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Holographic phase transition from dyons in an AdS black hole background

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Abstract

We construct a dyon solution for a Yang-Mills-Higgs theory in a 4 dimensional Schwarzschild-anti-de Sitter black hole background with temperature T. We then apply the AdS/CFT correspondence to describe the strong coupling regime of a $2 + 1$ quantum field theory which undergoes a phase transition exhibiting the condensation of a composite charge operator below a critical temperature T_c.

1 Introduction

It has recently been observed that the AdS/CFT correspondence [1]-[3] provides a powerful tool to study quantum critical dynamics in condensed matter models for superconductivity, superfluidity and Hall effect [4]-[10]. In this way, using the dual classical gravity description, correlation functions in strongly interacting systems can be calculated and the relevant physical

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1

properties can be determined in a relatively simple way, which has motivated an intense activity in the field (some works related to the present approach are listed in [11]).

Temperature is introduced in the gravity dual through the presence of a Schwarzschild-AdS black hole, either added as a background or resulting from back reaction. Apart from gauge fields, the models referred above include minimally coupled scalar matter fields. In the Abelian case the ansatz for gauge fields included an electrostatic potential A_0 which is spatially independent at the boundary as also is the case for the scalar. Spatially dependent droplet and vortex solutions have been also studied in [8]-[9] and in the non-Abelian case non-trivial gauge-field components were considered [7],[10]. In general, such classical solutions in the bulk exist below a certain critical temperature leading to thermodynamically favored phases in the the strongly coupled quantum field theory on the boundary.

Following the ideas described above, we consider in the present paper a model not yet studied: a dyon solution for Yang-Mills-Higgs theory in a 4 dimensional Schwarzschild-anti-de Sitter black-hole background metric. We shall be guided by our previous work on monopole and dyon solutions in AdS_4 space [12]-[13] where, due to the absence of singularities, the spherically symmetric solutions were constructed in the whole radial domain $\mathbb{R}^+$, while in the present case the radial variable extend from the black hole horizon to infinity. The black hole temperature, determined by the horizon, will become the temperature of the field theory defined on the boundary. The matter field will lead to the non-Abelian gauge symmetry breaking condensate while the field strength associated to the surviving $U(1)$ symmetry will correspond to that of a dyon with electric and magnetic charges.

The paper is organized as follows. In section 2 we present the gravity-Yang-Mills-Higgs model and specialize to the case in which the gravitational equations decouple from matter leading to an AdS-Schwarzschild black hole background for the gauge-matter system. Then in section 3 we analyze the properties of the dyon solution considering both the cases of an $S^2 \times S^1$ and $\mathbb{R}^2 \times S^1$ boundaries. In section 4 we proceed to describe the holography calculation of relevant quantities in the dual field theory defined on the boundary (taken as $\mathbb{R}^2 \times S^1$) and present our numerical results. We end with a summary and discussion in section 5.

2 The gravity-Yang-Mills-Higgs system

2.1 The model

We consider a gravity-Yang-Mills-Higgs system with gauge group $SU(2)$ and the scalar field in the adjoint representation, in a 4 dimensional space-time with Minkowski signature $(-, +, +, +)$. The action takes the form

$$S = S_G + S_{YM} + S_H = \int d^4x \sqrt{|g|} \, (L_G + L_{YM} + L_H) \tag{1}$$

with

$$L_G = \frac{1}{2\,\kappa^2} \, (R - 2\,\Lambda) \tag{2}$$

$$L_{YM} = -\frac{1}{4e^2} \, F_{\mu\nu}^a F^{a\,\mu\nu} \tag{3}$$

$$L_H = -\frac{1}{2} D_\mu H^a \, D^\mu H^a - V(H) \tag{4}$$

$$V(H) = \frac{\lambda}{4} \, (H^a H^a - H_0{}^2)^2 \tag{5}$$

$$\kappa^2 \equiv 8\,\pi\,G_N \tag{6}$$

Here G_N is the Newton constant, e the gauge coupling and Λ the cosmological constant (with our conventions $\Lambda < 0$ corresponds, in the absence of matter, to anti-de Sitter space). The field strength $F_{\mu\nu}^a$ ($a = 1, 2, 3$) is defined as

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + \varepsilon^{abc} A_\mu^b A_\nu^c \tag{7}$$

and the covariant derivative D_μ acting on the Higgs triplet H^a is given by

$$D_\mu H^a = \partial_\mu H^a + \varepsilon^{abc} A_\mu^b H^c \tag{8}$$

The equations of motion that follow from (1) are

$$\begin{aligned}
E_{\mu\nu} + \Lambda\,g_{\mu\nu} &= \kappa^2 \, (T_{\mu\nu}^{YM} + T_{\mu\nu}^H) \\
D_\rho D^\rho H^a &= \frac{\delta V(H)}{\delta H^a} \\
\frac{1}{e^2} D^\rho F_{\mu\rho}^a &= \varepsilon^{abc} \, (D_\mu H^b) \, H^c
\end{aligned} \tag{9}$$

3

where $E_{\mu\nu}$ is the Einstein tensor and the matter energy-momentum tensor $T_{\mu\nu}$,

$$T_{\mu\nu} = -2\frac{\delta S}{\delta g^{\mu\nu}} \quad , \tag{10}$$

is given by

$$
\begin{aligned}
T_{\mu\nu}^{YM} &= \frac{1}{e^2}F_{\mu\rho}^a F_\nu^{a\rho} + g_{\mu\nu}\ L_{YM} \\
T_{\mu\nu}^H &= D_\mu H^a\ D_\nu H^a + g_{\mu\nu}\ L_H
\end{aligned} \tag{11}
$$

The most general static spherically symmetric form for the metric in 3 spatial dimensions together with the t'Hooft-Polyakov-Julia-Zee ansatz for the gauge and Higgs fields in the usual vector notation reads [12]-[13]

$$
\begin{aligned}
g &= -\mu(x)\ A(x)^2\ d^2t + \mu(x)^{-1}\ d^2r + r^2\ d^2\Omega_2 \\
\vec{A} &= dt\ e\ H_0\ J(x)\ \check{e}_r - d\theta\ (1 - K(x))\ \check{e}_\varphi + d\varphi\ (1 - K(x))\ \sin\theta\ \check{e}_\theta \\
\vec{H} &= H_0\ H(x)\ \check{e}_r
\end{aligned} \tag{12}
$$

where we have introduced the dimensionless radial coordinate $x \equiv e\ H_0\ r$ and we denote the standard spherical unit vectors as $\check{e}_r, \check{e}_\varphi, \check{e}_\theta$. H_0 sets the mass scale ($[H_0] = m^1$).

Using this ansatz, the equations of motion take the form

$$
\begin{aligned}
(x\ \mu(x))' = 1 &+ 3\ \gamma_0^2\ x^2 - \kappa^2\ H_0^2\left(\mu(x)\ V_1 + V_2 + \frac{x^2}{2}\frac{J'(x)^2}{A(x)^2}\right. \\
&\left. + \frac{J(x)^2\ K(x)^2}{\mu(x)\ A(x)^2}\right)
\end{aligned}
$$

$$x\ A'(x) = \kappa^2\ H_0^2\left(V_1 + \frac{J(x)^2 K(x)^2}{\mu(x)^2 A(x)^2}\right)A(x)$$

$$
\begin{aligned}
(\mu(x)\ A(x)\ K'(x))' = A(x)\ K(x)&\left(\frac{K(x)^2 - 1}{x^2} + H(x)^2\right. \\
&\left. - \frac{J(x)^2}{\mu(x)\ A(x)^2}\right)
\end{aligned}
$$

$$\mu(x)\ A(x)\left(\frac{x^2\ J'(x)}{A(x)}\right)' = 2\ J(x)\ K(x)^2$$

$$\left(x^2\ \mu(x)\ A(x)\ H'(x)\right)' = A(x)\ H(x)\left(2\ K(x)^2 + \frac{\lambda}{e^2}\ x^2\ (H(x)^2 - 1)\right) \tag{13}$$

4

where, for convenience, we have defined the dimensionless parameter

$$\gamma_0^2 \equiv -\frac{\Lambda}{3\,e^2\,H_0{}^2} = \frac{1}{L^2\,e^2\,H_0{}^2} \qquad , \qquad L^2 \equiv -\frac{3}{\Lambda} \tag{14}$$

and

$$
\begin{aligned}
V_1 &= K'(x)^2 + \frac{x^2}{2} H'(x)^2 \\
V_2 &= \frac{(K(x)^2 - 1)^2}{2\,x^2} + \frac{\lambda}{4e^2}\,x^2\,(H(x)^2 - 1)^2
\end{aligned}
\tag{15}
$$

2.2 The AdS-Schwarzschild black hole background

In the $\kappa^2 \to 0$ limit the gravitational equations in system (13) decouple from the matter ones leading to a solution of the metric g of the form

$$g = -\mu_0(x)d^2t + \mu_0^{-1}(x)d^2r + r^2\,d^2\Omega_2 \tag{16}$$

with

$$\mu_0(x) = 1 + \gamma_0^2 x^2 - \frac{R^3}{\gamma_0 x} \tag{17}$$

Here R is a dimensionless parameter related to the black hole mass. For $R \neq 0$ the metric defined in (16) has an horizon at $x = x_h$, i.e.,

$$\mu_0(x_h) = 0 \tag{18}$$

with $\mu_0'(x_h) \neq 0$. Written in terms of the parameters of the model x_h reads

$$\gamma_0 x_h = \left(\sqrt{\frac{R^6}{4} + \frac{1}{27}} + \frac{R^3}{2}\right)^{1/3} - \left(\sqrt{\frac{R^6}{4} + \frac{1}{27}} - \frac{R^3}{2}\right)^{1/3} \tag{19}$$

