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# ON MOD $(2s+1)$-ORIENTATIONS OF GRAPHS*

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**Abstract.** An orientation of a graph $G$ is a mod $(2p+1)$-orientation if, under this orientation, the net out-degree at every vertex is congruent to zero mod $2p+1$. If, for any function $b : V(G) \to \mathbb{Z}_{2p+1}$ satisfying $\sum_{v\in V(G)} b(v) \equiv 0 \pmod{2p+1}$, $G$ always has an orientation $D$ such that the net out-degree at every vertex $v$ is congruent to $b(v)$ mod $2p+1$, then $G$ is strongly $\mathbb{Z}_{2p+1}$-connected. The graph $G'$ obtained from $G$ by contracting all nontrivial subgraphs that are strongly $\mathbb{Z}_{2s+1}$-connected is called the $\mathbb{Z}_{2s+1}$-reduction of $G$. Motivated by a minimum degree condition of Barat and Thomassen [*J. Graph Theory*, 52 (2006), pp. 135–146], and by the Ore conditions of Fan and Zhou [*SIAM J. Discrete Math.*, 22 (2008), pp. 288–294] and of Luo et al. [*European J. Combin.*, 29 (2008), pp. 1587–1595] on $\mathbb{Z}_3$-connected graphs, we prove that for a simple graph $G$ on $n$ vertices, and for any integers $s > 0$ and real numbers $\alpha, \beta$ with $0 < \alpha < 1$, if for any nonadjacent vertices $u, v \in V(G)$, $d_G(u) + d_G(v) \geq \alpha n + \beta$, then there exists a finite family $\mathcal{F}(\alpha, s)$ of nonstrongly $\mathbb{Z}_{2s+1}$-connected graphs such that either $G$ is strongly $\mathbb{Z}_{2s+1}$-connected or the $\mathbb{Z}_{2s+1}$-reduction of $G$ is in $\mathcal{F}(\alpha, s)$.

**Key words.** mod $(2p+1)$-orientations, strongly $\mathbb{Z}_{2p+1}$-connectedness, group connectivity of graphs, degree conditions

**AMS subject classifications.** 05C15, 05C40

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**1. The problem.** We consider finite graphs without loops, but permitting multiple edges, and follow [2] for undefined terms and notation. In particular, for a graph $G$, $\kappa(G)$, $\kappa'(G)$, and $\delta(G)$ denote the connectivity, the edge-connectivity, and the minimum degree of $G$, respectively. We write $H \subseteq G$ to mean that $H$ is a subgraph of $G$. If $X \subseteq E(G)$ is an edge subset, then the *contraction* $G/X$ is obtained by identifying the two ends of each edge in $X$ and then deleting all the resulting loops. If $H$ is a connected subgraph of $G$ and $v_H$ is the vertex in $G/H$ onto which $H$ is contracted, then $H$ is the *preimage* of $v_H$ and is denoted by $PI_G(v_H)$. We often use $G/H$ for $G/E(H)$.

Throughout this paper, $\mathbb{Z}$ denotes the set of all integers. For an $m \in \mathbb{Z}$, $\mathbb{Z}_m$ denotes the set of integers modulo $m$, as well as the additive cyclic group on $m$ elements. For a graph $G$, and for any integer $i \geq 0$, define

$$V_i(G) = \{v \in V(G) \ : \ d_G(v) = i\}.$$

Let $D$ denote an orientation of $G$. Following [2], we use $(u, v)$ to denote this arc (directed edge) oriented from $u$ to $v$. For each $v \in V(G)$, $d_D^+(v)$ and $d_D^-(v)$ denote the out-degree and the in-degree of $v$ under the orientation $D$, respectively. When the orientation $D$ is clear in the context, we use $d^+$ and $d^-$ to denote $d_D^+$ and $d_D^-$, respectively.

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For an integer $m > 1$, if a graph $G$ has an orientation $D$ such that at every vertex $v \in V(G)$, $d^+_D(v) - d^-_D(v) \equiv 0 \pmod m$, then we say that $G$ admits a *mod $m$-orientation*. The set of all graphs which have mod $m$-orientations is denoted by $M_m$. As it is immediate from the definition that if $m = 2s$ is an even integer, then a connected graph $G$ is in $M_{2s}$ if and only if $G$ is Eulerian, therefore, we are only interested in the case when $m = 2s+1$ is an odd integer.

