is in M_{2p+1}^o. Then $l' = l''$.*

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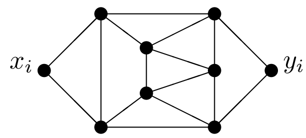

FIG. 2. *The building block H_i.*

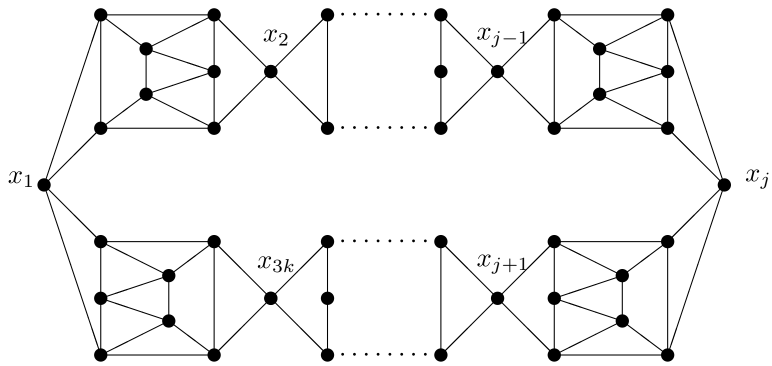

FIG. 3. *The graph $G = G(k)$.*

3. Planar graphs. When $p = 1$, graphs in M_3^o are also called $\mathbf{Z}_3$-connected graphs [8], [10], [11]. The following has been recently proved.

THEOREM 3.1 (Theorem 3 of [9]). *There exists a family of* 4*-edge-connected simple planar graphs that are not in M_3^o.*

In fact, the dual version of Theorem 3.1 is proved in [9]. The equivalence between Theorem 3 of [9] and Theorem 3.1 here was pointed out without a proof in [8], and a formal proof of this equivalence can be found in [6].

COROLLARY 3.2. *There exists a* 4*-edge-connected simple planar graph G with* $|E(G)| \equiv 0 \pmod 3$ *which does not have a claw decomposition.*

Proof. Suppose, to the contrary, that Conjecture 1.1 holds. Then by (ii) of Corollary 2.11, every 4-edge-connected simple planar graph must be in M_3^o, which contradicts Theorem 3.1. ∎

Corollary 3.2 disproves Conjecture 1.1. In fact, we can also directly construct an infinite family of 4-edge-connected simple planar graphs G with $|E(G)| \equiv 0 \pmod 3$ which does not have a claw decomposition. We present the construction as follows.

Let $k > 0$ be an integer. For each i with $1 \le i \le 3k$, define H_i to be the graph depicted below. See Figure 2.

A graph $G = G(k)$ can be constructed from the disjoint H_i's by identifying y_i and x_{i+1}, where $x_{3k+1} = x_1$ and where $i = 1, 2, \ldots 3k$.

EXAMPLE 3.3. *For each $k > 0$, $G = G(k)$ defined in Figure* 3 *is a* 4*-regular and* 4*-edge-connected simple planar graph with* $|E(G)| \equiv 0 \pmod 3)$*, and G has no claw decomposition.*

Proof. Suppose G has a claw decomposition $\mathcal{X} = \{X_1, X_2, \ldots, X_m\}$, and let $D = D(\mathcal{X})$. Since G is 4-regular, for all $v \in V(G)$, $|E_D^+(v)| \in \{0, 3\}$. Note that $|V(G)| = 24k$ and $|E(G)| = 48k$. Thus G has $m = 48k/3 = 16k$ edge-disjoint

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claws. Let W denote the set of vertices v with $|E_D^+(v)| = 0$. Then $|W| = |V(G)| - m = 24k - 16k = 8k$. Note that no two vertices in W are adjacent in G, and so, for each $i = 1, 2, \ldots, 3k \pmod{3k}$, $|W \cap V(H_i \cup H_{i+1} - \{y_{i+1}\})| \le 5$. It follows that $16k = 2|W| = \sum_{i=1}^{3k} |V(H_i \cup H_{i+1} - \{y_{i+1}\}) \cap W| \le 5 \times 3k = 15k$, a contradiction. □

It is an open problem whether $k_{c,2p+1}$, or, equivalently, $k^{c,2p+1}$, exists as a finite number. We conjecture that it does. In view of Corollary 3.2 and Example 3.3, we further conjecture that $k_{c,2p+1} = 4p + 1$.

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