In order to compute the black hole temperature one uses the standard recipe [16] leading to no conical singularity at the horizon after Wick rotation to the Euclidean time $\tau_E \equiv i\,t$. Hence, given (17) and imposing

$$\tau_E \sim \tau_E + \beta \tag{20}$$

one identifies the black hole temperature as

$$T \equiv \frac{1}{\beta} = \frac{|\mu_0'(x_h)|eH_0}{4\,\pi} = \frac{1}{4\pi L}\left(3\gamma_0 x_h + \frac{1}{\gamma_0 x_h}\right) \tag{21}$$

5

3 The dyon in the black hole background

3.1 Properties of the solution

From here on we shall consider, for simplicity, the BPS limit of the potential (5) which correspond to $\lambda/e^2 = 0$ with H_0 fixed. Taking the black hole metric (17) as a background we are left with the system

$$(\mu_0(x)\, K'(x))' = K(x)\left(\frac{K(x)^2 - 1}{x^2} + H(x)^2 - \frac{J(x)^2}{\mu_0(x)}\right) \tag{22}$$

$$\mu_0(x)\,\left(x^2\, J'(x)\right)' = 2\, J(x)\, K(x)^2 \tag{23}$$

$$\left(x^2\, \mu_0(x)\, H'(x)\right)' = 2H(x)K(x)^2 \tag{24}$$

We shall look for a solution to (22)-(24) regular at the horizon

$$K(x),\ H(x),\ J(x)/(x - x_h) \qquad \text{regular at } x_h \tag{25}$$

so that

$$
\begin{aligned}
H(x) &= h_0 + h_1(x - x_h) + \mathcal{O}[(x - x_h)^2] \\
K(x) &= k_0 + k_1(x - x_h) + \mathcal{O}[(x - x_h)^2] \\
J(x) &= j_h(x - x_h) + \mathcal{O}[(x - x_h)^2]
\end{aligned}
\tag{26}
$$

Concerning $x \to \infty$, the asymptotic behavior takes the form

$$|\vec{H}(x)| = H_0\left(1 - \frac{H_1}{x^3} + \frac{H_2}{x^5} + \frac{H_3}{x^{4+2\nu}} + \cdots\right) \tag{27}$$

$$K(x) = \frac{K_1}{x^{\nu+1}} + \frac{K_3}{x^{\nu+3}}, +\cdots \quad \nu \in \mathbb{R} \qquad \text{as x} \to \infty \tag{28}$$

$$J(x) = J_0 + \frac{J_1}{x} + \ldots \tag{29}$$

Such behavior is consistent with the equation of motion (22) provided that the following condition holds

$$\frac{1}{\gamma_0^2} = \nu(\nu + 1) \tag{30}$$

Equation (30) has two solutions,

$$\nu_\pm = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1 + \frac{4}{\gamma_0^2}} \tag{31}$$

6

Only the ν_+ root gives an acceptable asymptotic behavior for $K(x)$ so that

$$K(x) \sim \frac{K_1}{x^{\nu_+ + 1}} = \frac{K_1}{x^{\frac{1}{2}\left(1 + \sqrt{1 + \frac{4}{\gamma_0^2}}\right)}} \qquad \text{as } x \to \infty \qquad (32)$$

Being the $SU(2)$ gauge symmetry of the Lagrangian spontaneously broken by the expectation value of the scalar field, we shall define the field strength associated with the surviving $U(1)$ symmetry as in [15]

$$\mathcal{F}_{\mu\nu}^{U(1)} \equiv \frac{H^a}{H_0} F_{\mu\nu}^a \qquad (33)$$

From this expression one finds the U(1) magnetic and electric fields

$$B^i = \frac{1}{2} \frac{\varepsilon^{ijk}}{\sqrt{g^{(3)}}} \mathcal{F}_{jk}^{U(1)} , \quad E_i = \mathcal{F}_{i0}^{U(1)} \qquad (34)$$

and the magnetic and electric charges

$$Q_m = \int_V d^3x \sqrt{g^{(3)}} \nabla_i^{(3)} B^i \qquad (35)$$

$$Q_e = \int_V d^3x \sqrt{g^{(3)}} \nabla_i^{(3)} E^i \qquad (36)$$

where $\nabla_i^{(3)}$ is the 3-dimensional covariant derivative. In order to compute Q_m one just needs the asymptotic value of $\mathcal{F}_{ij}^{U(1)}$ since eqs.(34)-(35) imply

$$Q_m = \frac{1}{2} \int_{\partial V = S^2} \varepsilon^{ijk} \mathcal{F}_{jk}^{U(1)} dS_i \qquad (37)$$

Now, from the asymptotic conditions (29) one has

$$\varepsilon^{ijk} \mathcal{F}_{jk}^{U(1)} \sim \frac{1}{x^2} \check{e}_r \delta^{ir} \qquad \text{for } x \to \infty \qquad (38)$$

and then

$$Q_m = -\frac{1}{e} \int_{S^2} d\Omega = -\frac{4\pi}{e} \qquad (39)$$

As expected, the spherically symmetric solution has a quantized $n = -1$ magnetic charge (in units of $4\pi/e$).

Concerning the electric charge, one can see from the asymptotic behavior of J, eq.(29), that J_0 sets the charge scale and J_1 determines its value. One can write

$$Q_e = \frac{1}{e}\mathcal{Q}(e^2/\lambda, J_0) = -\frac{4\pi}{e}J_1 \tag{40}$$

with $\mathcal{Q}$ a dimensionless function to be determined numerically. As for the Julia-Zee flat space dyon, the electric charge is not quantized at the classical level. As we shall see, J_0 and J_1 will be identified, through holography, with the chemical potential and charge density of the strongly coupled system respectively.

Let us now find the expression for the dyon action which will be connected with the free energy of the field theory defined on the boundary. Inserting ansatz (12) in eqs.(3)-(4) one obtains for the Yang-Mills and Higgs Lagrangians

$$
\begin{aligned}
L_{YM} &= -\frac{e^2 H_0^4}{2}\left(-\frac{J'^2}{A^2} - 2\frac{J^2 K^2}{x^2\mu A^2} + 2\frac{\mu K'^2}{x^2} + \frac{(K^2-1)^2}{x^4}\right) \\
L_H &= -\frac{e^2 H_0^4}{2}\left(\mu H'^2 + 2\frac{K^2 H^2}{x^2}\right)
\end{aligned} \tag{41}
$$

Then, by analytic continuation, the Euclidean action takes the form

$$S_E = -\int_0^\beta d\tau_E \int_V d^3x \sqrt{g}\,(L_{YM} + L_H) \tag{42}$$

3.2   From $\mathbf{S^2}$ to $\mathbb{R}^2$

As stated in the introduction, our aim is to calculate the thermodynamics of the strongly interacting field theory defined on the boundary ∂M using the dual classical description of the gravity system governed by the action (1) in the 4 dimensional manifold M; now, according to (16), $\partial M = S^2 \times S^1$. From the condensed matter viewpoint, it is more relevant to study the finite temperature field theory defined on the plane $\mathbb{R}^2$ and not on the sphere S^2. To this end, one can follow the approach in [17] and proceed to a change coordinates $(t, x, \theta, \varphi) \to (\tau, y, x^1, x^2)$ or, equivalently, $(t, x, \theta, \varphi) \to (\tau, y, \rho, \varphi)$,

$$
\begin{aligned}
\tau &= \frac{R\,t}{\gamma_0} \\
y &= \frac{\gamma_0}{R}\,x
\end{aligned}
$$

$$x^1 = 2\,R\,\tan\frac{\theta}{2}\,\cos\varphi = \rho\,\cos\varphi$$
$$x^2 = 2\,R\,\tan\frac{\theta}{2}\,\sin\varphi = \rho\,\sin\varphi \qquad (43)$$

together with the field redefinitions

$$\begin{aligned}
\tilde{A}(y) &= A(x) \\
\tilde{\xi}(y) &= R^{-2}\,\xi(x) \\
\tilde{H}(y) &= H(x) \\
\tilde{K}(y) &= R^{-1}\,K(x) \\
\tilde{J}(y) &= R^{-1}\,J(x) \qquad (44)
\end{aligned}$$

where R was introduced in (17). After this change, ansatz (12) becomes

$$\begin{aligned}
g &= -\tilde{f}_R(y)\,\tilde{A}(y)^2\,d^2\tau + L^2\,\frac{d^2 y}{\tilde{f}_R(y)} + L^2\,y^2\frac{d^2\vec{x}}{(1+\frac{\rho^2}{4\,R^2})^2} \\
\vec{A} &= d\tau\,e\,H_0\,\tilde{J}(y)\,\left(\frac{4\,R\,\rho}{\rho^2+4\,R^2}\,\check{e}_\rho + \frac{4\,R^2-\rho^2}{\rho^2+4\,R^2}\,\check{e}_3\right) \\
&+ \frac{4\,R^2}{\rho^2+4\,R^2}\,\left(\frac{1}{R}-\tilde{K}(y)\right)\,\left[\left(\frac{4\,R^2-\rho^2}{\rho^2+4\,R^2}\,\check{e}_\rho - \frac{4\,R\,\rho}{\rho^2+4\,R^2}\,\check{e}_3\right)\,\rho\,d\varphi\right. \\
&- \left.\check{e}_\varphi\,d\rho\right] \\
\vec{H} &= H_0\,\tilde{H}(y)\,\left(\frac{4\,R\,\rho}{\rho^2+4\,R^2}\,\check{e}_\rho + \frac{4\,R^2-\rho^2}{\rho^2+4\,R^2}\,\check{e}_3\right) \qquad (45)
\end{aligned}$$

where
$$\tilde{f}_R(y) = f_R(y) + \tilde{\xi}(y) \qquad , \qquad f_R(y) \equiv \frac{1}{R^2} + y^2 - \frac{1}{y} \qquad (46)$$

and the equations of motion for the Einstein-Yang Mills-Higgs system, eqs.(13) take the form,

$$\left(y\,\tilde{\xi}(y)\right)' = -\kappa^2\,H_0^2\,\left(\tilde{f}_R(y)\,\tilde{V}_1 + \tilde{V}_2 + \frac{y^2}{2}\,\frac{\tilde{J}'(y)^2}{\tilde{A}(y)^2} + \frac{\tilde{J}(y)^2\,\tilde{K}(y)^2}{\tilde{f}_R(y)\,\tilde{A}(y)^2}\right)$$

$$y\,\tilde{A}'(y) = \kappa^2\,H_0^2\,\left(\tilde{V}_1 + \frac{\tilde{J}(y)^2\,\tilde{K}(y)^2}{\tilde{f}_R(y)^2\,\tilde{A}(y)^2}\right)\,\tilde{A}(y) \qquad (47)$$

$$\left(\tilde{f}_R(y)\,\tilde{A}(y)\,\tilde{K}'(y)\right)' = \tilde{A}(y)\,\tilde{K}(y)\,\left(\frac{\tilde{K}(y)^2 - \frac{1}{R^2}}{y^2} + \frac{1}{\gamma_0^2}\,\tilde{H}(y)^2\right.$$