Let $A$ be an (additive) abelian group and $G$ be a graph with an orientation $D = D(G)$. For any vertex $v \in V(G)$, let $E^+_D(v)$ denote the set of all edges directed out from $v$, and let $E^-_D(v)$ denote the set of all edges directed into $v$. For a function $f : E(G) \to A$, define $\partial f : V(G) \to A$, called the *boundary of $f$*, as follows:

$$\text{for any vertex } v \in V(G),\ \partial f(v) = \sum_{e \in E^+_D(v)} f(e) - \sum_{e \in E^-_D(v)} f(e).$$

A function $b : V(G) \to A$ is a *zero-sum function* on $A$ if $\sum_{v \in V(G)} b(v) \equiv 0$, where 0 denotes the additive identity. The set of all zero-sum functions on $A$ of $G$ is denoted by $Z(G, A)$. Let $A'$ be a subset of $A$. We define $F(G, A') = \{f : E(G) \to A'\}$. For any zero-sum function $b$ on $A$ of $G$, a function $f \in F(G, A')$ satisfying $\partial f = b$ is referred to as an $(A', b)$-flow. When $b = 0$, an $(A - \{0\}, 0)$-flow is known as a *nowhere-zero $A$-flow* in the literature (see [8, 9], among others). Following [9], if for any zero-sum function $b$ on $A$ of $G$, $G$ always has an $(A - \{0\}, b)$-flow, then $G$ is *$A$-connected*.

A graph G is *strongly $\mathbb{Z}_m$-connected* if, under a given orientation $D$, for any zero-sum function $b$ on $\mathbb{Z}_m$ of $G$, there exists a function $f \in F(G, \{\pm 1\})$ such that $\partial f = b$. Again, for a given $b \in Z(G, \mathbb{Z}_m)$ and an $f \in F(G, \{\pm 1\})$ with $\partial f = b$, one can keep the orientation of each edge with $f(e) = 1$ and reverse the orientation of each edge with $f(e) = -1$ to obtain a new orientation $D'$ of $G$ such that for any vertex $v \in V(G)$, $d^+_{D'}(v) - d^-_{D'}(v) = b(v) = \partial f(v)$. This orientation $D'$ will be referred to as a *$(\mathbb{Z}_m, b)$-orientation* of $G$. Thus a graph $G$ is strongly $\mathbb{Z}_m$-connected if and only if for any $b \in Z(G, \mathbb{Z}_m)$, $G$ always has a $(\mathbb{Z}_m, b)$-orientation. By definition, a graph $G$ is $\mathbb{Z}_3$-connected if and only if $G$ is strongly $\mathbb{Z}_3$-connected. But for an odd number $m \geq 5$, while every strongly $\mathbb{Z}_m$-connected graph is $\mathbb{Z}_m$-connected, not every $\mathbb{Z}_m$-connected graph is strongly $\mathbb{Z}_m$-connected. It has been proved [12, 14] that strongly $\mathbb{Z}_{2s+1}$-connected graphs must be $2s$-edge-connected and are precisely the graphs $H$ such that for any graph $G$ containing $H$ as a subgraph, $G$ is in $M_{2s+1}$ if and only if $G/H$ is in $M_{2s+1}$. Therefore, following the notation of Catlin [3] and Catlin, Hobbs, and Lai [4], the family of strongly $\mathbb{Z}_{2s+1}$-connected graphs is often denoted by $M^o_{2s+1}$.

Tutte and Jaeger proposed the following conjectures concerning mod $(2s+1)$-orientations. A conjecture on strongly $\mathbb{Z}_{2s+1}$-connected graphs has also been proposed recently.

CONJECTURE 1.1. *Let $s \geq 1$ denote an integer:*

(i) (Tutte [18]). *Every 4-edge-connected graph has a mod 3-orientation.*

(ii) (Jaeger [7] and [8]). *Every $4s$-edge-connected graph has a mod $(2s+1)$-orientation.*

(iii) (Jaeger [7, 8]). *Every 5-edge-connected graph is strongly $\mathbb{Z}_3$-connected.*

(iv) (Lai [11] and Lai et al. [13]). *Every $(4s+1)$-edge-connected graph is strongly $\mathbb{Z}_{2s+1}$-connected.*

Conjecture 1.1(i) is well known as Tutte's 3-flow conjecture. Conjecture 1.1(ii) is an extension of Tutte's 3-flow conjecture, which includes Conjecture 1.1(i) as the special case of $p = 1$. In [10], Kochol showed that to prove Conjecture 1.1(i), it suffices to prove that every 5-edge-connected graph has a mod 3-orientation. Consequently, Conjecture 1.1(iii) implies Conjecture 1.1(i). The following were also conjectured in [8, 11, 13] and have been proved by Thomassen [17] and Lovasz et al. [15]; see also [19].

Theorem 1.2. *Let $G$ be a graph, and let $m \geq 5$ be an odd integer:*

(i) (Thomassen [17]). *If $\kappa'(G) \geq 8$, then $G$ is strongly $\mathbb{Z}_3$-connected.*

(ii) (Thomassen [17]). *If $\kappa'(G) \geq 2m^2 + m$, then $G$ is strongly $\mathbb{Z}_m$-connected.*

Both bounds in Theorem 1.2 have recently been improved.

Theorem 1.3 (Lovasz et al. [15], Wu [19]). *Let $s > 0$ be an integer. If $\kappa'(G) \geq 6s$, then $G$ is strongly $\mathbb{Z}_{2s+1}$-connected.*

Barat and Thomassen presented the first degree condition to ensure a simple graph to be $\mathbb{Z}_3$-connected. This was later improved by Fan and Zhou [5] and Luo et al. [16].

Theorem 1.4 (Barat and Thomassen, Theorem 5.2 in [1]). *There exists a positive integer $n_1$ such that every simple graph on $n \geq n_1$ vertices with minimum degree at least $n/2$ is $\mathbb{Z}_3$-connected.*

Theorem 1.5. *Let $G$ be a simple graph on $n \geq 3$ vertices such that for every pair of nonadjacent vertices $u$ and $v$ in $G$, $d_G(u) + d_G(v) \geq n$. Then each of the following holds:*

(i) (Fan and Zhou [5]). *With six exceptional graphs, $G$ has a nowhere-zero 3-flow.*

(ii) (Luo et al., Theorem 1.8 of [16]). *With 12 exceptional graphs, $G$ is in $M_3^o$.*

These results in Theorems 1.4 and 1.5 have the format that if a simple graph satisfies the Direct condition or the Ore condition, then the graph has a nowhere-zero 3-flow or is $\mathbb{Z}_3$-connected, with finitely many exceptional cases. As checking whether a graph belongs to a finite list of graphs is computationally tractable, these results motivate the current research. The problem we propose to answer is, for any real numbers $a, b$ with $0 < a < 1$, if the degree sum over nonadjacent vertices of a simple graph $G$ is at least $an + b$, can we still conclude that the graph has a nowhere-zero 3-flow or is $\mathbb{Z}_3$-connected, with finitely many exceptional cases? We prove the following more general result.

Theorem 1.6. *Let $G$ be a simple graph on $n$ vertices. For any integers $s > 0$ and for any real numbers $\alpha$ and $\beta$ with $0 < \alpha < 1$, there exist an integer $N = N(\alpha, s)$ and a finite family $\mathcal{F}_1(\alpha, s)$ of graphs not in $M_{2s+1}^o$ such that if $n \geq N$ and if for every pair of nonadjacent vertices $u$ and $v$ in $G$, $d_G(u) + d_G(v) \geq \alpha n + \beta$, then either $G$ is strongly $\mathbb{Z}_{2s+1}$-connected or $G$ can be contracted to a member in $\mathcal{F}_1(\alpha, s)$.*

The following corollary of Theorem 1.6 is expected.

Corollary 1.7. *Let $G$ be a simple graph on $n$ vertices. For any integers $s > 0$ and for any real numbers $\alpha$ and $\beta$ with $0 < \alpha < 1$, there exist an integer $N = N(\alpha, s)$ and a finite family $\mathcal{F}_2(\alpha, s)$ of graphs not in $M_{2s+1}$ such that if $n \geq N$ and if $\delta(G) \geq (\alpha n + \beta)/2$, then either $G$ is strongly $\mathbb{Z}_{2s+1}$-connected or $G$ can be contracted to a member in $\mathcal{F}_2(\alpha, s)$.*

By definition, if $G$ is strongly $\mathbb{Z}_{2s+1}$-connected, then $G$ is in $M_{2s+1}$. Therefore, both Theorem 1.6 and Corollary 1.7 imply that the graph $G$ is either in $M_{2s+1}$ or contractible to a member in a finite family of graphs.