9

$$\left(y^2 \, \tilde{f}_R(y) \, \tilde{A}(y) \, \tilde{H}'(y) \right)' = 2\tilde{A}(y) \, \tilde{H}(y) \, \tilde{K}(y)^2$$

$$\tilde{f}_R(y) \, \tilde{A}(y) \, \left(\frac{y^2 \tilde{J}'(y)}{\tilde{A}(y)} \right)' = 2 \, \tilde{J}(y) \, \tilde{K}(y)^2 \qquad (48)$$

where

$$\tilde{V}_1 = \gamma_0{}^2 \, \tilde{K}'(y)^2 + \frac{y^2}{2} \, \tilde{H}'(y)^2$$

$$\tilde{V}_2 = \gamma_0{}^2 \, \frac{(\tilde{K}(y)^2 - \frac{1}{R^2})^2}{2 \, y^2} \qquad (49)$$

As explained above, our aim is to consider the case in which the gravitational equations decouple from the matter ($\kappa^2 \to 0$) leading to a background metric of the form (16). The relevant equations of motion for the gauge and Higgs field in the $R \to \infty$ limit are then

$$\left(\tilde{f}_\infty(y) \, \tilde{K}'(y) \right)' = \tilde{K}(y) \left(\frac{\tilde{K}(y)^2}{y^2} + \frac{1}{\gamma_0{}^2} \, \tilde{H}(y)^2 - \frac{1}{\gamma_0{}^2} \, \frac{\tilde{J}(y)^2}{\tilde{f}_\infty(y)} \right) \qquad (50)$$

$$\left(y^2 \, \tilde{f}_\infty(y) \, \tilde{H}'(y) \right)' = 2 \, \tilde{H}(y) \, \tilde{K}(y)^2 \qquad (51)$$

$$\tilde{f}_\infty(y) \left(y^2 \tilde{J}'(y) \right)' = 2 \, \tilde{J}(y) \, \tilde{K}(y)^2 \qquad (52)$$

$$\tilde{f}_\infty(y) = y^2 - \frac{1}{y}$$

Let us end this section by noting that the black hole temperature resulting from rescaling (21) and taking the $R \to \infty$ limit takes the form

$$\hat{T} \equiv \frac{T}{e \, H_0} = \frac{3\gamma_0}{4\pi} \qquad (53)$$

4 Holographic correspondence

According to the AdS/CFT correspondence [1]-[3], properties of the dual 3 dimensional field theory defined on the boundary can be read from the behavior of the solution of the system in the bulk. In this approach temperature $\hat{T}$ in (53) corresponds to the temperature of the $d = 3$ system.

10

Let us first consider the vacuum expectation value in the $d = 3$ field theory for the scalar operator $\mathcal{O}_H$, dual to the field H defined on the bulk. It follows from the identification $\langle \mathcal{O}_H \rangle \sim H_1$ with H_1 defined in eq.(27).

The asymptotic behavior of the A_0 component of the gauge field allows one to identify the chemical potential μ and the charge density ρ of the 3 dimensional system. Indeed, given the ansatz for the A_0 component of the gauge field, eq.(12), the $U(1)$ electric field as defined in (34) reads

$$E_i = \mathcal{F}_{i0}^{U(1)} = \frac{H^a(x)}{H_0} F_{i0}^a = eH_0 H(x)\partial_i J(x) \tag{54}$$

(the $\vec{H} \cdot \vec{A}_0 \wedge \vec{A}_i$ contribution coming from the field strength commutator vanishes because $\vec{H} \parallel \vec{A}_0$).

Since $\lim_{x \to \infty} H(x) = 1$, the electric field at infinity is given by

$$\lim_{x \to \infty} E_i = eH_0 \lim_{x \to \infty} \partial_i J(x) \tag{55}$$

We have now to expand $J(x)$ for large x and look for the $1/x$ term which is the one leading to a nontrivial electric flux (i.e., leading to the dyon charge). The corresponding coefficient defines the charge density:

$$\rho = -LH_0 \tilde{J}_1 \tag{56}$$

Here and in what follow we use the coefficients of the asymptotic expansions defined in $\mathbb{R}^2 \times S^1$. Following [18] we shall define a normal component ρ_n by analyzing the expansion for $\tilde{J}(y)$ at the horizon using eq.(26),

$$\rho_n = \tilde{j}_h \tag{57}$$

and a superconducting charge density in the form

$$\rho_s = \rho - \rho_n \tag{58}$$

The constant value $\tilde{J}_0$ that A_τ takes at the boundary at infinity is related to the chemical potential through the formula

$$\mu = H_0 \tilde{J}_0 \tag{59}$$

11

5 Numerical Analysis

We shall restrict the numerical calculations to the case in which the boundary is $S^1 \times \mathbb{R}^2$. We shall solve system (50)-(52) using the relaxation method [20] which determines the solution starting from an initial guess and improving it iteratively. The natural initial guess is the Prasad-Sommerfield solution [21] in flat space. Following this procedure, we have found regular solutions for different values of γ_0 i.e. of the temperature $\hat{T}$.

Figures 1 and 2 show a representative dyon solution that exists starting at the horizon. One can see that the scalar rapidly attains its symmetry breaking constant value while the magnetic and electric fields (associated to K and J respectively) concentrate in a spherical shell starting at the horizon. Figure 3 shows a remarkable property of the coupled system of non linear differential equations, namely the existence of several solutions satisfying the appropriate boundary conditions, each one corresponding to a different value of charge density ρ. This phenomenon was already encountered in [8] for the case of a $U(1)$ gauge field coupled to a complex scalar in an AdS$_4$-Schwarzschild background. As it also happens in the $U(1)$ case, we have found that for increasing values of ρ the solutions are distinguished by an increasing number of nodes n. Now, since evaluation of the free energy shows that it increases with the number of nodes (see below), we conclude that $n > 1$ solutions are thermodynamically unfavorable and therefore we shall concentrate hecenforth on the zero node solution.
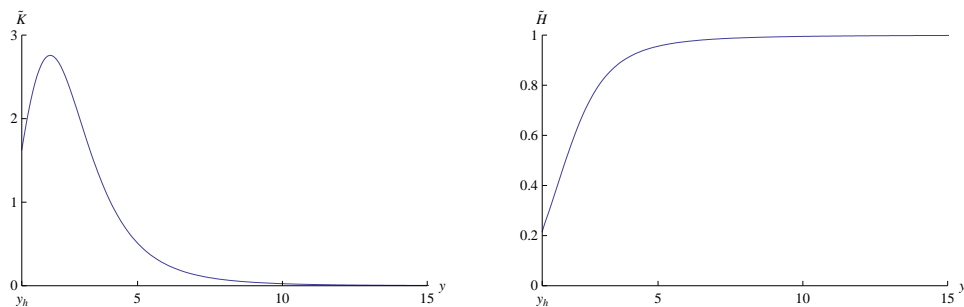

Figure 1: The solution for $\tilde{K}$ and $\tilde{H}$ as a function of y or $\tilde{J}_0 = 6$ and $\gamma_0 = 0.2$ ($\hat{T} = 3\gamma_0/(4\pi)$). The solution exists starting at the horizon $y = y_h = 1$.
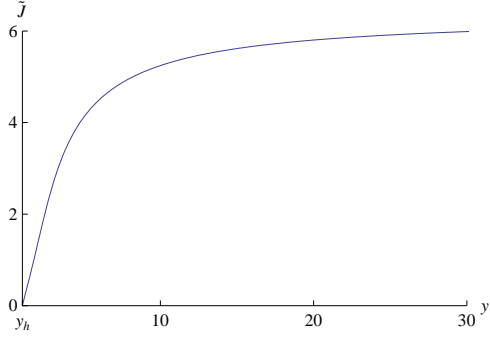
12

Figure 2: The solution for $\tilde{J}$ as a function of y for $\tilde{J}_0 = 6$ and $\gamma_0 = 0.2$ $(\hat{T} = 3\gamma_0/(4\pi))$.
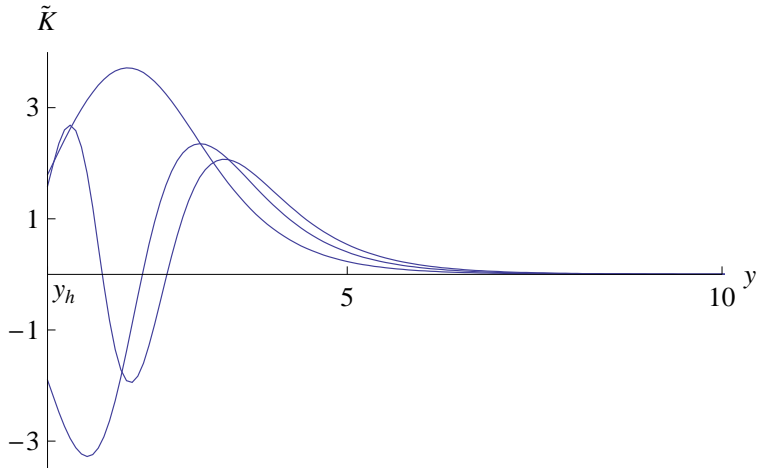

Figure 3: Different solutions for $\tilde{K}$ corresponding to different values of the charge density ρ. For increasing values of ρ the solutions are distinguished by the increasing number of nodes.