In the next section, we will present some of the basics on strongly $\mathbb{Z}_m$-connected graphs. The proof of Theorem 1.6 depends on an associate result (Theorem 2.8), to be proved in the next section. We shall apply this associate result to prove Theorem 1.6 in the last section.

## 2. Strongly $\mathbb{Z}_m$-connected and strongly $\mathbb{Z}_m$-reduced graphs.

Throughout this section, $s > 1$ is an integer. The statements (i) and (ii) of Proposition 2.1 below are proved in Proposition 2.2 of [11]. The proof for Proposition 2.1(iii) is similar to that for (i) and (ii), and so it is omitted.

PROPOSITION 2.1 (see [11]). *For any integer $s \geq 1$, each of the following holds:*

(i) *If $G$ is strongly $\mathbb{Z}_{2s+1}$-connected and $e$ is an edge of $G$, then $G/e$ is strongly $\mathbb{Z}_{2s+1}$-connected.*

(ii) *If $H$ is a subgraph of $G$ and both $H$ and $G/H$ are strongly $\mathbb{Z}_{2s+1}$-connected, then so is $G$.*

(iii) *If $H$ is a strongly $\mathbb{Z}_{2s+1}$-connected subgraph of $G$, then $G/H$ has a mod $(2s+1)$-orientation if and only if $G$ does.*

Let $K_2^{(m)}$ denote the loopless graph with two vertices and $m$ parallel edges. Some examples of strongly $\mathbb{Z}_{2s+1}$-connected graph have been found in [14].

LEMMA 2.2 (Lai et al. [12] and Liang [14]). *Let $G$ be a graph, and let $m, s \geq 1$ be integers. Each of the following holds:*

(i) *$K_2^{(m)}$ is strongly $\mathbb{Z}_{2s+1}$-connected if and only if $m \geq 2s$.*

(ii) *$K_n$ is strongly $\mathbb{Z}_{2s+1}$-connected if and only if $n \geq 4s+1$.*

DEFINITION 2.3. *Let $H$ be a subgraph of $G$, and let $s > 0$ be an integer. The* mod $(2s+1)$-closure *of $H$ in $G$, denoted by $cl_G^{2s+1}(H)$ or $cl(H)$ when the graph $G$ and the number $s$ are understood from the context, is the (unique) maximal subgraph of $G$ that contains $H$ such that $V(cl(H)) - V(H)$ can be ordered as a sequence $\{v_1, v_2, \ldots, v_t\}$ such that $|N_G(v_1) \cap V(H)| \geq 2s$ and for each $i$ with $1 \leq i \leq t-1$,*

$$|[N_G(v_{i+1}) \cap (V(H) \cup \{v_1, v_2, \ldots, v_i\})| \geq 2s. \tag{2.1}$$

PROPOSITION 2.4 (Lai et al. [12] and Liang [14]). *Let $H$ be a subgraph of $G$, and let $s > 0$ be an integer. If $H$ is strongly $\mathbb{Z}_{2s+1}$-connected, then each of the following holds:*

(i) *$cl(H)$ is strongly $\mathbb{Z}_{2s+1}$-connected.*

(ii) *The graph $G$ is strongly $\mathbb{Z}_{2s+1}$-connected if and only if $G/cl(H)$ is strongly $\mathbb{Z}_{2s+1}$-connected.*

For any graph $G$, every vertex lies in a maximal strongly $\mathbb{Z}_{2s+1}$-connected subgraph. Let $H_1, H_2, \ldots, H_c$ denote the collection of all maximal strongly $\mathbb{Z}_{2s+1}$-connected subgraphs. Then $G' = G/(\cup_{i=1}^c E(H_i))$ is the $\mathbb{Z}_{2s+1}$-*reduction* of $G$. A graph which does not have any nontrivial strongly $\mathbb{Z}_{2s+1}$-connected subgraph is $\mathbb{Z}_{2s+1}$-*reduced*. Thus the $\mathbb{Z}_{2s+1}$-reduction of a graph is always $\mathbb{Z}_{2s+1}$-reduced.

Define $\overline{\kappa}'(G) = \max\{\kappa'(H) : H \subseteq G \text{ with } |E(H)| > 0\}$. Even when $G$ is a simple graph, its $\mathbb{Z}_{2s+1}$-reduction may have multiple edges. Lemma 2.2 suggests that there are only finitely many nontrivial strong $\mathbb{Z}_{2s+1}$-reduced graphs with a given order. The following is a corollary of Theorem 1.3.

LEMMA 2.5. *Let $G'$ be the $\mathbb{Z}_{2s+1}$-reduction of a connected graph $G$ such that $G' \neq K_1$. Then $\overline{\kappa}'(G') \leq 6s-1$.*

For multigraphs with bounded values of $\overline{\kappa}'$, the number of edges will also be bounded.

LEMMA 2.6 (Gu et al. [6]). *Suppose that $G$ is a graph with order $n$ and $k > 0$ is an integer. If $\overline{\kappa}'(G) \leq k$, then $|E(G)| \leq (n-1)k$.*

DEFINITION 2.7. *Let $a, b$ be real numbers with $0 < a < 1$, let $k_1, k_2 > 0$ be integers, and let $G$ be a simple graph on $n$ vertices with a distinguished subset $S \subseteq V(G)$:*

(i) *A minimal edge cut $X$ of $G$ is an $(a, b, S)$-cut if for each component $L$ of $G - X$, either $|V(L)| \geq an + b$ or $V(L) \cap S \neq \emptyset$, and the induced subgraph $L_S := L[V(L) \cap S]$ of $L$ is either in $M_{2s+1}^o$ with $L \subseteq cl(L_S)$, or $L = L_S$ and $|V(L)| \leq 4s$.*

(ii) *A graph $G$ has property $(a, b; s, k_1, k_2)$ if $V(G)$ has a distinguished subset $S$ such that both of the following hold:*

(P1) *Every minimal edge cut $X$ of $G$ with $|X| \leq k_2$ is an $(a, b, S)$-cut.*

(P2) *In the $\mathbb{Z}_{2s+1}$-reduction $G'$ of $G$, $|\{v \in V(G') : PI_G(v) \cap S \neq \emptyset\}| \leq k_1$.*

THEOREM 2.8. *Let $a, b$ be real numbers with $0 < a < 1$. For any integers $k_2, k_1 \geq 0$ ad $s > 0$ with $k_2 \geq \max\{12s-1, k_1\}$, define $N = \lceil \frac{-2b}{a} \rceil + 1$ and $B = \lceil \left(\frac{2}{a} + k_2\right)(k_2+1) \rceil$. If a simple graph $G$ on $n \geq N$ vertices has property $(a, b; s, k_1, k_2)$, then one of the following must hold:*

(i) *$G$ is strongly $\mathbb{Z}_{2s+1}$-connected. (That is, $G \in M^o_{2s+1}$.)*

(ii) *$G \notin M^o_{2s+1}$, and the $\mathbb{Z}_{2s+1}$-reduction of $G$ has at most $B$ vertices.*

*Proof.* We assume that $a, b$, $k_1, k_2$, and $s$ are given and that $G$ has property $(a, b; s, k_1, k_2)$. Let $G'$ denote the $\mathbb{Z}_{2s+1}$-reduction of $G$, and define $S'$ to be the set of vertices in $G'$ whose preimages in $G$ contain vertices in $S$. By Definition 2.7(ii)(P2), $|S'| \leq k_1$. Let $n' = |V(G')|$. If $n' = 1$, then $G' = K_1$ is strongly $\mathbb{Z}_{2s+1}$-connected, and so by Conjecture 1.1, Theorem 2.8(i) holds.

Hence we assume that $n' > 1$. For integer $i > 0$, define $d_i = |V_i(G')|$. Then

$$n' = \sum_{i \geq 1} d_i. \tag{2.2}$$

Fix an integer $k'$ with $k_2 - 1 \geq k' \geq 6s - 1$. By Lemmas 2.5 and 2.6,

$$2(n'-1)k' \geq 2(n'-1)(6s-1) \geq 2|E(G')| = \sum_{i \geq 1} i d_i. \tag{2.3}$$