The behavior of the condensate is shown in figure 4 where the asymptotic coefficient $H_1 \sim \langle O_H \rangle$ in the asymptotic expansion (27) of the Higgs field is plotted as a function of the temperature (normalized in units of eH_0). In figures 5 and 6 we plot the coefficient $\tilde{K}_1$ in the asymptotic expansion of function $\tilde{K}$, while figure 7 shows the fraction ρ_s/ρ, as defined in eqs.(56)-(58), related to the behavior of the electric field at the horizon and at infinity. From these figures we conclude that a finite temperature continuous symmetry breaking transition takes place so that the system condenses below a critical

13

temperature $\hat{T}_c$. By fitting the curves we see that near T_c one has a typical second order phase transition with power behavior of the form

$$\begin{aligned}
\tilde{H}_1 &\propto (\hat{T}_c - \hat{T}) \\
\tilde{K}_1 &\propto (\hat{T}_c - \hat{T})^{1/2} \qquad \text{as } \hat{T} \to \hat{T}_c \\
\frac{\rho_s}{\rho} &\propto (\hat{T}_c - \hat{T})
\end{aligned} \qquad (60)$$

Notice that $\tilde{K}_1$ has a critical exponent $1/2$ typical of second order transitions.

One can see in figure 4 that at low temperature $\hat{H}_1$ appears to diverge. However, one should take into account that when the condensate becomes very large, the backreaction on the metric can no longer be neglected as we did by choosing a fixed Schwarzschild-AdS background metric.
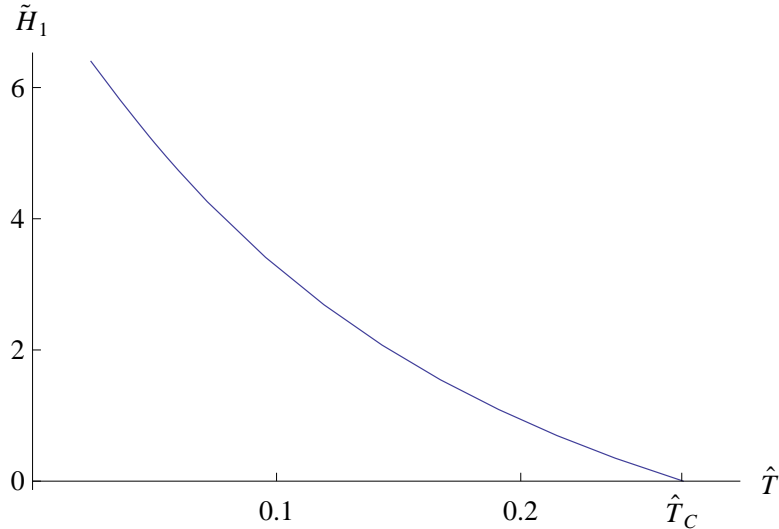

Figure 4: The $\tilde{H}_1$ coefficient in the Higgs field asymptotic expansion as a function of the temperature. We have taken $\tilde{J}_0 = 6$ and for this value the critical temperature is $\hat{T}_c = 0.26649....$
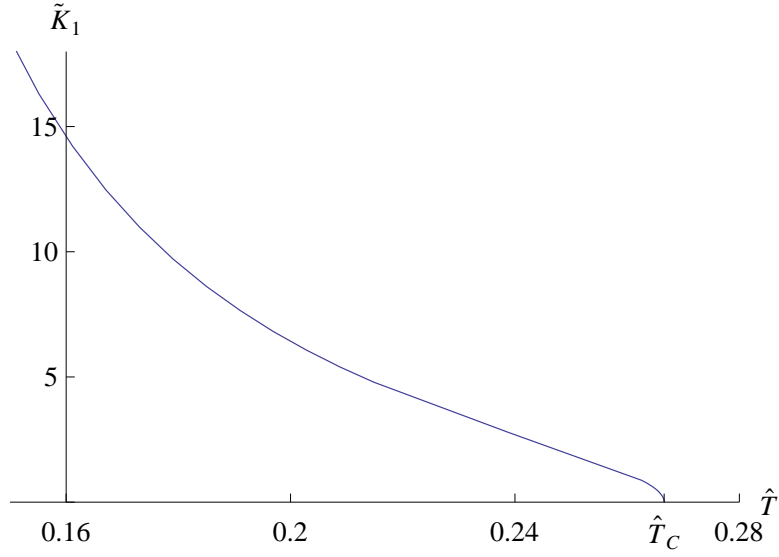
14

Figure 5: The $\tilde{K}_1$ coefficient in the gauge field asymptotic expansion as a function of the temperature for $\tilde{J}_0 = 6$.
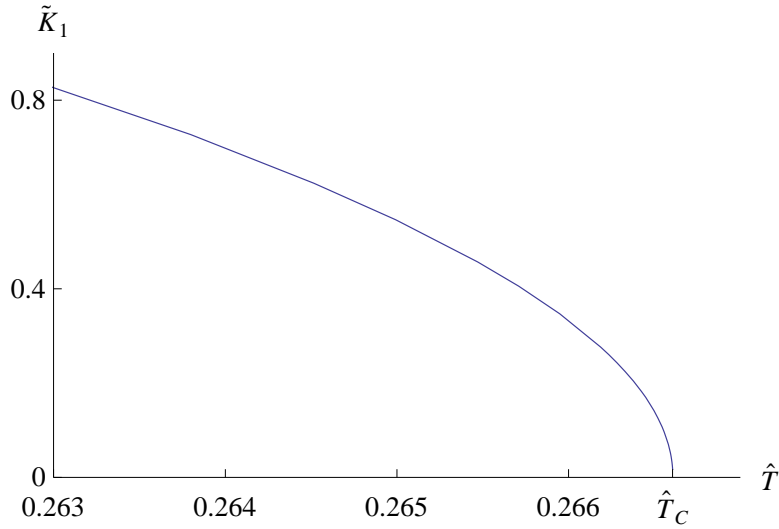

Figure 6: A zoomed view of $\tilde{K}_1$ near the critical point. One can see that its behavior near $T = T_c$ corresponds to $\sqrt{\hat{T}_c - \hat{T}}$.
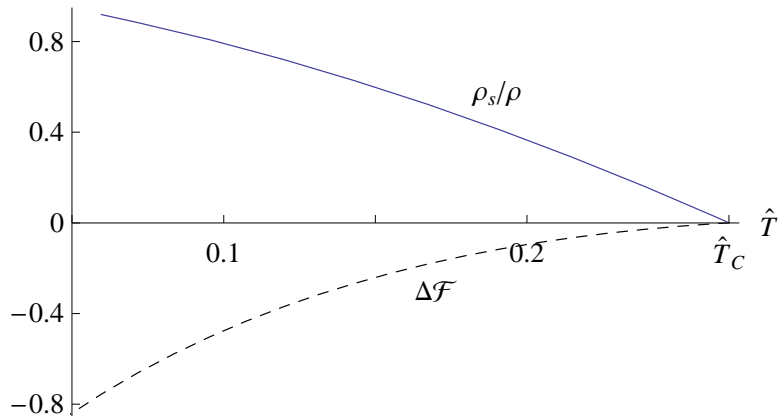
Figure 7: The solid line represents the fraction ρ_s/ρ of the superconducting and the total charge densities. The dashed line the free energy difference between the condensed and the uncondensed phases. At $\hat{T} = \hat{T}_c$, ρ_s/ρ vanishes linearly.

The free energy $\mathcal{F}$ is given by the on-shell action,

$$\mathcal{F} = TS_E|_{on \ shell} \tag{61}$$

Using the Euclidean action as given in eq.(42) we have numerically computed $\mathcal{F}$ for the dyon solution and compared it with the free energy in the uncondensed phase which corresponds to the solution $\tilde{H}(y) = 1, \tilde{K}(y) = 0, \tilde{J}(y) = \tilde{J}_0(1 - 1/y)$. We plot in figure 7 the free energy density difference $\Delta\mathcal{F}$, normalized in units of $H_0^2/(2L)$, which can be seen to be continuous at $\hat{T} = \hat{T}_c$.

6   Summary and discussion

We have applied the AdS/CFT correspondence to describe the strong coupling regime of a $2 + 1$ quantum field theory which undergoes a phase transition exhibiting the condensation of a composite charge operator below a critical temperature T_c. The dual gravity theory consists of an $SU(2)$ Yang-Mills-Higgs model in a Schwarzschild-AdS$_4$ black hole with temperature T.

Breaking the gauge symmetry down to $U(1)$ we started by constructing a spherically symmetric soliton solution in the bulk, having both magnetic and electric charge. By choosing appropriate conditions at the horizon and

16

at infinity, we found a family of dyon solutions leading to the existence of a condensate, below a critical temperature, in the dual field theory on the boundary which, after an appropriate scaling, becomes $\mathbb{R}^2 \times S^1$.

By calculating the free energy $\mathcal{F}$ we have seen that the condensed phase is thermodynamically favored and we were able to determine which one of the dyon family solutions had the lowest $\mathcal{F}$-value. We have also calculated the power behavior of the condensate, typical of a second order phase transition.

There are many issues that we have not discussed in our work and that deserve a thorough study. To begin with, one should consider the applicability of our results to describe the behavior of an actual condensed matter system. Also, since we have found on the gravity side classical solutions with quantized magnetic flux, we expect that the issue of confinement through the appearence of chromoelectric flux tubes in the dual strongly coupled QFT could take place. In this respect, it should be of interest to consider monopole solutions in Schwarzschild-AdS$_5$ black hole backgrounds so that the dual theory is defined in three spatial dimensions. We hope to discuss theses issues in a future publication.