Let $V_{\leq k'}(G')$ denote the set of vertices in $G'$ with degree at most $k'$. Then $|V_{\leq k'}(G')| = \sum_{i \leq k'} d_i$. Let $c(k') = |V_{\leq k'}(G') - S'|$. Then $D_{\leq k'}(G')$ has $c(k')$ vertices whose preimages, as connected subgraphs in $G$, do not contain vertices in $S$. Let $H_1, H_2, \ldots, H_{c(k')}$ denote these subgraphs, and let $H_{c(k')+1}, \ldots, H_h$ denote the subgraphs of $G$ that are preimages of vertices in $S' \cap V_{\leq k'}(G')$. Since $G$ has property $(a, b : s, k_1, k_2)$, for each $i = 1, 2, \ldots, c(k')$, as $V(H_i) \cap S = \emptyset$, by Definition 2.7(i), $|V(H_i)| \geq an + b$. This yields

$$n - \left| \bigcup_{j=c(k')+1}^{h} V(H_j) \right| \geq \sum_{i=1}^{c(k')} |V(H_i)| \geq c(k')(an+b). \tag{2.4}$$

As $n \geq N$, we have $\frac{n}{an+b} \leq \frac{2}{a}$. By (2.4), and by $k_2 \geq k_1$, it follows that

$$\begin{aligned} \frac{2}{a} + k_2 &\geq \frac{n}{an+b} + k_1 \geq \frac{n - |\cup_{j=c(k')+1}^{h} V(H_j)|}{an+b} + |S' \cap D_{\leq k'}(G')| \\ &\geq c(k') + |S' \cap D_{\leq k'}(G')| = \sum_{i=1}^{k'} d_i. \end{aligned} \tag{2.5}$$

Define $k = 6s - 1$. Substitute (2.2) in (2.3) with $k' = k$ to get

$$2k \sum_{i \geq 1} d_i - 2k \geq \sum_{i \geq 1} i d_i. \tag{2.6}$$

By (2.6), and applying (2.5) with $k' = 2k - 1 = 12s - 3 \leq k_2 - 1$, we have

$$\left(\frac{2}{a} + k_2\right) k_2 \geq 2k \sum_{i<2k} d_i \geq \sum_{i<2k} (2k - i) d_i - 2k \geq \sum_{i>2k} (i - 2k) d_i \geq \sum_{i>2k} d_i. \tag{2.7}$$

Applying (2.5) with $k' = 2k$, we have

$$\frac{2}{a} + k_2 \geq \sum_{i=1}^{2k} d_i. \tag{2.8}$$

It follows from combining (2.7) and (2.8) that

$$n' = \sum_{i=1}^{2k} d_i + \sum_{i>2k} d_i \leq \left(\frac{2}{a} + k_2\right)(k_2+1) = B.$$

This completes the proof of Theorem 2.8. ∎

**3. Proof of Theorem 1.6.** Throughout this section, $s \geq 1$ is an integer. We will prove the following version, which is equivalent to Theorem 1.6 with $\alpha = 2a$ and $\beta = 2b$.

THEOREM 3.1. *Let $G$ be a simple graph on $n$ vertices. For any integer $s > 0$ and for any real numbers $a$ and $b$ with $0 < a < 1$, there exist an integer $N \geq \max\{\frac{14s-b+1}{a} + 1, \lceil\frac{-2b}{a}\rceil + 1\}$ and a finite family $\mathcal{F}_1(a,s)$ of graphs not in $M^o_{2s+1}$ such that if $n \geq N$ and if for every pair of nonadjacent vertices $u$ and $v$ in $G$,*

$$d_G(u) + d_G(v) \geq 2(an+b), \tag{3.1}$$

*then either $G$ is strongly $\mathbb{Z}_{2s+1}$-connected or the $\mathbb{Z}_{2s+1}$-reduction of $G$ is in $\mathcal{F}_1(a,s)$.*

Define $k_1 = 4s$, $k_2 = 12s-1$. We shall show that any connected simple graph $G$ on $n$ vertices satisfying (3.1) will also have property $(a,b;s,k_1,k_2)$. Once this is done, we then will apply Theorem 2.8 to prove Theorem 1.6.

If $G$ is in $M^o_{2s+1}$, then Theorem 3.1 holds. Hence throughout the rest of the proof, we shall assume that $G$ is a connected simple graph on $n$ vertices satisfying the hypothesis of Theorem 3.1, but $G$ is not in $M^o_{2s+1}$; and we let $G'$ be the $\mathbb{Z}_{2s+1}$-reduction of $G$ such that $G' \neq K_1$. We then will prove the following lemmas to complete the proof.