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Holographic phase transition from dyons in an AdS black hole background

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Abstract

We construct a dyon solution for a Yang-Mills-Higgs theory in a 4 dimensional Schwarzschild-anti-de Sitter black hole background with temperature T. We then apply the AdS/CFT correspondence to describe the strong coupling regime of a $2 + 1$ quantum field theory which undergoes a phase transition exhibiting the condensation of a composite charge operator below a critical temperature T_c.

1 Introduction

It has recently been observed that the AdS/CFT correspondence [1]-[3] provides a powerful tool to study quantum critical dynamics in condensed matter models for superconductivity, superfluidity and Hall effect [4]-[10]. In this way, using the dual classical gravity description, correlation functions in strongly interacting systems can be calculated and the relevant physical

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properties can be determined in a relatively simple way, which has motivated an intense activity in the field (some works related to the present approach are listed in [11]).

Temperature is introduced in the gravity dual through the presence of a Schwarzschild-AdS black hole, either added as a background or resulting from back reaction. Apart from gauge fields, the models referred above include minimally coupled scalar matter fields. In the Abelian case the ansatz for gauge fields included an electrostatic potential A_0 which is spatially independent at the boundary as also is the case for the scalar. Spatially dependent droplet and vortex solutions have been also studied in [8]-[9] and in the non-Abelian case non-trivial gauge-field components were considered [7],[10]. In general, such classical solutions in the bulk exist below a certain critical temperature leading to thermodynamically favored phases in the the strongly coupled quantum field theory on the boundary.

Following the ideas described above, we consider in the present paper a model not yet studied: a dyon solutions for Yang-Mills-Higgs theory in a 4 dimensional Schwarzschild-anti-de Sitter black-hole background metric. We shall be guided by our previous work on monopole and dyon solutions in AdS_4 space [12]-[13] where, due to the absence of singularities, the spherically symmetric solutions were constructed in the whole radial domain $\mathbb{R}^+$, while in the present case the radial variable extend from the black hole horizon to infinity. The black hole temperature, determined by the horizon, will become the temperature of the field theory defined on the boundary. The matter field will lead to the non-Abelian gauge symmetry breaking condensate while the field strength associated to the surviving $U(1)$ symmetry will correspond to that of a dyon with electric and magnetic charges.

The paper is organized as follows. In section 2 we present the gravity-Yang-Mills-Higgs model and specialize to the case in which the gravitational equations decouple from matter leading to an AdS-Schwarzschild black hole background for the gauge-matter system. Then in section 3 we analyze the properties of the dyon solution considering both the cases of an $S^2 \times S^1$ and $\mathbb{R}^2 \times S^1$ boundaries. In section 4 we proceed to describe the holography calculation of relevant quantities in the dual field theory defined on the boundary (taken as $\mathbb{R}^2 \times S^1$) and present our numerical results. We end with a summary and discussion in section 5.

2 The gravity-Yang-Mills-Higgs system

2.1 The model

We consider a gravity-Yang-Mills-Higgs system with gauge group $SU(2)$ and the scalar field in the adjoint representation, in a 4 dimensional space-time with Minkowski signature $(-, +, +, +)$. The action takes the form

$$S = S_G + S_{YM} + S_H = \int d^4x \sqrt{|g|} \, (L_G + L_{YM} + L_H) \tag{1}$$

with

$$L_G = \frac{1}{2\,\kappa^2} \, (R - 2\,\Lambda) \tag{2}$$

$$L_{YM} = -\frac{1}{4e^2} \, F^a_{\mu\nu} F^{a\,\mu\nu} \tag{3}$$

$$L_H = -\frac{1}{2} D_\mu H^a \, D^\mu H^a - V(H) \tag{4}$$

$$V(H) = \frac{\lambda}{4} \, (H^a H^a - H_0{}^2)^2 \tag{5}$$

$$\kappa^2 \equiv 8\,\pi\,G_N \tag{6}$$

Here G_N is the Newton constant, e the gauge coupling and Λ the cosmological constant (with our conventions $\Lambda < 0$ corresponds, in the absence of matter, to anti-de Sitter space). The field strength $F^a_{\mu\nu}$ ($a = 1, 2, 3$) is defined as

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \varepsilon^{abc} A^b_\mu A^c_\nu \tag{7}$$

and the covariant derivative D_μ acting on the Higgs triplet H^a is given by

$$D_\mu H^a = \partial_\mu H^a + \varepsilon^{abc} A^b_\mu H^c \tag{8}$$

The equations of motion that follow from (1) are

$$
\begin{aligned}
E_{\mu\nu} + \Lambda \, g_{\mu\nu} &= \kappa^2 \, (T^{YM}_{\mu\nu} + T^H_{\mu\nu}) \\
D_\rho D^\rho H^a &= \frac{\delta V(H)}{\delta H^a} \\
\frac{1}{e^2} D^\rho F^a_{\mu\rho} &= \varepsilon^{abc} \, (D_\mu H^b) \, H^c
\end{aligned}
\tag{9}
$$

where $E_{\mu\nu}$ is the Einstein tensor and the matter energy-momentum tensor $T_{\mu\nu}$,

$$T_{\mu\nu} = -2\frac{\delta S}{\delta g^{\mu\nu}} \quad , \tag{10}$$

is given by

$$\begin{aligned}
T_{\mu\nu}^{YM} &= \frac{1}{e^2}F_{\mu\rho}^a F_\nu^{a\rho} + g_{\mu\nu} \, L_{YM} \\
T_{\mu\nu}^{H} &= D_\mu H^a \, D_\nu H^a + g_{\mu\nu} \, L_H
\end{aligned} \tag{11}$$

The most general static spherically symmetric form for the metric in 3 spatial dimensions together with the t'Hooft-Polyakov-Julia-Zee ansatz for the gauge and Higgs fields in the usual vector notation reads [12]-[13]

$$\begin{aligned}
g &= -\mu(x) \, A(x)^2 \, d^2t + \mu(x)^{-1} \, d^2r + r^2 \, d^2\Omega_2 \\
\vec{A} &= dt \, e \, H_0 \, J(x) \, \check{e}_r - d\theta \, (1 - K(x)) \, \check{e}_\varphi + d\varphi \, (1 - K(x)) \, \sin\theta \, \check{e}_\theta \\
\vec{H} &= H_0 \, H(x) \, \check{e}_r
\end{aligned} \tag{12}$$

where we have introduced the dimensionless radial coordinate $x \equiv e \, H_0 \, r$ and we denote the standard spherical unit vectors as $\check{e}_r, \check{e}_\varphi, \check{e}_\theta$. H_0 sets the mass scale ($[H_0] = m^1$).

Using this ansatz, the equations of motion take the form

$$(x \, \mu(x))' = 1 + 3 \, \gamma_0^2 \, x^2 - \kappa^2 \, H_0^2 \left(\mu(x) \, V_1 + V_2 + \frac{x^2}{2} \frac{J'(x)^2}{A(x)^2} \right.$$

$$\left. + \frac{J(x)^2 \, K(x)^2}{\mu(x) \, A(x)^2} \right)$$

$$x \, A'(x) = \kappa^2 \, H_0^2 \left(V_1 + \frac{J(x)^2 K(x)^2}{\mu(x)^2 A(x)^2} \right) A(x)$$

$$(\mu(x) \, A(x) \, K'(x))' = A(x) \, K(x) \left(\frac{K(x)^2 - 1}{x^2} + H(x)^2 \right.$$

$$\left. - \frac{J(x)^2}{\mu(x) \, A(x)^2} \right)$$

$$\mu(x) \, A(x) \left(\frac{x^2 \, J'(x)}{A(x)} \right)' = 2 \, J(x) \, K(x)^2$$

$$\left(x^2 \, \mu(x) \, A(x) \, H'(x) \right)' = A(x) \, H(x) \left(2 \, K(x)^2 + \frac{\lambda}{e^2} \, x^2 \, (H(x)^2 - 1) \right) \tag{13}$$

4

where, for convenience, we have defined the dimensionless parameter

$$\gamma_0^2 \equiv -\frac{\Lambda}{3\,e^2\,H_0{}^2} = \frac{1}{L^2\,e^2\,H_0{}^2} \qquad , \qquad L^2 \equiv -\frac{3}{\Lambda} \tag{14}$$

and

$$
\begin{aligned}
V_1 &= K'(x)^2 + \frac{x^2}{2}H'(x)^2 \\
V_2 &= \frac{(K(x)^2 - 1)^2}{2\,x^2} + \frac{\lambda}{4e^2}\,x^2\,(H(x)^2 - 1)^2
\end{aligned} \tag{15}
$$

2.2 The AdS-Schwarzschild black hole background

In the $\kappa^2 \to 0$ limit the gravitational equations in system (13) decouple from the matter ones leading to a solution of the metric g of the form

$$g = -\mu_0(x)d^2t + \mu_0^{-1}(x)d^2r + r^2\,d^2\Omega_2 \tag{16}$$

with

$$\mu_0(x) = 1 + \gamma_0^2 x^2 - \frac{R^3}{\gamma_0 x} \tag{17}$$

Here R is a dimensionless parameter related to the black hole mass. For $R \neq 0$ the metric defined in (16) has an horizon at $x = x_h$, i.e.,

$$\mu_0(x_h) = 0 \tag{18}$$

with $\mu_0'(x_h) \neq 0$. Written in terms of the parameters of the model x_h reads

$$\gamma_0 x_h = \left(\sqrt{\frac{R^6}{4} + \frac{1}{27}} + \frac{R^3}{2}\right)^{1/3} - \left(\sqrt{\frac{R^6}{4} + \frac{1}{27}} - \frac{R^3}{2}\right)^{1/3} \tag{19}$$

In order to compute the black hole temperature one uses the standard recipe [16] leading to no conical singularity at the horizon after Wick rotation to the Euclidean time $\tau_E \equiv i\,t$. Hence, given (17) and imposing

$$\tau_E \sim \tau_E + \beta \tag{20}$$

one identifies the black hole temperature as

$$T \equiv \frac{1}{\beta} = \frac{|\mu_0'(x_h)|eH_0}{4\,\pi} \tag{21}$$

3 The dyon in the black hole background

3.1 Properties of the solution

From here on we shall consider, for simplicity, the BPS limit of the potential (5) which correspond to $\lambda/e^2 = 0$ with H_0 fixed. Taking the black hole metric (17) as a background we are left with the system

$$(\mu_0(x)\, K'(x))' = K(x) \left(\frac{K(x)^2 - 1}{x^2} + H(x)^2 - \frac{J(x)^2}{\mu_0(x)} \right) \tag{22}$$

$$\mu_0(x)\, \left(x^2\, J'(x)\right)' = 2\, J(x)\, K(x)^2 \tag{23}$$

$$\left(x^2\, \mu_0(x)\, H'(x)\right)' = 2H(x)K(x)^2 \tag{24}$$

We shall look for a solution to (22)-(24) which is regular at the horizon

$$K(x),\ H(x),\ J(x)/(x - x_h) \qquad \text{regular at } x_h \tag{25}$$

so that

$$
\begin{aligned}
H(x) &= h_0 + h_1(x - x_h) + \mathcal{O}[(x - x_h)^2] \\
K(x) &= k_0 + k_1(x - x_h) + \mathcal{O}[(x - x_h)^2] \\
J(x) &= j_h(x - x_h) + \mathcal{O}[(x - x_h)^2]
\end{aligned} \tag{26}
$$

Concerning $x \to \infty$, the asymptotic behavior takes the form

$$|\vec{H}(x)| = H_0 \left(1 - \frac{H_1}{x^3} + \frac{H_2}{x^5} + \frac{H_3}{x^{4+2\nu}} + \cdots \right) \tag{27}$$

$$K(x) = \frac{K_1}{x^{\nu+1}} + \frac{K_3}{x^{\nu+3}}, + \cdots \qquad \nu \in \mathbb{R} \qquad \text{as x} \to \infty \tag{28}$$

$$J(x) = J_0 + \frac{J_1}{x} + \ldots \tag{29}$$

Such behavior is consistent with the equation of motion (22) provided that the following condition holds

$$\frac{1}{\gamma_0^2} = \nu(\nu + 1) \tag{30}$$

Equation (30) has two solutions,

$$\nu_\pm = -\frac{1}{2} \pm \frac{1}{2} \sqrt{1 + \frac{4}{\gamma_0^2}} \tag{31}$$

Only the ν_+ root gives an acceptable asymptotic behavior for $K(x)$ so that

$$K(x) \sim \frac{k_\nu}{x^{\nu_+ + 1}} = \frac{k_\nu}{x^{\frac{1}{2}\left(1 + \sqrt{1 + \frac{4}{\gamma_0^2}}\right)}} \qquad \text{as } x \to \infty \qquad (32)$$

Being the $SU(2)$ gauge symmetry of the Lagrangian spontaneously broken by the expectation value of the scalar field, we shall define the field strength associated with the surviving $U(1)$ symmetry as in [15]

$$\mathcal{F}_{\mu\nu}^{U(1)} \equiv \frac{H^a}{H_0} F_{\mu\nu}^a \qquad (33)$$

From this expression one finds the U(1) magnetic and electric fields

$$B^i = \frac{1}{2} \frac{\varepsilon^{ijk}}{\sqrt{g^{(3)}}} \mathcal{F}_{jk}^{U(1)}, \quad E_i = \mathcal{F}_{i0}^{U(1)} \qquad (34)$$

and the magnetic and electric charges

$$Q_m = \int_V d^3x \sqrt{g^{(3)}} \nabla_i^{(3)} B^i \qquad (35)$$

$$Q_e = \int_V d^3x \sqrt{g^{(3)}} \nabla_i^{(3)} E^i \qquad (36)$$

where $\nabla_i^{(3)}$ is the 3-dimensional covariant derivative. In order to compute Q_m one just needs the asymptotic value of $\mathcal{F}_{ij}^{U(1)}$ since eqs.(34)-(35) imply

$$Q_m = \frac{1}{2} \int_{\partial V = S^2} \varepsilon^{ijk} \mathcal{F}_{jk}^{U(1)} dS_i \qquad (37)$$

Now, from the asymptotic conditions (29) one has

$$\varepsilon^{ijk} \mathcal{F}_{jk}^{U(1)} \sim \frac{1}{x^2} \check{e}_r \delta^{ir} \qquad \text{for } x \to \infty \qquad (38)$$

and then

$$Q_m = -\frac{1}{e} \int_{S^2} d\Omega = -\frac{4\pi}{e} \qquad (39)$$

As expected, the spherically symmetric solution has a quantized $n = -1$ magnetic charge (in units of $4\pi/e$).

7

Concerning the electric charge, one can see from the asymptotic behavior of J, eq.(29), that J_0 which sets the charge scale and J_1, determines its value. One can write

$$Q_e = \frac{1}{e}\mathcal{Q}(e^2/\lambda, J_0) \tag{40}$$

with $\mathcal{Q}$ a dimensionless function to be determined numerically. As for the Julia-Zee flat space dyon, the electric charge is not quantized at the classical level. As we shall see, J_0 and J_1 will be identified, through holography, with the chemical potential and charge density of the strongly coupled system respectively.

Let us now find the expression for the dyon action which will be connected with the free energy of the field theory defined on the boundary. Inserting ansatz (12) in eqs.(3)-(4) one obtains for the Yang-Mills and Higgs Lagrangians

$$
\begin{aligned}
L_{YM} &= -\frac{e^2 H_0^4}{2}\left(-\frac{J'^2}{A^2} - 2\frac{J^2 K^2}{x^2 \mu A^2} + 2\frac{\mu K'^2}{x^2} + \frac{(K^2-1)^2}{x^4}\right) \\
L_H &= -\frac{e^2 H_0^4}{2}\left(\mu H'^2 + 2\frac{K^2 H^2}{x^2}\right)
\end{aligned}
\tag{41}
$$

Then, by analytic continuation, the Euclidean action takes the form

$$S_E = -\int_0^\beta d\tau_E \int_V d^3x \sqrt{g}\,(L_{YM} + L_H) \tag{42}$$

3.2   From $\mathbf{S}^2$ to $\mathbb{R}^2$

As stated in the introduction, our aim is to calculate the thermodynamics of the strongly interacting field theory defined on the boundary ∂M using the dual classical description of the gravity system governed by the action (1) in the 4 dimensional manifold M; now, according to (16), $\partial M = S^2 \times S^1$. From the condensed matter viewpoint, it is more relevant to study the finite temperature field theory defined on the plane $\mathbb{R}^2$ and not on the sphere S^2. To this end, one can follow the approach in [17] and proceed to a change coordinates $(t, x, \theta, \varphi) \rightarrow (\tau, y, x^1, x^2)$ or, equivalently, $(t, x, \theta, \varphi) \rightarrow (\tau, y, \rho, \varphi)$,

$$
\begin{aligned}
\tau &= \frac{R\,t}{\gamma_0} \\
y &= \frac{\gamma_0}{R}\,x
\end{aligned}
$$

8

$$x^1 = 2R\tan\frac{\theta}{2}\cos\varphi = \rho\cos\varphi$$
$$x^2 = 2R\tan\frac{\theta}{2}\sin\varphi = \rho\sin\varphi \tag{43}$$

together with the field redefinitions

$$\begin{aligned}
\tilde{A}(y) &= A(x) \\
\tilde{\xi}(y) &= R^{-2}\,\xi(x) \\
\tilde{H}(y) &= H(x) \\
\tilde{K}(y) &= R^{-1}\,K(x) \\
\tilde{J}(y) &= R^{-1}\,J(x)
\end{aligned} \tag{44}$$

where R was introduced in (17). After this change, ansatz (12) becomes

$$\begin{aligned}
g &= -\tilde{f}_R(y)\,\tilde{A}(y)^2\,d^2\tau + L^2\,\frac{d^2y}{\tilde{f}_R(y)} + L^2\,y^2\frac{d^2\vec{x}}{(1+\frac{\rho^2}{4R^2})^2} \\
\vec{A} &= d\tau\,e\,H_0\,\tilde{J}(y)\left(\frac{4\,R\,\rho}{\rho^2+4\,R^2}\,\breve{e}_\rho + \frac{4\,R^2-\rho^2}{\rho^2+4\,R^2}\,\breve{e}_3\right) \\
&\quad + \frac{4\,R^2}{\rho^2+4\,R^2}\left(\frac{1}{R}-\tilde{K}(y)\right)\left[\left(\frac{4\,R^2-\rho^2}{\rho^2+4\,R^2}\,\breve{e}_\rho - \frac{4\,R\,\rho}{\rho^2+4\,R^2}\,\breve{e}_3\right)\rho\,d\varphi \right. \\
&\quad \left. - \breve{e}_\varphi\,d\rho\right] \\
\vec{H} &= H_0\,\tilde{H}(y)\left(\frac{4\,R\,\rho}{\rho^2+4\,R^2}\,\breve{e}_\rho + \frac{4\,R^2-\rho^2}{\rho^2+4\,R^2}\,\breve{e}_3\right)
\end{aligned} \tag{45}$$

where
$$\tilde{f}_R(y) = f_R(y) + \tilde{\xi}(y) \qquad , \qquad f_R(y) \equiv \frac{1}{R^2} + y^2 - \frac{1}{y} \tag{46}$$

and the equations of motion for the Einstein-Yang Mills-Higgs system, eqs.(13) take the form,

$$\left(y\,\tilde{\xi}(y)\right)' = -\kappa^2\,H_0^{\,2}\left(\tilde{f}_R(y)\,\tilde{V}_1 + \tilde{V}_2 + \frac{y^2}{2}\,\frac{\tilde{J}'(y)^2}{\tilde{A}(y)^2} + \frac{\tilde{J}(y)^2\,\tilde{K}(y)^2}{\tilde{f}_R(y)\,\tilde{A}(y)^2}\right)$$

$$y\,\tilde{A}'(y) = \kappa^2\,H_0^{\,2}\left(\tilde{V}_1 + \frac{\tilde{J}(y)^2\,\tilde{K}(y)^2}{\tilde{f}_R(y)^2\,\tilde{A}(y)^2}\right)\tilde{A}(y) \tag{47}$$

$$\left(\tilde{f}_R(y)\,\tilde{A}(y)\,\tilde{K}'(y)\right)' = \tilde{A}(y)\,\tilde{K}(y)\left(\frac{\tilde{K}(y)^2-\frac{1}{R^2}}{y^2} + \frac{1}{\gamma_0^{\,2}}\,\tilde{H}(y)^2\right)$$