LEMMA 3.2. *Let $S = \{v \in V(G) | d_G(v) < an+b\}$. Each of the following holds:*

*(i) $G[S]$ is a complete graph. Moreover, if $|S| \geq 4s+1$, then $cl(G[S])$ is contained in a unique maximal strongly $\mathbb{Z}_{2s+1}$-connected subgraph of $G$.*

*(ii) $G$ has property $(a,b;s,k_1,k_2)$.*

*Proof.* (i). By (3.1), every distinct pair of vertices in $S$ must be adjacent. By Lemma 2.2 and Proposition 2.4, if $|S| \geq 4s+1$, then $cl(G[S])$ is strongly $\mathbb{Z}_{2s+1}$-connected.

(ii). We assume that $G$ does not have property $(a,b;s,k_1,k_2)$. Let $S'$ be the set of vertices in $G'$ whose preimages are singletons in $S$. If $|S| \geq 4s+1$, then by Lemma 3.2(i), $|S'| = 1 \leq k_1$. Otherwise, $|S'| \leq 4s = k_1$.

Thus by Definition 2.7(ii), $G$ must have a minimal edge cut $X$ with $|X| \leq 12s-1$, which is not an $(a,b,S)$-cut. This means that if $L_1, L_2$ are the two components of $G - X$, then one of the following holds:

(A) $|V(L_1)| < an+b$ and $V(L_1) - V(cl(L_1[V(L_1) \cap S])) \neq \emptyset$, or

(B) $|V(L_2)| < an+b$ and $V(L_2) - V(cl(L_2[V(L_2) \cap S])) \neq \emptyset$.

Assume (A) holds and we will find a contradiction. Since $V(L_1) - V(cl(L_1[V(L_1) \cap S])) \neq \emptyset$, $V(L_1) - V(cl(L_1[V(L_1) \cap S]))$ has a vertex $u$ with $d_G(u) \geq an+b$. Since $u \notin V(cl(L_1[V(L_1) \cap S]))$, $|N_G(u) \cap V(cl(L_1[V(L_1) \cap S]))| \leq 2s-1$. It follows from $n \geq N$ that

$$|N_G(u)| - |X| - |N_G(u) \cap V(cl(L_1[V(L_1) \cap S]))| \geq an + b - (12s - 1) - (2s - 1)) \geq 1.$$

This implies that $L_1$ has a vertex $w$ not in $V(cl(L_1[V(L_1) \cap S]))$ such that $w$ is not incident with any edges in $X$. Hence $N_G(w) \subseteq V(L_1)$, and so

$$|V(L_1)| \geq |N_G(w)| + 1 \geq d_G(w) + 1 > an + b,$$

contrary to (A). Hence (A) does not hold. Similarly, (B) does not hold. This proves (ii). □

LEMMA 3.3. *For any fixed number $B$, there exist only a finite number of nontrivial $\mathbb{Z}_{2s+1}$-reduced graphs with order at most $B$.*

*Proof.* For given integers $B > 0$ and $s \geq 1$, let $\mathcal{F}$ denote the collection of all $\mathbb{Z}_{2s+1}$-reduced graphs with order at most $B$. By Lemma 2.2(i), every graph $G \in \mathcal{F}$ has edge multiplicity at most $2s - 1$, and so $|\mathcal{F}|$ is a finite number. □

*Proof of Theorem* 3.1. By Lemma 3.2, $G$ has property $(a, b; s, k_1, k_2)$. Since $N \geq \lceil \frac{-2b}{a} \rceil + 1$, by Theorem 2.8 with $B = \lceil \left(\frac{2}{a} + k_2\right)(k_2 + 1) \rceil$ either $G$ is strongly $\mathbb{Z}_{2s+1}$-connected or the $\mathbb{Z}_{2k+1}$-reduction of $G$ is nontrivial and has order at most $B$. By Lemma 3.3, there are only a finite number of such strong $\mathbb{Z}_{2k+1}$-reduced graphs. Hence we can let $\mathcal{F}_1(a, s)$ be the family of nontrivial $\mathbb{Z}_{2s+1}$-reduced graphs with at most $B$ vertices. □

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