9

$$\left(y^2\, \tilde{f}_R(y)\, \tilde{A}(y)\, \tilde{H}'(y) \right)' = 2\tilde{A}(y)\, \tilde{H}(y)\, \tilde{K}(y)^2 \qquad \left. \begin{array}{c} -\dfrac{1}{\gamma_0{}^2}\, \dfrac{\tilde{J}(y)^2}{\tilde{f}(y)\, \tilde{A}(y)^2} \end{array} \right)$$

$$\tilde{f}_R(y)\, \tilde{A}(y)\, \left(\frac{y^2\, \tilde{J}'(y)}{\tilde{A}(y)} \right)' = 2\, \tilde{J}(y)\, \tilde{K}(y)^2 \tag{48}$$

where

$$\begin{aligned}
\tilde{V}_1 &= \gamma_0{}^2\, \tilde{K}'(y)^2 + \frac{y^2}{2}\, \tilde{H}'(y)^2 \\
\tilde{V}_2 &= \gamma_0{}^2\, \frac{(\tilde{K}(y)^2 - \frac{1}{R^2})^2}{2\, y^2}
\end{aligned} \tag{49}$$

As explained above, our aim is to consider the case in which the gravitational equations decouple from the matter ($\kappa^2 \to 0$) leading to a background metric of the form (16). The relevant equations of motion for the gauge and Higgs field in the $R \to \infty$ limit are then

$$\left(\tilde{f}_\infty(y)\, \tilde{K}'(y) \right)' = \tilde{K}(y) \left(\frac{\tilde{K}(y)^2}{y^2} + \frac{1}{\gamma_0{}^2}\, \tilde{H}(y)^2 - \frac{1}{\gamma_0{}^2}\, \frac{\tilde{J}(y)^2}{\tilde{f}_\infty(y)} \right) \tag{50}$$

$$\left(y^2\, \tilde{f}_\infty(y)\, \tilde{H}'(y) \right)' = 2\, \tilde{H}(y)\, \tilde{K}(y)^2 \tag{51}$$

$$\tilde{f}_\infty(y) \left(y^2\, \tilde{J}'(y) \right)' = 2\, \tilde{J}(y)\, \tilde{K}(y)^2 \tag{52}$$

$$\tilde{f}_\infty(y) = y^2 - \frac{1}{y}$$

Let us end this section by noting that the black hole temperature resulting from rescaling (21) and taking the $R \to \infty$ limit takes the form

$$\hat{T} \equiv \frac{T}{e\, H_0} = \frac{3\gamma_0}{4\pi} \tag{53}$$

4 Holographic correspondence

According to the AdS/CFT correspondence [1]-[3], properties of the dual 3 dimensional field theory defined on the boundary can be read from the behavior of the solution of the system in the bulk. In this approach temperature $\hat{T}$ in (53) corresponds to the temperature of the $d = 3$ system.

10

Let us first consider the vacuum expectation value in the $d = 3$ field theory for the scalar operator $\mathcal{O}_H$, dual to the field H defined on the bulk. It follows from the identification $\langle \mathcal{O}_H \rangle \sim H_1$ with H_1 defined in eq.(27).

The asymptotic behavior of the A_0 component of the gauge field allows one to identify the chemical potential μ and the charge density ρ of the 3 dimensional system. Indeed, given the ansatz for the A_0 component of the gauge field, eq.(12), the $U(1)$ electric field as defined in (34) reads

$$E_i = \mathcal{F}_{i0}^{U(1)} = \frac{H^a(x)}{H_0} F_{i0}^a = e H_0 H(x) \partial_i J(x) \tag{54}$$

(the $\vec{H} \cdot \vec{A}_0 \wedge \vec{A}_i$ contribution coming from the field strength commutator vanishes because $\vec{H} \parallel \vec{A}_0$).

Since $\lim_{x \to \infty} H(x) = 1$, the electric field at infinity is given by

$$\lim_{x \to \infty} E_i = e H_0 \lim_{x \to \infty} \partial_i J(x) \tag{55}$$

We have now to expand $J(x)$ for large x and look for the $1/x$ term which is the one leading to a nontrivial electric flux (i.e., leading to the dyon charge). The corresponding coefficient defines the charge density:

$$\rho = -\tilde{J}_1 \tag{56}$$

Here and in what follow we use the coefficients of the asymptotic expansions defined in $\mathbb{R}^2 \times S^1$. Following [18] we shall define a normal component ρ_n by analyzing the expansion for ρ at the horizon using eq.(26),

$$\rho_n = \tilde{j}_h \tag{57}$$

and a superconducting charge density in the form

$$\rho_s = rho - \rho_n \tag{58}$$

The constant value $\tilde{J}_0$ that A_τ takes at the boundary at infinity is related to the chemical potential through the formula

$$\mu = e H_0 \tilde{J}_0 \tag{59}$$

11

5 Numerical Analysis

We shall restrict the numerical calculations to the case in which the boundary is $S^1 \times \mathbb{R}^2$. We shall solve system (50)-(52) using the relaxation method [20] which determines the solution starting from an initial guess and improving it iteratively. The natural initial guess is the Prasad-Sommerfield solution [21] in flat space. Following this procedure, we have found regular solutions for different values of γ_0 i.e. of the temperature $\hat{T}$.

Figures 1 and 2 show a representative dyon solution that exists starting at the horizon. One can see that the scalar rapidly attains its symmetry breaking constant value while the magnetic and electric fields (associated to K and J respectively) concentrate in a spherical shell starting at the horizon. Figure 3 shows a remarkable property of the coupled system of non linear differential equations, namely the existence of several solutions satisfying the appropriate boundary conditions, each one corresponding to a different value of charge density ρ. This phenomenon was already encountered in [8] for the case of a $U(1)$ gauge field coupled to a complex scalar in an AdS$_4$-Schwarzschild background. As it also happens in the $U(1)$ case, we have found that for increasing values of ρ the solutions are distinguished by an increasing number of nodes n. Now, since evaluation of the free energy shows that it increases with the number of nodes (see below), we conclude that $n > 1$ solutions are thermodynamically unfavorable and therefore we shall concentrate hecenforth on the zero node solution.
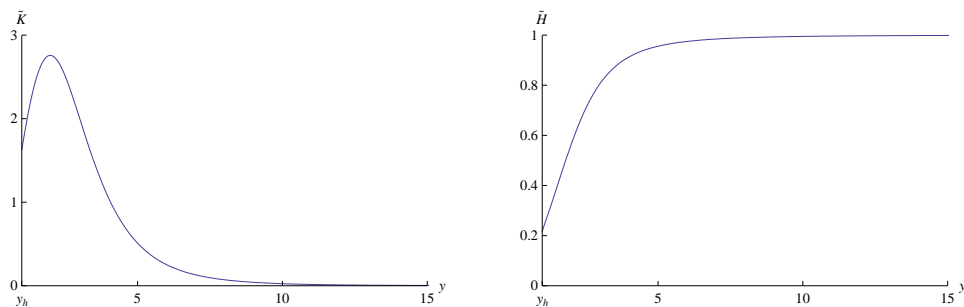

Figure 1: The solution for $\tilde{K}$ and $\tilde{H}$ as a function of y or $\tilde{J}_0 = 6$ and $\gamma_0 = 0.2$ ($\hat{T} = 3\gamma_0/(4\pi)$). The solution exists starting at the horizon $y = y_h = 1$.
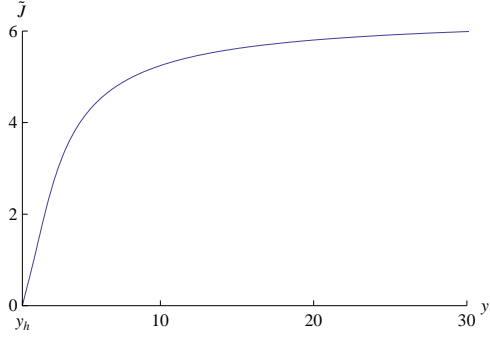
Figure 2: The solution for $\tilde{J}$ as a function of y for $\tilde{J}_0 = 6$ and $\gamma_0 = 0.2$ $(\hat{T} = 3\gamma_0/(4\pi))$.

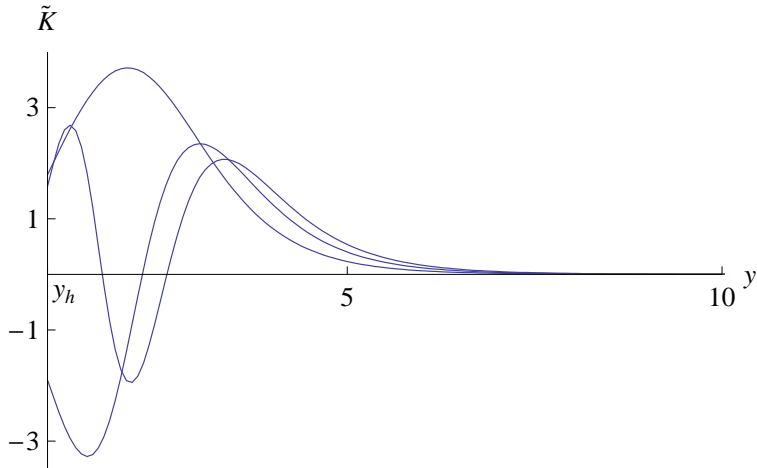
Figure 3: Different solutions for $\tilde{K}$ corresponding to different values of the charge density ρ. For increasing values of ρ the solutions are distinguished by the increasing number of nodes.

The behavior of the condensate is shown in figure 4 where the asymptotic coefficient $H_1 \sim \langle O_H \rangle$ in the asymptotic expansion (27) of the Higgs field is plotted as a function of the temperature (normalized in units of eH_0). In figures 5 and 6 we plot the coefficient $\tilde{K}_1$ in the asymptotic expansion of function $\tilde{K}$, while figure 7 shows the fraction ρ_s/ρ, as defined in eqs.(56)-(58), related to the behavior of the electric field at the horizon and at infinity. From these figures we conclude that a finite temperature continuous symmetry breaking transition takes place so that the system condenses below a critical

13

temperature $\hat{T}_c$. By fitting the curves we see that near T_c one has a typical second order phase transition with power behavior of the form

$$
\begin{aligned}
\tilde{H}_1 &\propto (\hat{T}_c - \hat{T}) \\
\tilde{K}_1 &\propto (\hat{T}_c - \hat{T})^{1/2} \qquad \text{as } \hat{T} \to \hat{T}_c \\
\frac{\rho_s}{\rho} &\propto (\hat{T}_c - \hat{T})
\end{aligned}
\tag{60}
$$

One can see in figure 4 that at low temperature $\hat{H}_1$ appears to diverge. However, one should take into account that when the condensate becomes very large, the backreaction on the metric can no longer be neglected as we did by choosing a fixed Schwarzschild-AdS background metric.
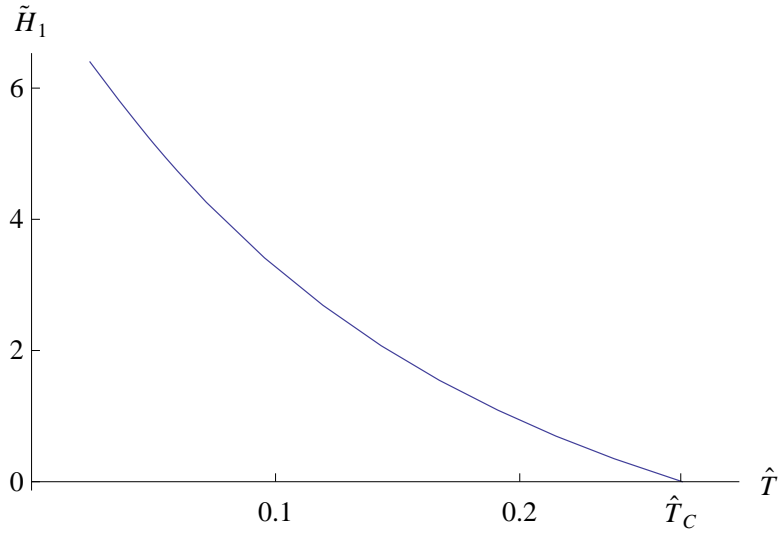

Figure 4: The $\tilde{H}_1$ coefficient in the Higgs field asymptotic expansion as a function of the temperature. We have taken $\tilde{J}_0 = 6$ and for this value the critical temperature is $\hat{T}_c = 0.26649....$
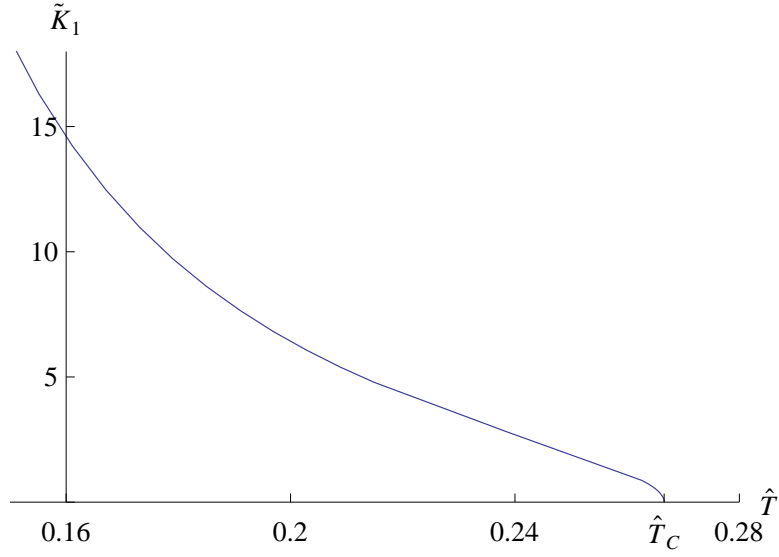
14

Figure 5: The $\tilde{K}_1$ coefficient in the in the gauge field asymptotic expansion as a function of the temperature for $\tilde{J}_0 = 6$.
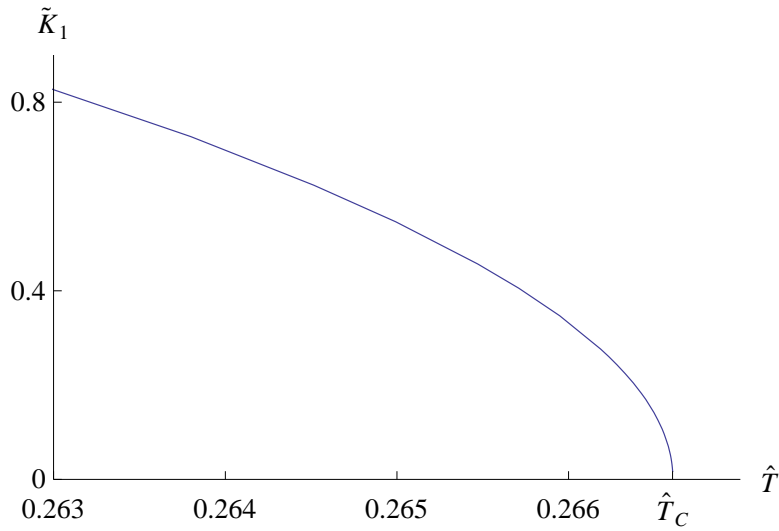

Figure 6: A zoomed view of $\tilde{K}_1$ near the critical point. One can see that its behavior near $T = T_c$ corresponds to $\sqrt{\hat{T}_c - \hat{T}}$.
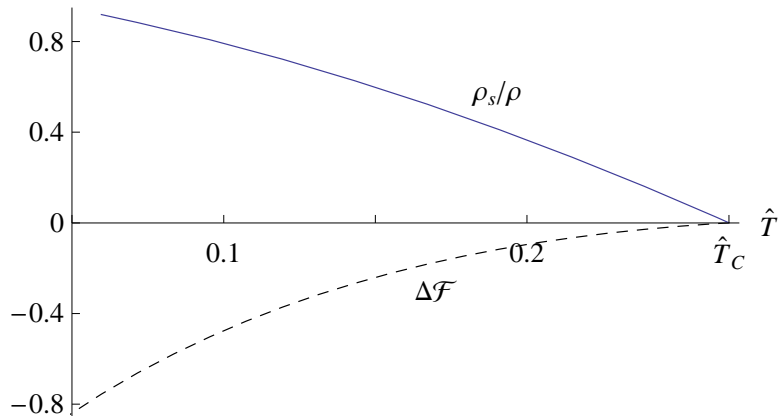
15

Figure 7: The solid line represents the fraction ρ_s/ρ of the superconducting and the total charge densities. The dashed line the free energy difference between the condensed and the uncondensed phases. At $\hat{T} = \hat{T}_c$, ρ_s/ρ vanishes linearly.

The free energy $\mathcal{F}$ is given by its on-shell action,

$$\mathcal{F} = TS_E|_{on\ shell} \tag{61}$$

Using the Euclidean action as given in eq.(42) we have numerically computed $\mathcal{F}$ for the dyon solution and compared it with the free energy in the uncondensed phase which corresponds to the solution $\tilde{H}(y) = 1, \tilde{K}(y) = 0, \tilde{J}(y) = \tilde{J}_0(1 - 1/y)$. We plot in figure 7 the free energy density difference $\Delta\mathcal{F}$, normalized in units of $e^2 H_0^4 L/2$, which can be seen to be continuous at $\hat{T} = \hat{T}_c$.

6 Summary and discussion

We have applied the AdS/CFT correspondence to describe the strong coupling regime of a $2 + 1$ quantum field theory which undergoes a phase transition exhibiting the condensation of a composite charge operator below a critical temperature T_c. The dual gravity theory consists of an $SU(2)$ Yang-Mills-Higgs model in a Schwarzschild-AdS$_4$ black hole with temperature T.

Breaking the gauge symmetry down to $U(1)$ we started by constructing a spherically symmetric soliton solution in the bulk, having both magnetic and electric charge. By choosing appropriate conditions at the horizon and

16

at infinity, we found a family of dyon solutions leading to the existence of a condensate, below a critical temperature, in the dual field theory on the boundary which, after an appropriate scaling, becomes $\mathbb{R}^2 \times S^1$.

By calculating the free energy $\mathcal{F}$ we have seen that the condensed phase is thermodynamically favored and we were able to determine which one of the dyon family solutions had the lowest $\mathcal{F}$-value. We have also calculated the power behavior of the condensate, typical of a second order phase transition.

There are many issues that we have not discussed in our work and that deserve a thorough study. To begin with, one should consider the applicability of our results to describe the behavior of an actual condensed matter system. Also, since we have found on the gravity side classical solutions with quantized magnetic flux, we expect that the issue of confinement through the appearence of chromoelectric flux tubes in the dual strongly coupled QFT could take place. In this respect, it should be of interest to consider monopole solutions in Schwarzschild-AdS$_5$ black hole backgrounds so that the dual theory is defined in three spatial dimensions. We hope to discuss theses issues in a future publication.

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